Non-perturbative Studies on Conformal theories and Strings

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Abstract

In this dissertation we discuss about some progress on understanding in super-conformal field theories and higher-spin theory.

We first study the basic properties of super-conformal symmetry in field theory and their implications on physical quantity. We review the AdS/CFT correspondence briefly and discuss the strong/weak nature.
Main part consist of three part. In terms of AdS/CFT correspondence, we study the anomalous dimension of operators in $\mathcal{N} = 4$ SYM with large spin and R-charge in the symmetric representaion. Explicit calculation of energy spectrum of D-branes in terms of spin and charge is performed.
Secondly, we study the infra-red finiteness of scattering amplitude in ABJM theory. In terms of Kinishita-Lee-Nauenberg theorem, we calculated soft radiations and check the cancellations of leading order divergence between loop corrections and soft radiations.
Finally asymptotic symmetry in super-symmetric higher-spin theory in AdS_3. We found that $shs^E(N|2,R)$ symmetry enhanced to the Super-W_∞ algebra.

keywords : Super-conformal theory, AdS/CFT, Anomalous dimension, Infra-red divergence, Asymptotic symmetry

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Introductions

A quantum field theory has been a successful framework of describing nature of fundamental forces in physics. After establishing the Quantum Electro-dynamics(QED) which describes electro-magnetic interactions between elementary particles, Other two fundamental forces - Weak and Strong interactions - unified in terms of Non-abelian Yang-Mills theory. These three fundamental forces are summarized in the theory called Standard Model and every experimental tests and discovery of Higgs particles support its validity.

Meanwhile, there are still remained problems in theoretical physics. Gravity theory which is the one of four fundamental forces in nature doesn't have any quantum mechanical descriptions yet. Any attempt to include gravity forces in quantum field theory framework has been unsuccessful. String theory is the one of the candidate for a consistent theory of quantum gravity.
Recently, duality between quantum field theory and gravity found in the string theory framework. It relates gauge theory with super-conformal symmetry and gravity theory in ten-dimensional curved space. Comprehensive studies has been made to understand the nature of each theory. Also explicit calculations are performed to check the duality called AdS/CFT correspondence. Super-conformal symmetry plays important role in these works.

Another issue is that because of confinement nature of strong forces, it is not easy to study QCD theory quantitatively. Usual perturbative expansion is not possible, so strong interaction phenomena in hadron energy scales has not solved yet. Super-conformal field theory which is the gauge theory with additional symmetry can be a simple toy model to overcome this difficulty. For example $\mathcal{N} = 4$ Super Yang-Mills theory has similar structure to QCD. Although theory is not same exactly, many aspect of QCD can be obtained from this super-conformal theory.

Along this line, we discuss the super-conformal field theory in this thesis. We discuss the general feature of super-conformal symmetry and AdS/CFT duality, we present three independent progress in super-conformal theories and higher-spin theory.
In terms of AdS/CFT correspondence, we study the anomalous dimension of operators in gauge theory. We focused on operators in symmetric representation with large spin and R-charge. Using duality, we calculated the scaling dimension by considering energy spectrum of spinning branes in $AdS_5 \times S^5$ space. We conclude that logarithmic scaling which was found in perturbative computations still holds in non-perturbative regions.
Next topic is the infra-red finiteness of scattering amplitude in three dimensional super-conformal theory. It is Chern-Simons-matter theory with gauge group $U(N) \times U(N)$. We review the general feature of IR divergences in terms of Kinishita-Lee-Nauenberg theorem. Then we calculated soft radiations explicitly and check the cancellations of leading order divergence between 2-loop corrections and soft radiations in ABJM theory.
Finally we study the asymptotic symmetry in super-symmetric higher-spin theory in AdS_3. Asymptotic symmetry is the coordinate transformation which preserves asymptotic boundary condition. We review the result of Brown-Henneaux original derivation in pure gravity theory. And we extend to the super-symmetric theory including higher-spin filds. We found asymptotic symmetry of super-symmetric higher-spin theory is Super-W_∞ algebra.

This thesis is organized as follows. In section 2, we briefly discuss the general feature of super-conformal symmetry. we give the definition of conformal symmetry and study its implications on field theory. We extend it to the super-conformal symmetry and give two examples of super-conformal theory in four, three dimensions. In section 3, we derive the AdS/CFT correspondence. Following the Maldacena's original argument, we study the strong/weak nature and AdS/CFT dictionary. In section 4, we focus on calculations on anomalous dimension. We introduce the BMN limit and correspondence between operators and spinning string(or branes). Explicit calculations of energy spectrum are performed. In section 5, we introduce the properties of scattering amplitude and structure of IR divergence in loop corrections in ABJM theory. From this we compute the IR divergence in soft emissions and verify the cancellations in the leading order divergence. In section 6, we study the asymptotic symmetry of higher-spin theory. We conclude the thesis by summarizing the results in section 7.

Chapter 2

(Super-)conformal field theory

In this section, we will review about basic properties of conformal field theories. Conformal symmetry plays important role in quantum field theory- for example, in the description of phase transitions in two-dimensional system and in string theory which is conformal theory of two-dimensional world-sheets. It also takes important role in the subject of AdS/CFT correspondence [15] or gauge/gravity duality which is the one of the main theme in this thesis.

Conformal symmetries are global coordinate transformation preserving angle of two vectors in spacetime. These transformations are represented by conformal group which is isomorphic to rotational group in higher dimension. The group structure is different according to the dimension of spacetime. For example in two dimension, it has infinite number of generators while there are only finite generators in other dimensions. In four dimensional theory which are relevant for particle physics, it has 15 generators.

Quantum field theory which have conformal symmetry can be roughly understood as it has no particular energy scale. Classically, this is achieved by writing the action of the theory with only dimensionless parameters. Classical field theory with dimensionless parameter like massless ϕ^4 theory or massless Quantum chromodynamics can be conformal theory. But in quantum theory, those conformal symmetries of lagrangian are broken by quantum corrections. It is because of ultraviolet divergence which introduces some energy scale and running coupling constant by renormalization procedure. To maintain the conformal symmetry in quantum theory, we need more constraints. With some other symmetry like super-symmetry, we can construct theories which conformal symmetries are preserved even at quantum level. In this section we will review two examples of those conformal theories - $\mathcal{N} = 4$ super Yang-Mills theory and ABJM theory. Each are in four, three dimension. These two theories are also important in the studies of AdS/CFT duality.

2.1 (Super-)conformal symmetry

To study the conformal field theories, we first review about conformal symmetry, their algebra structure and field representations. These are naturally extended to supersymmetric version which forms much larger group - super conformal group.

2.1.1 Definition of conformal transformaions

Conformal transformation are defined in the following way. Consider d dimensional spacetime with metric $g_{\mu\nu}$. Under coordinate transformation, conformal transformation leaves the metric up to a local scaling.

$$x^\mu \to x'^\mu, \qquad g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x) \tag{2.1}$$

Geometrical meaning of above expressions is that angle of two vector are preserved. We can easily see that in special case $\Lambda(x) = 1$, it is nothing but the usual Poincaré transformations which preserve the angles trivially. So conformal group has Poincaré group as a subgroup. We can find the other transformations by considering infinitesimal transformation.

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x) \tag{2.2}$$

The metric changes as $g'_{\mu\nu} = g_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu)$. So definition of conformal transformation requires

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = F(x) g_{\mu\nu} \tag{2.3}$$

where $F(x)$ is arbitrary function related to $\Lambda(x)$. By adding another derivative, taking permutations of indices and taking linear combinations of those equations, we can obtain one of the constraint equation

$$2\partial^2\epsilon_\mu = (2-d)\partial_\mu F \tag{2.4}$$

We can see that dimension 2 is special in the above constraint equation. In d=2, any holomorphic functions can be solutions of generator , so it has infinite number of symmetries. These properties are important in string theory, but in this section we will focus on dimension other than 2. By deriving several conditions of ϵ_μ from above expressions, we can solve the most general solution of ϵ_μ in $d > 2$ dimension

$$\epsilon_\mu = a_\mu + m_{\mu\nu}x^\nu + \lambda x_\mu + 2(b_\nu x^\nu)x_\mu - b_\mu x^2 \tag{2.5}$$

There are four kinds of parameters each represent different transformations. Poincaré transformation which are subgroup of conformal group is represented by parameters $a_\mu, m_{\mu\nu}$. Each corresponds translation and rotations. λ and b_μ correspond to dilation and special conformal transformation, respectively. In four dimensional spacetime where we are living in, conformal transformation has 15 generators - 4 translations, 6 rotations, 1 dilatation and 4 special conformal transforms.

The finite transformation can be obtained by exponentiate the above infinitesimal transformations.

$$\begin{aligned}
x'^\mu &= x^\mu + a^\mu && \text{-Translation} \\
x'^\mu &= \lambda x^\mu && \text{-Rotation} \\
x'^\mu &= M_{\mu\nu}x^\nu && \text{-Dilatation} \\
x'^\mu &= \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2x^2} && \text{-Special conformal transform}
\end{aligned} \tag{2.6}$$

For the comment, the last expression can be understood with the notion of inversion transformation $x^\mu \rightarrow \frac{x^\mu}{x^2}$. The complicate expression of special conformal transform can be made of inversion followed by translation and another inversion again. Inversion transformation is not connected to identity transform, so it can not be represented by infinitesimal transformations.

To study the algebra structure of conformal group, let us derive the generators of conformal transformation. We define the generators for the infinitesimal transformation as follows

$$x'^{\alpha} = (1 + ia^{\mu}P_{\mu} + im^{\mu\nu}M_{\mu\nu} + i\lambda D + ib^{\mu}K_{\mu})x^{\alpha} \tag{2.7}$$

Then from the solution of ϵ^{μ} in equation(2.5), we can easily obtain the expressions of generators of conformal group

$$\begin{aligned}
P_{\mu} &= -i\partial_{\mu} \\
M_{\mu\nu} &= i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}) \\
D &= -ix^{\mu}\partial_{\mu} \\
K_{\mu} &= -i(2x_{\mu}x^{\nu}\partial_{\nu} - x^{2}\partial_{\mu})
\end{aligned} \tag{2.8}$$

These generator obey the commutation relations which define the conformal group. The conformal algebra is written as follows [7]

$$\begin{aligned}
&[P_{\mu}, P_{\nu}] = 0\,, \qquad [M_{\mu\nu}, P_{\rho}] = i(\eta_{\nu\rho}P_{\mu} - \eta_{\mu\rho}P_{\nu})\,, \\
&[M_{\mu\nu}, M_{\rho\sigma}] = i(-\eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\sigma}M_{\mu\rho} + \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma})\,,
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
[D, P_{\mu}] &= iP_{\mu}\,, \qquad [K_{\rho}, M_{\mu\nu}] = i(\eta_{\rho\mu}K_{\nu} - \eta_{\rho\nu}K_{\mu})\,, \\
[D, K_{\mu}] &= -iK_{\mu}\,, \qquad [K_{\mu}, P_{\nu}] = 2i(\eta_{\mu\nu}D - M_{\mu\nu})\,.
\end{aligned} \tag{2.10}$$

Three relations in equation (2.9) are usual commutation relations of Poincaré algebra and other 4 relations in equation (2.10) are algebras of Dilatation and special conformal transformation.

To study further the structure of conformal group, it is better to combine the generators into another form J_{MN},

$$J_{\mu\nu} = M_{\mu\nu} \qquad J_{-1,0} = D \tag{2.11}$$

$$J_{-1,\mu} = \frac{1}{2}(P_{\mu} - K_{\mu}) \qquad J_{0,\mu} = \frac{1}{2}(P_{\mu} + K_{\mu}) \tag{2.12}$$

where $J_{MN} = -J_{NM}$ and $M, N \in -1, 0, 1, ...d$. So The commutation relations of these new generators can be summarized in simple form,

$$[J_{AB}, J_{CD}] = i(\eta_{AD}J_{BC} + \eta_{BC}J_{AD} - \eta_{AC}J_{BD} - \eta_{BD}J_{AC}) \tag{2.13}$$

with diagonal metric in $d+2$ dimension η_{AB} =diag(-1,1,1,..1,-1). Above commutation relations are algebra of group $SO(d,2)$. This shows that conformal group in d dimension is isomorphic to rotational group SO(d,2) in d+2 dimension. SO(d,2) has $\frac{1}{2}(d+2)(d+1)$ generators which are 15 generators in 4 dimensions. This isomorphism will be used in the study of gauge/gravity duality in the next chapter.

2.1.2 Field representations

So far we studied the definition of conformal symmetry and its action on coordinates. In field theory, we also have to define its action on fields so we can say the field theory is invariant under conformal symmetry.

Consider fundamental field $\phi_I(x)$ with conformal weight Δ and Lorentz index I. Acting of conformal generators to field gives

$$P^\mu \phi_I = \partial^\mu \phi_I \tag{2.14}$$

$$M^{\mu\nu}\phi_I = (x^\mu \partial^\nu - x^\nu \partial^\mu)\phi_I + (S^{\mu\nu})_I^J \phi_J \tag{2.15}$$

$$D^\mu \phi_I = x^\nu \partial^\nu \phi_I + \Delta\phi_I \tag{2.16}$$

$$K^{\mu\nu}\phi_I = (2x^\mu x^\nu \partial_\nu - x^2\partial^\mu)\phi_I + 2x^\mu \Delta\phi_I + 2x_\nu (S^{\mu\nu})_I^J \phi_J \tag{2.17}$$

where $S^{\mu\nu}$ is a spin generators. First 2 lines are familiar transformation of fields under Poincaré transformations and next 2 lines are new expressions under Dilatation and special conformal transformations.

In principle we can obtain the variation of fields under finite transformation. But for simplicity, we shall give the result for scalar field under finite conformal transformation. Under a conformal transformation $x \to x'$,

$$\phi(x) \to \phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/d} \phi(x) \tag{2.18}$$

The Jacobian factor is related with scale factor $\Lambda(x)$ as

$$\left|\frac{\partial x'}{\partial x}\right| = \Lambda(x)^{-d/2} \tag{2.19}$$

The field which transformed like above expressions called *primary* field.

Variation of primary fields under conformal transformation is encoded in conformal weight Δ. Usually, we can make the theory conformal if it has only dimensionless parameters and conformal weight is equal to the mass dimension of fields. For example, the usual massless ϕ^4 theory is invariant under conformal transformation if we choose the conformal weight $\Delta = 1$. But these conformal symmetry holds in only classical level and it start to be broken when we consider quantum corrections.

With the field representation of conformal symmetry we reviewed above, let us see the implication of conformal symmetries for correlation functions. In conformal theory, conformal covariance of primary fields give some restrictions on expression of correlation functions. We will see the implication on two, three, and four point functions respectively in the simple scalar fields case.

Two-point functions Consider two-point function of scalar primary operator ϕ_1, ϕ_2 with conformal weight Δ_1, Δ_2 respectively. Under conformal transformation, their two-point correlation function is transformed

$$\langle\phi_1(x_1), \phi_2(x_2)\rangle = \left|\frac{\partial x_1'}{\partial x_1}\right|^{\Delta_1/d} \left|\frac{\partial x_2'}{\partial x_2}\right|^{\Delta_2/d} \langle\phi_1(x_1'), \phi_2(x_2')\rangle \tag{2.20}$$

First if we specialize the transformation to a dilatation $x \to \lambda x$,

$$\langle\phi_1(x_1), \phi_2(x_2)\rangle = \lambda^{(\Delta_1+\Delta_2)}\langle\phi_1(\lambda x_1), \phi_2(\lambda x_2)\rangle \tag{2.21}$$

Second, from translation and rotation invariance, we can require

$$\langle\phi_1(x_1), \phi_2(x_2)\rangle = f((x_1 - x_2)^2) \tag{2.22}$$

From above two conditions, we can see that

$$\langle \phi_1(x_1), \phi_2(x_2) \rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \tag{2.23}$$

with unconstrained coefficient C_{12}. The last transformation remained is special conformal transformation. Under special conformal transformation, distance between two points is transformed as,

$$|x_1 - x_2| \to \frac{|x_1 - x_2|}{(1 - 2b \cdot x_1 + b^2 x_1^2)^{1/2}(1 - 2b \cdot x_1 + b^2 x_1^2)^{1/2}} \tag{2.24}$$

which implies,

$$\frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} = \frac{(\gamma_1\gamma_2)^{\frac{\Delta_1+\Delta_2}{2}}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}} \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \tag{2.25}$$

where $\gamma_i = (1 - 2b \cdot x_i + b^2 x_i^2)$. So we can arrive the result $\Delta_1 = \Delta_2$ that only fields which have same conformal dimension is correlated. In conclusion, the functional form of two point function of primary fields is completely fixed up to overall constant.

$$\langle \phi_1(x_1), \phi_2(x_2) \rangle = \frac{C_{12}}{|x_1 - x_2|^{2\Delta_1}} \tag{2.26}$$

Three-point functions A similar analysis can be performed in three-point functions. Just as previous case, a translation and rotation transformation make the functional form as function of Lorentz invariant distances $x_{ij}^2 = (x_i - x_j)^2$. And covariance under dilatation transform implies

$$\langle \phi_1(x_1), \phi_2(x_2), \phi_3(x_3) \rangle = \frac{C_{123}}{x_{12}^a x_{23}^b x_{13}^c} \tag{2.27}$$

with coefficient a,b,c such that

$$a + b + c = \Delta_1 + \Delta_2 + \Delta_3 \tag{2.28}$$

Under special conformal transformation, eq (2.27) becomes

$$\frac{(\gamma_1\gamma_2)^{a/2}(\gamma_2\gamma_3)^{b/2}(\gamma_1\gamma_3)^{c/2}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}\gamma_3^{\Delta_3}} \frac{C_{123}}{x_{12}^a x_{23}^b x_{13}^c} \tag{2.29}$$

which leads to

$$a + c = 2\Delta_1, \quad a + b = 2\Delta_2, \quad b + c = 2\Delta_3 \tag{2.30}$$

The solution of above equations are

$$a = \Delta_1 + \Delta_2 - \Delta_3 \tag{2.31}$$

$$b = \Delta_2 + \Delta_3 - \Delta_1 \tag{2.32}$$

$$c = \Delta_3 + \Delta_1 - \Delta_2 \tag{2.33}$$

So we arrived to conclusion that conformal symmetry also constrain the three-point function completely up to coefficient as

$$\langle \phi_1(x_1), \phi_2(x_2), \phi_3(x_3) \rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_2+\Delta_3-\Delta_1} x_{13}^{\Delta_3+\Delta_1-\Delta_2}} \tag{2.34}$$

Four-point functions So far conformal symmetry determines the functional form of two, three point function completely up to coefficient. But from four point correlation function, it stops. This property comes from the possibility of construction of conformal invariant cross-ratios,

$$\frac{x_{ij}^2 x_{k\ell}^2}{x_{ik}^2 x_{j\ell}^2} \tag{2.35}$$

which gives two independent cross-ratios in four point case.

$$\frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \tag{2.36}$$

Any function of these two cross-ratios are conformal invariant and cannot be restricted further. So the general form of four-point correlation function can be

$$\langle \phi_1(x_1), \phi_2(x_2), \phi_3(x_3), \phi_4(x_4) \rangle = f\left(\frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \right) \prod x_{ij}^{\Delta/3 - \Delta_i \Delta_j} \tag{2.37}$$

with $\Delta = \sum \Delta_i$.

As we have seen so far, in conformal theory, there is restrictions on correlation functions. It is one aspect that conformal symmetry constrain the structure of the theory. We can also find restrictions on correlation function made of fields other than scalars. We will not list particular example here but one can see some result in [7]

2.1.3 Super-symmetric extension

One can extend the ordinary Poincaré group by including fermionic generators which satisfies anti-commutation relations

$$\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\} = -2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \tag{2.38}$$

The fermionic generators $Q_\alpha, \overline{Q}_{\dot{\alpha}}$ is transformed under $(\frac{1}{2}, 0)$,$(0, \frac{1}{2})$ representation respectively and have roles of making connections between bosonic and fermionic fields. With these fermionic generators, new algebra between bosonic and fermionic generators arises and makes the larger group, Super-Poincaré group.

For the application in the following chapters, let us concentrate on $\mathcal{N} = 4$ super-conformal group. This is the super-conformal symmetry of gauge theory in four dimension which is called $\mathcal{N} = 4$ Super Yang-Mills theory. This theory plays the role of basic playground in the next two chapters. This theory is important because it can be considered as simplified version of Quantum chromo-dynamics which in not easy to solve directly. And it is also important as one of the concrete example of AdS/CFT duality which is the main theme in the next chapter.

In the $\mathcal{N} = 4$ super-conformal group, there are sixteen super-charges and sixteen super-conformal charges.

$$Q^a_\alpha, \qquad \overline{Q}_{\dot{\alpha}a}, \qquad S_{\alpha a}, \qquad \overline{S}^a_{\dot{\alpha}}, \qquad (\alpha, \dot{\alpha} = 1, 2, \quad a = 1, 2, 3, 4) \tag{2.39}$$

The $\mathcal{N} = 4$ super-conformal algebra which combine previous conformal algebra and super-symmetry algebra is written in the followings. [8]

$$[K^\mu, Q^a_\alpha] = (\sigma^\mu)_{\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma}\bar S^a_{\dot\gamma}\,, \qquad [K^\mu, \bar Q_{\dot\alpha a}] = (\sigma^\mu)_{\beta\dot\alpha}\epsilon^{\beta\gamma}S_{\gamma a}\,, \tag{2.40}$$

$$[P^\mu, S_{\alpha a}] = (\sigma^\mu)_{\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma}\bar Q_{\dot\gamma a}\,, \qquad [P^\mu, \bar S^a_{\dot\alpha}] = (\sigma^\mu)_{\beta\dot\alpha}\epsilon^{\beta\gamma}Q^a_\gamma\,, \tag{2.41}$$

$$[L^{\mu\nu}, Q^a_\alpha] = -i\,(\sigma^{\mu\nu})_{\alpha\beta}\,\epsilon^{\beta\gamma}Q^a_\gamma\,, \qquad [L^{\mu\nu}, \bar Q_{\dot\alpha a}] = -i\,(\bar\sigma^{\mu\nu})_{\dot\alpha\dot\beta}\,\epsilon^{\dot\beta\dot\gamma}\bar Q_{\dot\gamma a}\,, \tag{2.42}$$

$$[L^{\mu\nu}, S_{\alpha a}] = -i\,(\sigma^{\mu\nu})_{\alpha\beta}\,\epsilon^{\beta\gamma}S_{\gamma a}\,, \qquad [L^{\mu\nu}, \bar S^a_{\dot\alpha}] = -i\,(\bar\sigma^{\mu\nu})_{\dot\alpha\dot\beta}\,\epsilon^{\dot\beta\dot\gamma}\bar S^a_{\dot\gamma}\,, \tag{2.43}$$

$$[D, Q^a_\alpha] = -\tfrac{i}{2}\,Q^a_\alpha\,, \qquad [D, \bar Q_{\dot\alpha a}] = -\tfrac{i}{2}\,\bar Q_{\dot\alpha a}\,, \tag{2.44}$$

$$[D, S_{\alpha a}] = +\tfrac{i}{2}\,S_{\alpha a}\,, \qquad [D, \bar S^a_{\dot\alpha}] = +\tfrac{i}{2}\,\bar S^a_{\dot\alpha}\,, \tag{2.45}$$

$$\{Q^a_\alpha, \bar Q_{\dot\beta b}\} = (\sigma^\mu)_{\alpha\dot\beta}\delta^a{}_b P_\mu\,, \qquad \{S_{\alpha a}, \bar S^b_{\dot\beta}\} = (\sigma^\mu)_{\alpha\dot\beta}\delta_a{}^b K_\mu\,, \tag{2.46}$$

$$\{Q^a_\alpha, S_{\beta b}\} = (\sigma^{ij})^a{}_b\epsilon_{\alpha\beta}R_{ij} + i(\sigma^{\mu\nu})_{\alpha\beta}\delta^a{}_b L_{\mu\nu} - i\epsilon_{\alpha\beta}\delta^a{}_b D\,, \tag{2.47}$$

$$\{\bar Q_{\dot\alpha a}, \bar S^b_{\dot\beta}\} = -(\sigma^{ij})_a{}^b\epsilon_{\dot\alpha\dot\beta}R_{ij} + i(\bar\sigma^{\mu\nu})_{\dot\alpha\dot\beta}\delta_a{}^b L_{\mu\nu} - i\epsilon_{\dot\alpha\dot\beta}\delta_a{}^b D\,. \tag{2.48}$$

where the Pauli matrices are defines as

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad [\sigma^i, \sigma^j] = i\epsilon_{ijk}\sigma^k \tag{2.49}$$

$$\sigma^\mu = (-1, \sigma^i), \quad \overline\sigma^\mu = (-1, -\sigma^i), \quad \sigma^{\mu\nu} = \frac{1}{4}\sigma^{[\mu}\overline\sigma^{\nu]}, \quad \overline\sigma^{\mu\nu} = \frac{1}{4}\overline\sigma^{[\mu}\sigma^{\nu]} \tag{2.50}$$

All other commutation or anti-commutation relations are vanish. In the commutation relations between Q^a_α and S^b_β , new bosonic generators R_{ij} appear. These are $SO(6)$ R-symmetry generators which means it has extended super-symmetry $\mathcal{N} = 4$. This R-symmetry is used to label the local operators in terms of corresponding R charges.

The bosonic part of super-conformal symmetry are $SO(2,4) \approx SU(2,2)$ which is usual conformal symmetry and $SO(6) \approx SU(4)$ which is R-symmetry. These makes the full super-conformal group $SU(2,2|4)$. In matrix representation, this can be written schematically as

$$\begin{pmatrix} P_\mu, L_{\mu\nu}, D, K_\mu & Q^a_\alpha, \overline S^a_{\dot\alpha} \\ \overline Q_{\dot\alpha a}, S_{\alpha a} & R_{ij} \end{pmatrix} \tag{2.51}$$

Before finish this subsection, let us consider chiral primary operators which also called BPS operators. If some operator satisfy the relations

$$[Q^a_\alpha, \mathcal{O}] \;=\; 0 \qquad\qquad \text{for some } \alpha \text{ and a,} \tag{2.52}$$

$$[S_{\alpha a}, \mathcal{O}] \;=\; 0 \qquad\qquad \text{for all } \alpha \text{ and a} \tag{2.53}$$

we call this operator as chiral primary operator. According to the super-conformal algebra, we can see that there is nice property on conformal dimension of this operator. Consider scalar operator in zero point which is chiral primary,

$$[\{Q^a_\alpha, S_{\beta b}\}, \mathcal{O}(0)] = [(\sigma^{ij})^a{}_b \epsilon_{\alpha\beta} R_{ij} + i(\sigma^{\mu\nu})_{\alpha\beta}\delta^a{}_b L_{\mu\nu} - i\epsilon_{\alpha\beta}\delta^a{}_b D, \mathcal{O}(0)] \tag{2.54}$$

Because the definition of chiral primary operators, left hand side is zero. Then because $\mathcal{O}(0)$ is a scalar, $[L_{\mu\nu}, \mathcal{O}(0)] = 0$. So scaling dimension of chiral primary operator is directly determined from it R-symmetry charges. Because they do not receive quantum corrections, their scaling dimension is just the same as their classical values.

2.2 $\mathcal{N} = 4$ Super Yang-Mills theory in 4 dimension

So far we have been discussed about (super-)conformal symmetry of field theories. As a example of conformal theory which is still invariant at quantum level, let us introduce two maximally super-symmetric gauge theories. First one we will consider in this section is maximally super-symmetric Yang-Mills theory in four dimension which called $\mathcal{N} = 4$ Super Yang-Mills theory.

The natural way to derive $\mathcal{N} = 4$ Super Yang-Mills theory is starting from $\mathcal{N} = 1$ SYM in ten dimensions and taking dimensional reduction [6]. The Lagrangian of $\mathcal{N} = 1$ SYM in ten dimensions is

$$\mathcal{L} = Tr\left(-\frac{1}{4}F_{MN}F^{MN} + \frac{ig}{2}\overline{\Psi}\Gamma^N \mathcal{D}_N \Psi\right) \tag{2.55}$$

where the index M,N run from 0 to 9. The F_{MN} is field strength of ten-dimensional gauge field A_M and Γ_M,Ψ are Dirac matrices, Majorana-Weyl fermion respectively.

In dimensional reduction to four dimension, we split ten dimensional space into four dimensional Minkowski space and six dimensional Euclidean space. We write ten dimensional indices M,N into four dimensional indices μ and six dimensional one m. Then one can write the ten-dimensional Γ matrices in terms of four and six dimensional ones,

$$\Gamma_\mu = \gamma_\mu \otimes 1, \qquad \mu = 0,..,3 \tag{2.56}$$

$$\Gamma_{m+3} = \gamma_5 \otimes \widetilde{\Gamma}_m, \qquad m = 1,..,6 \tag{2.57}$$

where γ_μ and γ_5 are ordinary four dimensional gamma matrices and $\widetilde{\Gamma}_m$ are Dirac matrices in six dimensional Euclidean space,

$$\widetilde{\Gamma}_m = \begin{pmatrix} 0 & \widetilde{\sigma}_m \\ \widetilde{\sigma}_m^{-1} & 0 \end{pmatrix} \tag{2.58}$$

Now we require that fields depend only on four dimensions, i.e.

$$\partial^m A_N = 0 \tag{2.59}$$

$$\partial^m \Psi = 0 \qquad m = 4,5,..,9 \tag{2.60}$$

In four dimensional point of view, six dimensional component of gauge field behaves as scalars while other components are remained as vectors.

$$A_\mu = A_{M=\mu} \qquad \mu = 0,..,3 \tag{2.61}$$

$$\phi_m = A_{3+m} \qquad m = 1,..,6 \tag{2.62}$$

With these decompositions and reduction, we obtain four dimensional action which is $\mathcal{N} = 4$ SYM. Action for this in written as [7]

$$\mathcal{L} = Tr\left(-\frac{1}{4}F_{\mu\nu}^2 + i\lambda_i\sigma^\mu\mathcal{D}_\mu\overline{\lambda}^i + \frac{1}{2}(\mathcal{D}_\mu\phi)^2 + ig\lambda_i[\lambda_j,\phi^{ij}] + ig\overline{\lambda}^i[\overline{\lambda}^j,\phi_{ij}] + \frac{g^2}{4}[\phi_{ij},\phi_{k\ell}][\phi^{ij},\phi^{k\ell}]\right) \tag{2.63}$$

$\mathcal{N} = 4$ SYM is a Yang-Mills theory in four-dimension which is similar to QCD. Because of asymptotic freedom and confinement, it is not easy to analyze QCD quantitatively. By studying conformal theory in four-dimension, we can get many insight about the general behavior of QCD. This theory is also important as a concrete realization of AdS/CFT correspondence. We will study this aspect in the next chapter.

2.3 $\mathcal{N} = 6$ Chern-Simons matter theory in 3 dimension

Another example of super-conformal theory is $\mathcal{N} = 6$ Chern-Simons matter theory in 3 dimension which called ABJM theory [68]. It consists of two copies of Chern-Simons action which coupled to scalars and fermion fields. Its gauge group is $U(N) \times U(N)$. Action for ABJM theory is

$$\mathcal{S} = \mathcal{S}_{CS}(A_\mu) - \mathcal{S}_{CS}(\overline{A}_\mu) + \mathcal{S}_{matter} + \mathcal{S}_{int} \tag{2.64}$$

$$\mathcal{S}_{CS} = \int Tr\left(\frac{k}{4\pi} A \wedge dA + \frac{2i}{3} A \wedge A \wedge A\right) \tag{2.65}$$

$$\mathcal{S}_{matter} = \int Tr\left(\mathcal{D}_\mu \phi^A \mathcal{D}^\mu \overline{\phi}_A + i\psi_A \sigma^\mu \mathcal{D}_\mu \overline{\psi}^A\right) \tag{2.66}$$

$\mathcal{S}_{int}$ contains interactions between matters and gauge fields. Details of expression for this can be found in [68].

Because the action for gauge field is Chern-Simons theory, degrees of freedom of gauge fields are zero. They only participate as a interaction with matter fields. The representations of each fields under gauge group $U(N) \times U(N)$ is the followings.

$$A_\mu : (adj, 1) \qquad \overline{A}_\mu : (1, adj) \tag{2.67}$$

$$\phi^A : (N, \overline{N}; 4) \qquad \overline{\phi}_A : (\overline{N}, N, \overline{4}) \tag{2.68}$$

$$\psi_A : (N, \overline{N}; \overline{4}) \qquad \overline{\psi}^A : (\overline{N}, N, 4) \tag{2.69}$$

Each matter fields is in bi-fundamental representation under gauge group. To make a gauge invariant object, every fields and anti-fields come in alternate way. These properties constrains the structure of scattering amplitude. We will come to this point in the next section.

Like $\mathcal{N} = 4$ SYM theory, ABJM theory is also a example of AdS/CFT correspondence. In this AdS_4/CFT_3 case, dual gravity is ΠA string theory living in $AdS_4 \times \mathbb{CP}^3$ and ABJM theory is world-volume theory of M2-branes. To study further, let us review about AdS/CFT briefly.

Chapter 3

Gauge gravity duality

In this chapter we briefly review about the gauge-gravity duality. This has been one of the main theme in theoretical physics for last fifteen years. Gauge-gravity duality implies correspondence between two different theory. So studying the structure of each theories and reveal their correspondence relation will improves our understanding of quantum field theory and quantum gravity theory. Meanwhile because the physical observable in two different theories is related each other, gauge-gravity duality is important not only in pure theoretical point of view but also in practical point of view. If some physical quantity is hard to calculate in one theory, it could be obtained in a alternative way from a dual theory.

Gauge-gravity duality is also known as holographic theory because it relates gauge theory with gravity in higher dimension. There had been many argument about this duality [1, 2] and concrete example was found by J.Maldacena. We will review his derivation of AdS/CFT correspondence and briefly discuss about implication of it.

3.1 Large N expansion

Before we derive the AdS/CFT correspondence, let us review t'hooft large N planar limit. Consider Yang-Mills theory with gauge group $SU(N)$. Together with coupling parameter g_{YM}, we take N as a free parameter too. If we calculate the Feynman diagram in perturbative expansion, we can divide the diagrams according to the two parameters g_{YM}, N. Draw the Feynman diagram in the double line notation and consider it two-dimensional surfaces. Then we can see any Feynman diagram can be organized in terms of their topological nature. Generally, diagrams with vertices V, propagators E and loops F has dependence on parameters as

$$(g_{YM}^2 N)^{E-V} N^{F-E+V} \tag{3.1}$$

where $F - E + V = 2 - 2g$ with genus number g. Now, in the two parameter space g_{YM}, N we take the limit

$$N \to \infty, \qquad g_{YM}^2 N = \lambda = \text{fixed} \tag{3.2}$$

This is called **Large N limit** or **Planar limit** [5]. In planer limit, we organize it in the double expansion with $g_{YM}^2 N$ and $\frac{1}{N}$. We can easily check that $\frac{1}{N}$ expansion divide the diagram according to their topological property. The leading order in $\frac{1}{N}$ expansion can be seen a sphere and next sub-leading order becomes torus and so on. This implies double expansion of physical qunatity.

$$\mathcal{O}(\lambda, \frac{1}{N}) = \sum_{g=0} \left(\frac{1}{N}\right)^{2g-2} \sum_{n} \lambda^n \, \mathcal{O}_{g,n} \tag{3.3}$$

In the following chapters we will use this double expansion in large-N limit. In the leading order of N, we can concentrate on only planar diagrams. This makes the perturbative calculation simple.

3.2 AdS/CFT correspondence

Here we introduce the original derivation of AdS/CFT correspondence by J.Maldacena [15]. From the two different point of view of D-brane dynamics, we briefly show how correspondence arises. For detailed review, see [4].

Consider D3-branes in type IIB string theory in ten dimensional flat space time. We set the N D3-branes stacked together and choose a coordinate system as

$$x^0, x^1, x^2, x^3 : \text{coordinate of coincident D3-branes}, \qquad x^I = 0 \quad (I = 4, 5, ..9) \tag{3.4}$$

Now we analyze these D3-branes in two different point of view.

Low-energy dynamic of D-branes

In this setup, we need to consider open strings attached to D3-branes and closed strings living in ten dimensions. So this physical system can be divided into three part, brane dynamics including open strings, closed string propagating ten dimensional flat spacetime and interaction between brane and closed strings. Open string mode contains gauge field A_μ, fermionψ, scalar Φ and higher massive modes. Closed string also contains graviton $g_{\mu\nu}$, dilaton ϕ and two, four form fields $B_{\mu\nu}, C_{\mu\nu}, C_{\mu\nu\rho\sigma}$ and higher modes. Now consider low-energy and decoupling limit.

$$\alpha' \to 0, \qquad g_s = \text{fixed} \tag{3.5}$$

In this limit every massive excitation modes get away. Open string on D3-branes becomes $\mathcal{N} = 4$ SYM theory and closed string in the bulk becomes super-gravity. Moreover because

$$S_{int} \sim g_s \alpha'^2 \tag{3.6}$$

interaction between open and closed string switched off. Thus we have two decoupled system in this limit.

- $\mathcal{N} = 4$ Super Yang-Mills theory in 4 dimensions
- Free super-gravity in ten dimension

Black branes geometry

N coincident D3-branes can be seen in a geometric point of view. Like black-hole system, N d3-Dranes make the geometry curved. In this view point we can replace the D3-branes with curved geometry solution called black 3-brane geometry.

$$ds^2 = H^{-\frac{1}{2}}(r)dx_4^2 + H^{\frac{1}{2}}(r)(dr^2 + r^2 d\Omega_5^2) \tag{3.7}$$

with the harmonic function

$$H(r) = 1 + \frac{R^4}{r^4}, \qquad R^4 = 4\pi g_s \alpha'^2 N \tag{3.8}$$

In the same limit (3.5) we can see that two different region $r \ll R$, $r \gg R$ is decoupled. This can be understood by considering dilaton field going from $r \ll R$ to $r \gg R$ region. Due to the redshift, any finite energy cannot reach to $r \gg R$ region. In $r \ll R$ region, geometry becomes $AdS_5 \times S^5$

$$ds^2 \to \left(\frac{r^2}{R^2}dx_4^2 + \frac{R^2}{r^2}dr^2\right) + R^2 d\Omega_5^2 \tag{3.9}$$

and in $r \gg R$ region, geometry becomes flat space. So we have also two decoupled system in the limit.

- Type ΠB superstring theory in $AdS_5 \times S^5$
- Free super-gravity in ten dimension

Now by comparing the result of two different view point, we can arrive to the conclusion that two theories, $\mathcal{N} = 4$ Super Yang-Mills theory and Type PiB superstring theory in $AdS_5 \times S^5$ are same. This is the AdS_5/CFT_4 correspondence.

3.3 Implications of Duality

Let us study more about the duality and see what is the implication of duality we derived.

First we can compare the symmetry structure of both theory. In $\mathcal{N} = 4$ SYM theory, there are two kinds of bosonic symmetry

$$SO(2,4) : \text{conformal symmetry in 4d}, \qquad SO(6) : \text{R-symmetry} \tag{3.10}$$

These bosonic symmetries correspond isometry group of $AdS_5 \times S^5$ spacetime.

$$SO(2,4) : \text{isometry group of } AdS_5, \qquad SO(6) : \text{isometry of } S^5 \tag{3.11}$$

Beside this bosonic symmetry we can check the correspondence of fermionic generators and the full global symmetry $PSU(2,2|4)$.

Now let us see how the parameters of both theories are identified. According to the black p-brane solution, we can get the relation

$$4\pi g_s = g_{YM}^2, \qquad \frac{R^4}{\alpha'^2} = g_{YM}^2 N = \lambda \tag{3.12}$$

In large N planar limit(3.2), $\mathcal{N} = 4$ SYM theory has perturbative expansion with parameter is $g_{YM}^2 N = \lambda$. On the other hand, dual string theory has α' expansion which is in inverse relation with λ(3.12). If we keep the parpameter λ small, we can calculate perturbatively in field theory side, but cannot do it in gravity side, and vice versa. Because of this property, AdS/CFT duality is also called **strong/weak** duality.

This strong/weak nature of duality makes it difficult to test the duality. Because we cannot access to the both perturbative region, it is hard to calculate the physical quantity in both theory simultaneously. There has been many progress to overcome this difficulties. We will study about this in detail in the next chapter and calculate one physical quantity-anomalous dimension- explicitly to compare with dual theory calculations.

Part II

Studies on Super-Conformal field theories

Chapter 4

Anomalous dimensions of $\mathcal{N} = 4$ SYM

So far, we studied the basic properties of super-conformal theories and general argument about gauge gravity duality. After discovery of concrete example of holographic duality - the correspondence between $\mathcal{N} = 4$ Super Yang-Mills theory in 4 dimension and type IIB string theory in $AdS_5 \times S^5$, lots of explicit tests were delivered. One way to check the correspondence is computing physical quantities explicitly on both side and see if they agree each other. One of the object which is used to test the duality is anomalous dimension of composite operators in gauge theory. This is main theme of this chapter.

Let us see what kind of physical quantity in string side should be computed to compare with the anomalous dimension in gauge theory. According to the AdS/CFT dictionary, local operators in $\mathcal{N} = 4$ SYM correspond to string states. The AdS_5 space where the string states live in can be written in global coordinate system,

$$ds^2 = R^2(d\rho^2 - \cosh^2 \rho\, dt^2 + \sinh^2 \rho\, d\Omega_3) \tag{4.1}$$

with $\rho \in [0, \infty], t \in [-\infty, \infty]$. The gauge theory which is dual to string theory is supposed to live in the boundary space of AdS_5. This correspond to space located at $\rho \to \infty$. By conformal mapping, we can see that this boundary space is equal to

$\mathbb{R} \times S^3$,

$$\sinh\rho = \tan\alpha, \qquad \alpha \in [0, \frac{\pi}{2}] \tag{4.2}$$

$$ds^2 = \frac{R^2}{\cos^2\alpha}(d\alpha^2 - dt^2 + \sin^2\alpha \, d\Omega_3) \tag{4.3}$$

which corresponds to $\alpha \to \frac{\pi}{2}$ in this coordinate system. So the boundary theory which lives in $\mathbb{R} \times S^3$ is mapped to $\mathcal{N} = 4$ SYM in $\mathbb{R}^4$. This is done by usual radial quantization. In this conformal mapping, translation in time direction is mapped to dilatation generator in $\mathbb{R}^4$. In other word, Energy of string state correspond to conformal dimension of local operator.

$$E(\frac{R^4}{\alpha'^2}, g_s) \quad \text{of string} \quad = \quad \Delta(\lambda, \frac{\lambda}{N}) \quad \text{of local operator} \tag{4.4}$$

Next let us consider other quantum numbers in $\mathcal{N} = 4$ SYM. As we discussed earlier, underlying symmetry of $\mathcal{N} = 4$ SYM is $SU(2, 2|4)$ whose bosonic subgroup is $SO(2, 4) \times SO(6)$. So any local operators in gauge theory are labeled by six Cartan generators of $SO(2, 4) \times SO(6)$.

$$(\,\Delta,\, S_1,\, S_2,\, J_1,\, J_2,\, J_3\,) \tag{4.5}$$

where S_i are spins of Lorentz generators and J_i are R-charges of bosonic $SO(6)$ symmetries. From this we can guess which string state correspond to given local operator. Because the group $SO(2, 4)$ which is conformal symmetry in gauge theory is realized as isometry group of AdS_5 space in string theory, operators with Lorentz spin S_i is related to string state rotating in AdS_5 space. And by the fact that R symmetry $SO(6)$ is isometry group of S^5, operators with R-charge J_i correspond to sting state with spin J_i in S^5 space.

The strategy is now find the string state with given quantum numbers and compute its energy E.

$$E(S_1, S_2, J_1, J_2, J_3) \tag{4.6}$$

Then we compare it with the conformal dimension of operators in gauge theory side with same quantum charges.

$$\Delta(S_1, S_2, J_1, J_2, J_3) \tag{4.7}$$

Unfortunately, it is not easy to compare directly and check the relation (4) in general. As we have seen before, strong/weak nature of duality makes it difficult to check because we can not maintain both theory in perturbative regime. In gauge theory side, we can obtain perturbative expansion of conformal dimension of local operator in small-λ limit,

$$\Delta(\lambda) = \Delta_0 + \lambda\Delta_1 + \lambda^2\Delta_2 + \lambda^3\Delta_3 \dots \tag{4.8}$$

while in string theory side, energy of string state can be expanded in large-λ limit.

$$E(\lambda) = \sqrt{\lambda}E_0 + E_1 + \frac{1}{\sqrt{\lambda}}E_2 + \frac{1}{\sqrt{\lambda}^2}E_3 \dots \tag{4.9}$$

So unless we can calculate the all orders of perturbations and sum up to get exact result, we can not compare both quantity in ordinary perturbation expansions.

To overcome this problem we consider new expansion parameter and take some particular limit. This new region of parameter space was discovered by Berenstein, Maldacena and Nastase [13] in 2002. They consider R-charge J as a new expansion parameter and take it very large with certain conditions.

$$J, N \to \infty, \qquad \widetilde{\lambda} \equiv \frac{\lambda}{J^2} : \text{fixed}, \qquad \frac{N}{J^2} : \text{fixed} \tag{4.10}$$

With this BMN limit, even in the large-λ region which was not good perturbative region for gauge theory side, we can take the R-charge J much larger than $\sqrt{\lambda}$ and have a new small expansion parameter $\widetilde{\lambda} \equiv \frac{\lambda}{J^2}$. Therefore we can make both theories in the perturbative region and have a chance to compare them. More precisely, we expand the physical quantities in double expansion of $\frac{1}{J}, \frac{\lambda}{J^2}$

$$\Delta(J, \lambda) = J\left(1 + \frac{\lambda}{J^2}\delta_1 + \frac{\lambda^2}{J^4}\delta_2 + \dots\right) + O(J^0) \tag{4.11}$$

$$E(J, \lambda) = J\left(1 + \frac{\lambda}{J^2}\epsilon_1 + \frac{\lambda^2}{J^4}\epsilon_2 + \dots\right) + O(J^0) \tag{4.12}$$

and see if

$$\delta_k = \epsilon_k \quad ? \qquad\qquad k = 1, 2, 3 \ldots \tag{4.13}$$

So we compare the conformal dimension of long operator with large R-charge with the energy of sting state which is rotating very fast in S^5 space. In gauge theory side, such large R-charge limit is translated to thermodynamic limit of Bethe ansat equations and makes one possible to compute conformal dimension explicitly.

We can generalize the situation by turning on other quantum number, Lorentz spin S. Large spin S limit opens a new window of parameter space and make it possible to consider various limit. In the large S and J region, the local operators we consider takes the following forms

$$\mathcal{O}(\S) = Tr[\nabla^S \phi^J(x)] \tag{4.14}$$

The corresponding string state is spinning string in both AdS_5 and S^5 with large angular momentum S and J respectively. This is the main objet we will consider in this chapter. By obtaining solution of spinning strings or D-branes and computing their energy, we can compare them with conformal dimensions of corresponding local operators in gauge theory. In this way we can check the AdS/CFT conjecture explicitly. Another aspect of this study is that if we assume the duality hold, we can study the non-perturbative properties of conformal dimension by string dual method.

In the following sections, we will study how to compute the above physical quantities explicitly. In the gauge theory side, diagonalization of dilatation operator can be translated to spin chain problem and Bethe ansat equations. From this we can compute conformal dimension of operators we concern. We will not see this gauge theory aspect in details in this thesis. For reviews see [60, 61, 61–64] We will focus to string theory side and consider spinning string in AdS_5 and S^5. By solving equations of motion of string sigma model, we can get the expression of energy in terms of angular momentum S and J. we will explore the various parameter region and study several properties like logarithmic scaling on spin. We will conclude this section by considering generalize the result to operators in different representations - especially in symmetric representation. This can be done by replacing the string with D-branes.

4.1 Spinning strings in $AdS_5 \times S^5$

Local operator with Lorentz spin S and R-charge J corresponds to rotating string in $AdS_5 \times S^5$ with angular momentum S in AdS_5 and another angular momentum J in S^5. Here we will review the previous works of solving the string solution and computations their classical energy spectrum of it in large $\lambda = \frac{r^4}{\alpha'^2}$ limit. [14, 16, 65–67]

We shall use the global coordinate metric for AdS_5 and S^5 as followings [16].

$$ds^2_{AdS_5} = R^2(d\rho^2 - \cosh^2\rho\, dt^2 + \sinh^2\rho\, d\Omega_3) \tag{4.15}$$

$$d\Omega_3 = d\beta_1^2 + \cos^2\beta_1(d\beta_2^2 + \cos^2\beta_2 d\phi^2) \tag{4.16}$$

$$ds^2_{S^5} = R^2(d\psi_1^2 + \cos^2\psi_1(\, d\psi_2^2 + \cos^2\psi_2\, d\Omega_3')) \tag{4.17}$$

$$d\Omega_3' = d\psi_3^2 + \cos^2\psi_3(d\psi_4^2 + \cos^2\psi_4 d\varphi^2) \tag{4.18}$$

The relevant bosonic part of Polyakov string action is given by

$$S = -\frac{\sqrt{\lambda}}{4\pi}\int d^2\sigma\sqrt{-g}g^{ab}\left[G^{AdS_5}_{MN}(X)\partial_a X^M\partial_b X^N + G^{S^5}_{MN}(Y)\partial_a Y^M\partial_b Y^N\right] \tag{4.19}$$

where $\sigma_i = (\tau, \sigma)$, $M, N = 0, 1, .., 4$.

For the ansat $X^M(\tau,\sigma), Y^M(\tau,\sigma)$ for rotating closed string in AdS_5 and S^5, we take the following forms [16],

$$t = \kappa\tau, \quad \phi = \omega\tau, \quad \varphi = \nu\tau \tag{4.20}$$

$$\rho = \rho(\sigma), \quad \beta_i = 0, \quad \psi_i = 0 \tag{4.21}$$

with constants κ, ω, ν. Now we need to solve $\rho(\sigma)$ configuration by varying the string action. The result of equations of motion is

$$\rho'' = (\kappa^2 - \omega^2)\sinh\rho\cosh\rho \tag{4.22}$$

with the constraint from conformal gauge fixing,

$$\rho'^2 = \kappa^2 \cosh^2 \rho - \omega^2 \sinh^2 \rho - \nu^2 \tag{4.23}$$

The rotating closed string should satisfy the periodic condition $\rho(\sigma) = \rho(\sigma + 2\pi)$. The simplest solution for this condition is considering folded string configuration which splits into four segments. In $0 < \sigma < \frac{\pi}{2}$ region, as σ increase from 0 to $\pi/2$, the sting stretches from 0 to its maximum reach ρ_0. So we can obtain additional equations.

$$\begin{aligned} (\kappa^2 - \nu^2)\cosh^2 \rho_0 - (\omega^2 - \nu^2)\sinh^2 \rho_0 &= 0 & (4.24)\\ \int_0^{2\pi} d\sigma = 4\int_0^{\rho_0} \frac{d\rho}{\sqrt{(\kappa^2 - \nu^2)\cosh^2 \rho - (\omega^2 - \nu^2)\sinh^2 \rho}} &= 2\pi & (4.25)\end{aligned}$$

From these equations we have derived, we can obtain the solution of rotating string configuration.

Next things we need are the expressions for the conserved charges E, S and J. These can be obtained easily by Noether theorem and the results are [14, 16]

$$E = \sqrt{\lambda}\kappa \int_0^{2\pi} \frac{d\sigma}{2\pi} \cosh^2 \rho, \tag{4.26}$$

$$S = \sqrt{\lambda}\omega \int_0^{2\pi} \frac{d\sigma}{2\pi} \sinh^2 \rho, \tag{4.27}$$

$$J = \sqrt{\lambda}\nu \int_0^{2\pi} \frac{d\sigma}{2\pi} \tag{4.28}$$

Angular momentum J of S^5 is written trivially and other E and S are written in integral equations of string configuration. We can solve the integrals analytically and find the expression as

$$\frac{E}{\sqrt{\lambda}} = \frac{\kappa}{\sqrt{\kappa^2 - \nu^2}} \frac{1}{\sqrt{\eta}} F_{21}\left(-\frac{1}{2}, \frac{1}{2}; 1; -\frac{1}{\eta}\right) \tag{4.29}$$

$$\frac{S}{\sqrt{\lambda}} = \frac{\omega}{\sqrt{\kappa^2 - \nu^2}} \frac{1}{2\eta\sqrt{\eta}} F_{21}\left(\frac{1}{2}, \frac{3}{2}; 2; -\frac{1}{\eta}\right) \tag{4.30}$$

$$\sqrt{\kappa^2 - \nu^2} = \frac{1}{\sqrt{\eta}} F_{21}\left(\frac{1}{2}, \frac{1}{2}; 1; -\frac{1}{\eta}\right) \tag{4.31}$$

with the parameter η which is defined as

$$\coth^2 \rho_0 = \frac{\omega^2 - \nu^2}{\kappa^2 - \nu^2} = 1 + \eta \tag{4.32}$$

To study the meanings of this solution and compare with gauge theory side, now we consider several parameter limits.

Short string Let us first consider short string which means $\rho_0 \to 0$. When spin S is small, string can not be stretched in AdS space and shrink to point particle. To compare with BMN limit in gauge theory side, we keep the angular momentum J is large. In parameter space, this region is expressed as

$$\eta \gg 1, \qquad \nu \gg 1 \tag{4.33}$$

Under this limit, we can have the relations among parameters

$$\omega \approx \sqrt{1 + \nu^2 + \frac{1}{\eta}}, \qquad \kappa \approx \sqrt{\nu^2 + \frac{1}{\eta}} \tag{4.34}$$

and get the expression for energy spectrum.

$$E \approx J + S + \frac{\lambda}{J^2}\frac{S}{2} + \dots \tag{4.35}$$

In terms of conformal dimension of dual gauge theory side, first two terms are classical dimension which is trivial. For the next term, we can notice that this is exactly the BMN expansion(4.12) we have considered before. This is achieved because we are considering large J limit. This result can be directly compared with perturbative result of gauge theory side, and verified they agree. This is the one of the non-trivial test of AdS/CFT correspondence.

Long string More interesting case is long string case which is opposite limit $\rho_0 \to \infty$. We take the angular momentum S in AdS_5 large, so makes the string stretches. In this region ($\eta \ll 1$), we can get the parameter relations,

$$\omega \approx \sqrt{\nu^2 + \frac{1}{\pi^2}(1+\eta)\ln^2\frac{1}{\eta}}, \tag{4.36}$$

$$\kappa \approx \sqrt{\nu^2 + \frac{1}{\pi^2}\ln^2\frac{1}{\eta}} \tag{4.37}$$

Interesting point here is we now have logarithmic function of η which was just the polynomial in short string case. To go further in this large S region, we divide the case whether J is large or not.

In small angular momentum J case which can be expressed precisely,

$$\frac{J}{\sqrt{\lambda}} \ll \ln S \tag{4.38}$$

we obtain the energy spectrum as

$$E \approx S + \frac{\sqrt{\lambda}}{\pi} \ln \frac{S}{\sqrt{\lambda}} + \dots \tag{4.39}$$

This shows the famous logarithmic scaling in spin S of minimal twist operators in gauge theory [14]. In ordinary gauge theory like Quantum Chromo-Dynamics or $\mathcal{N} = 4$ SYM theory, twist 2 operators scales as

$$E = S + (a_1\lambda + a_2\lambda^2 + \dots) \ln S \tag{4.40}$$

This behavior is first obtained by perturbative calculations. Its logarithmic scaling is very non-trivial result - if we count naively $\ln^k S$ terms appear in each Feynman diagram but they are canceled each other in the summation of diagrams. We obtained same logarithmic scaling behavior in dual string theory side. Because we consider the case of small J with large S, the corresponding operator approaches to the minimal twist operator. Above result implies that even in the non-perturbative case (we keep the coupling constant λ large) logarithmic scaling continues to hold. This gives some evidence to the conjecture that for all λ value, scaling dimension of minimal twist operator with large spin S is written

$$E = S + f(\lambda) \ln S \tag{4.41}$$

Next, in the large J region with large S,

$$\ln S \ll \frac{J}{\sqrt{\lambda}} \ll \frac{S}{\sqrt{\lambda}} \tag{4.42}$$

We can obtain the BMN-like expansion of energy

$$E \approx S + J + \frac{\lambda}{2\pi^2 J} \ln^2 \frac{S}{J} + \dots \tag{4.43}$$

In this case, we can not see the large spin $\ln S$ term. Because we keep the angular momentum J is large enough(although it is still less than S) it doesn't correspond to minimal twist operators. Instead we can again compare the result with the conformal dimension of operators in gauge theory side, and see that it agrees again.

In the long string case, each result can be interpreted as follows. In considering the large J region we can give the non-trivial test of AdS/CFT duality. And in small J region, by just assuming the AdS/CFT duality, the non-perturbative calculation of scaling dimension of minimal twist operator can be performed using sting theory dual.

4.2 Spinning D-branes and higher representations

Fields in single trace operators which we considered so far was in fundamental representation of $SU(N)$ gauge group of $\mathcal{N} = 4$ SYM. This can be generalized to higher representation. In other word, we can consider the local operators of the form,

$$\mathcal{O} = Tr(D^S Z^J) \tag{4.44}$$

$$\text{where} \quad Z = Z^a T^a_R, \qquad D = n \cdot (\partial + [A^a T^a_R, *]) \tag{4.45}$$

with large spin S and R-charge J. As we have seen so far, conformal dimension of operators in fundamental representation can be related to the energy of rotating fundamental string in $AdS_5 \times S^5$ space. Then for operators in higher representation, what kind of object in string theory would be the corresponding one?

There are several studies on this problem with Wilson lines in AdS/CFT [47,48] which originally correspond to macroscopic fundamental strings. When the macroscopic string is replaced by a D-brane with electric flux [47, 49–52], it corresponds to a Wilson line of a higher representation; a D3-brane corresponds to the k-th symmetric representation while a D5-brane corresponds to the k-th anti-symmetric representation, where k is the string charge of the D-brane with electric flux.

Thus, replacing the rotating string by a rotating D3 or D5-brane, one could analyze the spectrum of the local operators in the k-th symmetric or anti-symmetric representation [1]. The spectrum of twist two operators in the k-th anti-symmetric representation is studied by Armoni [53] using rotating D5-brane.

In this section, we explicitly solve the equations of motion for D3 branes and find the expressions for energy spectrum in terms of angular momentums S and J of AdS_5 and S^5. To use the holographic dictionary for Wilson lines and the more symmetries, we study the D3-brane counterpart of the "long string". In the "long string" case, the folded string touches the boundary of AdS_5 (so it represents Wilson lines) and one more symmetry is enhanced (translation in χ. See the section 4.2.1). As a result, we find following scaling behavior in certain parameter regime.

$$(E-S)^2 - J^2 = T_3^2 f(\beta,\mu)\log^2\frac{S}{J}, \tag{4.46}$$

$$\text{where, } \beta = \frac{J}{T_3\log\frac{S}{J}}, \quad \mu = 2\pi\frac{\sqrt{\lambda}k}{N}, \quad T_3 = \frac{N}{2\pi^2}. \tag{4.47}$$

Which is valid when

$$\beta,\mu \text{ fixed}, \quad S \gg J, \quad N\to\infty, \quad \lambda\to\infty. \tag{4.48}$$

In small β and μ, the function f can be expanded as polynomial in β^2 and μ^2.

$$f(\beta,\mu) = \mu^2 + c_{2,0}\beta^4 + c_{1,1}\beta^2\mu^2 + c_{0,2}\mu^4 + \text{higher order terms}. \tag{4.49}$$

Thus from (4.46), the anomalous dimension $\gamma := E - S - J$ can be written as

$$\gamma = J\sum_{m=0}^{\infty}\beta^{2m}\gamma_m(x^2), \tag{4.50}$$

$$= J\left[(\sqrt{1+x^2}-1) + \beta^2\left(\frac{c_{2,0}+c_{1,1}x^2+c_{0,2}x^4}{2\sqrt{1+x^2}}\right) + o(\beta^4)\right], \tag{4.51}$$

$$\text{where, } x := \frac{\mu}{\beta} = \frac{k\sqrt{\lambda}\log(S/J)}{\pi J}. \tag{4.52}$$

[1] More precisely Tr in the fundamental representation is replaced by the character (or Schur polynomial) of the symmetric or the anti-symmetric representation.

Note that the double expansion in β^2 and x^2 has same structure with the double expansion in $\frac{1}{N^2}$ and λ in the gauge theory side. At planar order (zeroth in β^2), the anomalous dimension coincide with that of k noninteracting folded strings (compare with (4.43)).

In some region in (β, μ), there is no classical D3-brane solution. There exists a critical value for μ for each β above which the classical D3-brane solution does not exist (see Figure 4.1). This seems to be a similar phenomenon as the phase transition in a symmetric Wilson loop observed in [50, 58, 59].

4.2.1 Set up

Symmetry, ansatz and boundary condition

First we will consider the symmetries of "infinity strings" in [14] which are dual to twist two operators with large spin, $S \gg \sqrt{\lambda}$. From those symmetries, we will find the appropriate ansatz and boundary conditions for the D3-brane which wraps 4 dimensional submanifold in AdS_5 and ends on the two light-like segments in the AdS_5 boundary. Then we will generalize the ansatz by turning on the angular momentum along S^5.

The infinite string solution is given by [27] [2]

$$X_{-1}X_2 - X_0X_1 = 0, \quad X_3 = X_4 = 0. \tag{4.53}$$

Here $\{X_\mu\}$ are the Cartesian coordinates of $\mathbb{R}^{2,4}$ where the AdS_5 is embedded. In the global coordinates $\{\tilde{\tau}, \tilde{\rho}, \Omega_i\}$, $(i = 1, 2, 3, 4)$ for AdS_5,

$$X_{-1} = \cosh\tilde{\rho}\cos\tilde{\tau}, \quad X_0 = \cosh\tilde{\rho}\sin\tilde{\tau}, \quad X_i = \sinh\tilde{\rho}\Omega_i, \quad \sum_{i=1}^{4}\Omega_i^2 = 1, \tag{4.54}$$

the boundary is located at $\tilde{\rho} \to \infty$. The infinite string ends on the following two

[2] Actually the folded string world-sheet covers (4.53) twice. Thus quantum numbers of the folded string should be doubled if one calculates them using (4.53). This two-foldedness should be taken into account in calculating quantum numbers for folded D3-brane.

light-like lines at the boundary.

$$\tilde{\tau} = \tilde{\varphi} \quad \text{or} \quad \tilde{\tau} = \tilde{\varphi} + \pi, \quad \Omega_3^2 + \Omega_4^2 = 0. \quad \text{where } \tilde{\varphi} = \arctan \frac{\Omega_2}{\Omega_1}. \tag{4.55}$$

These two light-like Wilson lines preserve three symmetries of $SO(2,4)$. These symmetries are more manifest in the $AdS_3 \times S^1$ foliation of AdS_5.

$$\begin{aligned}
&(X_{-1}, X_0, X_1, X_2) = \cosh\zeta(x_{-1}, x_0, x_1, x_2), \quad -x_{-1}^2 - x_0^2 + x_1^2 + x_2^2 = -1, \\
&(X_3, X_4) = \sinh\zeta(x_3, x_4), \quad x_3^2 + x_4^2 = 1, \\
&ds^2(AdS_5) = \cosh^2\zeta ds^2(AdS_3) + \sinh^2\zeta d\psi^2 + d\zeta^2.
\end{aligned} \tag{4.56}$$

We will use two coordinate systems for AdS_3, $\{u, \chi, \sigma\}$ and $\{\tau, \rho, \varphi\}$. See Appendix A. The infinite string (4.53) stretches along u, χ directions and located at $\zeta = 0, \sigma = 0$. And the three symmetries correspond to translations in u, χ and ψ [26]. Besides these continuous symmetries, there is an additional $\mathbb{Z}_2$ symmetry, $\sigma \leftrightarrow -\sigma$.

We will consider the D3-brane motion described by the DBI+WZ action

$$\begin{aligned}
&S_{D3} = T_3 \int d^4y L = T_3 \int d^4y (L_{DBI} + L_{WZ}), \\
&L_{DBI} = -\sqrt{-\det H}, \qquad H_{\alpha\beta} := G_{MN}(Y) \frac{\partial Y^M}{\partial y^\alpha} \frac{\partial Y^N}{\partial y^\beta} + F_{\alpha\beta}, \\
&L_{WZ} = -aC_{M_1 \dots M_4} \frac{\partial Y^{M_1}}{\partial y^{\alpha_1}} \cdots \frac{\partial Y^{M_4}}{\partial y^{\alpha_4}} \epsilon^{\alpha_1 \dots \alpha_4} \frac{1}{4!},
\end{aligned} \tag{4.57}$$

where Y^M, $(M = 0, \dots, 9)$ denote the space-time coordinates and y^α, $(\alpha = 0, 1, 2, 3)$ are the D3-brane world-volume coordinates. $F_{\alpha\beta}$ is the world-volume gauge flux. a is ± 1 depending on the choice of the orientation. The D3-brane tension T_3 is related to N by $T_3 = \frac{N}{2\pi^2}$ in our unit (AdS radius)$= 1$.

We are going to find classical D3-brane solution which preserves the three symmetries and ends on the light-like segments (4.55) at the $\mathbb{R} \times S^3$ boundary. From three continuous symmetries, the ansatz for D3-brane is ($\{u, \chi, \psi, y\}$ are the world-volume coordinates)

$$F = b\, du\, d\chi, \quad \sigma = \sigma(y), \quad \zeta = \zeta(y). \tag{4.58}$$

To preserve the $\mathbb{Z}_2$ symmetry ($\sigma \leftrightarrow -\sigma$), we impose the following.

$$\frac{d\zeta}{d\sigma} = 0, \quad \text{when } \sigma = 0. \tag{4.59}$$

When $\zeta = 0$ where the size of S^1 shrinks, there may be conical singularity. To avoid this, we impose the following condition.

$$\frac{d\sigma}{d\zeta} = 0, \quad \text{when } \zeta = 0. \tag{4.60}$$

In the $AdS_3 \times S^1$ foliation, the $\mathbb{R} \times S^3$ boundary of AdS_5 is located at $\cosh^2 \zeta \cosh^2 \rho \to \infty$, or equivalently

$$\rho \to \infty \quad \text{or} \quad \zeta \to \infty. \tag{4.61}$$

And the two light-like lines at the boundary (4.55) becomes

$$\begin{aligned} &\tau = \varphi \quad \text{or} \quad \tau = \varphi + \pi, \\ &\frac{X_3^2 + X_4^2}{X_{-1}^2 + X_0^2} = \frac{\sinh^2 \zeta}{\cosh^2 \zeta \cosh^2 \rho} \to 0. \end{aligned} \tag{4.62}$$

Under the ansatz (4.58), the D3-brane ends on the two segments (4.62) at the boundary (4.61) if and only if

$$\sigma(y), \zeta(y) = \text{finite}. \tag{4.63}$$

Equations (4.58),(4.59),(4.60),(4.63) are the summary of ansatz and conditions for D3-brane rotating in AdS_5. These can be generalized by turning on the angular momentum along S^5:

$$\theta = \nu u. \tag{4.64}$$

Here θ is the coordinate of a great circle of S^5. Under these ansatz, the D3-brane action (4.57) becomes

$$\begin{aligned} S_{D3} &= T_3 \int du d\chi d\psi dy L, \qquad L = L_{DBI} + L_{WZ}, \\ L_{DBI} &= -\sqrt{\sinh^2 \zeta (\cosh^4 \zeta \cosh^2 2\sigma - b^2 - \nu^2 \cosh^2 \zeta)(\cosh^2 \zeta \sigma'^2 + \zeta'^2)}, \\ L_{WZ} &= -a(\cosh^4 \zeta - 1) \cosh 2\sigma \sigma'. \end{aligned} \tag{4.65}$$

Here a is ± 1 depending on the choice of the orientation. Under these ansatz, the equation of motion for the world-volume gauge field is automatically satisfied.

Quantum numbers

Here in this subsection we will obtain the expression of the conserved charges: the energy E, the spin S, the R-charge J, and the string charge k. First three charges E, S, J are calculated as the Noether charges from the spacetime isometry. Later we will calculate k by taking variation by NSNS B-field.

For a Killing vector ξ^M in $AdS_5 \times S^5$ and a small parameter ϵ, there is a symmetry of the action (4.57). Since this isometry also preserves the RR5-form field strength F_5, the variation of 4-form potential should be written as

$$\delta C_4 = \epsilon d\Lambda_3, \tag{4.66}$$

where Λ_3 is a 3-form. The variation of the Lagrangian becomes

$$\delta L = \epsilon\partial_\alpha \left[-a\frac{1}{3!}\epsilon^{\alpha\alpha_2\alpha_3\alpha_4}\partial_{\alpha_2}Y^{M_2}\partial_{\alpha_3}Y^{M_3}\partial_{\alpha_4}Y^{M_4}\Lambda_{M_2M_3M_4} \right] =: \epsilon\partial_\alpha R^\alpha. \tag{4.67}$$

The Noether current j^α and the Noether charge Q for this symmetry is written as

$$j^\alpha = \frac{\partial L}{\partial(\partial_\alpha Y^M)}\xi^M - R^\alpha, \tag{4.68}$$

$$Q = T_3 \int d^3y\, j^0. \tag{4.69}$$

We only need to consider DBI-term in the action because we are considering folded D3-brane solution. Actually the terms in eq. (4.68) which come from the WZ-term cancel since two D3-branes have the opposite sign of the WZ-term to each other. The derivative of the DBI-term is given by

$$\frac{\partial L_{DBI}}{\partial(\partial_\alpha Y^M)} = -\sqrt{-\det H}(H^{-1}_{sym})^{\alpha\beta}G_{MN}\partial_\beta Y^N, \tag{4.70}$$

where H^{-1}_{sym} is the symmetric part of the inverse matrix of H.

We take u as the world-volume time. For the R-charge J the Killing vector is $\xi_J = \partial/\partial\theta$. The Noether charge is given as

$$J = T_3 \int d\chi \int d\psi \int dy\, j^u_J = 2\chi_0 T_3 \beta, \tag{4.71}$$

$$\beta := \int dy \frac{4\pi\nu(\cosh^2\zeta\sigma'^2 + \zeta'^2)\sinh\zeta\cosh^2\zeta}{\sqrt{(\cosh^4\zeta\cosh^2 2\sigma - b^2 - \nu^2\cosh^2\zeta)(\cosh^2\zeta\sigma'^2 + \zeta'^2)}}, \tag{4.72}$$

where χ_0 is the cut-off of the χ integral; χ is limited to $-\chi_0 \leq \chi \leq \chi_0$. As the same way, the Killing vector for $E-S$ is $\xi_{E-S} = -\partial/\partial\tau - \partial/\partial\varphi = -\partial/\partial u$ (see eqs. (A) and (A)), and the Noether charge is obtained as

$$E - S = 2\chi_0 T_3 \alpha, \tag{4.73}$$

$$\alpha := \int dy \frac{4\pi(\cosh^2\zeta\sigma'^2 + \zeta'^2)\sinh\zeta\cosh^4\zeta\cosh^2 2\sigma}{\sqrt{(\cosh^4\zeta\cosh^2 2\sigma - b^2 - \nu^2\cosh^2\zeta)(\cosh^2\zeta\sigma'^2 + \zeta'^2)}}. \tag{4.74}$$

On the other hand, for the spin S, the components of the Killing vector behave as $\xi_S \sim e^{2\chi}$ in large χ (see eq. (A)). Thus the charge S after integral over χ behaves as

$$S \sim T_3 e^{2\chi_0}, \text{ or } 2\chi_0 \sim \log\frac{S}{J}. \tag{4.75}$$

As a result we obtain the scaling behavior

$$(E-S)^2 - J^2 = T_3^2(\alpha^2 - \beta^2)\log^2\frac{S}{J}. \tag{4.76}$$

Let us turn to the string charge k. For a variation of B-field $\delta B_{u\chi}$, the variation of the action and the string charge k are related as ($\alpha' = \frac{1}{\sqrt{\lambda}}$ is the slope parameter in our unit.)

$$\delta S_{D3} = \frac{k}{2\pi\alpha'}\int du d\chi \delta B_{u\chi}. \tag{4.77}$$

Hence the string charge k is expressed as

$$k = 2\pi\alpha' T_3 \int d\psi \int dy \frac{\partial L}{\partial b} = \frac{N}{2\pi\sqrt{\lambda}}\mu, \tag{4.78}$$

$$\mu := \int dy \frac{4\pi b \sinh\zeta(\cosh^2\zeta\sigma'^2 + \zeta'^2)}{\sqrt{(\cosh^4\zeta\cosh^2 2\sigma - b^2 - \nu^2\cosh^2\zeta)(\cosh^2\zeta\sigma'^2 + \zeta'^2)}}. \tag{4.79}$$

The scaling function $f(\beta, \mu)$ in eq. (4.46) is obtained from (4.76) by expressing $\alpha^2 - \beta^2$ in terms of β and μ.

4.2.2 Numerical analysis

So far we describe the general procedure for obtaining a D3-brane solution which is dual to the composite operator $Tr(D^S Z^J)$ in symmetric representations. In this

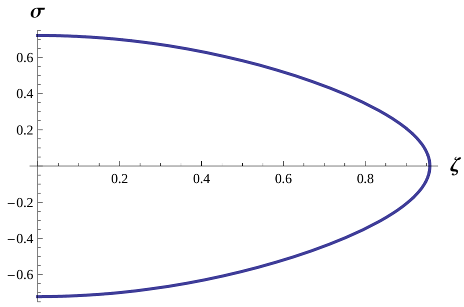

Figure 4.1: D3-brane example ($\nu = 0.999, b = 0.1$).

section, we will find numerical solutions and analyze its phase structure and energy spectrum.

Phase structure

The equation of motion derived from the action (4.65) is too complicated to solve it analytically. Thus, we find solutions numerically. The solution which satisfies the conditions (4.59),(4.60),(4.63) looks like an ellipse in (ζ, σ)-plane (see the Figure 4.1). For some values of (ν, b), there are several solutions. But if we impose stability condition[3], only one or no solution survives. And for some other values of (ν, b), there's no solution (even unstable one). Figure 4.2 shows the region in (ν, b) where the stable solutions exist. The region is surrounded by following three curves.

- $\nu^2 + b^2 = 1$.

  To avoid the Lagrangian (4.65) being an imaginary number, there's a lower bound for the size of solutions.

$$r := \sqrt{\zeta^2 + \sigma^2} \geq \frac{1}{2}\text{arccosh}(\sqrt{\nu^2 + b^2}) \tag{4.80}$$

  When $\nu^2 + b^2$ approaches to 1, the bound becomes smaller and stable solutions

[3]We check the stability numerically. We consider several small fluctuations $\{\delta\sigma, \delta\zeta\}$ around a solution. If the solution maximize the Lagrangian $\int d\chi d\psi dy L$ under the fluctuations, then it is considered as a stable one.

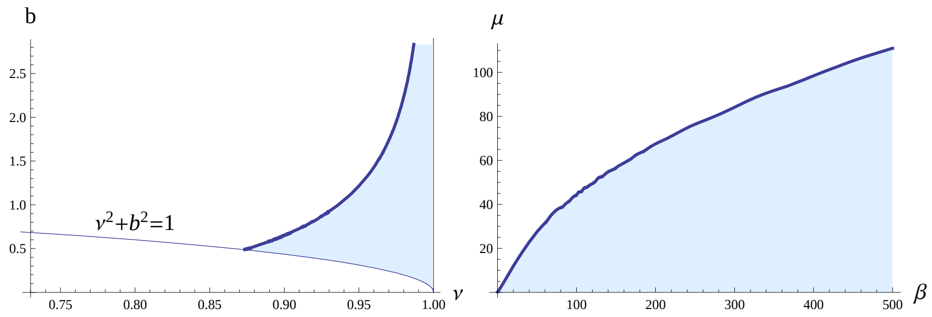

Figure 4.2: left: the region stable solutions exit, right: (β, μ) of the solutions fill the colored region.

tend to shrink to the point $r = 0$. Accordingly, the physical quantities (β, μ) of the solution become smaller.

- $\nu = 1$.

  If $\nu \geq 1$, there's no solution except unstable one. Stable solutions (and its β, μ) in the colored region become infinity when $\nu \to 1$. This bound for the angular velocity in S^5 direction also exists for the folded string solution case [16].[4]

- The upper curve.

  We cannot find analytic expression for this curve. Just below the curve there are two solutions (1 stable + 1 unstable). The two solutions get closer to each other when approaching the upper curve and disappear simultaneously above the curve. This curve is mapped to the upper curve in the (β, μ) plane via stable solutions. It suggests that there's some phase transition across the curve. This result requires further study to understand the phase transition in the gauge theory side.

[4] We fix $\frac{\omega}{\kappa} = 1$ and $\frac{\nu}{\kappa}$ in [16] corresponds to ν in this paper.

Expansion in small β, μ.

The energy spectrum (4.46), which is valid in the limit (4.48), is wholly determined if we find expression $f := \alpha^2 - \beta^2$ in terms of β, μ. Although we cannot find its full analytic expression, we suggest the form of series expansion and obtain the exact values of the coefficients at the lowest order. Higher order coefficients can be obtained numerically.

Consider the limit (ν, b) approaching to the curve $\nu^2 + b^2 = 1$. In the limit, as mentioned above, the stable solution (and its β, μ) become smaller. From the expression for α, β, μ in section 2 and the fact that ζ, σ is very small, one can see that

$$\frac{f}{\mu^2} = \frac{\alpha^2 - \beta^2}{\mu^2} \simeq \frac{1 - \nu^2}{b^2} \to 1 \tag{4.81}$$

in the limit. This result gives

$$(E - S)^2 - J^2 = k^2 \frac{\lambda}{\pi^2} \log^2(\frac{S}{J}) \quad \text{when } \beta, \mu \to 0. \tag{4.82}$$

This is nothing but the spectrum of k noninteracting folded strings!(cf. (4.43)).

Assuming the $f(\beta, \mu)$ is analytic near the origin $(\beta, \mu) = (0, 0)$, we propose following expansion

$$f(\beta, \mu) = \sum_{m,n} c_{m,n} \beta^{2m} \mu^{2n}, \quad m, n \geq 0. \tag{4.83}$$

Here we use the fact that $f(\beta, \mu)$ is even function in both β and μ.[5] And eq. (4.81) imply that

$$c_{0,0} = 0, \quad c_{1,0} = 0, \quad c_{0,1} = 1. \tag{4.84}$$

Numerically, we check the expansion (4.83) up to fourth power of β, μ and obtain the value of $c_{2,0}, c_{1,1}, c_{0,2}$.

[5]When $\nu \leftrightarrow -\nu$, e.o.m does not change and the stable solution remains same. So were α, μ. But β changes its sign (4.72). Similar argument hold for the $b \leftrightarrow -b$ case (in this case, (α, β) remains same but μ changes its sign.).

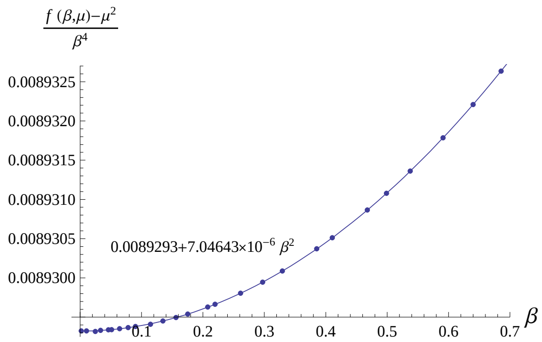

Figure 4.3: Along $\mu = 0.306857\beta$ we plot the graph. Its behavior agree with what we expected from (4.83).

$$c_{2,0} = 0.0084\ldots, \quad c_{1,1} = 0.0074\ldots, \quad c_{0,2} = -0.023\ldots. \tag{4.85}$$

For small (β, μ), $f(\beta, \mu)$ is well approximated by this expansion as shown in Figure 4.3.

Comments on gauge theory side

So far we have studied the properties of energy spectrum of spinning D3-brane. We can see that logarithmic scaling on spin S is recovered in the limit of $J \gg \sqrt{\lambda}\, log(S/J)$ which is related to minimal twist operators. And usual BMN expansion is obtained in opposite limit. Also we can check the consistency by taking planar order and see the coincidence of our result with k non-interacting fundamental strings.

Meanwhile if we are interested in the test of AdS/CFT correspondence, we need to compute the gauge theory side also. In order to calculate this anomalous dimension in the gauge theory side, one should consider the limit $N \to \infty$ while keeping β, μ finite and λ small finite instead of the limit (4.48). In this limit certain kinds of non-planer diagrams also contribute to the result since μ kept finite.

Chapter 5

Infra-red finiteness in 3 dimensions

In this chapter we consider infra-red divergences in 3 dimensional Chern-Simons theory which called ABJM theory. As we have seen in section 2, ABJM theory is studied extensively because of it is another example of holographic duality-AdS_4/CFT_3. In $\mathcal{N} = 4$ case, we concentrated on local operators and their scaling dimensions. These physical object is important in ABJM theory and have studied by many authors [69, 76–81]. Another important object in the study of field theory is scattering amplitudes. It is important because it reveals hidden symmetries in the theory. For examples, recursive structure in the loop amplitude can be made and connection with Wilson-loop value can be shown. These properties come from the rich structure of symmetries in the theory.

In the following chapters we will first review about some basic properties of scattering amplitudes in ABJM theory. We consider the result of two-loop amplitude and concentrate on Infra-red divergences. And we will see the implications from general argument of Kinoshita-Lee-Naunenberg theorem, which gives the guideline to deal with the IR divergences in ABJM theory. By calculating the four-point amplitude case explicitly, we will sketch the procedure of how IR divergences can be canceled and obtain the finite results.

5.1 Scattering amplitudes of ABJM theory

In section 2, we studied the general structure of ABJM theory. Field contents of the theory are 2 kinds of vector fields $A_\mu, \overline{A}_\mu$, four fermion fields ψ_A and four scalar fields ϕ^A. Because there is no degree of freedom in Chern-Simons theory, gauge fields A_μ does not participate as external particles. And because the matter fields ϕ^A, ψ_A transform in the $(N, \overline{N})$ representation of the gauge group $U(N) \times U(N)$, external particles should come in pairs. So scattering amplitudes in ABJM theory is composed of even number of matter field.

To go further in the study of scattering amplitudes, let us review some basic things which is used widely in this field

Spinor-helicity formalism In mass-less theory, on-shell condition $p^2 = 0$ gives the simple representation of momentum p. This is called Spinor-helicity formalism. Because the lorentz algebra in three dimension is $SO(1,2)$ which is isomorphic to $sl(2,\mathbb{R})$, we can express the momentum p as $sl(2,\mathbb{R})$ bispinor [71]. If we expand the momentum p in the basis of 2×2 matrices σ^μ

$$p^{ab} = (\sigma^\mu)^{ab} p_\mu \tag{5.1}$$

$$\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5.2}$$

on-shell condition $p^2 = 0$ is solved explicitly by 2 component spinor λ^a

$$p^{ab} = \lambda^a \lambda^b \tag{5.3}$$

Because we have one constraint on 3 dimensional momentum, 2 degrees of freedom is left. 2 component momentum spinor λ^a is the explicit solution of the constraint. We can see that solution λ^a is unique up to a overall sign. So in the calculations of scattering amplitudes, we use this spinor λ^a in the expressions of momentum invariants

$$\langle ij \rangle = \epsilon_{ab} \lambda^a \lambda^b, \qquad p_i \cdot p_j = -\frac{1}{2} \langle ij \rangle^2 \tag{5.4}$$

We can see in the following chapters that this representation makes the result simple.

Color ordered amplitude In non-abelian gauge theory, each scattering amplitudes carry color indices of external particles. In planar limit, this can be written in single trace forms

$$\mathcal{A}_n = \sum_{a(i)} A_n(p_{a(1)}, p_{a(2)}, ..., p_{a(n)}) Tr[T^{a(1)} T^{a(2)} ... T^{a(n)}] \tag{5.5}$$

The summation runs in permutations of n external lines. Coefficient of each trace of generators A_n is called color ordered amplitude. It doesn't have color structure anymore and represent the amplitudes in some fixed color ordering. Full amplitude can be obtained easily by summing up the given color-ordered amplitudes.

Cross section which is physical object is obtained from the square of full amplitudes. In general, there are cross-terms of color-ordered amplitudes with different color structure. But in the large N planar limit, we can see that they are negligible

$$\sigma_n = |\mathcal{A}_n(1,2,...,n)|^2 = N^{n-2}(N^2-1) \sum_{color} |A_n(1,2,...,n)|^2 + \mathcal{O}(\frac{1}{N}) \tag{5.6}$$

This is nice property. By computing color-ordered amplitudes, the cross-section in planar limit is obtained by squaring the color-ordered amplitudes. In the following chapters we will concentrate on how to compute the color-ordered amplitudes

5.1.1 Tree-level amplitudes

Tree level amplitude can be computed in various way. In Feynman diagram approach, Super-conformal symmetry can be used to relate the amplitudes in different matter contents. Because each external particles ϕ^A, ψ_A is related with R-symmetry, they are combined to super-fields $\Phi(\lambda, \eta)$ [71]

$$\Phi(\lambda,\eta) = \phi^4(\lambda) + \eta^A \psi_A(\lambda) + \frac{1}{2}\epsilon_{ABC}\eta^A\eta^B\phi^C(\lambda) + \frac{1}{3!}\epsilon_{ABC}\eta^A\eta^B\eta^C\psi_4(\lambda) \tag{5.7}$$

$$\overline{\Phi}(\lambda,\eta) = \overline{\psi}^4(\lambda) + \eta^A \overline{\phi}_A(\lambda) + \frac{1}{2}\epsilon_{ABC}\eta^A\eta^B\overline{\psi}^C(\lambda) + \frac{1}{3!}\epsilon_{ABC}\eta^A\eta^B\eta^C\overline{\phi}_4(\lambda) \tag{5.8}$$

where η^A is grassmann variables of $\mathcal{N}=3$ superspace. Using this super-field formalism we can consider super-amplitudes and make a connections between component amplitude of different external particles.

In the works [70,71], four, six point amplitudes is computed in Feynman diagram approach. In four point case, by computing one component amplitude- say four bosons amplitude - whole super-amplitude can be constructed. In six point case, they need two kinds of component amplitudes.

There is another approach to compute tree-level amplitudes. It is based on Grassmannian integral [73]. This is manifest realizations of super-conformal symmetry and it dual symmetry - Yangian symmetry. Compared to Feynman diagram method, this approach is rather simple and gives more compact results. In the works of [72] they computed eight-points amplitudes and gives recursion relations between tree-level amplitudes.

We will not follow the whole computations of tree-level amplitudes but just list some explicit four-point results for the use in the next chapters.

$$A_4(\phi^4, \bar{\phi}_4, \phi^4, \bar{\phi}_4) = \frac{\langle 24\rangle^3}{\langle 21\rangle\langle 14\rangle}, \qquad A_4(\psi_4, \bar{\psi}^4, \psi_4, \bar{\psi}^4) = \frac{\langle 13\rangle^3}{\langle 21\rangle\langle 14\rangle} \tag{5.9}$$

5.1.2 2-loop amplitudes

Now let us consider loop corrections of amplitudes in ABJM theory. One of the important result in loop amplitudes is that one-loop corrections are trivial. One-loop amplitude of four point amplitude vanishes [70]. In the six point case, they just gives trivial sign functions coming from collinear configuration [82, 83]. This vanishing properties in the one-loop order can be also found in local operators analysis [69] and light-like Wilson loop computations [84].

So first non-trivial correction comes from two-loop order. The result of two loop order of four-point amplitudes are [74, 75]

$$A_4^{(2)} = A_4^{tree} \cdot \left(\frac{N}{k}\right)^2 \left[-\frac{(s/\mu^2)^{-2\epsilon}}{(2\epsilon)^2} - \frac{(t/\mu^2)^{-2\epsilon}}{(2\epsilon)^2} + \frac{1}{2}\ln^2\left(\frac{s}{t}\right) + const\right] \tag{5.10}$$

where s, t are usual Mandelstam variables and ϵ is dimensional regularization parameter. We can see that it has $\frac{1}{\epsilon^2}$ IR divergences. This double poles (or double log divergences) are general properties of IR divergences. There are two kinds of limit which diverges.

- Soft limit

  If momentum goes to zero, denominators vanishes and amplitude diverges. In tree level amplitude, this corresponds to soft bremsstrahlung emissions.

- Collinear limit

  Although momentum is not small, two momentums which become collinear make the divergences. In conformal theory, this collinear emissions should be taken because there is no way to distinguish the parallel two particles state from one particle state.

In physical observable, these IR divergence should be away. To see if these divergence can be cured and get a finite observable, we need to review the general theorem about IR divergences in field theory.

5.2 Kinoshita-Lee-Nauenberg Theorem

Kinoshita-Lee-Nauenberg Theorem [85] gives general way to solve the IR divergences. Main result of it is we need to consider cross-sections and including additional soft or collinear radiations. Because the finiteness is required only in physical observable, scattering amplitude itself doesn't need to be finite. In the UV divergence case, scattering amplitude or even in the off-shell correlation function level, we can have UV finiteness. But IR divergence can not be canceled in the amplitudes level.

Cancellations of IR divergence of loop correction is achieved by including additional emissions in tree level amplitudes. In other word, to get a finite one-loop corrections of n-point scattering cross-sections, we need to consider n+1 tree-level amplitudes.

$$\sigma^0_{n+1} = |\lambda\, A^{(soft)}_{n+1}|^2 = \lambda^2 \sigma^{soft}_{n+1} \tag{5.11}$$

$$\sigma^{(2)}_n = |A^{(0)}_n + \lambda A^{(1)}_n|^2 = \sigma^{(0)}_n + \lambda^2 \sigma^{(2)}_n + \mathcal{O}(\lambda^3) \tag{5.12}$$

Those $\lambda^2\sigma^{soft}_{n+1}$ and $\lambda^2\sigma^{(2)}_n$ are canceled each other, we can get finite loop corrections. In conformal field theory, all particles are mass-less and there is no energy scales.

So asymptotic states cannot be defined properly and one-particle state becomes ambiguous. In other word, we cannot distinguish two parallel particles state from one particles. So, we have to consider additional externals in both initial states and final states carefully.

5.3 IR finiteness of ABJM theory

One interesting point of ABJM theory is that only even number of external particles are possible. So unlike the usual quantum field theory, we need to consider two particle emissions when considering IR cancellations. We can guess that this is related to the fact that first non-trivial corrections start from two loop order in ABJM theory. Technically two particle emissions imply double phase space integral. In this section we will carry explicit calculations of IR correction in the four-point amplitude. First by considering a few Feynman diagrams, we will show how double phase space integral give usual double pole divergences and the structure of cancellation procedure with loop correction diagrams. Next we will consider full four-point amplitude using spinor-helicity formalism, and check the finiteness of full four-point cross-sections.

5.3.1 Feynman diagram approach

Soft Bremsstrahlung

Consider the process that scalar particle with momentum p interact through gauge boson and then scattered with momentum p'. There are 2 types of diagrams emitting before interaction and emitting after interaction.

$$\begin{aligned}\mathcal{M}_s = {} & [i(2p - p_1 - p_2)^\mu] \frac{-i}{(p - p_1 - p_2)^2} \mathcal{M}_0(p - p_1 - p_2, p') \left(-\frac{2\pi}{k}\right) \frac{\epsilon_{\mu\nu\rho}(p_1 + p_2)^\rho}{(p_1 + p_2)^2} [i(p_2 - p_1)^\nu] \\ & + \mathcal{M}_0(p, p' + p_1 + p_2) \frac{-i}{(p' + p_1 + p_2)^2} [i(2p' + p_1 + p_2)^\mu] \left(-\frac{2\pi}{k}\right) \frac{\epsilon_{\mu\nu\rho}(p_1 + p_2)^\rho}{(p_1 + p_2)^2} [i(p_2 - p_1)^\nu]\end{aligned}$$

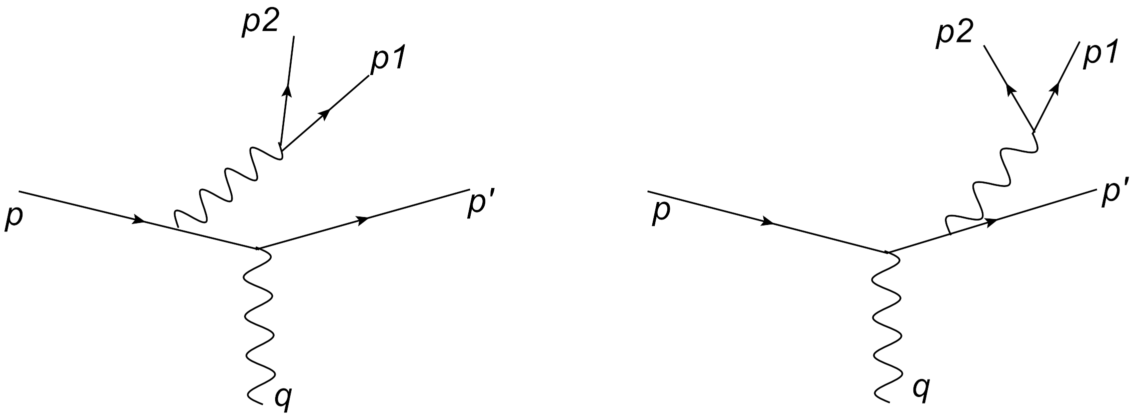

Figure 5.1: Bremsstrahlung diagrams

Since we are interested in the case of soft scalar limit $|p_1|, |p_2| \ll |p' - p|$, so we can treat

$$\mathcal{M}_0(p - p_1 - p_2, p') \sim \mathcal{M}_0(p, p' + p_1 + p_2) \sim \mathcal{M}_0(p, p') \tag{5.13}$$

and ignore the quadratic term of p_1, p_2 in the denominator

$$(p - p_1 - p_2)^2 = -2p \cdot (p_1 + p_2) + (p_1 + p_2)^2 \sim -2p \cdot (p_1 + p_2) \tag{5.14}$$

Under these soft scalar limit, the amplitude becomes

$$\mathcal{M}_s = \left(-i\frac{2\pi}{k}\right) \mathcal{M}_0(p, p') \left[\frac{\epsilon_{\mu\nu\rho} p^\mu p_1^\nu p_2^\rho}{p \cdot (p_1 + p_2)(p_1 \cdot p_2)} - (p \leftrightarrow p')\right] \tag{5.15}$$

and the cross-section has the following expression

$$d\sigma_s(p, p', p_1, p_2) = d\sigma_0(p, p') \cdot \int \frac{d^2 p_1}{(2\pi)^2} \frac{d^2 p_2}{(2\pi)^2} \frac{1}{2p_1} \frac{1}{2p_2} \frac{4\pi^2}{k^2} \left| \frac{\epsilon_{\mu\nu\rho} p^\mu p_1^\nu p_2^\rho}{p \cdot (p_1 + p_2)(p_1 \cdot p_2)} - (p \leftrightarrow p') \right|^2$$

Above correction term can be simplified as

$$d\sigma_s^{(1)} = \frac{1}{8\pi^2 k^2} \int d^2 p_1 d^2 p_2 \frac{1}{p_1 p_2} \Big[\frac{(p \cdot p_1)(p \cdot p_2)}{(p \cdot p_1 + p \cdot p_2)^2 (p_1 \cdot p_2)} + \frac{(p' \cdot p_1)(p' \cdot p_2)}{(p' \cdot p_1 + p' \cdot p_2)^2 (p_1 \cdot p_2)} - \frac{(p \cdot p_1)(p' \cdot p_2) + (p \cdot p_2)(p' \cdot p_1) - (p \cdot p')(p_1 \cdot p_2)}{(p_1 \cdot p_2)(p \cdot p_1 + p \cdot p_2)(p' \cdot p_1 + p' \cdot p_2)} \Big]$$

There can be a soft fermion emission also. Along the same procedure we get the result as

$$\mathcal{M}_f = \left(-i\frac{2\pi}{k}\right)\mathcal{M}_0(p,p')\left[\frac{\epsilon_{\mu\nu\rho}p^\mu(p_1+p_2)^\rho[\overline{u}(p_2)\gamma^\nu v(p_1)]}{2p\cdot(p_1+p_2)(p_1\cdot p_2)} - (p\leftrightarrow p')\right] \quad (5.13)$$

And after some Dirac matrices algebra, we get a fermion correction in cross-section

$$d\sigma_f^{(1)}(p,p') = -d\sigma_s^{(1)} + \frac{1}{8\pi^2k^2}\int d^2p_1d^2p_2\frac{1}{p_1p_2}\left[\frac{p\cdot p'}{(p\cdot p_1+p\cdot p_2)(p'\cdot p_1+p'\cdot p_2)}\right] \quad (5.14)$$

So the cancelation occurs between scalar and fermion part, the total cross-section becomes

$$d\sigma(p,p') = d\sigma_0(p\to p')\cdot\frac{1}{8\pi^2k^2}\int d^2p_1d^2p_2\frac{1}{p_1p_2}\left[\frac{p\cdot p'}{(p\cdot p_1+p\cdot p_2)(p'\cdot p_1+p'\cdot p_2)}\right] \quad (5.15)$$

We can see that there are well-known regions of integration which give IR divergence.

- Soft limit $\quad p_1, p_2 \to 0$ (5.16)
- Collinear limit $\quad p_1+p_2 \to \alpha p \quad \text{or} \quad \alpha p'$ (5.17)

Let's compute this correction term exactly. Using feynman parameter method, we can combine the denominator as

$$\frac{1}{8\pi^2k^2}\int d^2p_1d^2p_2\frac{1}{p_1p_2}\int_0^1 d\alpha\left[\frac{p\cdot p'}{(p_\alpha\cdot p_1+p_\alpha\cdot p_2)^2}\right] \quad (5.18)$$

where $p_\alpha = \alpha p + (1-\alpha)p'$. Let the θ_i be the angles between p_α and p_i. Then (5.3.1) becomes

$$\frac{1}{8\pi^2k^2}\int_0^1 d\alpha\int_0^\Lambda dp_1dp_2\int_0^{2\pi}d\theta_1d\theta_2\left[\frac{p\cdot p'}{(p_1(E_\alpha-p_\alpha\cos\theta_1)+p_2(E_\alpha-p_\alpha\cos\theta_2))^2}\right]$$

By setting $p_1 = p\cos\phi, p_2 = p\sin\phi$, we can rewrite above expression as

$$\frac{1}{8\pi^2k^2}\int_0^1 d\alpha\int_0^\Lambda dp\frac{1}{p}\int_0^{\frac{\pi}{2}}d\phi\int_0^{2\pi}d\theta_1d\theta_2\left[\frac{p\cdot p'}{(\cos\phi(E_\alpha-p_\alpha\cos\theta_1)+\sin\phi(E_\alpha-p_\alpha\cos\theta_2))^2}\right]$$

Integral of p variable gives well-known log divergence coming from soft radiation. And performing ϕ,θ_i integrals, we can get a simple result

$$\frac{1}{8\pi^2k^2}\int_0^\Lambda dp\frac{1}{p}\int_0^1 d\alpha\int_0^{2\pi}d\theta_1d\theta_2\left[\frac{p\cdot p'}{(E_\alpha-p_\alpha\cos\theta_1)(E_\alpha-p_\alpha\cos\theta_2)}\right]$$

$$= \frac{1}{4k^2}\left(\int_0^1 dp\frac{1}{p}\right)\left(\int_0^1 d\alpha\frac{1}{\alpha(1-\alpha)}\right) \tag{5.16}$$

which gives double log divergence. Because we have 4 kinds of matters, there are 4 possibilities of emitting soft matter particles. So final result for Bremsstrahlung correction is

$$d\sigma(p \to p' + p_1 + p_2) = d\sigma_0(p \to p') \cdot \frac{1}{k^2}\left(\int_0^1 dp\frac{1}{p}\right)\left(\int_0^1 d\alpha\frac{1}{\alpha(1-\alpha)}\right) \tag{5.17}$$

Vertex corrections

Now consider the correction of vertex form factor. One loop correction gives finite result and divergence occurs from two loop diagram.

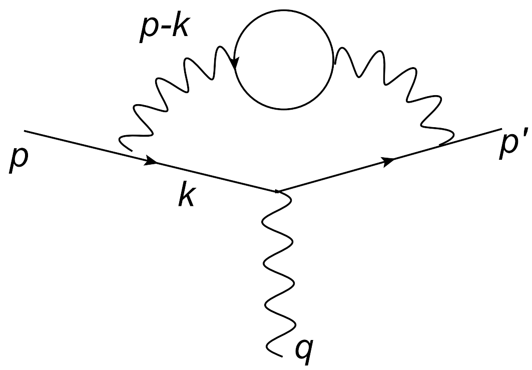

Figure 5.2: loop correction diagrams

$$\mathcal{M}^{(2)} = i\left(\frac{2\pi^2}{k^2}\right)\int\frac{d^3\ell}{(2\pi)^3}\left[\frac{Num}{[\ell^2+xyq^2]^{\frac{5}{2}}}\frac{z^{-\frac{1}{2}}\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})} - \frac{(2p+q)^\mu}{\ell^3}\right] \tag{5.18}$$

where $Num = (k^2 + 2k\cdot p' - q^2)(2k+q)^\mu$ with $k = \ell - yq + zp$.

We can see that second term in bracket gives UV and IR divergence which goes as Log function. Because ABJM theory is UV finite, this UV divergence will be canceled with other diagrams. So we will just drop all UV divergence term in this calculations and concentrate on IR divergent terms.

Now take a look at the Num in the first term in bracket. From the symmetry of ℓ variable in integral, we can see that there can be ℓ^2 and ℓ^0 term in Num. Because the ℓ^2 term gives UV divergence, we only concentrate on ℓ^0 term.

$$Num = (y^2q^2 - 2yzq\cdot p - 2yq\cdot p' + 2zp\cdot p' - q^2)((1-2y)q^\mu + 2zp^\mu) + O(\ell^2) \quad (5.19)$$

Performing the ℓ variable integral, we obtain

$$\mathcal{M}^{(2)} = \left(\frac{i}{4k^2}\right)\int dxdydz\frac{(y^2q^2 - 2yzq\cdot p - 2yq\cdot p' + 2zp\cdot p' - q^2)((1-2y)q^\mu + 2zp^\mu)}{z^{\frac{1}{2}}xyq^2} \quad (5.20)$$

Most divergent part comes from the region of $x, y \to 0$ and $z \to 1$. So leading term is

$$\mathcal{M}^{(2)} \sim \left(\frac{i}{4k^2}\right)\int dxdydz\delta(x+y+z-1)\frac{(2p\cdot p' - q^2)(2p^\mu + q^\mu)}{xyq^2} \quad (5.21)$$

$$= \left(-\frac{1}{2k^2}\right)[i(2p^\mu + q^\mu)]\int_0^1 dz\int_0^{1-z} dy\frac{1}{y(1-y-z)} \quad (5.22)$$

$$= \left(-\frac{1}{2k^2}\right)[i(2p^\mu + q^\mu)]\int_0^1 d\beta\frac{1}{\beta}\int_0^1 d\alpha\frac{1}{\alpha(1-\alpha)} \quad (5.23)$$

So the form factor has loop correction as

$$\mathcal{M} = \left(1 - \frac{1}{2k^2}\int_0^1 d\beta\frac{1}{\beta}\int_0^1 d\alpha\frac{1}{\alpha(1-\alpha)}\right)[i(2p^\mu + q^\mu)] \quad (5.24)$$

and gives correction on cross-section by

$$d\sigma = d\sigma_0\left(1 - \frac{1}{k^2}\int_0^1 d\beta\frac{1}{\beta}\int_0^1 d\alpha\frac{1}{\alpha(1-\alpha)}\right) \quad (5.25)$$

As a result we can see that 2 double log IR divergences each from soft Bremsstrahlung and loop correction are canceled each other.

5.3.2 Cross-section

Consider n point scattering cross-sections in ABJM theory. Because this is a physical observable, it should be IR finite if we calculate it properly.As we have seen in previous feynman diagram approach, we can predict that IR divergence coming from loop corrections would be canceled by including 2 soft particle emissions.

The cross-sections can be obtained by squaring well-known Color ordered amplitudes.

$$\mathcal{A}_n(1,2,...,n) = \sum_{color} Tr[T^{a_1}T^{a_2}..T^{a_n}]A_n(1,2,...,n) \tag{5.26}$$

$$\sigma_n = |\mathcal{A}_n(1,2,...,n)|^2 = N^{n-2}(N^2-1)\sum_{color}|A_n(1,2,...,n)|^2 + \mathcal{O}(\frac{1}{N}) \tag{5.27}$$

In large N limit, the cross terms of different color order are negligible. So by computing the square of color-ordered amplitude $|A_n(1,2,...,n)|^2$ with soft emission, we can study the IR finiteness of cross sections.

4 scalar scattering Cross-section

Let's see the most simple example - 4 point amplitude of bosons. In 2 loop order, it has the IR divergence of $\frac{1}{\epsilon^2}$ order.

$$A_4^{(2)} = A_4^{tree}\cdot\lambda^2\left[-\frac{(s/\mu^2)^{-2\epsilon}}{(2\epsilon)^2} - \frac{(t/\mu^2)^{-2\epsilon}}{(2\epsilon)^2} + finite\right] \tag{5.28}$$

This should be regularized by 4 point amplitude with additional 2 soft emissions. By considering 6 point amplitude with 2 momentum goes soft limit, we can obtained IR corrections. In 6 point amplitude, we checked every possible cases of soft emissions and concluded that divergence occurs only in the case that **2 soft momentums are in adjacent points**.

Let the 5,6 particles are soft matter in 6 point color-ordered amplitude.

$$|p_5|,|p_6| \ll \Lambda \sim |p_i| \tag{5.29}$$

$$|A_6(1,2,3,4,5,6)|^2 = |A_4(1,2,3,4)|^2 \cdot R^2(1,4,5,6) + \mathcal{O}(\frac{1}{\Lambda^3}) \tag{5.30}$$

$$R(1,4,5,6) = \frac{2\pi}{k}\langle 14\rangle \left(\frac{\langle 51\rangle\langle 54\rangle - \langle 61\rangle\langle 64\rangle}{(\langle 51\rangle^2 + \langle 61\rangle^2)(\langle 54\rangle^2 + \langle 64\rangle^2)} \right) \tag{5.31}$$

This result works in both 2 boson, 2 fermion soft emissions with the same $R(1,4,5,6)$ functions. The R^2 term diverge as $\frac{1}{\Lambda^4}$. Because we will perform 2 particle phase space integral which goes $\Lambda^3 d\Lambda$, terms of $\mathcal{O}(\frac{1}{\Lambda^3})$ become negligible in soft limit.

So our 4 point amplitude with 2 soft emissions becomes the following expression.

$$|A_4^{\text{soft}(56)}|^2 = 2 \cdot 4 \cdot |A_4|^2 \cdot \frac{4\pi^2}{k^2}\langle 14\rangle^2 \int \frac{d^2p_5}{(2\pi)^2}\frac{d^2p_6}{(2\pi)^2}\frac{1}{2p_5}\frac{1}{2p_6}\left(\frac{\langle 51\rangle\langle 54\rangle - \langle 61\rangle\langle 64\rangle}{(\langle 51\rangle^2 + \langle 61\rangle^2)(\langle 54\rangle^2 + \langle 64\rangle^2)} \right)^2 \tag{5.31}$$

where factor 2 is coming from the fact we have also fermions emissions and 4 is we have 4 kinds of matter related to R symmetry.

For given massless momentum p in 3 dimension, λ^α can be chosen uniquely up to overall sign.

$$p^0 = \frac{(\lambda^1)^2 + (\lambda^2)^2}{2}, \quad p^1 = \frac{-(\lambda^1)^2 + (\lambda^2)^2}{2}, \quad p^2 = \lambda^1\lambda^2 \tag{5.32}$$

$$\int_{-\infty}^{\infty} \frac{d^2p}{2p^0} = \int_{-\infty}^{\infty} d\lambda^1 \int_0^{\infty} d\lambda^2 \tag{5.33}$$

Interval for $d\lambda_2$ was chosen to positive region due to the overall sign ambiguity. Because integrand R^2 is symmetric under the $\lambda \to -\lambda$, interval can be written as $\frac{1}{2}\int_{-\infty}^{\infty} d\lambda^1 d\lambda^2$

$$|A_4^{\text{soft}(56)}|^2 = |A_4|^2 \cdot \frac{1}{2\pi^2 k^2}\langle 14\rangle^2 \int_{-\infty}^{\infty} d^2\lambda_5 d^2\lambda_6 \left(\frac{\langle 51\rangle\langle 54\rangle - \langle 61\rangle\langle 64\rangle}{(\langle 51\rangle^2 + \langle 61\rangle^2)(\langle 54\rangle^2 + \langle 64\rangle^2)} \right)^2 \tag{5.33}$$

To perform the above integral, we use the coordinate transformation in following way.

$$X = (x_1, x_2), \quad Y = (y_1, y_2) \tag{5.34}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 & -\lambda_1^1 & 0 & 0 \\ 0 & 0 & \lambda_1^2 & -\lambda_1^1 \\ 0 & 0 & \lambda_4^2 & -\lambda_4^1 \\ \lambda_4^2 & -\lambda_4^1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \lambda_5^1 \\ \lambda_5^2 \\ \lambda_6^1 \\ \lambda_6^2 \end{pmatrix} \tag{5.35}$$

In this coordinate, integral expression can be written in simple form.

$$\int_{-\infty}^{\infty} d^2\lambda_5 d^2\lambda_6 \left(\frac{\langle 51\rangle\langle 54\rangle - \langle 61\rangle\langle 64\rangle}{(\langle 51\rangle^2 + \langle 61\rangle^2)(\langle 54\rangle^2 + \langle 64\rangle^2)} \right)^2 = \frac{1}{\langle 14\rangle^2} \int d^2X d^2Y \left(\frac{X \times Y}{X^2 Y^2} \right)^2 \tag{5.36}$$

We can easily see that above integral gives double logarithm divergence. To regularize this, we perform the integral in $d = 3 - 2\epsilon$ dimension. ($\epsilon < 0$)

$$\frac{\mu^{4\epsilon}}{\langle 14\rangle^2} \int d^{2-2\epsilon}X d^{2-2\epsilon}Y \left(\frac{\vec{X} \times \vec{Y}}{X^2 Y^2} \right)^2 \tag{5.37}$$

where μ is the mass scale of dimensional regularization. Because we are performing soft particle phase integral, $|X|, |Y|$ should be less then ordinary energy scale of scattering. If we are considering the leading order of divergence, we can set the cutoff $\Lambda^2 \sim \langle 14\rangle^2$ because momentums participating are p_1, p_4. So we can obtain the following result.

$$|A_4^{\text{soft}(56)}|^2 = |A_4|^2 \cdot \frac{1}{2\pi^2 k^2} \cdot 2\pi^2 \frac{(t/\mu^2)^{-2\epsilon}}{(2\epsilon)^2} \tag{5.38}$$

For final expression of 4 point cross-section, we have to sum over all possible color permutation. Because 2 soft emissions should be in adjacent positions, there are 4 possible permutations in given ordered external particles. It is related to the possible insertions of soft matters between $i, i+1$ external particles.

$$\sigma_4^{\text{soft}} = N^4(N^2 - 1) \sum_{6 \text{ color}} |A_4^{\text{soft}(56)}|^2 \tag{5.39}$$

$$= N^2(N^2 - 1) \sum_{4 \text{ color}} \left[|A_4|^2 \cdot 2 \left(\frac{N^2}{k^2} \right) \frac{(s/\mu^2)^{-2\epsilon}}{(2\epsilon)^2} + |A_4|^2 \cdot 2 \left(\frac{N^2}{k^2} \right) \frac{(t/\mu^2)^{-2\epsilon}}{(2\epsilon)^2} \right] \tag{5.40}$$

$$= \sigma_4^{\text{tree}} \cdot \left(2\lambda^2 \frac{(s/\mu^2)^{-2\epsilon}}{(2\epsilon)^2} + 2\lambda^2 \frac{(t/\mu^2)^{-2\epsilon}}{(2\epsilon)^2} \right) \tag{5.41}$$

2 loop corrections in cross-section is

$$A_4 = A_4^{tree}\left[1-\lambda^2\left(\frac{(s/\mu^2)^{-2\epsilon}}{(2\epsilon)^2}+\frac{(t/\mu^2)^{-2\epsilon}}{(2\epsilon)^2}\right)+\mathcal{O}(\lambda^3)\right] \tag{5.42}$$

$$\begin{aligned} \sigma_4 &= \sum_{4\text{ color}} |A_4^{tree}|^2\left[1-\lambda^2\left(\frac{(s/\mu^2)^{-2\epsilon}}{(2\epsilon)^2}+\frac{(t/\mu^2)^{-2\epsilon}}{(2\epsilon)^2}\right)+\mathcal{O}(\lambda^3)\right]^2 && (5.43)\\ &= \sigma_4^{\text{tree}}\cdot\left[1-2\lambda^2\left(\frac{(s/\mu^2)^{-2\epsilon}}{(2\epsilon)^2}+\frac{(t/\mu^2)^{-2\epsilon}}{(2\epsilon)^2}\right)+\mathcal{O}(\lambda^3)\right] && (5.44)\end{aligned}$$

So we can see that 4 point cross-sections is IR finite in λ^2 order. We want to emphasize that this result is valid only in the leading order of divergence $\frac{1}{\epsilon^2}$. To consider the next sub-leading order $\frac{1}{\epsilon}$, we need to be careful in picking up the effective integrand and setting the momentum cutoff.

Part III

Studies on Higher-spin theories

Chapter 6

Asymptotic symmetry of higher-spin gravity

In the context of holographic duality which usually relates gravity theory and conformal field theory, asymptotic symmetry of given gravity system plays central role. Asymptotic symmetry is a residual symmetry which preserves asymptotic boundary conditions. Studying asymptotic symmetry reveals some hint of symmetry structure of dual conformal field theory.

For three dimensional AdS gravity, Brown-Henneaux [92] found that asymptotic symmetry is enlarged from $SO(2,2)$ isometry group to Virasoro algebra. According to AdS/CFT duality, AdS_3 gravity correspond to two dimensional CFT. As we have seen in chapter 2, conformal symmetry in two dimension has infinite generators and becomes Virasoro algebra.

In this chapter, we will study the asymptotic symmetry of three dimensional AdS gravity including higher-spin fields. Higher-spin(HS) theory is the extension of massless spin 2 gauge theory - gravity - including fields with spin higher than 2. HS theory in important in several reasons. It provides consistent framework of gravity system with arbitrary spin particles. It is also related to tension-less string theory which naturally contains higher-spin fields. If we consider HS gauge symmetry as unbroken symmetry, string theory can be taken as a theory under spontaneous symmetry breaking.

By no-go theorem [165–169], extension to HS fields in flat space is impossible. By series of works of Vasiliev [170–179], this difficulty can be overcome by considering curved space. In this chapter we will consider HS theory in AdS space in three dimension. In three dimension, there is a nice formulation for HS theory with two-copies of Chern-Simons actions.

We will first review of derivation of asymptotic symmetry in pure gravity. And then we will extend to AdS HS theory or super-symmetric HS theory and find the algebra of asymptotic symmetry is super W_∞ algebra.

6.1 Pure gravity in 3 dimension

Three-dimensional gravity with negative cosmological constant have vacuum solution of AdS_3 space. Isometry group of AdS_3 is $SO(2,2) \sim SL(2,R) \times SL(2,R)$. Let us see how this isometry group is enlarged in asymptotic boundary condition. The metric of AdS_3 is written as

$$ds^2 = \ell^2(-\cosh^2\rho d\tau^2 + \sinh^2\rho d\phi^2 + d\rho^2) \tag{6.1}$$

and asymptotic boundary space where ρ is large becomes

$$ds^2 \sim \ell^2(-e^{2\rho}d\tau^+ d\tau^- + d\rho^2) \tag{6.2}$$

Here, $\pm$ is the light-cone coordinate $\tau^\pm = \tau \pm \phi$. Asymptotic symmetry is the general coordinate transformation which preserve this boundary form. Under the general coordinate transformation $g_{\mu\nu} \to g_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$, solutions for ξ which preserve (6.2) are [4]

$$\xi^+ = f(\tau^+) + \frac{e^{-2\rho}}{2} g''(\tau^-) + \mathcal{O}(e^{-4\rho}) \tag{6.3}$$

$$\xi^- = g(\tau^-) + \frac{e^{-2\rho}}{2} f''(\tau^+) + \mathcal{O}(e^{-4\rho}) \tag{6.4}$$

$$\xi^\rho = -\frac{f'(\tau^+)}{2} - \frac{g'(\tau^-)}{2} + \mathcal{O}(e^{-2\rho}) \tag{6.5}$$

where $f(\tau^+), g(\tau^-)$ are arbitrary holomorphic and anti-holomorphic functions. From this, we can easily see that asymptotic symmetry forms Virasoro algebra. Brown-henneaux [92] calculated explicitly and found that result is Virasoro algebra with central charge

$$c = \frac{3\ell}{2G} \tag{6.6}$$

As we explained before, this Virasoro algebra implies that boundary theory is 2 dimensional conformal field theory.

For the extension to HS theory, let us review about Chern-Simons formulation of three-dimensional AdS gravity [100, 101]. For the vielbein e_a^μ and spin connection ω_a^μ, we can construct $SL(2,R)$ gauge field A_μ^a as

$$A_\mu^a = \omega_a^\mu + \frac{e_a^\mu}{\ell}, \qquad \widetilde{A}_\mu^a = \omega_a^\mu - \frac{e_a^\mu}{\ell} \tag{6.7}$$

where ℓ ia a radius of AdS curvature. Then Einstein-Hilbert action of three dimensional gravity with negative cosmological constant can be written as

$$S[A, \widetilde{A}] = S_{CS}[A] - S_{CS}[\widetilde{A}] \tag{6.8}$$

$$S_{CS}[A] = \frac{k}{4\pi} \int Tr \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \tag{6.9}$$

This is the two copies of Chern-Simons action of each gauge group $SL(2,R)$ with $k = \frac{\ell}{4G}$. We will use this formulation to extend to HS gravity and super-symmetric HS gravity.

6.2 Higher-spin super-gravity

6.2.1 The $\mathbf{shs^E}(N|2,\mathbb{R}) \oplus \mathbf{shs^E}(M|2,\mathbb{R})$ super-algebras

Extended supergravities and higher spins in $2+1$ dimensions

As we have seen in the last section, three-dimensional Einstein gravity with a negative cosmological constant (AdS_3 gravity) can be reformulated as a Chern-Simons gauge theory whose gauge connection take values in the isometry algebra

$sl(2,\mathbb{R})_{+k} \oplus sl(2,\mathbb{R})_{-k}$. Here $\pm k$ denotes the Chern-Simons levels of the chiral and anti-chiral sectors and is related to the gravitational coupling constant through formula (6.7) below.

The reformulation can be generalized to $\mathcal{N} = (N, M)$-extended AdS_3 supergravity [100], where M, N refer to the supersymmetry of the two gauge factors. Recall the isomorphism

$$\mathrm{sl}(2,\mathbb{R}) \simeq \mathrm{sp}(2,\mathbb{R}) \simeq \mathrm{so}(2,1) \simeq \mathrm{su}(1,1). \tag{6.10}$$

Then, AdS_3 supergravity theories are obtainable by taking an appropriate superalgebra containing (6.10) as a bosonic subalgebra.

For example, $\mathcal{N} = (1, 1)$ AdS_3 supergravity can be reformulated as a Chern-Simons super-gauge theory whose gauge super-connection takes values in the Lie superalgebra

$$\mathrm{osp}(1|2,\mathbb{R})_{+k} \oplus \mathrm{osp}(1|2,\mathbb{R})_{-k}.$$

Likewise, $\mathcal{N} = (N, M)$-extended supergravity is based on a super-connection taking values in the Lie superalgebra[1]

$$\mathrm{osp}(N|2,\mathbb{R})_{+k} \oplus \mathrm{osp}(M|2,\mathbb{R})_{-k}.$$

In all these cases, either chiral copy contains $\mathrm{sp}(2,\mathbb{R})$ as a bosonic subalgebra. The bosonic subalgebra of $\mathrm{osp}(N|2,\mathbb{R})$ is actually of the form $\mathrm{sl}(2,\mathbb{R}) \oplus \mathcal{G}$, where $\mathcal{G}$ is here $\mathrm{so}(N)$. The fermonic generators transform as spinors of $\mathrm{sp}(2,\mathbb{R})$ and vectors of $\mathrm{so}(N)$.

More generally, one can take the gauge superalgebra to be a direct sum of two simple superalgebras $\mathcal{A}_L, \mathcal{A}_R$:

$$\mathcal{A}_L \oplus \mathcal{A}_R, \tag{6.11}$$

with the conditions that (i) each superalgebra contains any of (6.10) as a bosonic subalgebra; and (ii) the fermionic generators transform in the **2** of (6.10). It has

[1]In what follows, we shall omit the Chern-Simons level specification in the gauge superalgebras. They can be reconstructed from the context.

$\mathcal{A}$	$\mathcal{G}$	ρ	D
$\mathrm{osp}(N\|2,\mathbb{R})$	$\mathrm{so}(N)$	$\mathbf{N}$	$\frac{N(N-1)}{2}$
$\mathrm{su}(1,1\|N)$ $(N\neq 2)$	$\mathrm{su}(N)\oplus\mathrm{u}(1)$	$\mathbf{N}+\overline{\mathbf{N}}$	N^2
$\mathrm{su}(1,1\|2)$ / $\mathrm{u}(1)$	$\mathrm{su}(2)$	$\mathbf{2}+\overline{\mathbf{2}}$	3
$\mathrm{osp}(4^*\|2M)$	$\mathrm{su}(2)\oplus\mathrm{usp}(2M)$	$(\mathbf{2M},\mathbf{2})$	$M(2M+1)+3$
$\mathrm{D}^1(2,1;\alpha)$	$\mathrm{su}(2)\oplus\mathrm{su}(2)$	$(\mathbf{2},\mathbf{2})$	6
$\mathrm{G}(3)$	G_2	$\mathbf{7}$	14
$\mathrm{F}(4)$	$\mathrm{spin}(7)$	$\mathbf{8}_{\mathrm{s}}$	21

Table 6.1: Superalgebras of extended anti-de Sitter supegravities in $2+1$ dimensions. Here, $\mathcal{A}$ is the (extended) superalgebra, $\mathcal{G}$ is the internal subalgebra, ρ is the representation of $\mathcal{G}$ in which the spinors transform, and D is the dimension of $\mathcal{G}$. The first four superalgebras belong to the $\mathrm{osp}(m,2n)$ and $\mathrm{spl}(m,n)$ infinite families, while the last three are the "exceptional" Lie superalgebras.

been shown [102–104] that this condition is satisfied in only seven classes, which are listed in Table 1. Thus, the most general (N,M)-extended AdS_3 supergravity can be defined as the Chern-Simons gauge theory whose gauge super-connections in the chiral and anti-chiral sectors take values in any two of the seven Lie superalgebras of Table 1.

Just as extended AdS_3 supergravity can be formulated as a Chern-Simons super-gauge theory, consistent higher-spin AdS_3 supergravity theories can also be formulated as Chern-Simons super-gauge theories [88]. This time, however, the gauge superalgebras $\mathcal{A}_L,\mathcal{A}_R$ are infinite-dimensional. Since the standard AdS_3 supergravity ought to be a consistent truncation of these theories, it must be that these infinite-dimensional gauge superalgebras contain the simple superalgebras in Table 1 as subalgebras. In other words, the higher-spin superalgebras are infinite-dimensional extensions of these simple superalgebras.

We shall mostly concentrate on the $\mathrm{osp}(N|2,\mathbb{R})$ class, since this is the class that encompasses uniformly all extended supersymmetries on each chiral sector. We first need an infinite-dimensional extension of $\mathrm{osp}(N|2,\mathbb{R})\oplus\mathrm{osp}(M|2,\mathbb{R})$ to a suitable higher-spin superalgebra. Fortunately, the construction of the relevant superalgebras were worked out already in [87, 88]. These superalgebras are denoted $\mathrm{shs}^{\mathrm{E}}(N|2,\mathbb{R}) \oplus \mathrm{shs}^{\mathrm{E}}(M|2,\mathbb{R})$[2]. In a nutshell, the higher-spin superalgebra so constructed corresponds to the universal enveloping superalgebra of the underlying finite-dimensional sub-superalgebras $\mathrm{osp}(N|2,\mathbb{R})\oplus$ $\mathrm{osp}(M|2,\mathbb{R})$ quotentized by certain ideals. This fits also with the requirement that the standard AdS_3 supergravity be a consistent truncation of the higher-spin AdS_3 supergravity.

In this section, we explain the higher-spin superalgebra $\mathrm{shs}^{E}(N|2,\mathbb{R})$ and its simplest realization in terms of "super-oscillators". In this realization, the minimal $N = 1$ case is special since it admits another equivalent formulation with a smaller number of oscillators. We shall mention this aspect along the way as we discuss the general N cases.

Polynomial realization of $\mathbf{shs^E}(N|2,\mathbb{R})$

In this part, we realize the Lie super-algebra $\mathrm{shs}^{\mathrm{E}}(N|2,\mathbb{R})$ in terms of "oscillator" polynomials.

General N Consider the following $N + 2$ Grassmann variables: two commuting ones, q_α ($\alpha = 1, 2$), together with N anticommuting ones, ψ_i ($i = 1, \ldots, N$). Adapting to the terminology used in the literature, we refer to the index i as the 'color' index. As such,

$$\begin{aligned}
q_\alpha q_\beta &= q_\beta q_\alpha \qquad && \forall\ \alpha, \beta = 1, 2 \\
\psi_i \psi_j &= -\psi_j \psi_i \qquad && \forall\ i, j = 1, \ldots, N \\
q_\alpha \psi_i &= \psi_i q_\alpha \qquad && \forall\ \alpha = 1, 2 \quad \& \quad i = 1, \ldots, N.
\end{aligned}$$

These variables are all taken to be real, $q^*_\alpha = q_\alpha$, $\psi^*_i = \psi_i$. We construct polynomials in these $N + 2$ variables, with coefficients that can be themselves commuting or anticommuting, i.e., that belong also to a different Grassmann algebra $\mathcal{G}$. Thus, we

[2] The two simple algebras describes each chiral sector and can be analyzed separately. From now on we shall focus on the first chiral piece

formally consider the (graded) tensor product $\mathcal{A} = \mathcal{G} \otimes \mathcal{P}$ of the polynomial algebra $\mathcal{P}$ in q_α, ψ_i with the Grassmann algebra $\mathcal{G}$. The sign in the commutation relations for the multiplication of elements in the graded tensor product is dictated by the total grading, so that odd elements of $\mathcal{G}$ and $\mathcal{P}$ anticommute. The Grassmann parity used below will always be the total grading. A complex conjugation is assumed to be defined in $\mathcal{G}$, and can be extended to $\mathcal{A}$ taking into account that q_α and ψ_i are real. We systematically use the convention $(ab)^* = b^* a^*$.

Let $\mathcal{A}^E$ be the subalgebra of Grassmann-even polynomials in q_α, ψ_i containing only monomials of even degree and no constant term. Thus, a general element of $\mathcal{A}^E$ reads

$$\begin{aligned} f &= f^{\alpha\beta} q_\alpha q_\beta + f^{\alpha,i} q_\alpha \psi_i + f^{ij} \psi_i \psi_j \\ &+ f^{\alpha\beta\gamma\delta} q_\alpha q_\beta q_\gamma q_\delta + f^{\alpha\beta\gamma,i} q_\alpha q_\beta q_\gamma \psi_i + f^{\alpha\beta,ij} q_\alpha q_\beta \psi_i \psi_j + \dots \\ &+ f^{\alpha\beta\gamma\delta\epsilon\eta} q_\alpha q_\beta q_\gamma q_\delta q_\epsilon q_\eta + \dots \\ &+ \dots , \end{aligned} \tag{6.12}$$

with finitely many terms. The coefficients in this expansion are completely symmetric (respectively, antisymmetric) in the Greek (respectively, Latin) indices. They are commuting (respectively, anticommuting) whenever they multiply an even (respectively, odd) number of ψ's. When we formulate higher-spin AdS_3 supergravity as a Chern-Simons super-gauge theory, the gauge super-connection will be taken to be of the form (6.12). The coefficients in the expansion will then be identified with commuting or anticommuting spacetime fields.

A $\star$-product is defined on $\mathcal{A}$ as follows:

$$(f \star g)(z'') \equiv \exp\left(i\, \epsilon_{\alpha\beta} \frac{\partial}{\partial q_\alpha} \frac{\partial}{\partial q'_\beta} + \delta_{ij} \frac{\overleftarrow{\partial}}{\partial \psi_i} \frac{\overrightarrow{\partial}}{\partial \psi'_j} \right) f(z) g(z') \Big|_{z=z'=z''} , \tag{6.13}$$

where $f(z) \equiv f(q_\alpha, \psi_i)$ and so on. In this expression, $f(z)g(z')$ is the standard Grassmann product. The operation (6.13) is called the $\star$-product. Left and right derivatives with respect to the anticommuting variables are defined by

$$\begin{aligned} \delta f &= \delta\psi_i \frac{\overrightarrow{\partial} f}{\partial\psi_i} \\ \delta f &= \frac{\overleftarrow{\partial} f}{\partial\psi_i}\delta\psi_i . \end{aligned} \tag{6.14}$$

The epsilon symbol is explicitly taken to be

$$(\epsilon^{\alpha\beta}) \equiv (\epsilon_{\alpha\beta}) \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \alpha,\, \beta \in \{1,2\}. \tag{6.15}$$

The above $\star$-product is well known to be associative. However, it does not preserve the reality condition, in the sense that $f \star g$ is not real even if f and g are so. On the other hand, one can check that if f and g are both real elements of $\mathcal{A}^E$, or both pure imaginary elements of $\mathcal{A}^E$, of respective order $2n$ and $2m$, then the homogenous polynomials appearing in the expansion of $f \star g$,

$$f \star g = \sum_{j=0}^{m+n} h_{2(m+n-j)} \,, \tag{6.16}$$

are alternatively real and imaginary. More precisely, the homogeneous polynomial $h_{2(m+n-j)}$ of degree $2(m+n-j)$ in q_α, ψ_i is:

- real and symmetric for the exchange of f and g when j is even;
- imaginary and antisymmetric for the exchange of f and g when j is odd.

We then define the $\star$-commutator (also called "$\star$-bracket"),

$$[f, g]_\star \equiv f \star g - g \star f \,, \tag{6.17}$$

which fulfills the Jacobi identity since the $\star$-product is associative. From what we have just seen, $[f, g]_\star$ is pure imaginary whenever f and g are both real or both pure imaginary.

The Lie superalgebra $\mathrm{shs}^{\mathrm{E}}(N|2,\mathbb{R})$ is the real subspace of pure imaginary elements of $\mathcal{A}^E$ equipped with the $\star$-bracket[3]. A general element of $\mathrm{shs}^{\mathrm{E}}(N|2,\mathbb{R})$ is

[3] One could equivalently insert a factor of i in the definition of the $\star$-bracket, which would no longer coincide with the star commutator, and define $\mathrm{shs}^{\mathrm{E}}(N|2,\mathbb{R})$ as the subspace of real polynomials equipped with that alternative bracket. Either convention has its own advantages.

thus of the above form

$$
\begin{aligned}
f = & f^{\alpha\beta} q_\alpha q_\beta + f^{\alpha,i} q_\alpha \psi_i + f^{ij} \psi_i \psi_j \\
& + f^{\alpha\beta\gamma\delta} q_\alpha q_\beta q_\gamma q_\delta + f^{\alpha\beta\gamma,i} q_\alpha q_\beta q_\gamma \psi_i + f^{\alpha\beta,ij} q_\alpha q_\beta \psi_i \psi_j + \dots \\
& + f^{\alpha\beta\gamma\delta\epsilon\eta} q_\alpha q_\beta q_\gamma q_\delta q_\epsilon q_\eta + \dots \\
& + \dots,
\end{aligned}
\tag{6.18}
$$

but the coefficients are further restricted so as to make f imaginary. So, for instance, the coefficient $f^{\alpha\beta}$ is imaginary while $f^{\alpha,i}$ and f^{ij} are real.

One can rewrite alternatively (6.17) as

$$
\begin{aligned}
[f,g]_\star(z'') = & \left(2i \sin\left(\epsilon_{\alpha\beta} \frac{\partial}{\partial q_\alpha} \frac{\partial}{\partial q'_\beta} \right) \cosh\left(\delta_{ij} \frac{\overleftarrow{\partial}}{\partial \psi_i} \frac{\overrightarrow{\partial}}{\partial \psi'_j} \right) \right. \\
& \left. + 2\cos\left(\epsilon_{\alpha\beta} \frac{\partial}{\partial q_\alpha} \frac{\partial}{\partial q'_\beta} \right) \sinh\left(\delta_{ij} \frac{\overleftarrow{\partial}}{\partial \psi_i} \frac{\overrightarrow{\partial}}{\partial \psi'_j} \right) \right) f(z) g(z') \Big|_{z=z'=z''}.
\end{aligned}
\tag{6.19}
$$

It should be stressed that the polynomial $[f,g]_\star$ starts at highest polynomial degree $2(n+m-1)$. Note also that the lowest polynomial degree term in the expansion (6.16) is $h_{2(|m-n|)}$ so that there is a term of degree zero in (6.16) only if $n = m$, in which case $j = 2m$ is even, which implies that the term of degree zero (when present) is symmetric for the exchange of f with g. This implies in particular that the constant term (when present in $f \star g$) drops from the $\star$-commutator so that $[f,g]_\star$ has indeed no constant term and belongs to $\mathrm{shs}^{\mathrm{E}}(N|2,\mathbb{R})$.

Supertrace and scalar product The supertrace of a polynomial in the q's and the ψ's is defined by its component of degree zero:

$$
\mathrm{STr} f(q,\psi) = 8 f(0). \tag{6.20}
$$

The normalization is chosen to match standard conventions in the normalization of the action below. Thus, elements in $\mathrm{shs}^{\mathrm{E}}(N|2,\mathbb{R})$ all have zero supertrace.

Although $\mathrm{STr} f = 0 \; \forall f \in \mathrm{shs}^{\mathrm{E}}(N|2,\mathbb{R})$, it turns out that $\mathrm{STr}(f \star g)$ may differ from zero even if $f, g \in \mathrm{shs}^{\mathrm{E}}(N|2,\mathbb{R})$. One thus defines a scalar product on $\mathrm{shs}^{\mathrm{E}}(N|2,\mathbb{R})$ by

$$
(f,g) \equiv \mathrm{STr}(f \star g). \tag{6.21}
$$

The scalar product is evidently bilinear, real and symmetric (given our discussions in the previous subsection). Using the symmetry together with the associativity of the $\star$-product, we further conclude that it is also invariant:

$$([f,g]_\star, h) = (f, [g,h]_\star). \tag{6.22}$$

In addition, it is non-degenerate. It is non-zero only when f and g have same degree in both the ψ_i's and the q_α's. It is this scalar product that will be used to define the Chern-Simons action below.

Basis A basis of $\mathrm{shs}^{\mathrm{E}}(N,2|\mathbb{R})$ is given by the monomials

$$X_{p,q;\,i_1,i_2,\cdots,i_N} \equiv \frac{i^{\lfloor\frac{K+1}{2}\rfloor}}{2\,i\,p!\,q!} q_1^p q_2^q \psi_1^{i_1}\dots\psi_N^{i_N}, \tag{6.23}$$

where $p, q \in \mathbb{N}$ and $i_k \in \{0,1\}$. The degree of $X_{p,q;\,i_1,i_2,\cdots,i_N}$, which is $p+q+K$, must be even and positive, where $K = \sum_{k=1}^N i_k$ is the degree in the ψ's. The power of i has been inserted in such a way that the elements of even Grassman parity are imaginary, while those of odd Grassman parity are real.

With this choice, a general element of $\mathrm{shs}^{\mathrm{E}}(N|2,\mathbb{R})$ is of the form

$$\sum \mu^{p,q;\,i_1,i_2,\cdots,i_N} X_{p,q;\,i_1,i_2,\cdots,i_N} \tag{6.24}$$

where the coefficients $\mu^{p,q;\,i_1,i_2,\cdots,i_N}$ are real and of Grassman parity $(-1)^K = (-1)^{p+q}$.

osp$(N|2,\mathbb{R})$ sub-superalgebra The subspace of quadratic polynomials is a subalgebra isomorphic to $\mathrm{osp}(N|2,\mathbb{R})$, as it is known from the familiar oscillator realization of $\mathrm{osp}(N|2,\mathbb{R})$ [104]. Renormalizing and relabeling[4] the quadratic basis elements as

$$Y_{\alpha\beta} = -\frac{i}{2} q_\alpha q_\beta, \quad X_{\alpha i} = \frac{1}{2} q_\alpha \psi_i, \quad X_{ij} = \frac{1}{2}\psi_i\psi_j \tag{6.25}$$

[4]Note that we have changed the letter X to Y for the generators with no ψ's since these differ from the corresponding X's by a factor.

one finds that the non-zero Lie superbrackets read explicitly

$$\begin{aligned}
[Y_{\alpha\beta}, Y_{\gamma\delta}] &= \epsilon_{\alpha\gamma} Y_{\beta\delta} + \epsilon_{\alpha\delta} Y_{\beta\gamma} + \epsilon_{\beta\gamma} Y_{\alpha\delta} + \epsilon_{\beta\delta} Y_{\alpha\gamma} \\
[X_{\alpha i}, Y_{\beta\gamma}] &= \epsilon_{\alpha\beta} X_{\gamma i} + \epsilon_{\alpha\gamma} X_{\beta i} \\
\{X_{\alpha i}, X_{\beta j}\} &= i\,(\epsilon_{\alpha\beta} X_{ij} - \delta_{ij} Y_{\alpha\beta}) \\
[X_{ij}, X_{\alpha k}] &= \delta_{jk} X_{\alpha i} - \delta_{ik} X_{\alpha j} \\
[X_{ij}, X_{kl}] &= \delta_{il} X_{jk} + \delta_{jk} X_{il} - \delta_{ik} X_{jl} - \delta_{jl} X_{ik}.
\end{aligned} \tag{6.26}$$

Hence, one goes from $\mathrm{shs^E}(N|2,\mathbb{R})$ to $\mathrm{osp}(N|2,\mathbb{R})$ by restricting the $\star$-algebra of polynomials of even degree in the q's and the ψ's to the $\star$-subalgebra of polynomials of second degree. Conversely, one goes from $\mathrm{osp}(N|2,\mathbb{R})$ to $\mathrm{shs^E}(N|2,\mathbb{R})$ by relaxing the condition that the polynomials should be quadratic, i.e., by allowing arbitrary (pure imaginary) polynomials of even degree modulo zero-degree term.

The $\mathrm{osp}(N|2,\mathbb{R})$ subsuperalgebra can also be realized in terms of matrices. In a matrix representation where imaginary elements are represented by anti-hermitian matrices for an appropriate (indefinite) hermitian product, the $Y_{\alpha\beta}$ are elements of $\mathrm{su}(1,1) \simeq \mathrm{sl}(2,\mathbb{R})$. Though we shall primarily use the oscillator polynomial realization, for comparison and completeness we collect the relevant results on the matrix representation in Appendix B.

As already mentioned, the infinite-dimensional higher-spin superalgebra corresponds to the universal enveloping superalgebra of the underlying finite-dimensional sub-superalgebra quotientized by certain ideals. The latter being generated by quadratic polynomials $\mathcal{A}^{(2)}$, this means that the polynomials of $\mathcal{A}^E$ can be reexpressed as polynomials in the generators of the finite-dimensional sub-superalgebra $\mathcal{A}^{(2)}$.

$\mathbf{hs}(2,\mathbb{R})$ subalgebra and internal subalgebra The polynomials that contain no ψ_i (degree K equal to zero) form a subalgebra, which is nothing but the algebra $\mathrm{hs}(2,\mathbb{R})$ that has been used for the description of the integer higher-spin gravity theory [88]. It is a subalgebra of the bosonic subalgebra containing the polynomials of even K-degree.

Another interesting subalgebra is the finite subalgebra of polynomials involving only ψ's and no q's. We call it the internal subalgebra U. The internal subalgebra U contains so$(N,\mathbb{R})$ as the subalgebra generated by the quadratic monomials X_{ij}. To identify U completely, we recall that the ψ_i's are the generators of a Clifford algebra. When N is even, the internal subalgebra is therefore the direct sum

$$\mathrm{U} = \mathrm{su}(2^{\frac{N-2}{2}}) \oplus \mathrm{su}(2^{\frac{N-2}{2}}) \oplus \mathrm{u}(1) \qquad (N \text{ even}), \tag{6.27}$$

while when N is odd, one gets

$$\mathrm{U} = \mathrm{su}(2^{\frac{N-1}{2}}) \qquad (N \text{ odd}). \tag{6.28}$$

6.2.2 Higher-Spin Chern-Simons Super-Gauge Theory

We now turn to the dynamics. The starting point is a doubled Chern-Simons gauge theory, whose super-connection one-forms are Γ taking values in $\mathrm{shs}^E(N|2,\mathbb{R})$ and $\overline{\Gamma}$ taking values in $\mathrm{shs}^E(M|2,\mathbb{R})$:

$$\Gamma(x;q,\psi) \;=\; \sum_{m,n,i_1,\cdots,i_N} \mathrm{d}x^\mu \Gamma_\mu^{m,n;i_1,\cdots,i_N}(x) X_{m,n;i_1,\cdots,i_N} \tag{6.1}$$

$$\overline{\Gamma}(x;q,\psi) \;=\; \sum_{m,n,i_1,\cdots,i_M} \mathrm{d}x^\mu \overline{\Gamma}_\mu^{m,n;i_1,\cdots,i_M}(x) \overline{X}_{m,n;i_1,\cdots,i_M}. \tag{6.2}$$

They can be decomposed further according to the spinor parity:

$$\Gamma_\mu^{m,n;i_1,\cdots,i_N}(x) \;=\; \begin{cases} A_\mu^{m,n;i_1,\cdots,i_N}(x) & (m+n = \text{even}) \\ \Psi_\mu^{m,n;i_1,\cdots,i_N}(x) & (m+n = \text{odd}) \end{cases},$$

$$\overline{\Gamma}_\mu^{m,n;i_1,\cdots,i_M}(x) \;=\; \begin{cases} \overline{A}_\mu^{m,n;i_1,\cdots,i_M}(x) & (m+n = \text{even}) \\ \overline{\Psi}_\mu^{m,n;i_1,\cdots,i_M}(x) & (m+n = \text{odd}) \end{cases}. \tag{6.2}$$

The even parity components are real spacetime Bose fields, while the odd parity components are real spacetime Fermi fields.

The super-gauge transformations of these super-connections are given in terms of a super-gauge 0-form $\Lambda(x;q,\psi)$:

$$\delta_\Lambda \Gamma(x;q,\psi) = \mathrm{d}\Lambda(x;q,\psi) + \Gamma(x;q,\psi) \star \Lambda(x;q,\psi) - \Lambda(x;q,\psi) \star \Gamma(x;q,\psi). \tag{6.3}$$

In accordance with the super-connection 1-form, the super-gauge 0-form is expandable as

$$\Lambda(x;q,\psi) = \sum_{m,n,i_1,\cdots,i_N} \Lambda^{m,n;i_1,\cdots,i_N}(x) X_{m,n;i_1,\cdots,i_N} \tag{6.4}$$

where the real coefficients

$$\Lambda^{m,n;i_1,\cdots,i_N}(x) = \begin{cases} \lambda^{m,n;i_1,\cdots,i_N}(x) & (m+n = \text{even}) \\ \eta^{m,n;i_1,\cdots,i_N}(x) & (m+n=\text{odd}) \end{cases} \tag{6.5}$$

parametrize respectively the bosonic and the fermionic gauge transformations.

The theory is defined by the action

$$S_{\text{HS}}[\Gamma, \overline{\Gamma}] = S_{\text{cs}}[\Gamma] - S_{\text{cs}}[\overline{\Gamma}]. \tag{6.6}$$

with a relative minus sign. The first part is referred to as the "chiral sector" whereas the second part is the "anti-chiral sector". The Chern-Simons action is given for the chiral part by

$$\begin{aligned} S_{\text{cs}}[\Gamma] &\equiv \frac{k}{4\pi}\int_{\mathcal{M}_3} \text{Str}\Big(\Gamma \wedge \text{d} \star \Gamma + \frac{2}{3}\Gamma \wedge \star\Gamma \wedge \star\Gamma\Big) \\ &= \frac{k}{4\pi}\int_{\mathcal{M}_3} \Big[\text{Tr}\big(A\wedge \text{d}A + \frac{2}{3}A\wedge A\wedge A\big) + i\text{Tr}\big(\overline{\Psi}\wedge \text{d}\Psi + \overline{\Psi}\wedge A \wedge \Psi\big)\Big] \end{aligned} \tag{6.6}$$

and similarly for the anti-chiral part. The coefficient k is a dimensionless, real-valued coupling constant of the theory. In the gravitational context considered here, it is related to the three-dimensional Newton's constant G and the AdS radius of curvature ℓ through

$$k = \frac{\ell}{4G}. \tag{6.7}$$

The cosmological constant is $\Lambda \equiv -\frac{1}{\ell^2}$. With k real, the action is real-valued.

As discussed in the previous section, the gauge algebra $\text{shs}^E(N|2,\mathbb{R})\oplus\text{shs}^E(M|2,\mathbb{R})$ contains various finite-dimensional subalgebras. When the gauge algebra is restricted to the $\text{so}(1,2,\mathbb{R}) \oplus \text{so}(1,2,\mathbb{R})$ bosonic algebra, the theory is reduced to the Chern-Simons formulation of the three-dimensional Einstein gravity with negative cosmological constant. When the gauge algebra is restricted to the $\text{osp}(1|2,\mathbb{R})\oplus\text{osp}(1|2,\mathbb{R})$

superalgebra, this theory is reduced to the Chern-Simons formulation of three-dimensional $\mathcal{N} = (1,1)$ Einstein supergravity with negative cosmological constant. When the gauge algebra is truncated to $\mathrm{sl}(3,\mathbb{R}) \oplus \mathrm{sl}(3,\mathbb{R})$ (which is not a subalgebra, but one can proceed along the lines explained in [86]), the theory is reduced to the Chern-Simons formulation of three-dimensional spin-3 gravity with negative cosmological constant, which describes the consistent interaction of a spin-3 field with Einstein gravity. In all these cases, the vacuum is the three-dimensional anti-de Sitter space. It is important to note that the isometry algebra $\mathrm{sl}(2,\mathbb{R})\oplus\mathrm{sl}(2,\mathbb{R})$ of the vacuum configuration coincides with the gravitational subalgebra $\mathrm{sl}(2,\mathbb{R}) \oplus \mathrm{sl}(2,\mathbb{R})$ of the respective gauge algebras. When Killing spinors are included in the context of (2+1)-supergravities [105], this gravitational algebra is enlarged to the corresponding superalgebras.

Though containing an infinite number of components, the Chern-Simons super-gauge theory has no propagating field degrees of freedom. The field equations

$$\begin{aligned} F(\Gamma) &\equiv \mathrm{d}\Gamma + \Gamma \wedge \star\Gamma = 0 \\ \overline{F}(\overline{\Gamma}) &\equiv \mathrm{d}\overline{\Gamma} + \overline{\Gamma} \wedge \star\overline{\Gamma} = 0 \end{aligned} \tag{6.7}$$

assert that the super-connections $\Gamma, \overline{\Gamma}$ are flat. This means that locally the connections can be put into a pure-gauge configuration:

$$\Gamma(x,\xi) = U^{-1}(x,\xi) \star \mathrm{d}U(x,\xi) \qquad \text{and} \qquad \overline{\Gamma}(x,\xi) = \overline{U}^{-1}(x,\xi) \star \mathrm{d}\overline{U}(x,\xi). \tag{6.8}$$

The configuration can still leave degrees of freedom describing global charges or holonomies, depending on the geometry and topology of the three-manifold $\mathcal{M}_3$ over which the theory is defined. Unraveling the global charges in the asymptotically AdS background is one main task of this paper.

6.2.3 Asymptotics symmetries

Asymptotics of $\mathbf{shs}^{\mathbf{E}}(N|2,\mathbb{R})$ connection

The spacetime manifold $\mathcal{M}_3$ is assumed to have topology $\mathbb{R}\times\mathcal{D}$, where $\mathbb{R}$ parametrizes the time coordinate and $\mathcal{D}$ is a two-dimensional spatial manifold, which we assume to have at least one boundary that we call "asymptotic infinity" or more loosely "infinity". This boundary is assumed to correspond to $r \to \infty$, where spacetime approaches AdS_3,

$$\mathrm{d}s^2 \to \frac{\ell^2}{r^2}\left[-\mathrm{d}x^+\mathrm{d}x^- + \mathrm{d}r^2\right], \tag{6.9}$$

We impose asymptotic conditions on the connection that simultaneously generalize those of [86] for higher spin bosonic models and those of [99, 106] for simple and extended supergravities.

In the case of minimal AdS_3 supergravity, the boundary conditions were [106] (after an appropriate gauge transformation that simplifies the form of the connection and its r-dependence [96])

$$\Gamma(x) \to [-1 \cdot X_{22} + B^1_+(x^+,x^-)X_1 + B^{11}_+(x^+,x^-)X_{11}]\mathrm{d}x^+ \tag{6.10}$$

$$\overline{\Gamma}(x) \to [+1 \cdot X_{11} + \overline{B}^2(x^+,x^-)X_2 + \overline{B}^{22}_-(x^+,x^-)X_{22}]\mathrm{d}x^-. \tag{6.11}$$

In the case of higher-spin AdS_3 gauge theory, the boundary conditions were [86]

$$\Gamma(x) \to [-1 \cdot X_{22} + \Delta^{11}(x^+,x^-)X_{11} + \Delta^{1111}(x^+,x^-)X_{1111} + \cdots]\mathrm{d}x^+ \tag{6.12}$$

$$\overline{\Gamma}(x) \to [+1 \cdot X_{11} + \overline{\Delta}^{22}(x^+,x^-)X_{22} + \overline{\Delta}^{2222}(x^+,x^-)X_{2222} + \cdots]\mathrm{d}x^-. \tag{6.13}$$

Combining these two limiting situations, it is fairly obvious that the correct boundary conditions for the $\mathrm{shs}(1|2,\mathbb{R})$-valued gauge connections of simple supergravity are

$$\Gamma(x) \to \big[-1 \cdot X_{22} + \sum_{\ell=1}^{\infty} \Delta^{(\ell,0)}(x^+,x^-)X_{(\ell,0)}\big]\mathrm{d}x^+ \tag{6.14}$$

$$\overline{\Gamma}(x) \to \big[+1 \cdot X_{11} + \sum_{\ell=1}^{\infty} \overline{\Delta}^{(0,\ell)}(x^+,x^-)X_{(0,\ell)}\big]\mathrm{d}x^-. \tag{6.15}$$

The boundary conditions for the theories with extended supersymmetry are similar but one does not impose a highest-weight or lowest-weight type of gauge condition along the internal symmetry algebra. Indeed, it was found in [99] that the boundary conditions for extended supergravities took the form

$$\Gamma(x) \to [-1 \cdot X_{22} + B^{1i}_+(x^+,x^-)X_{1i} + B^{11}_+(x^+,x^-)X_{11} + B^{ij}_+(x^+,x^-)X_{ij}]\mathrm{d}x^+ \tag{6.16}$$

$$\overline{\Gamma}(x) \to [+1 \cdot X_{11} + \overline{B}^{2i}_-(x^+,x^-)X_{2i} + \overline{B}^{22}_-(x^+,x^-)X_{22} + \overline{B}^{ij}_-(x^+,x^-)X_{ij}]\mathrm{d}x^- \tag{6.17}$$

with no restriction on the internal indices occurring asymptotically. Therefore, we impose

$$\Gamma(x) \to [-1 \cdot X_{22} + \sum \Delta^{p i_1 \cdots i_N}(x^+, x^-) X_{p,0;i_1\cdots i_N}]\mathrm{d}x^+ \tag{6.18}$$

$$\overline{\Gamma}(x) \to [+1 \cdot X_{11} + \sum \overline{\Delta}^{q i_1 \cdots i_N}(x^+, x^-) X_{0,q;i_1\cdots i_N}]\mathrm{d}x^- \tag{6.19}$$

where we sum on repeated indices over all their possible values (note in particular that the values $p = 0$ and $q = 0$ occur when the degree $K = i_1 + i_2 + \cdots + i_N$ does not vanish).

Even though there is no asymptotic restriction on the weights of the representations of the internal algebra, we continue to call the boundary conditions (6.18) and (6.19) the "highest-weight", respectively, the "lowest-weight" gauge boundary conditions, in analogy with the non-extended cases ($N = 0$ or $N = 1$).

Hamiltonian reduction The above boundary conditions on the currents coincide with the constraints that implement the familiar Drinfeld-Sokolov (DS) Hamiltonian reduction [94, 95] of WZWN models [97, 98, 107–110] – to which the Chern-Simons theory reduces on the boundary [111]. As it has been demonstrated in those references, the Virasoro algebra (or one of its appropriate extensions) emerges in the reduction procedure from the current algebra of the unreduced theory.

That the AdS_3 boundary conditions implement the DS Hamiltonian reduction was pointed out first in the case of pure AdS_3 gravity in [96], where the Virasoro algebra is generated from the affine $\mathrm{sl}(2,\mathbb{R})$ current algebra (one in each chiral sector). This was then extended to the case of $N = 1$ supergravity, where one gets after reduction the $N = 1$ superconformal algebra [106], and further to extended supergravity models in [99]. In that latter case, the extended superconformal algebras that arise contain nonlinearities in the Kac-Moody currents, realizing the algebraic structures uncovered in [112–117].

In all these cases, the conformal dimensions of the generators of the boundary superconformal algebras are ≤ 2 because the underlying bosonic algebras in the bulk are of the form $\mathrm{sl}(2,\mathbb{R}) \oplus \mathcal{G}$ and the $\mathrm{sl}(2,\mathbb{R})$-representations involve only spins ≤ 1.

The analysis was more recently generalized to include higher conformal dimensions in [86] and [91] with respective bulk algebras $\mathrm{hs}(2, \mathbb{R})$ and $\mathrm{sl}(N, \mathbb{R})$.

Because the boundary conditions (6.18) and (6.19) are precisely those that implement the Hamiltonian reduction of affine superalgebras, one can proceed along the well known DS reduction lines [94] to derive the corresponding asymptotic symmetry algebras. The precise steps adapted to an infinite number of AdS spins have been given in [86]. We shall follow this reference here, stressing the conceptual points rather than giving explicit formulas, which are rather cumbersome indeed. [The machinery to derive systematically the formulas will be, however, explained.]

Residual gauge transformations

Given the AdS boundary conditions (6.18) and (6.19), the next step is to look for the residual gauge transformations that act nontrivially at asymptotic infinity while leaving the boundary conditions intact. With gauge parameter $\Lambda(x)$, the infinitesimal gauge transformation of Γ reads

$$\Gamma \to \Gamma' = \Gamma + \delta\Gamma, \qquad \text{where} \qquad \delta\Gamma = d\Lambda + [\Gamma, \Lambda]. \tag{6.20}$$

We see that, in order for Γ' to retain the given asymptotics, Λ cannot possibly depend on r or x^- to leading order at infinity. Moreover, the gauge transformations should not generate any other components than the highest-weight ones already present. A similar argument goes for $\overline{\Gamma}$. With gauge parameter $\overline{\Lambda}(x)$, the infinitesimal gauge transformation of $\overline{\Gamma}$ reads

$$\overline{\Gamma} \to \overline{\Gamma}' = \overline{\Gamma} + \delta\overline{\Gamma}, \qquad \text{where} \qquad \delta\overline{\Gamma} = d\overline{\Lambda} + [\overline{\Gamma}, \overline{\Lambda}]. \tag{6.21}$$

Again, in order for $\overline{\Gamma}'$ to retain the boundary condition (6.19), $\overline{\Lambda}$ cannot possibly depend on r or x^+. Furthermore, the gauge transformations should not generate any other components than the lowest-weight ones already present in (6.19). Summarizing, we found that the gauge transformations $\Lambda(x^+)$ and $\overline{\Lambda}(x^-)$ must be chiral, respectively, antichiral at the least. These functions must be subject to further conditions in order to retain the boundary conditions. This is the task we will undertake next, treating explicitly for definiteness the positive chirality sector (the negative chirality sector is treated similarly).

To proceed further, we find it convenient to decompose the gauge transformations in stacks of successively higher sl$(2,\mathbb{R})$-spin layers. This is because, for each spin, the highest-weight or the lowest-weight components are the only ones that appear in the boundary conditions for the gauge connection. We thus write

$$\begin{aligned}\Lambda(x^+) &= \sum_{m,n,i_1,\cdots,i_N} \Lambda^{m,n;i_1,\cdots,i_N}(x^+)X_{m,n;i_1,\cdots,i_N} \\ &= \Lambda^{\mathrm{LW}} + \lambda \qquad (6.21)\end{aligned}$$

with

$$\begin{aligned}\Lambda^{\mathrm{LW}} &= \sum_{i_1+\cdots+i_N\geq 2} \Lambda^{0,0;i_1,\cdots,i_N}(x^+)X_{0,0;i_1,\cdots,i_N} + \sum_{i_1+\cdots+i_N\geq 1} \Lambda^{0,1;i_1,\cdots,i_N}(x^+)X_{0,1;i_1,\cdots,i_N} \\ &\quad + \sum_{\ell=2}^{\infty}\sum_{i_1,\cdots,i_N} \Lambda^{0,\ell;i_1,\cdots,i_N}(x^+)X_{0,\ell;i_1,\cdots,i_N} \qquad (6.21)\end{aligned}$$

and

$$\begin{aligned}\lambda &= \sum_{i_1+\cdots+i_N\geq 1} \Lambda^{1,0;i_1,\cdots,i_N}(x^+)X_{1,0;i_1,\cdots,i_N} + \sum_{\ell=2}^{\infty}\sum_{i_1,\cdots,i_N} \Lambda^{1,\ell-1;i_1,\cdots,i_N}(x^+)X_{1,\ell-1;i_1,\cdots,i_N} \\ &\quad + \sum_{\ell=2}^{\infty}\sum_{i_1,\cdots,i_N} \Lambda^{2,\ell-2;i_1,\cdots,i_N}(x^+)X_{2,\ell-2;i_1,\cdots,i_N} + \cdots \\ &\quad + \cdots + \sum_{\ell\geq s}^{\infty}\sum_{i_1,\cdots,i_N} \Lambda^{s,\ell-s;i_1,\cdots,i_N}(x^+)X_{s,\ell-s;i_1,\cdots,i_N} + \cdots . \qquad (6.20)\end{aligned}$$

In plain words, we collected all the lowest-weight states, which are the states involving $X_{0,s;i_1,\cdots,i_N}$ in Λ^{LW}. At the same time, all higher weight states, involving $X_{m,n;i_1,\cdots,i_N}$ with $m > 0$, are packaged together in λ. We should also stress that, although this is not written explicitly, the sums in the above expressions are always restricted to total even degree. So, for instance, $i_1 + \cdots + i_N$ must be even in the first term in the right-hand side of the expression for Λ^{LW}, while it must be odd in the second term. Such a convention will always be adopted in the sequel.

The reason for proceeding in that manner is that the requirement that the asymptotic boundary conditions be preserved determines λ in terms of Λ^{LW}. Indeed, let us compute $\delta\Gamma = \mathrm{d}\Lambda + [\Gamma, \Lambda]$. Structurally,

$$\delta\Gamma = \sum_{m,n;i_1,\cdots,i_N} \gamma^{m,n;i_1,\cdots,i_N}(x^+) X_{m,n;i_1,\cdots,i_N} \tag{6.21}$$

where

$$\gamma^{m,n;i_1,\cdots,i_N}(x^+) = \partial_+ \Lambda^{m,n;i_1,\cdots,i_N} + [\Gamma, \Lambda]^{m,n;i_1,\cdots,i_N}. \tag{6.22}$$

Since the only non-vanishing components of Γ at infinity are $\gamma^{m,0;i_1,\cdots,i_N}$ (apart from $\gamma^{0,2;0,\cdots,0}$, which is fixed to be equal to -1), the requirement that these global gauge transformations do not alter the boundary conditions is that

$$\gamma^{m,1;i_1,\cdots,i_N} = \gamma^{m,2;i_1,\cdots,i_N} = \cdots = 0 \qquad \text{for} \qquad m = 0, 1, 2, \cdots \tag{6.23}$$

or, equivalently,

$$\gamma^{s,\ell-s;i_1,\cdots,i_N} = 0 \qquad \text{for} \qquad \ell \geq s+1, \quad s = 0, 1, 2, \cdots . \tag{6.24}$$

The highest-weight terms $\gamma^{m,0;i_1,\cdots,i_N}$ are not constrained to be zero and are equal to $\Delta^{m i_1 \cdots i_N}$ according to (6.18).

Now, since

$$[X_{22}, X_{m,n;i_1,\cdots,i_N}] \sim X_{m-1,n+1;i_1,\cdots,i_N} \quad (m \geq 1)$$

one may solve recursively the conditions for the higher-weight coefficients $\Lambda^{1,n;i_1,\cdots,i_N}$, $\Lambda^{2,n;i_1,\cdots,i_N}$, ..., given the lowest-weight ones $\Lambda^{0,k;i_1,\cdots,i_N}$, along exactly the same lines as developed in [86]. One starts from the lowest-weight conditions $\gamma^{0,\ell;i_1,\cdots,i_N} = 0$ ($\ell \geq 1$) to determine the level-one coefficients $\Lambda^{1,\ell-1;i_1,\cdots,i_N}$. Then one proceeds to solving the level-one conditions $\gamma^{1,\ell-1;i_1,\cdots,i_N} = 0$ ($\ell \geq 2$)) to determine the level-two coefficients $\Lambda^{2,\ell-2;i_1,\cdots,i_N}$. One walks one's way up step by step in this fashion. The last set of conditions $\gamma^{\ell-1,1;i_1,\cdots,i_N} = 0$ ($\ell \geq 1$) determine the highest-weight coefficients $\Lambda^{\ell,0;i_1,\cdots,i_N}$. It should be stressed that the higher-weight coefficients depend not only on the lowest-weight coefficients but also on their derivatives. To emphasize this feature, we shall say that the higher-weight coefficients are *functionals* of the lowest-weight ones. The solutions depend also on the (non-zero) coefficients of the connection and their derivatives.

Collecting the results of the above structure analysis, we conclude that the gauge transformations that leave the boundary conditions intact are completely specified by the lowest-weight components $\Lambda^{0,k;i_1,\cdots,i_N}$ of the gauge function, while all higher weight components are determined functionally in terms of these lowest-weight components of the gauge function and the highest-weight components of the original gauge connection. Notice that, as in the higher-spin bosonic case as well as in the extended supergravity models, the solution for the higher-weight components of the gauge function Λ in terms of the lowest-weight ones, the free gauge potential components $\Delta^{mi_1\cdots i_N}$ and their derivatives is nonlinear. It is this feature that will render the resulting asymptotic algebra also nonlinear.

Asymptotic symmetry superalgebra

To identify the asymptotic symmetry superalgebra, one needs to extract the commutation relations for the superalgebra of asymptotic gauge transformations induced by the gauge function Λ. In the canonical formalism, these commutation relations are realized as the Poisson brackets of the generators of these asymptotic symmetries (up to possible central charges [118]), and we shall focus on these here.

Consider a phase-space observable $\mathcal{O}$. Under the global gauge transformation parametrized by Λ, this observable transforms according to

$$\mathcal{O} \to \mathcal{O} + \delta\mathcal{O} \quad \text{with} \quad \delta\mathcal{O} = \{\mathcal{O}, G[\Lambda]\}_{\text{PB}}. \tag{6.25}$$

On an equal-time slice Σ_2, the functional of gauge transformation $G[\Lambda]$ is given by

$$G[\Lambda] = \int_{\Sigma_2} \sum_{m,n,i_1,\cdots,i_N} \Lambda^{m,n;i_1,\cdots,i_N} \mathcal{G}_{m,n;i_1,\cdots,i_N} + S_\infty, \tag{6.26}$$

where $\mathcal{G}_{m,n;i_1,\cdots,i_N}$ are the Gauss law constraints and S_∞ is a boundary term at asymptotic infinity defined by the requirement that $G[\Lambda]$ must have well-defined functional derivatives with respect to the connection components, i.e., $G[\Lambda]$ must be such that $\delta G[\Lambda]$ contains only undifferentiated field variations under the given boundary conditions for Γ [119].

In the present case, the on-shell configuration is

$$\mathcal{G} = 0, \tag{6.27}$$

so that the generator reduces on-shell to the surface term S_∞. On the other hand, S_∞ just follows from straightforward integration by part and is proportional to the angular components of the connection along the highest weight basis vectors times the components of the gauge parameters along the lowest weight basis vectors (to leading order), as it was already found for supergravity [99, 106]. Explicitly,

$$S_\infty = \oint \sum_{s,i_1,\cdots,i_N} \Lambda^{0,s;i_1,\cdots,i_N} \Delta^{s}{}_{i_1,\cdots,i_N}\,, \tag{6.28}$$

where we have redefined the Δ's through the absorption of the factors that appear in front of the integral, which we denote by $\alpha_{s,0;i_1,\cdots,i_N}$,

$$\Delta^{s;i_1,\cdots,i_N} = \Gamma^{s,0;i_1,\cdots,i_N} \alpha_{s,0;i_1,\cdots,i_N}\,.$$

We thus see that (up to those factors) the generators of the asymptotic symmetries are indeed nothing but the leading terms in the asymptotic expansion of the highest-weight components $\Gamma^{s,0;i_1,\cdots,i_N}$ of the gauge connection.

The algebra of the asymptotic symmetry generators $\Delta^{s;i_1,\cdots,i_N}$ can be read off by equating their variations under an arbitrary asymptotic symmetry transformation, computed in two different ways. First, $\delta\Delta^{s;i_1,\cdots,i_N}$ can be derived from the gauge variation formula,

$$\delta\Delta^{s;i_1,\cdots,i_N}(\theta) = \delta\Gamma^{s,0;i_1,\cdots,i_N}(\theta)\,\alpha_{s,0;i_1,\cdots,i_N} = \left(\partial\Lambda^{s,0;i_1,\cdots,i_N} + [\Gamma,\Lambda]^{s,0;i_1,\cdots,i_N}\right)\alpha_{s,0;i_1,\cdots,i_N} \tag{6.29}$$

with the $\Lambda^{m,n;i_1,\cdots,i_N}$ determined from the lowest-weight $\Lambda^{0,s;i_1,\cdots,i_N}$ along the lines explained in the previous subsection. Second, $\delta\Delta^{s;i_1,\cdots,i_N}$ can be obtained directly from the Hamiltonian expression (6.25),

$$\delta\Delta^{s;i_1,\cdots,i_N}(\theta) = \{\Delta^{s;i_1,\cdots,i_N}(\theta), \oint \sum_{s',j_1,\cdots,j_N} \Lambda^{0,s';j_1,\cdots,j_N} \Delta^{s'}{}_{j_1,\cdots,j_N}(\theta)\}_{\text{PB}}\,. \tag{6.30}$$

Here, θ denotes the angular coordinate of the asymptotic infinity. Comparison of these two ways of computing $\delta\Delta^{s;i_1,\cdots,i_N}$ yields the Poisson brackets

$$\{\Delta^{s;i_1,\cdots,i_N}(\theta), \Delta^{s';j_1,\cdots,j_N}(\theta')\}_{\rm PB} \qquad \text{for} \qquad s, s' \in \mathbb{N}\,. \tag{6.31}$$

It is evident that this algebra is closed, since the variations $\delta\Gamma^{s,0;i_1,\cdots,i_N} = \gamma^{s,0;i_1,\cdots,i_N}$, determined through the recursive procedure explained above, are functionals of $\Gamma^{s,0;i_1,\cdots,i_N} \sim \Delta^{s;i_1,\cdots,i_N}$ only (in addition to depending linearly on the independent gauge parameters $\Lambda^{0,s;i_1,\cdots,i_N}$). The functional dependence of $\gamma^{s,0;i_1,\cdots,i_N}$ on $\Delta^{s;i_1,\cdots,i_N}$ is nonlinear, which implies that the algebra of the Δ's is nonlinear. The terms independent of Δ and linear in the gauge parameters corresponds to the central charges. Although nonlinear, the algebra obeys of course the Jacobi identity since the Poisson bracket does[5].

6.2.4 Nonlinear Super-W_∞ Algebra

The actual computation of the algebra $\mathcal{SW}$ of the $\Delta^{s;i_1,\cdots,i_N}$'s is rather cumbersome but it can be identified to be a super-W_∞ by following a general argument similar to the one given in [86] for the bosonic case. We consider first the $N = 1$ case, i.e., $\text{shs}^E(1|2,\mathbb{R})$:

1. By computing the general solution to the equations for the $\Lambda^{(m,n)}$'s ($m > 0$) when only the free gauge parameter $\Lambda^{(0,2)}$ is non zero, one observes (i) that the generators $L \equiv \Delta^2$ form a Virasoro algebra with central charge $k/4\pi$:

$$\{L(\theta), L(\theta')\}_{\rm PB} = \frac{k}{4\pi}\delta'''(\theta-\theta') - \big(L(\theta)+L(\theta')\big)\,\delta'(\theta-\theta'); \tag{6.0}$$

and (ii) that the generators $M_{\frac{j}{2}+1} \equiv \Delta^j$ have conformal dimension $(\frac{j}{2}+1)$:

$$\{L(\theta), M_{\frac{j}{2}+1}(\theta')\}_{\rm PB} = -\left(M_{\frac{j}{2}+1}(\theta) + \frac{j}{2}M_{\frac{j}{2}+1}(\theta')\right)\delta'(\theta-\theta')\,. \tag{6.0}$$

[5] Upon gauge fixing, the Poisson algebra becomes the Dirac algebra. However, the asymptotic algebra does not depend on the gauge choice because the constraints of the theory are all first class.

2. By computing the general solution to the equations for the $\Lambda^{(m,n)}$'s $(m>0)$ when only the free (fermionic) gauge parameter $\Lambda^{(0,1)}$ is non-zero, one observes that the generator $Q \equiv M_{\frac{3}{2}} \equiv \Delta^1$ is the supercharge,

$$\begin{aligned} i\{Q(\theta), M_{\frac{s}{2}+1}(\theta')\}_{\rm PB} &= -\frac{k}{\pi}\delta_{s,1}\delta''(\theta-\theta') \\ &+ (s+1)M_{\frac{s+3}{2}}(\theta) \quad (s \text{ odd}) \end{aligned} \tag{6.0}$$

and

$$\{Q(\theta), M_{\frac{s}{2}+1}(\theta')\}_{\rm PB} = -\delta'(\theta-\theta')\left(\frac{1}{s}M_{\frac{s+1}{2}}(\theta) + M_{\frac{s+1}{2}}(\theta')\right) \quad (s \text{ even}). \tag{6.1}$$

The relations are linear at these levels. They start displaying the nonlinear structure of the algebra at higher levels. For instance, one finds explicitly

$$\begin{aligned} \{M_{\frac{5}{2}}(\theta), M_{\frac{5}{2}}(\theta')\}_{\rm PB} &= \frac{\alpha^3}{6}\delta''''(\theta-\theta') + \frac{\alpha^3}{12\alpha^6}(N^6(\theta)+N^6(\theta'))\delta(\theta-\theta') \\ &+ \frac{3\alpha^3}{2(\alpha^2)^2}L(\theta)L(\theta')\delta(\theta-\theta') - \frac{5\alpha^3}{6\alpha^2}\delta''(\theta-\theta')(L(\theta)+L(\theta')) \\ &- \frac{\alpha^3}{3\alpha^2}\delta'(\theta-\theta')(L'(\theta)-L'(\theta')) + \frac{i\alpha^3}{6(\alpha^1)^2}Q(\theta)Q(\theta')\delta'(\theta-\theta') \end{aligned} \tag{6.0}$$

and

$$\begin{aligned} \{M_3(\theta), M_3(\theta')\}_{\rm PB} &= \frac{\alpha^3}{24}\delta'''''(\theta-\theta') - \frac{5\alpha^3}{12\alpha^2}(L(\theta)+L(\theta'))\delta'''(\theta-\theta') \\ &+ \frac{\alpha^3}{6\alpha^6}(N^6(\theta)+N^6(\theta'))\delta'(\theta-\theta') + \frac{2i\alpha^3}{3(\alpha^1)^2}Q(\theta)Q(\theta')\delta''(\theta-\theta') \\ &- \frac{i\alpha^3}{2(\alpha^1)^2}(Q'(\theta)Q(\theta)+Q'(\theta')Q(\theta'))\delta'(\theta-\theta') \\ &+ \frac{\alpha^3}{(\alpha^2)^2}\delta'(\theta-\theta')(L^2(\theta) + \frac{2L(\theta)L(\theta')}{3} + L^2(\theta')) \\ &+ \frac{\alpha^3}{4\alpha^2}\delta''(\theta-\theta')(L'(\theta')-L'(\theta)). \end{aligned} \tag{6.-3}$$

The numerical factors α^i appearing in these expressions read

$$\alpha^i = \frac{k(-)^{n+1}i^n}{\pi n!}. \tag{6.-3}$$

In the extended case, the derivation proceeds in the same way. The salient new features that arise are:

1. There are now fields $\Delta^{0;i_1,\cdots,i_N}$ of conformal dimension 1. These are the currents of the internal symmetry, and they form an affine subalgebra. Their brackets with the other generators reflect how these other generators transform under the internal symmetry. Indeed, the solution for Λ when the only non-vanishing lowest-weight free components are $\Lambda^{0,0;i_1,i_2,\cdots,i_N}$ $(i_1+\cdots i_N \geq 2)$ is easily seen to be simply $\Lambda = \Lambda^{\text{LW}} = \sum_{i_1+\cdots i_N \geq 2} \Lambda^{0,0;i_1,\cdots,i_N} X_{0,0;i_1,\cdots,i_N}$.

2. A Sugawara redefinition of the Virasoro generator L must actually be performed, as already found in [99] (see that reference for details).

3. While there is a single generator M_j at each conformal dimension > 1 for $N = 1$, this is not any more the case for extended models. The degeneracies of each conformal dimension > 1 is equal to 2^{N-1}, while the degeneracy of conformal dimension 1 is $2^{N-1}-1$. In particular, the Virasoro generator is not the only field with conformal dimension 2 for extended models.

We stress that our construction guarantees automatically that the brackets among the generators fulfill the Jacobi identity since these are just Poisson brackets (or Dirac brackets if one fixes the gauge). This is worth emphazising since other methods for constructing super W-algebras met with difficulties with the Jacobi identity.

Although there is no consistent truncation of $\text{shs}(N|2,\mathbb{R})$ to finite dimensional superalgebras that can be made beyond $\text{osp}(N|2,\mathbb{R})$, the Hamiltonian reduction procedure is very similar to that encountered for the finite-dimensional super-algebras $\text{sl}(n+1|n)$, which yields $N = 2$ models with generators M_s of higher conformal dimensions up to $s = \frac{2n+1}{2}$ [120–123].

Part IV

Conclusions and outlook

Chapter 7

Conclusions

We discussed the progress in several topics in super-conformal field theory and higher-spin theory. Let us summarize the main result of each topic we discussed in the last chapters.

In $\mathcal{N} = 4$ SYM theory, we calculated anomalous dimensions of local operators using spinning D3-branes. In ordinary perturbative calculations it has been known that anomalous dimension of twist 2 operator scales logarithmic way. This is not trivial but is the general feature in Yang-Mills theories. By studying spinning D-branes we can verified that even in the non-perturbative region, logarithmic scaling still holds. By considering large R-charge limit we can compare the result with gauge theory side in BMN limit. This can be a consistent check or non-trivial test of AdS/CFT duality.

In $\mathcal{N} = 6$ Chern-Simons matter theory, we focused on infra-red divergence in scattering amplitude. In ABJM theory one-loop correction gives trivial result and first non-trivial corrections arise in the two-loop order. The IR divergence in the two loop order gives $\frac{1}{\epsilon^2}$ pole. Due to the gauge group structure, any scattering amplitude has even number of external particles and this implies soft emission of two particles. We performed explicit double phase space integral in spinor-helicity formalism. We checked that two-particles emissions give $\frac{1}{\epsilon^2}$ pole and canceled exactly with 2-loop corrections. This guaranteed the infra-red finiteness of cross-sections at least in the leading order of ϵ. It would be nice if one can show that divergences are cancelled in $\frac{1}{\epsilon^2}$ order.

Asymptotic symmetry of super-symmetric higher-spin theory was calculated. Like AdS gravity has $sl(2,R) \oplus sl(2,R)$ symmetry, super-symmetric higher-spin theory has original symmetry $shs^E(N|2,R) \oplus shs^E(N|2,R)$. Under the AdSboundary condition, we found the general transformation preserving boundary. It turned out the super-W_∞ algebra. We can summarize the symmetry extension in each AdS theory. In pure gravity case, $sl(2,R) \oplus sl(2,R)$ extends to Virasoro algebra which is conformal algebra in two dimension. In higher-spin theory, $hs(2,R) \oplus hs(2,R)$ becomes W_∞ algebra. So our result is the natural super-symmetric extension of higher-spin gravity case.

So far we discussed the progress on super-conformal field theory and string theory. Our goal is understanding the nature of real world. We believe that studying this theories would give hints on unraveling the mysteries of nature. Let us end this thesis with the hope that our results on each topics could be small blocks of future progress in theoretical physics.

Appendices

Appendix A

Coordinates of AdS$_3$

We mainly use the coordinates of the $AdS_3 \times S^1$ foliation of AdS_5 (4.56). The coordinates of AdS_3 appeared in [26] is convenient for our purpose. We summarize its relation to the usual global coordinates. The coordinates (u, χ, σ) are given by

$$\begin{aligned} x_{-1} &= \cos u \cosh\sigma \cosh\chi - \sin u \sinh\sigma \sinh\chi, \\ x_0 &= \sin u \cosh\sigma \cosh\chi + \cos u \sinh\sigma \sinh\chi, \\ x_1 &= \cos u \cosh\sigma \sinh\chi - \sin u \sinh\sigma \cosh\chi, \\ x_2 &= \cos u \sinh\sigma \cosh\chi + \sin u \cosh\sigma \sinh\chi. \end{aligned} \tag{A.0}$$

The metric in this coordinates is written as

$$ds^2(AdS_3) = -du^2 + d\chi^2 - 2\sinh 2\sigma\, du d\chi + d\sigma^2.$$

On the other hand, the global coordinates (τ, ρ, φ) parametrize the AdS_3 as

$$\begin{aligned} x_{-1} &= \cosh\rho \cos\tau, \qquad x_0 = \cosh\rho \sin\tau, \\ x_1 &= \sinh\rho \cos\varphi, \qquad x_2 = \sinh\rho \sin\varphi. \end{aligned} \tag{A.0}$$

The metric in this global coordinates is written as

$$ds^2(AdS_3) = -\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\varphi^2.$$

These two coordinate systems are related as

$$
\begin{aligned}
\sinh\rho &= \sqrt{\cosh^2\sigma\sinh^2\chi + \sinh^2\sigma\cosh^2\chi},\\
\tan\tau &= \frac{\tan u + \tanh\sigma\tanh\chi}{1 - \tan u\tanh\sigma\tanh\chi},\\
\tan\varphi &= \frac{\tanh\sigma + \tan u\tanh\chi}{\tanh\chi - \tan u\tanh\sigma},
\end{aligned}
$$

or

$$
\begin{aligned}
\sinh 2\sigma &= \sinh 2\rho\sin(\varphi - \tau),\\
\sinh 2\chi &= \frac{\sinh 2\rho\cos(\varphi-\tau)}{\sqrt{1 + \sinh^2 2\rho\sin^2(\varphi-\tau)}},\\
e^{4iu} &= e^{2i(\tau+\varphi)}\frac{\cos(\varphi-\tau) - i\cosh 2\rho\sin(\varphi-\tau)}{\cos(\varphi-\tau) + i\cosh 2\rho\sin(\varphi-\tau)}.
\end{aligned}
$$

The Killing vectors corresponding to the energy and the angular momentum are given in the (u, χ, σ) coordinates as

$$
\begin{aligned}
\partial_\tau =& \frac{1}{2}\left(1 + \frac{\cosh 2\chi}{\cosh 2\sigma}\right)\partial_u + \frac{1}{2}\cosh 2\chi\tanh 2\sigma\partial_\chi - \frac{1}{2}\sinh 2\chi\partial_\sigma,\\
\partial_\varphi =& \frac{1}{2}\left(1 - \frac{\cosh 2\chi}{\cosh 2\sigma}\right)\partial_u - \frac{1}{2}\cosh 2\chi\tanh 2\sigma\partial_\chi + \frac{1}{2}\sinh 2\chi\partial_\sigma.
\end{aligned}
$$

Appendix B

Asymptotic symmetry in HS theory

B.1 Conventions and Notations

• sl(2, $\mathbb{R}$)

A commuting spinor q of sl(2, $\mathbb{R}$) is a two-component, real-valued column vector

$$q \equiv (q^{\alpha}) = \begin{pmatrix} q^1 \\ q^2 \end{pmatrix} \qquad (\alpha = 1, 2). \tag{B.1}$$

The spinor indices are raised and lowed with the spinor metric

$$(\epsilon^{\alpha\beta}) = (\epsilon_{\alpha\beta}) = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}, \qquad (\alpha, \beta = 1, 2) \tag{B.2}$$

in the North-West/South-East convention:

$$A_{\alpha} = A^{\beta}\epsilon_{\beta\alpha}, \qquad A^{\alpha} = \epsilon^{\alpha\beta}A_{\beta}. \tag{B.3}$$

• AdS_3

Denote AdS_3 radius as ℓ. We adopt the global coordinates of AdS_3:

$$(x) = (x^0, x^1, x^2) = (t, \ell\theta, r). \tag{B.4}$$

in which the metric reads

$$ds^2 = -\left(1 + \left(\frac{x^2}{\ell}\right)^2\right)(dx^0)^2 + \left(1 + \left(\frac{x^2}{\ell}\right)^2\right)^{-1}(dx^2)^2 + \left(\frac{x^2}{\ell}\right)^2(dx^1)^2 \quad \text{(B.5)}$$

To leading order at infinity, the "1" is negligible and one can replace asymptotically the metric by that of the zero mass black hole [164],

$$ds^2 = -\left(\frac{x^2}{\ell}\right)^2(dx^0)^2 + \left(\frac{x^2}{\ell}\right)^{-2}(dx^2)^2 + \left(\frac{x^2}{\ell}\right)^2(dx^1)^2 \quad \text{(B.6)}$$

The light-cone coordinates are defined by

$$(x) = (x^{\pm}, x^2) = (t \pm \ell\theta, r). \quad \text{(B.7)}$$

B.2 Matrix realization of osp$(N|2,\mathbb{R})$ superalgebra

We collect useful result for matrix realization of osp$(N|2,\mathbb{R})$ superalgebra.

B.2.1 The non-extended case

The orthosymplectic osp$(1,2|\mathbb{R})$ superalgebra can be realized as the real vector space of even (grading-preserving) 3×3 supermatrices acting on 1 commuting real Grassmann variable x and 2 anticommuting real Grassmann variables θ^1 and θ^2 and which preserve the quadratic form

$$x^2 + 2i\theta^1\theta^1 2 = x^2 + i\epsilon_{\alpha\beta}\theta^\alpha\theta^\beta \quad \text{(B.7)}$$

as well as the real character of the coordinates, with the usual Lie bracket

$$[\Gamma, \Gamma'] \equiv \Gamma\Gamma' - \Gamma'\Gamma, \quad \text{(B.7)}$$

where the multiplication is the matrix multiplication. Such supermatrices have the form

$$\begin{pmatrix} 0 & i\mu & -i\lambda \\ \lambda & a & b \\ \mu & c & -a \end{pmatrix} \quad \text{(B.7)}$$

with a, b, c real and commuting and λ, μ real and anticommuting. We identify the generators:

$$H \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad E \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad F \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$
$$R^+ \equiv \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad R^- \equiv \begin{pmatrix} 0 & 0 & -i \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{B.7}$$

according to which we find the supercommutators

$$\begin{aligned} &[H,E] = 2E, && [H,F] = -2F, && [E,F] = H, \\ &[H,R^+] = R^+, && [E,R^+] = 0, && [F,R^+] = R^-, \\ &[H,R^-] = -R^-, && [E,R^-] = R^+, && [F,R^-] = 0, \\ &\{R^+,R^+\} = -2iE && \{R^-,R^-\} = 2iF && \{R^+,R^-\} = iH \end{aligned}$$

where the supercommutator is defined in the usual way

$$[\Gamma, \Gamma'\} \equiv \Gamma\Gamma' - (-)^{\pi_\Gamma \pi_{\Gamma'}} \Gamma'\Gamma. \tag{B.7}$$

The supertrace and scalar product are defined as

$$\mathrm{STr}(\Gamma) \equiv \Gamma_{11} - \mathrm{Tr}(\Gamma_{\mathrm{sp}(2)}) = \Gamma_{11} - \Gamma_{22} - \Gamma_{33} = -\Gamma_{22} - \Gamma_{33}, \tag{B.7}$$

$$(\Gamma, \Gamma') \equiv \mathrm{STr}(\Gamma\Gamma'), \tag{B.7}$$

where $\Gamma_{\mathrm{sp}(2)}$ is the submatrix generated by E, F and H ("spacetime" algebra), and there is no internal algebra because $N = 1$. In our representation, the fermionic sector is thus encoded in the F_{1a} and F_{a1} components of the matrices and the $\mathrm{sp}(2|\mathbb{R})$ subalgebra of $\mathrm{osp}(1,2|\mathbb{R})$ thus lies in the F_{ab} components, with $a, b = 1, 2$.

B.2.2 The extended case

The orthosymplectic $\mathrm{osp}(N,2|\mathbb{R})$ superalgebra can be realized as the real vector space of even (grading-preserving) $(N+2) \times (N+2)$ supermatrices acting on N

commuting real Grassmann variables x^i and 2 anticommuting real Grassmann variables θ^1 and θ^2 and which preserve the quadratic form

$$\sum_{i=1}^{N}(x^i)^2 + 2i\theta^1\theta^2 = \delta_{ij}x^i x^j + i\epsilon_{\alpha\beta}\theta^\alpha\theta^\beta \tag{B.7}$$

as well as the real character of the coordinates, with the usual Lie bracket

$$[\Gamma, \Gamma'] \equiv \Gamma\Gamma' - \Gamma'\Gamma, \tag{B.7}$$

where the multiplication is the matrix multiplication. Such supermatrices have the form

$$\begin{pmatrix} & & & i\mu_1 & -i\lambda_1 \\ & O_{ij} & & \vdots & \vdots \\ & & & i\mu_N & -i\lambda_N \\ \lambda_1 & \cdots & \lambda_N & a & b \\ \mu_1 & \cdots & \mu_N & c & -a \end{pmatrix} \tag{B.7}$$

with $O_{ij} = -O_{ji}$ and a, b, c real and commuting and λ_i, μ_i real and anticommuting. We identify the generators:

$$H \equiv \begin{pmatrix} & & & 0 & 0 \\ & 0 & & \vdots & \vdots \\ & & & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & -1 \end{pmatrix}, \qquad E \equiv \begin{pmatrix} & & & 0 & 0 \\ & 0 & & \vdots & \vdots \\ & & & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix}, \qquad F \equiv \begin{pmatrix} & & & 0 & 0 \\ & 0 & & \vdots & \vdots \\ & & & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

$$R_i^+ \equiv \begin{pmatrix} & & & & & 0 & 0 \\ & & & & & \vdots & \vdots \\ & & 0 & & & 0 & -i \\ & & & & & \vdots & \vdots \\ & & & & & 0 & 0 \\ 0 & \cdots & 1 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}, \quad R_i^- \equiv \begin{pmatrix} & & & & & 0 & 0 \\ & & & & & \vdots & \vdots \\ & & 0 & & & i & 0 \\ & & & & & \vdots & \vdots \\ & & & & & 0 & 0 \\ 0 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 1 & \cdots & 0 & 0 & 0 \end{pmatrix},$$

$$J_{ij} \equiv \begin{pmatrix} 0 & & 1 & 0 & 0 \\ & \ddots & & \vdots & \vdots \\ \bar{1} & & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix}, \tag{B.7}$$

where in R_i^+ and R_i^- (odd generators) the i factors sit in the i-th line and the 1 factors in the i-th column, and in J_{ij} the 1 (resp. -1) factors sit in the position (i, j) (resp. (j, i)). We find the supercommutators

$$\begin{array}{lll} [H, E] = 2E, & [H, F] = -2F, & [E, F] = H, \\ [H, R_i^+] = R_i^+, & [E, R_i^+] = 0, & [F, R_i^+] = R_i^-, \\ [H, R_i^-] = -R_i^-, & [E, R_i^-] = R_i^+, & [F, R_i^-] = 0, \\ i\{R_i^+, R_j^+\} = 2\delta_{ij} E & i\{R_i^-, R_j^-\} = -2\delta_{ij} F & i\{R_i^+, R_j^-\} = J_{ij} - \delta_{ij} H \\ [J_{ij}, E] = 0, & [J_{ij}, F] = 0, & [J_{ij}, H] = 0 \end{array}$$

$$[J_{ij}, R^+_k] = \delta_{jk} R^+_i - \delta_{ik} R^+_j \qquad [J_{ij}, R^-_k] = \delta_{jk} R^-_i - \delta_{ik} R^-_j \tag{B.7}$$

$$[J_{ij}, J_{kl}] = \delta_{jk} J_{il} + \delta_{il} J_{jk} - \delta_{ik} J_{jl} - \delta_{jl} J_{ik}, \tag{B.7}$$

where the supercommutator is defined in the usual way

$$[\Gamma, \Gamma'\} \equiv \Gamma\Gamma' - (-)^{\pi_\Gamma \pi_{\Gamma'}} \Gamma'\Gamma. \tag{B.7}$$

The supertrace and scalar product are defined as

$$\mathrm{STr}(\Gamma) \equiv \mathrm{Tr}(\Gamma_{\mathrm{so}(N)}) - \mathrm{Tr}(\Gamma_{\mathrm{sp}(2)}), \tag{B.7}$$

$$(\Gamma, \Gamma') \equiv \mathrm{STr}(\Gamma\Gamma'), \tag{B.7}$$

where $\Gamma_{\mathrm{so}(N)}$ is the submatrix of Γ generated by the J_{ij} basis elements (internal algebra) and $\Gamma_{\mathrm{sp}(2)}$ is the submatrix generated by E, F and H ("spacetime" algebra).

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요약 (국문 초록)

이 논문에서는 초대칭 등각장론과 3차원 중력 이론에 대한 세 가지 분야의 연구를 한다.

4차원 시공간의 초대칭 등각장론에서 국소적 연산자의 scaling-dimension 을 홀로그래피 대응성을 이용하여 계산한다. 이를 통하여 섭동적 전개를 통해 발견된 스핀에 대한 로그 스케일링이 비섭동적 영역에서도 여전히 유효함을 확인한다.
또한 천-사이먼즈 초대칭 등각장론의 산란 진폭에 대한 발산 문제를 다룬다. 섭동 전개의 보정항에서 나타나는 적외선 영역에서의 발산이 Soft-Bremsstrahlung 의 보정항과 서로 상쇄되는 과정을 보인다.
마지막으로 높은 스핀 입자를 가지는 3차원의 중력이론에서 점근적 대칭성이 확장됨을 보인다. 안티 드 시터 공간에 대한 경계조건을 통해 점근적 대칭성이 수퍼-W 대칭성으로 주어짐을 확인한다.

주요어 : 초대칭 등각장론 , 중력/장론 대응성, 적외선 발산, 점근적 대칭성

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