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Aspects of vacuum moduli in string theory

MATTHEW MAGILL

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Abstract
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In this thesis we explore geometric aspects of the space of vacua in supersymmetric string compactifications. More specifically, we will look at heterotic strings compactified on Calabi-Yau manifolds and, separately, G2 structure manifolds. After reviewing the basic concepts underlying the thesis, specifically dimensional reduction of supergravity theories and G structures on manifolds, we begin looking at the vacua leading to supersymmetric three dimensional theories. The underlying compact manifold has a G2 structure. We investigate some structures that further refine the G2 structure: almost contact metric three-structures. We report on a relation between these structures and (co)associative cycles in the G2 structure manifold, and initiate an exploration of the topological properties of the moduli space of such structures. Next, we identify a superpotential whose critical locus is precisely the moduli space of supersymmetry preserving backgrounds, including both gauge and geometric fields. Finally, we study the Yukawa couplings of heterotic string compactifications on Calabi-Yau manifolds. Using sheaf cohomological tools, we find widely applicable vanishing theorems, which can constrain the Yukawa couplings of the effective four dimensional theory.

Keywords: string compactifications, g structures, homological algebra

Matthew Magill, Department of Physics and Astronomy, Theoretical Physics, Box 516, Uppsala University, SE-751 20 Uppsala, Sweden.

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List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I X. de la Ossa, M. Larfors, M. Magill, E.E. Svanes, *Superpotential of three dimensional N=1 heterotic supergravity*, JHEP 01 (2020) 195 [arxiv:1904.01027]

II X. de la Ossa, M. Larfors, M. Magill, *Almost contact structures on manifolds with a G_2 structure*, ATMP, Accepted, [arxiv: 2101.12605]

III L.B. Anderson, J. Gray, M. Larfors, M. Magill, R. Schneider, *Generalized vanishing theorems for Yukawa couplings in heterotic compactifications*, JHEP 05 (2021) 085, [arxiv:2103.10454]

Contents

1. Introduction

String theory aims to unite the two biggest successes of twentieth century physics, general relativity and quantum mechanics, into a unified theory of quantum gravity.

This unification has proved to be a difficult problem, which is not just technical but also conceptual. The foundation of general relativity [1] is that spacetime is malleable and reacts to the presence of matter, while quantum mechanics has particles living on a fixed spacetime, but in which the position of such particles is ambiguously defined [2–4]. In fact, even merging special relativity with quantum mechanics was a Herculean task that eventually lead to quantum field theory (see Ch. 1 of Weinberg [5] and references therein), whose mathematical foundation is still lacking (see, however, [6–12] and references therein for a sample of mathematically rigourous approaches). Despite this lack of foundations, quantum field theory as represented by the Standard Model has unprecedented success in predicting experimental results.

General relativity is to special relativity as a smooth manifold is to a tangent space: indeed, symmetries of general relativity are the manifold diffeomorphisms, while the symmetries of special relativity form the Lorentz group, the tangent group of a Lorentzian manifold.

In this sense, given the technical and conceptual difficulties combining the special relativistic and quantum mechanical world, an endeavor which lead to the theory of quantum fields, it is not so surprising that a quantum theory incorporating general relativity has been difficult to construct.

Superstring theories attempt to solve the problem by replacing "particles" with microscopic strings, (we will rely on the standard texts, [13–16], for the elementary string theoretic results cited here). Consistency puts strong constraints on the possibilities, leading to the five known theories.[1] One such constraint leads to the conclusion that the spacetime described by string theory is ten dimensional, a prediction valid in all five string theories. This is in stark contrast with the observation that we live in four spacetime dimensions.

One popular method of attacking this problem is a so-called dimensional reduction [19–21]. Indeed, it turns out that although there are only five string theories, they possess a large[2] number of vacua. One can find vacua in which

[1] We are ignoring the evidence that points towards a unifying non-perturbative theory which has the five possible string theories as perturbative expansions in different corners of moduli space, [17, 18].

[2] Long believed, however, to be finite [22], the modern approach to understanding this space is related to the swampland program [23].

some of the ten dimensional spacetime dimensions are rolled up, or “compactified”, whilst leaving some directions extended. At certain energy scales, the compact directions are essentially invisible, but leave a signature in the spectrum and couplings of the effective theory on the remaining spacetime. General reviews of different aspects of this huge topic include [24–26] as well as the standard string theory texts. A more precise, though still concise, review is also in Chapter 2 of this thesis. The shadow of the compact geometry is encoded in a certain moduli space; in the settings of interest for this thesis it corresponds to the moduli of a globally supersymmetric effective theory, but its higher dimensional origins reveal that it is described as the moduli of certain geometric features of the compact manifold.

This thesis is concerned with the study of these, and closely related, moduli problems and the concommitant effective theories.

Thesis outline

The thesis begins in Chapter 2 with a review of the compactification problems that are studied: heterotic string compactifications.

In Chapter 3, the mathematics behind a major component of the geometric moduli problem, that of G structures and their connections, is reviewed.

Chapter 4 is a summary of Paper II. In this paper, we investigated certain refinements that are present whenever one studies a supersymmetric compactification on seven manifolds. The focus of this chapter is on the space of the “maximal” guaranteed refinement that exists. We explain what is meant by this statement, and review some of the novel results of that paper.

Chapter 5 summarises another paper on supersymmetric, seven dimensional compactifications, Paper I. In that paper, a superpotential is found, whose critical locus describes the moduli space of such compactifications. We review the reasoning behind this and demonstrate the claimed behaviour.

The final chapter of substance is Chapter 6, reviewing Paper III. In this paper, we investigated the Yukawa couplings of an extremely popular class of compactifications that lead to effective four dimensional physics, Calabi-Yau compactifications. We utilised homological algebra to find vanishing theorems that control these couplings. Here, we briefly review the mathematical background needed for the statement and proof of these theorems, and work our way up to a streamlined proof.

2. Moduli and dimensional reduction

String theory demands a ten dimensional spacetime.[1] This appears to be at odds with basic observations and therefore needs to be addressed. Dimensional reduction is one such attempt. The putative solution is the assertion that, although spacetime is ten dimensional, six of those dimensions are very small. As a consequence, all measurements at a sufficiently low energy will probe the wrapped up directions in only the most rudimentary fashion. Although making contact with measurement is probably the most important motivation for introducing this procedure, it has also been fruitful to study the theories obtained by compactifying to different dimensions. In particular, compactifying allows one to probe different sectors of the moduli space and a number of the interesting dualities that link the various string theories to each other and their non-perturbative cousins, M- and F-theory. For a small sample of such results, see [17, 18, 27–35].

A dimensional reduction occurs in two steps. Firstly, one *compactifies* the theory, by placing it on a vacuum that takes a product form, with one compact factor. Secondly, one scales out the modes that propagate along the compact factor by going to low enough energies that such excitations are essentially irrelevant. This results in an effective theory on the large directions and, so long as the compact volume is small enough, it is valid at energy scales that cover particle experiments for the foreseeable future. In string theory models, the internal manifold must also have length scale much longer than the string, to avoid modes coming from string wrapping modes.

In a little more detail, the starting point is a ten dimensional supergravity that approximates a string theory at low energies, with fields the massless excitations of the string [16]. Next, we need to find a vacuum of this theory. In full generality, this is a difficult problem, but asking that the solution be supersymmetric simplifies things. The models that one obtains had promising phenomenological properties, for instance an explanation of the hierarchy problem and naturally appearing GUT groups, right from their inception in [20] as discussed in [16].

By elementary supersymmetry arguments, reviewed in Section 2.1, a state that is annihilated by the fermionic supercharges is necessarily a vacuum state and it is generally easier to find a state annihilated by a supercharge in contrast to a general ground state. In the compactification setting, this advantage extends further. We will further restrict the problem of looking for supersymmetric vacua by looking for vacua that factorize into a product of a maximally

[1] The standard texts [13–16] provide excellent reviews of these basic string theoretic facts.

symmetric space and a compact space, see Section 2.3. In that case, the preserved supercharges assemble into a full fermionic symmetry on the symmetric space, provided that the compact geometry solves equations derived from the supersymmetry variation. The ten dimensional variations are described in Section 2.2, and the induced six and seven dimensional equations are presented in Section 2.4. The space of solutions to these equations form a geometric moduli space which, by construction, describes the moduli space of the supersymmetric field theory on the maximally symmetric space. Again, the foundations of supersymmetry ensure that the associated formal moduli problem encodes the effective theory around a given vacuum. The problems that the papers around which this thesis is built are concerned with determining different aspects of this moduli problem. This chapter is intended as a review of the basic logic underlying the constructions that are studied in this thesis; see the relevant Chapters 4, 5 and 6 for more details regarding these studies.

2.1 Supersymmetry

A supersymmetric field theory is a theory whose symmetry algebra contains a fermionic generator. It thereby evades the Coleman-Mandula theorem, [36], and defines a nontrivial extension of the Poincaré algebra. Supersymmetry leads to sufficiently tight constraints that field theories possessing it are comparatively well understood, even in the strongly coupled regime (for instance, [37]), and at a greater depth of mathematical rigour than can usually be achieved.

It is, consequentially, a subject about which much has been written and many good introductions exist, for instance [38–40]. In this section, we will content ourselves with the very minimum needed for the thesis, in particular, the fact that a state preserved by supercharges is necessarily a vacuum.

Mathematically, a supersymmetry algebra is, in particular, a $\mathbb{Z}_2$-graded Lie algebra, $L = L_0 \oplus L_1$. The supercharges form a basis of the degree 1 component, L_1 and the degree 0 component, L_0 is an ordinary Lie algebra that acts on L_1. In particular, the span of the bosonic generators contain the Poincaré algebra[2] as a subalgebra and the supercharges must therefore be in a representation of this algebra. As a consequence of spin-statistics, this must be a spin representation, [41]. We will use the convention that the amount of supersymmetry is counted by the number of *minimal* spinors in any given dimension. In particular, $N = 1$ in four dimensions means that there is a single Majorana or Weyl generator, which corresponds to four real supercharges. In dimension three, $N = 1$ means there is one Majorana spinor generator, so two real supercharges. In dimension ten, there is a Majorana-Weyl spinor representation,

[2]This is slightly imprecise; the local supersymmetry that we are interested in will have a copy of the Poincaré algebra fibrewise, but we will in fact have vacua with an AdS geometry and therefore the global superalgebra of such a case will be an extension of the isometries of AdS geometry, rather than the Poincaré algebra

which means that $N = 1$ has 16 real supercharges. Appendix B of Polchinski, [14], records many explicit calculations and results.

A theory which possesses any global symmetry has the property that the states of the theory come in representations of the algebra underlying the symmetry. In the case of supersymmetry, the generators have non-trivial Poincaré charge which means that the representations contain particles in different spin representations. These representations form the so-called supermultiplets.

As a graded Lie algebra, we know that the bracket preserves degree, i.e. $[L_i, L_j] \subset L_{i+j}$, and is graded antisymmetric, $[L_i, L_j] = (-1)^{ij+1}[L_j, L_i]$. In particular, the bracket of two supercharges must be bosonic. In fact, it can be shown that it commutes with the generators of the translation subalgebra, [40] so is necessarily a linear combination of momenta and internal symmetry generators. As an example, the minimal super extension of the Poincaré algebra in dimension three has a Majorana fermion supergenerator, $(Q_\alpha)^\dagger = Q_\alpha$. The only nontrivial bracket that is not immediately determined by the Poincaré algebra and its spinor representation is [39]

$$\{Q_\alpha, Q_\beta\} = 2(\Gamma^\mu \Gamma_0)_{\alpha\beta} P_\mu \,. \tag{2.1}$$

Since, in particular, P_0 is the Hamiltonian and $P_0 = Q_1^2 + Q_2^2$, we know that for any state, $|\Omega\rangle$, we have:

$$\begin{aligned}\sum_{\alpha=1,2} ||Q_\alpha|\Omega\rangle||^2 &= \sum \langle\Omega|Q_\alpha^2|\Omega\rangle \\ &= \langle\Omega|H|\Omega\rangle,\end{aligned}$$

so the Hilbert space is bounded below and states of minimal energy preserve supersymmetry. The general form of such an argument leads to the BPS inequalities that bound the masses of particles in terms of central charges. States that saturate the bound are known as BPS states and are necessarily annihilated by some combination of supercharges. They are thus absolute minima and must solve the equations of motion. For this thesis, it suffices to observe that states annihilated by one (or more) supercharge solve the equations of motion, [39]. These kind of arguments have been used to great effect in many different settings, including from Witten's supersymmetry approach to Morse theory, [42], understanding electric-magnetic duality, [37] and so on.

We will be interested in configurations in supergravity preserving some supersymmetry and, consequentially, annihilated by some supercharges. This implies that the configuration space locally factorises into directions with the preserved symmetry, and the transverse, symmetry breaking directions. By a supersymmetric analogue of the Higgs mechanism, the transverse directions become massive, while the supersymmetry preserving directions are massless. In the cases relevant to this thesis, this space of supersymmetric vacua will have a direct correspondence with a space of geometric structures on certain manifolds. Thus, the infinitesimal neighbourhood around a point in that moduli space will correspond to massless directions of an effective field theory.

2.2 Heterotic supergravity

In this section we review the theory of supergravity that is relevant to this thesis. A theory of supergravity is to global supersymmetry as a gauge theory is to a theory with global symmetry. In particular, rather than having a finite family of constant supercharges, a supergravity has local supersymmetry so there are infinitely many supercharges corresponding, roughly, to k spinors at each spacetime point in the case of $N = k$ supergravity. From the basic superalgebra, local supersymmetry implies that the space of vector fields forms part of the bosonic algebra and since this algebra is the Lie algebra of the diffeomorphism group this implies it must be a gravity theory.

This thesis is concerned with dimensional reductions of the ten dimensional heterotic supergravity, specifically with gauge group $E_8 \times E_8$. We will begin by recalling the field content and then give an explicit action.

The field content consists of the massless fields of the heterotic string. From string worldsheet computations (see for instance [13–15]), the massless sector of the heterotic string on flat spacetime $\mathbb{R}^{1,9}$ consists of:

1 x Yang-Mills supermultiplet The multiplet containing the gauge boson consists of (A, χ), where A is the local, adjoint-valued one form describing a connection and χ is an adjoint-valued, left-handed Majorana-Weyl fermion.

1 x Supergravity supermultiplet The gravity multiplet consists of bosonic fields (g, B, ϕ), corresponding to the graviton, antisymmetric B-field and scalar dilaton, along with their superpartners, (Ψ, λ), the gravitino and dilatino respectively. More precisely, Ψ is a left-handed Majorana-Weyl spinor valued in the tangent bundle, λ a right-handed Majorana-Weyl spinor.

The action contains both super Yang-Mills and supergravity actions, derived from symmetry considerations in [43–46] and we use the explicit conventions from the presentation of [47]:

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{bos} + \mathcal{L}_{ferm} \\
\mathcal{L}_{bos} &= \frac{1}{2\kappa^2} e^{-2\phi}(R + 4d\phi \wedge *d\phi + \frac{1}{2} H \wedge *H) \\
\mathcal{L}_{ferm} &= -\frac{1}{2\kappa^2} e^{-2\phi}\Big(\bar{\Psi}_M \Gamma^{MNP} D_N \Psi_P + \\
&\quad - \frac{1}{24}\big(\bar{\Psi}_M \Gamma^{MNPQR} \Psi_R + 6\bar{\Psi}^N \Gamma^P \Psi^Q\big) H_{NPQ}\Big) + \cdots
\end{aligned} \tag{2.2}$$

where κ^2 is the ten dimensional Newton's constant. This is the effective lagrangian for the heterotic string at next-to-leading order in α', missing terms with four fermions. These α' corrections are all hidden in the field strength H:

$$H = dB + \tfrac{\alpha'}{4}(\omega_L - \omega_{YM}) \tag{2.3}$$

where ω_L, ω_{YM} is the Chern-Simons forms for the spin connection and the Yang-Mills connection respectively, e.g.

$$\omega_{YM} = \mathrm{tr}(A \wedge dA - \tfrac{2}{3}A^3)\,. \tag{2.4}$$

The ten dimensional action is used explicitly in Paper I.

The supersymmetry variations of the fermionic fields are [48]:

$$\delta_\epsilon \Psi_M = (\nabla^{LC}_M + \tfrac{1}{4}H_M\cdot)\epsilon + (\mathrm{fermi})^2 \tag{2.5}$$
$$\delta_\epsilon \lambda = \tfrac{1}{2}\big(-d\Phi \cdot + \tfrac{1}{2}H\cdot\big)\epsilon + (\mathrm{fermi})^2 \tag{2.6}$$
$$\delta_\epsilon \chi = -\tfrac{1}{2}F\cdot\epsilon + (\mathrm{fermi})^2\,. \tag{2.7}$$

The action of a differential form on the spinor, ϵ, is induced by Clifford multiplication, e.g. $F\cdot\epsilon = \frac{1}{2}F_{MN}\Gamma^{MN}\epsilon$ and the three form, H defines a one form-valued two form via $H_M := \frac{1}{2}H_{MNP}\,dx^N \wedge dx^P$. We will consider vacua in which all Fermi condensates vanish so that the $(\mathrm{fermi})^2$- terms can be safely neglected (see [49], for one place they are included). The symbol ∇^{LC} denotes the spin connection lifting the Levi-Civita connection.

Only the supersymmetry variations of the fermionic fields are presented because it is only these that will be needed for the purposes of this thesis. Since a supersymmetric variation of a bosonic field is fermionic, these variations automatically vanish in vacuum so are irrelevant to our purposes. Therefore, a vacuum configuration is supersymmetric only if the variations (2.5), (2.6) and (2.7) vanish. These conditions must be supplemented by the Bianchi identities:

$$[D_A, F] = 0 \tag{2.8}$$
$$dH = \frac{\alpha'}{4}(\mathrm{tr}(R^2) - \mathrm{tr}(F^2))\,, \tag{2.9}$$

which ensure that the differential forms F, H are not arbitrary, but are field strengths of the appropriate geometric object.

2.3 Product vacua

In the above, Section 2.1, we sketched out the definition of a BPS state of a general superalgebra and roughly argued that a configuration of fields that preserve some number of supersymmetries must in fact be a solution to the equations of motion. In this section, we will continue to narrow in on the configurations of most relevance to the papers in this thesis. In particular, we will consider product geometries, involving a maximally symmetric factor and a compact factor. This will ensure that the supercharges that annihilate the state can be assembled into a full spinor on the maximally symmetric space.

Suppose that the ten dimensional geometry is of the form $M_d \times Y$, where M_d is a maximally symmetric d-dimensional space, either Minkowski or AdS.[3]

The condition that M_d be maximally symmetric is motivated in two ways: firstly, the dimension of the space of Killing spinors is equal to the appropriate spinor representation, [51], which will eventually mean that the effective theory on M_d will have unbroken supersymmetry and, therefore, unbroken Poincaré symmetry[4]; secondly, maximally symmetric spaces are precisely the local models of solutions to Einstein's equations in the absence of matter or gravitational radiation, and consequentially are good models for relativistic vacua, [52]. The effective theories that we obtain can, therefore, be sensibly interpreted as vacua in a conventional sense.

In order to preserve these properties, we will need to constrain the background values of all fields, not just the metric. This means that we will need homogeneous and isotropic background values for all fields on M_d and the product structure must be preserved. To be more concrete, we introduce the following convention that will be followed throughout: coordinates on the full ten dimensional spacetime are denoted by (X^M), with indices, $M, N, P, \ldots$; coordinates on the maximally symmetric space, M_d will be denoted (x^μ), with indices $\mu, \nu, \kappa, \ldots$; coordinates on the internal geometry will be denoted by (y^i), with indices $i, j, k, \ldots$.

Isotropy means that the only non-trivial background values on M_d must be in the trivial representation of the tangent group $SO(1, d-1)$, and homogeneity implies they must be independent of spacetime point, i.e. constant. This means that, unless $d = 3$, we must have $H(x, y) = \frac{1}{3!} H_{ijk}(y) dy^i \wedge dy^j \wedge dy^k$, while in dimension three we are also allowed to turn on a constant, h, such that $H_{\mu\nu\kappa} \sim *_3 h$. We will similarly turn off any background values for the gauge field strength, $F_{\mu\nu}$.

Observe that we can not demand that fluctuations about the vacuum preserve these structures. Nevertheless, we can always expand a fluctuation into a sum of fields that have definite spin with respect to the $SO(1, d-1)$ tangent group and with separated variables. As an example, a gauge field (of a trivial gauge bundle for simplicity), transforms in the vector representation of of the full spacetime tangent group, $SO(1, 9)$. Under the product group embedding $SO(1, d-1) \times SO(10-d) \hookrightarrow SO(1, 9)$, this irreducible representation splits into a sum of vector representations of the relevant groups. This expresses the fact that we can write $A_M(X) dX^M = A_\mu(x, y) dx^\mu + A_i(x, y) dy^i$. We can

[3]We restrict to these cases because they are the only possibilities that the constructions that we use support. The question of whether or not string theory supports dS solutions is currently a very active field, see [50] and references therein.

[4]Analogous to footnote 2, this is the fibrewise statement, the global group may be the isometry group of AdS space.

further separate the spacetime dependence and expand:

$$A_\mu(x,y) = \sum_m (A_\mu(x))_m \, \alpha_m(y) \tag{2.10}$$

$$A_i(x,y) = \sum_m \beta_m(x)(A_i(y))_m \tag{2.11}$$

where neither expansion need be a finite sum. The functions $\alpha_m(y), \beta_m(x)$ are smooth real-valued functions on Y and X respectively.

Similar expansions exist for all of the fields in the supermultiplets, see [16] for instance. We can, in particular view each of the fields $(A_\mu(x))_m$ as a gauge field on M_d, so that a single field on the ten dimensional theory will naively contribute infinitely many fields (labelled by m here) to an effective theory on M_d. We are saved from this unpleasantness by using that the spectra of the relevant differential operators on Y are gapped and there exist only a finite number of modes that contribute at a given energy scale (see for instance [53]). In particular, so long as the manifold, Y, has characteristic length scale sufficiently small, we can restrict to the zero modes since the first excited states will be extremely heavy. Note that supersymmetry arguments imply that the zero modes will be precisely the tangent space to the supersymmetric moduli space at the vacuum. We will now interpret this moduli problem purely in terms of the internal manifold.

This is possible precisely because of the fact that M_d is maximally symmetric. We can revisit the supersymmetry variations, (2.5)-(2.7), in the context of the product ansatz. Analogous to the expansion of the vector representation, the spinor representation decomposes as a tensor product, that is $S_{1,9} \to S_{1,d-1} \otimes S_{10-d}$. This can be checked directly, though we should be careful to identify the correct minimal spin representations, see [16] for a detailed discussion in $d = 4$, and [54] for details in the case of $d = 3$, the cases of interest for this thesis.

With this in mind, one rewrites the spinor parametrising the supersymmetry generator $\epsilon = \sigma \otimes \eta$. Letting σ_a denote a basis of the Killing spinors on M_d, it is relatively straightforward to confirm that the terms in the supersymmetry variations that depend on σ drop out and one is left with equations on the internal manifold, schematically written:

$$\delta_\sigma \Psi_i = \nabla_i^{LC}\eta - \tfrac{1}{4}H_i^Y \cdot \eta \stackrel{!}{=} 0 \tag{2.12}$$

$$\delta_\sigma \lambda = \tfrac{1}{2}(-d_Y\phi \cdot + \tfrac{1}{2}H^Y \cdot)\eta \stackrel{!}{=} 0 \tag{2.13}$$

$$\delta_\sigma \chi = -\frac{1}{2}F^Y \cdot \eta \stackrel{!}{=} 0\,. \tag{2.14}$$

The solution space of these *Killing spinor* equations is rich and interesting, both physically and mathematically, see for instance [20, 21, 48, 55–68]

It might be worth observing that the condition induced from the gaugino variation, (2.14), depends on the metric through the Clifford multiplication. In particular, it means that the geometric and gauge theoretic moduli are necessarily intertwined.

2.4 Examples

In this section we will introduce examples of supersymmetry-preserving compactification geometries that are directly relevant to this thesis.

2.4.1 Calabi-Yau manifolds

We will consider a dimensional reduction down to four dimensions, which means the internal manifold is six dimensional.

The original examples of heterotic compactifications are those on Calabi-Yau manifolds, [20] (in Subsection 3.5.3 of this thesis we will review the mathematical definition of a Calabi-Yau manifold). They are still, more than 35 years later, being studied and are a necessary ingredient of Paper III.

Dimensionally reducing on a Calabi-Yau manifold demands a constant dilaton and vanishing flux, $d_Y\phi = H^Y = 0$. These conditions are not strictly necessary for a six dimensional compactification, but they simplify the problem. The study of the general problem was independently initiated in [19] and [21] (see [26], and references therein, for a recent discussion).

Focusing on the Calabi-Yau case, vanishing of the gravitino supersymmetry transformation (2.12) reduces to the statement that η is covariantly constant with respect to the spin connection, implying the manifold has special holonomy $SU(3)$. G structures and holonomy are reviewed in Chapter 3. It can be shown that the gauge bundle must be a polystable holomorphic bundle, as a consequence of the Donaldson-Uhlenbeck-Yau theorem, [69, 70]. These results were shown for a specific choice of bundle, the so-called standard embedding, in [20] and are reviewed in the standard string theory texts. More general choices of bundle are reviewed in [26].

The infinitesimal moduli are classes in the bundle cohomology group $H^1(Y, Q)$, [26, 71–73] (see also [74, 75] for a similar discussion in the setting of the Strominger system) where Q is defined as an extension bundle:

$$0 \to \mathrm{End}_0(V) \to Q \to TY \to 0\,. \tag{2.15}$$

This exact sequence is known as the Atiyah sequence. The cohomology group $H^1(Y, Q)$ includes both bundle deformations and geometric deformations. This is really just the first order deformations of the moduli problem, meaning that these directions may not be precisely flat and there may exist higher order obstructions.

Another approach to this moduli problem utilises a superpotential, studied in a more general setting in [76]. In a Taylor expansion of a superpotential, the cohomology group is identified with the directions that have vanishing quadratic terms, but there may still exist terms of higher order.

2.4.2 G_2 structure manifolds

We will now consider dimensionally reducing down to three dimensions, with a seven dimensional internal space. A particular novelty arises in such scenarios, which is that the effective spacetime supports a nontrivial vacuum expectation value for the three form H, whilst remaining homogeneous and isotropic. As was mentioned earlier, this is because a three form on an oriented three manifold is a scalar. A nontrivial expectation value for this three form, or more precisely its dual zero form, gives the space a constant, negative curvature. We will also allow for a dilaton that varies and a nontrivial three form in the internal space, analogous to the Strominger-Hull system.

Demanding the vanishing of the supersymmetry variations (2.5)-(2.7) leads to the Killing spinor equations [47, 48, 60, 61, 66]

$$\nabla_i^{LC}\epsilon + \frac{1}{8}H_{ijk}\Gamma^{jk}\cdot\epsilon = 0 \tag{2.16}$$

$$(\partial_i\phi\Gamma^i + \frac{1}{12}H_{ijk}\Gamma^{ijk})\epsilon = 0 \tag{2.17}$$

$$\Gamma^{ij}F_{ij}\epsilon = 0 \tag{2.18}$$

which, as always, must be supplemented by the Bianchi identity:

$$dH = \tfrac{\alpha'}{4}(\mathrm{tr}F^2 - \mathrm{tr}R^2)\,. \tag{2.19}$$

The dilatino and gravitino equations will be geometrically interpreted in the context of G structures in Chapter 3.

The first order deformations of a solution to these equations are, similar to the Calabi-Yau case, encoded in a cohomology group [77, 78], $H^1(Y, Q)$ (or see [79] for another approach). In contrast to the Calabi-Yau case, the bundle with connection, Q, is not an extension.

A superpotential that reproduces these first order deformations is derived from the ten dimensional theory in Paper I and reviewed in Chapter 5.

3. G structures

The condition that a compactification be supersymmetric imposes conditions on the tangent bundle of the internal manifold [48, 59–61]. These conditions come from requiring the existence of a non-vanishing spinor that satisfies a partial differential equation coming from the ten dimensional supersymmetry variations as we discussed in Chapter 2. In particular, the Killing spinor equations (2.5)-(2.7) can be rephrased in terms of a reduced structure group of the tangent bundle: a G structure. Therefore, in this chapter we will review the basics of these structures.

Note that a manifold that *admits* a non-vanishing spinor is a candidate compactification manifold - then one must *choose* a spinor satisfying the equation and this choice is part of the data defining the vacuum. The fact that this choice exists and is non-trivial is a statement about the non-triviality of the moduli of such structures. In Chapter 2 the relationship between the moduli of a supersymmetric internal manifold and the low energy physics on the non-compact manifold was highlighted. This explains the relevance of the moduli of G structures to compactifications. To clarify: the supersymmetric moduli problem that we consider will not be precisely the moduli of G structures - it will interweave with gauge theoretic moduli - nevertheless, it is important to understand all the ingredients, including these geometric ones.

In this chapter we will review elementary notions revolving around G structures. All results and a more leisurely discussion can be found in [80] or [81], for instance.

3.1 Reduced structure group

In this thesis, we will be particularly interested in manifolds that admit a nowhere vanishing spinor, for reasons reviewed in the previous chapter, Chapter 2. The existence of such a spinor indicates that the manifold has a reduced structure group. In this section, we will explain what this means and how it relates to the geometry of the manifold. This classical material can be found in [80, 81].

A first attempt at a definition of a manifold with a reduced structure group, say G, is that the tangent bundle admits a covering by local trivialisations such that the transition functions take values in the group G. This gets at the right intuition, but is not very practical. With this definition, it can be hard to determine when such structures exist, and whether two given presentations are really the same G structure or not.

We will work up to a more covariant formulation for this notion, and see via examples that the notion of a reduced structure group is generally encoded as a geometric structure that is easier to work with. In the general formulation that we work with, the reduction of the structure group is phrased as the existence of a G principal subbundle of the frame bundle. In this way, the tangent bundle becomes a vector bundle associated to this G bundle. Since associated vector bundles can be locally trivialised over any cover that locally trivialises the principal bundle, with transition functions deduced from the principal bundle and the defining representation, the earlier "definition" is subsumed by this one.

We begin by recalling the definition of the frame bundle of a smooth manifold. Let (M, g) an arbitrary n dimensional, smooth, oriented Riemannian manifold, and $\pi : TM \to M$ denote the tangent bundle.

The frame bundle is defined to be a principal $GL(n)$ bundle, $\mathcal{F} \to M$, whose fibre over a point x is given by the space of all bases of T_xM. Linear algebra shows that any basis can be obtained by a unique $GL(n)$ transformation from any given initial basis, which is how the fibre is identified with the group $GL(n)$. Given a local coordinate chart, one can construct a local frame by pulling back the obvious basis on $\mathbb{R}^n$ and this shows that the construction is a locally trivialisable bundle. Further, given a coordinate atlas on M and the associated local frames, the transition functions are given by left multiplication with the Jacobian matrix, demonstrating that it is a principal $GL(n)$ bundle.

Notice, however, that this only uses the data of a smooth structure on M. It is assumed, however, that there was extra geometric data, namely an orientation and metric, and these can be used to give the first examples of a reduced structure group.

Recall, firstly, that a manifold is orientable if and only if one can choose a coordinate atlas such that the determinants of the Jacobian of the transition functions are positive. An orientation of an orientable manifold is a maximal atlas such that the coordinate charts satisfy this property. The manifold, M, has been assumed to have an orientation and yet the construction of the frame bundle did not use this geometric information at all.

This situation can be improved upon. Recall that a local frame is an ordered basis and consequentially defines an orientation on each tangent space. A principal bundle can be constructed whose fibres are all the bases over the point whose orientation agrees with the global orientation over the manifold. Note that once a single basis is chosen, all the other oriented bases can be uniquely obtained by acting with the group of invertible matrices whose determinant is larger than zero, $GL^+(n)$. By choosing any subatlas of the maximal atlas corresponding to the orientation, it is guaranteed that the transition functions preserve the orientation and, consequentially, the logic of the general frame bundle goes through and a principal $GL^+(n)$ bundle is obtained. The total space of this bundle shall be denoted $\mathcal{F}^+$.

This still neglects the Riemannian structure, so the next task is to include this geometry into the oriented frame bundle. This is achieved by further restricting the allowed frames to those orthonormal frames whose orientation agrees with the manifold's global orientation. This construction leads to a principal $SO(n)$ bundle, $\mathcal{F}^{SO}$. There is a slight subtlety here, which is that the argument for the frame and oriented frame bundle used local coordinates as a starting point. In the case at hand, it may not be possible to find any local coordinates that produce an orthonormal frame. The construction is easily saved by applying the Gram-Schmidt procedure to a coordinate frame, but the fact that local coordinate frames tend to be absent from the $SO(n)$ structure is an indication of something geometric. We will be particularly interested in the interplay between G structures and geometry as reviewed in Section 3.4.1.

The first lesson we learn, then, is that (some) geometric structures give a natural restriction to the kind of frames considered and thereby a restriction to the group of the principal bundle. Each of the groups that appeared have a canonical representation on the vector space, $\mathbb{R}^n$. Indeed, by the very construction there is a group morphism $\iota_G : G \to GL(n)$. Using this representation, there are associated vector bundles and unravelling definitions immediately shows that

$$TM \cong \mathcal{F} \times_{\iota_{GL(n)}} \mathbb{R}^n \cong \mathcal{F}^+ \times_{\iota_{GL^+(n)}} \mathbb{R}^n \cong \mathcal{F}^{SO} \times_{\iota_{SO(n)}} \mathbb{R}^n \,. \tag{3.1}$$

For completeness, recall that the vector bundles associated to a G-principal bundle, P, and representation $\rho : G \to GL(n)$ has total space $P \times \mathbb{R}^n / \sim$, where $(pg, v) \sim (p, \rho(g^{-1})v)$. In other words, it is the quotient of $P \times \mathbb{R}^n$ by the diagonal action. A more hands on definition, using a covering by local trivialisations (U_i, τ_i) of P, with transition functions $g_{ij} : U_i \cap U_j \to G$, is that the associated bundle is also trivial over U_i, with transition functions $\rho(g_{ij})$.

With this motivation out of the way, let us now recall the definition of a G structure on manifold, M.

Definition 1. Let M be a manifold of dimension n. A G structure on M is a principal G subbundle of the principal frame bundle, $\mathcal{F}$.

In other words, a G structure consists of

- A principal G bundle, $P \to M$; and
- A faithful representation $\rho : G \hookrightarrow GL(n)$;

with the property that $P \times_\rho \mathbb{R}^n \cong TM$.

In this definition, a G structure is a *sub*bundle and the group G is a subgroup of the general linear group. This agrees with the definition used by Joyce in [80], but observe that a spin structure is not a G structure under this definition.

Now is a good time to consider when different G structures are the same. The correct notion of an equivalence of G structures is that the principal subbundles of the frame bundles are isomorphic as subbundles.

Definition 2. Let M be a smooth manifold and P_G, P'_G two principal G-bundles encoding a G structure on M. Then P_G is equivalent to P'_G if there exists a G-bundle isomorphism, ϕ, such that the diagram commutes:

$$\begin{tikzcd} P_G \arrow[rr, "\phi"] \arrow[dr] & & P'_G \arrow[dl] \\ & \mathcal{F} & \end{tikzcd} \tag{3.2}$$

The arrows $P_G \to \mathcal{F}$ and $P'_G \to \mathcal{F}$ are part of the data defining the G structure.

3.2 Examples of G structures

In this section we will introduce several examples of G structures that will be relevant to this thesis. The examples that are of most relevance will be assumed to be oriented and Riemannian and the groups will have an injection into $SO(n)$, so in most cases it is most practical to start with the $SO(n)$ structure, as opposed to $GL(n)$.

3.2.1 Almost complex structures

In this section, the notion of an almost complex structure will be introduced and interpreted as a G structure. Good sources for this material are plentiful, see for instance [80, 82, 83].

The starting point is the notion of a complex structure on a vector space, $\mathbb{R}^k$ for some k. Recall that a complex structure is given by $J : \mathbb{R}^k \to \mathbb{R}^k$ such that $J^2 = -\text{Id}$. If J were diagonalisable, its eigenvalues would also square to -1 which means that the best we can do in the world of real numbers is to find a block diagonal form with blocks:

$$\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.3}$$

Note that this implies that $k = 2n$. In essence, we will want to consider the subspace of $\mathbb{R}^{2n}$ frames given by those that preserve this form of J. That is, bases that are like $(e_1, Je_1, \cdots, e_n, Je_n)$. The group that preserve such bases is the subgroup of $GL(2n, \mathbb{R})$ that commutes with J. This defines an embedding of $GL(n, \mathbb{C}) \hookrightarrow GL(2n, \mathbb{R})$, the group of $n \times n$ invertible matrices with complex coefficients.

A more convenient presentation, which is most often encountered in complex geometry, is to complexify $\mathbb{R}^k$, i.e. $\mathbb{R}^k \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}^k$. Extend J complex linearly to this larger vector space. We are now able to diagonalize J, to get a diagonal matrix with $\pm i$ on the diagonal. Note that since J has real coefficients, its eigenvalues must come in conjugate pairs, so the dimension of the $+i$

eigenspace is equal to that of the $-i$ eigenspace. Decompose the complexified space into eigenspaces, $\mathbb{C}^{2n} = \Lambda_+ \oplus \Lambda_-$, where $\dim_{\mathbb{C}} \Lambda_\pm = n$ and observe that complex conjugation maps $\Lambda_\pm \to \Lambda_\mp$. In terms of this splitting, consider the inclusion $GL(n, \mathbb{C}) \to GL(2n, \mathbb{C})$ given by

$$g \in GL(n, \mathbb{C}) \mapsto \begin{pmatrix} g & 0 \\ 0 & g^* \end{pmatrix} \tag{3.4}$$

By construction, under the obvious inclusion $GL(2n, \mathbb{R}) \subset GL(2n, \mathbb{C})$, the copy of $GL(n, \mathbb{C}) \subset GL(2n, \mathbb{R})$ defined above is precisely equal to the image of $GL(n, \mathbb{C})$ just defined. In particular, we have a commutative diagram:

$$\begin{tikzcd} & GL(2n, \mathbb{C}) \\ GL(n, \mathbb{C}) \arrow[ur] \arrow[r] & GL(2n, \mathbb{R}) \arrow[u] \end{tikzcd} \tag{3.5}$$

In terms of the complexified tangent space, a $GL(n, \mathbb{C})$ structure is encoded by a complex basis of Λ^+, say. We now have sufficient background linear algebra to consider the situation on a manifold. We start with a smooth manifold, M, equipped with a fibrewise linear endomorphism $J : TM \to TM$ satisfying $J^2 = -\mathrm{Id}$. The complexified tangent bundle, $TM^{\mathbb{C}}$ decomposes into a direct sum of eigenspaces, $T^+ \oplus T^-$. A local frame of T^+ complex conjugates to a local frame of T^- and such a frame can be seen as arising from TM. The collection of all local frames of T^+ is, once a starting point is chosen, isomorphic to $GL(n, \mathbb{C})$. This gives, by the same yoga as previously, a commutative diagram of frame bundles:

$$\begin{tikzcd} & \mathcal{F}^{\mathbb{C}} \\ \mathcal{C} \arrow[ur] \arrow[r] & \mathcal{F} \arrow[u] \end{tikzcd} \tag{3.6}$$

where $\mathcal{F}^{\mathbb{C}}$ is a principal $GL(2n, \mathbb{C})$ bundle comprising of frames in the complexified tangent bundle and $\mathcal{C}$ is the collection of bases of T^+.

3.2.2 Almost hermitian structures

We now want to include metrics in this structure. The appropriate notion of metric in complex linear algebra is Hermitian, as opposed to Riemannian. Since we will be interested in Riemannian manifolds, we will start with a Riemannian metric and use an almost complex structure to construct an hermitian metric in a standard fashion.

First, working linearly, let g an inner product on $\mathbb{R}^{2n}$ and J a complex structure. An hermitian metric is not symmetric, instead $h(x, y) = \overline{h(y, x)}$, where $\overline{(-)}$ indicates complex conjugation. Since J and g were chosen arbitrarily, they may not be compatible in the sense that $g(J-, J-) \neq g(-,-)$. But, there is a brute-force way to construct a metric with this property, i.e. $g' = \frac{1}{2}(g + g(J-, J-))$, so we assume this has been done. There is, now, a two-form that we can construct that will be used in constructing the hermitian form, h: $\omega(x, y) := g(Jx, y)$. This is indeed antisymmetric since $J^2 = -\mathrm{Id}$. We now pass to the complexification, $\mathbb{C}^{2n}$ and extend all structures $\mathbb{C}$-linearly. The hermitian metric is defined to be:

$$h := g - i\omega \,. \tag{3.7}$$

It can be checked that this is indeed an hermitian structure, and that J is anti-hermitian with respect to h, such that its eigenspaces are orthogonal.

We can then add the condition that the bases considered in the previous section should be orthonormal, analogous to the SO structures, to obtain a $U(n)$-structure, $\mathcal{F}^U$.

3.2.3 $SU(n)$ structure

Suppose that M is endowed with a $U(n)$ structure, $\mathcal{F}^U \to M$. By the inclusion $SU(n) \to U(n)$, we can try to define an $SU(n)$ structure. Observe that *if* there exists an $SU(n)$ structure, then the transition functions can be chosen to be $U(n)$ transformations with the extra condition that the determinant is $+1$. This implies that the bundle $\det T^+ = \Lambda^n T^+$ is trivial. In the case that this condition is satisfied, we can choose a unit section of this hermitian bundle, say χ^+ and define an $SU(n)$ structure to consist of those frames of $(e_1, \cdots, e_n) : \mathbb{R}^n \to T_x^+ M$ which satisfy $e_1 \wedge \cdots \wedge e_n = \chi^+$.[1]

3.2.4 Almost quaternionic structures

We saw above that a $GL(n, \mathbb{C})$ structure enabled us to the think of the tangent fibres as copies of $\mathbb{C}^n$, rather than $\mathbb{R}^{2n}$. In this section, we will consider a similar story, but now with the quaternions. Again, we will start by understanding the fibrewise story in linear algebra. We will follow Joyce [80], Chapter 7.1 and, in particular, include the notion of a metric from the beginning so the story is more closely analogous to that of $U(n)$ structures.

The algebra of quaternions is the third of the four possible normed division algebras over $\mathbb{R}$, after $\mathbb{R}$ and $\mathbb{C}$. The fourth will be important in the next subsection. The quaternions will be denoted $\mathbb{H}$ and is the algebra generated by i, j, k satisfying $ij = k = -ji$, $i^2 = j^2 = k^2 = -1$. A general element in $\mathbb{H}$ is of the form $x = x_0 1 + x_1 i + x_2 j + x_3 k$ with each $x_i \in \mathbb{R}$, so choosing generators

[1] It is more to common to impose this condition in terms of the cotangent bundle and a holomorphic top form, but using the metric, the notions are equivalent.

i, j, k induces an isomorphism $\mathbb{R}^4 \cong \mathbb{H}$. The algebra is noncommutative, but associative. Given an element $x \in \mathbb{H}$, define $\bar{x} = x_0 1 - x_1 i - x_2 j - x_3 k$, defining an anti-homomorphism of the algebra: $\overline{xy} = \bar{y}\bar{x}$. The inner product is defined by $\langle x, y \rangle := \bar{x}y$.

We can consider $\mathbb{H}^m$ to have coordinates $(w^1, \cdots, w^m)$ where each coordinate is of the form $w^a = x_0^a + x_1^a i + x_2^a j + x_3^a k$. In this basis $\mathbb{H}^m$ has a naturally induced quadruple of structures, the metric g and two-forms $\omega_i,\ i = 1, 2, 3$ that can be extracted from the expression:

$$g + i\omega_1 + j\omega_2 + k\omega_3 = \sum_{a=1}^{m} d\bar{w}^a \otimes dw^a \tag{3.8}$$

We can also define three complex structures induced by left-multiplication with the three generators $J_1 = L_i,\ J_2 = L_j,\ J_3 = L_k$. In fact, more generally there is an S^2 worth of complex structures induced by $J_x := x_i L_i$ for any $x_i x^i = 1$ in $\mathbb{R}^3$. Furthermore it can be checked that J_x and $\omega_x := x_i \omega_i$ are compatible in the sense of section 3.2.2. By the same argument that we saw there, each triple (g, J_x, ω_x) is overdetermined and it suffices to specify two-out-of-three structures.

Define the group $Sp(m)$ to be the subgroup of $GL(4m, \mathbb{R})$ that preserves $(g, \omega_1, \omega_2, \omega_3)$. It can be more explicitly presented as

$$Sp(m) \cong \{A \in M_m(\mathbb{H}) \,:\, A\bar{A}^t = I\}\,. \tag{3.9}$$

An almost quaternionic structure on a manifold M^{4m} will be defined by the sub-frame bundle induced by the sub-collection $\mathbb{H}$ bases, analogous to the way that almost complex structures were defined. Evidently, this defines an $Sp(m)$-structure on M.

3.2.5 G_2 structures

We will now turn to examples of G structures, important to this thesis: G_2 structures. Good sources on the definition of G_2 structures include [80,84,85].

The group G_2 is an exceptional Lie group that can be described as the group of automorphisms of the octonions. Recall that the octonions are the fourth and final example of a normed division algebra over $\mathbb{R}$. It might be worth beginning with a comment on the similarities and differences of a G_2 structure with the almost-complex and -quaternionic cases.

In complex, respectively quaternionic structures, the tangent space is identified with modules of a matrix algebra with values in $\mathbb{C}$ or $\mathbb{H}$ respectively. Such a strategy will not work on the octonions because of the nonassociativity: in particular, the one-by-one octonionic matrix algebra should just be the octonions, but it is not a module over itself, by the failure of associativity. Roughly speaking, a G_2 structure will circumvent this problem by identifying the tangent space of a manifold with the algebra itself. We will now make this more precise.

We begin with a very brief reminder of the algebraic structure of the octonions, following Baez [86]. The octonionic algebra is unital, non-commutative and also non-associative. It is eight dimensional over the reals and for a given choice of generators, $\{1, e_1, \cdots, e_7\}$ has the multiplication table:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_7	e_5	$-e_6$	$-e_3$	e_4	$-e_2$
e_2	$-e_7$	-1	$-e_6$	$-e_5$	e_4	e_3	e_1
e_3	$-e_5$	e_6	-1	e_7	e_1	$-e_2$	$-e_4$
e_4	e_6	e_5	$-e_7$	-1	$-e_2$	$-e_1$	e_3
e_5	e_3	$-e_4$	$-e_1$	e_2	-1	e_7	$-e_6$
e_6	$-e_4$	$-e_3$	e_2	e_1	$-e_7$	-1	e_5
e_7	e_2	$-e_1$	e_4	$-e_3$	e_6	$-e_5$	-1

(3.10)

The group of automorphisms of this algebra, identified as G_2, is simply connected and fourteen dimensional.

By construction, G_2 has an action on the octonions, $\mathbb{O} \cong \mathbb{R}^8$. However, since G_2 respects the algebra structure, it must also respect the unit, which means $1 \in \mathbb{O}$ is G_2 invariant and the orthogonal complement, $(1 \cdot \mathbb{R})^\perp = \operatorname{Im} \mathbb{O}$, also forms a G_2 representation. It can be checked that this is irreducible; it is in fact the smallest non-trivial representation of G_2.

Separately, observe that a map of the octonions that preserves the algebraic structure must also preserve the inner product as this is defined with the multiplication: $\langle x, y \rangle = \operatorname{Re}(\bar{x} y)$. As a consequence, $G_2 \subset O(8)$. In fact, using without proof that G_2 is connected, $G_2 \subset SO(8)$. Combining with the earlier observation that the eight dimensional representation decomposes as a sum $(1 \cdot \mathbb{R})^\perp \oplus \operatorname{Im} \mathbb{O}$, with the first summand trivial, it follows that $G_2 \subset SO(7)$. This is the inclusion that will be of most relevance for this thesis. In particular, G_2 structure manifolds must be seven dimensional. There is another representation that will be used and discussed later, namely $G_2 \subset \operatorname{Spin}(7)$. This is the unique lift of the $SO(7)$ embedding, which exists because G_2 is simply connected. This viewpoint will be used in Subsection 3.6.6.

Returning to the representation on the imaginary octonions, we will characterize the algebraic structure induced from the octonionic operation in terms of a three form

$$\phi(a, b, c) = \langle a, bc \rangle \, ; \; a, b, c \in \operatorname{Im} \mathbb{O} \, . \tag{3.11}$$

Evidently, G_2 preserves this structure and, in fact, any $SO(7)$ transformation that preserves ϕ must preserve the full structure of the octonions, so that the subgroup of $SO(7)$ that preserves ϕ is precisely G_2.

Consulting the table for the octonionic multiplication, (3.10), we can deduce the explicit form:

$$\phi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245} \, . \tag{3.12}$$

To construct a G_2 structure entails defining a G_2 orbit in the space of frames. A convenient way to encode a basis on a 7d vector space, V, is an isomorphism $\mathbf{e} : \mathbb{R}^7 \to V$. Given a three form on V, we can consider the collection of isomorphisms that preserve this three form. By construction, this forms a G_2 subgroup of $GL(7, \mathbb{R})$. So, analogous to the almost complex case, for instance, where we had to choose an endomorphism J and consider the group that commutes with it, in the G_2 case we have to choose a three form that takes the form (3.12) in some frame and then consider the G_2 orbit, considered pointwise as the isomorphisms $\mathbb{R}^7 \to T_xM$ that preserve the given three forms.

Passing to the global situation, a G_2 structure on a seven manifold, Y, is encoded in a three form $\varphi \in \Omega^3(Y)$, for which there exists local frames $(e^1, \cdots, e^7)$ in which it takes the form

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245} . \tag{3.13}$$

Such a three form is called *positive*, [80]. One should note that the precise presentation is convention dependent. This convention conforms with that of Paper II, which was itself chosen to conform with [65]. There exist other conventions in the literature notably that used by Joyce in his constructions [87,88] and, perhaps most commonly, that used by Bryant [89].

The principal subbundle of frames in which φ takes the indicated form exhibits the G_2 principal bundle as a subbundle of the full frame bundle.

We now address the question of when a G_2 structure exists. The easiest way to answer this question [84], involves the second presentation of the group G_2 that was alluded to above.

Begin by observing that, since a G_2 structure is a refinement of an $SO(7)$ structure, the manifold must be orientable. It follows that a G_2 structure carries with it a choice of orientation.

Next, use that, since G_2 is simply connected, it has a unique lift to the $\mathrm{Spin}(7)$ cover of $SO(7)$, so a manifold that admits a G_2 structure must also be spinnable.

Seeing that these are the only obstructions is where the second presentation is preferred: The group G_2 is the subgroup of $\mathrm{Spin}(7)$ that leaves invariant a given non-zero element in the spinor representation (a good explanation of this fact can be found in Baez' notes, [86]). Consequentially, a G_2 structure on a spin seven manifold is specified by a choice of nowhere vanishing spinor. The spin representation of $\mathrm{Spin}(7)$ is eight dimensional, while the base manifold is only seven dimensional, and therefore, a generic section will not intersect the zero section (see e.g. [90] for a discussion of these transversality arguments) and there must exist a G_2 structure.

This perspective makes it extremely plausible that a G_2 structure is necessary for supersymmetric compactifications on seven manifolds. Recalling from Chapter 2, demanding supersymmetry preservation forced the internal manifold to possess a nowhere vanishing spinor (parametrizing the supercharge) that further satisfies differential equations, e.g. equations (2.5)-(2.7). To deal with this differential condition and interpret it as a condition on the G_2 structure we need to look closer at the differential geometry of G structures, which is the subject of the next section.

3.3 Connections

In this section we introduce the notion of a connection on a principal bundle. A connection is necessary to express the Killing spinor equations in the geometric language of G structures.

A connection on a vector bundle is a familiar notion in modern physics, while the notion of a connection on a principal bundle may be less familiar to some readers. This notion is important, at least for conceptual clarity, for the way G structures are used in this thesis, and therefore this section is dedicated to these elementary notions. A principal bundle connection also induces a connection on associated vector bundles, and this construction is also recalled. Both Joyce [80] and Kobayashi-Nomizu [81], are good examples of texts covering this material.

Roughly speaking, a principal bundle connection gives a notion of parallel transport, i.e. a distinguished way of lifting paths from the base to the total space of the bundle. We will ask that these lifts be equivariant with respect to the group action on the total space.

Definition 3. Let $\pi : P \to M$ a G-principal bundle on a manifold Y. A principal bundle connection on P is a smooth splitting of the tangent bundle $TP \cong H \oplus V$, where $V = \ker(\pi_*)$ and where H is preserved by the right G action, i.e. $(R_g)_* : H_p \to H_{pg}$ for each $p \in P$ and $g \in G$. We call the subbundle, V the *vertical* subbundle and its complement, H, the *horizontal* subbundle.

Notice that the vertical subbundle is completely canonical, unlike its complement, so the real content of a principal bundle connection is in the horizontal subbundle. Further, the kernel of π_*, at any point p is, by definition, the tangent to the fibre of the principal bundle. Since the right G-action is free and transitive on fibres, this gives a natural isomorphism of V_p with the Lie algebra of G, denoted $\mathfrak{g}$. The data of this complement can also be captured by a one form with values in G's Lie algebra, $\mathfrak{g}$, satisfying conditions that are extracted from the invariance of H.

By definition, the differential of the projection at each $p \in P$ induces an isomorphism $\pi_* : H_p \to T_{\pi(p)}M$. In fact, the horizontal subbundle, H, is naturally isomorphic to the pullback of the base tangent bundle $H \cong \pi^*TM$ and, consequentially, having chosen a connection, vector fields on the base can be canonically lifted to a vector field in H. This lift of $X \in \Gamma_M(TM)$ will be denoted $\lambda(X)$. In the language of principal bundle connections, the curvature can be defined by:

$$F(X,Y) = [\lambda(X), \lambda(Y)] - \lambda[X,Y]\,. \tag{3.14}$$

That is, the curvature can be seen as a measure of the difference between the Lie bracket in the base and in the horizontal distribution. F takes values in the vertical distribution and, therefore, is naturally viewed as $\mathfrak{g}$-valued.

It is probably no surprise that any principal bundle admits a connection, a fact whose proof will not be reviewed here but is contained in standard references, including [80, 81]. We will now review how connections interact with G structures, in particular recalling the definition of the holonomy of a connection, and the relationship between a covariant derivative and a connection.

Definition 4. Let $\pi : P \to M$ a principal G bundle, H a principal bundle connection on P and $\gamma : [0,1] \to M$ a smooth curve. Choose a point in the fibre $p \in \pi^{-1}(\gamma(0))$. A *horizontal lift* of γ at p is a path, $\tilde{\gamma} : [0,1] \to P$ such that

- $\tilde{\gamma}$ starts at p: $\tilde{\gamma}(0) = p$;
- $\tilde{\gamma}$ is a lift of γ: $\pi \circ \tilde{\gamma} = \gamma$;
- The tangent vectors are horizontal: $\frac{d\tilde{\gamma}(t)}{dt} \in H_{\tilde{\gamma}(t)}$ for all $t \in [0,1]$.

Proposition 1 ([81], Vol. 1, Prop 3.1)**.** Let $\pi : P \to M$ a principal G bundle, H a principal bundle connection on P and $\gamma : [0,1] \to M$ a piecewise-smooth curve. For any point p in the fibre of $\gamma(0)$, there exists a unique horizontal lift.

A horizontal lift is used to define the holonomy of the connection. Observe that a closed loop in the base need not lift to a closed lift in the total space, although the end point of the lift will be in the same fibre. In particular, looking at the horizontal lift of a curve that starts at p, the endpoint will necessarily be $p' = p \cdot g$ for a unique $g \in G$. The holonomy group of the connection, H, based at p, is a subgroup of G consisting of those elements obtained through the horizontal lift of a curve. We will denote this group by $Hol_p(P,H)$.

The holonomy group is an obstruction to reducing the principal bundle with connection. Indeed, if $Q \subset P$ is a principal subbundle with group $G' \subset G$, with the property that the connection restricts, then a horizontal lift of a curve in the base must surely stay in Q and, consequentially, the holonomy subgroup of $q \in Q$ must satisfy $Hol_q(P,H) = Hol_q(Q,H') \subset G' \subset G$. In fact, it is the only obstruction, as the next theorem demonstrates:

Theorem 1 ([80] Thm 2.3.6)**.** Let $\pi : P \to M$ be a principal G bundle, H a principal bundle connection on P. Fix an element $p \in P$ and define $G' = Hol_p(P,H)$. Let Q be the subspace of P whose points are those that are connected to p by a horizontal path, i.e. $q \in Q$ if and only if there exists $\gamma : [0,1] \to M$, $\gamma(0) = \pi(p)$ whose unique horizontal lift, $\tilde{\gamma}$ has endpoint equal to q, $\tilde{\gamma}(1) = q$. Then, Q is a G'-principal subbundle of P and H restricts to Q.

As a consequence, a principal G bundle with connection and holonomy, H, can be restricted to a principal G' bundle, only if $H \subset G'$. As a converse statement, we have [80, 81]:

Theorem 2. Let M a manifold with dimension $n \geq 2$, P a principal bundle over M with fibre G. Then, for each connected Lie subgroup $G' \subset G$ there exists a connection, H, on P with holonomy group $Hol(P, H)$ if and only if P reduces to a principal G' bundle.

In particular, if a manifold has a G structure, encoded in a principal subbundle $P \subset \mathcal{F}$, then a connection on $\mathcal{F}$ can be found that also has holonomy P. We will next want to understand the relation between the principal bundle connections we have been dealing with and the more familiar covariant derivatives of vector bundles. We briefly review this story, relying on Joyce [80] or Kobayashi-Nomizu [81] for proofs.

Let us fix a group, G and a linear representation $\rho : G \to GL(W)$. Given a principal G bundle, P, we can therefore define an associated vector bundle $E = P \times_\rho W$. Recall that the elements of the associated bundle are equivalence classes of pairs $[p, v]$ where $[pg, v] = [p, \rho(g)v]$. Recall further that for any connection on $\pi : P \to M$, the subbundle H is isomorphic to the pullback $\pi^* TM$. The extra data that the connection gives is an inclusion $\pi^* TM \to TP$. These observations lead to a means of endowing E with a covariant derivative. Before reproducing the construction, we recall the definition of a covariant derivative of a vector bundle.

Definition 5. Let $E \to M$ be a smooth vector bundle over a smooth manifold. A covariant derivative, ∇ is a map $\nabla : \Gamma_M(E) \to \Gamma_M(T^*M \otimes E)$ which is $\mathbb{R}$-linear and satisfies the Leibniz rule:

$$\nabla(fe) = df \otimes e + f\,\nabla e\,. \tag{3.15}$$

Restrict to the case that E is the associated bundle, $E = P \times_\rho W$. Let $e \in \Gamma_M(E)$, a smooth section, and consider the projection $\mathrm{p} : P \times W \to E$. The section, e, can be pulled back along this map to give a G-invariant section of $P \times W$, considered as a trivial vector bundle over P. Indeed, we must have $e_x = [p, v]$ and observe $\mathrm{p}^* e(p) = (p, v)$. For $p' = gp$, then $e_x = [pg, \rho(g)v]$ so that $\mathrm{p}^* e(p') = (pg, \rho(g)v)$. In this way, we have a bijection between smooth sections of the associated bundle, E, and smooth, G-invariant sections of the trivial bundle $P \times W \to P$. Consider the pullback as a map $\mathrm{p}^* e : P \to W$ and take its differential, $(\mathrm{p}^* e)_{*,p} : T_p P \to V$. This is interpreted as a smooth section of the vector bundle $W \otimes T^* P \to P$. By dualising the connection, we have that $T^*P \cong V^* \oplus H^*$. Using the natural isomorphism $V_p \cong \mathfrak{g}$ and $H_p \cong \pi^*(T_{\pi(p)}M)$, we have the natural splitting

$$W \otimes T^*P \cong W \otimes \mathfrak{g}^* \oplus W \otimes \pi^*(T^*M)\,. \tag{3.16}$$

It is now sensible to project the section $(\mathrm{p}^*e)_*$ to the horizontal component, $W \otimes \pi^*(T^*M)$. Borrowing some of Joyce's notation, we write this section as $\pi_H(\mathrm{p}^*e_*) \in \Gamma_P(W \otimes \pi^*T^*M)$. By construction, this is G-invariant and consequentially is equivalent to a section of $E \otimes T^*M$ over M. Then define $\nabla e \in \Gamma_M(E \otimes T^*M)$ to be the section uniquely defined by $\pi_H(\mathrm{p}^*e_*)$. It can be checked using the definition that this is indeed $\mathbb{R}$-linear and satisfies the Leibniz rule with respect to multiplication by smooth functions. The following definition records this construction.

Definition 6. Let $\pi : P \to M$ a principal G-bundle, $\rho : G \to GL(W)$ a representation and $E := P \times_\rho W$ the associated bundle. Let $H \subset P$ a principal bundle connection. Define the associated connection to be the map $\nabla^E : \Gamma_M(E) \to \Gamma_M(E \otimes T^*M)$:

$$\nabla^E \,:\, e \mapsto [\pi_H(\mathrm{p}^*e_*)]\,. \tag{3.17}$$

This gives a map of sets

$$\left\{\begin{matrix}\text{Principal bundle connections}\\ \text{on } P\end{matrix}\right\} \to \left\{\begin{matrix}\text{covariant derivative on}\\ \text{an associated bundle}\end{matrix}\right\}.$$

In general, this map need not be a bijection, but it is for $G = GL(k,\mathbb{R})$ and ρ the fundamental representation, [80].

In particular, a covariant derivative is equivalent to a connection on the principal frame bundle. If the connection on the principal bundle has holonomy contained in some group, G, and the manifold admits a G structure, then the given covariant derivative induces a connection on the G structure.

Example 1. Recall that an $SO(n)$ structure on M is equivalent to an isometry class of metric and an orientation. A covariant derivative on TM will restrict to connection on $\mathcal{F}^{SO}$ only if the covariant derivative is metric.

In this context, Theorem 2 expresses the fact that there always exists a non-empty family of metric connections. The famous Levi-Civita connection is a metric connection that satisfies an extra condition: it is "torsion-free". As is well known, the Levi-Civita connection is the *unique* connection that is torsion-free and metric. We will be interested in analogous problems: given a G structure, we will need to look for connections that reduce to the G structure *and* satisfy some constraint on their torsion. All the groups, G, that we will be interested in are in fact subgroups of $SO(n)$, not just $GL(n)$, so the connections will be metric. In general, demanding that the torsion also vanishes will be too strong a constraint - it imposes that the Levi-Civita connection restricts to the G structure. Manifolds whose Levi-Civita connection has restricted holonomy in this sense are quite special, but are not the main focus of this thesis.

3.4 Torsion

In this section we review the definition of the torsion of a connection. Ultimately, the existence of spinor satisfying the Killing spinor equations, specifically the gravitino equations (2.12), can be rephrased in terms of the existence of a G structure connection with specific torsion and this relationship is reviewed in this section.

3.4.1 Intrinsic torsion of a G structure

Let M be a smooth manifold. Recall that a covariant derivative on a tangent bundle can be thought of as a map of sections $\Gamma_M(TM \otimes TM) \to \Gamma_M(TM)$, cf. Definition 5. The torsion of such a covariant derivative is a tensor that gives a measure of how symmetric this map is in the inputs.

Definition 7. Let M a smooth manifold and ∇ a covariant derivative on TM. The *torsion* of ∇ to defined to be the tensor, $T(\nabla) \in \Gamma_M(TM \otimes \Lambda^2 T^* M)$ by:

$$T(\nabla)(v, w) := \nabla_v w - \nabla_w v - [v, w] \in \Gamma_M(TM)\,, \qquad (3.18)$$

for all $v, w \in \Gamma_M(TM)$.

Our aim will be to understand whether a given G structure will admit a compatible connection satisfying a constraint on its torsion. We will therefore need to understand both how torsion behaves with a change of connection and to what extent the intrinsic G bundle constrains the torsion.

Towards the first goal, suppose that ∇, ∇' are distinct G-connections on TM. Recall that the difference between any two covariant derivatives is a globally defined one form valued in the endomorphism bundle, in this case:

$$\nabla - \nabla' \in \Omega^1(M, \mathrm{End}(TM))\,. \qquad (3.19)$$

We can therefore regard this difference as a three index tensor $\alpha_{ab}^{\ \ c}$. Further:

$$\begin{aligned}[T(\nabla) - T(\nabla')](v, w) &= \nabla_v w - \nabla_w v - \nabla'_v w + \nabla'_w v \\ &= \alpha(v, w) - \alpha(w, v) \in \Gamma_M(TM)\,. \end{aligned} \qquad (3.20)$$

Thus, $T(\nabla)$ will have the required torsion, say τ, only if any other connection satisfies

$$T(\nabla') = \tau + \alpha(v, w) - \alpha(w, v)\,. \qquad (3.21)$$

To make systematic use of this formula, we recall the notion of intrinsic torsion, [80].

Let $W \cong \mathbb{R}^n$ a faithful representation of a group, G, with induced inclusion $G \subset GL(n)$. Since $\mathfrak{gl}(n) = W \otimes W^*$, we can embed G's Lie algebra into this space of endomorphisms, $\mathfrak{g} \subset W \otimes W^*$.

Use this embedding to define a linear map $\sigma : \mathfrak{g} \otimes W^* \to W \otimes \Lambda^2 W^*$, by:

$$\sigma(\alpha^a_{bc}) = \alpha^a_{bc} - \alpha^a_{cb} \,. \tag{3.22}$$

We get a four-term exact sequence:

$$0 \to \ker \sigma \to \mathfrak{g} \otimes V^* \to V \otimes \Lambda^2 V^* \to (V \otimes \Lambda^2 V^*)/\mathrm{Im}\,\sigma \to 0 \,. \tag{3.23}$$

In the notation of Joyce, [80], let $W_1 := W \otimes \Lambda^2 W^*$ and consider the spaces $W_2 = \mathrm{Im}\,\sigma$, $W_3 = W_1/W_2$ and $W_4 = \ker \sigma$. Each of these has an induced G representation, $\rho_i : G \to GL(W_i)$ and, consequentially, given a G structure we can form the associated bundle to each representation, say E_i.

Evidently, the torsion of any G structure connection forms a section of E_1 while the equivalence class of the torsion of any given connection, i.e. $[T(\nabla)] \in \Gamma_M(E_3)$ is independent of the connection.

Definition 8. Let M a smooth manifold, $P \to M$ a G structure on M, H a connection on P and ∇ the associated covariant derivative on TM. The *intrinsic torsion* of P is defined to be the section $[T(\nabla)] \in \Gamma_M(E_3)$. The intrinsic torsion depends only on the G structure.

Note that sections of the bundle associated with the kernel, E_4 can be regarded as deformations of any given connection that preserve the torsion. That is, if ∇ a connection and $\alpha \in \Gamma_M(E_4)$, then $\nabla + \alpha$ is a new connection and $T(\nabla + \alpha) = T(\nabla)$.

We will now specialise to the case that $G \subset SO(n)$, and take up the story using Bryant's notes [84]. The Lie algebra of $SO(n)$, denoted $\mathfrak{so}(n)$ has the property that

$$\sigma : \mathfrak{so}(n) \otimes W^* \to W \otimes \Lambda^2 W^* \tag{3.24}$$

is an isomorphism. The above machinery then tells us that there is a unique torsion-free metric connection and, more generally, the triviality of the kernel implies that the torsion of a metric connection uniquely specifies the connection itself.

Since $\mathfrak{g} \subset \mathfrak{so}(n)$, the triviality of the kernel is inherited by σ_G, but it generally fails to be surjective. A convenient means of measuring this failure is to recall that $\mathfrak{so}(n)$ has an essentially unique inner product which induces a canonical splitting

$$\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{g}^\perp \,. \tag{3.25}$$

Evidently, the quotient W_2 is simply isomorphic to $\mathfrak{g} \otimes W^*$, while the quotient $W_3 \cong \mathfrak{g}^\perp \otimes W^*$, both isomorphisms as G-spaces. We will therefore interpret the intrinsic torsion of a G structure, P, for $G \subset SO(n)$, as a section of the associated bundle $E_3 \cong P \times_\rho (\mathfrak{g}^\perp \otimes \mathbb{R}^n)$.

Still following Bryant, consider the Levi-Civita connection on TM, distinguished as the unique torsion-free metric connection. We can think of this connection as a globally defined one-form on the orthonormal frame bundle, $\mathcal{F}^{SO}$, e.g. $\Psi \in \Omega^1(\mathcal{F}^{SO}, \mathfrak{so}(n))$. The connection restricts to a given G structure, P, if pulling-back along $P \to \mathcal{F}^{SO}$ gives a one-form with values in $\mathfrak{g} \subset \mathfrak{so}(n)$. More generally, after pulling back, there is a unique decomposition $\Psi = \Theta + \tau$ where Θ takes values in $\mathfrak{g}$ and τ has values in $\mathfrak{g}^\perp$.

Then, τ represents the intrinsic torsion of the G structure, P.

It can be observed that the connection specified by Θ need not be unique. In fact, the space of G-equivariant homomorphisms,

$$\mathrm{Hom}^G(\mathfrak{g}^\perp \otimes W^*, \mathfrak{g} \otimes W^*)$$

describes the modifications on Θ, first order in the torsion functions, such that the modification defines a connection on M that is compatible with the G structure.

3.5 Examples of G structures and torsion constraints

We will now use these formal deliberations to consider some examples of interesting geometries, relevant to this thesis, that are encoded in terms of a G structure with constrained torsion.

3.5.1 Complex structure

Recall from Section 3.2.1 that an almost complex structure on a $2n$ dimensional manifold is a $GL(n, \mathbb{C})$ structure. A complex structure is a *torsion-free* $GL(n, \mathbb{C})$ structure.

By the Newlander-Nirenberg theorem, [91], this is equivalent to the statement that the manifold has a holomorphic atlas compatible with the endomorphism J. Complex geometry is a rich and interesting topic, with many introductory accounts that deal with this theorem, including [80, 82, 83].

3.5.2 Kähler structure

In Subsection 3.2.2 we saw that an hermitian metric on an almost complex structure induced a $U(n)$ structure and that the metric and almost complex structure combined define a two form, ω. A torsion-free $U(n)$ structure is a Kähler structure. In this case, the two form is closed and defines a symplectic structure.

3.5.3 Calabi-Yau manifold

In Subsection 3.2.3, the notion of an $SU(n)$ structure was defined. A torsion-free $SU(n)$ structure is a *Calabi-Yau* manifold. By the Calabi conjecture [92, 93], famously proven by Yau, [94], a compact Kähler manifold with vanishing first Chern class admits a unique metric with two form in the same cohomology class that is Ricci-flat. This ensures that many manifolds that admit a Calabi-Yau structure have been found, though explicitly computing the metric is almost always out of reach (though see e.g. [95, 96] and references therein for recent work in the area). This fact plays a background in the discussion of Paper III.

Calabi-Yau manifolds are of great interest in the physics community, being one of the first semi-realistic compactifications [20].[2] They continue to play a distinguished role in the physics literature, with numerous interesting applications, including mirror symmetry and topological strings, for instance.

3.5.4 G_2 manifolds

A manifold with torsion-free G_2 structure is often called a G_2 manifold. In this case, the three form that describes the G_2 structure is closed and co-closed. Similar to the Calabi-Yau case, explicit metrics are difficult to come by, but there exist several compact constructions, firstly the orbifold constructions of Joyce [87, 88], and more recently the twisted (and extra twisted) connected sum (TCS) constructions due initially to Kovalev [98] and elaborated upon in [99–101].

G_2 manifolds continue to arouse a great deal of interest in the physics literature, see for instance [33–35, 102–105] for a small sample.

Despite their importance and interest, G_2 manifolds do not play a significant role in this thesis. The G_2 structure manifolds that are of most relevance to this thesis are those considered in the next subsection.

3.5.5 Integrable G_2 structures

The main focus for the G_2 structures considered in this thesis are those G_2 structure manifolds whose torsion is not necessarily vanishing, but is totally antisymmetric. In this section, we will recall the constraints on the intrinsic torsion imposed by this requirement, following Bryant [84]. Given the results of the previous section, this basically amounts to a little G_2 representation theory. These results can be found in many sources, including the standard sources for this chapter, [84] and [80]. Alternatively, one can use a computer package e.g. LieART [106] and replicate the computations.

We have already implicitly encountered two irreducible representations of the group G_2, namely the seven dimensional vector representation inherited

[2]For a physics-aimed introduction, see for instance [97].

from $\mathfrak{so}(7)$ and the Lie algebra $\mathfrak{g}_2$, which is fourteen dimensional. The irreducible representations that are relevant for us are characterised by their dimension, so this how we will label them. We will be particularly interested in decomposing the representations arising from the decomposition of tensor products of the $\mathbf{7}$, especially the endomorphism representation, $\mathrm{End}(\mathbf{7})$ and the exterior powers $\Lambda^k\mathbf{7}$. It is important to note that the fundamental representation, $\mathbf{7}$, is self-dual, i.e. $\mathbf{7}^* \cong \mathbf{7}$. Therefore, there is an isomorphism $\mathrm{End}(\mathbf{7}) \cong \mathbf{7}\otimes\mathbf{7}$. The $\mathbf{7}\otimes\mathbf{7}$ contains all the representations that are relevant to us: in particular:

$$\mathbf{7}\otimes\mathbf{7} \cong (\mathbf{1}\oplus\mathbf{27})\oplus(\mathbf{7}\oplus\mathbf{14})\,. \tag{3.26}$$

The parentheses group the summands into the symmetric and antisymmetric parts, respectively. In particular, the $\mathbf{27}$ is the traceless symmetric part and the $\mathbf{1}$ is the trace. The antisymmetric part is isomorphic to $\mathfrak{so}(7)$ and the further decomposition corresponds to the splitting $\mathfrak{so}(7) = \mathfrak{g}^{\perp}\oplus\mathfrak{g}$ of (3.25).

The antisymmetric powers decompose as:

$\Lambda^0\mathbf{7}$	$\mathbf{1}$
$\Lambda^1\mathbf{7}$	$\mathbf{7}$
$\Lambda^2\mathbf{7}$	$\mathbf{7}\oplus\mathbf{14}$
$\Lambda^3\mathbf{7}$	$\mathbf{1}\oplus\mathbf{7}\oplus\mathbf{27}$
$\Lambda^4\mathbf{7}$	$\mathbf{1}\oplus\mathbf{7}\oplus\mathbf{27}$
$\Lambda^5\mathbf{7}$	$\mathbf{7}\oplus\mathbf{14}$
$\Lambda^6\mathbf{7}$	$\mathbf{7}$
$\Lambda^7\mathbf{7}$	$\mathbf{1}$

In particular, for a manifold with G_2 structure, Y, differential forms have a comparable fibrewise decomposition, e.g. $\Omega^2(Y) = \Omega^2_7(Y)\oplus\Omega^2_{14}(Y)$. Importantly, the space of sections depends only on the representation so that $\Omega^3_1(Y)\cong\Omega^0_1(Y)$ and so on. The final tensor decomposition required is given by $\mathbf{14}\otimes\mathbf{7}\cong\mathbf{7}\oplus\mathbf{27}\oplus\mathbf{64}$. It will not be necessary to know anything about the 64 dimensional representation.

Using these decompositions, we can investigate the class of connections that were singled out at the end of Subsection 3.4.1. Recall that, starting from the Levi-Civita connection, the family of connections that can be obtained at first order in the torsion functions is [84]:

$$\mathrm{Hom}^{G_2}(\mathfrak{g}_2^{\perp}\otimes W^*, \mathfrak{g}_2\otimes W^*) \cong \mathrm{Hom}^{G_2}(\mathbf{1}\oplus\mathbf{27}\oplus\mathbf{7}\oplus\mathbf{14}, \mathbf{7}\oplus\mathbf{27}\oplus\mathbf{64})\,. \tag{3.27}$$

Since there are no equivariant morphisms between non-isomorphic irreducible representations, it follows that there is an isomorphism:

$$\mathrm{Hom}^{G_2}(\mathfrak{g}_2^{\perp}\otimes W^*, \mathfrak{g}_2\otimes W^*) \cong \mathrm{Hom}^{G_2}(\mathbf{27},\mathbf{27})\oplus\mathrm{Hom}^{G_2}(\mathbf{7},\mathbf{7}) \cong \mathbb{R}^2\,. \tag{3.28}$$

Note that by construction, the deformations that are parametrized by these families are proportional to the torsion, τ, defined in terms of the Levi-Civita connection. Consequentially, if the **7** and **27** part of τ vanishes then the family collapses, [84].

We will need our torsion to be totally antisymmetric, which means that it must be valued in the $\mathbf{1} \oplus \mathbf{7} \oplus \mathbf{27}$. Since we can not remove the **14**, at least at first order, that means that we must have the **14** part of τ vanishes. This is called an *integrable* G_2 structure by Fernández and Ugarte [107] where it used to construct a canonical chain complex associated to the G_2 structure. A more explicit proof that the **14** part of the torsion must vanish in order to find antisymmetric torsion is given in [48, 78].

3.6 Constant tensors

We will conclude this review chapter on G structures by giving another characterization in terms of differential forms. The appeal of this approach is that it is very concrete and, from the physics perspective, quite natural. The essential idea is contained in the following Proposition:

Proposition 2 ([80] Proposition 2.5.2)**.** Let M be a manifold and ∇ a connection on M. Fix $x \in M$ and define $H = \mathrm{Hol}_x(\nabla)$, so $H \subset GL(T_xM)$. Let E be a tensor bundle on M, i.e. $E = \bigotimes^k TM \otimes \bigotimes^l T^*M$ for some $k, l \in \mathbb{Z}_{\geq 0}$. Then, the connection, ∇ induces a connection on E, ∇^E, and H has a natural representation on the fibre E_x of E at x.

Suppose that $S \in \Gamma_M(E)$ is covariantly constant, i.e. $\nabla^E S = 0$. Then $S|_x$ is fixed by the action of H on E_x. Conversely, if $S_x \in E_x$ is fixed by the action of H, then there exists a unique tensor $S \in \Gamma_M(E)$ such that $\nabla^E S = 0$ and $S|_x = S_x$.

Recall that if $H = \mathrm{Hol}_x(\nabla)$ and M admits a G structure, then the connection only restricts to G if $H \subset G$. The above proposition tells us that at any given point in M, the H invariants of any tensor are equivalent to covariantly constant sections.

In many circumstances, this proposition expresses ideas that we are already familiar with. We will have a look at the invariants characterising a subset of the collection of G structures that we have studied thus far.

3.6.1 $SO(n)$ structure

The invariant of an $SO(n)$ structure is the metric $g \in \Gamma_M(S^2TM)$. We have observed earlier, in Example 1, that a connection has holonomy in $SO(n)$ if and only if it preserves the metric.

3.6.2 $U(n)$ structure

Recall from 3.2.2 that $U(n)$ structures are redundantly described by a triple of tensors (J, h, ω), where any two out of the three fixes the third. These invariants are tensors with the following structure

$$
\begin{aligned}
h &\in \Gamma_M((T^+ \otimes T^-)^*) \subset \Gamma(S^2 T^* M^{\mathbb{C}}) \\
J &\in \Gamma_M((T^+ \otimes (T^+)^* \oplus (T^- \otimes (T^-)^*)) \\
\omega &\in \Gamma_M((T^- \otimes T^+)) \subset \Omega^2(M)^{\mathbb{C}}, .
\end{aligned}
$$

As a consequence of Proposition 2, any covariant derivative on TM that preserves the $U(n)$ structure must have the property that each of these are covariantly constant.

3.6.3 $SU(n)$-structure

An $SU(n)$ structure has the same invariants as a $U(n)$ structure, in addition to the top-form $\Omega^+ \in \Gamma_M(\Lambda^n T^+) \subset \Omega^n(M)^{\mathbb{C}}$. A $U(n)$ connection will only restrict to the $SU(n)$ structure if the new form, Ω is also covariantly constant.

3.6.4 Quaternionic structures

As we saw in Subsection 3.2.4, a quaternionic structure is characterized by a triple of non-degenerate two-forms, $\omega_1, \omega_2, \omega_3$, and a Riemannian metric, g. By construction, these are invariants of the structure group, so a tangent bundle connection will respect the reduction only if these defining forms are covariantly constant.

3.6.5 G_2 structure

We have already seen that a G_2 structure is characterised by a three form, φ from which we can extract a metric and a four form $\psi = *\varphi$. These are all invariants so are covariantly closed by Proposition 2. Since this class of manifolds is deeply relevant to two-out-of-three presented papers, we will here explore this in a little more depth. We aim to review the relationship between the intrinsic torsion of a G_2 structure, as defined in Subsection 3.4.1, and a conceptually simpler invariant of the G_2 structure. In the next example, we will relate this to the spinor defining a G_2 structure, which will provide a conceptually crucial piece in relating the geometric considerations of this chapter with supersymmetric compactifications.

The starting point is to consider the exterior derivatives of φ and ψ, giving a four- and five- form respectively. This can be decomposed into components in the irreducible representations:

$$d\varphi = \tau_0\psi + 3\tau_1 \wedge \varphi + *\tau_3 \tag{3.29}$$

$$d\psi = 4\tau_1 \wedge \psi + *\tau_2\,. \tag{3.30}$$

Here, $\tau_0 \in \Omega^0(Y)$, $\tau_1 \in \Omega^1_7(Y)$, $\tau_2 \in \Omega^2_{14}(Y)$ and $\tau_3 \in \Omega^3_{27}(Y)$. The only remarkable aspect of equations (3.29) and (3.30) are that the same τ_1 appears twice. This is the content of Proposition 1 in [84].

That there is a relation to the intrinsic torsion that we met previously is hinted at by the fact that the degrees of freedom match, i.e. the intrinsic torsion was valued in the $\mathbf{7}\otimes\mathbf{7} \cong \mathbf{1}\oplus\mathbf{7}\oplus\mathbf{14}\oplus\mathbf{27}$. If it weren't for the fact that the $\mathbf{7}$ part of $d\varphi$ and $d\psi$ agree (up to constant factor and under the isomorphism $\Omega^4_7 \cong \Omega^5_7$), this assertion would fail.

The correspondence can be checked explicitly, by following through the logic of Bryant [84] as reviewed in 3.4.1, and explicit computations along these lines are in [48]. A different presentation of a related approach can be found in [85]. The explicit relationship can be found in [48, 78] where the τ classes are combined into a three-form

$$H = \frac{1}{6}\tau_0\varphi - \tau_1 \lrcorner \psi - \tau_3\,. \tag{3.31}$$

In the case that $\tau_2 = 0$, then the authors of [48, 78] show there is a unique G_2 connection with torsion $T = H$. Explicitly, the G_2 connection has connection symbol:

$$\Gamma_{ab}{}^{c} = (\Gamma^{LC})_{ab}{}^{c} + \frac{1}{2}H_{ab}{}^{c} \tag{3.32}$$

3.6.6 G_2 structure via a spinor

Recall that an alternative description of a G_2 structure on Y was a choice of spin structure on Y along with a non-zero spinor. In the choice of spin structure there is implicitly a fixed metric and orientation, so the G_2 invariants are now a spinor, λ, and the metric. A G_2 connection on the tangent bundle induces a spin connection on the spin structure, but Proposition 2 does not immediately imply that the spinor is covariantly constant, because it was stated only for tensor bundles. The fact that the spinor is indeed covariantly constant can be seen via the following reasoning.

Let λ a unit spinor defining a G_2 structure. Then, the corresponding positive three form, φ has an explicit, local expression as a spinor bilinear [77, 108]:

$$\varphi_{abc} = -i\lambda^T\Gamma_{abc}\lambda, \tag{3.33}$$

where Γ_a indicates a Clifford matrix, taken to be purely imaginary, and Γ_{abc} is the antisymmetrized product.

We will again restrict to an integrable G_2 structure and the compatible connection with totally antisymmetric torsion, whose connection symbol is explicitly given by (3.32). The corresponding spin connection is given by

$$D_i^{G_2} = D_i^{LC} - \frac{1}{8} H_{ijk} \Gamma^{jk}, \tag{3.34}$$

where D^{LC} is the spin connection associated to the Levi-Civita connection.

It can then be directly computed (a package like GAMMA, [109], is very helpful) that λ is covariantly constant with respect to D^{G_2}, [48, 61, 108].

This fact lies at the crux of the relationship between supersymmetric compactifications and the geometry of G structures: it corresponds precisely to the vanishing of the gravitino equation, (2.12).

4. Almost contact (3-) structures

In Chapter 3 we reviewed the fundamental notions of G structures and saw in Example 3.6.6 how these are related to the Killing spinor equations derived from supersymmetry preservation of a supergravity vacuum. It is clear, then, that it would be desirable to have as complete an understanding of the space of G structures and connections as is possible.

In Paper II, we investigated possible tools that may aid in this understanding in the case of G_2 structure manifolds. The basic premise is that G_2 structure manifolds always admit a further refinement of structure group [108,110–114], or more precisely, it always admits an infinite dimensional family of refinements.

These further reduction of structure groups can be captured by an almost contact metric structure (ACMS) [115,116], which reduces the group to $SU(3)$, and almost contact metric 3-structures [114], which further reduce the structure group to $SU(2)$. These are related to complex and quaternionic structures, but not precisely in the sense of Subsection 3.2.3 respectively 3.2.4, due to the mismatch in dimensions: complex structures must be even dimensional and quaternionic require a multiple of four; seven dimensions satisfies neither of these requirements. More details will be given below, but essentially these structures trivialise part of the tangent bundle and the remaining, non-trivial part will have the correct rank for an almost complex, resp. quaternionic structure.

Although these structures always exist and always admit connections that preserve the restricted group, by Theorem 2 a given G_2 connection will restrict to an $SU(3)$, resp. $SU(2)$ structure, if and only if its holonomy is already a subgroup of the relevant group. Nevertheless, we can use the refined G structure to express elements of the geometry, for instance the G_2 torsion, into irreducible representations of the reduced structure group. For the case of ACMS, this was carried out for the G_2 connection with totally antisymmetric torsion that is relevant for heterotic compactifications. In fact, we decomposed the entire system of Killing spinor equations of the heterotic string. This enabled us to recognise that a certain solution from the literature in fact had enhanced, $N = 2$ supersymmetry, [65].

For more details on this side of things, the reader is referred to Paper II. In this summary, we will focus on the ACM3S story, i.e. the reduction from G_2 to $SU(2)$ structures. Whilst it is, in principle, possible to carry out the same computations here as we did for the ACMS case, the number of terms makes it impractical [117]. The study focused instead on the space of ACM3S, i.e. all the possible reductions of the G_2 structure to an $SU(2)$ structure group.

4.1 Definitions

We begin by recalling the precise definitions of an almost contact (metric) structure [115, 116] and almost contact (metric) 3-structure, [114].

Definition 9. An almost contact structure (ACS) on an odd dimensional Riemannian manifold is a triple $(R, J, \sigma) \in \Gamma_Y(TY \oplus \mathrm{End}\, TY \oplus T^*Y)$, such that

1. $|R|^2 = 1$;
2. $\sigma(R) = 1$;
3. $J^2 = -\mathbf{1} + R \otimes \sigma$.

An almost contact metric structure (ACMS) is an ACS that satisfies, in addition:

$$g(Ju, Jv) = g(u, v) - \sigma(u)\sigma(v)\,, \quad \forall\, u, v \in \Gamma_Y(TY)\,. \tag{4.1}$$

We can remark that, in general, a choice of a nowhere vanishing vector field can be regarded as defining a reduction of structure from $GL(n, \mathbb{R})$ to $GL(n-1, \mathbb{R})$. In case the manifold was oriented and metric, then this can instead be regarded as reducing $SO(n)$ to $SO(n-1)$. In our case, $n = 7$ and, having fixed R, the rest of the structure can be seen as endowing the transverse bundle, defined to be the kernel of σ, with an almost complex structure, see Section 3.2.1. The metric condition ensures that it is, in fact, an almost hermitian structure as reviewed in Section 3.2.2.

The fact that we end up with structure group $SU(3)$ instead of $U(3)$ is a consequence of the fact that we started with a G_2 structure, [110–112].

Definition 10. An almost contact three-structure (AC3S) on a Riemannian manifold, M, of dimension $4k+3$ is given by a triple of almost contact structures, $(R_i, J_i, \sigma_i) \in \Gamma_Y(TY \oplus \mathrm{End}\, TY \oplus T^*Y)$, satisfying (in addition to the conditions of Definition 9)

1. $\sigma_i(R_j) = \delta_{ij}$;
2. $\sigma_i \circ J_j = -\sigma_j \circ J_i = \sigma_k$;
3. $J_i \circ J_j - R_i \otimes \sigma_j = -J_j \circ J_i - R_j \otimes \sigma_i = J_k$;

for (i, j, k) a cyclic permutation of $(1, 2, 3)$.

An almost contact metric 3-structure (ACM3S) is an AC3S that satisfies, in addition:

$$g(J_i v, J_i w) = g(v, w) - \sigma_i(v)\sigma_i(w)\,, \quad \forall\, i \in \{1, 2, 3\}\,. \tag{4.2}$$

Similar to the comment on ACS structures, we can regard an ACM3S structure as inducing an almost quaternionic structure (see Subsection 3.2.4) on the rank-4 bundle that is transverse to the trivial bundle spanned by $(R_1\,, R_2,\ R_3)$.[1]

[1] Since we only dealt with metric quaternionic structures in Subsection 3.2.4, we need the metricity condition to compare these notions.

In fact, part of the stated condition of being metric is redundant: it suffices to check the metric condition on any two of the three almost contact structures.

Todd [110, Thm 5.1], shows that a G_2 structure, along with an orthonormal pair vector fields, allows one to construct an ACM3S. We will now review this construction. Let R_1, R_2 denote the vector fields and σ_1, σ_2 denote their duals with respect to the given G_2 metric. Define J_i by the property that

$$g_\varphi(J_i(X), Y) = \varphi(R_i, X, Y)\,. \tag{4.3}$$

The results of Todd show that these pair satisfy the necessary conditions $\sigma_i \circ J_j = -\sigma_j \circ J_i$ and $J_i \circ J_j - R_i \otimes \sigma_j = -J_j \circ J_i - R_j \otimes \sigma_i$ and are metric, thus defining an ACM3S.

4.2 Space of G_2-ACM3S

Given the above construction, it is natural to ask: On which G_2 structure manifolds can we find a pair of vector fields, from which we can can construct an ACM3S? and secondly, what is the space of such structures?

The first question was implicitly answered by Thomas in 1969 [113], where he showed that on any spinnable seven manifold, there exists such a pair of everywhere linearly independent vector fields. A seven manifold admits a G_2 structure if and only if it is spinnable, as reviewed in Subsection 3.2.5, and consequentially every G_2 structure manifold admits an ACM3S.

To the best of the author's knowledge, the second question was not answered until Paper II, although the argument is not a new one and closely related to the obstruction theoretic arguments of Thomas [113]. We notice that an ACM3S that is constructed in this fashion is uniquely specified by an orthonormal pair of vector fields. At each point on the seven manifold, the space of orthonormal pairs is a Stiefel manifold, which has a convenient presentation as a G_2 homogeneous space due to [118]

$$V_2 = G_2/SU(2)\,. \tag{4.4}$$

This just means that $SU(2)$ is the subgroup of G_2 that preserves a pair of vectors.

Therefore, one can define the fibre bundle associated to the principal G_2 frame bundle whose fibre is G_2, call it $\mathcal{V}_2$, say. The sections of this bundle is precisely the space of orthonormal two-frames on our original space, which is bijective with the space of ACM3S of the form constructed above. That is, $\Gamma_Y(\mathcal{V}_2)$ is the space of interest to us.

There is a fibrewise-action of $SO(3)$ on this space. Indeed, using the G_2 cross product, we have the orthonormal triple $(R^1, R^2, R^3 := R^1 \times_\varphi R^2)$, and $SO(3)$ acts by rotation.

We can view the data of (R^1, R^2, R^3) as a trivialised rank three subbundle of TY, and quotienting by the above $SO(3)$ action has the effect of forgetting the trivialisation, which means the resulting space is the space of trivial rank three subbundles. We can, therefore, view the space of 3-structures as a bundle, whose fibre is Maps$(Y, SO(3))$. It turns out that this need not be trivial, which was shown in an example in subsection 5.4.3 of Paper II.

4.3 Integrability

The discussion thus far has been quite topological, while any physics applications will have to include more differential geometric information, as encoded in a connection, for instance. However, the brute force decomposition into $SU(2)$ irreducible components has too many pieces to be practical or, presumably, enlightening [117].

We would therefore like to find some other means of accessing at least some differential geometric information. The fact the ACM3S that we are interested in are specified by the vector fields and the G_2 structure suggests that we investigate the Lie bracket of these vector fields.

There is certainly no reason to expect that the Lie algebra will close, but we want to understand what happens when it does. The generic answer is given by Frobenius theorem [119–122], and says that closure of the Lie algebra is equivalent to integrability of the distribution they span. We will consider two distributions: that spanned by (R^1, R^2, R^3), say $\mathcal{T} \subset TY$, and the rank four orthogonal complement $\mathcal{T}^\perp \subset TY$. Frobenius' theorem tells us that a distribution is tangent to a foliation if and only if it is closed under the Lie bracket. A weaker condition is that there exists a cycle such that the distribution is tangent to this cycle. This is possible only if the distribution is closed under the Lie bracket at each point in the cycle. Note that, crucially, we can choose an arbitrary local frame for the distribution and check on this basis, the results are independent of the frame. That means, for instance, that involutivity is independent of the specific framing (R^1, R^2, R^3) and depends only on the spanned vector space. In other words, it is invariant under the Maps$(Y, SO(3))$-action identified above.

The above discussion is standard and goes for an arbitrary distribution. What is interesting about the distributions that arise in the ACM3S context, is the following.

Suppose that $X \subset Y$ is a three-cycle with the property that $TX = \mathcal{T}|_X$. Then, the volume form of X is the three form $\sigma_1 \wedge \sigma_2 \wedge \sigma_3$. On the other hand, $\sigma_3 = \iota_{R^2}\iota_{R^3}\varphi$, which implies that this volume form is $\varphi|_X$. In the context of G_2 geometry, a three cycle whose volume form is the G_2 structure three form is called associative. These are particularly interesting when the three form is closed, in which case they are calibrated geometries and, for instance, volume minimising in their class. More generally, these cycles are energy minima of wrapped branes in the presence of flux [60, 123, 124].

Similarly, one can show that a four cycle, $W \subset Y$ has $TW = \mathcal{T}^{\perp}|_W$ has volume form given by the four form, ψ. In that case, one says that W is coassociative and, in the case that ψ is closed, W is a calibrated cycle.

In particular, when $\mathcal{T}$, respectively $\mathcal{T}^{\perp}$ are integrable, the manifold is foliated by associative, respectively coassociative manifolds. Manifolds with these geometries have attracted interest from both physicists and mathematicians, e.g. [34, 35, 105, 125, 126].

4.4 Examples

4.4.1 Barely G_2 holonomy

We here consider a class of G_2 manifolds that are built out of Calabi-Yau manifolds. This construction gives a torsion free G_2 structure, but its holonomy is not the full group G_2, rather it is twisted product, $SU(3) \rtimes \mathbb{Z}_2$, whence the name "barely" G_2. This class of examples is discussed in a physics context in [127], for instance, and a subset of these are classified in [128].

Let X a Calabi-Yau manifold that admits a fixed point free, antiholomorphic involution, $\sigma : X \to X$. Not every Calabi-Yau admits such a map, but Grigorian classifies all the complete intersection Calabi-Yaus (CICYs) that do, [128].

Consider the product $X \times S^1$, equipped with the involution

$$\sigma \times -1 : X \times S^1 \to X \times S^1 \,.$$

This is fixed point free, because σ is, so the quotient is a smooth manifold, $Y = (X \times S^1)/\mathbb{Z}_2$, with the induced metric. It is not hard to check that this has the claimed structure group. What about ACM3S?

To make things a little more concrete, let us restrict to the case that the Calabi-Yau manifold admits a vector field. This is true if and only if the Euler number vanishes and there are two CICYs in Grigorian's list which have this property, [128].

Since the Euler characteristic X/σ is also zero, there exists at least one unit vector field which is invariant under this involution. Choose one such vector field, i.e. $v \in \Gamma_X(S(TX))$ satisfying $\sigma_* v = v$.

Using the complex structure on X, denoted by I say, we can define a new vector field, $w := Iv$, that is everywhere linearly independent of v. This does not descend to the quotient, since σ is antiholomorphic and therefore anticommutes with the complex structure. Similarly, the circle factor admits a vector field that anticommutes with the symmetry action and we will be able to use this to construct a triple on the resulting barely G_2 manifold, Y.

We will construct a pair of vector fields; the third is then fixed by the G_2 structure. The first vector field is the image of the invariant vector field assumed above, $R^1 := v$.

The second can be constructed out of w and ∂_t:

$$R^2 = \cos(2\pi t)w - \sin(2\pi t)\partial_t \,. \tag{4.5}$$

We need to show that this is invariant under the $\mathbb{Z}_2$ action which sends the point $e^{2\pi i t} \in S^1$ to $-e^{2\pi i t}$, i.e. $t \mapsto t + \frac{1}{2}$. Since both w and ∂_t acquire a sign under the $\mathbb{Z}_2$ action, this cancels out the sign picked up by the trigonometric prefactors and R^2 is invariant.

It remains to understand the G_2 structure three form. This is constructed out of the Calabi-Yau invariants $\omega \in \Omega^{1,1}(X)$ and $\Omega \in \Omega^{(3,0)}(X)$ by

$$\varphi := dt \wedge \omega + \operatorname{Re}\Omega \,. \tag{4.6}$$

Note that, since σ is anti-holomorphic it acts by $\Omega \leftrightarrow \overline{\Omega}$ and thus leaves the real part of Ω invariant. Similarly, since σ is an isometry but acts with a sign on the complex structure, the Kahler form $\omega(-,-) = g(J-,-)$ is acted on by a sign. This is cancelled by the sign of dt, so φ is indeed an invariant.

We can then concretely compute that the natural guess,

$$R^3 = -\sin(2\pi t)w + \cos(2\pi t)\partial t \tag{4.7}$$

is indeed the third component of our triple of vectors.

As we have highlighted in the above, this is by no means unique. It would be interesting to understanding if it is unique up to homotopy, which means understanding the connected components of the space of sections $\Gamma_Y(\mathcal{V}_2)$.

4.4.2 A class of examples modelled on associative cycles

The example considered in this section is inspired by the relation between associatives and integrability of the tangent subbundle, $\mathcal{T}$. The motivating question is: if X is an associative cycle in a compact G_2 manifold, Y, can we find an ACM3S which is tangent to X? Recall from Subsection 3.5.4 that a G_2 manifold is a G_2 structure manifold such that the Levi-Civita connection has holonomy contained in G_2. It is not strictly necessary to restrict to this class of G_2 structures, but doing so offers one advantage in terms of the ease of semi-explicitly constructing an ACM3S. In this section, it will be argued that there may exist global obstructions to extending an ACM3S off an associative cycle. It would be interesting to understand how these obstructions relate to topological features on the manifold Y.

The argument is essentially as follows: first, we show that we can extend a trivialisation of the associative submanifold's tangent bundle to an ACM3S on a small neighbourhood of the submanifold. This is achieved by direct construction. Next, we imagine that we fix an ACM3S on the boundary of a small neighbourhood as well as a trivialisation of the associative submanifold's tangent bundle. We ask if there will always exist an ACM3S that exists on the full open neighbourhood, that restricts to the given structures. We find that this is not true by studying the disconnected components of the space of ACM3S that we described above in Section 4.2.

As a preliminary step, recall that a compact, smooth submanifold of a Riemannian manifold, $X \subset Y$ admits a tubular neighborhood $T(X) \subset Y$, which is open in Y and diffeomorphic to the normal bundle of X in Y, via the exponential map, see for instance [122]. Of course, a tubular neighbourhood is not unique, we can always shrink it, but for our purposes we will simply fix a choice. We will regard the tubular neighbourhood of X as a fibre bundle with finite radius disk fibres and express a point in the tubular neighbourhood with a normal vector via the exponential map. That is, a point $y \in T(X) \subset Y$ is uniquely expressible as $\exp_x(n_x)$ for some $x \in X$ and $n_x \in (T_xX)^{\perp} \subset T_xY$.

Now we will show that we can extend a framing of the associative submanifold, X to a tubular neighbourhood in Y.

Let (R^1_X, R^2_X) an orthonormal two-framing of X; by construction, the vector field $R^1_X \times R^2_X =: R^3_X$ is orthonormal to this two-frame and tangent to X, so this triple defines a trivialisation of X, which will be called an associative framing. This three-frame needs to be extended over the disk bundle, whilst preserving that $R^3 = R^1 \times_\varphi R^2$. Since the value of the three form at a given point in the disk bundle is given by parallel transporting along a geodesic from the base of the fibre (this is the only part where we use that Y is a G_2 manifold), we can simply parallel transport the three frame. That is, if P denotes parallel transport, then

$$R^i_{(x,n_x)} := \mathsf{P}_{\exp_x(n_x)} R^i_{X,x} \,. \tag{4.8}$$

As an aside: this will *not* preserve closure of the Lie bracket, generically, which can be checked by direct computation (see Eqn 5.39 of Paper II, for instance).

We have thus shown that an associative trivialisation of X extends to an ACM3S on a neighbourhood of X in Y. If we now choose an ACM3S on the rest of Y, we can ask if it restricts to the one just constructed at the boundary. We will view this as a trial-and-error problem, i.e. imagining that an arbitrary ACM3S is chosen on $Y \setminus T(X)$, an arbitrary ACM3S is chosen on $T(X)$ extending a framing on X, and ask if they glue together smoothly at the mutual boundary. If not, can we smoothly perturb them so that they *do* patch together? In other words, are their restrictions to the common boundary homotopic in the space of ACM3S?

Unfortunately, we do not have such good control of the space of ACM3S on $Y \setminus T(X)$ and not know which boundary values can be so obtained. We therefore simply ask if there are *any* boundary conditions on $T(X)$ that are not connected to a trivialisation on X and, indeed, even in the simplest case, there are.

In this setup, the tubular neighbourhood, regarded as a disk bundle, is trivial, $T(X) \cong D^4 \times X$. One can use this to see that the tangent bundle of the tubular neighbourhood is also trivialisable and we will assume that a trivialisation is chosen once and for all. Everything we do will be relative to this trivialisation.

We can now regard an ACM3S on $T(X)$ as a section of a trivial fibre bundle, $G_2/SU(2) \times T(X) \to T(X)$, and the space of these sections is given by $\mathrm{Maps}(T(X), G_2/SU(2))$. The space of sections that have the right behaviour at the boundary is just the space of maps relative to a fixed map at the boundary and over X.

Using the expression of the tubular neighbourhood as $X \times D^4$, we can choose polar coordinates on the disk to write $X \times S^3 \times [0,1]$. Our boundary conditions are encoded in

$$\mathbf{R}_\partial : X \times S^3 \times \{1\} \to G_2/SU(2) \tag{4.9}$$

$$\mathbf{R}_X : X \times S^3 \times \{0\} \to G_2/SU(2)\,. \tag{4.10}$$

Thus, our question is neatly wrapped up in determining the connected components of $\mathrm{Maps}(X \times S^3, G_2/SU(2))$.

That there are, in fact, disconnected components for $X = S^3$ is determined using elementary techniques from algebraic topology [129, 130], combined with the known structure of the homotopy groups of G_2 and $SU(2) \cong S^3$ [131–133].

4.5 Summary

In this chapter, we have reviewed some of the new results that were obtained in Paper II. Some of the results of that paper that were not presented here include a study of how heterotic supergravity interacts with a further refinement to an almost contact metric structure. In doing so, one recovers equations in terms of $SU(3)$ structures and so, in a certain sense, interpretable in terms of the more familiar four dimensional systems. We also used this perspective to find that a specific model of G_2 heterotic system, [65], possessed an extra supersymmetry.

It would be interesting to also explore such aspects of the ACM3S, especially with respect to their interesting relationship to associative and coassociative cycles. We reviewed here some of the results concerning the space of such 3-structures, but a good understanding of how its topological and geometrical features translate into physics is still missing. This would be something interesting to explore further.

5. Superpotential

In this chapter we will summarise the results of Paper I. This paper grew out of attempts to understand the finite moduli of heterotic systems, i.e. the seven dimensional configurations that preserve a single supersymmetry on a three dimensional, maximally symmetric spacetime. The background in Chapter 2 emphasised that this moduli space corresponds to the moduli of a supersymmetric theory, specifically in three dimensions. This means that the first order moduli, i.e. the tangent space, parametrized the effective field content and that higher order obstructions appear as couplings in the effective theory.

Since our spacetime is four dimensional, not three, a reader may question the physical relevance of such a construction. The interest in such models comes from particular supersymmetry breaking structures in four dimensions: domain walls (see [54, 66] and references for a discussion in the heterotic setting). A domain wall in the four dimensional space breaks translation invariance in the directions parallel to the wall, so must also break supersymmetry. By including the direction transverse to the domain wall in the internal manifold, one obtains a ten dimensional space of the form $M_3 \times Y$, where Y is a non-compact seven manifold. Assuming that the effective theory far from the domain wall is obtained by compactifying on an $SU(3)$ structure manifold and asking that the four dimensional $N = 1/2$ supersymmetry is preserved, one must conclude that Y is a noncompact G_2 structure manifold with two (possibly different) $SU(3)$ structure manifolds at its boundary. In this way, aspects of four dimensional theories get mapped to the moduli of, albeit non-compact, G_2 structures. On the other hand, we can imagine stretching some specific compact G_2 structure manifold into a domain wall solution so that the four dimensional effective theory is presumably realised at infinite distance in the G_2 structure moduli space. It is therefore interesting to ask about how physics at this boundary relates to the genuinely three dimensional physics. When looking at the dimensionally reduced theories we are unable to connect these regions of moduli space directly, because we must leave the domain of validity of our approximations, but if we can achieve a full understanding of a perturbative neighbourhood in each region then comparisons could perhaps be made.

The first order deformations of this moduli space were studied in [78, 79]. In both papers, it was found that the space of first order deformations was finite dimensional (when the G_2 structure manifold is compact), using ellipticity of certain operators.

In Paper I, a different approach was taken. There, a functional was extracted from the ten dimensional action (2.2), whose critical locus is exactly the space of solutions to the BPS equations. In the $SU(3)$ case, where the effective theory is four dimensional, a superpotential was similarly constructed and used to show that the all-order expansion truncates at order three in a given choice of coordinates on the configuration space, [76]. The choice of "good" coordinates was motivated by supersymmetry considerations, in particular the famous holomorphy of 4d $N = 1$ theories. This is not a property that is shared by the three dimensional theories that arise from G_2 heterotic systems. Nevertheless, the functional that we construct captures, in principle, the full perturbation expansion and therefore the full perturbative effective action, to first order in α', in the large volume, weak coupling region of the stringy moduli space.

In this chapter, we will very briefly review the strategy taken in deriving the superpotential and the computations that prove its locus reproduces the Killing spinor equations. More details can be found in Paper I.

5.1 The background fields

We will be looking for compact seven manifolds equipped with fields whose background values solve the Killing spinor equations (2.12)-(2.14) as well as the Bianchi identities. In particular, the geometric backgrounds that we are interested in will be comprised of:

$$(Y, \varphi, (V, A), (TY, \Theta), H) \tag{5.1}$$

where:

- Y is a seven manifold;
- φ is a positive three form on Y, which is taken to define a G_2 structure;
- A is a connection on the gauge bundle, V; the curvature will be denoted F;
- Θ is a metric connection on the tangent bundle, TY; the curvature will be denoted $\tilde{R}$;
- H is a three form that satisfies the Bianchi identity:

$$dH = \frac{\alpha'}{4}(\mathrm{tr}\tilde{R}^2 - \mathrm{tr}F^2)\,. \tag{5.2}$$

Ignoring the H field for the moment, the data consists of an arbitrary G_2 structure, an arbitrary gauge bundle and an arbitrary metric connection on the bundle TY. The H field restricts the possible isomorphic class of gauge bundle as well as instating a relation between Θ and A. The Killing spinor equations can then be regarded as defining a subspace of this background data.

We saw in Example 3.6.6 that a spinor defines an integrable G_2 structure if it satisfies the gravitino's Killing spinor equation and the torsion of the G_2 structure is identified with the three form, H.

In terms of the three form, φ and its dual four form, ψ, the Killing spinor equations can be re-expressed as [48, 57, 61, 66]:

$$d(e^{-2\phi}\varphi) = - * H + \tfrac{7}{2}h\psi \tag{5.3}$$
$$d(e^{-2\phi}\psi) = 0 \tag{5.4}$$
$$2H \wedge \psi = h\varphi \wedge \psi \tag{5.5}$$
$$F \wedge \psi = 0 = \tilde{R} \wedge \psi\,. \tag{5.6}$$

These are the form of the supersymmetry-preserving equations that we will require the superpotential to reproduce.

5.2 Deriving the functional

Our aim in Paper I was to find a functional - the superpotential - whose locus is explicitly the configurations that have unbroken $N = 1$ supersymmetry in three dimensions. This was achieved using the explicit ten dimensional action (2.2). Since this action is already an approximation, neglecting terms of order $(\alpha')^2$ and above, the superpotential that is obtained is also only approximate. Specifically, our results only hold true in the large volume, weak coupling limit.

Motivated by [47, 56], the proposed functional will come from the mass term of the three dimensional gravitino. This is sensible, because the space of vacua we are interested in is precisely the locus of unbroken supersymmetry and the gravitino is massless if and only if the corresponding supersymmetry is unbroken.

It is also efficient, because it is relatively simple to read off the terms that will contribute to the mass term, after which it is simply a matter of being careful with conventions and numerical factors in the explicit dimensional reduction.

A three dimensional gravitino has action functional (κ_3^2 is the three dimensional Newton's constant):

$$S = \frac{1}{2\kappa_3^2}\int \mathrm{dvol}\left(\bar{\psi}_\mu \Gamma^{\mu\nu\kappa} D_\nu \psi_\kappa + m\bar{\psi}_\mu \Gamma^{\mu\kappa}\psi_\kappa\right) \tag{5.7}$$

so the only relevant terms from the 10d action (2.2) are of the form

$$S_{rel} = -\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}e^{-2\phi}(\bar{\Psi}_M \Gamma^{MNP} D_N \Psi_P + \tag{5.8}$$
$$-\frac{1}{24}\left(\bar{\Psi}_M \Gamma^{MNPQR}\Psi_R + 6\overline{\psi}^N \Gamma^P \Psi^Q\right) H_{NPQ}\,. \tag{5.9}$$

The single Γ-matrix term is kept because it is dual to a two Γ-matrix term in three dimensions, see [14] for instance. Therefore, the presence of this term is something unique to three dimensions.

A related peculiarity that three dimensions offers is the possibility of a non-trivial three form flux on the maximally symmetric space, which, as mentioned above, sources a constant negative scalar curvature on the effective spacetime. This means that the Killing spinors are not annihilated by the spin connection, but are instead eigenspinors of the spin connection with eigenvalue proportional to the square of the curvature. One can define a shifted spin connection by absorbing this constant term so that the Killing spinors are indeed annihilated by this operator. Thus, the correct meaning of "massless" in this context is the zero modes of this *shifted* connection, not the naive spin connection. This is discussed in [39], for instance, and we apply this reasoning in Paper I.

To be a little more precise, the scalar curvature of the AdS_3 space in a background with three form $H^{(3)} := \frac{1}{3!} H_{\mu\nu\kappa} dx^\mu \wedge dx^\nu \wedge dx^\kappa$ is given by, [134]:

$$\Lambda = -\frac{1}{2}(*_3 H^{(3)}) \,, \tag{5.10}$$

where $*_3$ is the Hodge star of the three dimensional metric. Define the operator:

$$\mathcal{D}_\mu = \nabla^{LC}_\mu + \frac{1}{2}\Lambda\Gamma_\mu \,, \tag{5.11}$$

then by explicit computation, the Killing spinors are annihilated by $\mathcal{D}_\mu$. If one does not take into account this shift, than the gravitational mass term would be shifted relative to the result that we will find, and this causes inconsistencies.[1]

Observe that the operator $\mathcal{D}_\mu$ can be directly obtained from the ten dimensional gravitino Killing spinor equation, (2.5), with index $M \in \{0, 1, 2\}$. Using that a single Γ matrix is dual to an antisymmetrised pair in two dimensions (see e.g. App. B of [14]), we deduce the relationship between Λ and $H_{\mu\nu\kappa}$ shown in (5.10). This duality between single Γ matrices and antisymmetrised pairs is important in the interpretation of $\mathcal{D}_\mu$ as a shifted spin connection. In dimensions other than three, it would not be true. This is because, *a priori*, a term like Γ_μ is *not* in the algebra $\mathfrak{spin}(1,2)$, so $\mathcal{D} - \nabla^{LC}$ is not, naively, a one form valued in $\mathfrak{spin}(1,2)$ and we would erroneously conclude that $\mathcal{D}$ is not a spin connection. However, since $\mathfrak{spin}(1,2)$ is identifiable with the subspace of antisymmetrized products of two Γ matrices (see [135] for instance), this conclusion is incorrect in dimension three. In other dimensions, the correct interpretation is that the operator $\mathcal{D}$ comes from a connection on the metric cone, where the spinor is in fact parallel, [136–138].

[1] Since it is the superpotential's critical locus that is relevant, this may be surprising. The reason is that the dilaton's equations of motion will always set the functional to zero on the critical locus and this fact is also true for the "shifted" superpotential, but the quantities set to zero are manifestly different. A component of the geometric variations are comprised of the same terms as the dilaton's, but with different proportionalities. By demanding that these are consistent one will get the dilatino BPS equation (2.6) in our case, but not in the unshifted case.

Explicitly reducing the action on a compact manifold Y leads to:

$$W = \int_Y e^{-2\phi}\left((H + h\varphi) \wedge \psi - \frac{1}{2} d\varphi \wedge \varphi\right) \tag{5.12}$$

where:

- H is the three form flux restricted to the internal manifold,

$$H = \frac{1}{3!} H_{ijk} dy^i \wedge dy^j \wedge dy^k\,. \tag{5.13}$$

  Recall that H is first order in α', due to the Chern-Simons-like terms.
- $\phi = \phi(y)$ is the dilaton;
- φ is the three form that defines a G_2 structure, related to the spinor that solves the Killing spinor equations, η, by:

$$\varphi_{ijk} = -i\eta^T \Gamma_{ijk} \eta\,; \tag{5.14}$$

- ψ is the Hodge dual of φ, i.e. $\psi = *\varphi$;
- h is a convenient scaling of the AdS_3 curvature, defined by:

$$h = -\tfrac{2}{7} *_3 H^{(3)}\,. \tag{5.15}$$

5.3 Verifying the critical locus

Having obtained a proposal for the superpotential, it is important to verify that its critical locus reproduces the expected moduli space. The means of computing the critical locus is logically the same as deriving the equations of motion of an ordinary action functional. The full presentation is in Paper I using the efficient notation of [78]. The techniques involved are standard in both mathematics and physics, so we will not linger too long over it here. Before beginning, however, it is best to be sure that one knows the tangent directions of the parameter space, so we begin by reviewing this. This is slightly nontrivial in our current setting because of the B-field and its nontrivial gauge dependency deriving from the Green-Schwarz mechanism, as well as nonlinearities inherent in the description of G_2 structures.

Recall that anomaly cancellation leads to the Bianchi identity:

$$dH = \tfrac{\alpha'}{4}(\mathrm{tr}\tilde{R}^2 - \mathrm{tr}F^2)\,, \tag{5.16}$$

where the equality is one of differential forms. It implies an equality in cohomology of certain characteristic classes, but is much more rigid. The three form, H, is not a fundamental field of the supergravity theory, but is instead analogous to the field strength of a gauge field. The variations that we consider are over the local two form, B. Since B is itself not globally defined, one might expect these variations to also be over some space of locally defined two forms. Fortunately, this is not true. Using the inherent gauge freedom of the B field, one can choose a representative of a deformation of the B field that is a globally defined two form, see [78, 139, 140] for instance.

We can therefore identify the relevant independent field variations that we must compute:

Gauge fields A variation of the gauge field is a globally defined one form valued in the associated adjoint bundle $\alpha \in \Omega^1(Y, \mathrm{End}(V))$. The same can be said for the tangent bundle connection that appears in ω_L;

B **field** As discussed above, the B field variations are globally defined two forms, $\mathcal{B} \in \Omega^2(Y)$. The B field also depends on the gauge field variations, in a combination that cancels out the exact terms in $\delta_A \omega_{CS}(A)$ in H (and similar in terms of Θ);

Dilaton The dilaton is a smooth, $\mathbb{R}$-valued function, so it can be deformed by any smooth, $\mathbb{R}$-valued function;

G_2 **three form** A variation of the three form that defines the G_2 structure must be such that the positivity condition remains satisfied. However, this is an open condition, [80, 84], so any infinitesimal three form variation preserves the G_2 condition.

For convenience we state the general expression for a variation of the three form:

$$\delta_{\alpha,\zeta,\mathcal{B}} H = d\mathcal{B} + \tfrac{\alpha'}{4}\big(\mathrm{tr}(F \wedge \alpha) - \mathrm{tr}(\tilde{R} \wedge \zeta)\big) \tag{5.17}$$

where α is a variation of the gauge field, ζ a variation of the tangent bundle gauge field, and $\mathcal{B}$ a globally defined two form.

This covers all the fundamental fields, but we will also need to know how the four form, $\psi = *\varphi$ varies with φ. This is subtle because the metric, and thus Hodge star, also depends on φ. Fortunately, at first order this is simplified, [77, 78]. In these articles, a variation is written in terms of a tangent bundle valued one form, $M \in \Omega^1(Y, TY)$, by contracting the indices:

$$\delta\varphi = \iota_M \varphi = M^a \wedge \varphi_a = \frac{1}{2!} M_i{}^a \varphi_{ajk} dy^i \wedge dy^j \wedge dy^k\,. \tag{5.18}$$

At first order the variation in the four form can also be written using M:

$$\delta\psi = \iota_M \psi = \frac{1}{3!} M_i{}^a \psi_{ajkl} dy^i \wedge dy^j \wedge dy^k \wedge dy^l\,. \tag{5.19}$$

With these ingredients in hand, one can perform the variation of the superpotential. The aim is to make contact with the set of equations (5.3)-(5.6).

We will start with the easy variations, which are those corresponding to the dilaton, B-field and gauge connection. Since the appearance of A and Θ in the functional is completely symmetric, we will only explicitly deal with A.

Dilaton variation
The dilaton is an overall factor, so demanding that its variations vanish implies:

$$(H + h\varphi) \wedge \psi - \frac{1}{2} d\varphi \wedge \varphi = 0\,. \tag{5.20}$$

B-field variation
By the above argument, the pure B field variation is a global two form, and therefore:

$$\delta_{\mathcal{B}} W = \int_Y e^{-2\phi} d\mathcal{B} \wedge \psi \stackrel{!}{=} 0\,. \tag{5.21}$$

Integration by parts imposes that $d(e^{-2\phi}\psi) = 0$, in agreement with (5.4).

Gauge variation
The behaviour of the gauge connection and the connection on the tangent bundle is identical with respect to this computation, so to avoid redundancy we only present the gauge computation. The gauge variations only appear in the H term, (5.17), so it follows that

$$\delta_A W = \int_Y \mathrm{tr}(F \wedge \delta A) \wedge \psi\,. \tag{5.22}$$

We immediately conclude that $F \wedge \psi = 0$, in agreement with (5.6). Recalling the principle that anything that we say for A goes through for ζ, we do recover both equations in (5.6).

Geometric variations
Now we come to the more involved calculation. Let us imagine, hypothetically, that the variation commutes with the Hodge star. We could then write an arbitrary geometric variation of the superpotential as

$$\delta W = \int_Y \big(e^{-2\phi}(*H + h\psi - d\varphi + d\phi \wedge \varphi)\big) \wedge \delta\varphi) \stackrel{!}{=} 0 \tag{5.23}$$

and, since $\delta\varphi$ is an arbitrary three form, we would deduce that the factor in big brackets must vanish, and be done.

Unfortunately, however, the geometric variation, which includes metric variations, does not commute with the star and we must take a longer path. We will split the variation into fibrewise G_2 irreducible representations. By orthogonality, a variation in some representation will pick out the corresponding piece of the equation of motion. We will then glue these back into the general equation.

We continue to utilise the language of [77, 78] to parametrise the variations in the three- and four- form, i.e. utilising a vector-valued one form.

Let $M \in \Omega^1(Y, TY)$ be such a form and consider the variation

$$\delta\varphi = \iota_M \varphi\,. \tag{5.24}$$

As noted above, this fixes the first order variation of ψ to be:

$$\delta\psi = \iota_M \psi\,. \tag{5.25}$$

We use irreducible G_2 representations to analyse this further. To that end, let us now recall some representation theory; whenever necessary we use the package LieART, [106], though one can also consult [84], for instance, to check the relevant decompositions. In the range of dimensions that we are interested in, an irreducible G_2 representation is uniquely specified by its dimension[2]. By construction of the G_2 structure, the tangent space of Y has the induced 7 dimensional G_2 representation, so a fibre of $T_pY \otimes T_p^*Y$ is in the $\mathbf{7}\otimes\mathbf{7}$, which decomposes as $\mathbf{1}\oplus\mathbf{7}\oplus\mathbf{14}\oplus\mathbf{27}$. The wedge product $\Lambda^3 T_p^*Y$ has the direct sum decomposition given by $\mathbf{1}\oplus\mathbf{7}\oplus\mathbf{27}$. Consequentially, the relevant variations will not depend on the $\mathbf{14}$ part of M and we may as well take it to be zero for the purposes of this computation.

We will think of M as a matrix by lowering an index with the G_2-invariant metric, and from there decompose it into irreducible $SO(7)$-representations, consisting of the trace, the symmetric-traceless and the antisymmetric pieces. Since $G_2 \subset SO(7)$ these are not-necessarily-irreducible representations of G_2 and we can try to decompose further. In fact, the trace and symmetric representations are also irreducible represents of G_2 and correspond to $\mathbf{1}\oplus\mathbf{27}$. The antisymmetric product $\Lambda^2\mathbf{7}$ is not irreducible, but since we assume that M has no $\mathbf{14}$, only the $\mathbf{7}$ component of $\Lambda^2\mathbf{7}$ will appear. In particular, we can write $M = \frac{1}{7}(\mathrm{tr}M)\mathrm{Id} + m + h$, m is the antisymmetric $\mathbf{7}$ part and h is the traceless, symmetric $\mathbf{27}$. Note that the symmetric representation parametrises variations of the G_2 structure that change the metric, while the $\mathbf{7}$ part leaves the metric invariant. This is checked explicitly in [77].

We these conventions we can explicitly compute the variation [78]:

$$\delta\varphi = \iota_M\varphi = \tfrac{3}{7}\mathrm{tr}M\varphi - (m\lrcorner\varphi)\lrcorner\psi + \iota_h\varphi \tag{5.26}$$

$$\delta\psi = \iota_M\psi = \tfrac{4}{7}\mathrm{tr}M\psi + (m\lrcorner\varphi)\wedge\varphi + \iota_h\psi\,. \tag{5.27}$$

We can immediately see (by looking at the singlet part, for instance) that there is a difference: $*\delta\varphi \neq \delta\psi$. The symbol $\lrcorner$ denotes a contraction of differential forms, defined by dualising with a metric. More explicitly, on a seven manifold[3] with arbitrary $\alpha \in \Omega^k(Y)$ and $\beta \in \Omega^{p+k}(Y)$

$$\alpha\lrcorner\beta = (-1)^{pk} * (\alpha \wedge *\beta) = \frac{1}{k!p!}\alpha^{i_1\cdots i_k}\beta_{i_1\cdots i_k j_1\cdots j_p} dy^{j_1}\wedge\cdots\wedge dy^{j_p}\,. \tag{5.28}$$

[2]This stops being true when the dimensions are large enough, for instance there are two 77 dimensional representations, [84].

[3]The dimension is only relevant for the middle expression; on a general d dimensional manifold the correct sign factor is $(-1)^{p(d-p-k)}$.

We will now use the expressions (5.26) and (5.27) to compute the relevant pieces of the superpotential variation.

Let us start with the singlet, which immediately gives

$$\delta_{\mathbf{1}}W = \int_Y \mathrm{tr}M\left(e^{-2\phi}\big(\frac{4}{7}H\wedge\psi + h\varphi\wedge\psi - \frac{3}{7}\varphi\wedge d\varphi\big)\right). \tag{5.29}$$

This is not independent from the equation obtained by varying the dilaton.

Indeed, we have

$$\frac{4}{7}H\wedge\psi + h\varphi\wedge\psi - \frac{3}{7}\varphi\wedge d\varphi \overset{!}{=} 0 \overset{!}{=} (H+h\varphi)\wedge\psi - \frac{1}{2}d\varphi\wedge\varphi\,. \tag{5.30}$$

Consistency requires that:

$$H\wedge\psi = \frac{h}{2}\varphi\wedge\psi\,, \tag{5.31}$$

in agreement with the expected equation, (5.5). We can use this to rearrange:

$$d\varphi\wedge\varphi = 2(H+h\varphi)\wedge\psi \tag{5.32}$$

$$\iff d\varphi\wedge\varphi = 3h\varphi\wedge\psi \tag{5.33}$$

$$\iff d\varphi\wedge\varphi = \frac{7}{2}h\varphi\wedge\psi - *H\wedge\varphi \tag{5.34}$$

$$\iff (d\varphi)_{\mathbf{1}} = \frac{7}{2}(h\psi)_{\mathbf{1}} - (*H)_{\mathbf{1}}\,. \tag{5.35}$$

For the **7** part, we *do* have that $\delta\psi_7 = *\delta\varphi_7$ and can use this to conclude:

$$\int_Y e^{-2\phi}(*H - d\varphi + d\phi\wedge\varphi)\wedge\delta\varphi_7 \tag{5.36}$$

leading to the equation $(*H+d\phi\wedge\varphi)_{\mathbf{7}} = (d\varphi)_{\mathbf{7}}$. Now, the exterior differential of the G_2 structure constants, (φ,ψ) is related to the torsion and their **7** parts are related (see Subsection 3.5.5)

$$(d\psi)_{\mathbf{7}} = 4\tau_1\wedge\psi \tag{5.37}$$

$$(d\varphi)_{\mathbf{7}} = 3\tau_1\wedge\varphi\,. \tag{5.38}$$

Since we have already found that $(d\psi)_{\mathbf{7}} = 2d\phi\wedge\psi$, we must have that $d\phi = 2\tau_1$. On the other hand, we just found that $(d\varphi)_{\mathbf{7}} = d\phi\wedge\varphi + *H_{\mathbf{7}}$. Writing $H_{\mathbf{7}} = \lambda\wedge\varphi$, for a one form λ, we must have that $3(d\phi+\lambda) = 2d\phi$, or after rearranging $d\phi\wedge\varphi = 2*H_{\mathbf{7}}$. We can therefore conveniently rewrite the equation of motion as

$$(d\varphi)_{\mathbf{7}} = 2d\phi\wedge\varphi - *H_{\mathbf{7}}\,. \tag{5.39}$$

Since the **27** variation affects the metric, like that of the singlet, it is not true that $*\delta\varphi_{\mathbf{27}} = \delta\psi_{\mathbf{27}}$. However, it can be directly computed that

$$H\wedge\iota_h\psi = -*H\wedge\iota_h\varphi\,.$$

We can therefore put all our variations onto φ and conclude that

$$\int_Y e^{-2\phi}(-d\varphi - *H) \wedge \iota_h\varphi \stackrel{!}{=} 0\,. \tag{5.40}$$

where we used that that $(d\phi \wedge \varphi)_{\mathbf{27}} = 0$.

This gives us that $(d\varphi)_{\mathbf{27}} = - * H_{\mathbf{27}}$ and comparing with the **7** part, we see that the our earlier manipulations conveniently fit together. Gathering the work, we have

$$(d\varphi)_{\mathbf{1}} = \tfrac{7}{2}(h\psi)_{\mathbf{1}} - (*H)_{\mathbf{1}} \tag{5.41}$$

$$(d\varphi)_{\mathbf{7}} = 2d\phi \wedge \varphi - *H_{\mathbf{7}} \tag{5.42}$$

$$(d\varphi)_{\mathbf{27}} = - * H_{\mathbf{27}} \tag{5.43}$$

$$\iff d\varphi = 2d\phi \wedge \varphi - *H + \tfrac{7}{2}h\psi \tag{5.44}$$

recovering the one remaining equation (5.3).

5.4 Summary

In this chapter we reviewed the core arguments from Paper I, in particular that there is a functional, naturally obtained by dimensional reduction, whose critical locus reproduces the Killing spinor equations in the form of (5.3)-(5.6). In principle, this ought to recover the full perturbative expansion around a supersymmetric vacuum. It should be noted, however, that the vacuum only preserves two real supercharges and therefore we do not have the nonrenormalisation theorems that are familiar from $N = 1$, $d = 4$. This means that the results are only trustworthy at first order in α' and in the large volume, weak coupling regime.

In future work, it would be interesting to utilise the superpotential to express the finite deformations of the heterotic G_2 systems, analogous to the superpotential's usage in the $SU(3)$ case [76]. The lack of supersymmetry of G_2 heterotic systems in comparison to $SU(3)$ heterotic systems is problematic. In the $SU(3)$ case, there is enough supersymmetry to guarantee holomorphy and this was utilised in [76] to find convenient coordinates for the expansion. In the G_2 case, we do not have holomorphy and overcoming this lack is challenging.

6. Yukawa couplings

The aim of this chapter is to motivate and summarise Paper III. We begin with physics motivation and then review the necessary mathematical background. We then state and give a streamlined proof of the main theorems that were obtained. Paper III was concerned with understanding the behaviour of certain couplings - the Yukawa couplings - in four dimensional effective theories obtained from dimensionally reducing heterotic supergravity on a compact Calabi-Yau manifold (see subsections 2.4.1 and 3.5.3 for a review of these structures in compactification scenarios). In Calabi-Yau compactifications the internal flux associated to the B field must vanish, $H_{ijk} = 0$, and the gauge bundle must be holomorphic and polystable, [16, 20].[1]

6.1 Motivation

In particle physics, Yukawa couplings describe an interaction between a scalar and two fermions, [141]. Such a coupling appears in the Standard model to give mass to the leptons and quarks via the Higgs mechanism, [142]; elementary discussions can be found in Srednicki, [143] or Weinberg [144], for instance. The strength of the couplings have an interesting feature, a mass hierarchy: a discrepancy in mass scale between the generations, as discussed in [145]. Since, in the Standard Model, the Yukawa couplings are free parameters that must be fitted to experiment, [146], there is no way to understand this behaviour purely within the standard model context and, although it may be conceptually unsettling, it can be accepted as a fact of life. Within string model building, this is no longer true and one can hope that the model-building process itself will explain the hierarchical structure.

Work in computing Yukawa couplings has been on-going since almost the very beginning of compactification models [147–157]. There *are* interesting textures in the couplings of models coming from compactifying the heterotic string [155, 156, 158–160]: in many cases couplings that are allowed *a priori* are vanishing, at least perturbatively. Such zeros in the couplings may lead to the observed hierarchy after supersymmetry breaking, for instance. In some cases, particularly [155, 161], this has its origin in a symmetry that is present in a particular phase of the theory, while in other cases, e.g. [158, 159], the four

[1] For readers unfamiliar with polystability of a vector bundle, it will suffice to know that it admits a connection that solves the gaugino Killing spinor equation (2.14), by [69, 70]

dimensional explanation is lacking. There appears to be two alternatives, then. Either, there is an hitherto overlooked symmetry in such models, or the vanishing couplings can only be explained from a higher dimensional perspective. Given that the observed couplings exhibit such textures, both possibilities are interesting and it would be desirable to decide between them.

In Paper III, we add to the list of constraints on the Yukawa couplings that have an explicit origin in the compactification construction. Our results are very similar to, and inspired by, [158, 159] but have a much wider domain of applicability. In principle, our results should extend to any Calabi-Yau that is constructed as an embedded submanifold of an ambient space and with any suitable gauge bundle. At present, the results are stated for a holomorphic $SU(3)$ bundle to avoid complicated theorem statements, but there is no reason that they could not be extended. As such, these theorems can be useful as a testing ground for understanding genericness of textures in the couplings and looking for four dimensional explanations.

The theorems that we derive apply to the so-called holomorphic Yukawa couplings, which correspond to cubic terms in the superpotential and depend holomorphically on the moduli space [150, 155]. These are not the physical Yukawa couplings, [157], since they are defined without properly normalising the fields. Nevertheless, if a given coupling vanishes, than the normalisation is clearly irrelevant and the zeros of this coupling are physically relevant.

The physical Yukawa couplings are generally hard to get a hold of, as the Kähler potential is not explicitly known, [158], except for the case of the standard embedding [157]. As a consequence, many (but not all, see [145]) of the observed textures in Yukawa couplings, including [153, 158–160], utilise the holomorphic coupling.

6.2 The holomorphic Yukawa couplings

The quantity that we wish to constrain is the holomorphic Yukawa coupling, so we will begin by giving an explicit definition. These couplings correspond to cubic terms in the superpotential of the form $\phi\psi\psi$ with the fields ϕ and ψ being the Bose, respectively Fermi, components of a chiral field [16]. The coupling can be deduced from the ten dimensional theory, by first identifying the ten dimensional supermultiplet which descends to give a chiral supermultiplet in the effective theory, and then looking for an appropriate term in the ten dimensional action. This is the approach carried out in [157] and followed in the text [16], which we now review.

In particular, the origin for the fermionic ψ is in the gaugino, χ, and ϕ comes from the gauge field, A (see Section 2.2 where the ten dimensional supermultiplets are reviewed). The single multiplet (A, χ) induces a family of chiral multiplets in the effective theory, parametrised by the moduli. More precisely, the scalar will be valued in a representation of the commutant of the structure group of the holomorphic bundle, V. Focusing on one factor of the $E_8 \times E_8$ heterotic group, we let the structure group of V be G and $H \subset E_8$ its commutant. This induces an embedding $H \times G \subset E_8$. The commutant group is the gauge group of the effective theory in four dimensions. The ten dimensional gauge multiplet is valued in the adjoint of E_8 and the embedding $H \times G \subset E_8$ induces a decomposition of this representation into irreducible representations of $H \times G$, i.e.

$$\mathfrak{e}_8 = \bigoplus_i R_i \otimes T_i \,. \tag{6.1}$$

Just as the branching rules of the tangent group led to fields of different spin (cf. equation (2.10) and (2.11)), the gauge group branching rules induce multiplets valued in different representations. The scalar that we are interested in, $\phi(x)$ is, then, R_i valued for some R_i, and comes from $A(x, y) \to \phi(x)^{R_i} A^{T_i}(y)$. Its multiplicity comes from the zero modes in A^{T_i}. Of course, supersymmetry ensures similar statements can be made for the fermionic components.

In principle, we can have couplings like $\phi^{R_1}\psi^{R_2}\psi^{R_3}$ so one has a representation theoretic problem of determining when this can be a scalar, i.e. when $R_1 \otimes R_2 \otimes R_3 = \mathbf{1} \oplus \cdots$. Similarly, we can have both left- and right-handed chiral multiplets in the four dimensional theory, which the Yukawa couplings can, in principle, mix. Determining if this occurs is again a question in representation theory, now with respect to the tangent group. The results when the gauge bundle has structure group $SU(3)$ can be found in [157] or any of the standard texts, [14–16]. For more general structure groups see [154], for instance.

The theorems of Paper III are derived in the context of an $SU(3)$ bundle, so we will focus on that case here. The commutant of $SU(3)$ in E_8 is E_6. The adjoint of E_8 decomposes as:

$$\mathfrak{e}_8 \cong \mathfrak{su}(3) \otimes \mathbf{1} \oplus \mathbf{1} \otimes \mathfrak{e}_6 \oplus \mathbf{3} \otimes \mathbf{27} \oplus \bar{\mathbf{3}} \otimes \overline{\mathbf{27}} \tag{6.2}$$

where the non-adjoint representations are denoted by their dimension.

This representation theoretic decomposition can be thought of as a fibrewise decomposition, which is promoted to a global decomposition using the background gauge bundle, V. In particular, if V is an $SU(3)$ bundle, then a field valued in the **3** of $SU(3)$ is precisely a section of this bundle; a field valued in the adjoint is valued in the endomorphism bundle of V. The trivial representation corresponds to an ordinary scalar without gauge indices.

The interesting couplings correspond to the components in the **3** or $\bar{\mathbf{3}}$ of $SU(3)$ and since the only singlets that can be formed with any combination of three of these two representations is to take them all the same and antisymmetrise, these will be the couplings of relevance to us.

The zero modes corresponding to the **3** are [20] harmonic form $(0,1)$ forms valued in V and to the $\bar{\mathbf{3}}$ are $(0,1)$ forms valued in V^*.

Since the structure of the antigenerations valued in V^* is completely analogous to that of the V-valued forms, we will only explicitly make statements concerning the latter.

For any triple of such V-valued forms, ν_1, ν_2, ν_3 the coupling is proportional to [16, 150, 157]

$$\lambda(\nu_1, \nu_2, \nu_3) = \int_Y \Omega \wedge \mathrm{tr}(\nu_1 \wedge \nu_2 \wedge \nu_3) \tag{6.3}$$

where Ω is the $(3, 0)$ form that exists and is fixed as part of the $SU(3)$ structure on the Calabi-Yau, Y. The symbol, tr denotes the fibrewise projection from $\mathbf{3}^{\otimes 3} \to \Lambda^3\mathbf{3} = \mathbf{1}$, inducing a bundle morphism $V^{\otimes 3} \to \underline{\mathbb{C}}$.

Unfortunately, the coupling λ is defined as a functional on the set of harmonic forms, which are difficult to obtain and depend on intricacies of the Calabi-Yau structure. Fortunately, none of this dependence appears in the coupling λ [157]. Indeed, harmonic forms are in bijective correspondence with the Dolbeault cohomology and since W vanishes on Dolbeault exact forms, we can replace the precise moduli space with the equivalent Dolbeault cohomology vector space. That is, $\lambda(\nu_1, \nu_2, \nu_3) = \lambda([\nu_1], [\nu_2], [\nu_3])$.

This is a vast improvement, because Dolbeault cohomology is relatively easy to access. However, the structure of the cubic superpotential depends on algebraic data of the cohomology that is much more refined than the simple structure of a vector space.

Our aim is to push the above reasoning further: if harmonic forms can be replaced with Dolbeault cohomology, then perhaps we can replace this group with a different cohomology theory where the algebraic structure is more accessible. Indeed, it is well known that Dolbeault cohomology computes sheaf cohomology, [83, 162] just as de Rham cohomology computes singular cohomology with real coefficients and sheaf cohomology of the real-valued constant sheaf. We will consider the Yukawa coupling as defined on sheaf cohomology classes, and therefore give a very brief review of this topic in the next section.

6.3 Sheaf cohomology and its product

In this section, we will review the definition of sheaf cohomology and its product using the methods of Godement [163] in the form presented in [154]. The presentation will be brief. Good introductions to sheaves and their cohomology can be found in [162–166], to name just a few. To aid brevity, we will appeal to some basic categorical constructions, which can be found in, for instance, [167].

This section is intended as theoretical background to justify that the structures that appear in the Yukawa coupling can be reliably computed in sheaf cohomology, as opposed to Dolbeault cohomology. This will be used in the next section, Section 6.4 where the main results of Paper III will be presented and justified. All discussion before Section 6.4, is review.

6.3.1 Sheaves

In this review subsection we will briefly introduce the notion of a sheaf.

Definition 11 (Set-valued sheaf)**.** Let X a topological space. Denote by C_X the category that has as objects the open sets of X and morphism sets:

$$C_X(U,V) = \begin{cases} \{*\} & \text{if } U \subset V \\ \emptyset & \text{else} \end{cases} . \tag{6.4}$$

A presheaf on X valued in sets, P_X, is a contravariant functor from C_X to the category of sets, $P : C_X^{op} \to \mathsf{Sets}$.

A map of presheaves is a natural transformation of functors.

In other words, a set-valued presheaf on X consists of a set $P(U)$ assigned to each open of X, along with a *restriction* map, $\rho_{UV} : P(V) \to P(U)$ for each inclusion $U \subset V$ and compatibility between the maps. The elements of $P(U)$ are called local sections.

A sheaf is a presheaf that possesses properties that make it possible to glue local data into global data.

Definition 12 (Set-valued sheaf)**.** Let X be a topological space. A set-valued presheaf on X, P, is a set-valued *sheaf* if for any open set, $V \subset X$, and any open cover of $V, \mathcal{U} = \{U_i\}_{i \in I}$, P satisfies:

Identity If $\sigma, \tau \in P(V)$ such that $\rho_{U_i V}\sigma = \rho_{U_i V}\tau$, then $\sigma = \tau$.

Gluability If $\{\sigma_i \in U_i\}_{i \in I}$ is a collection of local sections in each set of the open cover, and $\rho_{(U_i \cap U_j)U_i}\sigma_i = \rho_{(U_i \cap U_j)U_j}\sigma_j$ for each $i, j \in I$, then there exists a section, $\sigma \in V$ with $\rho_{U_i V}\sigma = \sigma_i$ for all $i \in I$.

A map of sheaves is the same as a map of the underlying presheaves, i.e. a natural transformation.

We will be interested in sheaves of commutative algebras and modules, rather than sets. Fortunately, these are just sets with extra structure, so the right notion of sheaf will just replace the target category. This means that each collection of local sections will give a commutative algebra, and the restriction maps are algebra morphisms.

Definition 13 (Module of a sheaf)**.** Let X be a topological space and $\mathcal{O}_X$ an algebra-valued sheaf. An $\mathcal{O}_X$-module is a sheaf, P, such that $P(U)$ is an $\mathcal{O}_X(U)$ module for each open $U \subset X$ and such that the restriction maps respect the module structure. That is, if $U \subset V$, $\sigma \in P(V)$ and $f \in \mathcal{O}_X(V)$, then $\rho^P_{UV}(f\sigma) = \rho^{\mathcal{O}_X}_{UV}(f)\rho^P_{UV}(\sigma) \in \mathcal{O}_X(U)$.

A map of $\mathcal{O}_X$-modules is a map of the underlying sheaves, whose components are module morphisms.

We will be interested in sheaves of algebras that describe the ring of functions of a space, particularly holomorphic functions or the structure sheaf of a scheme and, at least initially, the modules will correspond to sections of bundles.

Example 2. Let X be a smooth manifold. The *sheaf of smooth functions*, C^∞_X, assigns to each open set, U, a smooth, real valued function on U, i.e. $f \in C^\infty_X(U)$ is a smooth map $f : U \to \mathbb{R}$. To each inclusion $U \subset V$, the functor assigns the map $C^\infty_X(V) \to C^\infty_X(U)$ given by restricting the domain. This is a sheaf of $\mathbb{R}$-algebras.

Example 3. Let X be a complex manifold. The *sheaf of holomorphic functions*, $\mathcal{O}_X$, assigns to each open set, U, the holomorphic maps into $\mathbb{C}$, i.e. $f \in \mathcal{O}_X(U)$ implies $f : U \to \mathbb{C}$ is holomorphic. As in the case of smooth functions, the restriction maps are given by restricting the domain. This is a sheaf of $\mathbb{C}$-algebras.

Example 4. Let X be a smooth manifold and $V \to X$ a vector bundle. Define the *sheaf of sections* of V, to be the sheaf, $C^\infty_X(-, V)$, that sends an open set, U, to the set of smooth sections of $V|_U$ and any inclusion $U \subset V$ to the restriction of domain of the sections. This is a C^∞_X-module.

A special case is the trivial $\mathbb{R}$-line bundle. Its sheaf of sections is identical with the sheaf of smooth functions.

Example 5. Let X a complex manifold and $V \to X$ a holomorphic bundle. Define the *sheaf of holomorphic sections* of V to the be the sheaf $\Gamma(-, V)$ that sends an open set, U, to the set of holomorphic sections of $V|_U$ and any inclusion $U \subset V$ to the restriction of domain of the sections. This is an $\mathcal{O}_X$-module.

Example 6. Another interesting class of sheaves are the locally constant sheaves. For a fixed abelian group, G, (or ring, or algebra), define a sheaf valued in groups (or rings, or algebras) to be the functor $\underline{G}$ that assigns to any open set the set of locally constant functions into G, along with the obvious restriction functors.

A concept that it will be convenient to know is that of a stalk of a sheaf.

Definition 14. Let X be a topological space and $\mathcal{F}$ a sheaf on X. Fix a point in X, $x \in X$, and consider the collection of equivalence classes of pairs

$$\mathcal{F}_x := \{(\sigma, U) \,:\, \sigma \in \mathcal{F}(U)\}/\sim$$

where $(\sigma, U) \sim (\sigma', V)$ whenever there exists an open $W \subset U \cap V$ such that $\rho_{WU}\sigma = \rho_{WV}\sigma'$. $\mathcal{F}_x$ is the *stalk* of $\mathcal{F}$ at x.

It can be explicitly checked that if $\mathcal{F}$ is a sheaf of groups, rings, etc. then the stalk has a natural structure of a group, ring, etc. Further, a morphism of sheaves induces a morphism of stalks.

6.3.2 Sheaf cohomology

We will now describe the basic definitions of sheaf cohomology, but will not attempt to justify the construction. Good sources that contain more details and discussion include [162, 164, 165, 168].

Definition 15 (Injective modules)**.** Let X be a topological space and $\mathcal{O}_X$ a sheaf of commutative rings. We say that an $\mathcal{O}_X$ module, $\mathcal{I}$, is *injective* if every inclusion of $\mathcal{O}_X$ modules $f : \mathcal{A} \hookrightarrow \mathcal{B}$ extends along any module morphism $g : \mathcal{A} \to \mathcal{I}$, i.e. there always exists a dashed arrow such that the following diagram:

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook, "f"] \arrow[d, "g"] & \mathcal{B} \arrow[dl, dashed, "h"] \\ \mathcal{I} & \end{tikzcd} \tag{6.5}$$

commutes.

In other words, we can *extend* morphisms into $\mathcal{I}$.

Observe that all objects and arrows in the above definition are sheaves; in particular the injection $\mathcal{A} \to \mathcal{B}$ is a monomorphism in the category of $\mathcal{O}_X$-modules, for a sheaf $\mathcal{O}_X$, which essentially means that each component of the natural transformation is an injective module morphism.

Definition 16 (Complex of abelian groups)**.** A complex of abelian groups is a collection of groups, $\{A_i\}_{i\in\mathbb{Z}}$, along with maps

$$\cdots \to A_{i-1} \overset{\alpha_{i-1}}{\to} A_i \overset{\alpha_i}{\to} A_{i+1} \to \cdots \tag{6.6}$$

such that each composition $\alpha_i \circ \alpha_{i-1} = 0$.

A complex is said to be *exact* if $\ker \alpha_i = \operatorname{Im} \alpha_{i-1}\ \forall\, i \in \mathbb{Z}$.

Definition 17 (Complex of sheaves)**.** A complex of sheaves of abelian groups on X is a collection $\{\mathcal{A}_i\}_{i\in\mathbb{Z}}$ along with maps:

$$\cdots \to \mathcal{A}_{i-1} \stackrel{\alpha_{i-1}}{\to} \mathcal{A}_i \stackrel{\alpha_i}{\to} \mathcal{A}_{i+1} \to \cdots \tag{6.7}$$

such that for every $x \in X$ the diagram of abelian groups defined by passing to the stalk at x is a complex.

A complex of sheaves is *exact* if the complex is exact at the level of stalks.

A special example of an exact sequence is a "short exact sequence", which is of the form:

$$0 \to \mathcal{A}_0 \to \mathcal{A}_1 \to \mathcal{A}_2 \to 0\,. \tag{6.8}$$

A functor is said to be *exact* if the image of any short exact sequence is short exact.

The injective modules are useful because of the following lemma.

Lemma 1 (Lemma 2.3.4 [168])**.** The functor $\mathrm{Hom}(-, \mathcal{I})$ is exact when $\mathcal{I}$ is injective.

One says that a category has enough injectives when every object in the category admits a monomorphism into an injective object. This is the categorical condition that one can resolve our objects using injectives.

We will work in the abelian category of $\mathcal{O}_X$-modules and rely on the theorem:

Theorem 3 (Prop II.2.2 [165])**.** The category of $\mathcal{O}_X$-modules has enough injectives.

Definition 18. An *injective resolution* of a sheaf, $\mathcal{F}$, is a complex of injective sheaves with support in nonnegative degrees, $\mathcal{I}^\bullet$, along with a monomorphism $\mathcal{F} \to \mathcal{I}^0$ such that the resulting diagram

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots \tag{6.9}$$

is exact.

Since $\mathcal{O}_X$ modules have enough injectives, every sheaf admits an injective resolution.

In essence, we regard the injective resolution as an approximation to $\mathcal{F}$, which behaves better with respect to the hom functor. We use it to define sheaf cohomology.

Definition 19 (Sheaf cohomology)**.** The sheaf cohomology of a sheaf, $\mathcal{F}$, is a graded abelian group $H^\bullet(X, \mathcal{F})$ which is obtained by taking the cohomology of the chain complex of global sections, i.e.

$$\ker\left(\Gamma(X,\mathcal{I}^i)\to\Gamma(X,\mathcal{I}^{i+1})\right)/\mathrm{Im}\left(\Gamma(X,\mathcal{I}^{i-1})\to\Gamma(X,\mathcal{I}^i)\right). \tag{6.10}$$

One checks that the choice of resolution does not affect the result. The main result of this section is:

Proposition 3 (Corollary 4.38, [83])**.** The sheaf cohomology of the sheaf of sections of a holomorphic vector bundle is isomorphic to the Dolbeault cohomology of the bundle itself.

As a consequence, the vector space $H^1(X,V)$ can be computed using techniques from sheaf cohomology.

6.3.3 Long exact sequences

In this section, we fix an arbitrary topological space, X, and sheaf of algebras $\mathcal{O}_X$. All sheaves will be $\mathcal{O}_X$-modules on X.

A key property of a cohomology theory is that short exact sequences get converted to long exact sequences in cohomology. In sheaf cohomology, a short exact sequences of sheaves induces a long exact sequence of sheaf cohomology groups.

In the case of interest, exact sequences of sheaves will yield a long exact sequence in sheaf cohomology:

$$0\longrightarrow\mathcal{A}\longrightarrow\mathcal{B}\longrightarrow\mathcal{C}\longrightarrow 0$$

$$\implies\ \cdots\to H^i(X,\mathcal{A})\to H^i(X,\mathcal{B})\to H^i(X,\mathcal{C})\to H^{i+1}(X,\mathcal{A})\to\cdots$$

The coboundary map, i.e. $\delta: H^i(X,\mathcal{C})\to H^{i+1}(X,\mathcal{A})$ will play an important role in the reasoning that leads to the theorems of Paper III, so we will briefly review this construction, then explain how the construction can be extended to longer sequences via slicing and splicing. These techniques are standard homological algebra, so this explication will be brief and skip some of the necessary consistency checks. All the details can be found in [168].

First, the coboundary map. In order to construct this map, one must ensure that it is possible to construct injective resolutions over the short exact sequence in a compatible fashion. That is, it must be possible to construct a commuting diagram:

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \mathcal{I}^2(\mathcal{A}) \arrow[r] \arrow[u] & \mathcal{I}^2(\mathcal{B}) \arrow[r] \arrow[u] & \mathcal{I}^2(\mathcal{C}) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{I}^1(\mathcal{A}) \arrow[r] \arrow[u] & \mathcal{I}^1(\mathcal{B}) \arrow[r] \arrow[u] & \mathcal{I}^1(\mathcal{C}) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{I}^0(\mathcal{A}) \arrow[r] \arrow[u] & \mathcal{I}^0(\mathcal{B}) \arrow[r] \arrow[u] & \mathcal{I}^0(\mathcal{C}) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{A} \arrow[r] \arrow[u] & \mathcal{B} \arrow[r] \arrow[u] & \mathcal{C} \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
\tag{6.11}
$$

in which both columns and rows are exact. One can argue that this is possible by fixing a choice of resolution for $\mathcal{A}$ and $\mathcal{C}$ and then checking that $\mathcal{I}^j(\mathcal{B}) := \mathcal{I}^j(\mathcal{A}) \oplus \mathcal{I}^j(\mathcal{C})$ works. This is essentially the only sensible construction to make, because any exact sequence $0 \to I \to B \to C \to 0$ in which I is injective, splits (see [169], for instance). The map from $\mathcal{B}$ to $\mathcal{I}^0(\mathcal{C})$ can be constructed by composing the given maps, while the map to $\mathcal{I}^0(\mathcal{A})$ is constructed using the universal property of injectives, and use induction for higher degrees. The argument in full can be found in [162], for instance.

Now, utilise (6.11) to construct the coboundary as follows. Let the class $[\gamma] \in H^k(X, \mathcal{C})$ be represented by $\gamma \in \mathcal{I}^k(\mathcal{C})(X)$. It admits a preimage by surjectivity (which uses that this is an injective object), which we can call β. Applying the vertical differential, we obtain something in the kernel of the horizontal map $d\beta \in \ker\left(\mathcal{I}^{k+1}(\mathcal{B})(X) \to \mathcal{I}^{k+1}(\mathcal{C})(X)\right)$. By exactness, $d\beta$ is therefore in the image of an element $\alpha \in \mathcal{I}^{k+1}(\mathcal{A})(X)$ and this represents the coboundary: $[\alpha] = \delta[\gamma]$. it is a straightforward extension of these arguments to verify that α is closed and the cohomology class is independent of choices made in the construction, so that the map is well-defined.

We will also need to deal with longer exact sequences, e.g.

$$
0 \longrightarrow \mathcal{A}_{-l} \xrightarrow{\alpha_{-l}} \mathcal{A}_{-l+1} \xrightarrow{\alpha_{-l+1}} \cdots \xrightarrow{\alpha_1} \mathcal{A}_0 \longrightarrow 0 \ . \tag{6.12}
$$

The approach taken is to slice the exact sequence into $(l-2)$ short exact sequences:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{A}_{-l} & \xrightarrow{\alpha_{-l}} & \mathcal{A}_{-l+1} & \xrightarrow{\beta_{-l+1}} & \mathcal{B}_{-l+1} & \longrightarrow & 0 \\
\\
0 & \longrightarrow & \mathcal{B}_{-l+1} & \xrightarrow{\gamma_{-l+1}} & \mathcal{A}_{-l+2} & \xrightarrow{\beta_{-l+2}} & \mathcal{B}_{-l+2} & \longrightarrow & 0 \\
\\
 & & \vdots & & \vdots & & \vdots & & \\
\\
0 & \longrightarrow & \mathcal{B}_{-2} & \xrightarrow{\gamma_{-1}} & \mathcal{A}_{-1} & \xrightarrow{\alpha_{-1}} & \mathcal{A}_{0} & \longrightarrow & 0\,.
\end{array}
\tag{6.13}
$$

where each $\mathcal{B}_i$ is defined as a cokernel of α_{i-1} and thus injects into $\mathcal{A}_{i+1}$ due to exactness of the original sequence.

Now, suppose one has a class $[\nu] \in H^i(\mathcal{A}_0)$. By the earlier construction, we have a sequence of maps:[2]

$$
H^i(\mathcal{A}_0) \longrightarrow H^{i+1}(\mathcal{B}_{-2}) \longrightarrow \cdots \longrightarrow H^{i+l-2}(\mathcal{A}_l)\,. \tag{6.14}
$$

Note that all of the intermediate stages take values in the auxilliary sheaves $\mathcal{B}$, so only after applying all of the coboundary maps and arriving at the other end of the exact sequence does one get a cohomology class that lies in a sheaf in the original exact sequence $\mathcal{A}$.

However, it may happen that some class $[\nu] \in H^i(\mathcal{A}_0)$ has the property that $[\delta^r \nu] \in H^{i+r}(\mathcal{B}_{-r-1})$ vanishes, while $[\delta^{r-1}\nu] \neq 0$. In this case, we use the long exact sequence from the short exact sequence $\mathcal{B}_{-r-1} \to \mathcal{A}_{-r} \to \mathcal{B}_{-r}$:

$$
\cdots \to H^{i+r-1}(\mathcal{A}_{-r}) \to H^{i+r-1}(\mathcal{B}_{-r}) \xrightarrow{\delta} H^{i+r}(\mathcal{B}_{-r-1}) \to \cdots \tag{6.15}
$$

and observe that, since $[\delta^{r-1}\nu]$ is in the kernel of δ it is the image of a class in $H^{i+r-1}(\mathcal{A}_{-r})$. More precisely, for any choice of class $[\mu] \in H^{i+r-1}(\mathcal{A}_{-r})$ we have that $[\mu] + \gamma_{-r+1,*}H^{i+r-1}(\mathcal{B}_{-r-1})$ is a space of valid classes.

This suffices to define a filtration of $H^i(\mathcal{A}_0)$ with $F^j H^i(\mathcal{A}_0)$ being those classes, $[\nu]$ such that $[\delta^j \nu] = 0$. Thus we have:

$$
0 = F^0 H^i(\mathcal{A}_0) \subset F^1 H^i(\mathcal{A}_0) \subset \cdots \subset F^{i+l-2} H^i(\mathcal{A}_0) = H^i(\mathcal{A}_0)\,. \tag{6.16}
$$

Let us now return to the precise setup for Yukawa couplings. We have a Calabi-Yau manifold, X, equipped with an (holomorphic and polystable) $SU(3)$ bundle, V. The sheaf of sections, $\mathcal{V}$, is equipped with a resolution:

$$
0 \longrightarrow \mathcal{F}_{-l} \longrightarrow \mathcal{F}_{-l+1} \longrightarrow \cdots \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{V} \longrightarrow 0\,. \tag{6.17}
$$

[2] Since all cohomology groups are over the same space, we drop the explicit dependence for conciseness, i.e. $H^i(\mathcal{A}) := H^i(X, \mathcal{A})$

In fact, we will assume that X is embedded in some ambient complex manifold, $\varepsilon : X \hookrightarrow Y$. This is because our results will only give nontrivial conditions in this case. We will abusively let $\mathcal{V}$ stand in for pushforward along the inclusion map. This abuse is not too bad, since $H^*(X, \mathcal{V}) = H^*(Y, \varepsilon_* \mathcal{V})$, [164].

The above construction gives a filtration on $H^1(X, \mathcal{V})$. Since this is the only instance that we apply the construction to, we give it a simplified notation $F^j H^1(X, \mathcal{V}) := F^j$. A class $[\nu] \in F^1$ is in the kernel of the first coboundary, i.e. $\delta[\nu] = 0 \in H^2(Y, \mathcal{B}_{-1})$ (where $\mathcal{B}_*$ denotes the relevant cokernels, as above) and, consequentially $[\nu]$ has a non-unique lift to $H^1(Y, \mathcal{F}_0)$. More generally, a class in F^i that is not in F^{i-1} has the property that $[\delta^{i-1}\nu] \neq 0$, while $[\delta^i \nu]$ does equal 0.

Definition 20. A class, $[\nu] \in H^1(X, \mathcal{V})$ is said to be of *type a* if it represents a non-zero class in the quotient $[\nu] \neq 0 \in F^a / F^{a-1}$, with $a \geq 1$.

The notion of type is essentially just a convenient terminology, borrowed from [158,159], which simplifies the statements of the results presented below.

6.3.4 Cup products

Twisted Dolbeault cohomology has an algebraic structure akin to the wedge product of differential forms, which featured in the Yukawa coupling (6.3). The main difference is that, in the twisted setting, the target of the product differs from the source. A concise introduction to the sheaf cup product can be found in the appendix of [154]. For the notion in Dolbeault cohomology, one can see Voisin [83] or Huybrechts [82], amongst others.

The general form of the twisted Dolbeault product is of the form:

$$H^p_{\bar{\partial}}(X, E) \otimes H^q_{\bar{\partial}}(X, F) \to H^{p+q}_{\bar{\partial}}(X, E \otimes F) \tag{6.18}$$

for holomorphic bundles E, F. The product is a combination of wedge product in the differential form indices, and tensor product in the bundle indices.

Sheaf cohomology has a comparable product, which combines classes in the cohomology of two sheafs, $\mathcal{E}, \mathcal{F}$ and outputs a class in the cohomology of $\mathcal{E} \otimes \mathcal{F}$. Ultimately, these two products agree when they are both defined and this fact will allow us to use sheaf cohomology and its broader domain of applicability in place of Dolbeault cohomology to study the structure of Yukawa couplings. Thus, it is necessary to understand the product in sheaf cohomology.

The approach used here and in Paper III utilises a very specific resolution, the so-called Godement resolution, in order to define the product. This resolution can always be constructed and is always locally free, but in the case that the structure sheaf, $\mathcal{O}_X$ is an algebra over a field (as in our case), the construction yields a resolution in which the objects are both injective *and* locally free. Being locally free is crucial for the product: just as $\mathcal{I}$ being injective implies the hom-functor into $\mathcal{I}$ is exact, being locally free ensures that the tensor product is exact.

Definition 21 ([154, 163]). Let $\mathcal{F}$ be an arbitrary $\mathcal{O}_X$-module. Define a new sheaf under $\mathcal{F}$ as follows: For each open $U \subset X$, set $\mathcal{G}^0(U) := \prod_{x \in U} \mathcal{F}_x$, where $\mathcal{F}_x$ is the stalk of $\mathcal{F}$ at x. It is straightforward to see that this is indeed a sheaf and it is injective. There is a canonical injection $\iota : \mathcal{F}(U) \hookrightarrow \mathcal{G}_0(U)$ and direct computation shows that this commutes with the structure maps, therefore defining a map of sheafs.

The Godement sheaf $\mathcal{G}^0$ is the first step in the Godement resolution, which is a functorially defined injective resolution. To continue, we take the cokernel of the map ι to obtain a short exact sequence:

$$0 \to \mathcal{F} \to \mathcal{G}^0 \to K^0 \to 0\,. \tag{6.19}$$

Taking the Godement sheaf of K^0, we obtain $\mathcal{G}^1$, and iterating, we obtain the *Godement resolution* $\mathcal{G}^\bullet$:

$$\begin{tikzcd}
\mathcal{F} \arrow[r] & \mathcal{G}^0 \arrow[r] \arrow[dr] & K^0 \arrow[d] & & \\
& & \mathcal{G}^1 \arrow[r] \arrow[dr] & K^1 \arrow[d] & \\
& & & \mathcal{G}^2 \arrow[r] \arrow[dr] & K^2 \\
& & & & \ddots
\end{tikzcd} \tag{6.20}$$

It is vital for us that $\mathcal{G}^\bullet$ is functorial with respect to $\mathcal{F}$, and it is an exact functor, [164]. Consequentially, given an exact sequence of sheaves, one can apply the Godement resolution degree-wise to obtain a bicomplex that is exact in both directions.

In particular, a cohomology class, $[\nu] \in H^i(X, \mathcal{F})$ is represented by a global section $\nu \in \mathcal{G}^i(\mathcal{F})(X)$ that is closed under the map of global sections induced by $\mathcal{G}^i \to \mathcal{G}^{i+1}$.

How to represent the cohomological product? Suppose that $\mathcal{E}, \mathcal{F}$ are sheaves and let $\mathcal{G}^\bullet(\mathcal{E})$ and $\mathcal{G}^\bullet(\mathcal{F})$ denote their respective Godement resolution. A cohomology class $[\mu] \in H^j(X, \mathcal{E})$ or $[\nu] \in H^k(X; \mathcal{F})$ will be represented by a global section $\mu \in \mathcal{G}^j(\mathcal{E})(X)$, $\nu \in \mathcal{G}^k(\mathcal{F})(X)$ respectively. The tensor product lands in $\mathcal{G}^j(\mathcal{E})(X) \otimes \mathcal{G}^k(\mathcal{F})(X)$, so does not yet represent a class in $H^{j+k}(X, \mathcal{E} \otimes \mathcal{F})$. Fortunately, the Godement resolution *does* admit a map from the tensor product of the Godement resolution into an arbitrary injective resolution over the tensor product of sheaves [154] (for simplicity, we will take the injective resolution of the tensor product to also be given by the Godement resolution):

$$c : \mathrm{Tot}^\bullet(\mathcal{G}(\mathcal{E}) \otimes \mathcal{G}(\mathcal{F})) \to \mathcal{G}^\bullet(\mathcal{E} \otimes \mathcal{F})\,. \tag{6.21}$$

Here, we have converted the tensor product bicomplex to the "total" complex: $\mathrm{Tot}^i(\mathcal{G}(\mathcal{E}) \otimes \mathcal{G}(\mathcal{F})) := \bigoplus_{j+k=i}(\mathcal{G}^j(\mathcal{E}) \otimes \mathcal{G}^k(\mathcal{F}))$, with total differential $d := d_{\mathcal{E}} + (-1)^j d_{\mathcal{F}}$.

Definition 22 ([163])**.** The *cup product* of two sheaf cohomology classes, $[\nu] \in H^j(\mathcal{E})$, $[\mu] \in H^k(\mathcal{F})$ is defined to be the class

$$[\mu] \smile [\nu] = [c(\mu \otimes \nu)] \tag{6.22}$$

In order to relate this cup product with the Yukawa coupling, one must check that the isomorphism between sheaf cohomology and Dolbeault cohomology respects the multiplicative structure (at least up to signs and numerical prefactors that can be safely ignored). Fortunately, this is true and is the content of Theorem 5.29 in [83], for instance. In essence, when the sheaf is given by the holomorphic sections of a holomorphic bundle (such that Dolbeault cohomology is defined), one can use the same formula as for the Godement product, only now using the Dolbeault resolutions, so they are formally the same.

6.4 Vanishing theorems for Yukawa couplings

We now have a notion of cup product in sheaf cohomology, which does not rely on any intrinsic properties of the underlying sheaf. We will use this freedom to re-express the bundle over a Calabi-Yau in terms of an arbitrary, sheafy resolution and obtain the main theorems from Paper III. The presentation here is slightly streamlined.

Let V denote the $SU(3)$ bundle and $\mathcal{V}$ its sheaf of sections. Suppose that $\mathcal{F}^\bullet$ denotes a complex of sheaves, bounded below and such that

$$0 \longrightarrow \mathcal{F}^{-k} \longrightarrow \mathcal{F}^{-k+1} \longrightarrow \cdots \longrightarrow \mathcal{F}^0 \longrightarrow \mathcal{V} \longrightarrow 0 \tag{6.23}$$

is an exact sequence of sheaves. To aid in the next steps, introduce notational convention that $[\mathcal{V}]$ denotes the complex of sheaves with $\mathcal{V}$ in degree zero and zeros elsewhere; likewise for any complex $[\mathcal{F}^\bullet]$ indicates adding zeros away from the $\mathcal{F}^i$-entries. Then, the exact sequence above can be written as a quasi-isomorphism of chain complexes $[\mathcal{F}^\bullet] \overset{\sim}{\to} [\mathcal{V}]$.

The aim to construct the class $[\nu_1 \wedge \nu_2 \wedge \nu_3] \in H^3(\Lambda^3\mathcal{V})$ in terms of classes in $\mathcal{G}^{i+1}(\mathcal{F}_{-i})$ that represent the classes ν. Recall the filtration defined at the end of subsection 6.3.3: the ith level in the filtration, $F^i \subset H^1(\mathcal{V})$ consists of the cohomology classes, $[\nu]$ such that $[\delta^i\nu] = 0$.

Now, the $\mathcal{F}$ sheaves can not be assumed to be flat and consequentially, taking the tensor product does not preserve exactness. Therefore we will run into issues if we attempt to combine this filtration with the naive tensor product. In Paper III, we avoided this problem by sticking to a small portion of the complex that is exact, the truncated exterior power sequence exact [168]:

$$\mathcal{F}_1 \otimes \Lambda^2 \mathcal{F}_0 \longrightarrow \Lambda^3 \mathcal{F}_0 \longrightarrow \Lambda^3 \mathcal{V} \longrightarrow 0 \tag{6.24}$$

This suffices to give some control of low type classes.

Case (type 1)^3
Consider three classes that are type 1, i.e. $[\nu_1]$, $[\nu_2]$, $[\nu_3] \in F^1$. The aim is to show that the triple cup product of these objects also appears in the first non-trivial stage of the analogous filtration of $H^3(\Lambda^3\mathcal{V})$. To emphasise: the exact sequence (6.24) only gives the first two stages of a filtration, but it is enough to handle low types as considered here.

Let $\mathcal{K}_1$ denote the sheaf kernel of $\Lambda^3\mathcal{F}_0 \to \Lambda^3\mathcal{V}$, so that there is an exact sequence:

$$0 \longrightarrow \mathcal{K}_1 \longrightarrow \Lambda^3 \mathcal{F}_0 \longrightarrow \Lambda^3 \mathcal{V} \longrightarrow 0 \tag{6.25}$$

along with the Godement resolution over it.

Choose representatives $\nu_i \in \mathcal{G}^1(\mathcal{V})(X)$ of the class $[\nu_i]$ for each $i = 1, 2, 3$. Since each $[\nu_i]$ is in the first level of the filtration, F^1, it is in the kernel of the first coboundary $H^1(\Lambda^3\mathcal{V}) \to H^2(\mathcal{K})$ and therefore each section ν_i admits a closed lift, say $\alpha_i \in \mathcal{G}^1(\mathcal{F}_0)$. The aim is to show the same property holds for the class $[\nu_1 \wedge \nu_2 \wedge \nu_3]$. To see this, observe that this class is represented by a choice of section $c(\nu_{[1} \otimes \nu_2 \otimes \nu_{3]})$ where the square brackets on indices indicates the antisymmetrization. Then, the choices, α_i, can be used to make a choice of lift for the product:

$$c(\alpha_{[1} \otimes \alpha_2 \otimes \alpha_{3]})\,, \tag{6.26}$$

and this is evidently closed. Consequentially, the class has *a* closed lift and thus *a* representative of the coboundary is the zero class. Since, however, the coboundary is independent of these choices, it follows that the class is in the kernel of the coboundary. The long exact sequence induced by the short exact sequence, (6.25), therefore shows that the class of the cup product is in the kernel, $[\nu_1 \wedge \nu_2 \wedge \nu_3] \in \ker(H^3(\Lambda^3\mathcal{V}) \to H^4(\mathcal{K}))$, so is in the image of $H^3(\Lambda^3\mathcal{F}_0)$. By constraining this group, and consequentially the image, one obtains a constraint on the coupling.

Theorem 4. Assume there is a quasi-isomorphism $[\mathcal{F}_\bullet] \to [\mathcal{V}]$ such that the sheaf cohomology group vanishes: $H^3(\Lambda^3\mathcal{F}_0) = 0$. Let the three classes $[\nu_1], [\nu_2], [\nu_3] \in H^1(X, \mathcal{V})$ each be of type one Then, their product (and consequentially the associated Yukawa coupling), $[\nu_1 \wedge \nu_2 \wedge \nu_3]$ vanishes.

Case (type 1)^2(type 2)

It is natural to attempt to push this further, so suppose that $[\nu_1], [\nu_2] \in F^1$, $[\nu_3] \in F^2/F^1$ and choose representatives $\nu_i \in \mathcal{G}^1(\mathcal{V})(X)$.

Just as previously, the sections ν_1, ν_2 admit closed lifts that are again denoted by $\alpha_1, \alpha_2 \in \mathcal{G}^2(\mathcal{F}_0)(Y)$. The lift of the third class, however, will not be closed since it is assumed to have a non-trivial coboundary $\delta[\nu_3] \in H^2(\mathcal{B}_{-1})$. Since its second coboundary vanishes, this class admits a lift to $H^3(\mathcal{F}_{-1})$. Running through the machinery, let $\alpha_3 \in \mathcal{G}^1(\mathcal{F}_0)(X)$ a choice of section that lifts ν_3, $\delta\nu_3 \in \mathcal{G}^2(\mathcal{B}_{-1})(Y)$ a representative of the non-trivial coboundary of ν_3 and $\beta_3 \in \mathcal{G}^2(\mathcal{F}_{-1})(Y)$ a choice of closed section lifting $\delta\nu_3$. Then, the pair of classes $(\alpha_3, \beta) \in \mathcal{G}^1(\mathcal{F}_0)(Y) \oplus \mathcal{G}^2(\mathcal{F}_{-1})(Y)$ represents the class, $[\nu_3]$ in the presentation determined by the exact sequence $\mathcal{F}_\bullet$. These explicit choices can be used to make comparable choices for the cup product.

Indeed, recall that the cup product of interest is represented by the section $c(\nu_{[1} \otimes \nu_2 \otimes \nu_{3]}) \in \mathcal{G}^3(\Lambda^3\mathcal{V})(X)$. We can extract two short exact sequences of sheaves from (6.24):

$$\begin{array}{c} 0 \longrightarrow \mathcal{K}_1 \longrightarrow \Lambda^3\mathcal{F}_0 \longrightarrow \Lambda^3\mathcal{V} \longrightarrow 0 \\ 0 \longrightarrow \mathcal{K}_2 \longrightarrow \Lambda^2\mathcal{F}_0 \otimes \mathcal{F}_{-1} \longrightarrow \mathcal{K}_1 \longrightarrow 0 \end{array} \tag{6.27}$$

and now a pair of sections represents the product class:

$$\big(c(\alpha_{[1}\otimes\alpha_2\otimes\alpha_{3]}), c(\alpha_{[1}\otimes\alpha_{2]}\otimes\beta)\big) \in \mathcal{G}^3(\Lambda^3\mathcal{F}_0)\oplus\mathcal{G}^4(\Lambda^2\mathcal{F}_0\otimes\mathcal{F}_{-1}) \tag{6.28}$$

and it is immediate that the second class $c(\alpha_{[1} \otimes \alpha_{2]} \otimes \beta)$ is closed. Suppose, then, that $H^4(\Lambda^2\mathcal{F}_0 \otimes \mathcal{F}_{-1})$ vanishes, from which it can be concluded that this section is exact. Let $\xi \in \mathcal{G}^3(\Lambda^2\mathcal{F}_0 \otimes \mathcal{F}_{-1})$ such that $d\xi = dc(\alpha_{[1} \otimes \alpha_{2]} \otimes \beta)$. By construction, $d\xi$ pushes forward to $c(\alpha_{[1} \otimes \alpha_{2]} \otimes \beta)$ and, by commutativity, it follows that:

$$\xi - c(\alpha_{[1} \otimes \alpha_2 \otimes \alpha_{3]}) \in \mathcal{G}^3(\Lambda^3\mathcal{F}_0)_{closed}\,. \tag{6.29}$$

Consequentially, in the case that $H^4(\Lambda^2\mathcal{F}_0 \otimes \mathcal{F}_{-1}) = 0$, the cup product can be simplified and represented by $[\xi - c(\alpha_{[1} \otimes \alpha_2 \otimes \alpha_{3]})] \in H^3(\Lambda^3\mathcal{F}_0)$. In order to universally constrain this coupling, then, sequence chasing forces the assumption that $H^3(\Lambda^3\mathcal{F}_0) = 0$, in addition to the earlier assumption on H^4.

This gives the theorem:

Theorem 5. Assume there is a quasi-isomorphism $[\mathcal{F}_\bullet] \to [\mathcal{V}]$ such that the cohomology group $H^3(\Lambda^3\mathcal{F}_0)$ vanishes and $H^4(\mathcal{F}_1 \otimes \Lambda^2\mathcal{F}_0) = 0$. Let the three cohomology classes $[\nu_1], [\nu_2], [\nu_3] \in H^1(X, \mathcal{V})$ be such that $[\nu_1], [\nu_2]$ are type 1 and $[\nu_3]$ is type 2. Then, their product (and consequentially the associated Yukawa coupling) $[\nu_1 \wedge \nu_2 \wedge \nu_3]$ vanishes.

These theorems are applied to several examples in Paper III and it is found that in some examples we find more constraints than other methods, e.g. stability walls [155, 161], and in others we find less.

6.5 Summary

In this chapter, we reviewed the arguments that lead to the new vanishing theorems of Paper III.

The main idea behind Theorems 4 and 5 was that any resolution of the bundle leads to a filtration of the cohomology and that this plays well with the operation of tensor product *if* the resolution itself plays well with the tensor product. The right language to extract information from such a filtration is that of spectral sequences, [162, 168, 170], specifically multiplicative spectral sequences.

In order to get the generality that we wished for, we essentially dropped the assumption that the arbitrarily constructed resolution plays well with the tensor product. This introduced the main weakness of the paper, which is that it only tells us about very low levels in the filtration and it would be desirable to improve on that result. In fact, the world of homological algebra gives us tools to handle a tensor product that fails to be exact, but things quickly became unmanageable in the hands-on approach that was taken in this paper. These are the kind of problems that the machinery of spectral sequences is designed to deal with. For these reasons, it may be possible to extend the results of Paper III by making use of this heavier machinery.

There is another weakness that could be addressed (and would also be improved by extending the range of types that can be constrained): it is currently unclear how to quantify the number of couplings that are constrained. Indeed, in principle, *any* finite resolution will give the possibility of a constraint on the Yukawa couplings. It is straightforward to see that if two resolutions are chain homotopic then the filtrations are equivalent, so they contain the same constraints. On the other hand, by construction, any two resolutions will be quasi-isomorphic [168], but need not be chain homotopic and thus could give a different filtration and different constraints. This means that the effect that we rely on is visible in the homotopy category of chain complexes, but not the derived category (for the sake of this discussion, the difference between these two constructions is precisely that the quasi-isomorphisms are invertible in the latter and not the former, see [171] for precise definitions). It would be very interesting and important for the physics applications described in Section 6.1 to follow this up and determine the maximum amount of constraints that our theorems can obtain for a given construction, as well the generic behaviour.

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Svensk Sammanfattning

Under 1900-talet formulerade forskare teorier som noggrant förklarar universum både i mycket stora skalor och i mycket små skalor. Einsteins allmänna relativitetsteori förklarar gravitationens kraft mellan stora och tunga objekt, som planeter, stjärnor och galaxer. Å andra sidan förklarar kvantmekanik beteendet av små objekt som elektroner, fotoner och andra partiklar som inte går att se med blotta ögat. Båda teorier är verifierade i experiment med mycket hög precisionsnivå. Problemet är att teorierna är motstridiga. Detta innebär bland annat att särskilda platser i rumtiden, som inuti ett svart hål eller tidigt i universums existens, faller utanför teoriernas ramar. Följaktligen är utvecklingen av en teori för kvantgravitation, som fångar både relativitetsteori och kvantmekanik, en av dagens viktigaste frågor i teoretisk fysik.

Strängteori är just en sådan, matematiskt motsägelsefri teori för kvantgravitation. Dessvärre är en av de första slutsatserna som kan dras av strängteorin att universum har tio dimensioner. Att teorin motsäger praktiska experiment förklaras av det faktum att, trots att rumtiden har tio dimensioner, så är formen för dessa mindre uppenbar. I själva verket verkar det finnas många alternativa former för universum, inklusive rumtid där vissa dimensioner inte går att se med blotta ögat. Dessa småskaliga dimensioner lämnar efter sig en skugga som märks av vid låga energier; partiklarna och sättet de interagerar är relaterat till dimensionerna som de är kopplade till. I denna doktorsavhandling undersöks aspekter av relationen mellan dessa geometrier och den tillhörande fysiken.

Doktorsavhandlingen består av tre artiklar. I artikel I tittar vi på universum med sju små dimensioner och tre utsträckta riktningar som återfinns i så kallade "heterotisk strängteori". Vi introducerar en särskild funktional som kallas "superpotential" vilken visar att locus där funktionen når sitt maximala och minimala värde förklarar formerna för ett universum med denna dimensionsstruktur. Typen av sju-dimensionella geometrier som beskrivs ovan har en särskild struktur gemensamt och kallas för en "G2-struktur". I artikel II studeras matematiska egenskaper anknutna till dessa strukturer och hur de kan relatera till fysiken av dessa modeller. I artikel III är fokus istället på teorier med fyra utsträckta dimensioner, som är en möjlig modell för vårt universum. Vi tittar på sättet som geometrierna kontrollerar några specifika interaktioner mellan partiklar som ges av så kallade "Yukawa kopplingar". I standardmodellen är sådana kopplingar fria och måste fixeras med hjälp av experiment. Dessa kopplingar har en oförklarad inbördes hierarki som vore angeläget att förklara. Detta är teoretiskt sätt möjligt att åstadkomma med strängteorins modeller och vårt mål var att utforska hur sannolikt sådana hierarkier är i vissa typer av konstruktioner. Detta gjordes genom att studera mekanismer som förbjuder vissa kopplingar och etablerade därmed allmänt applicerbara teorem.

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