Review

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Abstract: This is a review/research paper on anomalies applied in a bottom–up approach to standard model and gravity. It is divided into two parts. The first consists of a proper review of anomalies in quantum field theories. Anomalies are analyzed according to three different methods: a perturbative one based on a Feynman diagram, a non-perturbative one relying on the Schwinger–DeWitt approach, and, third, one hinging on the Atiyah–Singer family's index theorem. The three methods are applied both to chiral gauge anomalies and trace anomalies. The fundamental distinction, which our presentation leads to, is between obstructive (O) and non-obstructive (NO) anomalies. The former is tied to the non-existence of fermion propagators, which fatally maim the corresponding theory. In the second part, we apply this analysis to the SM and a variety of its extensions, which are immersed in a gravitational background, and we find that they are all plagued by a residual chiral trace anomaly. To completely eliminate all kinds of dangerous anomalies in SM-like theories, we propose a somewhat unconventional scheme and exemplify it by means of an explicit model. The latter is a left–right symmetric model. We embed it in a Weyl geometry to render it a conformal invariant. We then deal with some of its quantum aspects, particularly its even (NO) trace anomalies and the means to preserve its conformal invariance at the quantum level. We briefly review renormalization and unitarity in the framework of similar models discussed in the existing literature. Finally, we present a possible (conjectural) application of the model to describe the junction between cosmology and quantum field theory.

Keywords: particle physics; quantum gravity; conformal symmetry

1. Introduction

The quantum field theory, called the standard model (SM) of particle physics, describes the 5% of the energy/matter of the universe as being made of baryons and light. Its neutrino sector is still under construction; the dark matter and dark energy sectors are out of its reach, but there is no doubt that the strong, weak and electromagnetic interactions of standard matter can find a satisfactory explanation within it. This is fine as long as gravity is ignored or, at least, recedes to a remote enough distance, which is assumed not to disturb these interactions and forms a realm of its own, which, in turn, is described by the so-called standard model of cosmology (ΛCDM). This situation is clearly unsatisfactory, and the attempts at bridging the gap between the two pictures have abounded since earlier times: clearly, it is, at present, the main theoretical physics puzzle, and it includes in particular, the challenge of quantizing gravity. There are two general attitudes in this regard: the top–down one (superstring theories being the prominent examples) and the bottom–up.

For example, in the approach based on a given superstring theory, one starts from a ten-dimensional geometric configuration, which possibly includes branes. One considers the field theory (low energy) limit and compactifies to four dimensions. The final field theory includes matter and gravity. The bet is to be able to carefully choose the compactification in such a way as to reproduce the SM spectrum in the final theory. The latter turns out to be automatically UV complete. A bottom–up approach means, instead, starting from the SM and its classical interaction with gravity to build up, step by step, a larger effective quantum theory that encompasses all the interactions. Here, the key word is 'effective'. We imagine the ultimate theory as a complete quantum field theory, which describes the four basic interactions as being unitary—that is, it conserves probability—and renormalizable (possibly finite), or more generically, predictive; but being also able to explain those aspects of the universe which we are so far able to describe phenomenologically, but whose origin and link with ordinary matter and gravity remain mysterious: big bang, inflation, dark matter, dark energy, baryon asymmetry, etc. One such quantum field theory is not yet available. And we have to make do with approximations of it, that is, with quantum field theories larger than the SM, which is comprehended in some form gravity and is able to explain one or more of the above mentioned pieces of cosmology.

The bottom–up approach in constructing such an effective field theory, starting from the SM, parallels the inverted time evolution of the universe. The SM and classical gravity are probably satisfactory descriptions of the universe below the TeV energy or after a time of around 10^{-12} s from the beginning. While we do not know when the quantum regime of gravity began, it must have been much earlier than this. There is a vast stretch in logarithmic time where many phenomena must have taken place, such as baryogenesis, dark matter generation, and inflation. This part of the history of the universe is still largely conjectural. Trying to reconstruct it is like driving a car in reverse in a dark night while relying only on the information gathered from the headlights. Any tiny bit of information is precious. The purpose of the present paper, which is, in part, a review of the existing literature and, in part, an original elaboration, is to direct the headlights to and scrutinize the conditions that make a matter theory like the SM (or an extension thereof) consistent with the (minimal) coupling to gravity. We will find information that has not been sufficiently considered so far and will inject it in an attempt to construct an effective scheme (rather than a specific model) of quantum matter in interaction with quantum gravity. This extra input comes from the analysis of anomalies, which arise when matter like that of the SM couples to gravity. As the title says, we wish to expose how anomalies can help us in the bottom–up approach. In quantum field theories, there appear to be two large classes of anomalies, which we, being short on imagination, dub type O (O stands for obstructive) and type NO (non-obstructive). Type O is a dangerous kind of anomaly in that, when present, it compromises renormalization and/or unitarity. In fact, as we shall see, the reason is very down to earth: these anomalies appear in theories with Weyl fermions (with chiral fermions, in general); they are all odd-parity and include the so-called consistent chiral gauge and diffeomorphism anomalies and the consistent odd-parity trace anomalies. They signal the fact that the corresponding fermion propagators do not exist. Type NO anomalies are not dangerous; they do not signal any irreparable flaw of the theory and, in general, need not be canceled. In particular, the relevant propagators do exist (possibly after gauge fixing). They include the even parity trace anomalies and the ABJ anomalies in theories of Dirac fermions.

The first main point stressed by this paper is that the minimal standard model (MSM), along with its extensions (see below), is free of type O anomalies as long as we can disregard gravity. But this is not true anymore if the MSM is coupled to gravity in a minimal way. Some type O anomalies, which seem to have been disregarded so far, do not vanish. They

are odd-parity (chiral) trace anomalies belonging to the type O group. The second main purpose is to present a possible type O-anomaly-free scheme, which includes extensions of the SM, and discuss in this setup the role of even-parity trace anomalies, which, instead, belong to the set of type NO set. Accordingly, the paper is divided into two parts, I and II.

The distinction between type O and type NO anomalies is a basic one, particularly in relation to the SM and its extensions. For this reason, we have felt the necessity to return to the basics with a concise but (hopefully) thorough presentation of anomalies and their calculations. This covers Sections 2–11. We first review the definitions and calculations of the chiral gauge anomalies with perturbative methods based on Feynman diagrams, then with non-perturbative methods, of which we present the SDW (Schwinger–DeWitt) one in particular. But these two methods, which are fundamental for reasons that do not need to be stressed, are unable to unmask the true nature of type O anomalies and explain why and in what sense they differ from the type NO ones. A third approach is necessary, the one based on the family's index theorem. The latter tells us that type O anomalies are obstructions to the existence of the relevant Weyl fermion propagators, and this reveals their true nature. In particular, that is why they are 'dangerous': if the fermion propagators do not exist, the concept of (perturbative) quantization does not make sense. Our subsequent analysis of the SM and its extensions is based primarily on this fact and on the complementary fact that type NO anomalies do not imply any lack of propagators. This second fact is sanctioned by the family's index theorem, in which Euclidean self-adjoint operators, which are the sources of type NO anomalies, have vanishing family's index and so do not represent obstructions.

This is the basic distinction to always keep in mind. It is, therefore, of paramount importance to recognize type O and type NO anomalies. The well-known consistent gauge anomalies in theories of chiral fermions are type O, and their absence in a theory is the first requisite for its consistency. But there are other type O anomalies that correspond to non-vanishing obstructions exposed by the family's index theorem: they are the chiral trace anomalies in the theories of Weyl fermions. We devote a sizable portion of the first part to derive, by means of different methods, these kinds of trace anomalies. Once this is done, we apply the results to the minimal version of the SM and to a set of its extensions, which are minimally coupled to gravity. It turns out that in all these models, some residual chiral trace anomalies survive; more precisely, the chiral trace anomalies generated by the $SU(2)$ gauge fields. This result concludes the first part.

In the second part of the paper (Sections 12–20), we propose a scheme in order to cancel the above anomalies for SM extensions. The cancelation of all type O anomalies requires a left–right symmetry of the spectrum of fermions, which is, however, quite different from the L-R extension models present in the literature. The fermionic L-R symmetry is accompanied by also doubling the $SU(3)$ and $U(1)$ gauge fields (but not of the $SU(2)$ ones), by a doubling of the scalars and also by a doubling of the metric (which, incidentally, makes it a bimetric theory). The resulting theory is denoted $\mathcal{T}$. Its formulation becomes remarkably simple if one uses, as a bookkeeping device, the AE (axially extended) formalism, which is based on a version of the so-called hypercomplex analysis. This is based on a doubling of the variables analogous to the passage from the real to the complex numbers, with the imaginary unit i being replaced by the matrix γ_5. The action of $\mathcal{T}$, without Yukawa couplings, is invariant not only under diffeomorphisms and gauge transformations, like any ordinary extension of the SM minimally coupled to gravity, but also under doubled AE diffeomorphisms, i.e., diffeomorphisms, which have two components: the ordinary one and the other represented by the coefficient of γ_5. The chiral symmetry of $\mathcal{T}$ splits it into two, $\mathcal{T} = \mathcal{T}_R \cup \mathcal{T}_L$, which are formulated in terms of multiplets of right-handed and left-handed fermions, respectively, and also on sets of 'right' and 'left' metrics and $SU(3)$ and $U(1)$ gauge fields and scalar fields, respectively, with the only exception being the $SU(2)$ gauge fields, which are in

common to both $\mathcal{T}_R$ and $\mathcal{T}_L$. It is important to bear in mind that the pervasive presence of the chiral projectors does not allow the 'right' fields to interact directly with the 'left' fields, except through the $SU(2)$ gauge fields.

At this stage of the construction of an anomaly-free version of the SM coupled to gravity, which is invariant under gauge transformations and diffeomorphisms, we face a problem common to all analogous attempts. The SM above the electroweak symmetry-breaking energy is formulated in terms of massless fermion and gauge fields, which, as such, are naturally coupled to a metric. It is true that the scalars are massive, but at high enough energy, these masses can be considered irrelevant. In this situation, the absence of dimensionful parameters, the coupling to a metric and the addition of a very simple term make the action invariant under local Weyl transformations. The question, therefore, is whether Weyl invariance has to be subsumed under the local symmetry of the theory by enlarging it beyond gauge transformations and diffeomorphisms. We believe that this has to be decided in terms of the unitarity and renormalizability (predictivity) of the theory. In that sense, we do not have a definite answer to the question. After formulating $\mathcal{T}$ as $\mathcal{TW}$, i.e., as a Weyl conformal field theory, which is a rather simple matter, we observe that Weyl invariance is a rather peculiar gauge symmetry in that it admits a minimal realization that does not require the presence of a corresponding vector gauge field; the function of the latter can be carried out by a scalar field, which can be identified with the dilaton. We perform the quantization without fixing the conformal gauge; there is no obstruction as no propagator is missing without gauge fixing. Then, we study the Ward identity of the Weyl symmetry and connect it with the study of trace anomalies. We determine, at least in principle, all the latter and show that they can be canceled by means of Wess–Zumino terms. The obstacles, however, come from renormalization and unitarity. If we want to oblige the renormalization protocol, we must also introduce into the action terms like the square Weyl tensor, which carries in the particle spectrum a physical ghost, i.e., a particle with a wrong sign of the kinetic term, thus affecting unitarity. It would seem that Weyl invariance is an excessive request for our theory $\mathcal{TW}$, at least as it is formulated in this paper.

An alternative possibility is to consider the Weyl symmetry an accidental symmetry that may appear in approximate form at some stage of the history of the universe, which, in any case, is a non-necessary symmetry in the process of quantization. In this case, it seems that the process of renormalization can go on while preserving unitarity.

But, perhaps, these two alternatives should be thought of in sequence, in the sense that they describe two ranges of energies: First, at lower energy (anyhow, higher than the electroweak symmetry breaking scale); second, at higher energy, which is a satisfactory description. We have to resort to the first, which requires Weyl invariance. Its lack of unitarity could signal that some degrees of freedom are missing for the theory to be UV complete.

More research is certainly necessary for finding a precise answer. But if we do not have an answer in the direction of higher energies, we can perhaps speculate further in the opposite direction, that is, toward lower energies and the SM in its familiar form. First of all, we fix the conformal gauge in the right part of the theory, i.e., in $\mathcal{TW}_R$, by fixing the corresponding dilaton to a constant value. At this stage, the theory has the aspect of the SM minimally coupled to gravity except for a couple of additional (non-minimal) couplings of the Ricci tensor to the scalars. Next, we postulate an early breaking of the chiral symmetry (a *primitive chiral symmetry breaking*) by means of a Higgs mechanism generated by the non-zero vev (vacuum expectation value) of the real scalar field present in $\mathcal{TW}_R$. This endows all the 'right' fermions and other 'right' fields with a large mass, while the $\mathcal{TW}_L$ remains with all the original symmetries. Clearly, this determines different evolutions for the 'left' and 'right' theories, which are, however, not completely unrelated. The point is

that they keep interacting, although weakly, via the common $SU(2)$ gauge fields and Higgs. The 'right' one becomes a theory of heavy quarks and leptons, while the 'left' evolves toward a theory encompassing classical gravity and the SM, together with a real scalar field, which can realistically play the role of inflation. Due to the weak interaction among the two theories, it is natural to think of the 'right' as a candidate for the dark matter world.

As long as all the above are not supported by concrete calculations, they are nothing more than speculations. But they are fascinating speculations, and they are not unreasonable.

1.1. The Standard Model

In the absence of gravity, the world of (visible) matter and light is well described, up to around the TeV energy, by the standard model (SM) of elementary particles. It is a gauge theory with a gauge group $SU(3) \times SU(2) \times U(1)$. The latter is the symmetry in the unbroken phase when all the quarks and leptons are massless. The $SU(3)$ gauge bosons mediate the strong interactions. The $SU(2) \times U(1)$ ones carry the electroweak interactions. In the symmetric phase, the spectrum of massless quarks and leptons is specified by three families like the one below,

$$
\begin{array}{cccc}
\mathrm{G}/fields & SU(3) & SU(2) & U(1) \\
\begin{pmatrix} u \\ d \end{pmatrix}_L & 3 & 2 & \frac{1}{6} \\
u_R & 3 & 1 & \frac{2}{3} \\
d_R & 3 & 1 & -\frac{1}{3} \\
\begin{pmatrix} v_e \\ e \end{pmatrix}_L & 1 & 2 & -\frac{1}{2} \\
e_R & 1 & 1 & -1
\end{array}
\tag{1}
$$

The second column specifies the relevant representations of $SU(3)$, the third the ones of $SU(2)$ and the last is the list of $U(1)$ representations, which are denoted by the corresponding hypercharge eigenvalues [A frequent alternative notation is to use the Lorentz covariant conjugates $(u_R)^c$, $(d_R)^c$ and $(e_R)^c$ instead of u_R, d_R and e_R, in order to collect all the fields in a unique left-handed multiplet. In this case, one has to reverse the signs of their $U(1)$ charges and replace the representation 3 of $SU(3)$ of u_R, d_R with the $\bar{3}$ of $(u_R)^c, (d_R)^c$]. We recall that the hypercharge is defined by

$$
Y = Q - T_3, \qquad T_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\tag{2}
$$

here Q is the electromagnetic charge and T_3 the third generator of $SU(2)$ in the doublet representation. The SM spectrum is completed by the sector of Higgs scalars.

The fermion spectrum in (1) is not the only possible one, it refers to the original formulation of the SM, referred to as the minimal standard model (MSM), which is anyhow the basis of the subsequent (attempted) modifications. We shall discuss several of the latter in the course of the paper, but for the moment, we focus on (1). At suitable low energies, the gauge symmetry of the model breaks down to the local $U(1)$ of electromagnetism, and this is achieved thanks to the presence in the model of the Higgs doublet of scalars, which is responsible for the breakdown of symmetry and simultaneous generation of the fermion masses. For the moment we restrict our consideration to the symmetric phase and to the fermionic spectrum (1).

One of the most important criteria followed in the construction of the SM has been the absence of chiral gauge anomalies. This is a well-known fact, and if we review here

some well-known material, it is because it is necessary for a full appreciation of other less known aspects. Needless to say, the subject of gauge anomalies, after more than fifty years of analyses, papers and attempts, still seems to preserve a halo of mystery. But, as anticipated in the introduction, the reason why anomalous theories, i.e., theories infected by type O anomalies, are sick and cannot be taken into serious consideration is not due to a mysterious spell or curse of some malignant jinn, the reason is instead very clear and simple: in such theories, which are theories of Weyl fermions [The same problem also exists for other chiral fermions, for instance for the gravitino, but in this paper, we limit ourselves to spin 1/2 chiral fermions]: the corresponding fermion propagators do not exist. Anomalies are simply an indirect manifestation of this fact. This is the origin of the conundrum, which explains why the construction of consistent quantum field theories in such cases is impossible. But let us proceed step by step. Needless to say, the reader familiar with this subject may skip this part. Refs.: [1–15].

1.2. A Summary

The paper is divided into two parts. The first part is a review of anomalies in field theories. Section 2 is devoted first to the simplest calculation of chiral gauge anomalies in 4d with perturbative methods (Feynman diagrams). In the next subsection we review the method of Bardeen to arrive at the same result by introducing an auxiliary axial potential. Then, we introduce the Wess–Zumino consistency conditions and the BRST cohomology. In the final subsection, we show how the previously computed chiral anomalies cancel in the SM. In Section 3 we obtain the same Bardeen's anomaly derived before with a non-perturbative method, the Schwinger–DeWitt (SDW) method. Section 4 is devoted to a third method of computing the same chiral anomalies based on the Atiyah–Singer family's index theorem. We start with a geometrical presentation of anomalies relying on the geometry of principal fiber bundles. We show that their full comprehension requires the introduction of the relevant universal bundles and classifying spaces. Then, we introduce the Atiyah–Singer theorem and the Atiyah–Singer family's index theorem, which is required by the geometry of gauge theories. Next, we apply this theorem to the problem of anomalies: the latter turn out to be obstructions to the existence of fermion propagators in theories of chiral fermions, as it clearly appears using the Quillen determinant. This section contains the crucial results for our identification of the meaning of chiral anomalies (both gauge and trace). Section 5 is an exposé of gravitational anomalies. In Section 6, we introduce and compute gauge and gravity-induced odd-parity trace anomalies in theories of Weyl fermions. This first computation is carried out with perturbative methods. At this stage, in Section 7, we evaluate the odd-parity trace anomalies in the standard model minimally coupled to gravity and show that the gauge-induced ones do not cancel completely. In Section 8 we recalculate the same trace anomalies by means of the SDW method. To this end, we introduce the MAT (metric-axial-tensor) formalism, wherein we redefine the usual metric tensor $g_{\mu\nu}$ by adding to it another symmetric tensor $f_{\mu\nu}$ multiplied by γ_5, which makes use of the axial-complex analysis (a form of hypercomplex analysis), see Section 8.1 below. By this method, we compute various types of anomalies that may appear in matter theories coupled to gravity. In Section 9, we return to the meaning of anomalies and explain the difference between O (obstructive) anomalies and NO (non-obstructive) anomalies. Section 10 is devoted to our critical assessment of the existing literature on trace anomalies. The first part ends with the application of the previous results to the SM and its extensions minimally coupled to gravity. In Section 11 we compute the (gauge and trace) anomalies in some of the extensions of the SM and verify that while they are all free of chiral gauge anomalies they all contain a residue of chiral trace anomalies.

The second part begins with a summary of this problem and an enumeration of a few way-outs, which are not, however, our preferred choice. In Section 13 we propose a scheme, illustrated with a concrete left–right (or chirally or parity) symmetric model [We use throughout *left–right* (L-R) and *chirally symmetric* as synonyms, with some abuse of language because the latter should refer only to fermions, while the former refers to all the fields], that completely cancels all the O anomalies. We then construct the full theory, dubbed $\mathcal{T}$, which includes the SM extension and gravity. It turns out convenient to use the axial-complex calculus so that all pieces of the action are in the AE (axially extended) form, i.e., left–right symmetric form. Section 14 is devoted to a short review of Weyl geometry and Higgs mechanisms. In Section 15 we embed our model $\mathcal{T}$ in Weyl geometry and render it Weyl invariant. Its conformal invariant action is called $\mathcal{TW}$. In Section 16 we start discussing aspects of $\mathcal{TW}$ as a quantum theory. Section 17 is devoted to a discussion of even trace anomalies, an introduction of WZ terms and how to use the latter in order to eliminate all trace anomalies and restore the conformal symmetry at a quantum level. Section 18 is a short review based on the existing literature on renormalization and unitarity in theories similar to $\mathcal{TW}$. Finally, in Section 19, we argue about the fate of the two halves, $\mathcal{TW}_L$ and $\mathcal{TW}_R$, after a spontaneous L-R symmetry breaking and how we may recover the familiar minimal SM at lower energy.

A few appendices concern the notations, our rules for the Wick rotation, gauge fixings, an example of trace anomaly due to the scalar fields and a discussion concerning the dilaton.

2. Review of Anomalies (Without Gravity)

In order to study the chiral anomalies in the massless SM, it is not necessary to consider the full Lagrangian. The fermion kinetic terms are enough. Since anomalies are additive, it is enough to consider one type of fermion at a time. Let us consider, for instance, the classical kinetic action for a right-handed Weyl fermion coupled to an external gauge field $V_\mu = V_\mu^a T^a$, T^a being Hermitean generators $[T^a, T^b] = i f^{abc} T^c$ (in the Abelian case $T = 1$, $f^{abc} = 0$) in the fundamental representation of a compact group G:

$$S_R[V] = \int d^4 x\, i\overline{\psi_R}(\slashed{\partial} - i\slashed{V})\psi_R. \tag{3}$$

Here, ψ_R is a right-handed Weyl spinor, which can be projected out from a Dirac spinor ψ by means of the chiral projector P_R, $\psi_R = P_R \psi$ (see Appendix A for the notation). This action is invariant under the gauge transformation $\delta \psi_R = -\lambda \psi_R$ and $\delta_\lambda V_\mu = D_\mu \lambda \equiv \partial_\mu \lambda - i[V_\mu, \lambda]$, where $\lambda = \lambda^a(x) T^a$, which implies the classical conservation of the non-Abelian current $J_{R\mu}^a = \bar{\psi}_R \gamma_\mu T^a \psi_R$, i.e.,

$$(D \cdot J_R)^a \equiv (\partial^\mu \delta^{ac} + i f^{abc} V^{b\mu}) J_{R\mu}^c = 0. \tag{4}$$

This is, in general, not true after quantization. Our purpose here is to compute such violations. This type of calculation has been performed in a number of different ways. First, we focus on the perturbative method, in which we treat V as a perturbation and adopt the Feynman diagram technique.

2.1. The Perturbative Approach

The quantum effective action for this theory is given by the generating functional of the connected Green's functions of such currents in the presence of the source $V^{a\mu}$

$$W[V] = W[0] + \sum_{n=1}^{\infty} \frac{i^{n-1}}{n!} \int \prod_{i=1}^{n} d^4 x_i V^{a_i \mu_i}(x_i) \langle 0 | \mathcal{T} J_{R\mu_1}^{a_1}(x_1) \dots J_{R\mu_n}^{a_n}(x_n) | 0 \rangle_c \tag{5}$$

and the full 1-loop 1-point function of $J^a_{R\mu}$ is

$$\langle\langle J^a_{R\mu}(x)\rangle\rangle = \frac{\delta W[V]}{\delta V^{a\mu}(x)} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \prod_{i=1}^{n} d^4x_i\, V^{a_i\mu_i}(x_i) \langle 0|\mathcal{T} J^a_{R\mu}(x) J^{a_1}_{R\mu_1}(x_1)...J^{a_n}_{R\mu_n}(x_n)|0\rangle_c \quad (6)$$

We wish to calculate the odd parity anomaly of the divergence $D\cdot\langle\langle J^a_R\rangle\rangle$, where D represents the covariant derivative defined in (4). The RHS of (6) contains an infinite series of current amplitudes. The one-point amplitude vanishes because of translational invariance; the two-point amplitude is non-vanishing but does not contribute to the odd-parity anomaly. Therefore, the first non-trivial contribution comes from the three-point function, which, in terms of Feynman diagrams, is represented by the famous triangle diagram, whereby each side represents a fermion propagator and each vertex an interaction vertex.

Here comes the complication: The Weyl fermion propagator, the inverse of the kinetic operator $i\slashed{\partial}P_R$, simply does not exist. For P_R is a projector and it is not invertible, while $i\slashed{\partial}$ is, but contains both a right-handed and a left-handed part [It isworth spending a few words to avoid misunderstandings on this issue: this is not the non-invertibility due to the presence of some zero modes of the kinetic operator, a case which is anyhow relevant only to nonperturbative calculations and can be handled by subtracting the zero modes; in the present case, we have to do with an operator whose domain and image are different spaces and for which the eigenvalue problem is not even defined].

We can dodge this obstacle by an escamotage: instead of $i\slashed{\partial}P_R$ we invert $i\slashed{\partial}$, i.e., we replace the missing propagator with the Dirac propagator. We remark, however, that this corresponds to adding a term $i\psi_L\slashed{\partial}\psi_L$ to the integrand of (3), which corresponds to adding a *free* left-handed Weyl fermion. Whether we have to pay the price or not for this escamotage, it allows us to proceed in the calculation. In momentum space, the propagator of a fermion with momentum p_μ is $\frac{i}{\slashed{p}+i\epsilon}$, and the vertices have the form $i\gamma_\mu P_R T^a$. We associate an entering momentum q to the current $j^a_{R\mu}T^a$ to be differentiated and exiting momenta k_1 and k_2 to $j^b_{R\lambda}T^b$ and $j^c_{R\rho}T^c$; of course, $q = k_1 + k_2$. The three-current amplitude factorizes into the $\text{Tr}(T^aT^bT^c)$ factor times the amplitude for three Abelian currents. We forget for the moment the trace factor and proceed with the computation of the Abelian amplitude. In momentum space, it is

$$q^\mu \widetilde{F}^{(R)}_{\mu\lambda\rho}(k_1,k_2) = \int \frac{d^4p}{(2\pi)^4} \text{tr}\left\{ \frac{1}{\slashed{p}}\gamma_\lambda \frac{1+\gamma_5}{2} \frac{1}{\slashed{p}-\slashed{k}_1}\gamma_\rho \frac{1+\gamma_5}{2} \frac{1}{\slashed{p}-\slashed{q}}\slashed{q} \frac{1+\gamma_5}{2} \right\} \quad (7)$$

The integral over the momentum p running around the loop is UV divergent. It can be dealt with, for instance, by transforming it into a Euclidean integral by means of a Wick rotation, then dimensionally regularizing it and, finally, rotating it back to a Minkowski expression (see Appendix B for our conventions concerning Wick rotations). All of this is well known. Here, we limit ourselves to writing down the result for the odd parity part.

$$\widetilde{F}^{(R,odd)}_{\lambda\rho}(k_1,k_2) + \widetilde{F}^{(R,odd)}_{\rho\lambda}(k_2,k_1) = \frac{1}{12\pi^2}\varepsilon_{\mu\nu\lambda\rho}k_1^\mu k_2^\nu. \quad (8)$$

where the second term in the LHS comes from the crossed diagram, which has to be added according to the Feynman rules.

From (8), and ignoring again the group theoretical factor, we can obtain the final result for the purely Abelian case. It is enough to Fourier anti-transform it and insert it in (6)

$$
\begin{aligned}
\partial^\mu \langle\!\langle J_{R\mu}(x) \rangle\!\rangle &= \int \frac{d^4 q}{(2\pi)^4} e^{-iqx}(-iq^\mu) \langle\!\langle \tilde{J}_{R\mu}(q) \rangle\!\rangle \\
&= -\frac{1}{24\pi^2} \int \frac{d^4 q\, d^4 k_1\, d^4 k_2}{(2\pi)^{12}} \int d^4 y\, d^4 z\, e^{i(qx - k_1 y - k_2 z)} \delta(q - k_1 - k_2) \varepsilon_{\mu\nu\lambda\rho} k_1^\mu k_2^\nu V^\lambda(y) V^\rho(z) \\
&= \frac{1}{24\pi^2} \varepsilon_{\mu\nu\lambda\rho} \partial^\mu V^\nu(x) \partial^\lambda V^\rho(x).
\end{aligned}
\tag{9}
$$

In the Abelian case, the amplitudes (8) are all one needs in order to derive the complete result. The analogous anomaly in the non-Abelian case would require the calculation of the four-current correlators, but it can be obtained in a simpler way from the Abelian term multiplied by the group-theoretical factor, by means of the Wess–Zumino consistency conditions (see below). In the non-Abelian case, however, it is more expedient to use anti-Hermitean generators, $(T^a)^\dagger = -T^a$ and $[T^a, T^b] = f^{abc} T^c$, and from now on, we will stick to this convention (although the Hermitean convention is more familiar in particle physics) unless otherwise specified. As pointed out before, in the non-Abelian case, the three-point correlators are multiplied by

$$
\mathrm{Tr}(T^a T^b T^c) = \frac{1}{2}\mathrm{Tr}(T^a [T^b, T^c]) + \frac{1}{2}\mathrm{Tr}(T^a \{T^b, T^c\}) = f^{abc} + d^{abc}
\tag{10}
$$

where the normalization used is $\mathrm{Tr}(T^a T^b) = \delta^{ab}$. Since the three-point function is the sum of two equal pieces with $\lambda \leftrightarrow \rho, k_1 \leftrightarrow k_2$, the first term in the RHS of (10) drops out and only the second remains. For the right-handed current $J_{R\mu}^a$, we have

$$
(D \cdot \langle\!\langle J_R \rangle\!\rangle)^a = \frac{1}{24\pi^2} \varepsilon_{\mu\nu\lambda\rho} \mathrm{Tr}\left[T^a \partial^\mu \left(V^\nu \partial^\lambda V^\rho + \frac{1}{2} V^\nu V^\lambda V^\rho \right) \right].
\tag{11}
$$

Remark 1. *If the fermion is left-handed, instead of right-handed, the anomalies (9) and (11) simply change sign. It follows that for a Dirac fermion, which can be written as the sum of a left-handed and a right-handed Weyl fermion, the consistent gauge anomalies vanish. The same is true for a Majorana fermion, which can also be expressed as the sum of a left-handed and a right-handed Weyl fermion.*

These anomalies affecting Weyl fermions (the consistent chiral gauge anomalies) are dangerous, in fact deadly, for a theory. A simple example of such a catastrophe is represented by a right-handed fermion coupled to an Abelian gauge field, thus with the anomaly (9). To see it, recall that the Lorentz invariant quantum theory of a gauge vector field inevitably involves a Fock space of states with indefinite norm. In order to select a physical Hilbert subspace of the Fock space, a subsidiary condition is necessary. In the Abelian case, when the fermion current satisfies the continuity equation, the equation of motion leads to $\Box(\partial \cdot V) = 0$, so that a subspace of states of non-negative norm can be selected through the auxiliary condition

$$
\partial \cdot V^{(-)}(x) |\mathrm{phys}\rangle = 0
\tag{12}
$$

$V^{(-)}(x)$ being the annihilation operator, the positive frequency part of the quantum field. On the contrary, in the above-mentioned chiral model, one finds

$$
\Box(\partial \cdot V) \sim \varepsilon_{\mu\nu\lambda\rho} \partial^\mu V^\nu(x) \partial^\lambda V^\rho(x) \neq 0
\tag{13}
$$

This is an obstruction to selecting a physical subspace of states with non-negative norm.

2.2. Bardeen's Method

Equations (9) and (11) are the general form of the consistent Abelian and non-Abelian chiral gauge anomalies in 4d. But before discussing the Wess–Zumino consistency conditions, let us return to the question of the non-existence of the Weyl propagator, which we have bypassed above by adding to the action of a right-handed Weyl fermion interacting with a gauge potential the action of a left-handed non-interacting fermion. There is another way to overcome this obstacle, Bardeen's way. It consists of a stratagem often resorted to in field theory: when a calculation is too contrived, one had better enlarge the parameter space of the theory. Bardeen's way, ref. [16], consists in adding new degrees of freedom by considering a theory of Dirac fermions coupled to a vector potential as well as to an axial one, making the calculation in such a larger setup and taking a specific limit to return to the initial configuration. Thus, let us consider a theory of Dirac fermions coupled to non-Abelian vector and axial potential

$$S_B[V, A] = \int d^4x i \overline{\psi} \mathcal{D} \psi. \tag{14}$$

where $\mathcal{D} = \partial\!\!\!/ + V\!\!\!\!/ + \gamma_5 A\!\!\!\!/$. The evident advantage is that now the propagator exists and, perturbatively, corresponds to the usual Dirac propagator. Moreover, taking the limit $V \to V/2, A \to V/2$ in (14) we recover the action of a right-handed Weyl fermion coupled to V_μ, plus the action of a free left-handed Weyl fermion.

The action (14) is invariant under the gauge transformations

$$\delta\psi = -(\lambda + \rho\gamma_5)\psi \tag{15}$$

and

$$\delta V_\mu = \partial_\mu \lambda + [V_\mu, \lambda] + [A_\mu, \rho], \qquad \delta A_\mu = \partial_\mu \rho + [V_\mu, \rho] + [A_\mu, \lambda] \tag{16}$$

where the local functions λ and ρ are Lie algebra-valued, $\lambda = \lambda^a(x)T^a, \rho = \rho^a(x)T^a$. This implies the classical conservation of the vector and axial current, $j_\mu^a = i\overline{\psi}\gamma_\mu T^a\psi, j_{5\mu}^a = i\overline{\psi}\gamma_\mu\gamma_5 T^a\psi$

$$(D_V^\mu j_\mu + [A^\mu, j_{5\mu}])^a = 0, \qquad (D_V^\mu j_{5\mu} + [A^\mu, j_\mu])^a = 0 \tag{17}$$

where D_V denotes the covariant derivative with respect to the vector potential.

To compute the anomalies in this case, we can proceed as above, via Feynman diagrams, i.e., compute the triangle diagrams $\langle j_5 jj \rangle$ and $\langle j_5 j_5 j_5 \rangle$ (the other distinct three-current amplitudes cannot contribute to the odd parity anomaly), determine the lowest order contribution to the anomaly and then use the consistency conditions to reconstruct its full expression. It can be performed, but it certainly is not the quickest way. Another possibility is to use a non-perturbative method, see below. Here, we anticipate the result. After subtracting trivial terms, if need be, we obtain vector current conservation

$$\left(D_V^\mu \langle\!\langle j_\mu(x) \rangle\!\rangle + [A^\mu(x), \langle\!\langle j_{5\mu}(x) \rangle\!\rangle] \right)^a = 0 \tag{18}$$

while the axial conservation equation is anomalous:

$$\left(D_V^\mu \langle\!\langle j_{5\mu}(x) \rangle\!\rangle + [A^\mu(x), \langle\!\langle j_\mu(x) \rangle\!\rangle] \right)^a = \frac{1}{4\pi^2} \varepsilon_{\mu\nu\lambda\rho} \text{tr} \left[T^a \left(\frac{1}{4} F_V^{\mu\nu} F_V^{\lambda\rho} + \frac{1}{12} F_A^{\mu\nu} F_A^{\lambda\rho} \right. \right.$$
$$\left. \left. - \frac{1}{6} F_V^{\mu\nu} A^\lambda A^\rho - \frac{1}{6} A^\mu A^\nu F_V^{\lambda\rho} - \frac{2}{3} A^\mu F_V^{\nu\lambda} A^\rho - \frac{1}{3} A^\mu A^\nu A^\lambda A^\rho \right) \right] \tag{19}$$

where $F_V^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu + [V^\mu, V^\nu]$, and $F_A^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + [V^\mu, A^\nu] + [A^\mu, V^\nu]$. This is Bardeen's anomaly.

From this expression, we can derive two results in particular. Setting $A_\mu = 0$, we obtain the so-called covariant or ABJ anomaly

$$[D_V^\mu j_{5\mu}]^a = \frac{1}{16\pi^2} \varepsilon_{\mu\nu\lambda\rho} \mathrm{tr}\left(T^a V^{\mu\nu} V^{\lambda\rho}\right) \tag{20}$$

Taking the chiral limit $V \to \frac{V}{2}, A \to \frac{V}{2}$, and summing (18) and (19), we obtain

$$[D_{V\mu} j_R^\mu]^a = \frac{1}{24\pi^2} \varepsilon_{\mu\nu\lambda\rho} \mathrm{tr}\left[T^a \partial^\mu \left(V^\nu \partial^\lambda V^\rho + \frac{1}{2} V^\nu V^\lambda V^\rho\right)\right] \tag{21}$$

where $j_{R\mu}^a = i\bar{\psi}_R \gamma_\mu T^a \psi_R$ with $\psi_R = \frac{1+\gamma_5}{2}\psi$, which is the same consistent non-Abelian gauge anomaly as (11).

Remark 2. *It is worth recalling that both (19) and (21) satisfy the appropriate WZ consistency conditions (see below), while (20) does not. The latter is called covariant because the RHS transforms in the adjoint representation of the gauge Lie algebra. Of course, the two anomalies (21) and (20) collapse to the same expression in the Abelian case, but the coefficients in front of them are always different.*

Bardeen's anomaly, the covariant anomaly (20) and the consistent anomaly (21) are all proportional to the third-order ad-invariant symmetric tensor $d^{abc} = \frac{1}{2}\mathrm{tr}\left(T^a\{T^b, T^c\}\right)$. Therefore, they all vanish when the latter vanishes.

The covariant anomaly (20) is the anomaly of the chiral current $j_{5\mu}^a$ in a theory of a Dirac fermion coupled to a vector potential in response to the gauge variation $\delta\psi = \rho\gamma_5\psi$. This anomaly cannot show up in the internal lines of Feynman diagrams; therefore, it cannot endanger unitarity or renormalizability. [We reserve the term ABJ anomaly only to this type. It is rather frequent in the literature (and a source of confusion) to use the ABJ tag for any kind of chiral anomaly, consistent or covariant].

2.3. WZ Consistency Conditions and BRST Cohomology

At this point, we owe the reader a few explanations. We have already mentioned several times consistent and covariant anomalies and their vanishing. It is time to clarify this terminology. The gauge invariance of the effective action can be expressed by means of the functional operator $X^a(x)$ defined by

$$X^a(x) = \partial_\mu \frac{\delta}{\delta V_\mu^a(x)} + f^{abc} V_\mu^b(x) \frac{\delta}{\delta V_\mu^c(x)}, \tag{22}$$

as follows

$$X^a(x) W[V] = 0 \tag{23}$$

This is the full one-loop Ward identity (WI) for gauge symmetry. It can be obtained by expressing the invariance of the effective action as follows

$$0 = \delta_\lambda W[V] = \int d^d x\, \delta_\lambda V_\mu^a(x) \frac{\delta}{\delta V_\mu^a(x)} W[V], \tag{24}$$

where we integrate by parts and note that $\lambda^a(x)$ are arbitrary functions. Equation (23) is equivalent to the one-loop covariant conservation law

$$[D^\mu \langle\langle j_\mu(x) \rangle\rangle]^a = 0 \tag{25}$$

In a number of cases, the WI (23) is violated

$$X^a(x)W[V] = \kappa\Delta^a(x) \tag{26}$$

where $\Delta^a(x)$ is a local expressions of the fields and κ is a small dimensionless expansion parameter for the one-loop calculation. From now on, for simplicity, we shall set it = 1.

Applying $X^b(y)$ to both sides of (26) and then inverting the order of the two operations, we find a remarkable relation of group-theoretical nature

$$X^a(x)\Delta^b(y) - X^b(y)\Delta^a(x) + f^{abc}\Delta^c(x)\delta(x-y) = 0, \tag{27}$$

that $\Delta^a(x)$ must satisfy. These are the Wess–Zumino (WZ) consistency conditions.

Now, it may happen that

$$\Delta^a(x) = X^a(x)\mathcal{C} \tag{28}$$

where $\mathcal{C}$ is the spacetime integral of a local expression of $V^a(x)$, in short, a local counterterm. In this case, we can redefine the effective action as $W'[V] = W[V] - \mathcal{C}$ so that

$$X^a(x)W'[V] = 0 \tag{29}$$

and recover invariance. If (28) is not possible for any local counterterm, then $\Delta^a(x)$ is a true anomaly. Saturating $\Delta^a(x)$ with $\lambda^a(x)$ and integrating over spacetime, we obtain the integrated anomaly

$$\mathcal{A}_\lambda = \int d^d x \lambda^a(x)\Delta^a(x) \tag{30}$$

where the sum over a is understood.

The WZ consistency condition (27) is not very simple to be verified in practice. It can, however, be recast into a form that is easier to deal with and, at the same time, it establishes a well-defined mathematical formulation by means of the BRST formalism. In QFT the BRST formalism is formulated by promoting the gauge variables to anticommuting fields, the FP ghosts. Here, we do the same by promoting the gauge parameters $\lambda^a(x)$ to anticommuting fields $c^a(x)$: $\delta_c V_\mu = \partial_\mu c + [V_\mu, c]$, and endowing them with the transformation properties

$$\delta_c c^a = -\frac{1}{2}f^{abc}c^b c^c, \qquad \text{or} \qquad \delta_c c = -\frac{1}{2}[c, c] \tag{31}$$

Then it is easy to see that the functional operator

$$\delta_c = \int d^d x \left(\delta_c V_\mu^a(x) \frac{\partial}{\partial V_\mu^a(x)} + \delta_c c^a(x) \frac{\partial}{\partial c^a(x)} \right). \tag{32}$$

is nilpotent

$$\delta_c^2 = 0 \tag{33}$$

Using this, we can rewrite the WZ consistency condition as follows

$$\delta_c \mathcal{A}_c = 0 \tag{34}$$

In order to recover (27), one can simply differentiate (34) with respect to $c^a(x)$ and $c^b(y)$. In this language, the anomalous WI (26) can be rewritten

$$\delta_c W[V] = \mathcal{A}_c, \tag{35}$$

from which (34) is obtained by simply applying δ_c to both sides. Similarly, we can rewrite the triviality condition (28) as

$$\mathcal{A}_c = \delta_c \mathcal{C} \tag{36}$$

What we have defined in this way is a cohomology problem. δ_c is the coboundary operator of this problem. The cochain space is the space of local polynomials in V_μ^a and c^a and their derivatives with the right canonical dimension. A cochain satisfying (35) is a cocycle, and a cocycle satisfying (36) is a coboundary. In this way, we have transformed the anomaly problem into a cohomology problem. Anomalies are non-trivial (non-exact) cocycles of the operator δ_c.

This formulation has made it possible to solve the problem of finding all the non-trivial solutions of Equation (34) for all dimensions and all gauge Lie groups.

To construct such solutions, we start from an order n symmetric polynomial in some representation of the Lie algebra, $P_n(T^{a_1}, ..., T^{a_n})$, invariant under the adjoint transformations:

$$P_n([X, T^{a_1}], ..., T^{a_n}) + ... + P_n(T^{a_1}, ..., [X, T^{a_n}]) = 0 \tag{37}$$

for any element X of a Lie algebra $\mathfrak{g}$. In many cases, these polynomials are symmetric traces of the generators in the corresponding representation

$$P_n(T^{a_1}, ..., T^{a_n}) = Str(T^{a_1}...T^{a_n}) = d^{a_1...a_n} \tag{38}$$

(*Str* stands for symmetric trace). For instance

$$Tr(T^a T^b) = c_2(R)\delta^{ab} \tag{39}$$

$$\frac{1}{2}Tr(T^a\{T^b, T^c\}) = c_3(R)d^{abc} \tag{40}$$

where $c_2(R), c_3(R)$ are representation-dependent numerical coefficients. As these two examples show, the tensors $d^{a_1...a_n}$ are universal; they are characteristic of the Lie algebra; changing the representation only modifies the numerical coefficients in front of them.

Now, it is convenient to use the differential form notation: $V = V_\mu dx^\mu = V_\mu^a T^a dx^\mu$ and $F = dV + V \wedge V = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$, where d is the exterior differential operator $d = \frac{\partial}{\partial x^\mu}dx^\mu$ and $F^{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu + [V_\mu, V_\nu]$. Then, one can construct the 2n-form

$$\Delta_{2n}(V) = P_n(F, F, ... F) \tag{41}$$

In this expression, the product of forms is understood to be the exterior product. It is easy to prove [17], that

$$P_n(F, F, ... F) = d\left(n \int_0^1 dt P_n(V, F_t, ..., F_t)\right) = d\Delta_{2n-1}^{(0)}(V) \tag{42}$$

where we have introduced the symbols $V_t = tV$ and its curvature $F_t = dV_t + \frac{1}{2}[V_t, V_t]$, where $0 \leq t \leq 1$. One can prove Equation (42) by noting that $\frac{d}{dt}F_t = dV + [V_t, V] \equiv d_{V_t}V$

and $d_{V_t}F_t = 0$, and exploiting the symmetry of P_n and the graded commutativity of the exterior product of forms.

Equation (42) is the first of a sequence of equations that can be shown to hold

$$\Delta_{2n}(V) - d\Delta_{2n-1}^{(0)}(V) = 0 \tag{43}$$

$$\delta_c \Delta_{2n-1}^{(0)}(V) - d\Delta_{2n-2}^{(1)}(V,c) = 0 \tag{44}$$

$$\delta_c \Delta_{2n-2}^{(1)}(V,c) - d\Delta_{2n-3}^{(2)}(V,c) = 0 \tag{45}$$

$$\cdots \cdots$$

$$\delta_c \Delta_0^{(2n-1)}(c) = 0 \tag{46}$$

These are known as descent equations. All the expressions $\Delta_k^{(p)}(V,c)$ are polynomials of c, dc, V, dV and their commutators, whose explicit forms are known. The lower index k is the form degree, and the upper one p is the ghost number, i.e., the number of c factors. The last polynomial $\Delta_0^{(2n-1)}(c)$ is a 0-form and a function of c alone. What matters here is that the $\Delta_{2n-2}^{(1)}(V,c)$ is the general expression of the consistent gauge anomaly in d $= 2n - 2$ dimensions; for integrating (45) over spacetime, one gets

$$\delta_c \mathcal{A}_c[V] \;=\; 0 \tag{47}$$

$$\mathcal{A}_c[V] \;=\; \int d^{\mathsf{d}}x \Delta_{\mathsf{d}}^1(c,V), \qquad \text{where}$$

$$\Delta_{\mathsf{d}}^{(1)}(c,V) \;=\; n(n-1)\int_0^1 dt(1-t)P_n(dc,V,F_t,\dots F_t) \tag{48}$$

$\mathcal{A}_c[V]$ identifies the anomaly up to an overall numerical coefficient. If $n = 3$, i.e., d $= 4$, is chosen, the RHS of (48) is proportional to the RHS of (11) or (21) multiplied by c^a and summed over a. This is a sleek way to show that the chiral anomaly (21) satisfies the WZ consistency conditions.

Thus, the existence of chiral gauge anomalies relies on the existence of the adjoint-invariant symmetric polynomials P_n. The latter are completely classified. The space of symmetric ad-invariant polynomials of the simple group G is usually denoted by $\mathbb{I}(\mathsf{G})$. It is a commutative algebra. If G has rank r, then $\mathbb{I}(\mathsf{G})$ has r algebraically independent generators of order $m_1, \dots, m_r$. In other words, we have r algebraically independent polynomials P_n, or symmetric tensors $d^{a_1 \dots a_r}$, of order $m_1, \dots, m_r$. The values $m_1, \dots, m_r$ for the simple Lie algebras are as follows

$$
\begin{array}{cl}
Lie \quad algebra & m_1, \dots, m_r \\
A_r & 2, 3, \dots r+1 \\
B_r & 2, 4, \dots, 2r \\
C_r & 2, 4, \dots, 2r \\
D_r & 2, 4, \dots 2r-2, r \\
G_2 & 2, 6 \\
F_4 & 2, 6, 8, 12 \\
E_6 & 2, 5, 6, 8, 9, 12 \\
E_7 & 2, 6, 8, 10, 12, 14, 18 \\
E_8 & 2, 8, 12, 14, 18, 20, 24, 30
\end{array}
\tag{49}
$$

For product groups, we have simply to consider all possible products of the corresponding symmetric polynomials. When the gauge group contains a $U(1)$ factor, the general form (41) can contain as factors any products of the Abelian curvature two-form F.

The previous table contains a lot of information about anomalies. If a symmetric tensor is absent in this table, the corresponding cocycle, and thus the corresponding anomaly, does not exist. This is the case, for instance, for SU(2) in 4d. The corresponding Lie algebra is A_1, which has only the second-order symmetric tensor, while in order to construct a consistent chiral anomaly in 4d, one needs the third-order tensor. Analogously, the third-order symmetric tensor does not exist for D_2, which is the Lie algebra of SO(4), the compact version of the Lorentz group in 4d. But the Lorentz Lie algebra generators are the same as for SO(4), except that a few of them are multiplied by i. In fact, the local Lorentz symmetry is not anomalous in 4d.

Other examples of vanishing anomalies may come up even when the adjoint-invariant symmetric tensor in question is non-vanishing, but the fermion representations are real. If a representation is real $T^{a\dagger} = -T^a$. For instance in 4d, taking the hermitean conjugate of d^{abc} we find $d^{abc} = -d^{abc}$, therefore $d^{abc} = 0$.

2.4. Gauge Anomaly Cancelation in the SM

At this point, we have enough information to discuss the cancelation of gauge anomalies in the MSM Table 1.

Table 1. MSM gauge.

- $T^{\mathfrak{su}(3)} \times T^{\mathfrak{su}(3)} \times T^{\mathfrak{su}(3)}$: there are two left-handed and two right-handed triplets whose anomalies cancel one another.
- $T^{\mathfrak{su}(2)} \times T^{\mathfrak{su}(2)} \times T^{\mathfrak{su}(2)}$, which vanishes because the tensor d^{abc} vanishes in general for the Lie algebra $\mathfrak{su}(2)$.
- $T^{\mathfrak{su}(2)} \times T^{\mathfrak{su}(2)} \times T^{\mathfrak{u}(1)}$, in which case we have the trace of two $\mathfrak{su}(2)$ generators in two doublet representations. These traces are non-vanishing because $\mathrm{tr}(T^a T^b) \sim \delta^{ab}$, but they are multiplied by the corresponding $\mathfrak{u}(1)$ charges, whose total value is $6\left(\frac{1}{6}\right) - 2\left(\frac{1}{2}\right) = 0$.
- $T^{\mathfrak{su}(3)} \times T^{\mathfrak{su}(3)} \times T^{\mathfrak{u}(1)}$, in which case we have the trace of two $\mathfrak{su}(3)$ left triplet generators and two right triplet generators. These traces are again non-vanishing, but they are multiplied by the corresponding $\mathfrak{u}(1)$ charge, whose total value is $3\left(2\left(\frac{1}{6}\right) - \frac{2}{3} + \frac{1}{3}\right) = 0$.
- $T^{\mathfrak{u}(1)} \times T^{\mathfrak{u}(1)} \times T^{\mathfrak{u}(1)}$; in this case, the tensor is proportional to the overall sum of the charge products:
$$6\left(\frac{1}{6}\right)^3 - 3\left(\frac{2}{3}\right)^3 - 3\left(-\frac{1}{3}\right)^3 + 2\left(-\frac{1}{2}\right)^3 - (-1)^3 = 0.$$

As we know, the chiral consistent anomaly is determined by the tensor $d^{abc} = \frac{1}{2}\mathrm{tr}\left(T^a\{T^b, T^c\}\right)$, where T^a denotes the total antihermitean generator of the Lie algebra $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$: dropping labels we can generically write $T = T^{\mathfrak{su}(3)} \oplus T^{\mathfrak{su}(2)} \oplus T^{\mathfrak{u}(1)}$. The tensor d^{abc} decomposes into various independent components, which we list hereafter:

This completes the analysis of gauge anomalies in the standard model. There are no local anomalies in a flat background theory.

The results illustrated so far are generally considered completely satisfactory: the SM is anomaly-free, that is that, and let us move on to other things, notably to its renormalization or to its physics. But there are aspects of the analysis carried out so far that deserve more attention and, in view of the coupling of the SM to gravity, would be superficial to ignore. First, as already promised, we need a non-perturbative method to justify the results shown above beyond the perturbative level. This will turn out to be indispensable for trace anomalies. Second, we need a deeper understanding of why consistent chiral gauge anomalies are so deadly. The obstruction to selecting a physical Hilbert space for a quantum

field in the quantization of a Weyl fermion coupled to a gauge potential is a consequence rather than the origin of the problem. More insight is necessary. Refs.: [3,5,16,18–43].

3. Non-Perturbative Approach: The Heat-Kernel-Like Methods

There are several different non-perturbative (a.k.a. functional) methods to calculate anomalies: Schwinger's proper-time method, the Seeley–DeWitt and the zeta-function regularization method and Fujikawa's method. We comprise them under the term of heat-kernel-like methods. The purpose of all of them is to represent in a mathematically affordable way the path integral corresponding to the quadratic part of a theory, and in particular of a fermionic theory, interacting with external potentials: the central object is, therefore, the determinant of the full kinetic operator, let us call it D, (or the square thereof in the case of Dirac-like operators). It is crucial for these methods to work that the operator in question, after passing from a Minkowski to a Euclidean background metric, be a quadratic elliptic self-adjoint operator. Below, we give a synthetic presentation of one of them, which we call SDW (Schwinger–DeWitt).

Our problem is to represent $\log \det D = \mathrm{Tr}\log D$. We start with a familiar formula in theoretical physics

$$\frac{i}{D + i\epsilon} = \int_0^\infty ds\, e^{is(D+i\epsilon)} \tag{50}$$

with infinitesimal $\epsilon > 0$. It certainly holds when D is a real number . We extend it to an operator D by representing it via its eigenvalues. By formally integrating, we write

$$\mathrm{Tr}\log D = -i\int_0^\infty \frac{ds}{is}\mathrm{Tr}\left(e^{iDs}\right) + \mathrm{const} \tag{51}$$

where the $i\epsilon$ prescription is understood, as we will do from now on for the s integrals.

The Dirac operator $\slashed{D} = i\gamma^\mu \partial_\mu$, even after the addition of a vector and an axial potential, in a Euclidean background, is an example of an elliptic operator, but the application of the SDW method to it in the linear form is not known. The best we can do is to apply this method to its square and extract the square root of the determinant. The choice of the square Dirac operator is not always univocal and requires some care. The minimal precaution is that it preserve the same symmetries as the linear operator and become a self-adjoint operator after a Wick rotation, see below. Let us call $\mathcal{F}$ such a square Dirac operator.

We identify the effective action for Dirac fermions with

$$W = -\frac{i}{2}\mathrm{Tr}(\ln\mathcal{F}) \tag{52}$$

Tr includes all the traces plus the spacetime integration. Any variation of (52) is given by

$$\delta W = \frac{i}{2}\mathrm{Tr}(\mathcal{G}\delta\mathcal{F}) \tag{53}$$

where, in symbols, $\mathcal{F}\mathcal{G} = -1$. Thus we can formally write

$$\delta W = -\frac{1}{2}\mathrm{Tr}\left(\int_0^\infty ds\, e^{i\mathcal{F}s}\delta\mathcal{F}\right). \tag{54}$$

and, like in (51), W can be represented as

$$W = -\frac{1}{2}\int_0^\infty \frac{ds}{is}\mathrm{Tr}\left(e^{i\mathcal{F}s}\right) + \mathrm{const} \equiv \int d^d x L(x) + \mathrm{const}, \tag{55}$$

where

$$L(x) = -\frac{1}{2} \lim_{x' \to x} \text{tr} \int_0^\infty \frac{ds}{is} K(x, x', s),$$

(56)

and the kernel K is defined by

$$K(x, x', s) = \langle x | e^{i\mathcal{F}s} | x' \rangle$$

(57)

Inserted under the symbol Tr, (56) means integrating over x after taking the limit $x' \to x$.

Remark 3. *We recall that in all the above s-integrals the $i\epsilon$ prescription, i.e., the substitution $\mathcal{F} \to \mathcal{F} + i\epsilon$, is understood. It is clear that such integrals converge only if $\mathcal{F}$ is a self-adjoint operator. This is possible in a Euclidean background, i.e, after a Wick rotation. In the sequel, we make the substitution $\mathcal{F} \to \widetilde{\mathcal{F}}$, where the tilde represents a Wick rotation, and anti-Wick-rotate is the final result. However, for the sake of simplicity, we understand this substitution and continue with the Minkowski notation.*

The SDW method proceeds with the definition of the amplitude

$$\langle x, s | x', 0 \rangle = \langle x | e^{i\mathcal{F}s} | x' \rangle,$$

(58)

which satisfies the (heat kernel) differential equation

$$i\frac{\partial}{\partial s} \langle x, s | x', 0 \rangle = -\mathcal{F}_x \langle x, s | x', 0 \rangle,$$

(59)

$\mathcal{F}_x$ is a quadratic elliptic differential operator, typically, $\mathcal{F}_x = \eta^{\mu\nu}\partial_\mu\partial_\nu + \ldots$, where ellipses denote non-leading terms.

The SDW method now introduces an ansatz for the amplitude (58)

$$\langle x, s | x', 0 \rangle = -\lim_{m \to 0} \frac{i}{(4\pi s)^{\frac{d}{2}}} e^{i\left(\frac{(x-x')^2}{4s} - m^2 s\right)} \Phi(x, x', s),$$

(60)

$\Phi(x, x', s)$ is a function to be determined. The mass parameter m is introduced to guarantee convergence for the s integration. In the limit $s \to 0$, the RHS of (60) becomes the definition of a delta function multiplied by Φ. More precisely, since it must be $\langle x, 0 | x', 0 \rangle = \delta(x, x')$ and

$$\lim_{s \to 0} \frac{i}{(4\pi s)^{\frac{d}{2}}} e^{i\left(\frac{(x-x')^2}{4s} - m^2 s\right)} = \delta^{(\text{d})}(x, x'),$$

(61)

we must have

$$\lim_{s \to 0} \Phi(x, x', s) = \mathbf{1}.$$

(62)

Looking at (60), in dimension d, we can make the identification

$$K(x, x, s) = \frac{i}{(4\pi i s)^{\frac{d}{2}}} \sqrt{g} e^{-im^2 s} [\Phi](x, x, s).$$

(63)

From now on, the symbol $[H](x, \ldots)$ will denote the coincidence limit $\lim_{x' \to x} H(x, x', \ldots)$.

The heat-kernel Equation (59) becomes a quadratic partial differential equation for Φ in x_μ and s. To find useful solutions, one has to expand Φ as a series of s:

$$\Phi(x, x', s) = \sum_{n=0}^{\infty} a_n(x, x')(is)^n \tag{64}$$

so that (59) becomes a series of recursive differential relations for the coefficients a_n. Starting with the position $[a_0] = 1$ they permit us to compute the other coefficients up to the relevant order (which, of course, depends on the dimension d).

Bardeen's Anomaly with SDW

Let us apply the just illustrated method to the calculation of Bardeen's anomaly in d = 4. In this case, the relevant operator is the square of $\mathcal{D} = \slashed{\partial} + \slashed{V} + \gamma_5 \slashed{A}$, which has the properties

$$\mathcal{D}^\dagger = \gamma_0 \mathcal{D} \gamma_0, \qquad \left(\mathcal{D}^2\right)^\dagger = \gamma_0 \mathcal{D}^2 \gamma_0, \tag{65}$$

but after a Wick rotation (see Appendix B for conventions) we get

$$\tilde{\mathcal{D}}^\dagger = -\tilde{\mathcal{D}}, \qquad \left(\tilde{\mathcal{D}}^2\right)^\dagger = \tilde{\mathcal{D}}^2 \tag{66}$$

The relevant quadratic Dirac operator is

$$\mathcal{D}^2 = -\eta^{\mu\nu} \overline{\mathscr{D}}_\mu \mathscr{D}_\nu - \Sigma^{\mu\nu} \left(\mathcal{B}_{\mu\nu} + \left(\mathcal{C}_{\mu\nu} + 2(A_\nu \partial_\mu - A_\mu \partial_\nu) \right) \gamma_5 \right) \tag{67}$$

where $\Sigma_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu]$, $\overline{\mathscr{D}}_\mu$ is $\mathscr{D}_\mu$ with the change $\gamma_5 \to -\gamma_5$, and

$$\mathcal{B}_{\mu\nu} = F_{V\mu\nu} - [A_\mu, A_\nu], \qquad \mathcal{C}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \{V_\mu, A_\nu\} - \{A_\mu, V_\nu\} \tag{68}$$

We use also the notation

$$\mathcal{E} = \Sigma^{\mu\nu} \mathcal{B}_{\mu\nu} + \gamma_5 \Sigma^{\mu\nu} \mathcal{C}_{\mu\nu}, \tag{69}$$

with similar expressions for the Euclidean version.

Under the transformations (16), we find

$$\delta_\lambda \mathcal{D} = [\mathcal{D}, \lambda], \qquad \delta_\rho \mathcal{D} = [\mathcal{D}, \rho] + 2\gamma_5 \rho \mathcal{D} \tag{70}$$

The relevant operator is not invariant under the ρ transformation. Therefore an anomaly is to be expected in the divergence of the chiral current. For taking the variation with respect to ρ of the path integral Z, we obtain, formally,

$$\delta_\rho \ln Z = 2i \text{Tr}(\gamma_5 \rho) \equiv \mathcal{A}^{(unreg)}, \tag{71}$$

where Tr includes all traces plus spacetime integration. $\mathcal{A}^{(unreg)}$ is the unregulated anomaly. We regularize it with the Schwinger proper time approach. This means using the procedure outlined before with $\mathcal{F} = -\mathcal{D}^2$ with regard in particular Equation (54) [The square Dirac operator must respect the properties of the linear one. For instance, in the present case, an alternative quadratic operator would be $\overline{\mathcal{D}}\mathcal{D}$, with the transformation property $\delta_Y \left(\overline{\mathcal{D}}\mathcal{D} \right) = [\overline{\mathcal{D}}\mathcal{D}, Y]$, with $Y = \lambda + \gamma_5 \rho$, which is also an elliptic operator when turned Euclidean.

This would exclude any anomaly. But this would contradict the linear transformation $\delta_Y \mathscr{D} = [\mathscr{D}, Y] + 2\gamma_5 \rho \mathscr{D}$],

$$
\begin{aligned}
\mathcal{A} \equiv \delta_\rho W &= \frac{1}{2}\mathsf{Tr}\left(\int_0^\infty ds e^{-i\mathscr{D}^2 s}\delta_\rho \mathscr{D}^2\right) = 2i\mathsf{Tr}\left(\rho\gamma_5 \int_0^\infty ds e^{-i\mathscr{D}^2 s}\mathscr{D}^2\right) \\
&= -2\mathsf{Tr}\left(\rho\gamma_5 \int_0^\infty ds \frac{\partial}{\partial s}e^{-i\mathscr{D}^2 s}\right) = 2\,\mathsf{Tr}\left(\rho\gamma_5 e^{-i\mathscr{D}^2 s}\right)\bigg|_{s=0}
\end{aligned}
\tag{72}
$$

In the last integration convergence at infinity is guaranteed by the factor $e^{-im^2 s}$ (with the prescription $m^2 \to m^2 - i\epsilon$). A remark is in order here: looking at Equations (70) and (72), we note that these manipulations are possible because the operator $\mathscr{D}$ in it commutes with $e^{i\mathscr{D}^2 s}$.

Now, we write (72) in coordinate representation

$$
\mathcal{A} = -2\lim_{x'\to x}\mathsf{Tr}\left(\rho\gamma_5\langle x|e^{-i\mathscr{D}^2 s}|x'\rangle\right)\bigg|_{s=0}
\tag{73}
$$

and insert

$$
\langle x, s|x', 0\rangle = \lim_{m\to 0}\frac{i}{(4\pi s)^2}e^{i\left(\frac{(x-x')^2}{4s} - m^2 s\right)}\Phi(x, x', s),
\tag{74}
$$

Next, we expand $\Phi(x, x', s)$ in powers of is, as in (64), and integrate over s. Setting $s = 0$ suppresses all higher order terms in the expansion, except the 0-th, first and second order term. The 0-th order term is annihilated by the trace over γ_5. The first order term, involving $[a_1]$, diverges for $s \to 0$. $[a_1]$ can be explicitly calculated: its odd parity part vanishes due to the gamma matrix trace, while the even parity part survives the tracing but is trivial and can be subtracted with a counterterm.

Therefore, the finite part of (73) becomes

$$
\mathcal{A} = -\frac{2i}{16\pi^2}\int d^4 x \mathsf{Tr}(\gamma_5\rho[a_2](x)), \qquad [a_2(x)] = \lim_{x'\to x}a_2(x, x')
\tag{75}
$$

To compute $[a_2]$, we resort to the above-mentioned recursive relation, which is obtained from the heat-kernel equation:

$$
(n+1)a_{n+1} + (x-x')^\mu\left(\mathcal{D}_\mu a_{n+1} + 2\gamma_5\Sigma^{\mu\nu}A_\nu a_{n+1}\right) - \overline{\mathscr{D}}^\mu\mathscr{D}_\mu a_n - 4\gamma_5\Sigma^{\mu\nu}A_\nu\partial_\nu a_n - \mathcal{E}a_n = 0
\tag{76}
$$

Starting from $[a_0] = 1$, one can derive $[a_1]$ and $[a_2]$. The calculation is lengthy, but it is not hard to extract the odd parity part of the trace in (75). It coincides with the RHS of (19) saturated with ρ^a and integrated over spacetime (Alternatively, one can arrive directly at the anomaly (21) by applying the SDW method to the operator (94) below, see [5]). Refs.: [5,35,44–52].

4. Anomalies as Obstructions

Non-perturbative methods, like the one just illustrated, are very useful because they yield the full expressions of consistent anomalies, not just their lowest-order approximations. This is extremely useful in cases where the lowest order terms are not enough to unambiguously determine the anomalies. However, returning to the discussion on fermion propagators after Equation (6), they do not differ substantially from the perturbative methods in that they rely on fermion propagators, which are obtained from the original Weyl theory by adding a free Weyl fermion of opposite chirality (or resorting to Dirac fermions, like in Bardeen's approach). The aim of this section is to illustrate a different approach in

which there is no need to introduce additional degrees of freedom. In this new approach, anomalies appear as obstructions to the existence of fermion propagators. The method is based on the family's index theorem of Atiyah and Singer. The new approach requires quite a lot of preliminary material. Here, we try to squeeze it into a few pages. In this section, the framework is entirely Euclidean. We will assume that our conclusions can be transferred to the corresponding Minkowski formulation.

4.1. A Geometrical Description of Anomalies

The first element we need is a geometrical description of consistent gauge anomalies. The algebraic formalism of Section 2.3, from this point of view, is unsatisfactory. Looking at Equation (42) we see that in the case $n = 3$, i.e., d = 4, both sides of the equation vanish because they are forms of degree 6. One could suspect that the descent equations that follow from this one are built on sand. It is not so, but an appropriate elaboration is no doubt required. It is well known that the fitting framework for a gauge theory is provided by the geometry of principal and associated fiber bundles. The underlying geometry of a gauge theory on a spacetime M with gauge group G is grounded on a principal fiber bundle P(M, G),

$$G \hookrightarrow P$$
$$\downarrow \pi$$
$$M$$

π denotes the projection $\pi : P \to M$. The inverse image of a point in M is a fiber of P, and is a copy of G. A gauge potential $V_\mu = V_\mu^a T^a$, where T^a are the generators of $\mathfrak{g} \equiv Lie(G)$, represents a connection form $V = V_\mu dx^\mu$, with curvature is $F = dV + \frac{1}{2}[V, V]$. A connection V splits the vector fields of P into vertical and horizontal ones [$V_\mu(x)$ comes from the pull-back of $\sigma_U^* V = V_\mu dx^\mu$ by a local section $\sigma_U : U \to P$, where $\pi \circ \sigma_U(x) = x$ for $\forall x \in U \subset M$], and is properly defined on the total space P. Thus, the six-form polynomial $P_3(F, F, F)$ is also defined on P, whose dimension is d times the group dimension. This, however, does not prevent this form from vanishing because it is an invariant form that descends on M (a basic form) and thus cannot but vanish.

The geometry of the total space is not yet enough. We have to enlarge our horizons.

In this setup, gauge transformations are represented by vertical automorphisms of P. An automorphism is a diffeomorphism of P, $\psi : P \to P$, such that $\psi(pg) = \psi(p)g$, for any $p \in P$ and any $g \in G$. Automorphisms form a group, which will be denoted Aut(P). For any automorphism ψ, the map $\pi \circ \psi$ is a diffeomorphism of the base space, i.e., $\in$ Diff(M). Vertical automorphisms are those that do not move the base point. They form a subgroup of Aut(P) denoted by $\mathrm{Aut}_v(P)$. The corresponding Lie algebras will be denoted by aut(P) and $\mathrm{aut}_v(P)$, respectively; the latter are spaces of vector fields in P generated by one-parameters subgroups of Aut(P) and $\mathrm{Aut}_v(P)$, respectively. For the time being, we disregard diffeomorphisms of the base space and focus on $\mathrm{Aut}_v(P)$. The latter is to be identified with the group $\mathcal{G}$ of gauge transformations, whose Lie algebra $Lie\mathcal{G}$ is, in turn, identified with $\mathrm{aut}_v(P)$, the Lie algebra of $\mathrm{Aut}_v(P)$. The reason for this identification is clear from the way a connection transforms under vertical automorphisms. Let V be a connection with curvature F and let ψ be a vertical automorphism: we can associate to it a map $\gamma : P \to G$ defined by $\psi(p) = p\gamma(p)$ satisfying $\gamma(pg) = g^{-1}\gamma(u)g$. Then, one can show that the pull-back by ψ is

$$\psi^* V = \gamma^{-1} V \gamma + \gamma^{-1} d\gamma, \qquad \psi^* F = \gamma^{-1} F \gamma$$

Thus, we set $\mathsf{Aut}_v(\mathsf{P}) \equiv \mathcal{G}$ and introduce the evaluation map

$$ev : \mathsf{P} \times \mathcal{G} \to \mathsf{P}, \qquad ev(p, \psi) = \psi(p) \tag{77}$$

$\mathsf{P} \times \mathcal{G}$ is a principal fiber bundle over $\mathsf{M} \times \mathcal{G}$, with group G. This means, in particular, that, by pulling back a connection V in P, we obtain a connection $\mathbf{V} = ev^*V$ in $\mathsf{P} \times \mathcal{G}$. This connection splits into a one-form in P plus a one-form in $\mathcal{G}$ and contains all the information about the properties of the FP ghosts and BRST transformations. First, it can be shown that

$$\mathbf{V} \equiv ev^*V = V + i_{(\cdot)}V \tag{78}$$

where i is the interior product and $i_{(\cdot)}V$ denotes the map $Z \to i_Z V$, that associates to every $Z \in Lie(\mathcal{G})$ the map $\xi_Z = V(Z) : \mathsf{P} \to Lie(\mathsf{G})$. ξ_Z is, in fact, an infinitesimal gauge transformation since the action of Z over the connection V is given by the Lie derivative L_Z, which takes the following form:

$$L_Z V = d\xi_Z + [V, \xi_Z] \tag{79}$$

The symbol $i_{(\cdot)}V$ has an anticommuting nature and plays the same role as the FP ghost field c introduced in Section 2.3. For instance, if $\hat{\delta}$ represents the exterior differential in $\mathcal{G}$, one can show that

$$\hat{\delta}V = -di_{(\cdot)}V - \frac{1}{2}[V, i_{(\cdot)}V] - \frac{1}{2}[i_{(\cdot)}V, V] \tag{80}$$

$$\hat{\delta}i_{(\cdot)}V = \frac{1}{2}[i_{(\cdot)}V, i_{(\cdot)}V] \tag{81}$$

Therefore, if we identify $\hat{\delta} = (-1)^k \delta_c$, where k is the degree of the form in P it acts on and δ_c is the BRST nilpotent operator, we reproduce the BRST transformations introduced in Section 2.3.

In other words, BRST transformations and coboundary coincide with gauge transformations along the fibers of P and de Rham differential in $\mathcal{G}$. Next, let us apply these new notions to gauge anomalies. Let P_n be an ad-invariant polynomial with n entries, $n = \frac{d}{2} + 1$. The expression

$$TP_n(V) = n \int_0^1 dt P_n(V, F_t, \dots, F_t) \tag{82}$$

introduced in Section 2.3 is called *transgression formula*, and

$$dTP_n(V) = P_n(F, \dots, F) = 0 \tag{83}$$

because of dimensional reasons, for $P_n(F, \dots, F)$ is a $d + 2$ basic form in a d dimensional base spacetime M. Let us pull back (83) through the evaluation map to $\mathsf{P} \times \mathcal{G}$. We obtain

$$(d + \hat{\delta})ev^*TP_n(V) = (d + \hat{\delta})TP_n(ev^*V) = 0 \tag{84}$$

This splits into a set of equations of decreasing degree

$$\hat{\delta}TP_n(V) + di_{(\cdot)}TP_n(V) = 0 \tag{85}$$

$$\hat{\delta}i_{(\cdot)}TP_n(V) + di_{(\cdot)}i_{(\cdot)}TP_n(V) = 0 \tag{86}$$

$$\dots \dots$$

$$\dots \dots$$

These are the descent equations written in geometrical language, and, unlike the algebraic descent equations of Section 2.3, they are all well defined in P. Focusing on $i_Z TP_n(V)$, it can be rewritten

$$i_Z TP_n(V) = -n(n-1) \int_0^1 dt(t-1) \Big(P_n(di_Z V, V, F_t, \dots, F_t) - dP_n(V, i_Z V, F_t, \dots, F_t) \Big)$$

The first term on the RHS is precisely, up to global sign, the anomaly (48) for d generic:

$$\Delta_d^1(i_{(.)}V, V) = n(n-1) \int_0^1 dt(t-1) P_n(di_{(.)}V, V, F_t, \dots, F_t). \tag{87}$$

The second piece is a total differential which drops out upon spacetime integration.

A cocycle $\Delta_{2n-2}^{(1)}(V, c)$ is trivial when there are local cochains $C_{2n-2}^{(0)}(V, c)$ and $C_{2n-3}^{(1)}(V, c)$ such that

$$\Delta_d^1(V, c) = \delta_c C_{2n-2}^{(0)}(V, c) + dC_{2n-3}^{(1)}(V, c) \tag{88}$$

where $C_k^{(p)}(V, c)$ denotes a polynomial k-form of ghost number p, built with V, c, their commutators, wedge products, and exterior differentials. If such local cochains do not exist, the cocycle is said to be non-trivial and defines a non-trivial class of the BRST cohomology, a true anomaly. The latter characterizes local quantum field theories: how does it relate to the familiar de Rham cohomology of M and P ?

Let us return to the descent Equation (84) and, in particular, to the consistency conditions (86). Since $dTP_n(V) = 0$, anomalies are to be found among the representatives of the de Rham cohomology group of order d + 1 in P. But do they coincide with them? Now, it is known that the de Rham cohomology group of order d + 1 in P, $H_{deRham}^{d+1}(P)$, is spanned by forms $TP_i(V) \wedge \beta_i$, where β_i are basic and closed forms of order $h_i = 2(n - i)$, such that $2i - 1 + h_i = $ d + 1. The β_i may be forms of the type $P_{n-i}(F, \dots, F)$ (this is the case of a reducible polynomial $P_n(F) = P_i(F) P_{n-i}(F)$), or of the type $P'_{n-i}(R, \dots, R)$, where R is the Riemann (or spin) curvature 2-form on M, and P'_i are the corresponding ad-invariant symmetric polynomials. But the set of β_i may also contain forms that cannot be written in local form like in the previous examples. Such non-local β_i do not give rise to anomalies in local field theory.

On the other hand, let us suppose that $\chi = TP_n(V)$ be cohomologically trivial, i.e., $TP_n(V) = d\eta$. Then, decomposing (84), we get, in particular,

$$i_{(.)}TP_n(V) = i_{(.)}d\eta = -di_{(.)}\eta + \hat{\delta}\eta \tag{89}$$

Since $\hat{\delta}\eta = -\delta_c \eta$, this is a triviality condition for the anomaly $i_{(.)}TP_n(V)$ when compared with (88). However, this does not correspond automatically to a trivial field theory anomaly. The condition $TP_n(V) = d\eta$ means that $TP_n(V)$ is exact in the de Rham cohomology of P . But this does not automatically mean that it is trivial with respect to the local BRST cohomology. For this to be the case it must be that η is a local expression of V. If such a local η does not exist, we are in the presence of a true non-trivial anomaly originating from a cocycle $d\eta$ in P, which is trivial (a coboundary) in the de Rham cohomology. On the same footing, if χ and χ' are two closed d + 1 forms in P such that $\chi - \chi' = d\eta$, we can say the corresponding anomalies are the same only if η is a local form.

The previous rather lengthy and somewhat convoluted discussion is to stress the fact that the BRST cohomology (i.e., the true anomalies of gauge field theories) does not coincide with the ordinary cohomology of P, and even less with that of M. In fact, the topology of M does not play a role in determining the local anomalies on it. To geometrically characterize

the BRST cohomology of local quantum field theory a further effort is necessary. We need to introduce the the concept of classifying space and universal bundle.

Given a compact Lie group G, any principal fiber bundle $P(M, G)$ can be obtained as the pull-back of the so-called *universal principal bundle* $EG(BG, G)$,

$$
\begin{array}{ccc}
G & \hookrightarrow & EG \\
& & \downarrow{\scriptstyle \pi} \\
& & BG
\end{array}
$$

BG, the basis, is called a *classifying space*. An important property of EG, the total space, is that it is a contractible space, that is, it is continuously deformable to a point. Therefore its cohomology is that of a single point.

For any principal bundle $P(M, G)$, there exist bundle morphisms $(\hat{f}, f)$ such that the following diagram is commutative

$$
\begin{array}{ccc}
P & \xrightarrow{\hat{f}} & EG \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
M & \xrightarrow{f} & BG
\end{array}
$$

f is unique up to homotopy [Two maps f_1, f_2 are homotopic if there exists a continuous map $F : M \times I \to BG$ such that $F(x, 0) = f_1(x)$ and $F(x, 1) = f_2(x), \forall x \in \mathsf{m}$] and is called a classifying map. Both EG and BG are generally infinite dimensional, but in many practical applications, one can use finite-dimensional approximants for them.

A very important fact is the existence of a universal connection $\mathfrak{v}$ in EG from which any connection on P can be derived via pull-back: for any V in P there exists a bundle morphisms $(\hat{f}, f)$ such that $V = \hat{f}^* \mathfrak{v}$. For convenience, we shall call, with some abuse of language, *universal* the forms χ in P constructed with the connection form V if they can be derived by pulling back an analogous form in EG constructed with $\mathfrak{v}$:

$$
\chi(V) = \chi(\hat{f}^* \mathfrak{v}) = \hat{f}^*(\chi(\mathfrak{v})), \tag{90}
$$

for a bundle map $\hat{f}$. Therefore, forms like $TP_n(V)$ and $P_n(F)$ are universal. But, of course, in P, there may be non-universal forms. In particular, the forms β_i that appear in the cohomology of P, see above, may not be universal. We have already remarked that not all the forms that span the nontrivial cohomology classes of $H^{d+1}_{\text{deRham}}(P)$ necessarily correspond to local anomalies.

Let us now connect the anomalies in quantum field theory to their universal origin. We pull-back $\mathfrak{v}$ and forms constructed with it to $P \times \mathcal{G}$ by combining the evaluation map and bundle maps

$$
P \times \mathcal{G} \xrightarrow{ev} P \xrightarrow{\hat{f}} EG \tag{91}
$$

Now, starting with local expressions of $\mathfrak{v}$ in EG and pulling them back with $ev \circ \hat{f}$, we create universal expressions in $P \times \mathcal{G}$. We focus in particular on the universal forms of order $d + 1$ and state the result of the previous discussion: *non-trivial local field theory anomalies are identified by the quotient*

$$
\frac{closed\ universal\,(\mathsf{d}+1)-forms\ in\ \mathsf{P}}{differentials\ of\ universal\ \mathsf{d}-forms\ in\mathsf{P}} \tag{92}
$$

For instance, consider $TP_{n_1}(V)P_{n_2}(F)$ and suppose that $P_{n_2}(F) = d\eta$ with η a basic form in some manifold M of dimension $\mathsf{d} = 2(n_1 + n_2) - 2$. Then $TP_{n_1}(V)P_{n_2}(F) = d(TP_{n_1}(V)\eta)$

(because $P_{n_1}(F)\eta$ is basic and therefore vanishes on M for dimensional reasons). Nevertheless, the anomaly corresponding to $TP_{n_1}(V)P_{n_2}(F)$ is universal because, even though $P_{n_2}(F) = d\eta$ in M, the form η cannot be universal. To explain the reason for this we need the concept of Weil homomorphism.

Chern–Weil homomorphism. For any group G we have introduced earlier the space of symmetric ad-invariant polynomials, usually denoted by $\mathbb{I}(G)$, which is a commutative algebra. Given a principal fiber bundle $P(M, G)$ there is a homomorphism $w : \mathbb{I}(G) \to H^*(M)$ between the algebra of ad-invariant symmetric polynomials and the de Rham cohomology of M, which associates to any polynomial P_n a cohomology class $w(P_n) = P_n(F, \ldots, F)$. The homomorphism is constructed by filling in the curvature F of $P(M, G)$ in all the entries of P_n, so as to build the forms $P_n(F, \ldots, F)$, which are closed in M. They may or may not be exact according to the topology of M. But in the case when the bundle is the universal one with compact group G and the base manifold is the classifying space, the Weil homomorphism is an isomorphism, i.e., no form $P_n(F, \ldots, F)$ is exact.

Then, let us explain why the anomaly corresponding to the above-mentioned form $TP_{n_1}(V)P_{n_2}(F)$ is universal. This is so because, even if $P_{n_2}(F) = d\eta$ in M, the form η cannot be universal. The reason is that for BG the Weil homomorphism is an isomorphism when the group G is compact. This means that $P_{n_2}(F)$ is not the differential of any universal form and, thus, $TP_{n_1}(V)P_{n_2}(F)$ cannot be written itself as a differential of any universal form. From the field theory point of view, this means that $P_{n_2}(F)$ cannot be written as the total differential of a local expression in the fields. In other words, the anomaly originated from $TP_{n_1}(V)P_{n_2}(F)$ is a true local field theory anomaly.

A comment is in order to explain this at first sight surprising correspondence. The origin of it lies in the circumstance that standard perturbative field theory is formulated in a local geometry. Propagators and vertices are evaluated on plane wave configurations of fields, thus in a unique local patch isomorphic to a Minkowski (or Euclidean) spacetime. In this way, the cohomological properties of the base space M, reflected also in the topology of P, are irrelevant. Therefore, as far as local anomalies are concerned, the results of perturbative field theories are independent of the topological features of the base manifold M. The BRST cohomology of local field theories reflects the de Rham cohomology of the classifying space, which is the same for all spacetimes of a given dimension. Once again, the roots of local gauge anomalies in QFT are not in the topology of the spacetime M but in the topology of the classifying space BG, [This is not anymore true if we consider global anomalies, in which the topology of M may play a role. Global anomalies call into play the global aspects of gauge transformations, i.e., the global aspects of $\mathcal{G}$. We will not analyze global anomalies in this paper. Enough is to say that the gauge transformations involved in the various SMs considered here do give rise to global anomalies [5]].

Now, one step remains before closing the circle: what is the connection of this geometry with the bugs of the corresponding theories? It is the lack of a fermion propagator, as is made clear by the family's index theorem.

4.2. The Atiyah–Singer Index Theorems

The family's index theorem is the index theorem appropriate for consistent local chiral anomalies. In general, an (ordinary) index theorem counts the difference between the graded zero modes of an elliptic operator. For instance, zero modes of a Dirac operator with a fixed potential and metric. [From now on, we also introduce in our analysis the dependence on a non-trivial metric, which we have so far ignored]—they are labeled by $+$ and $-$, which denote opposite chiralities—and the index counts the difference between the $+$ and the $-$ zero modes. The family's index theorem does the same, except that the operator, in the case of gauge theories, varies on a Hausdorff space, the moduli space of

connections (i.e., the space of orbits under the action of the gauge group). The index is formally represented by the space of zero mode eigenvectors (kernel of the operator) of one chirality *minus* the space of zero mode eigenvectors of the opposite chirality (cokernel). The *difference* of two vector spaces varying from point to point of the parameter space can be given a mathematical sense in terms of the K-theory of vector bundles. What really matters to us in relation to the anomaly problem is not as much the number of zero modes, but the existence or non-existence of the inverse of the Dirac operator. It is of paramount importance that one appreciates the difference between the two situations. The presence of even a finite number of zero modes for a Dirac operator with (twisted by) a fixed potential or metric prevents its inversion, but this problem can be easily fixed by excluding the (finite number of) zero modes from the spectrum. The problem, however, becomes unwieldy when we face a continuous spectrum of gauge potentials (or, better, their orbit space). Only K-theory can and does provide a sensible treatment.

We recall that this is the same problem we have already met in the perturbative approach to anomalies, as well as in the non-perturbative approach à la Schwinger–DeWitt. In those approaches, the inverse of the relevant elliptic operator (or, which is the same, the existence of an effective action for Weyl fermions) is required, and one is obliged, for instance, to replace the Dirac–Weyl operator

$$\mathcal{D} = i(\slashed{\partial} + \slashed{V})P_+, \qquad P_\pm = \frac{1 \pm \gamma_5}{2} \tag{93}$$

by

$$\mathcal{D} = i(\slashed{\partial} + \slashed{V}P_+) \tag{94}$$

in order to guarantee the invertibility of the kinetic operator, i.e., the existence of the fermion propagator. In those approaches, anomalies show up as non-conservation of the chiral gauge currents. The family's index theorem, instead, calculates for us the obstructions to the invertibility of the Dirac–Weyl operator. In other words it tells us what are the topological conditions for the Weyl fermion propagator to exist.

Although these obstructions manifest themselves in different ways in the two approaches, their origin is the same and can be traced to the geometry of universal bundles and classifying spaces. The added value of the family's index theorem approach is that it provides compact formulas of the obstructions for all dimensions in terms of ad-invariant polynomials, gauge curvatures and/or Riemann curvatures, from which one can derive the consistent gauge anomalies by the transgression formulas and the descent equations - and not only those but also other anomalies, as we shall see. It also provides the numerical coefficients (depending on the fermion field representations) in front of the anomalies, based on which one can study the conditions for their cancelation in general.

In this limited presentation, it is, of course, impossible to expound all the mathematics underlying the family's index theorem, but we try to provide a summary presentation that is hopefully enough to capture the essential points of the subject.

The family's index theorem applies to elliptic operators. Given two vector bundles $\mathcal{E}$ and $\mathcal{F}$ over a Riemannian manifold X and the corresponding spaces of sections $\Gamma(\mathcal{E})$ and $\Gamma(\mathcal{F})$, respectively, an elliptic operator is a partial differential operator that maps sections of $\Gamma(\mathcal{E})$ to sections of $\Gamma(\mathcal{F})$ in a way which is fiberwise invertible. In practice, we are interested in sections representing matter fields and, in particular, spinors. For instance, an operator with a local representative $A_{ij}\partial_{x_i}\partial_{x_j}$ is elliptic if the matrix $A_{ij}(x)$ is symmetric and positive definite; a Euclidean Dirac operator is elliptic. With reference to field theory, a linear or quadratic elliptic operator locally has an inverse.

An elliptic operator P on a compact space X is endowed with an index *(the analytic index)*

$$indP = dim(kerP) - dim(cokerP) \tag{95}$$

where the kernel of P is the space of its zero eigenvectors and the cokernel is formed by the sections of the target bundle that do not belong to the image of P. Some remarkable properties of an elliptic operator are as follows: the index of the operator depends on its leading term; if P is self-adjoint (same source and target space) it has real discrete eigenvalues with finite eigenspace consisting of smooth sections.

The Dirac operator is the central object for us now. In physics it is usually (but not exclusively) defined on a Minkowski spacetime (a pseudo-Riemannian manifold), say X. After a Wick rotation, i.e., in a Euclidean background, it is an example of a linear elliptic operator. Dirac operators act on spinors, which are used in physics to represent fermions. A spinor is a section of a vector bundle with two simultaneous properties: it belongs both to a representation space of the appropriate Clifford algebra and a representation of the appropriate spin group; in addition, it may also transform according to the representation of some compact group (the gauge group), in which case the corresponding Dirac operator is called twisted.

To be more precise, let TX be the tangent space of X. We call $C\ell(X)$ the Clifford algebra, i.e., the gamma matrix algebra, constructed over TX on the basis of the metric inherited from X. And let $\mathsf{P}_{SO}(X)$ be the *orthonormal tangent frame bundle* over X, i.e., the principal fiber bundle built with the orthonormal frames of TX. We need also the spin bundle $\mathsf{P}_{Spin}(X)$, which is the double covering of $\mathsf{P}_{SO}(X)$ (a spin structure):

$$\xi : \mathsf{P}_{Spin}(X) \longrightarrow \mathsf{P}_{SO}(X)$$

where $\xi(p,g) = \xi(p)\xi_0(g)$, for any $p \in \mathsf{P}_{Spin}(X)$ and any $g \in Spin_n$, $n = dim(X)$, and $\xi_0 : Spin_n \longrightarrow SO(n)$ is the universal covering map (fermions see a doubly ample world with respect to us). A spinor bundle S is a bundle associated to $\mathsf{P}_{Spin}(X)$

In fact, we will deal with complex spinor bundles. A *complex spinor bundle* $S_{\mathbb{C}}(X)$ is an associated bundle

$$S_{\mathbb{C}}(X) = \mathsf{P}_{Spin}(X) \times_{\mu} M_{\mathbb{C}} \tag{96}$$

where $M_{\mathbb{C}}$ is a left module of $C\ell(X) \otimes \mathbb{C}$ whose action is represented by μ.

In this context, the (Euclidean) Dirac operator, denoted by $\slashed{D}$ (we drop in this section the $\widetilde{\slashed{D}}$ notation), is

$$\slashed{D} = i\gamma^{\mu}(\partial_{\mu} + \frac{1}{2}\omega_{\mu}) \tag{97}$$

where $\gamma^{\mu} = e_a^{\mu}\gamma^a$ (e_a^{μ} are the inverse vielbein, $\eta^{ab}e_a^{\mu}e_b^{\nu} = g^{\mu\nu}$) and $\omega_{\mu} = \omega_{\mu}^{ab}\Sigma_{ab}$ is the spin connection, with $\Sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$.

If n is even, $C\ell(n)(X)$ posseses an element $\omega_{\mathbb{C}}$ (the chirality operator) such that

$$\omega_{\mathbb{C}}^2 = 1, \qquad \omega_{\mathbb{C}}e = -e\omega_{\mathbb{C}} \tag{98}$$

for any element $e \in C\ell(n)(X)$. As a consequence, any $S_{\mathbb{C}}$ splits according to

$$S_{\mathbb{C}} = S_{\mathbb{C}}^+ \oplus S_{\mathbb{C}}^-, \qquad S_{\mathbb{C}}^{\pm} = \frac{1}{2}(1 \pm \omega_{\mathbb{C}})S_{\mathbb{C}} \tag{99}$$

Accordingly $\not{D}$ splits as

$$\not{D} = \begin{pmatrix} 0 & \not{D}^- \\ \not{D}^+ & 0 \end{pmatrix} \tag{100}$$

Therefore, we have $ker\not{D} = ker\not{D}^+ \oplus ker\not{D}^-$. Now since $\not{D}$ is self-adjoint, it follows that $(\not{D}^{\pm})^{\dagger} = \not{D}^{\mp}$. And since, for any elliptic operator T, $coker T = ker T^{\dagger}$, it follows that

$$Ind\not{D}^+ = dim(ker\not{D}^+) - dim(ker\not{D}^-) \tag{101}$$

In gauge field theory, we have to do with spinors that transform according to definite representations of the gauge group G. In this case we generalize the previous setup and consider the tensor product of a spinor bundle $S_{\mathbb{C}}^{\pm}$ with a vector bundle E corresponding to a representation ρ of the structure group G of $P(X, G)$: $S_{\mathbb{C}}^{\pm} \otimes E$. The relevant connection will be the spin connection plus a gauge connection $V = V_{\mu} dx^{\mu}$ valued in the representation ρ of the Lie algebra of G with antihermiten generators. The corresponding Dirac operator

$$\mathcal{D} = \not{D} + i\not{V} \tag{102}$$

acts on the space of sections of $S_{\mathbb{C}}^{\pm}(E) \equiv S_{\mathbb{C}}^{\pm} \otimes E$ and maps it to itself. In this case Equation (99) is replaced by

$$S_{\mathbb{C}}(E) = S_{\mathbb{C}}^+(E) \oplus S_{\mathbb{C}}^-(E) \tag{103}$$

and Equation (100) by

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix} \tag{104}$$

with obvious replacements in the definition of the index.

4.2.1. The Atiyah–Singer Index Theorem

In this context the *Atiyah-Singer index theorem* says:

$$Ind(\mathcal{D}^+) = \int_X ch(E)\hat{A}(X) \tag{105}$$

The symbol $ch(E)$ indicates the Chern character of the E bundle, i.e., the rational characteristic class given in terms of the curvature F of V by

$$ch(E) = r + \frac{i}{2\pi}\text{tr}F + \frac{i^2}{2(2\pi)^2}\text{tr}F^2 + \ldots + \frac{i^n}{n!(2\pi)^n}\text{tr}F^n + \ldots \tag{106}$$

where r is the dimension of the representation ρ. The symbol $\hat{A}(X)$ denotes the $\hat{A}$-genus, which is the (rational) Pontryagin characteristic class of X. It can be expressed in terms of the Riemann curvature R as follows

$$\begin{aligned} \hat{A}(X) \quad = \quad & 1 + \frac{1}{(4\pi)^2}\frac{1}{12}\text{tr}R^2 + \frac{1}{(4\pi)^4}\left[\frac{1}{288}\text{tr}(R^2)^2 + \frac{1}{360}\text{tr}R^4\right] \\ & + \frac{1}{(4\pi)^6}\left[\frac{1}{10368}\text{tr}(R^2)^3 + \frac{1}{4320}\text{tr}R^2\text{tr}R^4 + \frac{1}{5670}\text{tr}R^6\right] + \ldots \end{aligned} \tag{107}$$

This famous theorem holds for a *fixed background*, i.e., for a fixed metric and a fixed gauge potential. But, although it features in several physical applications, it is not what

we need in relation to anomalies. As already pointed out, we are not interested in the numerical counting of zero modes but in the invertibility of the Dirac operator over the whole space of metric and potential backgrounds.

The field theory path integral is a functional of the fields, in particular of the potentials or the metrics or both. Let us focus simply on a gauge theory defined on a principal fiber bundle $P(M, G)$, with a space $\mathcal{A}$ of connections and a symmetry group $\mathcal{G}$ of gauge transformations. We denote by $\mathcal{Z}[V]$ the fermion determinant, i.e., the partial path integral in which we suppose that only the quadratic part of the action is considered and that fermions have been integrated out. $\mathcal{Z}[V]$ is a functional of the gauge potential, which can be computed by means of perturbative and non-perturbative methods. But it also contains the gauge degrees of freedom, which we have to get rid of since the relevant information for physics is stored in the orbit space $\mathcal{A}/\mathcal{G}$. The full path integral is obtained by integrating over the orbit space

$$\mathcal{Z} = \int_{\mathcal{A}/\mathcal{G}} d[V]\,\mathcal{Z}[V] \tag{108}$$

Here $[V]$ denotes the orbit passing through V, and $\mathcal{G} = \mathrm{Aut}_v(\mathsf{P})$. The issue here is to guarantee that $\mathcal{Z}[V]$ is a well-defined function, integrable over $\mathcal{A}/\mathcal{G}$.

The geometry is represented by the following bundle diagrams:

$$\begin{array}{ccc} \mathcal{G} \lhook\joinrel\longrightarrow & \mathcal{A} \\ & \downarrow \\ & \mathcal{A}/\mathcal{G} \end{array} \tag{109}$$

which is a principal fiber bundle with structure group $\mathcal{G}$. Since the actions of G and $\mathcal{G}$ commute, another relevant principal fiber bundle is

$$\begin{array}{ccc} \mathsf{G} \lhook\joinrel\longrightarrow & \frac{\mathsf{P} \times \mathcal{A}}{\mathcal{G}} \\ & \downarrow \\ & \frac{\mathsf{M} \times \mathcal{A}}{\mathcal{G}} \end{array} \tag{110}$$

with structure group G. In turn, $\frac{\mathsf{M} \times \mathcal{A}}{\mathcal{G}}$ is a fiber bundle over $\frac{\mathcal{A}}{\mathcal{G}}$ with fiber M. Given a complex vector space E, which is a representation space of G, we have also two associated fiber bundles

$$E = \mathsf{P} \times_\mathsf{G} \mathsf{E}, \qquad \text{and} \qquad \mathcal{E} = \frac{\mathsf{P} \times \mathcal{A}}{\mathcal{G}} \times_\mathsf{G} \mathsf{E} \tag{111}$$

From the above, we see that what is required in a gauge theory is an index theorem over the entire orbit space because we have to deal with an operator that varies from point to point in it. The dimensions of its kernel and cokernel may jump from point to point in the parameter space provided by the orbit space. The relevant object in this case is the difference between the kernel and cokernel, which are two vector spaces, as they vary from point to point, i.e., the difference between two vector bundles over the parameter space. The appropriate mathematical theory in this situation is K-theory. K-theory is precisely the theory of formal differences of vector bundles. Denoting by $[E]$ and $[F]$ the classes of vector bundles isomorphic to E and F, respectively, via the Grothendieck construction, K-theory assigns a precise meaning to the difference $[E] - [F]$. Such objects are called virtual bundles.

Let us first define the index for families. The geometrical environment is the one corresponding to (109) and (110). We have a Hausdorff space $\mathcal{B}$ which is the basis of a

principal fiber bundle (like $\mathcal{A}/\mathcal{G}$ in (109)). A continuous family $\mathcal{V}$ of smooth vector bundles over X, which is parameterized by $\mathcal{B}$, is a fiber bundle $\mathcal{V} \to \mathcal{B}$ whose fiber is a smooth vector bundle V over X, which is associated to a principal fiber bundle $\mathsf{P}(X, \mathsf{G})$, on which the group $\mathsf{Aut}_v(\mathsf{P})$ acts ($\mathcal{V}$ is like $\mathcal{E}$, defined above). Now, let us consider two families $\mathcal{V}$ and $\mathcal{W}$ with the respective spaces of sections $\Gamma(\mathcal{V})$ and $\Gamma(\mathcal{W})$ and a differential operator $P : \Gamma(\mathcal{V}) \to \Gamma(\mathcal{W})$.

P defines a family of elliptic operators which associates at each point of $\mathcal{B}$ an elliptic operator between $\Gamma(\mathcal{E})$ and $\Gamma(\mathcal{W})$. The kernel and cokernel of P are vector spaces that vary from point to point in $\mathcal{B}$. Therefore, their difference defines an element of $K(\mathcal{B})$:

$$indP = [kerP] - [cokerP] \in K(\mathcal{B}) \tag{112}$$

Even though the dimensions of these vector bundles may vary in $\mathcal{B}$, the definition makes sense. Atiyah and Singer have provided a formula for it in terms of characteristic classes.

4.2.2. The Family's Index and the Quillen Determinant

In the case outlined above, in which P is the Dirac operator $\mathcal{D}^+$ between two families of sections $\Gamma(S_{\mathbb{C}}^+ \otimes \mathcal{V})$ and $\Gamma(S_{\mathbb{C}}^- \otimes \mathcal{V})$, the index theorem can be expressed in the form

$$ch\left(ind(\mathcal{D}^+)\right) = \int_{\mathsf{M}} ch(\mathcal{V}) \cdot \hat{A}(T\mathcal{Q}) \tag{113}$$

In this case, $X = \mathsf{M}$ is the spacetime manifold, $\mathcal{Q} \equiv \frac{\mathcal{A}}{\mathcal{G}}$ is the orbit space of connections, $T\mathcal{Q}$ is its tangent space and $\mathcal{V}$ is the gauge bundle. The Chern character of the index measures the extent to which $ker\mathcal{D}^+$ differs from $coker\mathcal{D}^+ = ker\mathcal{D}^-$. It is intuitive that, as long as, $ind(\mathcal{D}^+)$ differs from 0, the inverse of $\mathcal{D}^+$ cannot exist (see below for a more cogent argument). Therefore, *the only way to ensure the existence of this inverse (and, so, the existence of the fermion propagator and, in turn, the existence of the fermion path integral) is that the RHS of* (113) *corresponds to the 0 class*. This is particularly visible with the first Chern class, i.e., the first nontrivial term of the Chern character. Let us recall that, for a vector bundle V,

$$ch(V) = rank(V) + c_1(V) + \frac{1}{2}\left(c_1(V)^2 - 2c_2(V)\right) + \dots \tag{114}$$

Therefore,

$$c_1\left(ind(\mathcal{D}^+)\right) = \int_{\mathsf{M}} ch(\mathcal{V}) \cdot \hat{A}(T\mathcal{Q})\Big|_{\mathsf{d},2} \tag{115}$$

where $\mathsf{d} = dim(M)$; in the RHS, we indicate the $(\mathsf{d}, 2)$ component in $M \times \mathcal{Q}$. The RHS is a two-form in $\mathcal{Q}$, therefore it belongs to a class in $H^2(\mathcal{Q}, \mathbb{Z})$. It represents the first Chern class of the so-called Quillen determinant bundle. Given the importance of the latter it is worth spending some time to introduce it in more detail.

Let us focus on the case of a twisted Dirac operator, for instance $\mathcal{D}$ considered above. It is an elliptic operator on a compact space, and so a Fredholm operator, which splits according to Equation (104). If we wish to define the determinant of either $\mathcal{D}^+$ or $\mathcal{D}^-$, as the product of its eigenvalues, there may be obstacles. As pointed out in Section 2.1, the first obstacle is the very definition of eigenvalues of a Dirac–Weyl operator. In perturbative and SDW approaches we have bypassed this problem by modifying the Dirac–Weyl operator like in (94). Here, instead, we consider the quadratic operator $(\mathcal{D}^+)^\dagger\mathcal{D}^+$. [We use this operator only in order to discuss the eigenvalue splitting, not for a direct calculation—for instance, by the SDW method—because, for that purpose, it is inapplicable for an obvious reason: it is not invertible!] It has the same eigenvalues as $(\mathcal{D}^-)^\dagger\mathcal{D}^-$, which are the square

of the eigenvalues of the Dirac operator $\mathcal{D}$, but of course the eigenvectors are different. More precisely, the nonzero eigenvalues of $(\mathcal{D}^+)^\dagger \mathcal{D}^+$ are paired with the non-zero eigenvalues of $(\mathcal{D}^-)^\dagger \mathcal{D}^-$, in the sense that to each common eigenvalue there correspond two distinct eigenvectors of the two operators with the same multiplicity. But this may not be true anymore for the zero eigenvalues. Of course this is the crucial point in regard to the index theorem. But it is also crucial for defining the path integral of the two Dirac–Weyl operators. If we wish to define the relevant path integrals as the square root of the product of their eigenvalues, the non-zero eigenvalues do not pose a problem because they are gauge invariant, but for the zero eigenvalues, this may not be true. Therefore, we have to settle the question of the zero modes in such a way as to produce a well-defined integrable $\mathcal{Z}[V]$.

Let us denote by $\psi_1, \ldots, \psi_r$ a basis for the zero modes of $\mathcal{D}^+$. In field theory, a familiar object, due to the Pauli principle, is the Slater determinant, i.e., wedge product $\psi_1 \wedge \ldots \wedge \psi_r$. It is a complex number (a determinant) that varies as $\mathcal{D}^+$ varies over $\mathcal{X} \equiv \frac{M \times \mathcal{A}}{\mathcal{G}}$, but, in general, does not form a complex line bundle over $\mathcal{X}$. In the same way, one can construct an analogous wedge product for $coker\mathcal{D}^+ = ker\mathcal{D}^-$, which also forms a complex line but, in general, not a line bundle over $\mathcal{Q}$. However, the formal difference of the two, i.e.

$$\coprod_{[V] \in \mathcal{X}} \overset{dim\ ker\mathcal{D}_V^+}{\bigwedge} ker(\mathcal{D}_V^+)^\dagger \otimes \overset{dim\ coker\mathcal{D}_V^+}{\bigwedge} coker(\mathcal{D}_V^+) \tag{116}$$

where we have made explicit the dependence on the connection V, constitutes an element of $K(\mathcal{X})$, which is called the Quillen determinant line bundle. This complex line bundle has remarkable topological properties. For us, it is a fundamental object because it is trivial, i.e., it has a global non-vanishing section, precisely when $\mathcal{D}^+$ is invertible. [This is what D. Quillen has proved for a Cauchy–Riemann operator over a Riemann surface. We are not aware of a general proof of this theorem extending to all the cases considered here. We are, therefore, assuming its validity in the present context, validity which is, on the other hand, rather plausible [53], and heuristically supported by the results in quantum field theory].

In the case of the Dirac–Weyl operator $\mathcal{D}^+$, the Quillen determinant bundle is a bundle over the base space $\mathcal{X} \equiv \frac{M \times \mathcal{A}}{\mathcal{G}}$. It is a non-trivial bundle as long as $H^2(\mathcal{X}, \mathbb{Z}) \neq 0$. Upon integrating along the fiber M, the relevant cohomology group is $H^2(\mathcal{Q}, \mathbb{Z})$ and coincides with $c_1\left(ind(\mathcal{D}^+)\right)$. If the index vanishes, the Quillen bundle is trivial; that is, it has a global section, which is a non-vanishing function over $\mathcal{A}/\mathcal{G}$ and can be integrated over. Returning now to the definition (108) of the path integral over the orbit space, we conclude that the existence of this section implies that $\mathcal{Z}[V]$ is well defined.

The formal equation

$$\delta\det A = \det A\ \mathrm{tr}\left(A^{-1}\delta A\right). \tag{117}$$

when applied to $A = \mathcal{D}^+$, implies that, in order for $\mathcal{Z}[V]$ to be well defined, the inverse of $\mathcal{D}^+$, i.e., the fermion propagator, must be well defined. The inverse operator is represented by the inverse eigenvalues. For the non-zero eigenvalues, invertibility is obvious. For the 0 eigenvalue, the inverse is represented by the inverse of the global section of the Quillen determinant bundle. We expect, therefore, that, if $H^2(\mathcal{Q}, \mathbb{Z})$ is trivial, this global section is invertible. On the contrary, a nonzero $c_1\left(ind(\mathcal{D}^+)\right)$, given by (115), is an obstruction to defining the inverse of $\mathcal{D}^+$, i.e., the Weyl fermion propagator. The vanishing of $c_1\left(ind(\mathcal{D}^+)\right)$ is, therefore, a necessary condition for the existence of a well-defined partition function.

4.3. Closing the Circle: Obstructions and Anomalies

We have just seen that, in a Euclidean d-dimensional spacetime M, the obstructions to the existence of a Weyl fermion propagator (i.e., the existence of the inverse of $\mathcal{D}^+$, for instance) is contained in the class

$$ch(\mathcal{V}) \cdot \hat{A}(T\mathcal{X}) \tag{118}$$

where $\mathcal{X} \equiv \frac{M \times \mathcal{A}}{\mathcal{G}}$. The question is now: where is the connection, if any, between the just outlined mathematical approach, based on the obstruction concept, and chiral gauge anomalies? The answer is that the connection is in the topology of the classifying space. It is now time to establish this relationship. The exposition becomes now inevitably more abstract and we will have to assume some results without (even an attempt of a) proof.

For simplicity, we consider the pure gauge case. It corresponds to taking (118) with $\hat{A}(T\mathcal{X}) = 1$. Then recall that $\mathcal{G} = \mathsf{Aut_v}(\mathsf{P})$ in reference to the principal fiber bundle $\mathsf{P}(\mathsf{M}, \mathsf{G})$, that $\mathcal{A} \to \mathcal{A}/\mathcal{G}$ is a principal fiber bundle with structure group $\mathcal{G}$, (109), that $\frac{\mathsf{P} \times \mathcal{A}}{\mathcal{G}} \to \frac{\mathsf{M} \times \mathcal{A}}{\mathcal{G}}$ is a principal fiber bundle with group G, (110), and, finally, that $\frac{M \times \mathcal{A}}{\mathcal{G}}$ is a fiber bundle over $\mathcal{Q} = \frac{\mathcal{A}}{\mathcal{G}}$ with fiber M (in the case of $\mathcal{G} = \mathsf{Aut_v}(\mathsf{P})$, the bundle $\frac{M \times \mathcal{A}}{\mathcal{G}}$ reduces to $M \times \frac{\mathcal{A}}{\mathcal{G}}$ because the basepoint is not moved by a vertical automorphism). Let us endow $\mathcal{A}$ with a connection ω in the fiber bundle $\mathcal{A} \to \mathcal{A}/\mathcal{G}$ (see Appendix 12A of [5] for further details). Then we can define a connection η on the G bundle $\mathsf{P} \times \mathcal{A}$ over $\mathsf{M} \times \mathcal{A}$ as follows:

$$\eta_{p,V}(X,Y) = V_p(X) + V(\omega(Y))_p, \qquad X \in T_p\mathsf{P}, \quad Y \in T_V\mathcal{A} \tag{119}$$

where V is a connection in P and ω a connection in $\mathcal{A} \to \mathcal{A}/\mathcal{G}$. To render this construction less abstract let us notice that, looking at the definition (119), if we restrict η to the orbit of $\mathcal{G}$ we obtain $\eta = ev^*V$, where $ev : \mathsf{P} \times \mathcal{G} \to \mathsf{P}$ is the evaluation map introduced in Section 4.1.

This connection descends to a connection η' on the fiber bundle (110)

$$\mathsf{G} \lhook\joinrel\longrightarrow \frac{\mathsf{P} \times \mathcal{A}}{\mathcal{G}} \tag{120}$$
$$\Big\downarrow$$
$$\frac{\mathsf{M} \times \mathcal{A}}{\mathcal{G}}$$

We can, therefore, introduce the curvature $\mathcal{F}_{\eta'}$ of η' and express $ch(\mathcal{V})$ in terms of it. Since the Chern character can be expressed as a polynomial in the curvature, the relevant piece for dimension d will take the form $P_n(\mathcal{F}_{\eta'}, \ldots, \mathcal{F}_{\eta'})$, where $n = \frac{\mathsf{d}}{2} + 1$ and P_n is an ad-invariant (reducible or irreducible) polynomial for the group G. Now let us recall that, in the definition (119) of the connection η, the connection V in P, can be obtained from the universal connection $\mathfrak{v}$ (with curvature $F_\mathfrak{v}$) via a map $\hat{f} : \mathsf{P} \to \mathsf{EG}$, $V = \hat{f}^*\mathfrak{v}$. In the same way also ω, the connection in $\mathcal{A}$, can be pulled back from the same universal connection via a map $\hat{\mathsf{f}} : \mathcal{A} \to \mathsf{EG}$: $\omega = \hat{\mathsf{f}}^*\mathfrak{v}$. Therefore, there exists an overall map, $\hat{f}_\times = (\hat{f}, \hat{\mathsf{f}}) : \mathsf{P} \times \mathcal{A} \to \mathsf{EG}$, such that $\eta = \hat{f}_\times^*\mathfrak{v}$ (for a more detailed discussion of η and η', see Appendix 12A of [5]). Therefore,

$$P_n(\mathcal{F}_\eta, \ldots, \mathcal{F}_\eta) = f_\times^* P_n(F_\mathfrak{v}, \ldots, F_\mathfrak{v}) \tag{121}$$

The LHS then descends to $P_n(\mathcal{F}_{\eta'}, \ldots, \mathcal{F}_{\eta'})$ on (120). Remember that $P_n(F_\mathfrak{v}, \ldots, F_\mathfrak{v})$ is a closed form in EG that descends to a closed form in BG, which represents a nontrivial cohomology class of the classifying space.

Now, we know that, via the evaluation map and the transgression formula, we can extract from this class a corresponding chiral anomaly, which is given by $i_{(.)}TP_n(V)$. This can be seen directly from the transgression formula ensuing from (121)

$$TP_n(\eta) = n \int_0^1 dt P_n(\eta, \mathcal{F}_\eta, \ldots, \mathcal{F}_\eta), \tag{122}$$

which splits into components of order $(2n - p - 1, p)$ in P and $\mathcal{A}$. The $(2n - 1, 0)$ is precisely $TP_n(V)$. We also know that any local chiral anomaly takes this form and originates from a class in BG. Therefore, we see that *consistent gauge anomalies and obstructions to the existence of a Weyl fermion propagator coming from the index theorem are rigidly connected*. [The reader should not be misled by the fact that the family's index theorem is formulated for a compact manifold M while local field theory is formulated in a local patch. As explained above in Section 4.1, the link between the two is natural in the framework of classifying space. The BRST cohomology, that is, the cohomology of anomalies, is the same as the deRham cohomology of the classifying space, whose geometry encompasses all compact manifolds.] The added bonus of the family's index theorem is that it supplies the precise numerical coefficients of the irreducible polynomials and the product of reducible ones (see (106) for the case of spin $1/2$ [For other cases, for instance, for spin $3/2$, we need appropriate modifications.]). This allows us to cope with the problem of anomaly cancelation in general.

4.4. Anomaly Cancelation: Another Look

In the light of the previous results it is worth at this point to return to the issue of anomaly cancelation. In general, the absence or cancelation of local anomalies takes place:

- (A) In gauge field theories or sigma models in which the gauge group G has vanishing ad-invariant tensors in the relevant representations. Consistent anomalies are determined by ad-invariant polynomials P_n, or by the corresponding ad-invariant tensors $t^{a_1 \ldots a_n}$ with a specific coefficient depending on the representation of the matter fields involved. In many simple groups, some of these tensors vanish identically. This is the case, as far as the polynomial P_3 is concerned, for all simple groups except $SU(N)$, for $N \geq 3$. This cancelation occurs at the very origin, in the sense that the form $P_3(F(\mathfrak{v}), F(\mathfrak{v}), F(\mathfrak{v}))$, where $\mathfrak{v}$ is the universal connection, identically vanishes in the classifying space. Even in groups where these tensors are non-trivial it may happen they vanish depending on the representation of the fundamental fields. A well-known example is the case of matter in the adjoint representation in 4d.
- (B) In gauge field theories or sigma models with different species of fermion fields, which are separately anomalous, but when put together in the same theory, the coefficients of the various consistent anomalies sum up to 0. The standard model of particle physics is an example (see above).
  The absence of anomalies in the SM is due to a combination of (A) and (B).
- (C) There is also a third case, where chiral anomalies due to elementary fermion fields do not cancel completely, but cancelation may take place thanks to other fields in the theory. Such fields are endowed with transformation properties that allow them to cancel the original anomalies of the theory. This is the case of the anomalies canceled by Wess–Zumino terms or the case of the Green–Schwarz mechanism, which is vastly used in target space field theories derived from superstrings.

Other Anomalies?

The family's index theorem of an elliptic operator (like the Dirac–Weyl operator) signals the obstructions to the existence of an inverse (the propagator). So far, we have considered the obstructions related to the $(d, 2)$ component in $P \times \mathcal{A}$, see (115). We have

seen that this obstruction is linked to the non-triviality of the Quillen determinant. There are other components in the RHS of (115). In (118) there are forms of any even degree 2, 4, 6, 8, ... In d = 4 it is reasonable to disregard the forms of order 2, 8, 10, ... for dimensional reasons [But a careful investigation of this point is still lacking]. We know that chiral gauge anomalies come from the form of degree 6. But what about the degree 4 forms? These forms may, in principle, couple to the theory. What is their meaning? There is no literature on this subject, but this does not mean that we are allowed to casually shrug off an explanation.

Then, let us focus on the $(d, 0)$ component for d = 4. There are components coming from the $\widehat{A}$-genus and from $ch(\mathcal{V})$. The first gives a term $\frac{1}{192\pi^2} \text{tr}(R^2)$, the second (in the non-Abelian case) a term $\frac{1}{8\pi^2} \text{tr} F^2$. What can be the meaning of these two terms? The geometrical meaning of the $(d, 0)$ component in $P \times \mathcal{A}$ could be the 'virtual' rank of the index bundle, as one can deduce from the formulas (113) and (114). Let us elaborate a bit on this argument. The index, (112), is a local mathematical object that represents an asymmetry between the kernel of $\mathcal{D}^+$ and the kernel of $\mathcal{D}^-$. As long as this asymmetry is non-vanishing, it represents a threat to the existence of the propagator (and the theory). The above-mentioned densities (of the Chern and Pontryagin classes) are an additional hazard besides the ones contained in the $(d, 2)$ component (in $P \times \mathcal{A}$) of the index, which give rise to the consistent chiral anomalies. [The integration over M in (113) should not mislead the reader: for some M this integral may vanish, but, we should never forget that, in the spirit of locality, such formulas as (113) hold for any M (see discussion after (92))]. This is the mathematical side of the problem. It is to be expected that such a problem will show up also on the field theory side. A physical interpretation is needed.

In specific gauge field theories that couple to a metric, besides the gauge potentials, there is a quantity that naturally couples to the above-mentioned odd parity densities: it is the trace of the energy–momentum tensor of a theory of chiral fermions. The corresponding trace anomalies are not obtained via the transgression formula from the universal bundle like the chiral consistent anomalies (the way we have illustrated above), but are pulled back directly to the spacetime M via the classifying map. On the basis of our experience with quantum theories, the correspondence of quantum numbers (form degree, dimension, parity, Lorentz covariance) is perfect, and, barring the presence of a symmetry that forbids it, a non-vanishing coupling has to be expected.

This will be confirmed in the sequel, but before proceedings along this line of analysis, we need an introduction to diffeomorphisms, Weyl transformations, and related anomalies. Refs.: [53–59].

5. Gravitational Anomalies

When gravity comes into play, the main character becomes the metric $g_{\mu\nu}(x)$ and a fundamental symmetry becomes the symmetry under diffeomorphisms. The latter corresponds to the general coordinate transformations $x^\mu \to x^\mu + \xi^\mu(x)$, where $\xi^\mu(x)$ are generic infinitesimal smooth functions of the coordinates. They act on the metric as follows

$$\delta_\xi g_{\mu\nu} = D_\mu \xi_\nu + D_\nu \xi_\mu, \qquad \xi_\mu = g_{\mu\nu} \xi^\nu \tag{123}$$

D_μ is the covariant derivative: $D_\mu \xi_\nu = \partial_\mu \xi_\nu - \Gamma^\lambda_{\mu\nu} \xi_\lambda$, and $\Gamma^\lambda_{\mu\nu}$ are the Christoffel symbols.

The basic covariant geometrical object in this context is the Riemann tensor

$$R^\rho{}_{\lambda\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\lambda} - \partial_\nu \Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} - \Gamma^\rho_{\nu\sigma} \Gamma^\sigma_{\mu\lambda} \tag{124}$$

It is convenient to introduce a compact matrix-form notation for the Christoffel symbols

$$\Gamma \equiv \{\Gamma_\nu{}^\lambda\}, \qquad \Gamma_\nu{}^\lambda = dx^\mu \Gamma_{\mu\nu}{}^\lambda \tag{125}$$

and for the Riemann curvature tensor

$$\mathcal{R} = d\Gamma + \Gamma \wedge \Gamma, \qquad \mathcal{R} = \{\mathcal{R}^\rho{}_\lambda\}, \qquad \mathcal{R}^\rho{}_\lambda = \frac{1}{2} dx^\mu \wedge dx^\nu R_{\mu\nu}{}^\rho{}_\lambda \tag{126}$$

The product between adjacent entries is understood to be the matrix product.

The BRST transformations for the gravitational case are obtained by promoting the parameters ξ^μ to anticommuting fields. The BRST transformations of $g_{\mu\nu}$ are the same as above, (123), but now ξ^μ are anticommuting fields, which transform themselves as

$$\delta_\xi \xi^\mu = \xi^\lambda \partial_\lambda \xi^\mu \tag{127}$$

It is not hard to show that, due to this transformation, δ_ξ becomes a nilpotent operation:

$$\delta_\xi^2 = 0 \tag{128}$$

This is the coboundary operator for diffeomorphisms. It defines the corresponding cohomology problem, analogous to the one defined above for gauge theories.

In the sequel, we will be interested in fermions ψ interacting with gravity. The relevant action is

$$S[g, \psi] = \int d^{\mathrm{d}} x \sqrt{g} \, i\overline{\psi} \gamma^\mu (D_\mu + \frac{1}{2}\omega_\mu)\psi \tag{129}$$

where $\gamma^\mu = e_a^\mu \gamma^a$, and e_a^μ are the vielbein ($\mu, \nu, \ldots$ are world indices, $a, b, \ldots$ are flat indices). ω_μ is the spin connection:

$$\omega_\mu = \omega_\mu^{ab} \Sigma_{ab}$$

where $\Sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$ are the Lorentz generators. In Equation (129), ψ is a generic fermion (Dirac, Weyl or Majorana). A right-handed Weyl fermion will be more explicitly denoted $\psi_R = \frac{1+\gamma_*}{2}\psi$, where ψ is a Dirac field and γ_* denotes the chirality matrix (in 4d it is γ_5).

A fermion field transforms, under diffeomorphisms, is

$$\delta_\xi \psi = \xi^\mu \partial_\mu \psi, \tag{130}$$

and it is easy to prove that (129) is invariant under diffeomorphisms.

Classically, the energy–momentum tensor

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{S[g]}{\delta g^{\mu\nu}} = \frac{i}{4}\overline{\psi}\gamma_\mu \overset{\leftrightarrow}{\nabla}_\nu \psi + \{\mu \leftrightarrow \nu\} - \frac{i}{2}\eta_{\mu\nu}\overline{\psi}\gamma^\mu (D_\mu + \frac{1}{2}\omega_\mu)\psi, \qquad \nabla_\mu = D_\mu + \frac{1}{2}\omega_\mu, \tag{131}$$

is both conserved on shell and traceless, i.e.,

$$D^\mu T_{\mu\nu}(x) = 0, \qquad T^\mu_\mu(x) = 0 \tag{132}$$

The first condition follows from diffeomorphism invariance, the second from invariance of (129) under the Weyl transformation

$$\delta_\omega g_{\mu\nu}(x) = 2\omega(x)g_{\mu\nu}(x) \tag{133}$$

where $\omega(x)$ is an arbitrary infinitesimal parameter.

The presence of fermions in a theory carries in the game also another local symmetry, the local Lorentz symmetry. Since it will play a marginal role in this review, we limit ourselves to a short mention. Local Lorentz transformations on a spinor ψ are defined by

$$\delta_\Lambda \psi = -\frac{1}{2}\Lambda\psi, \qquad \Lambda = \Sigma_{ab}\Lambda^{ab} \tag{134}$$

where $\Lambda^{ab}(x)$ are arbitrary infinitesimal parameters, antisymmetric in a, b. This implies

$$\delta_\Lambda \omega_\mu = \partial_\mu\Lambda + \frac{1}{2}[\omega_\mu, \Lambda] \tag{135}$$

On the vielbein e_μ^a the transformation is $\delta_\Lambda e_\mu^a = \Lambda^a{}_b e_\mu^b$. Like for other symmetries, we will also promote $\Lambda_{ab}(x)$ to an anticommuting field, which is denoted by the same symbol and endowed with the transformation property $\delta_\Lambda \Lambda_{ab} = -\Lambda_a{}^c \Lambda_{cb}$. With this property, δ_Λ becomes a nilpotent operation: $\delta_\Lambda^2 = 0$. Consistency requires that $\delta_\xi \Lambda_{ab} = \xi^\mu \partial_\mu \Lambda_{ab}$ and $\delta_\Lambda \xi^\mu = 0$, which implies that

$$(\delta_\xi + \delta_\Lambda)^2 = 0. \tag{136}$$

A pause is necessary at this point to make an important recall. In quantum field theory, we normally have to do with several symmetries of the action simultaneously, and it does not make sense to analyze the anomalies of a single symmetry disregarding the others. For the sake of simplicity, let us consider two symmetries, which we shall refer to as R and S (for instance, vector and axial gauge transformations, or diffeomorphisms and local Lorentz or Weyl transformations). Let δ_R and δ_S be the corresponding coboundary operators. They are functional differential operators linear in the relevant ghost fields and satisfy

$$(\delta_R + \delta_S)^2 = 0 \tag{137}$$

(see, for instance, (136)), which splits into

$$\delta_R^2 = 0 \qquad \delta_S^2 = 0, \qquad \delta_R\delta_S + \delta_S\delta_R = 0 \tag{138}$$

A cocycle of $\delta_R + \delta_S$ is, in general, an overall cocycle, i.e., the sum of two integrated local expressions Δ_R and Δ_S such that

$$(\delta_R + \delta_S)W = \Delta_R + \Delta_S \tag{139}$$

Therefore

$$(\delta_R + \delta_S)(\Delta_R + \Delta_S) = 0 \tag{140}$$

This equation splits into three distinct ones

$$\delta_R\Delta_R = 0, \qquad \delta_S\Delta_R + \delta_R\Delta_S = 0, \qquad \delta_S\Delta_S = 0 \tag{141}$$

In all known cases so far, one can find a local expression of the fields (excluding ghosts), $\mathcal{C}$, such that, for instance, $\Delta_R = \delta_R\mathcal{C}$, and we can redefine $W \to W' = W - \mathcal{C}$ so that

$$\begin{aligned}
\delta_R W' &= \Delta_R' = 0 \tag{142} \\
\delta_S W' &= \Delta_S - \delta_S\mathcal{C} \equiv \Delta_S' \tag{143}
\end{aligned}$$

That is, in all known cases, the anomaly cocycle can be cast in the form that violates only one symmetry.

After the previous technical introduction, it is perhaps useful to give a glance at the anomalies related to diffeomorphisms and Weyl transformations that are in store for us.

Consistent diffeomorphisms and local Lorentz anomalies have the same form as the gauge anomalies introduced in Section 2.3, in particular in Formula (48), with V replaced by Γ or ω, F_t replaced by $\mathcal{R}_t$ or R_t, and λ by $\Xi^\mu{}_\nu = \partial_\nu \xi^\mu$ or Λ, respectively [R is defined by $R = d\omega + \frac{1}{2}[\omega, \omega]$, it represents the matrix-form $R = \{R^{ab}\}$ and it is related to $\mathcal{R}$ by $\mathcal{R}^\mu{}_\nu = e_a^\mu R^{ab} e_{b\nu}$]. It is not worth spending more time on these anomalies because in 4d, they almost all vanish due to the mechanism (A) explained above: the corresponding irreducible ad-invariant symmetric polynomials vanish identically, as already pointed out after Equation (49). When the relevant polynomial is reducible, however, we may come across a non-trivial anomaly. This may happen with mixed gauge-gravity anomalies. We shall deal with them in due time.

There is another type of diffeomorphism anomaly, which arises as follows. The transformation of the Christoffel symbol can be written in the following compact form

$$\delta_\xi \Gamma^\rho{}_\lambda \;=\; (i_\xi d + d i_\xi) \Gamma^\rho{}_\lambda + d\Xi^\rho{}_\lambda + [\Gamma, \Xi]^\rho{}_\lambda \tag{144}$$

From this and similar formulas, it is evident that any diffeomorphism splits into two parts: one looks like an ordinary gauge transformation, the other corresponds to the Lie derivative $\mathcal{L}_\xi = i_\xi d + d i_\xi$. The first transformation gives rise to the just mentioned consistent chiral anomalies, formally similar to the gauge anomalies. The second can also give rise to cocycles of the form

$$\mathcal{A}_{\mathsf{d}}[\xi, B] = \int d^{\mathsf{d}} x \sqrt{g} \, \partial \cdot \xi B \tag{145}$$

where B is a local expression of the fields that transforms like a scalar: $\delta_\xi B = \xi \cdot \partial B$. This cocycle is consistent and, usually, has a Weyl anomaly partner and forms with it a non-trivial cocycle of the coupled cohomology; see below.

Finally, we have Weyl anomalies. We have already introduced the Weyl transformation, Equation (133). For instance, it induces the following transformation for the Christoffel symbols

$$\delta_\omega \Gamma^\lambda_{\mu\nu} \;=\; \partial_\mu \omega \delta^\lambda_\nu + \partial_\nu \omega \delta^\lambda_\mu - g_{\mu\nu} g^{\lambda\rho} \partial_\rho \omega \tag{146}$$

Promoting $\omega(x)$ to an anticommuting field the Weyl transformation becomes nilpotent, $\delta_\omega^2 = 0$, and defines a coboundary operator with relative cohomology. We cannot tire to stress again the utmost importance of studying the joint cohomology defined by the coboundary operator $\delta_\xi + \delta_\omega$.

The possible nontrivial cocycles of δ_ω with vanishing diffeomorphism partners in 4d are well-known; they take the form

$$\Delta[g, \omega] = \int d^4 x \sqrt{g} \, \omega \, T[g], \qquad \delta_\omega \Delta[g, \omega] = 0 \tag{147}$$

where the density $T[g](x)$ can be the quadratic Weyl density

$$\mathcal{W}^2 = R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} - 2 R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2, \tag{148}$$

the Gauss–Bonnet (or Euler) density,

$$E = R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \tag{149}$$

and the Pontryagin density,

$$P = \frac{1}{2}\left(\varepsilon^{\mu\nu\mu'\nu'}R_{\mu\nu\lambda\rho}R_{\mu'\nu'}{}^{\lambda\rho}\right). \tag{150}$$

Other possible cocycles have densities

$$T_e[V] = F_{\mu\nu}F^{\mu\nu}, \tag{151}$$

and

$$T_o[V] = \varepsilon^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho}. \tag{152}$$

where $F_{\mu\nu}$ is an Abelian field strength, and analogous expressions for the non-Abelian case (a gauge potential is invariant under Weyl transformations in 4d). Other even trace anomalies will be introduced and discussed in Section 17.

A Weyl (or conformal) anomaly is the nonvanishing response of the path integral under a Weyl variation:

$$\delta_\omega W[g] = -\mathcal{A}_\omega, \qquad \mathcal{A}_\omega = \int d^{\mathrm{d}}x \sqrt{g}\,\omega(x)T(x) \tag{153}$$

where $T(x)$ is the trace of the energy–momentum tensor

$$T(x) = g^{\mu\nu}(x)\frac{\delta}{\delta g^{\mu\nu}(x)}W[g]. \tag{154}$$

$\mathcal{A}_\omega$ must take the form of a linear combination of the above cocycles.

In a theory like (129), all the above trace anomalies may show up. But in the sequel, we are principally interested only in those that signal the non-existence of the fermion propagator (the dangerous ones). For instance, the even parity anomalies show up in many theories (for instance, in a Dirac fermion theory) in which no doubt the fermion propagator exists. These anomalies are interesting for other reasons, but they do not signal any breakdown of the well-definiteness of the theory. For the time being we will focus only on the others, analogous to those that impair gauge theories. On the basis of the above discussion on the family's index theorem, we expect them to appear in chirally asymmetric theories (because the family's index for a self-adjoint operator vanishes identically). Refs.: [56,60–62].

6. Trace Anomalies

Historically, chiral anomalies were born out of ambiguities. The conservation of a chiral current turned out to be tied to an ambiguous integral. Finally it was decided that it is not wise to freely shift an integration variable in a divergent integral. And the ABJ anomaly was born. For trace anomalies, ambiguities are even more abundant and subtle than in the chiral case. It is imperative to have them in mind and to try to resolve them before plunging into calculations.

One first ambiguity is in the classical definition of the e.m. tensor. As an example, consider the action of a Dirac fermion coupled to a metric and an Abelian vector field

$$S = \int d^4x \sqrt{g}\, i\overline{\psi}\gamma^\mu\left(D_\mu + \frac{1}{2}\omega_\mu - iV_\mu\right)\psi \tag{155}$$

with the usual notation. The vector current is $j_\mu = \overline{\psi} \gamma_\mu \psi$ and for the stress–energy tensor, we may take

$$T_{\mu\nu} = \frac{i}{4} \overline{\psi} \gamma_\mu \overset{\leftrightarrow}{\nabla}_\nu \psi + \{\mu \leftrightarrow \nu\}, \qquad \nabla_\mu = D_\mu + \frac{1}{2}\omega_\mu - iV_\mu \tag{156}$$

They are both conserved on shell, and $T_{\mu\nu}$ is also traceless on shell. There is, in fact, another definition of the e.m. tensor, which corresponds to the general formula

$$T_{\mu\nu}(x) = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}}, \tag{157}$$

and, from the differentiation of the $\sqrt{g}$, in (155), this leads to

$$\widehat{T}_{\mu\nu} = \frac{i}{4} \overline{\psi} \gamma_\mu \overset{\leftrightarrow}{\nabla}_\nu \psi + (\mu \leftrightarrow \nu) - g_{\mu\nu} \frac{i}{2} \overline{\psi} \gamma^\lambda \overset{\leftrightarrow}{\nabla}_\lambda \psi = T_{\mu\nu} - g_{\mu\nu} T_\lambda{}^\lambda, \tag{158}$$

which is also conserved and traceless on shell. This ambiguity in the definition of the e.m. tensor gives rise to an ambiguity in the definition of the trace anomaly (as well as of the diffeomorphism anomaly). Such an uncertainty is, in fact, resolved by the definition (161) below: thanks to it, the second term of $\widehat{T}_{\mu\nu}$ drops out. For this reason, in the sequel, we will proceed with (156). But let us comment on (161).

The perturbative effective action corresponding to the classical action (129) is expressed in terms of the metric fluctuation $h_{\mu\nu}$, where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$:

$$W[g] = W[0] + \sum_{n=1}^{\infty} \frac{i^{n-1}}{2^n n!} \int \prod_{i=1}^{n} d^d x_i \sqrt{g(x_i)} h^{\mu_i \nu_i}(x_i) \langle 0| \mathcal{T} T_{\mu_1\nu_1}(x_1) \ldots T_{\mu_n\nu_n}(x_n)|0\rangle \tag{159}$$

from which we can derive the full one-loop expression of the e.m. tensor

$$\langle\!\langle T_{\mu\nu}(x)\rangle\!\rangle = 2\frac{\delta W[h]}{\delta h^{\mu\nu}(x)} = \sum_{n=1}^{\infty} \frac{i^n}{2^n n!} \int \prod_{i=1}^{n} d^d x_i \sqrt{g(x_i)} h^{\mu_i \nu_i}(x_i) \langle 0| \mathcal{T} T_{\mu\nu}(x) T_{\mu_1\nu_1}(x_1) \ldots T_{\mu_n\nu_n}(x_n)|0\rangle_c \tag{160}$$

To eliminate the above ambiguity as well as others, our definition of trace anomaly *in the perturbative case* will be the following: if $T_{\mu\nu}$ is the stress–energy tensor of a theory, the trace anomaly is given by the difference, [63,64],

$$g^{\mu\nu} \langle\!\langle T_{\mu\nu}(x)\rangle\!\rangle - \langle\!\langle g^{\mu\nu} T_{\mu\nu}(x)\rangle\!\rangle = T[g](x) \tag{161}$$

Thanks to this formula, the second term of $\widehat{T}_{\mu\nu}$ drops out. In addition, the field operator $T_\mu^\mu(x)$ vanishes on shell, while in the case when a theory contains a conformal soft breaking term (a mass term, for instance) $T_\mu^\mu(x) \neq 0$, even on shell. The second term of (161) is certainly present in such a case and the subtraction in (161) is needed in order to exclude this unwanted term from the anomaly.

This is also connected to a notation problem: the difference between the two definitions of the e.m. tensor can be reproduced in the definition of the corresponding effective action by inserting or not the factor $\sqrt{g(x_i)}$ in the integral $\int d^d x_i$ of such formulas as (159) and (160) and similar ones. Precisely, when this factor is inserted, one should use $\widehat{T}_{\mu\nu}$, while, if it is not inserted, the definition is valid for $T_{\mu\nu}$. Henceforth, we shall be using such a simpler version of the effective action.

But this is not all concerning the definition (161). As a matter of fact, as we shall see, the just mentioned term is non-vanishing in many other instances and in subtler ways.

The trace anomaly is a violation of the classical tracelessness condition $T^\mu_\mu(x) = 0$. It should be kept in mind that these equations are all valid on shell, while off-shell, they do not hold in general. Off-shell $T^\mu_\mu(x)$ is a non-vanishing quantum operator. Another important point to be kept in mind is that, in terms of representations of the Lorentz group, $T_{\mu\nu}(x)$ is a reducible tensor of which the trace $T^\mu_\mu(x)$ is an irreducible component. In the expression of the effective action, the latter is coupled to the field $h(x) = h^\mu_\mu(x)$. Likewise, the amplitude $\langle 0|\mathcal{T}T^\mu_\mu(x)\Phi(y)\Psi(z)|0\rangle$, where Φ and Ψ are generic Lorentz covariant fields, is an irreducible component of $\langle 0|\mathcal{T}T_{\mu\nu}(x)\Phi(y)\Psi(z)|0\rangle$. Much as we do in field theory when we compute quantum form factors, the former amplitude cannot be computed as a particular case of the latter but must be computed independently. There are several examples where, when calculated with Feynman diagrams, the amplitudes $\langle 0|\mathcal{T}T^\mu_\mu(x)\Phi(y)\Psi(z)|0\rangle$ and $\eta^{\mu\nu}\langle 0|\mathcal{T}_{\mu\nu}\Phi(y)\Psi(z)|0\rangle$ are different, see [5].

There are also other reasons that forbid us to merge the two terms of (161), due to the connection with other amplitudes. For instance, the Feynman amplitude corresponding to the second term is the same as the amplitude of the Kimura–Delbourgo–Salam anomaly, which appears in the current divergence of chiral fermions. Therefore, we have to live with the trace anomaly (perturbatively) defined (at the three-point level) by the difference

$$\eta^{\mu\nu}\langle 0|\mathcal{T}T_{\mu\nu}(x)\Phi(y)\Psi(z)|0\rangle - \langle 0|\mathcal{T}T^\mu_\mu(x)\Phi(y)\Psi(z)|0\rangle \tag{162}$$

This difference means in particular that the (regularized) effective action has discontinuities: differentiating it with respect to $h_{\mu\nu}$ and then saturating the result with $\eta_{\mu\nu}$ may not be the same as differentiating it with respect to $h(x) = h^\mu_\mu(x)$, which is the conjugate source of $T^\mu_\mu(x)$.

One may ask what its physical meaning is. The reason for taking the difference in the LHS of (161) is that two correlators may, in general, contain extra terms, which have nothing to do with the anomaly. These terms are as follows:

- Possible soft terms that classically violate conformal invariance.
- The term $i\eta^{\mu\nu}\bar\psi\overleftrightarrow{\partial}\psi$ in the modified definition of the e.m. tensor.
- The semi-local terms in the conformal WI.
- Possible off-shell contributions to the anomaly: The derivative of the effective action with respect to $h_{\mu\nu}$ contracted with $\eta^{\mu\nu}$, or the derivative with respect to h^μ_μ, do not automatically vanish off-shell. In fact, the operator T^μ_μ identically vanishes on shell. Therefore, its contribution can only be off-shell. This means that in Formula (161), the off-shell contributions to the anomaly are subtracted away. In other words, the trace anomaly (161) receives only on shell contributions.

All these extra terms cancel out in (161).

The definition (161) of the trace anomaly refers to the perturbative approach. In the non-perturbative approaches of the heat kernel type, the trace anomaly is simply the response of the effective action to a Weyl rescaling of the metric [In the heat kernel method, we differentiate the square kinetic operator and ignore the $\sqrt{g}$ factor in the action. Thus, the e.m. tensor we are dealing with is (156)]. The connection of the two definitions is not simple, in the non-perturbative cases it is hard if not impossible to separately evaluate the two terms of this definition. However, at least intuitively, we know that the anomaly, if any, appears precisely when the kinetic operator is not invertible, that is precisely when the theory is exactly on shell, which corresponds to our previous interpretation. In all cases considered the two approaches lead to the same (final) results. We may nevertheless ask: why is it that in the perturbative calculation we need to compute two separate terms? As we have noted before, the perturbative approach starts from the lowest order of the perturbative cohomology because higher-order calculations are more difficult and often inaccessible.

Now, the lowest-order perturbative cohomology is a much looser mathematical structure than the full BRST cohomology. The former has plenty of nontrivial cocycles, while in the latter, non-trivial cocycles are very limited in number. The definition (161) is designed to channel the lowest order perturbative results in the right track so as to coincide with the non-perturbative approaches. In a more formal language, one can say that in perturbative cohomology, each term of (161) is separately unstable, while their difference is stable.

After this long introduction, let us show a few examples of odd-parity trace anomalies.

6.1. Gauge-Induced Odd Parity Trace Anomaly of a Weyl Fermion

We refer to the action (155) with the fermion ψ being a right-handed Weyl fermion, $\psi = \psi_R$, coupled to an Abelian vector potential. This time we wish to compute the trace of the quantum e.m. tensor on the basis of the definition (161). To this end, the relevant objects are the classically conserved and traceless (on-shell) energy–momentum tensor

$$T_{\mu\nu}^{(R)} = \frac{i}{4}\overline{\psi_R}\gamma_\mu \overset{\leftrightarrow}{\nabla}_\nu \psi_R + \{\mu \leftrightarrow \nu\}, \qquad \nabla_\mu = D_\mu + \frac{1}{2}\omega_\mu + V_\mu \tag{163}$$

and the classically conserved current $J_{R\mu} = i\bar\psi_R\gamma_\mu\psi_R$. For a quantum evaluation, we must refer to the effective action

$$\begin{aligned}
W[h,V] &= W[0] + \sum_{n,r=1}^{\infty} \frac{i^{n+r-1}}{2^n n! r!} \int \prod_{i=1}^{n} dx_i h^{\mu_i\nu_i}(x_i) \prod_{l=1}^{r} dy_l e_{a_l}^{\lambda_l}(y_l) V_{\lambda_l}(y_l) \\
&\quad \cdot \langle 0|\mathcal{T} T_{\mu_1\nu_1}^{(R)}(x_1) \dots \widehat{T}_{\mu_n\nu_n}^{(R)}(x_n) j_R^{a_1}(y_1) \dots j_R^{a_r}(y_r)|0\rangle
\end{aligned} \tag{164}$$

The one- and two-point functions do not contribute to the trace anomaly we wish to compute. What we have to find, therefore, is the relation between the (odd parity) trace $\eta^{\mu\nu}\langle 0|\mathcal{T} T_{R\mu\nu}(x)j_{R\lambda}(y)j_{R\rho}(z)|0\rangle$ and the (odd parity) correlator $\langle 0|\mathcal{T} T_{R\mu}^{\mu}(x)j_{R\lambda}(y)j_{R\rho}(z)|0\rangle$. The Feynman rule for the vector-fermion-fermion vertex V_{vff} is $i\gamma_\mu P_R$ and the graviton-fermion-fermion vertex V_{hff} is

$$-\frac{i}{8}\big[(p-p')_\mu\gamma_\nu + (p-p')_\nu\gamma_\mu\big]P_R \tag{165}$$

where p, p' are the fermion momenta: one entering and the other exiting. The fermion propagator is the usual one [It is understood that we have added to the action a free left-handed fermion in order to have a well-defined propagator]. We have to compute a triangle diagram in which we associate an incoming momentum q to $T_{R\mu}^{\mu}(x)$ and outgoing momenta k_1, k_2 to the potentials coupled to the two chiral currents; thus, $q = k_1 + k_2$.

The expression for $\langle 0|\mathcal{T} T_{R\mu}^{\mu}(x)j_{R\lambda}(y)j_{R\rho}(z)|0\rangle$ is

$$\begin{aligned}
\widetilde{T}_{\mu\lambda\rho}^{(R)\mu}(k_1,k_2) &= \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\text{tr}\left[\frac{1}{\not p}\gamma_\lambda P_R \frac{1}{\not p - \not k_1}\gamma_\rho P_R \frac{1}{\not p - \not k_1 - \not k_2}(2\not p - \not q)P_R\right] \\
&= \frac{1}{2}\int \frac{d^4p\,d^\delta\ell}{(2\pi)^{4+\delta}}\text{tr}\left\{\frac{\not p}{p^2-\ell^2}\gamma_\lambda \frac{\not p - \not k_1}{(p-k_1)^2-\ell^2}\gamma_\rho \frac{\not p - \not q + \not\ell}{(p-q)^2-\ell^2}(2\not p + 2\not\ell - \not q)P_R\right\}
\end{aligned} \tag{166}$$

The second line is the regularized integral obtained with the adjunction to p of a momentum ℓ along δ extra dimensions. To it, the cross term must be added. Evaluating the integral after a Wick rotation yields

$$\widetilde{T}_{\mu\lambda\rho}^{(R)\mu}(k_1,k_2) + \widetilde{T}_{\mu\rho\lambda}^{(R)\mu}(k_2,k_1)\Big|_{odd} = \frac{1}{24\pi^2}\varepsilon_{\mu\nu\lambda\rho}k_1^\mu k_2^\nu, \tag{167}$$

This is the result for $\langle 0|\mathcal{T} T_{R\mu}^{\mu}(x)j_{R\lambda}(y)j_{R\rho}(z)|0\rangle$. In an analogous way, we obtain

$$\frac{1}{2}\int\frac{d^4p d^\delta\ell}{(2\pi)^{4+\delta}}\text{tr}\left\{\frac{\not{p}}{p^2-\ell^2}\gamma_\lambda\frac{\not{p}-\not{k}_1}{(p-k_1)^2-\ell^2}\gamma_\rho\frac{\not{p}-\not{q}}{(p-q)^2-\ell^2}(2\not{p}-\not{q})P_R\right\}\bigg|_{odd}+cross$$

$$=\frac{1}{48\pi^2}\varepsilon_{\mu\nu\lambda\rho}k_1^\mu k_2^\nu \tag{168}$$

which gives the odd parity part of $\eta^{\mu\nu}\langle 0|\mathcal{T}T_{R\mu\nu}(x)j_{R\lambda}(y)j_{R\rho}(z)|0\rangle$. Therefore, after returning to a Lorentzian background, we obtain

$$g^{\mu\nu}\langle\langle T_{R\mu\nu}(x)\rangle\rangle\Big|_{odd}-\langle\langle g^{\mu\nu}T_{R\mu\nu}(x)\rangle\rangle\Big|_{odd}=-\frac{i}{96\pi^2}\varepsilon_{\mu\nu\lambda\rho}\partial^\mu V^\nu(x)\partial^\lambda V^\rho(x) \tag{169}$$

The non-Abelian version of this result is

$$\mathcal{A}_\omega^{(odd,R)}=-\frac{i}{384\pi^2}\varepsilon_{\mu\nu\lambda\rho}\text{tr}(F^{\mu\nu}F^{\lambda\rho}) \tag{170}$$

This is the gauge-induced odd parity trace anomaly for a right-handed Weyl fermion. For a left-handed Weyl fermion, the only difference is the sign in front of γ_5 in the initial Formula (166). Therefore, the only change for the odd parity part is an overall sign:

$$\mathcal{A}_\omega^{(odd,L)}=\frac{i}{384\pi^2}\varepsilon_{\mu\nu\lambda\rho}\text{tr}(F^{\mu\nu}F^{\lambda\rho}) \tag{171}$$

In fact, we have not justified the cubic and quartic terms in (170) and (171), which would require the evaluation of higher-order amplitudes. They will be derived more comfortably with non-perturbative methods.

6.2. Gravity-Induced Odd Parity Trace Anomaly of a Weyl Fermion

Let us consider the case of the action (155) for a right-handed Weyl fermion $\psi\equiv\psi_R$ coupled only to a metric ($V_\mu=0$). The effective action is

$$W^{(R)}[g]=W^{(R)}[0]+\sum_{n=1}^\infty\frac{i^{n-1}}{2^n n!}\int\prod_{i=1}^n d^{\text{d}}x_i h^{\mu_i\nu_i}(x_i)\langle 0|\mathcal{T}T_{\mu_1\nu_1}^{(R)}(x_1)\dots T_{\mu_n\nu_n}^{(R)}(x_n)|0\rangle \tag{172}$$

Our task is to compute multi-point amplitudes of the e.m. tensor. To start with, from the action we have to extract the Feynman rules. We write it down more explicitly as follows

$$S=\int d^4x\sqrt{|g|}\left[\frac{i}{2}\overline{\psi_R}\gamma^\mu\overleftrightarrow{\partial}_\mu\psi_R-\frac{1}{4}\varepsilon^{\mu abc}\omega_{\mu ab}\overline{\psi_R}\gamma_c\gamma_5\psi_R\right] \tag{173}$$

where it is understood that the derivative applies to ψ_R and $\overline{\psi_R}$ only. We have used the relation $\{\gamma^a,\Sigma^{bc}\}=i\varepsilon^{abcd}\gamma_d\gamma_5$.

Now we expand

$$e_\mu^a=\delta_\mu^a+\chi_\mu^a+...,\qquad e_a^\mu=\delta_a^\mu+\hat{\chi}_a^\mu+...,\qquad\text{and}\quad g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}+... \tag{174}$$

and make a local Lorentz gauge choice by dropping the antisymmetric part of the vierbein. Inserting these expansions in the defining relations $e_\mu^a e_b^\mu=\delta_b^a$, $g_{\mu\nu}=e_\mu^a e_\nu^b\eta_{ab}$, we find

$$\hat{\chi}_\nu^\mu=-\chi_\nu^\mu\qquad\text{and}\qquad h_{\mu\nu}=2\chi_{\mu\nu}. \tag{175}$$

Let us expand accordingly the spin connection:

$$\omega_{\mu ab}\varepsilon^{\mu abc}=-\varepsilon^{\mu abc}\partial_\mu\chi_{a\lambda}\chi_b^\lambda+... \tag{176}$$

Therefore, up to second order, the action can be written (by incorporating $\sqrt{|g|}$ in a redefinition of the ψ_R field.)

$$S \approx \int d^4x \left[\frac{i}{2}\left(\delta_a^\mu - \chi_a^\mu + \frac{3}{2}(\chi^2)_a^\mu \right) \overline{\psi_L} \gamma^a \overset{\leftrightarrow}{\partial}_\mu \psi_L + \frac{1}{4} \varepsilon^{\mu abc} \partial_\mu \chi_{a\lambda} \chi_b^\lambda \overline{\psi}_L \gamma_c \gamma_5 \psi_L \right]$$

As a consequence, the Feynman rules are as follows (momenta are incoming, and the external gravitational field is assumed to be $h_{\mu\nu}$). The fermion propagator is the usual one as before. The two-fermion-one-graviton vertex (V_{ffh}) is

$$-\frac{i}{8}\left[(p-p')_\mu \gamma_\nu + (p-p')_\nu \gamma_\mu \right] \frac{1+\gamma_5}{2} \tag{177}$$

There are two two-fermion-two-graviton vertices: V_{ffhh}

$$\frac{3i}{64}\left[\left((p+p')_\mu \gamma_{\mu'} \eta_{\nu\nu'} + (p+p')_\mu \gamma_{\nu'} \eta_{\nu\mu'} + \{\mu \leftrightarrow \nu\} \right) \right. \tag{178}$$
$$\left. + \left((p+p')_{\mu'} \gamma_\mu \eta_{\nu\nu'} + (p+p')_{\nu'} \gamma_\nu \eta_{\mu\nu'} + \{\mu' \leftrightarrow \nu'\} \right) \right] \frac{1+\gamma_5}{2}$$

and V_{ffhh}^ε

$$\frac{1}{64} t_{\mu\nu\mu'\nu'\kappa\lambda}(k-k')^\lambda \gamma^\kappa \frac{1+\gamma_5}{2} \tag{179}$$

where

$$t_{\mu\nu\mu'\nu'\kappa\lambda} = \eta_{\mu\mu'} \varepsilon_{\nu\nu'\kappa\lambda} + \eta_{\nu\nu'} \varepsilon_{\mu\mu'\kappa\lambda} + \eta_{\mu\nu'} \varepsilon_{\nu\mu'\kappa\lambda} + \eta_{\nu\mu'} \varepsilon_{\mu\nu'\kappa\lambda} \tag{180}$$

and the graviton momenta k, k' are incoming.

Let us come next to the explicit calculations. Like in the previous calculations, the lowest order contribution to the odd-parity trace anomaly can be proven to come from the three-point e.m. amplitude. In a careless approach, this does not seem to make sense because a well-known result of CFT states that a conformal odd parity three-point function $\langle 0|\mathcal{T} T_{\mu\nu}(x) T_{\mu'\nu'}(y) T_{\alpha\beta}(z)|0\rangle^{(odd)}$ vanishes identically for algebraic reasons. Therefore, at the lowest perturbative order, we can write

$$\eta^{\mu\nu} \langle\!\langle T_{\mu\nu}(x) \rangle\!\rangle^{(odd)} = 0, \tag{181}$$

as can be proven also by a direct calculation. However according to the definition (161) we must compute also the second term with one insertion of a trace of the e.m. tensor.

Here, at variance with the previous examples in this review, the Feynman diagrams to be calculated do not reduce to a single triangle diagram. Due to the existence of the four-legs vertices (178) and (179), there are also two bubble diagrams. They are obtained by joining two vertices, V_{ffh} (on the left) and V_{ffhh}^ε or V_{ffhh} (on the right), with two fermion propagators. The incoming graviton in V_{ffh} has momentum q and the two outgoing gravitons in V_{ffhh}^ε or V_{ffhh} are specified by $k_1, \{\mu, \nu\}$ and $k_2, \{\mu', \nu'\}$, respectively, with $q = k_1 + k_2$. The two fermion propagators form a loop. However these diagrams turn out to give a vanishing contribution to the anomaly. Therefore, let us focus on the triangle diagram.

The triangle diagram with three e.m. insertions is constructed by joining three vertices V_{ffh} with three fermion lines. The external momenta are q (incoming) with labels α and β, and k_1, k_2 (outgoing), with labels μ, ν and μ', ν', respectively. Of course $q = k_1 + k_2$. The internal momenta are p, $p - k_1$ and $p - k_1 - k_2$, respectively.

Employing the above Feynman rules of the chiral fermion coupled to an external gravitational field, we can write down the Fourier transform of the three-point e.m. tensor amplitude $\widetilde{\mathcal{T}}^{(odd)}_{\mu\nu\mu'\nu'\alpha\beta}(k_1, k_2)$. Contracting α and β and moving P_R to the rightmost position [This is, in this case, the simplest prescription, but by no means the only one. In [5] it is shown that the position of γ_5 is irrelevant for the final result provided we use the same prescription also in computing the diffeomorphism anomaly. In other words, its position is, in general, irrelevant to the joint Weyl-diffeomorphism cohomology], the correlator we are looking for is

$$
\widetilde{\mathsf{T}}_{\mu\nu\mu'\nu'}(k_1, k_2) \equiv \widetilde{\mathcal{T}}^{(odd)\alpha}_{\mu\nu\mu'\nu'\alpha}(k_1, k_2) = -\frac{1}{256} \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left\{ \left[\frac{\slashed{p}}{p^2}(2p - k_1)_\mu \gamma_\nu + (\mu \leftrightarrow \nu) \right] \frac{(\slashed{p} - \slashed{k}_1)}{(p - k_1)^2} \right.
$$
$$
\left. \times \left[(2p - 2k_1 - k_2)_{\mu'} \gamma_{\nu'} (\mu' \leftrightarrow \nu') \right] \frac{(\slashed{p} - \slashed{k}_1 - \slashed{k}_2)}{(p - k_1 - k_2)^2}(2\slashed{p} - \slashed{k}_1 - \slashed{k}_2)\left(\frac{1 + \gamma_5}{2} \right) \right\}. \tag{182}
$$

which is evidently divergent. We dimensionally regularize it in the usual way. The calculation is lengthy. Finally, after adding the cross term, one obtains

$$
\widehat{\mathsf{T}}^{(tot)}_{\mu\nu\mu'\nu'}(k_1, k_2) = -\frac{i}{3072\pi^2} k_1^\alpha k_2^\beta \left(\left(k_1^2 + k_2^2 + k_1 \cdot k_2 \right) t_{\mu\nu\mu'\nu'\alpha\beta} - t^{(21)}_{\mu\nu\mu'\nu'\alpha\beta} \right). \tag{183}
$$

where

$$
t^{(21)}_{\mu\nu\mu'\nu'\kappa\lambda} = k_{2\mu}k_{1\mu'}\varepsilon_{\nu\nu'\kappa\lambda} + k_{2\nu}k_{1\nu'}\varepsilon_{\mu\mu'\kappa\lambda} + k_{2\mu}k_{1\nu'}\varepsilon_{\nu\mu'\kappa\lambda} + k_{2\nu}k_{1\mu'}\varepsilon_{\mu\nu'\kappa\lambda} \tag{184}
$$

To simplify the derivation, we shall set the external lines on shell, $k_1^2 = k_2^2 = 0$. This requires a comment.

Inserted in the formula for the effective action, the term $k_1^\alpha k_2^\beta (k_1^2 + k_2^2) t_{\mu\nu\mu'\nu'\alpha\beta}$ gives rise to the cocycle Δ_ω

$$
\Delta_\omega \sim \int d^4x \omega \varepsilon^{\mu\nu\lambda\rho} \partial_\mu \Box h_\nu^\alpha \partial_\lambda h_{\rho\alpha} \tag{185}
$$

This is a consistent Weyl cocycle (at the lowest significant order), but it is not invariant under diffeomorphism transformations, and, of course, there is no covariant extension of it to all orders. In such kind of situation this usually means that there is a partner cocycle of the diffeomorphisms, of the type (145), that, together with (185), form a complete cocycle of the overall Weyl-diffeomorphism cohomology. When this happens, one can usually find a local counterterm that cancels one of the two partner cocycles and modifies the other. There are plenty of such examples. But, in this case, if we compute the odd parity part of the divergence of the e.m. tensor, we find zero, as expected because of the remark around Equation (181). We can also view the problem from another point of view; we can cancel the cocycle (185) by subtracting from the effective action a counterterm

$$
\sim \int d^4x h \varepsilon^{\mu\nu\lambda\rho} \partial_\mu \Box h_\nu^\alpha \partial_\lambda h_{\rho\alpha} \tag{186}
$$

where $h = h_\mu^\mu$. But this counterterm is not invariant under diffeomorphisms, therefore it would generate a diffeomorphism anomaly. This means that at the lowest perturbative order, the cross-consistency conditions for diffeomorphisms and Weyl transformations are not satisfied. This is not surprising because one can prove that on algebraic grounds, the odd part of the three-point conformal amplitude of the e.m. tensor vanishes identically. In other words, due to algebraic constraints, the usual cohomology machinery for the two joint symmetries cannot work, at least at the lowest perturbative order (the three-point function).

A way out is to compute higher order diagrams (the four-, five-, ... point function may have a nonvanishing odd part) or resort to a nonperturbative method. The first alternative is rather impervious, the second more accessible, and, in fact, we will exhibit a nonperturbative derivation of this trace anomaly later on. But there is also a way to utilize the result (183) relying on the fact that should we be able to go beyond the lowest order calculation the rupture of the machinery would disappear. What we have to do is render the term (185) irrelevant for the derivation of the final result. This is the rationale for putting the external lines on shell.

6.2.1. On Shell Conditions

Putting the external lines on shell means that the corresponding fields have to satisfy the eom of Einstein–Hilbert gravity $R_{\mu\nu} = 0$. In the linearized form, this means

$$\Box\chi_{\mu\nu} - \partial_\mu\partial_\lambda\chi_\nu^\lambda + \partial_\nu\partial_\lambda\chi_\mu^\lambda - \partial_\mu\partial_\nu\chi_\lambda^\lambda = 0 \tag{187}$$

We also choose the De Donder gauge: $\Gamma_{\mu\nu}^\lambda g^{\mu\nu} = 0$, which at the linearized level becomes $2\partial_\mu\chi_\lambda^\mu - \partial_\lambda\chi_\mu^\mu = 0$. In this gauge (187) becomes

$$\Box\chi_{\mu\nu} = 0 \tag{188}$$

In momentum space, this implies that $k_1^2 = k_2^2 = 0$. But this is not simply an *ad hoc* trick. We are, in fact, defining a restricted cohomology of the diffeomorphisms and the Weyl transformations: a cohomology defined up to terms $\Box h_{\mu\nu}$ and $\Box\xi^\mu$. This is a well-defined cohomology, under which we have, in particular,

$$\delta_\xi\left(2\partial_\mu\chi_\lambda^\mu - \partial_\lambda\chi_\mu^\mu\right) = 2\,\Box\xi_\lambda \approx 0 \tag{189}$$

i.e., in this restricted cohomology, the De Donder gauge fixing is irrelevant. Similarly, the term (185) remains null after a restricted diffeomorphism transformation, so it does not play any role in cohomology. The restricted cohomology has the same odd class (the Pontryagin one) as the unrestricted one, i.e., it completely determines it (this is not true, for instance, for the even parity classes).

6.2.2. Overall Contribution

The overall one-loop contribution to the trace anomaly in momentum space, *as far as the parity-violating part is concerned*, is given by (183). After returning to the Minkowski metric and Fourier-anti-transforming it, we can extract the local expression of the trace anomaly by replacing the results found so far in (160). The result, to the lowest order, is

$$\langle\langle T_\mu^\mu(x)\rangle\rangle^{(odd)} \approx -\frac{i}{768\pi^2}\varepsilon^{\mu\nu\lambda\rho}\left(\partial_\mu\partial_\sigma h_\nu^\tau\partial_\lambda\partial_\tau h_\rho^\sigma - \partial_\mu\partial_\sigma h_\nu^\tau\partial_\lambda\partial^\sigma h_{\tau\rho}\right) \tag{190}$$

The factor in front is due to a factor of 8 that comes from the coefficient $\frac{1}{2^n n!}$ with $n = 2$ in the denominator and a factor of 8 in the numerator because the Fourier transform of the three-point function of the e.m. tensor is 8 times $\widetilde{\mathcal{T}}_{\mu\nu\mu'\nu'\alpha\beta}(k_1, k_2)$; another factor of 4 in the numerator is due to the symmetry of the tensors t and $t^{(21)}$ in (183), which yields four times the same term. Comparing with the expansion

$$\varepsilon^{\mu\nu\lambda\rho}R_{\mu\nu}{}^{\sigma\tau}R_{\lambda\rho\sigma\tau} = \varepsilon^{\mu\nu\lambda\rho}\left(\partial_\mu\partial_\sigma\chi_\nu^a\partial_\lambda\partial_a\chi_\rho^\sigma - \partial_\mu\partial_\sigma\chi_\nu^a\partial_\lambda\partial^\sigma\chi_{a\rho}\right) + ... \tag{191}$$

we obtain

$$\langle\langle T_\mu^\mu(x)\rangle\rangle^{(odd)} = -\frac{i}{768\pi^2}\frac{1}{2}\varepsilon^{\mu\nu\lambda\rho}R_{\mu\nu}{}^{\sigma\tau}R_{\lambda\rho\sigma\tau}$$ (192)

Now applying the definition (161) and recalling (181), we obtain the covariant expression of the parity-violating part of the trace anomaly

$$T[g](x) = \frac{i}{768\pi^2}\frac{1}{2}\varepsilon^{\mu\nu\lambda\rho}R_{\mu\nu}{}^{\sigma\tau}R_{\lambda\rho\sigma\tau}.$$ (193)

It goes without saying that a left-handed Weyl fermion has the same anomaly with the opposite sign in front.

Remark 4. *On the basis of a not uncommon prejudice in the literature, the previous result sounds unexpected. It is sometimes stated that the theory of gravity is chirally blind, meaning that the relevant charge, the mass, is positive and is thus different from the typical case of a U(1) interaction. However, this is based on a misunderstanding. The coupling between gravity and matter is given by the juxtaposition of the metric and the energy–momentum tensor, and the energy–momentum tensors of fermions with opposite chiralities are different. Therefore, it is to be expected that at some stage, differences might emerge between fermions with opposite chiralities in their interaction with gravity. An obvious place where such differences may show up are the anomalies, and in particular, the trace anomaly, because it involves precisely the coupling between the metric and the energy–momentum tensor.*

Remark 5. *Contrary to what may seem at first sight, the anomalies (271) and (273) violate parity but, due to the imaginary coefficient in front, do not violate time reversal. Therefore, assuming CPT invariance, they do not violate CP either. The imaginary coefficient means, however, that the quantum effective Hamiltonian becomes complex, which thus becomes a source of possible violation of unitarity.*

6.3. The KDS Anomaly

It is not irrelevant to notice at this point a fact that has not been given the right importance in the literature. Consider a Dirac fermion ψ coupled to a metric and vector potential V_μ like in (155). It is invariant under the Abelian local transformation $\psi \to e^{i\gamma_5\eta}\psi$ and $V_\mu \to V_\mu + \gamma_5\partial_\mu\eta$. The corresponding axial current $j_{5\mu} = \overline{\psi}\gamma_\mu\gamma_5\psi$ is classically conserved. Is it conserved in the quantum theory? Or is it violated due to the gravitational interaction? To answer this question, one has to consider the amplitude $\langle\partial\cdot j_5 TT\rangle$. The amplitude in question is the same as (182) with the factor $(2\not{p} - \not{q})$ replaced by $\not{q}$ and P_R replaced by γ_5, and an overall coefficient $\frac{1}{64}$ instead of $\frac{1}{256}$. If we rewrite $\not{q} = 2\not{p} - (2\not{p} - \not{q})$, we see that the second term reproduces the calculation of the previous subsection multiplied by a suitable coefficient, while the term corresponding to $2\not{p}$, once regularized, is easily seen to vanish. Therefore, we can conclude that

$$\partial^\mu j_{5\mu} = \frac{1}{768\pi^2}\varepsilon^{\mu\nu\lambda\rho}R_{\mu\nu}{}^{\sigma\tau}R_{\lambda\rho\sigma\tau}.$$ (194)

This is the Kimura–Delbourgo–Salam anomaly, [65]. It is an anomaly of a Dirac fermion theory, that is, of the ABJ type [ABJ-type anomalies appear in theories involving Dirac fermions. A Dirac fermion can be decomposed into a sum of one left-handed and one right-handed Weyl fermion. If the two Weyl fermions have consistent anomalies, the latter have opposite signs. The consistent anomaly of the fermion is the sum of these two; thus, it vanishes. The ABJ anomaly is instead the difference between the two (the right-handed minus the left-handed one)]. This derivation of the KDS anomaly is an example of the rigid

link that connects chiral ABJ-type anomalies to odd parity trace anomalies. The same holds also for the gauge induced trace anomaly considered previously, see Section 6.1, in a Weyl fermion theory, which is rigidly linked in the same way to the standard ABJ gauge anomaly. Refs.: [65–70].

7. The Standard Model in a Metric Background Geometry

We can now analyze the overall panorama of the new anomalies in the SM due to its immersion in a non-trivial metric background. As mentioned before, the dangerous consistent diffeomorphism anomalies, which might occur in the divergence of the e.m. tensor, vanish identically for group theoretical reasons. But, as already pointed out, there may appear a mixed gauge-gravitational anomaly.

7.1. Mixed Gauge-Gravity Anomaly

The action (155), with $\psi \equiv \psi_R$, is invariant under the local $U(1)$ transformation $V_\mu \to V_\mu + \partial_\mu \lambda$. The relevant current $j_\mu = \bar{\psi}_R \gamma_\mu \psi_R$ is classically conserved, and to find the corresponding anomaly, if any, we have to evaluate the amplitude $\langle \partial \cdot j_R T_R T_R \rangle$. The latter can be easily calculated with the usual Feynman diagram. The final result is

$$\partial^\mu j_{R\mu} = \frac{1}{1536\pi^2} \varepsilon^{\mu\nu\lambda\rho} R_{\mu\nu}{}^{\sigma\tau} R_{\lambda\rho\sigma\tau}. \tag{195}$$

Its integrated form is $\sim \Delta_G(\lambda, g) = \int d^4 x \lambda \varepsilon^{\mu\nu\lambda\rho} R_{\mu\nu}{}^{\sigma\tau} R_{\lambda\rho\sigma\tau}$, which is a diffeomorphism invariant (trivially) consistent Abelian gauge cocycle. This cocycle can take different forms, for instance, $\Delta_G(\lambda, g)$ is equivalent to

$$\Delta_d(\xi, g, V) = \int d^4 x \sqrt{g} \varepsilon^{\mu\nu\lambda\rho} \mathrm{tr}(\partial_\mu \Xi \Gamma_\nu) F_{\lambda\rho} \tag{196}$$

where $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$, Ξ represents the matrix $\Xi_\tau{}^\sigma = \partial_\tau \xi^\sigma$ and Γ_μ represents the matrix $\Gamma^\tau_{\mu\sigma}$. They can be obtained from each other by subtracting a suitable conterterm.

This anomaly can also take the form of a (gauge+diffeomorphism invariant) Lorentz anomaly. The reason is that all these cocycles descend from the same 6-form $\mathrm{tr}(RR)F$ of the classifying space of the group $SO(4) \times U(1)$.

Let us see the effect of these mixed anomalies in the SM Table 2. Adopting the style of Section 2.4 and the notation Σ for the generic Lorentz generator, it is easy to see that the only a priori non-vanishing possibility is

Table 2. MSM mixed.

- $\Sigma \times \Sigma \times T^{u(1)}$, the trace $\mathrm{tr}\left(\Sigma^{ab}\Sigma^{cd}\right)$ is non-vanishing, but it is multiplied by the total $U(1)$ charge: $6\left(\frac{1}{6}\right) - 3\left(\frac{2}{3}\right) - 3\left(-\frac{1}{3}\right) + 2\left(-\frac{1}{2}\right) + 1 = 0$.

The addition of sterile neutrinos to (1) does not alter this conclusion.

7.2. Do Odd-Parity Trace Anomalies Cancel?

Next, let us analyze the situation for trace anomalies in the SM. First, it is important to clarify that these anomalies are relevant only when gravity is effectively coupled to the model. For, looking at the effective action (164) one can see that any quantum contribution to the trace of the e.m. tensor is multiplied by the metric fluctuation $h = h^\mu_\mu$. If the latter vanishes, no quantum contribution from the trace anomaly can affect the effective action. Therefore, the previous anomaly analysis is completely satisfactory if gravity can be disregarded. On the other hand, if gravity is assumed to interact with the standard model

via the minimal covariant couplings, odd trace anomalies may have significant fallouts, as was pointed out before. Therefore it makes sense to study the conditions under which also these anomalies cancel. Hereafter, we broach this subject.

For odd parity trace anomalies, the cancelation takes place in any case if there is a perfect balance between opposite chiralities, between, say, left-handed and right-handed components. From the above we see that in the multiplet (1) there is a balance between the left-handed and right-handed components except for the left-handed ν_e.

This breakdown (Table 3) is naturally avoided if we add to the above SM multiplet other Weyl fermions (for instance, a right-handed sterile neutrino) so as to produce a chirally symmetric model without compromising the cancelation of the chiral gauge and gravity anomalies.

Table 3. MSM trace-gravity.

• Therefore, the multiplet (1), when weakly coupled to gravity, will produce an overall non-vanishing (imaginary) coefficient for the Pontryagin density in the trace anomaly.

The analysis concerning gauge-induced odd trace anomalies, see (170), is more complex. First of all, we have three types of such anomalies (Table 4), constructed with $SU(3), SU(2)$ and $U(1)$ gauge fields, respectively.

Table 4. MSM trace-gauge.

• We have six units of the anomaly (170) with curvature $F \equiv F^{\mathfrak{su}(3)}$ and six units with opposite signs. Therefore, the multiplet (1) is free of these anomalies.
• Instead, we have 4 units of the same anomaly with gauge field $F \equiv F^{\mathfrak{su}(2)}$ and positive sign, see (171), computed in the doublet representation of $\mathfrak{su}(2)$.
• Finally, we have a $U(1)$ gauge-induced trace anomaly with a vanishing total coefficient: $6\left(\frac{1}{6}\right)^2 - 3\left(\frac{2}{3}\right)^2 - 3\left(-\frac{1}{3}\right)^2 + 2\left(-\frac{1}{2}\right)^2 - (-1)^2 = 0$

The addition of sterile neutrinos does not change these conclusions. *The $SU(2)$ gauge-induced odd trace anomalies do not cancel in the MSM.* Refs.: [5,15].

8. A Metric-Axial-Tensor Background

Before passing to the second part of the paper, we would like to complete our description of the anomalies relevant to the SM by showing how to compute the previous trace anomalies in a non-perturbative way via the SDW method. To this end, we enlarge the background geometry by introducing the metric-axial-tensor (MAT) gravity. It is a generalization of Bardeen's method for gauge fields to the gravitational environment. It must be said that this extension is not strictly necessary to derive the final results, but, in view of the ambiguities scattered in this type of computations, as pointed out before, this expedient is a good bookkeeping device, certainly less exposed to blunders than the direct method, and, anyhow, it introduces in gravity theories a richer formalism susceptible of new unexplored applications, as we shall see in the second part of the paper.

8.1. Axial-Complex Analysis and Geometry

Axial-complex numbers are defined by

$$\hat{a} = a_1 + \gamma_5 a_2 \tag{197}$$

where a_1 and a_2 are real numbers (a_1 is called axial-real and a_2 axial-imaginary). Arithmetic is defined in the obvious way. We have a natural conjugation operation $\bar{\hat{a}} = a_1 - \gamma_5 a_2$.

We shall consider functions $\hat{f}(\hat{x})$ of the axial-complex variable $\hat{x} = x_1 + \gamma_5 x_2$, as well as functions of several axial-complex variables $\hat{x}^\mu = x_1^\mu + \gamma_5 x_2^\mu$. We can easily define derivatives:

$$\frac{\partial}{\partial \hat{x}^\mu} = \frac{1}{2}\left(\frac{\partial}{\partial x_1^\mu} + \gamma_5 \frac{\partial}{\partial x_2^\mu}\right), \qquad \frac{\partial}{\partial \bar{\hat{x}}^\mu} = \frac{1}{2}\left(\frac{\partial}{\partial x_1^\mu} - \gamma_5 \frac{\partial}{\partial x_2^\mu}\right) \tag{198}$$

Notice that for axial-analytic functions (that is, functions defined by their Taylor series expansion),

$$\frac{d}{d\hat{x}} = \frac{\partial}{\partial x_1} \equiv \frac{\partial}{\partial \hat{x}}, \tag{199}$$

whereas $\frac{\partial}{\partial \bar{\hat{x}}}\hat{f}(\hat{x}) = 0$.

As for integrals, since we will always deal with fluctuating fields or parameters rapidly decreasing at infinity, we define $\int d\hat{x}\hat{f}(\hat{x})$ as the rapidly decreasing primitive $\hat{g}(\hat{x})$ of $\hat{f}(\hat{x})$. Therefore, in particular, the property

$$\int d\hat{x}\frac{\partial}{\partial \hat{x}}\hat{f}(\hat{x}) = 0 \tag{200}$$

follows immediately.

In this axial-spacetime we introduce an axial-Riemannian geometry as follows. We generalize the metric $\hat{g}_{\mu\nu} = g_{\mu\nu} + \gamma_5 f_{\mu\nu}$, by adding to the usual metric an axial symmetric tensor. Their background values are $\eta_{\mu\nu}$ and 0, respectively. So, we write as usual $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Likewise, for the vierbein we write $\hat{e}_\mu^a = e_\mu^a + \gamma_5 c_\mu^a$ and $\hat{e}_a^\mu = \tilde{e}_a^\mu + \gamma_5 \tilde{c}_a^\mu$. This implies

$$\eta_{ab}\left(e_\mu^a e_\nu^b + c_\mu^a c_\nu^b\right) = g_{\mu\nu}, \qquad \eta_{ab}\left(e_\mu^a c_\nu^b + e_\nu^a c_\mu^b\right) = f_{\mu\nu} \tag{201}$$

The Christoffel symbols are defined by

$$\hat{\Gamma}_{\mu\nu}^\lambda = \frac{1}{2}\hat{g}^{\lambda\rho}\left(\frac{\partial}{\partial \hat{x}^\mu}\hat{g}_{\rho\nu} + \frac{\partial}{\partial \hat{x}^\nu}\hat{g}_{\mu\rho} - \frac{\partial}{\partial \hat{x}^\rho}\hat{g}_{\mu\nu}\right) \tag{202}$$

They split as $\hat{\Gamma}_{\nu\lambda}^\mu = \Gamma_{\nu\lambda}^{(1)\mu} + \gamma_5 \Gamma_{\nu\lambda}^{(2)\mu}$, and are such that the metricity condition is satisfied

$$\frac{\partial}{\partial \hat{x}^\mu}\hat{g}_{\nu\lambda} = \hat{\Gamma}_{\mu\nu}^\rho\hat{g}_{\rho\lambda} + \hat{\Gamma}_{\mu\lambda}^\rho\hat{g}_{\nu\rho}, \tag{203}$$

Proceeding the same way, one can define the MAT Riemann tensor via $\hat{R}_{\mu\nu\lambda}^\rho$:

$$\hat{R}_{\mu\nu\lambda}^\rho = -\partial_\mu\hat{\Gamma}_{\nu\lambda}^\rho + \partial_\nu\hat{\Gamma}_{\mu\lambda}^\rho - \hat{\Gamma}_{\mu\sigma}^\rho\hat{\Gamma}_{\nu\lambda}^\sigma + \hat{\Gamma}_{\nu\sigma}^\rho\hat{\Gamma}_{\mu\lambda}^\sigma \equiv R_{\mu\nu\lambda}^{(1)\,\rho} + \gamma_5 R_{\mu\nu\lambda}^{(2)\,\rho} \tag{204}$$

The MAT spin connection is introduced in analogy

$$\hat{\Omega}_\mu^{ab} = \hat{e}_\nu^a\left(\partial_\mu\hat{e}^{\nu b} + \hat{e}^{\sigma b}\hat{\Gamma}_{\sigma\mu}^\nu\right) = \Omega_\mu^{(1)ab} + \gamma_5\Omega_\mu^{(2)ab} \tag{205}$$

In a MAT background, one must also introduce axially extended (AE) diffeomorphisms. They are defined by

$$\hat{x}^\mu \to \hat{x}^\mu + \hat{\xi}^\mu(\hat{x}^\mu), \qquad \hat{\xi}^\mu = \xi^\mu + \gamma_5\zeta^\mu \tag{206}$$

Operationally, these transformations act in the same way as the usual diffeomorphisms, therefore, for the non-covariant part of the transformation

$$\delta_{\widehat{\xi}}^{(n.c.)}\widehat{\Gamma}^\lambda_{\mu\nu} = \widehat{\partial}_\mu\widehat{\partial}_\nu\widehat{\xi}^\lambda \tag{207}$$

where the derivatives are understood with respect to $\widehat{x}^\mu$ and $\widehat{x}^\nu$. We also have

$$\delta_{\widehat{\xi}}\widehat{g}_{\mu\nu} = \widehat{D}_\mu\widehat{\xi}_\nu + \widehat{D}_\nu\widehat{\xi}_\mu \tag{208}$$

where $\widehat{\xi}_\mu = \widehat{g}_{\mu\nu}\widehat{\xi}^\nu$ and $\widehat{D}_\mu$ is the covariant derivative with respect to $\widehat{\Gamma}$.

There are two types of Weyl transformations that can be compactly written as

$$\widehat{g}_{\mu\nu} \longrightarrow e^{2\widehat{\omega}}\widehat{g}_{\mu\nu}, \tag{209}$$

where $\widehat{\omega} = \omega + \gamma_5\eta$.

The SDW method is based on point-splitting along a geodesic. Therefore it is crucial to define geodesics in a MAT background. Their defining equations are

$$\ddot{\widehat{x}}^\mu + \widehat{\Gamma}^\mu_{\nu\lambda}\dot{\widehat{x}}^\nu\dot{\widehat{x}}^\lambda = 0 \tag{210}$$

where a dot denotes a derivative with respect to $\widehat{t} = t_1 + \gamma_5 t_2$. Since $\widehat{g}_{\mu\nu}\dot{\widehat{x}}^\mu\dot{\widehat{x}}^\nu$ is constant for geodesics, we can write the following for the arc length parameter $\widehat{s}$

$$\frac{d\widehat{s}}{d\widehat{t}} = \sqrt{\widehat{g}_{\mu\nu}\dot{\widehat{x}}^\mu\dot{\widehat{x}}^\nu}, \tag{211}$$

The quantity

$$\widehat{\sigma}(\widehat{x}, \widehat{x}') = \frac{1}{2}(\widehat{s} - \widehat{s}')^2 \tag{212}$$

is called the *world function*. It has remarkable properties, starting with

$$\widehat{\sigma}_{;\mu} = \widehat{\partial}_\mu\widehat{\sigma} = (\widehat{t} - \widehat{t}')\widehat{g}_{\mu\nu}\dot{\widehat{x}}^\nu \equiv -\widehat{g}_{\mu\nu}\widehat{y}^\nu \tag{213}$$

$\widehat{y}^\mu$ are the *normal coordinates* based at $\widehat{x}$. One can prove that

$$\frac{1}{2}\widehat{\sigma}_{;\mu}\widehat{\sigma}_{;\mu} = \widehat{\sigma} \tag{214}$$

Covariantly differentiating (214), we obtain

$$\widehat{\sigma}_{;\nu} = \widehat{\sigma}_{;\mu\nu}\widehat{\sigma}_{;\mu} \tag{215}$$

In the coincidence ($x' \to x$) limit, $[\widehat{\sigma}_{;\nu}] = 0$. Therefore (215) is trivial in the coincidence limit. But differentiating further, one can show that

$$[\widehat{\sigma}_{;\mu\lambda}] = \widehat{g}_{\mu\lambda} \tag{216}$$

Now the game consists of repeatedly differentiating these relations and computing the coincidence limit. For instance, one can prove

$$[\widehat{\sigma}_{;\nu\lambda\rho\tau}] = -\frac{1}{3}\left(\widehat{R}_{\nu\tau\lambda\rho} + \widehat{R}_{\nu\rho\lambda\tau}\right) \tag{217}$$

and so on.

In the sequel, we need the Van Vleck–Morette determinant

$$\widehat{\mathcal{D}}(\widehat{x}, \widehat{x}') = \det(-\widehat{\sigma}_{;\mu\nu'}) \tag{218}$$

and the geodetic parallel displacer for spinors, $\widehat{I}(\widehat{x}, \widehat{x}')$: the object $\widehat{I}(\widehat{x}, \widehat{x}')\widehat{\psi}(\widehat{x}')$ is the spinor $\psi(\widehat{x})$ obtained by parallel displacing $\widehat{\psi}(\widehat{x}')$ along the geodesic from $\widehat{x}'$ to $\widehat{x}$. It is a bispinor quantity satisfying

$$\widehat{\sigma}_{;\mu}\widehat{I}_{;\mu} = 0, \qquad [\widehat{I}] = \mathbf{1} \tag{219}$$

Among its remarkable properties, we have the following concidence limit

$$[\widehat{I}(x, x')_{;[\mu,\nu]}] = [\widehat{I}(x, x')_{;\mu\nu}] = -\frac{1}{4}\widehat{\mathcal{R}}_{\mu\nu} \tag{220}$$

where $\widehat{\mathcal{R}}_{\mu\nu} = \widehat{R}_{\mu\nu}{}^{ab}\Sigma_{ab}$.

8.2. Fermions in a MAT Background

The action of a fermion interacting with a metric and an axial tensor is

$$\begin{aligned}
\widehat{S} &= \int d^4\widehat{x}\left(i\overline{\psi}\sqrt{\widehat{g}}\gamma^a\widehat{e}_a^\mu\left(\partial_\mu + \frac{1}{2}\widehat{\Omega}_\mu\right)\psi\right)(\widehat{x}) \tag{221} \\
&= \int d^4\widehat{x}\left(i\overline{\psi}\sqrt{\widehat{g}}\gamma^a(\widetilde{e}_a^\mu + \gamma_5\widetilde{c}_a^\mu)\left(\partial_\mu + \frac{1}{2}\left(\Omega_\mu^{(1)} + \gamma_5\Omega_\mu^{(2)}\right)\right)\psi\right)(\widehat{x}) \tag{222}
\end{aligned}$$

It must be noticed that this action takes on axial-real values. The field $\psi(\widehat{x})$ can be understood, classically, as a series of powers of $\widehat{x}$ applied to constant spinors on their right, and the symmetry transformations act on it from the left. The analogous definitions for $\psi^\dagger$ are obtained via hermitean conjugation.

Remark the position of the density $\sqrt{\widehat{g}}$: it must be inserted between $\overline{\psi}$ and ψ, due to the presence in it of the γ_5 matrix. Of course one has to remember that the kinetic operator contains a γ matrix that anticommutes with γ_5.

Let us consider AE (axially extended) diffeomorphisms first, (206). It is not hard to prove that the action (221) is invariant under these transformations, provided $\delta_{\widehat{\xi}}\psi = \widehat{\xi}^\mu\frac{\partial\psi}{\partial\widehat{x}^\mu}$. Also, for the axial complex Weyl transformation, one can prove that, assuming for the fermion field, the transformation rule

$$\psi \to e^{-\frac{3}{2}(\omega+\gamma_5\eta)}\psi, \tag{223}$$

(221) is invariant.

Now, define the full MAT e.m. tensor by means of

$$\mathbf{T}_{\mu\nu} = \frac{2}{\sqrt{\widehat{g}}}\frac{\overleftarrow{\delta}\,\widehat{S}}{\delta\widehat{g}^{\mu\nu}} \tag{224}$$

This formula needs a comment since $\sqrt{\widehat{g}}$ contains γ_5. To give a meaning to it, we understand that the operator $\frac{2}{\sqrt{\widehat{g}}}\frac{\overleftarrow{\delta}}{\delta\widehat{g}^{\mu\nu}}$ in the RHS acts on the operatorial expression, say $\mathcal{O}\sqrt{\widehat{g}}$, which is inside the spinor scalar product, $\overline{\psi}\mathcal{O}\sqrt{\widehat{g}}\psi$. Moreover, the functional derivative acts from the right of the action.

The explicit e.m. formula (on shell) is

$$\mathbf{T}_{\lambda\rho} = -\frac{i}{2}\overline{\psi}\widehat{\gamma}_\lambda\left(\partial_\rho + \frac{1}{2}\widehat{\Omega}_\rho\right)\psi + (\lambda \leftrightarrow \rho) = -\frac{i}{2}\overline{\psi}\widehat{\gamma}_\lambda\widehat{\nabla}_\rho\psi + (\lambda \leftrightarrow \rho) \tag{225}$$

where $\widehat{\gamma}_\lambda = \gamma_a \widehat{e}^a_\lambda$. This expression splits into a vector and an axial part, $T_{\mu\nu}$ and $T_{5\mu\nu}$, which in the flat limit are given on shell by

$$T_{\mu\nu} \approx T^{(flat)}_{\mu\nu} = -\frac{i}{4}\left(\overline{\psi}\gamma_\mu \overleftrightarrow{\partial}_\nu \psi + \mu \leftrightarrow \nu\right), \tag{226}$$

and

$$T_{5\mu\nu} \approx T^{(flat)}_{5\mu\nu} = \frac{i}{4}\left(\overline{\psi}\gamma_5\gamma_\mu \overleftrightarrow{\partial}_\nu \psi + \mu \leftrightarrow \nu\right), \tag{227}$$

Weyl invariance leads to the classical Ward identities:

$$\mathcal{T}(x) \equiv T_{\mu\nu}g^{\mu\nu} + T_{5\mu\nu}f^{\mu\nu} = 0, \tag{228}$$

$$\mathcal{T}_5(x) \equiv T_{\mu\nu}f^{\mu\nu} + T_{5\mu\nu}g^{\mu\nu} = 0, \tag{229}$$

8.3. The Dirac Operator and Its Square

In the action (221) the Dirac operator is

$$\widehat{F} = i\widehat{\gamma}\cdot\widehat{\nabla} = i\widehat{\gamma}^\mu\widehat{\nabla}_\mu = i\gamma^a\widehat{e}^\mu_a\widehat{\nabla}_\mu \equiv \gamma^a\widehat{F}_a \tag{230}$$

where $\widehat{\nabla} = \widehat{D} + \frac{1}{2}\widehat{\Omega}$ and satisfies $\widehat{\nabla}_\mu \widehat{e}^a_\nu = 0$.

Under AE diffeomorphisms, ψ transforms as follows: $\delta_{\widehat{\xi}}\psi = \widehat{\overline{\xi}}\cdot\partial\psi$, while

$$\delta_{\widehat{\xi}}\left(i\widehat{\gamma}\cdot\widehat{\nabla}\psi\right) = \widehat{\overline{\xi}}\cdot\partial\left(i\widehat{\gamma}\cdot\widehat{\nabla}\psi\right) \tag{231}$$

Under AE Weyl transformation, $\widehat{F}$ transforms as

$$\delta_{\widehat{\omega}}\widehat{F} = -\frac{1}{2}\gamma^a\{\widehat{F}_a, \widehat{\omega}\} \tag{232}$$

and it has the following hermiticity property

$$\widehat{F}^\dagger = \gamma_0 \widehat{F} \gamma_0 \tag{233}$$

where γ_0 is the non-dynamical (flat) gamma matrix.

In order to be able to apply the SDW method we have to select the squared Dirac operator. As pointed out earlier, see Section Bardeen's Anomaly with SDW, there are a priori a few possibilities. However, the request of respecting the basic (AE diffeomorphism) symmetry of the theory [We recall once more that this invariance is needed for the point-splitting technique, which underlies the SDW method, to work properly. We stress that invariance under ordinary diffeomorphisms alone is not enough because, in the chiral limit, a violation of the axial diffeomorphisms would precisely affect also the ordinary diffeomorphism conservation] and the self-adjointness after the Wick rotation identifies it uniquely. In ordinary gravity, from the diffeomorphism invariance of the fermion action, we can extract the transformation rule

$$\delta_\xi\left(i\gamma^\mu\nabla_\mu\psi\right) = \xi\cdot\partial(i\gamma\cdot\nabla\psi) \tag{234}$$

while $\delta_\xi\psi = \xi\cdot\partial\psi$. Therefore, it makes sense to apply $\gamma\cdot\nabla$ to $\gamma\cdot\nabla\psi$, because the latter transforms as ψ. This allows us to define the square of the Dirac operator:

$$F^2\psi = (i\gamma\cdot\nabla)^2\psi \tag{235}$$

The expression $\bar{\psi}F^2\psi$ is well-defined and invariant, and it makes sense to extract the square root of the path integral constructed by exponentiating it. If we want to preserve general covariance, it is not possible to repeat the same for MAT because of (231), from which we see that $\left(i\widehat{\gamma}\cdot\widehat{\nabla}\psi\right)$ does not transform like ψ, and an expression like $\left(i\widehat{\gamma}\cdot\widehat{\nabla}\right)^2\psi$ would break general covariance. Noting that

$$\delta_{\widehat{\xi}}\left(i\overline{\widehat{\gamma}}\cdot\overline{\widehat{\nabla}}\psi\right) = \widehat{\xi}\cdot\partial\left(i\overline{\widehat{\gamma}}\cdot\overline{\widehat{\nabla}}\psi\right) \tag{236}$$

when $\delta_{\overline{\widehat{\xi}}}\psi = \overline{\widehat{\xi}}\cdot\partial\psi$, we are led to choose, instead, the *covariant* quadratic operator

$$\widehat{\mathcal{F}} = \left(i\overline{\widehat{\gamma}}\cdot\overline{\widehat{\nabla}}\right)\left(i\widehat{\gamma}\cdot\widehat{\nabla}\right) \tag{237}$$

This is the first reason that justifies the choice of (237).

Let us come next to the Euclidean version of $\widehat{\mathcal{F}}$. To deal with it it is easier to remark that

$$\left(i\widetilde{\widehat{\gamma}}\widetilde{\widehat{\nabla}}\right)^{\dagger} = -i\widetilde{\overline{\widehat{\gamma}}}\widetilde{\overline{\widehat{\nabla}}} \tag{238}$$

Therefore

$$\left(\widetilde{\widehat{\mathcal{F}}}\right)^{\dagger} = \widetilde{\overline{\widehat{\mathcal{F}}}} \tag{239}$$

This is the second fundamental reason for using $\widehat{\mathcal{F}}$.

The operator $\widehat{\mathcal{F}}$ is the main intermediate result we need. It is natural to assume that its inverse $\widehat{\mathcal{G}}$ exists. The differential operator $\widehat{\mathcal{F}}$ (after a Wick rotation) is an axial-elliptic operator. In fact its quadratic part can be cast in the form $\partial_i A_{ij}(x)\partial_j$, where A_{ij} is an invertible matrix and its leading term is symmetric and positive definite. Moreover $\widetilde{\overline{\widehat{\mathcal{F}}}}$ is self-adjoint. Therefore, the conditions for the application of the SDW method are satisfied. As we have done before, we will always work with Minkowski quantities, paying attention, however, that no relation is used that cannot be mapped from the Minkowski to the Euclidean by a Wick rotation, and back by an inverse Wick rotation.

8.4. The SDW Method for the Trace Anomaly

There are significant changes with respect to the flat background application of Section Bardeen's Anomaly with SDW. Again, we define the amplitude

$$\langle\widehat{x},\widehat{s}|\widehat{x}',0\rangle = \langle\widehat{x}|e^{i\widehat{\mathcal{F}}\widehat{s}}|\widehat{x}'\rangle \tag{240}$$

which satisfies the (heat kernel) differential equation

$$i\frac{\partial}{\partial\widehat{s}}\langle\widehat{x},\widehat{s}|\widehat{x}',0\rangle = -\widehat{\mathcal{F}}_{\widehat{x}}\langle\widehat{x},\widehat{s}|\widehat{x}',0\rangle \equiv K(\widehat{x},\widehat{x}',\widehat{s}) \tag{241}$$

where $\widehat{\mathcal{F}}_{\widehat{x}}$ is the differential operator

$$\widehat{\mathcal{F}}_{\widehat{x}} = \widehat{\nabla}_{\mu}\widehat{g}^{\mu\nu}\widehat{\nabla}_{\nu} - \frac{1}{4}\widehat{R} \tag{242}$$

In Formula (240), as usual, we understand the $i\epsilon$ prescription. But now, instead of (60), we have the ansatz

$$\langle \widehat{x}, \widehat{s} | \widehat{x}', 0 \rangle = -\lim_{m \to 0} \frac{i}{16\pi^2} \frac{\sqrt{\widehat{D}(\widehat{x}, \widehat{x}')}}{\widehat{s}^2} e^{i\left(\frac{\widehat{\sigma}(\widehat{x}, \widehat{x}')}{2\widehat{s}} - m^2 \widehat{s}\right)} \widehat{\Phi}(\widehat{x}, \widehat{x}', \widehat{s}) \tag{243}$$

where $\widehat{D}(\widehat{x}, \widehat{x}')$ is the VVM determinant and $\widehat{\sigma}$ is the world function. $\widehat{\Phi}(\widehat{x}, \widehat{x}', \widehat{s})$ is to be determined. Since it must be $\langle \widehat{x}, 0 | \widehat{x}', 0 \rangle = \delta(\widehat{x}, \widehat{x}')$ and instead of (61) we have

$$\lim_{\widehat{s} \to 0} \frac{i}{(4\pi)^2} \frac{\sqrt{\widehat{D}(\widehat{x}, \widehat{x}')}}{\widehat{s}^2} e^{i\left(\frac{\widehat{\sigma}(\widehat{x}, \widehat{x}')}{2\widehat{s}} - m^2 \widehat{s}\right)} = \sqrt{\widehat{g}(\widehat{x})} \delta(\widehat{x}, \widehat{x}'), \tag{244}$$

like before it follows that

$$\lim_{\widehat{s} \to 0} \widehat{\Phi}(\widehat{x}, \widehat{x}', \widehat{s}) = \mathbf{1} \tag{245}$$

Equation (241) becomes an equation for $\widehat{\Phi}(\widehat{x}, \widehat{x}', \widehat{s})$:

$$i \frac{\partial \widehat{\Phi}}{\partial \widehat{s}} + \frac{i}{\widehat{s}} \widehat{\nabla}^\mu \widehat{\Phi} \widehat{\nabla}_\mu \widehat{\sigma} + \frac{1}{\sqrt{\widehat{D}}} \widehat{\nabla}^\mu \widehat{\nabla}_\mu \left(\sqrt{\widehat{D}} \widehat{\Phi}\right) - \left(\frac{1}{4} \widehat{R} - m^2\right) \widehat{\Phi} = 0 \tag{246}$$

Again, we expand

$$\widehat{\Phi}(\widehat{x}, \widehat{x}', \widehat{s}) = \sum_{n=0}^{\infty} \widehat{a}_n(\widehat{x}, \widehat{x}')(i\widehat{s})^n \tag{247}$$

with the boundary condition $[\widehat{a}_0] = 1$. The $\widehat{a}_n$ must satisfy the recursive relations:

$$(n+1)\widehat{a}_{n+1} + \widehat{\nabla}^\mu \widehat{a}_{n+1} \widehat{\nabla}_\mu \widehat{\sigma} - \frac{1}{\sqrt{\widehat{D}}} \widehat{\nabla}^\mu \widehat{\nabla}_\mu \left(\sqrt{\widehat{D}} \widehat{a}_n\right) + \left(\frac{1}{4} \widehat{R} - m^2\right) \widehat{a}_n = 0 \tag{248}$$

Using these relations, it is possible to compute each coefficient a_n at the coincidence limit. We are, in particular, interested in $[\widehat{a}_2]$, which turns out to be

$$\begin{aligned}
[\widehat{a}_2] &= \frac{1}{2} m^4 - \frac{1}{12} m^2 \widehat{R} + \frac{1}{288} \widehat{R}^2 - \frac{1}{120} \widehat{R}_{;\mu}{}^\mu - \frac{1}{180} \widehat{R}_{\mu\nu} \widehat{R}^{\mu\nu} + \frac{1}{180} \widehat{R}_{\mu\nu\lambda\rho} \widehat{R}^{\mu\nu\lambda\rho} \\
&+ \frac{1}{48} \widehat{\mathcal{R}}_{\mu\nu} \widehat{\mathcal{R}}^{\mu\nu}
\end{aligned} \tag{249}$$

We recall that $\widehat{\mathcal{R}}_{\mu\nu} = \widehat{R}_{\mu\nu}{}^{ab} \Sigma_{ab}$.

Next, we continue as in Section 2.1. In particular, Equations (55)–(57) remain the same and lead to effective action

$$\frac{\widehat{L}(x)}{\mu^{\mathsf{d}}} = -\frac{i}{2}(4\pi\mu^2)\text{tr} \int_0^\infty d\widehat{s}(4\pi i \mu^2 \widehat{s})^{-\frac{\mathsf{d}}{2}-1} \sqrt{\widehat{g}} e^{-im^2 \widehat{s}} [\widehat{\Phi}(\widehat{x}, \widehat{x}, \widehat{s})] \tag{250}$$

where tr denotes the trace over gamma matrices. Our purpose is to analytically continue in d. But we can do this only for dimensionless quantities. For this reason $\widehat{L}$ is multiplied by $\mu^{-\mathsf{d}}$, where μ is a mass parameter.

Now, we make the assumption that $\lim_{s \to \infty} e^{-im^2 \widehat{s}} [\widehat{\Phi}(\widehat{x}, \widehat{x}, \widehat{s})] = 0$. As a consequence, we can integrate by parts

$$\frac{\widehat{L}(x)}{\mu^{\mathsf{d}}} = -\frac{4i}{\mathsf{d}(2-\mathsf{d})(4-\mathsf{d})} \frac{1}{(4\pi\mu^2)^2} \text{tr} \int_0^\infty d\widehat{s}(2\pi i \mu^2 \widehat{s})^{2-\frac{\mathsf{d}}{2}} \sqrt{\widehat{g}} \frac{\partial^3}{\partial(i\widehat{s})^3} \left(e^{-im^2 \widehat{s}} [\widehat{\Phi}(\widehat{x}, \widehat{x}, \widehat{s})]\right) \tag{251}$$

Now we use $[\widehat{\Phi}(\widehat{x}, \widehat{x}, \widehat{s})] = 1 + [\widehat{a}_1]i\widehat{s} + [\widehat{a}_2](i\widehat{s})^2 + \dots$.

To expand around d = 4, we use $\frac{1}{d(d-2)(d-4)} \approx \frac{1}{8}\left(\frac{1}{d-4} - \frac{3}{4}\right)$. With reference to (251), we differentiate twice $[\widehat{\Phi}(x, x, s)]$ and integrate the third derivative by parts. The result is

$$
\begin{aligned}
\widehat{L}(\widehat{x}) \quad \approx \quad & \frac{1}{32\pi^2}\left(\frac{1}{d-4} - \frac{3}{4}\right)\mathrm{tr}\left(m^4 - 2m^2[\widehat{a}_1] + 2[\widehat{a}_2]\right)\sqrt{\widehat{g}} \\
& + \frac{i}{64\pi^2}\mathrm{tr}\int_0^\infty d\widehat{s}\ln(4\pi i\mu^2\widehat{s})\sqrt{\widehat{g}}\frac{\partial^3}{\partial(i\widehat{s})^3}\left(e^{-im^2\widehat{s}}[\widehat{\Phi}(\widehat{x}, \widehat{x}, \widehat{s})]\right)
\end{aligned}
\tag{252}
$$

The last line depends explicitly on the parameter μ and represents a part, which cannot contribute to the anomaly for dimensional reasons. In the first line of (252) one can ignore m^2 or m^4 terms (they can be subtracted away because they are trivial). Therefore, we can limit ourselves to

$$
\widehat{L} \quad = \quad \frac{1}{16\pi^2}\left(\frac{1}{d-4} - \frac{3}{4}\right)\int d^d\widehat{x}\,\mathrm{tr}\left([\widehat{a}_2]|_{m=0}\sqrt{\widehat{g}}\right)
\tag{253}
$$

We now act with $\delta_{\widehat{\omega}} = \int d^d\widehat{x}\,2\mathrm{tr}\left(\widehat{\omega}\widehat{g}_{\mu\nu}\frac{\delta}{\delta\widehat{g}_{\mu\nu}}\right)$. An explicit calculation gives, for example,

$$
\delta_{\widehat{\omega}}\mathrm{tr}\left(\sqrt{\widehat{g}}\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}\right) \quad = \quad (d-4)\mathrm{tr}\left(\widehat{\omega}\sqrt{\widehat{g}}\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}\right) + 2\mathrm{tr}\left(\sqrt{\widehat{g}}\widehat{R}\Box\widehat{\omega}\right)
\tag{254}
$$

Finally, we obtain

$$
\delta_{\widehat{\omega}}\mathrm{tr}\left(\sqrt{\widehat{g}}[\widehat{a}_2]|_{m=0}\right) = (d-4)\mathrm{tr}\left(\sqrt{\widehat{g}}\widehat{\omega}[\widehat{a}_2]|_{m=0}\right) - \frac{d-4}{120}\mathrm{tr}\left(\sqrt{\widehat{g}}\widehat{R}\Box\widehat{\omega}\right)
\tag{255}
$$

The second piece can be canceled, e.g., by a counterterm proportional to $\mathrm{tr}\left(\sqrt{\widehat{g}}\widehat{R}^2\right)$.

Now, the variation of the effective action under the $\widehat{\omega}$ transformation defines the integrated anomaly. Therefore, defining $\frac{2}{\sqrt{\widehat{g}}}\frac{\delta}{\delta\widehat{g}^{\mu\nu}}\widehat{L} = \Theta_{\mu\nu}$ and recalling the definition (153), and using the fact that the second line of (252) is Weyl invariant, we obtain the following in the d $\rightarrow$ 4 limit

$$
\int d^4\widehat{x}\,\mathrm{tr}\left(\widehat{\omega}\sqrt{\widehat{g}}\widehat{g}^{\mu\nu}\widehat{\Theta}_{\mu\nu}\right) \quad = \quad \frac{1}{16\pi^2}\int d^4\widehat{x}\,\mathrm{tr}\left(\sqrt{\widehat{g}}\widehat{\omega}[\widehat{a}_2]|_{m=0}\right)
\tag{256}
$$

where the d $-$ 4 factor in (255) has canceled the pole $\frac{1}{d-4}$ in (253).

Clearly, the odd parity anomaly can come only from the term $\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}$ contained in $[\widehat{a}_2]$. For the odd part, we have

$$
\int d^4\widehat{x}\,\mathrm{tr}\sqrt{\widehat{g}}\widehat{\omega}\widehat{\mathcal{T}} = \frac{1}{768\pi^2}\int d^4x\,\mathrm{tr}\sqrt{\widehat{g}}\widehat{\omega}\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}\Big|_{\text{odd}}
\tag{257}
$$

where we denoted $\widehat{\mathcal{T}} = \widehat{g}^{\mu\nu}\widehat{\Theta}_{\mu\nu} = \widehat{g}^{\mu\nu}\langle\langle\widehat{T}_{\mu\nu}\rangle\rangle$.

8.5. The Chiral Limit and the Chiral Trace Anomaly

Let us return to the original problem, that is the trace anomaly of a Weyl tensor in a chiral fermion theory coupled to ordinary gravity. To this end we need to take the chiral limit. But before we rewrite the anomaly (257) by splitting it into the chiral and anti-chiral parts. Setting $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$ we can write

$$
\sqrt{\widehat{g}} = P_+\sqrt{\det g_+} + P_-\sqrt{\det g_-}, \qquad g_{\pm\mu\nu} = g_{\mu\nu} \pm f_{\mu\nu}
\tag{258}
$$

and

$$\widehat{\omega} = P_+\omega_+ + P_-\omega_-, \qquad \widehat{\mathcal{R}}_{\mu\nu} = P_+\mathcal{R}_{\mu\nu}^{(+)} + P_-\mathcal{R}_{\mu\nu}^{(-)} \qquad (259)$$

where

$$\omega_\pm = \omega \pm \eta, \qquad \mathcal{R}_{\mu\nu}^\pm = \mathcal{R}_{\mu\nu}^{(1)} \pm \mathcal{R}_{\mu\nu}^{(2)} \qquad (260)$$

Then, we can rewrite (257) as follows

$$\int d^d\widehat{x}\,\mathrm{tr}\sqrt{\widehat{g}}\widehat{\omega}\widehat{\mathfrak{T}}\Big|_{\mathrm{odd}} \qquad (261)$$
$$= \frac{1}{1536\pi^2}\int d^4x\,\mathrm{tr}\left(P_+\sqrt{\det g_+}\,\omega_+\mathcal{R}_{\mu\nu}^{(+)}\mathcal{R}^{(+)\mu\nu} + P_-\sqrt{\det g_-}\,\omega_-\mathcal{R}_{\mu\nu}^{(-)}\mathcal{R}^{(-)\mu\nu}\right)\Big|_{\mathrm{odd}}$$

The chiral limit is defined by making the replacements

$$g_{\mu\nu} \to \eta_{\mu\nu} + \frac{h_{\mu\nu}}{2}, \qquad f_{\mu\nu} \to \frac{h_{\mu\nu}}{2} \qquad (262)$$

in the previous formulas. With this choice, one has

$$\widehat{g}_{\mu\nu} = P_-\eta_{\mu\nu} + P_+\mathsf{g}_{\mu\nu}, \qquad \mathsf{g}_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu} \qquad (263)$$

From this we see that the left-handed part couples to the flat metric, while the right-handed part couples to the (generic) metric $\mathsf{g}_{\mu\nu}$. As a consequence, we also have

$$\widehat{e}_m^a \to \delta_m^a P_- + e_m^a P_+, \qquad \widehat{e}_a^m \to \delta_a^m P_- + e_a^m P_+ \qquad (264)$$

as well as $\sqrt{\widehat{g}} \to P_- + P_+\sqrt{\mathsf{g}}$. Similarly, for the Christoffel symbols,

$$\Gamma_{\mu\nu}^{(1)\lambda} \to \frac{1}{2}\Gamma_{\mu\nu}^\lambda, \qquad \Gamma_{\mu\nu}^{(2)\lambda} \to \frac{1}{2}\Gamma_{\mu\nu}^\lambda, \qquad (265)$$

for the spin connections,

$$\Omega_\mu^{(1)ab} \to \frac{1}{2}\omega_\mu^{ab}, \qquad \Omega_\mu^{(2)ab} \to \frac{1}{2}\omega_\mu^{ab}, \qquad (266)$$

and for the curvatures,

$$R_{\mu\nu\lambda}^{(1)\ \rho} \to \frac{1}{2}R_{\mu\nu\lambda}{}^\rho, \qquad R_{\mu\nu\lambda}^{(2)\ \rho} \to \frac{1}{2}R_{\mu\nu\lambda}{}^\rho, \qquad (267)$$

where all the quantities on the RHS of these limits are built with the metric $\mathsf{g}_{\mu\nu}$.

As a consequence, the action (221) becomes

$$\widehat{S} \quad \to \quad S' \qquad (268)$$
$$S' \quad = \quad \int d^4x\, i\overline{\psi}\gamma^a P_-\partial_a\psi + \int d^4x\sqrt{\mathsf{g}}\, i\overline{\psi}\gamma^a e_a^\mu\left(\partial_\mu + \frac{1}{2}\omega_\mu\right)P_+\psi$$

where γ^a are the flat (non-dynamical) gamma matrices while the vierbein e_a^μ and the connection ω_μ are compatible with the metric $\mathsf{g}_{\mu\nu}$. The action S' is the action of a right-handed Weyl fermion coupled to ordinary gravity, except for the term that represents a decoupled left-handed fermion in the flat spacetime.

Moreover, in the chiral limit, we have $\mathcal{R}^{(-)}_{\mu\nu} \to 0, \mathcal{R}^{(+)}_{\mu\nu} \to 2\mathcal{R}_{\mu\nu}$. Therefore (261) becomes

$$
\int d^d\widehat{x}\,\text{tr}\sqrt{\widehat{g}}\widehat{\omega}\widehat{\mathfrak{T}} \to -\frac{1}{768\pi^2}\int d^4x\text{tr}\left(P_+\sqrt{g}\omega_+\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}\right)
$$
$$
= \frac{i}{2\times 768\pi^2}\int d^4x\sqrt{g}\omega_+\varepsilon^{\mu\nu\lambda\rho}R_{\mu\nu\alpha\beta}R_{\lambda\rho}{}^{\alpha\beta} \tag{269}
$$

Now, notice that an extended Weyl transformation split as follows

$$
e^{2\widehat{\omega}}\widehat{g} = P_+e^{2\omega_+}g_+ + P_-e^{2\omega_-}g_- \tag{270}
$$

therefore, in the chiral limit, and for consistency with (268), we must have that η coincides with ω, $\omega \to \omega/2, \eta \to \omega/2$. We conclude that in the chiral limit, the e.m. trace is

$$
\mathfrak{T}(x) = \frac{i}{1536\pi^2}\varepsilon^{\mu\nu\lambda\rho}R_{\mu\nu\alpha\beta}R_{\lambda\rho}{}^{\alpha\beta} \equiv \mathfrak{T}_R(x) \tag{271}
$$

which coincides with (193). If, instead of (262), we take the following chiral limit

$$
g_{\mu\nu} \to \eta_{\mu\nu} + \frac{h_{\mu\nu}}{2}, \qquad f_{\mu\nu} \to -\frac{h_{\mu\nu}}{2} \tag{272}
$$

we obtain the Pontryagin Weyl anomaly for the left-handed Weyl fermion

$$
\mathfrak{T}_L(x) = -\frac{i}{1536\pi^2}\varepsilon^{\mu\nu\lambda\rho}R_{\mu\nu\alpha\beta}R_{\lambda\rho}{}^{\alpha\beta}. \tag{273}
$$

8.5.1. The Full MAT Trace Anomalies

From Equation (261), it is easy to compute the $\mathfrak{T}$ and $\mathfrak{T}_5$ anomalies in the general MAT background. They are the coefficients of ω and η, respectively.

$$
\mathfrak{T}(x) = \frac{i}{1536\pi^2}\varepsilon^{\mu\nu\lambda\rho}\left(\sqrt{g_+}R^{(+)}_{\mu\nu\alpha\beta}R^{(+)\alpha\beta}_{\lambda\rho} + \sqrt{g_-}R^{(-)}_{\mu\nu\alpha\beta}R^{(-)\alpha\beta}_{\lambda\rho}\right) \tag{274}
$$
$$
\mathfrak{T}_5(x) = \frac{i}{1536\pi^2}\varepsilon^{\mu\nu\lambda\rho}\left(\sqrt{g_+}R^{(+)}_{\mu\nu\alpha\beta}R^{(+)\alpha\beta}_{\lambda\rho} - \sqrt{g_-}R^{(-)}_{\mu\nu\alpha\beta}R^{(-)\alpha\beta}_{\lambda\rho}\right) \tag{275}
$$

We have Wick-rotated back the result: this is the origin of i in the anomaly coefficient.

At this point, we can safely replace $\widehat{x}^\mu$ with x^μ everywhere.

8.5.2. Other Results

The previous trace anomalies can be obtained with different non-perturbative methods, for instance, with the Seeley–DeWitt method, see [5]. In the wake of these derivations, one can easily obtain other interesting results. For instance the even parity trace anomalies for Weyl and Dirac fermions, which we do not report here.

Another interesting result is the analog of the ABJ anomaly, which is the response of the effective action of a Dirac fermion ψ coupled to ordinary metric $g_{\mu\nu}$ under the transformation

$$
\delta_\eta g_{\mu\nu} = 2\gamma_5\eta g_{\mu\nu}, \qquad \delta_\eta\psi = -\frac{3}{2}\gamma_5\eta\psi \tag{276}
$$

This response can be extracted from (261) in the limit $g_{\mu\nu} \to g_{\mu\nu}, f_{\mu\nu} \to 0$. In this limit $\Gamma^{(1)\lambda}_{\mu\nu} \to \Gamma^\lambda_{\mu\nu}$ and $\Gamma^{(2)\lambda}_{\mu\nu} \to 0$, $R^{(1)}_{\mu\nu\lambda}{}^\rho \to R_{\mu\nu\lambda}{}^\rho, R^{(2)}_{\mu\nu\lambda}{}^\rho \to 0$. Therefore $g_{\pm\mu\nu} \to g_{\mu\nu}$ and $R^\pm_{\mu\nu\lambda\rho} \to R_{\mu\nu\lambda\rho}$. As a consequence, the odd parity (ABJ-like) trace anomaly is

$$
\mathfrak{T}_5(x) = \frac{i}{768\pi^2}\varepsilon^{\mu\nu\lambda\rho}R_{\mu\nu\alpha\beta}R_{\lambda\rho}{}^{\alpha\beta}. \tag{277}
$$

Contrary to the anomalies (271) and (273), the anomaly (277) is not a risk for the consistency of a theory of Dirac fermions coupled to gravity. Rather, its absorbitive nature suggests a possible phenomenological application to a decay of a neutral bound state into two gravitons, similar to the one of the ABJ anomaly for a double photon decay of π^0.

Finally, with a small supplementary effort, we can obtain the gauge-induced trace anomaly, for instance, a Weyl fermion coupled to a vector potential V_μ. The kinetic operator, in this case, is $i\widehat{\mathbb{D}}$ with

$$\widehat{\mathbb{D}}_\mu = \widehat{D}_\mu + \frac{1}{2}\widehat{\Omega}_\mu + V_\mu \tag{278}$$

Proceeding exactly as in the gauge-less case, we find

$$\delta_\omega L = -\frac{i}{32\pi^2}\int d^4x\sqrt{g}\omega\mathrm{tr}(P_+[a_2](x)) \tag{279}$$

The odd part of $\mathrm{tr}(P_+[a_2](x))$ produces precisely the density $-\varepsilon^{\mu\nu\lambda\rho}\partial_\mu V_\nu\partial_\lambda V_\rho$. Therefore, (279), together with the definition (153), yields a right-handed fermion

$$\mathcal{A}_\omega = -\frac{1}{96\pi^2}\int d^4x\sqrt{g}\omega\varepsilon^{\mu\nu\lambda\rho}\partial_\mu V_\nu\partial_\lambda V_\rho, \tag{280}$$

which coincides with Equation (169).

9. Recap of Anomalies

The main subdivision of local anomalies is in two types (see Table 5):

- Type O comprises all the anomalies that correspond to the non-existence of a (fermion) propagator. These anomalies show up only in theories of chiral fermions. These anomalies are a threat to the consistency of the theory.
- Type NO includes all the others. They have nothing to do with the lack of any propagator and do not signal by themselves any inconsistency of a theory.

Type O anomalies are odd-parity, they are present in parity violating theories of fermions. They may appear as a violation of the classical conservation of the gauge (example. Equation (21)) or diffeomorphisms symmetries, and of the conformal invariance in the non-vanishing trace of the e.m. tensor (examples: (150) and (152)). From a formal point of view, they satisfy WZ consistency conditions. In the physical approach and mathematical approach, they appear in different, complementary ways. In the former case one faces the problem of the lack of the propagator in a chiral fermion theory and overcomes it by adding a free fermion of opposite chirality. The trick works in general, but it fails if (in an originally classical conformal theory) the gauge current or the e.m. tensor are non-conserved or traceful, in which case inconsistency may manifest itself as a violation of unitarity (through a breakdown of the BRST symmetry in the first case, and a complex effective Hamiltonian in the second). In the mathematical approach, type O anomalies appear as obstructions to the existence of a chiral fermion propagator, as represented by a non-trivial family's index. Between the two approaches there is a perfect correspondence. A non-trivial family's index exactly signals anomalies that appear in the divergence of the chiral gauge current or of the e.m. tensor and in the trace of the latter, and vice-versa.

Type NO anomalies are, first, the even parity anomalies that appear in the trace of the e.m. tensor (see, Equations (148), (149) and (151) and Sections 17 and 18 below). They satisfy the appropriate WZ consistency conditions. But type NO also includes the odd-parity anomalies of the ABJ type, for instance, the anomalies of a chiral current in a Dirac fermion theory (example: Equation (20)), or, anyhow, in theories free of type O anomalies. They

are usually called covariant anomalies and appear in the conservation of chiral currents that do not propagate in the internal lines of the Feynman diagrams. Formally, they may be viewed as anomalies of chiral currents coupled to auxiliary axial potentials that are subsequently made to vanish. All type NO anomalies are generated from a (Euclidean) self-adjoint kinetic operator and do not signal any absence of propagators (their family's index vanishes, thus no obstructions!). Therefore, they are no threat to the consistency of the theory and usually play a role in phenomenological applications.

From the calculation point of view, the two types of anomalies in the perturbative approach do not differ. As we have explained several times, the method based on the Feynman diagram is the same, but one may face the difficulties pointed out in ref. [70]: the perturbative calculations are rather accessible at one loop of approximation, but may be quite unwieldy at higher loops, especially when gravitons are involved; this is a source of trouble if the one loop calculation has problems like the ones pointed out in Section 6. For this reason it is important to have at hand other methods, in particular non-perturbative ones, like the SDW method. The advantage of these method is that it allows us to compute in one stroke the full expression of the anomaly, non just the lowest perturbative orders. There is a disadvantage with the SDW method: since it is based on the point splitting along a geodesic, it is automatically diffeomorphism-invariant. Therefore, it cannot reproduce the consistent diffeomorphism anomalies, but it reproduces all the anomalies that preserve diffeomorphisms. This problem is irrelevant in 4d because such anomalies vanish identically. The SDW method does not distinguish between O and NO anomalies, what makes a difference between the two is the quadratic Dirac (or other) operator. When computing NO anomalies, it is of tremendous help to enlarge the space of fields with the adjunction of an axial gauge potential (Bardeen's method) or an axial metric

The calculation of obstructions—corresponding to O anomalies—from the family's index theorem are straightforward. It is enough to consult a textbook such as [59] and copy the formulas of the index there. As far as the chiral gauge anomalies are concerned, one further step is needed. One must apply the appropriate transgression Formula (82) to the polynomials provided by the family's index theorem in order to obtain the descent Equation (85) and companions. Instead, in what concerns the NO anomalies, the family's index theorem yields an identically vanishing result, meaning that there are no obstructions to the existence of the relevant propagator.

A difference between the last and the previous two methods has to be mentioned. In the latter, we start from a theory in a Minkowski metric background and carry on all manipulations with this metric while we turn to a Euclidean background in order to give meaning to divergent Feynman integrals or the $s-$integrals in the SDW method. The family's index theorem instead is entirely formulated in the Euclidean: the Dirac (or other) operator is analyzed in Euclidean spacetime.

The O and NO anomalies are studied in quantum field theories for different purposes. The obstructive ones need to be known in order to guarantee that in a well-defined theory, they are absent. In this sense, the O anomalies do not have any physical application, but they are crucial in determining the physical content of a theory. The NO anomalies, on the contrary, make their appearance (also) in well-defined theories and may have important physical applications. A typical example is the application of the ABJ anomaly of a chiral current, such as (20) in the Abelian case, to the decay π^0 into two γ's. Another application is in the strong $U(1)$ problem, where the same current (20) in the non-Abelian case, but with $T^a = 1$ has a crucial role.

Table 5. Anomaly recap.

- Site of local anomalies: $\begin{cases} \text{divergenceofcurrentorofe.m.tensor} \\[1em] \text{traceofe.m.tensor} \end{cases}$

- Local Anomalies are of two types: $\begin{cases} \text{typeO} \quad \begin{cases} \text{preventexistenceofpropagators,} \\ \text{dangerous : mustbecanceled} \end{cases} \\[1.5em] \text{typeNO} \quad \begin{cases} \text{noobstructionforpropagators,} \\ \text{neednotbecanceled} \end{cases} \end{cases}$

- Where $\begin{cases} \text{typeO} \quad \text{onlyinchiraltheories} \\ \text{typeNO} \quad \text{inanytheory} \end{cases}$

- Examples: $\begin{cases} \text{typeO} \quad \begin{cases} \text{gauge : Equations (11) and (21),} \quad cons. \\[1em] \text{trace : Equations (150), (152), (170), (192) and (271),} \quad cons. \end{cases} \\[2em] \text{typeNO} \quad \begin{cases} \text{ABJ : Equation (20),} \quad cov. \\[1em] \text{trace : Equations (148), (149) and (151) and Sections 17 and 18, Equations (457) and (464),} \quad cons. \\[1em] \text{ABJ-liketrace : Equation (277),} \quad cons. \end{cases} \end{cases}$

- Cancelation: $\begin{cases} \text{typeO} \quad \begin{cases} \text{A : grouptheoretical(unavailablefortraceanomalies)} \\[1em] \text{B : coefficientmatching(unlikelyfortraceanomalies)} \\[1em] \text{C : Wess-ZuminotermsorGreen-Schwarzmechanisms} \end{cases} \\[3em] \text{typeNO} \quad \begin{cases} \text{ingeneralnotrequired} \\[1em] \text{foreventraceanomalies} \begin{cases} \text{B : coefficientmatchingunlikely} \\[1em] \text{C : withWess-Zuminoterms} \end{cases} \end{cases} \end{cases}$

cons. means they satisfy WZ consistency conditions, *cov.* means covariant.

As for the even trace anomalies, they share the same characteristics as all the NO anomalies. Therefore, they do not endanger any theory free of O anomalies, and they might have phenomenological applications similar to ABJ gauge anomalies. One has mentioned after Equation (277). But this is a ground largely still to be explored. On the other hand even trace anomalies have a special status, because, although they may appear in perfectly well defined theories, they signal the breakdown of Weyl symmetry, therefore they pose the problem of whether that symmetry have to be restored or not. We will discuss this issue at length in the second part of the paper. Refs.: [5,48,60,71–76].

10. Comments

The results illustrated above concerning trace anomalies are not universally accepted. Some research has led to the conclusion that the odd-parity trace anomalies do not show up in field theory models. One must recognize that the calculation of these trace anomalies is, in general, a subtle task, subtler than for other anomalies. This is not totally surprising because field theory is not an axiomatic theory that allows one to deduce results by means of a unique formalism. It is rather a theory under construction, which often requires different approaches. In particular, for anomalies, it is important to use different methods and compare them in order to obtain reliable results. Using one single method of calculation may not be safe. One can, in fact, envisage procedures that yield zero when searching for the odd part of the trace anomaly and even for ordinary chiral gauge anomalies—one example is signaled in the footnote before Equation (72). But they are to be traced back to

not infrequent wrong prejudices or misunderstood statements in the literature. Below are some more examples.

- Type O anomalies may occur only in theories of Weyl fermions, not in theories of Majorana fermions. A quick method for obtaining a vanishing odd parity trace anomaly is searching for type O anomalies in the latter case. The wrong use of Majorana fermions in the derivation of anomalies arises from a misunderstanding: the misplaced credence that massless Majorana fermions are equivalent to Weyl fermions. A Majorana fermion can be represented as a superposition, satisfying the Majorana condition, of two Weyl fermions of opposite chirality. We have repeated many times that Weyl fermions of opposite chirality have opposite type O anomalies, with consequent null overall result for Majorana fermions (no need to do any calculation) [On the other hand, we are not aware of any attempt to formulate the SM in terms of Majorana fermions alone].

- There are several ways to obtain a vanishing odd parity trace anomaly via a perturbative calculation, and the calculation in itself may be correct, depending on the regularization procedure. The reason has been explained above. It does not make sense to study conformal anomalies separately from diffeomorphism invariance. The calculation of a conformal anomaly is safe when we can guarantee that diffeomorphisms are conserved. If diffeomorphisms are anomalous, their anomaly (in 4d) is, in general, trivial and can be eliminated by a counterterm in the effective action, which, in general, modifies the trace anomaly. This can be verified in a number of cases, see [5]. But this mechanism does not work in the case of 4d odd trace anomalies at the lowest perturbative order because the odd part of the conformal 3-point function of the e.m. tensor vanishes identically for algebraic reasons, [77,78]; therefore, its divergence also vanishes identically and no (trivial) diffeomorphism anomaly can show up. In other words, this problem is *undecidable* at the lowest perturbative order; one must go to higher orders or use non-perturbative methods.

- In the heat-kernel-like methods applied to, say, a right-handed Weyl fermion one cannot simply use the Dirac kinetic operator $\slashed{D}$ multiplied by the chiral projector P_R, because the overall operator $\slashed{D}P_R$ is not invertible: using formally this operator invalidates all the manipulations necessary to derive anomalies. Its square is zero. Then, one may be tempted to use in the SDW method its product with its adjoint, making the (Euclidean) operator $\slashed{D}^2 P_R$, and inserting it, for instance, in Equation (73). Using mechanically the SDW formulas one gets 0 for the odd parity trace anomaly. However, the operator $\slashed{D}^2 P_R$ is not invertible, therefore, in particular, not elliptic, and cannot be used in the SDW (or in any heat kernel-like) approach.

- An often confusing issue is Wick rotation. In Appendix B, we explain our rules for applying Wick rotation. In perturbative and non-perturbative (SDW) approaches it does not make sense to start from a Euclidean action. Any attempt to Wick-rotate the Minkowski action for Dirac fermions leads to a doubling of degrees of freedom, thus changing the nature of the problem (in the approach with the family's index theorem, this difficulty is avoided because the action is not needed, one simply studies the inversion of the linear Dirac–Weyl operator). The right attitude toward Wick rotation is to consider it a way to make sense of perturbative Feynman diagram integrals or the *s*-integrals in the SDW case.

- Another confusing issue is that of unitarity. A tenet in physics is that any calculation has to guarantee unitarity. However, this is not the correct attitude when searching for type O anomalies. The reason is that the latter are not physical quantities. As we know from above, type O anomalies (contrary to type NO ones) are not only not physical but, in fact, they disrupt unitarity. Type O anomalies signal a clash between correct

mathematics and consistent physics (unitarity). As a consequence, requiring unitarity in this case is nonsense: searching for type O anomalies means investigating possible breakdowns of unitarity, and requiring unitarity from the start prevents the possibility of finding out if there is any such breach. On the contrary, we have to proceed in the most rigorous way to discover if any mathematical obstruction exists.

- The previous case may be connected with the choice of the square Dirac operator in the SDW approach. It has been suggested that the choice should be $\widehat{\mathcal{F}}_s = \left(i\overline{\widehat{\gamma}} \cdot \overline{\widehat{\nabla}} \right) \left(i\widehat{\gamma} \cdot \widehat{\nabla} \right) + \left(i\widehat{\gamma} \cdot \widehat{\nabla} \right) \left(i\overline{\widehat{\gamma}} \cdot \overline{\widehat{\nabla}} \right)$, instead of our choice (237): $\widehat{\mathcal{F}} = \left(i\overline{\widehat{\gamma}} \cdot \overline{\widehat{\nabla}} \right) \left(i\widehat{\gamma} \cdot \widehat{\nabla} \right)$. The choice $\widehat{\mathcal{F}}_s$ is symmetric in the exchange $\gamma_5 \leftrightarrow -\gamma_5$, thus it excludes any odd-parity result. But it is a forbidden choice because it breaks diffeomorphism invariance, which is the basis of the SDW method and thus disrupts mathematical consistency. The choice of $\widehat{\mathcal{F}}_s$ is a perfect example of what one should not do when computing anomalies.

- However, above all, what is missing in all the 'vanishing odd trace anomaly' papers is a plausible explanation of why these anomalies should be absent. The densities of the Pontryagin and Chern classes have the right quantum numbers (dimensions, form degree, odd parity, Lorentz covariance) to couple to the trace of the e.m. tensor in a theory of Weyl fermions. Why should this coupling vanish? Usually, this requires a protecting symmetry. For instance, a Dirac fermion is free of this anomaly (and of the consistent gauge anomaly as well) because the trace of e.m. tensor (and the divergence of the vector current) is protected by parity conservation. But in the case of a Weyl fermion theory, parity is violated. What other reason is there for such terms not to couple?

- Our final consideration may sound paradoxical, but it is the gist of our anomaly analysis: no doubt eventually an agreement will be reached about odd parity trace anomalies, but the proof of their existence is only relatively important. The very important issue is the obstructions they represent and are only a symptom of. As explained above, the obstructions are those defined by the family's index theorem. Even when not believing in the validity of the derivations we have presented, one stumbles anyhow against the existence of the obstructions signaled by the family's index theorem (not a minor result in modern mathematics). Only when these obstructions are removed does the Weyl fermion propagator (and the theory) exist. The problem represented by such obstructions exists independently of the anomaly problem.

Refs.: [5,79–89].

11. Extensions of the MSM

The vanishing of type O anomalies for the MSM (1) immersed in a gravitational background have been analyzed in Table MSM gauge, mixed, trace-gravity and trace-gauge. When gravity-fluctuating fields become relevant, one gravity-induced trace anomaly and one $SU(2)$ gauge-induced trace anomaly survive cancelation. This may change if we add to the MSM spectrum a right-handed sterile neutrino: it cancels the gravity-induced trace anomaly but leaves the gauge-induced one unaltered.

We would like to examine now various generalizations of the MSM, see [90], in relation to the anomaly cancelation when in the presence of a non-trivial background metric. For some of them, nothing changes because they do not involve modifications of the fermion spectrum. This is the case of the multi-Higgs-doublet models or similar models in which only the spectrum of scalars is modified with respect to the original SM. They have been introduced mostly to describe explicit and spontaneous CP violation. The fermion spectrum is the same as in the original SM. But plenty of extensions of the ordinary SM have been proposed which involve modification of the fermion spectrum. Hereafter, we wish to examine some of them.

11.1. Models with Vector-Like Quarks

Models with vector-like quarks have been proposed in order to produce non-vanishing but naturally suppressed flavor-changing neutral currents and to simplify the mechanisms of spontaneous CP violation. They also represent an attempt to solve the strong CP problem. The quark spectrum, together with an ordinary lepton spectrum, are summarized in the following table

$$
\begin{array}{ccccc}
\mathsf{G}/fields & \begin{pmatrix} u \\ d \end{pmatrix}_L & SU(3) & SU(2) & U(1) \\
n_g & \begin{pmatrix} u \\ d \end{pmatrix}_L & 3 & 2 & \frac{1}{6} \\
n_g & u_R & \bar{3} & 1 & \frac{2}{3} \\
n_g & d_R & \bar{3} & 1 & -\frac{1}{3} \\
n_u & U_L & 3 & 1 & \frac{2}{3} \\
n_d & D_L & 3 & 1 & -\frac{1}{3} \\
n_g + n_u & U_R & \bar{3} & 1 & \frac{2}{3} \\
n_g + n_d & D_R & \bar{3} & 1 & -\frac{1}{3} \\
n_g & \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L & 1 & 2 & -\frac{1}{2} \\
n_g & e_R & 1 & 1 & -1
\end{array}
\tag{281}
$$

where n_g is the number of generations (that is, for instance, u_L represents a set of fields numbered from 1 to n_g: u_{Li}, $i = 1, \ldots, n_g$, and so on). Analogously n_u and n_d are the numbers of left-handed singlets $U_{L\alpha}$, $\alpha = 1, \ldots, n_u$, etc. These models are the old SM to which a bunch of vector-like $SU(2)$ singlet quarks have been added. By vector-like quarks it is meant that they are quarks whose left- and right-handed components appear in a symmetric form. In other words, the kinetic term of the left and right components of each of them conjures up the kinetic term of a Dirac fermion.

On the basis of our previous conclusions, the addition of chiral symmetric fermions does not add any type O anomaly, neither gauge nor trace. Therefore, the balance of anomalies is the same as in the original MSM (with or without sterile neutrinos). Refs.: [6,91].

11.2. Left–Right (Parity) Symmetric Extensions of the SM Model

The MSM is CP invariant but P non-invariant. Some generalizations have been considered with the aim of realizing parity invariant models. The models are built in such a way that parity is spontaneously broken in the vacuum so that in the broken phase, the original MSM turns up. A parity invariant SM can be realized by introducing a second $SU(2)$ group, realizing in this way a left–right symmetry. The gauge group is $SU(3) \times SU(2)_L \times SU(2)_R \times U(1)$ and the spectrum of fields is

$$
\begin{array}{ccccc}
\mathsf{G}/fields & SU(3) & SU(2)_L & SU(2)_R & U(1) \\
\begin{pmatrix} u \\ d \end{pmatrix}_L & 3 & 2 & 1 & \frac{1}{3} \\
\begin{pmatrix} u \\ d \end{pmatrix}_R & 3 & 1 & 2 & \frac{1}{3} \\
\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L & 1 & 2 & 1 & -1 \\
\begin{pmatrix} \nu_e \\ e \end{pmatrix}_R & 1 & 1 & 2 & -1
\end{array}
\tag{282}
$$

The relation (2), in this case, is

$$\widetilde{Y} = Q - T_{3L} - T_{3R},\qquad(283)$$

and $\widetilde{Y}$ is interpreted as $\frac{B-L}{2}$ (B is the baryon, L thelepton number).

Warning: In this model, left and right fermions couple to different (left and right) gauge fields, say $W_{R\mu}$ and $W_{L\mu}$. To make this point more clear we notice that this situation is the same as considered in connection with Bardeen's method in Section 2.2. For the action term

$$\overline{\psi_R}i\gamma^\mu\left(\partial_\mu + W_{R\mu}\right)\psi_R + \overline{\psi_L}i\gamma^\mu\left(\partial_\mu + W_{L\mu}\right)\psi_L\qquad(284)$$

can be easily rewritten in the form

$$\overline{\psi}i\gamma^\mu\left(\partial_\mu + V_\mu + \gamma_5 A_\mu\right)\psi\qquad(285)$$

where $W_{R/L\mu} = V_\mu \pm A_\mu$.

The analysis of gauge and diffeomorphism anomalies in this model leads to the following Table 6:

Table 6. LR gauge and mixed.

- $T^{\mathfrak{su}(3)} \times T^{\mathfrak{su}(3)} \times T^{\mathfrak{su}(3)}$: there are two left-handed and two right-handed triplet, whose anomalies cancel one another.
- $T_L^{\mathfrak{su}(2)} \times T_L^{\mathfrak{su}(2)} \times T_L^{\mathfrak{su}(2)}$ and $T_R^{\mathfrak{su}(2)} \times T_R^{\mathfrak{su}(2)} \times T_R^{\mathfrak{su}(2)}$ both vanish because the tensor d^{abc} vanishes in general for the Lie algebra $\mathfrak{su}(2)$.
- $T^{\mathfrak{su}(3)} \times T^{\mathfrak{su}(3)} \times T^{\mathfrak{u}(1)}$, in which case we have the trace of two $\mathfrak{su}(3)$ left triplet generators and two right triplet generators. These traces are again non-vanishing, but they are multiplied by the corresponding $\mathfrak{u}(1)$ charge, whose total value is $3\left(2\left(\frac{1}{3}\right) - 2\left(\frac{1}{3}\right)\right) = 0$.
- $T_L^{\mathfrak{su}(2)} \times T_L^{\mathfrak{su}(2)} \times T^{\mathfrak{u}(1)}$, in which case we have the trace of two $\mathfrak{su}(2)$ generators in two doublet representations. These traces are non-vanishing because $\mathrm{tr}(T^a T^b) \sim \delta^{ab}$, but they are multiplied by the corresponding $\mathfrak{u}(1)$ charges, whose total value is $6\left(\frac{1}{3}\right) + 2(-1) = 0$.
- Analogously, $T_R^{\mathfrak{su}(2)} \times T_R^{\mathfrak{su}(2)} \times T^{\mathfrak{u}(1)}$ leads to the following: $-6\left(\frac{1}{3}\right) - 2(-1) = 0$
- $T^{\mathfrak{u}(1)} \times T^{\mathfrak{u}(1)} \times T^{\mathfrak{u}(1)}$: in this case, the tensor is proportional to the overall sum of the charge products $6\left(\frac{1}{3}\right)^3 + 2(-1)^3 - 6\left(\frac{1}{3}\right)^3 - 2(-1)^3 = 0$.
- $\Sigma \times \Sigma \times T^{\mathfrak{u}(1)}$: the trace $\mathrm{tr}\left(\Sigma^{ab}\Sigma^{cd}\right)$ is non-vanishing, but it is multiplied by the total $U(1)$ charge: $6\left(\frac{1}{3}\right) + 2(-1) - 6\left(\frac{1}{3}\right) - 2(-1) = 0$.

The analysis of gravity and gauge-induced trace anomalies yields, Table 7.

Table 7. LR trace-gravity and trace-gauge

- The multiplet (282), when weakly coupled to gravity, will produce (6 + 2) units of trace anomaly with Pontryagin density with + sign and (6 + 2) units with - sign, with vanishing total result.
- We have six units of the anomaly (170) with curvature $F \equiv F^{\mathfrak{su}(3)}$ and six units with opposite sign. Therefore, the multiplet (1) is free of these anomalies.
- We have instead 4 units of the same anomaly with gauge field $F \equiv F_L^{\mathfrak{su}(2)}$ and positive sign, see (171), computed in the doublet representation of $\mathfrak{su}(2)$.
- We also have 4 units of the same anomaly with gauge field $F \equiv F_R^{\mathfrak{su}(2)}$ and negative sign, see (171), computed in the doublet representation of $\mathfrak{su}(2)$.
- Finally, we have a $U(1)$ gauge-induced trace anomaly with vanishing total coefficient: $6\left(\frac{1}{3}\right)^2 + 2(-1)^2 - 6\left(\frac{1}{3}\right)^2 - 2(-1)^2 = 0$

Refs.: [92–95].

11.3. Other Hypothetical Extensions of the MSM

All the extensions of the SM considered so far have a residual trace anomaly. Let us try to realize a combination of elementary fields which are completely anomaly-free. To this end, we start from the generation (1) and add doublets or singlets of leptons.

$$
\begin{array}{ccccc}
\text{G}/fields & & SU(3) & SU(2) & U(1) \\
1 & \begin{pmatrix} u \\ d \end{pmatrix}_L & 3 & 2 & \frac{1}{6} \\
1 & u_R & 3 & 1 & \frac{2}{3} \\
1 & d_R & 3 & 1 & -\frac{1}{3} \\
1 & \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L & 1 & 2 & -\frac{1}{2} \\
1 & e_R & 1 & 1 & -1 \\
n_L & \begin{pmatrix} \nu \\ e \end{pmatrix}_L & 1 & 2 & y_L \\
n_R & \begin{pmatrix} \nu \\ e \end{pmatrix}_R & 1 & 2 & y_R \\
m_L & \nu_L & 1 & 1 & 0 \\
m_R & \nu_R & 1 & 1 & 0
\end{array}
\tag{286}
$$

where n_L, n_R, m_L, m_R are integers and y_L, y_R are rationals to be determined. We notice that the modifications involve only the leptonic sector. Therefore, all the data involving $SU(3)$ are unchanged. Below, we report only the differences Δ_i with respect to the entries in Tables MSM and the equations that have to be satisfied in order to cancel all anomalies, see Table 8

Table 8. Table of differences with MSM.

- $T^{\mathfrak{su}(2)} \times T^{\mathfrak{su}(2)} \times T^{\mathfrak{u}(1)}$: $\Delta_1 = 2(n_L y_L - n_R y_R)$, $\Delta_1 = 0$
- $T^{\mathfrak{u}(1)} \times T^{\mathfrak{u}(1)} \times T^{\mathfrak{u}(1)}$: $\Delta_2 = n_L(y_L)^3 - n_R(y_R)^3$, $\Delta_2 = 0$
- $\Sigma \times \Sigma \times T^{\mathfrak{u}(1)}$: $\Delta_3 = n_L y_L - n_R y_R$, $\Delta_3 = 0$
- $\Sigma \times \Sigma \times$ numberofunits: $\Delta_4 = n_L - n_R + m_L - m_R$, $\Delta_4 + 1 = 0$
- $T^{\mathfrak{su}(2)} \times T^{\mathfrak{su}(2)} \times$ numberofunits: $\Delta_5 = n_L - n_R$, $\Delta_5 + 4 = 0$
- $T^{\mathfrak{u}(1)} \times T^{\mathfrak{u}(1)} \times$ numberofunits: $\Delta_6 = n_L(y_L)^2 - n_R(y_R)^2$, $\Delta_6 = 0$

The simplest solution is $n_L = 0$, $n_R = 4$, $y_R = 0$ and $m_R = 0$, $m_L = 3$. i.e., we should add 4 right-handed $U(1)$-sterile doublets and 3 left-handed sterile neutrinos. $y_R = 0$ means that the two components of the doublet have charge $1/2$ and $-1/2$, very exotic animals.

11.4. GUT Models

Among the extensions of the SM a particular role is reserved to the grand-unified theories. They have the major merit of unifying strong, weak and electromagnetic interactions. The most prominent among them is probably the $SO(10)$ GUT. In what concerns the problem of the anomaly analysis, it turns out to be particularly simple because all fermions of the MSM of each family plus a right-handed neutrino are collected in one single multiplet, the **16** irreducible representation of $SO(10)$. Using the same notation as for the MSM, the multiplet is as follows:

$$(u_L, d_L, (u_R)^c, (d_R)^c, \nu_L, e_L, (\nu_R)^c, (e_R)^c) \tag{287}$$

where X^c represents the conjugate of X (and u and d carry three components each). This left-handed multiplet is potentially anomalous for a consistent gauge anomaly, but such a gauge anomaly is proportional to $Str\left(T^{\mathfrak{so}(10)}T^{\mathfrak{so}(10)}T^{\mathfrak{so}(10)}\right)$, the symmetric trace of three generators $T^{\mathfrak{so}(10)}$ of the Lie algebra of $SO(10)$. As is clear from the table (49) from the entry D_5, this trace vanishes identically. When we couple this model to gravity, the consistent diffeomorphism anomalies are absent for an analogous and usual reason. It is free of any mixed gauge-gravity anomaly because the latter is proportional to either $\mathrm{tr}\left(T^{\mathfrak{so}(10)}T^{\mathfrak{so}(10)}\right)\mathrm{tr}\Sigma$ or to $\mathrm{tr}\left(T^{\mathfrak{so}(10)}\right)\mathrm{tr}(\Sigma\Sigma)$, where Σ denotes a generator of the Lorentz group.

The situation for the odd trace anomalies is different. They do not cancel completely. We may have two types of trace anomalies, one proportional to the Pontryagin density $\epsilon^{\mu\nu\lambda\rho}R_{\mu\nu}{}^{\sigma\tau}R_{\lambda\rho\sigma\tau}$ and the other to the Chern class density $\epsilon^{\mu\nu\lambda\rho}\mathrm{tr}_{\mathbf{16}}\left(F_{\mu\nu}^{\mathfrak{su}(10)}F_{\lambda\rho}^{\mathfrak{su}(10)}\right)$. There are 16 units of the former and one of the latter (corresponding to the **16** representation of $SO(10)$). The former cancels out; let us recall why: the symbol such as $(\psi_R)^c$ (for instance u_R) in a generic metric can be rewritten as

$$(\psi_R)^c = \gamma^0 C \psi_R^* = \gamma^0 C P_R^* \psi^* = P_L \gamma^0 C \psi^* = P_L \psi^c = (\psi^c)_L. \tag{288}$$

Inserted into the kinetic term, it gives

$$\sqrt{g}\overline{(\psi^c)_L}\gamma^\mu(\partial_\mu + \tfrac{1}{2}\omega_\mu)(\psi^c)_L = \sqrt{g}\overline{(\psi_R)^c}\gamma^\mu(\partial_\mu + \tfrac{1}{2}\omega_\mu)(\psi_R)^c = \sqrt{g}\overline{\psi_R}\gamma^\mu(\partial_\mu + \tfrac{1}{2}\omega_\mu)\psi_R \tag{289}$$

where $\omega_\mu = \omega_\mu^{ab}\Sigma_{ab}$ and Σ_{ab} are anti-hermitean. The last passage requires an overall transposition and a partial integration. Therefore, the kinetic term of the multiplet (287), coupled only to the metric, splits into 16 independent Weyl fermion kinetic terms, 8 left-handed and 8 right-handed, with opposite contributions to the trace anomaly.

As for the other anomaly, the Chern class one, this 'sleight of hand' is not possible because the multiplet (287) couples to a unique potential valued in the Lie algebra of $SO(10)$, and it is not possible to split the kinetic term in individual components like in the metric case.

12. What Should We Do Then?

The conclusion from the previous analysis is that it is possible to cancel all the anomalies in the extended SMS only at the price of introducing new exotic particles, which make up a rather unlikely spectrum. Barring these possibilities, which are all comprised in the scheme (A) and (B) of Section 4.4, there remains the scheme (C), i.e., the introduction in the

theory of fields with suitable transformation properties. The troubling anomalies are those related to the densities of the Chern and Pontryagin classes:

$$\mathcal{A}_\omega^{\mathfrak{su}(2)} = i \int d^4x \sqrt{g}\,\omega \mathrm{tr}\left(F^{\mathfrak{su}(2)} * F^{\mathfrak{su}(2)}\right) \equiv \int d^4x \sqrt{g}\,\omega \epsilon^{\mu\nu\lambda\rho} \mathrm{tr}\left(F_{\mu\nu}^{\mathfrak{su}(2)} F_{\lambda\rho}^{\mathfrak{su}(2)}\right) \qquad (290)$$

and

$$\mathcal{A}_\omega^{grav} = i \int d^4x \sqrt{g}\,\omega \mathrm{tr}(\mathcal{R} * \mathcal{R}) \equiv \int d^4x \sqrt{g}\,\omega \epsilon^{\mu\nu\lambda\rho} R_{\mu\nu}{}^{\sigma\tau} R_{\lambda\rho\sigma\tau} \qquad (291)$$

The latter can be canceled, for instance, by adding to the MSM a right-handed sterile neutrino. The former anomalies are more 'resistant'. In any case, both can be canceled by a WZ term [For a discussion of WZ terms, see Section 17.2], that is a term proportional to

$$\mathcal{C}_{WZ}^{\mathfrak{su}(2)} = i \int d^4x \sqrt{g}\,\sigma \mathrm{tr}\left(F^{\mathfrak{su}(2)} * F^{\mathfrak{su}(2)}\right) \qquad (292)$$

and

$$\mathcal{C}_{WZ}^{grav} = i \int d^4x \sqrt{g}\,\sigma \mathrm{tr}(\mathcal{R} * \mathcal{R}) \qquad (293)$$

respectively. σ is a scalar field which transforms as

$$\delta_\omega \sigma = \omega \qquad (294)$$

under a Weyl transformation. This scheme looks very much like the Peccei-Quinn solution for the strong CP problem. But the two schemes refer to different energy scales. The Peccei-Quinn solution applies at the electroweak scale ($\sim$200 GeV), corresponding to 10^{-11} s in the history of the universe when gravity was already classical. The cancelation of trace anomalies by (292) and (293) should take place at much higher energies and earlier times. But the trouble here is with the i which renders both counterterms imaginary. It is very likely that this term may backreact and endanger unitarity.

This imaginary unit is important and it is worth spending a comment on it. It comes out explicitly from the calculation. But there is also an indirect argument that supports its presence. As was said above the i guarantees the invariance under time reversal. If the i were not there, time reversal symmetry would be violated, together with parity. But while parity violation is expected since the classical theory is not parity invariant, T symmetry violation is totally unexpected because the classical theory is T symmetric, and there is no time reversal violation in the process of deriving the anomaly by, for instance, dimensional regularization.

This imaginary unit also marks the difference with the even parity trace anomalies, where it is absent. As we have explained above, even trace anomalies do not affect unitarity, and a WZ term, like above, can be easily implemented in order to restore conformal invariance (but see below for a more accurate discussion).

It is perhaps not useless to recall once again the difference between type O and type NO anomalies. The just mentioned odd parity anomalies are type O, they are a spy of the lack of a Weyl fermion propagator, which hinders the idea itself of quantization. They are unveiled by obstructions that manifest themselves in the family's index theorem. However—this is the important point—the corresponding anomalies are the symptoms of a disease, not the disease itself. The topological obstructions are. We can suppress the symptoms, i.e., these anomalies, with clever WZ terms, but they do not heal the disease.

Resuming the previous discussion, one possible attitude is to assume that gravity couples to matter in a non-minimal way, which does not give rise to anomalies. This implies a reformulation of the SM and its coupling to gravity, which is not covered by this article.

Finally, a more radical possibility is that gravity is entirely classical. As explained above the coupling of the anomalous e.m. trace in the effective action is via the fluctuating field $h = h_\mu^\mu$. This field is ineffective if gravity is classical. As a consequence, the anomalies in question can be disregarded.

Summarizing, we have the following alternatives for odd trace anomaly cancelation:

- Minimal coupling matter-gravity: cancelation by exotic SM spectrum;
- Minimal coupling matter-gravity: cancelation by WZ terms, unitarity in danger;
- Non-minimal coupling gravity-matter: to be investigated;
- Classical gravity: trace anomalies irrelevant.

Altogether, none of these possibilities is appealing. The present paper is clearly inspired by a bottom–up attitude. It is an analysis of the consistency conditions that arise as soon as we try to immerse the SM, in one of its versions, into a background of dynamical gravity. In such a context, we meet new anomalies which are not present when gravity is irrelevant. These new anomalies are both of type O and of type NO. As we have explained, the former may endanger unitarity. Therefore, one has to make sure that they are absent. The possibilities we have presented above are unconvincing. Even freely playing with the still unknown SM sector of neutrinos, it does not seem possible to meet the requirement of full freedom from type O anomalies by simply enlarging the spectrum of the MSM. If our anomaly analysis is correct, it is clear that we have to change something in the previous pictures of the SM. In the rest of this paper, we would like to present an (unconventional) proposal in this direction. It certainly does not solve all the problems, but it is a hint of where the anomaly analysis, if taken seriously, may lead to.

13. A Chirally Symmetric Model

As pointed out above the L-R model is the one that comes closer to the cancelation of all the anomalies. That would be the case if, instead of two $SU(2)$ gauge fields, one left and one right, we had a unique $SU(2)$ gauge symmetry and a unique gauge field $V_\mu = V_\mu^a T^a$. However, this would lead to a theory of Dirac fermions, and at that point, we would not have a mechanism to break parity and recover the MSM in the broken phase.

We would like to construct an L-R symmetric (or chirally symmetric, or parity invariant) model, which is certainly anomaly-free. It is based on the same multiplet as the MSM with the addition of a right-handed sterile neutrino. In the usual SM notation, it is

$$
\begin{array}{cccc}
\text{G}/fields & SU(3) & SU(2) & U(1) \\
\begin{pmatrix} u \\ d \end{pmatrix}_L & 3 & 2 & \frac{1}{6} \\
(u_R)^c & \bar{3} & 1 & -\frac{2}{3} \\
(d_R)^c & \bar{3} & 1 & \frac{1}{3} \\
\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L & 1 & 2 & -\frac{1}{2} \\
(e_R)^c & 1 & 1 & 1 \\
(\nu_R)^c & 1 & 1 & 0
\end{array}
\tag{295}
$$

where X^c represents the Lorentz conjugate spinor of X, i.e., $X^c = \gamma_0 C X^*$. This multiplet couples to a left gravitational metric and connection and to the $SU(3)_L \times SU(2) \times U(1)_L$ gauge fields. We have seen that all the anomalies cancel out except for 4 units of the trace

anomaly due to the gauge field $F \equiv F^{\mathfrak{su}(2)}$, (171), computed in the doublet representation of $\mathfrak{su}(2)$.

The multiplet (295) describes left-handed particles and right-handed antiparticles.

There is also a right-handed multiplet

$$
\begin{array}{cccc}
\mathsf{G}/fields & SU(3) & SU(2) & U(1) \\
\begin{pmatrix} u' \\ d' \end{pmatrix}_R & 3 & 2 & \frac{1}{6} \\
(u'_L)^c & \bar{3} & 1 & -\frac{2}{3} \\
(d'_L)^c & \bar{3} & 1 & \frac{1}{3} \\
\begin{pmatrix} v'_e \\ e' \end{pmatrix}_R & 1 & 2 & -\frac{1}{2} \\
(e'_L)^c & 1 & 1 & 1 \\
(v'_L)^c & 1 & 1 & 0
\end{array}
\tag{296}
$$

coupled to a right gravitational metric and connection. This multiplet couples to the $SU(3)_R \times SU(2) \times U(1)_R$ gauge fields. The anomaly analysis of this right-handed multiplet is the same as for the left-handed one except for the sign of the trace anomaly due to the gauge field $F \equiv F^{\mathfrak{su}(2)}$, see (171), which is opposite. Therefore the overall sum of the anomalies of the system vanishes.

The multiplet (296) describes right-handed particles and left-handed antiparticles.

We shall call these two intertwined theories, with field content (295) and (296), $\mathcal{T}_L$ and $\mathcal{T}_R$, respectively. The overall theory is free of type O anomalies. We denote it simply by $\mathcal{T} = \mathcal{T}_L \cup \mathcal{T}_R$.

Important. Both multiplets couple to the same $SU(2)$ gauge fields. Only in this case do all anomalies cancel! This is the reason why we use the symbol $\cup$ intead of $+$.

Let us see explicitly in the sequel the various possible pieces of the relevant actions. The MAT (hypercomplex) formalism turns out to be a very effective bookkeeping device to write them down and we will use it as broadly as possible.

13.1. The Quadratic Fermion Action

The quadratic fermionic action is modeled on the one already introduced above, (221),

$$
\begin{aligned}
\widehat{S}_f &= \int d^4\widehat{x} \left(i\overline{\psi}\sqrt{\widehat{\overline{g}}}\gamma^a \widehat{e}^\mu_a \left(\mathcal{D}_\mu + \frac{1}{2}\widehat{\Omega}_\mu \right)\psi \right)(\widehat{x}) \\
&= \int d^4\widehat{x} \left(i\overline{\psi}\sqrt{\widehat{\overline{g}}}\gamma^a (\widetilde{e}^\mu_a + \gamma_5\widetilde{c}^\mu_a) \left(\mathcal{D}_\mu + \frac{1}{2}\left(\Omega^{(1)}_\mu + \gamma_5\Omega^{(2)}_\mu \right) \right)\psi \right)(\widehat{x})
\end{aligned}
\tag{297}
$$

where ψ is a 16 component spinor field, which encompasses the two left and right-handed multiplets above, as will be explained shortly. In (297), we have introduced the covariant derivative $\widehat{\mathcal{D}}_\mu = \widehat{\partial}_\mu + \mathcal{V}_\mu$, where $\widehat{\partial}_\mu$ here and below has to be understood as $\frac{\partial}{\partial\widehat{x}^\mu}$, and $\mathcal{V}_\mu$ is valued in the Lie algebra $SU(3)_L \times SU(3)_R \times SU(2) \times U(1)_L \times U(1)_R$ in the relevant fundamental representations of $SU(3)$ and $SU(2)$ and in the representations of $U(1)$ specified by their hypercharges. Now let us recall that

$$
\widehat{g} = \{\widehat{g}_{\mu\nu}\} = g + \gamma_5 f, \qquad \widehat{g}_{\mu\nu} = g_{\mu\nu} + \gamma_5 f_{\mu\nu} = g_{+\mu\nu}P_+ + g_{-\mu\nu}P_-, \qquad g_{\pm\mu\nu} = g_{\mu\nu} \pm f_{\mu\nu}
\tag{298}
$$

and

$$
\widehat{g}^{-1} = \{\widehat{g}^{\mu\nu}\}, \qquad \widehat{g}^{-1} = \mathsf{g} + \gamma_5\mathsf{f}, \qquad \widehat{g}^{-1}\widehat{g} = 1, \qquad \widehat{g}^{\mu\lambda}\widehat{g}_{\lambda\nu} = \delta^\mu_\nu
\tag{299}
$$

so that [Careful! $g^{\mu\nu}, f^{\mu\nu}$ are *not* the inverse of $g_{\mu\nu}, f_{\mu\nu}$, respectively, while $g_{\pm}^{\mu\nu}$ is the inverse of $g_{\pm\mu\nu}$.]

$$\widehat{g}^{\mu\nu} = g^{\mu\nu} + \gamma_5 f^{\mu\nu} = g_+^{\mu\nu} P_+ + g_-^{\mu\nu} P_-, \qquad g_\pm^{\mu\nu} = g^{\mu\nu} \pm f^{\mu\nu}, \qquad g_\pm^{\mu\lambda} g_{\pm\lambda\nu} = \delta_\nu^\mu \qquad (300)$$

Moreover

$$\sqrt{\widehat{g}} = \sqrt{\det(g_+)} P_+ + \sqrt{\det(g_-)} P_-, \qquad (301)$$

$$\begin{aligned}
\widehat{e}_\mu^a &= e_\mu^{(+)a} P_+ + e_\mu^{(-)a} P_-, & e_\mu^{(\pm)a} &= e_\mu^a \pm c_\mu^a, & e_\mu^{(\pm)a} e_a^{(\pm)\nu} &= \delta_\mu^\nu \\
\widehat{e}_a^\mu &= \tilde{e}_a^{(+)\mu} P_+ + e_a^{(-)\mu} P_-, & e_a^{(\pm)\mu} &= e_a^\mu \pm c_a^\mu, & e_a^{(\pm)\mu} e_\mu^{(\pm)b} &= \delta_a^b
\end{aligned} \qquad (302)$$

$$\widehat{\Omega}_\mu^{ab} = \Omega_\mu^{(+)ab} P_+ + \Omega_\mu^{(-)ab} P_-, \qquad \Omega_\mu^{(\pm)ab} = \Omega_\mu^{(1)ab} \pm \Omega_\mu^{(2)ab} \qquad (303)$$

Using these we can split $\widehat{S}_f$ into $\widehat{S}_f = S_f^{(+)} + S_f^{(-)}$ where

$$S_f^{(\pm)} = \int d^4\widehat{x} \left(\sqrt{g_\pm} i\overline{\psi} P_\mp \gamma^a e_a^{(\pm)\mu} P_\pm \left(\mathcal{D}_\mu^{(\pm)} + \frac{1}{2}\Omega_\mu^{(\pm)} P_\pm \right) \psi \right)(\widehat{x}) \qquad (304)$$

In more detail,

$$S_f^{(+)} \equiv S_{fR} = \int d^4\widehat{x} \left(\sqrt{g_+} i\overline{\psi'}_R \gamma^a e_a^{(+)\mu} \left(\mathcal{D}_\mu^{(+)} + \frac{1}{2}\Omega_\mu^{(+)} \right) \psi'_R \right)(\widehat{x}) \qquad (305)$$

where ψ'_R represents the right-handed multiplet (296), and

$$\mathcal{D}_\mu^{(+)} = \partial_\mu + g_X^+ X_\mu^{(+)} + g_W W_\mu + g_B^+ B_\mu^{(+)} \qquad (306)$$

while

$$S_f^{(-)} \equiv S_{fL} = \int d^4\widehat{x} \left(\sqrt{g_-} i\overline{\psi}_L \gamma^a e_a^{(-)\mu} \left(\mathcal{D}_\mu^{(-)} + \frac{1}{2}\Omega_\mu^{(-)} \right) \psi_L \right)(\widehat{x}) \qquad (307)$$

where ψ_L represents the left-handed multiplet (295), and

$$\mathcal{D}_\mu^{(-)} = \partial_\mu + X_\mu^{(-)} + W_\mu + B_\mu^{(-)} \qquad (308)$$

The symbols $X_\mu^{(\pm)}, W_\mu, B_\mu^{(\pm)}$ refer to the $SU(3)_{R/L}, SU(2)$ and $U(1)_{R/L}$ potentials, respectively. Of course, each potential has its own distinct coupling to the fermions, which can be made explicit through a rededinition.

Remark 6. *Here and below, once the hypercomplex variable $\widehat{x}$ has performed its job as a bookkeeping device, we can safely replace it everywhere with x.*

Since parity (see Appendix A) exchanges $g_{\pm\mu\nu}$ with $g_\mp^{\mu\nu}$, $X_\mu^\pm$ with $X^{\mp\mu}$, $B_\mu^\pm$ with $B^{\mp\mu}$ and $\psi_{L/R}$ with $\psi_{R/L}$, and W_μ with W^μ, the sum $S_f^{(-)} + S_f^{(+)}$ is parity invariant.

13.2. The Gauge Field Action

The action for the gauge fields is

$$\widehat{S}_g = -\frac{1}{8} \int d^4\widehat{x} \text{Tr}\left(\frac{1}{\widehat{g}^2} \sqrt{\widehat{g}}\widehat{g}^{\mu\mu'}\widehat{g}^{\nu\nu'} F_{\mu\nu} F_{\mu'\nu'}\right), \qquad F_{\mu\nu} = F_{\mu\nu}^a T^a \tag{309}$$

where Tr denotes the trace over all matrices, including γ_5, and T^a are the generators of the overall gauge Lie algebra. The coupling constant can be split into left and right parts. This can be made explicit in the action as follows: define

$$\widehat{g} = g + \gamma_5 h = g_+ P_+ + g_- P_-, \qquad g_\pm = g \pm h \tag{310}$$

then the factor $\widehat{g}^{-2}$ inside the round brackets of (310) becomes

$$\frac{1}{\widehat{g}^2} = \frac{1}{g_+^2} P_+ + \frac{1}{g_-^2} P_- \tag{311}$$

Therefore, (309) splits as follows

$$S_g^{(\pm)} = -\frac{1}{4g_\pm^2} \int d^4\widehat{x}\sqrt{g_\pm}\text{tr}\left(g_\pm^{\mu\mu'} g_\pm^{\nu\nu'} F_{\mu\nu} F_{\mu'\nu'}\right) \tag{312}$$

The action $\widehat{S}_g$ is appropriate for the $SU(2)$ gauge fields, which are the same both for $\mathcal{T}_L$ and $\mathcal{T}_R$.

13.3. AE Gauge Field Action

We can generalize the previous action by extending it to a field $\widehat{V}_\mu = V_\mu + \gamma_5 A_\mu$

$$\widehat{S}_{aeg} = -\frac{1}{8} \int d^4\widehat{x} \text{Tr}\left(\frac{1}{\widehat{g}^2} \sqrt{\widehat{g}}\widehat{g}^{\mu\mu'}\widehat{g}^{\nu\nu'} \widehat{F}_{\mu\nu} \widehat{F}_{\mu'\nu'}\right), \qquad \widehat{F}_{\mu\nu} = \widehat{F}_{\mu\nu}^a T^a \tag{313}$$

where $\widehat{F} = \widehat{d}\widehat{V} + \frac{1}{2}[\widehat{V}, \widehat{V}]$, and Tr and T^a are as above. The action (313) splits as

$$S_{aeg}^{(\pm)} = -\frac{1}{4g_\pm^2} \int d^4\widehat{x}\sqrt{g_\pm}\text{tr}\left(g_\pm^{\mu\mu'} g_\pm^{\nu\nu'} F_{\mu\nu}^{(\pm)} F_{\mu'\nu'}^{(\pm)}\right) \tag{314}$$

where $F_{\mu\nu}^{(\pm)} = \widehat{d}V^{(\pm)} + \frac{1}{2}[V^{(\pm)}, V^{(\pm)}]$ and $V_\mu^{(\pm)} = V_\mu \pm A_\mu$, and $\widehat{g} = g + \gamma_5 h = g_+ P_+ + g_- P_-$. As an example, in the case of $SU(2)_R \times SU(2)_L$ considered in Section 11.2 above, $V_\mu^{(\pm)}$ has been denoted $W_{R/L\mu}$.

In the case of $\mathcal{T} = \mathcal{T}_L + \mathcal{T}_R$, the action S_{aeg} is appropriate for the groups $SU(3)_L \times SU(3)_R$ and $U(1)_L \times U(1)_R$, each with its own coupling $\widehat{g}$, since we have distinct left and right potentials. As before, the action splits in the sum $S_{aeg}^{(+)} + S_{aeg}^{(-)}$ with

$$S_{aeg}^{(\pm)} = -\frac{1}{4g_\pm^2} \int d^4\widehat{x}\sqrt{g_\pm}\text{tr}\left(g_\pm^{\mu\mu'} g_\pm^{\nu\nu'} F_{\mu\nu}^{(\pm)} F_{\mu'\nu'}^{(\pm)}\right) \tag{315}$$

where $F^\pm$ denote the curvatures of the $SU(3)_R$ and $U(1)_R$, and $SU(3)_L$ and $U(1)_L$ potentials, respectively. Of course, the factors $g_\pm^{-2}$ can be absorbed, as usual, in a redefinition of the gauge potentials.

$\widehat{S}_g$ and $\widehat{S}_{aeg}$ are parity invariant.

13.4. AE Real Scalar Field

We will also need the action for an axially extended (AE) real scalar field $\widehat{\Phi} = \phi + \gamma_5 \pi$, where ϕ is an ordinary real and π a pseudoreal scalar field. We write the action as

$$\widehat{S}_{aes} = \int d^4 \widehat{x}\, \mathrm{tr}\left[\sqrt{\widehat{g}}\left(\widehat{g}^{\mu\nu} \partial_\mu \widehat{\Phi} \partial_\nu \widehat{\Phi} - \widehat{m}^2 \widehat{\Phi}^2 - \frac{\widehat{\lambda}}{4} \widehat{\Phi}^4 \right) \right] \tag{316}$$

We can decompose $\widehat{\Phi}$ as $\widehat{\Phi} = \Phi_+ P_+ + \Phi_- P_-$ where $\Phi_\pm = \phi \pm \pi$, as well as $\widehat{m}^2 = m_+^2 P_+ + m_-^2 P_-$ and $\widehat{\lambda} = \lambda_+ P_+ + \lambda_- P_-$. Then we have the splitting $\widehat{S}_{aes} = S_{aes}^{(+)} + S_{aes}^{(-)}$, with

$$S_{aes}^{(\pm)} = \int d^4 \widehat{x} \sqrt{g_\pm}\left(g_\pm^{\mu\nu} \partial_\mu \Phi_\pm \partial_\nu \Phi_\pm - m_\pm^2 \Phi_\pm^2 - \frac{\lambda_\pm}{4} \Phi_\pm^4 \right) \tag{317}$$

13.5. AE Scalar Doublet Action

We will employ later also an AE complex scalar field $\widehat{H} = \mathsf{h} \pm \gamma_5 \mathsf{k}$ which couples to a metric $\widehat{g}_{\mu\nu}$ and is a doublet under $SU(2)$. The action is

$$\widehat{S}_{aed} = \int d^4 \widehat{x}\, \mathrm{tr}\left[\sqrt{\widehat{g}}\left(\widehat{g}^{\mu\nu} \mathcal{D}_\mu \widehat{H}^\dagger \mathcal{D}_\nu \widehat{H} - \widehat{M}^2 \widehat{H}^\dagger \widehat{H} - \frac{\widehat{\lambda}}{4}\left(\widehat{H}^\dagger \widehat{H} \right)^2 \right) \right] \tag{318}$$

where $\mathcal{D}_\mu = \partial_\mu - ig W_\mu$, and W_μ is an $SU(2)$ gauge field (notice that W_μ is not AE). We can decompose $\widehat{H}$ as $\widehat{H} = H_+ P_+ + H_- P_-$ where $H_\pm = \mathsf{h} \pm \mathsf{k}$, as well as $\widehat{M}^2 = M_+^2 P_+ + M_-^2 P_-$ and $\widehat{\lambda} = \lambda_+ P_+ + \lambda_- P_-$. Then $\widehat{S}_{aed}$ splits as $\widehat{S}_{aed} = S_{aed}^{(+)} + S_{aed}^{(-)}$ with

$$S_{aed}^{(\pm)} = \int d^4 \widehat{x} \sqrt{g_\pm}\left[g_\pm^{\mu\nu} \mathcal{D}_\mu H_\pm^\dagger \mathcal{D}_\nu H_\pm - M_\pm^2 H_\pm^\dagger H_\pm - \frac{\lambda_\pm}{4}\left(H_\pm^\dagger H_\pm \right)^2 \right] \tag{319}$$

$S_{aes}^{(+)} + S_{aes}^{(-)}$ and $S_{aed}^{(+)} + S_{aed}^{(-)}$ are parity invariant.

13.6. The EH-Like Action

The Einstein–Hilbert-like action in this context takes the form

$$\widehat{S}_{EH} = \frac{1}{4} \int d^4 \widehat{x}\, \mathrm{Tr}\left(\frac{1}{\widehat{\kappa}} \sqrt{\widehat{g}} \widehat{R} \right), \tag{320}$$

where $\widehat{R}$ is the Ricci scalar, obtained from (204) by contracting ν with ρ to obtain the Ricci tensor

$$\widehat{R}_{\mu\lambda} = R_{\mu\lambda}^{(+)} P_+ + R_{\mu\lambda}^{(-)} P_-, \qquad R_{\mu\lambda}^{(\pm)} = R_{\mu\lambda}^{(1)} \pm R_{\mu\lambda}^{(2)} \tag{321}$$

and then

$$\widehat{R} = \widehat{g}^{\mu\lambda} \widehat{R}_{\mu\lambda} = \widehat{g}^{\mu\lambda}\left(\widehat{R}_{\mu\lambda}^{(1)} + \gamma_5 R_{\mu\lambda}^{(2)} \right) = R^{(+)} P_+ + R^{(-)} P_- \tag{322}$$

where $R^{(\pm)} = g_\pm^{\mu\nu} R_{\mu\lambda}^{(\pm)}$. Moreover $\frac{1}{\widehat{\kappa}} = \frac{1}{\kappa_+} P_+ + \frac{1}{\kappa_-} P_-$.

The action (320) splits as $\widehat{S}_{EH} = S_{EH}^{(+)} + S_{EH}^{(-)}$ with

$$S_{EH}^{(\pm)} = \frac{1}{2\kappa_\pm} \int d^4 \widehat{x} \sqrt{g_\pm} R^{(\pm)} \tag{323}$$

Remark 7. *The action $S_{EH}^{(+)} + S_{EH}^{(-)}$ is parity invariant. The previous formalism works even if there is only one metric, that is $g_+ = g_-$. Here and in the sequel, we wish to keep track of the more general possibility of a bimetric theory.*

13.6.1. AE Diffeomorphisms

Axially extended diffeomorphisms, see Section 8.1, are defined by

$$\widehat{x}^\mu \rightarrow \widehat{x}^\mu + \widehat{\xi}^\mu(\widehat{x}^\mu), \qquad \widehat{\xi}^\mu = \xi^\mu + \gamma_5 \zeta^\mu = \xi_+^\mu P_+ + \xi_-^\mu P_-, \qquad \xi_\pm^\mu = \xi^\mu \pm \zeta^\mu \tag{324}$$

The way the various fields transform under these transformations is formally the same as for the corresponding ordinary fields, for instance,

$$\delta_{\widehat{\xi}} \psi = \widehat{\xi}^\mu \partial_\mu \psi, \qquad \delta_{\widehat{\xi}} \widehat{\Phi} = \widehat{\xi}^\mu \partial_\mu \widehat{\Phi}, \qquad \delta_{\widehat{\xi}} \widehat{g}_{\mu\nu} = \widehat{D}_\mu \widehat{\xi}_\nu + \widehat{D}_\nu \widehat{\xi}_\mu, \qquad \text{etc.} \tag{325}$$

In particular $\sqrt{\widehat{g}}$ has the basic property that, under AE diffeomorphisms:

$$\delta_{\widehat{\xi}} \sqrt{\widehat{g}} = \widehat{\xi}^\lambda \widehat{\partial}_\lambda \sqrt{\widehat{g}} + \sqrt{\widehat{g}} \widehat{\partial}_\lambda \widehat{\xi}^\lambda \tag{326}$$

Since the form of the actions (297), (309), (316) and (320) above are formally the same as that of the corresponding ordinary actions, it follows that the former are invariant under the extended diffeomorphisms.

In turn, we have

$$\delta_{\xi_\pm} \psi = \xi_\pm^\mu \partial_\mu \psi, \qquad \delta_{\xi_\pm} \Phi_\pm = \xi_\pm^\mu \partial_\mu \Phi_\pm, \qquad \text{etc.} \tag{327}$$

It follows that the actions $S_f^{(\pm)}, S_g^{(\pm)}, S_{aeg}^{(\pm)}, S_{aes}^{(\pm)}$ and $S_{EH}^{(\pm)}$ are invariant under the (ordinary) diffeomorphisms spanned by the parameters $\xi_\pm^\mu$, respectively.

A system invariant under AE diffeomorphisms is automatically *chirally symmetric*, but need not be parity invariant because parity invariance also requires the equality of the left and right couplings. The systems defined by the actions (297), (309), (318), (316) and (320) are chirally symmetric.

The actions (309) and (318) deserve a special comment. Since they involve an $SU(2)$ gauge field, say W_μ, valued in the Lie algebra $\mathfrak{su}(2)$, and a complex scalar field H, both without any axial counterpart, it would seem inappropriate to consider the transformations

$$\delta_{\widehat{\xi}} W_\mu = \widehat{\xi}^\lambda \widehat{\partial}_\lambda W_\mu + \widehat{\partial}_\mu \widehat{\xi}^\lambda W_\lambda \tag{328}$$

because $\widehat{\xi}^\lambda$ splits according to Equation (324). But, first, one should recall that W_μ are functions of $\widehat{x}$, and, second, the operation (328) is akin to a similar and very familiar operation in quantum field theory when, in an ordinary theory of Dirac fermions coupled to an Abelian vector potential V_μ, we consider a chiral transformation $\delta V_\mu = i\gamma_5 \partial_\mu \lambda$. This transformation formally does not make any sense because V_μ is not a 4×4 matrix, but it is accepted with the understanding that we can slyly introduce a phantom axial companion of V_μ and eventually drop it, as we have explained several times.

13.6.2. AE Weyl Transformations

The axially extended Weyl transformations are defined by

$$\widehat{g}_{\mu\nu} \longrightarrow e^{2\widehat{\omega}} \widehat{g}_{\mu\nu}, \qquad \widehat{\omega} = \omega + \gamma_5 \eta = \omega_+ P_+ + \omega_- P_- \tag{329}$$

where $\omega_\pm = \omega \pm \eta$. Under these transformations, we have

$$\delta_{\widehat\omega}\sqrt{\widehat{g}} \;=\; \mathrm{d}\widehat\omega\sqrt{\widehat{g}} \tag{330}$$

$$\delta_{\widehat\omega}\widehat{R} \;=\; -2\widehat\omega\widehat{R} - 2(\mathrm{d}-1)\widehat\square\widehat\omega \tag{331}$$

$$\delta_{\widehat\omega}\widehat{R}_{\mu\nu\lambda}{}^{\rho} \;=\; -\delta_{\nu}^{\rho}\widehat{D}_{\mu}\widehat{D}_{\lambda}\widehat\omega + \delta_{\mu}^{\rho}\widehat{D}_{\nu}\widehat{D}_{\lambda}\widehat\omega + \widehat{D}_{\mu}\widehat{D}_{\sigma}\widehat\omega\widehat{g}^{\rho\sigma}\widehat{g}_{\nu\lambda} - \widehat{D}_{\nu}\widehat{D}_{\sigma}\widehat\omega\widehat{g}^{\rho\sigma}\widehat{g}_{\mu\lambda} \tag{332}$$

The fermions transform as

$$\psi \to e^{-\frac{3}{2}(\omega+\gamma_5\eta)}\psi, \tag{333}$$

the scalars as

$$\widehat\Phi \to e^{-\widehat\omega}\widehat\Phi, \tag{334}$$

while the gauge fields are invariant

$$\widehat{V}_\mu \to \widehat{V}_\mu. \tag{335}$$

Under AE Weyl transformations, the actions (297), (309) and (313) are invariant, while (318), (316) and (320) are not.

$S_f^{(\pm)}, S_g^{(\pm)}, S_{aeg}^{(\pm)}$ are invariant under the Weyl transformations with parameter $\omega_\pm$, respectively.

13.7. Yukawa Couplings

The Yukawa coupling between a fermion ψ, an AE complex scalar $\widehat\Phi$ and another fermion χ is defined as follows. Set

$$\widehat{S}_Y = S_Y + \overline{S}_Y \tag{336}$$

Then

$$S_Y = \int d^4\widehat{x}\left(\overline\psi\widehat{y}\sqrt{\overline{\widehat{g}}\widehat\Phi}\chi^c\right) \tag{337}$$

where $\widehat{y}$ is the Yukawa coupling constant. Recall that $\overline{\widehat\Phi} = \phi - \gamma_5\pi$ denotes the axial conjugate; therefore,

$$\delta_{\widehat\xi}\overline{\widehat\Phi} = \overline{\widehat\xi}^{\mu}\widehat\partial_\mu\overline{\widehat\Phi} \tag{338}$$

Moreover,

$$\delta_{\widehat\xi}\chi^c = (\xi^\mu - \gamma_5\zeta^\mu)\gamma_0 C\widehat\partial_\mu\chi^* = \overline{\widehat\xi}^{\mu}\widehat\partial_\mu\chi^c \tag{339}$$

Therefore,

$$\delta_{\widehat\xi}\left(\overline\psi\sqrt{\overline{\widehat{g}}\widehat\Phi}\chi^c\right) = \widehat\partial_\mu\left(\overline\psi\sqrt{\overline{\widehat{g}}}\overline{\widehat\xi}^{\mu}\overline{\widehat\Phi}\chi^c\right) \tag{340}$$

Thus, (336) is invariant under AE diffeomorphisms. Its Hermitean conjugate is

$$\overline{S}'_Y = \frac{1}{2}\int d^4\widehat{x}\left(\overline{\chi^c}\sqrt{\widehat{g}}\widehat\Phi^{\dagger}\psi\right) \tag{341}$$

Proceeding as above, one can prove that

$$\delta_{\widehat{\xi}}\left(\overline{\chi^c}\sqrt{\widehat{g}}\widehat{\Phi}^\dagger\psi\right) = \widehat{\partial}_\mu\left(\overline{\chi^c}\widehat{\xi}^\mu\sqrt{\widehat{g}}\widehat{\Phi}^\dagger\psi\right) \tag{342}$$

which proves that (341) is invariant under AE diffeomorphisms.

Similarly, applying the transformation rules (333) and (334), one can prove that both (336) and (341) are invariant under AE Weyl transformations.

Remark 8. *Notice that such terms as $\int d^4\widehat{x}\left(\overline{\psi}\sqrt{\overline{\widehat{g}}\widehat{\Phi}}\chi\right)$ and the conjugate are not AE diffeomorphism invariant.*

Now, we specify the previous results for our case. If ψ_L is the (reducible) left multiplet of $\mathcal{T}_L$, then we define

$$S_{YL} = \frac{1}{2}\int d^4\widehat{x}\left(\overline{\psi_L}\widehat{y}\sqrt{\overline{\widehat{g}}\widehat{\Phi}}(\psi_L)^c\right) = \frac{y_-}{2}\int d^4\widehat{x}\sqrt{g_-}\overline{\psi_L}\Phi_-(\psi_L)^c \tag{343}$$

Its hermitean conjugate is

$$\overline{S}_{YL} = \frac{y_-^*}{2}\int d^4\widehat{x}\sqrt{g_-}\overline{(\psi_L)^c}\Phi_-^\dagger\psi_L \tag{344}$$

In much the same way as before, we can introduce the Yukawa couplings for the model $\mathcal{T}_R$, i.e, for the multiplet (296):

$$S_{YR} = \frac{y_+}{2}\int d^4\widehat{x}\sqrt{g_+}\overline{\psi_R'}\Phi_+(\psi_R')^c, \qquad \overline{S}_{YR} = \frac{y_+^*}{2}\int d^4\widehat{x}\sqrt{g_+}\overline{(\psi_R')^c}\Phi_+^\dagger(\psi_R') \tag{345}$$

In the sequel, we shall need these formulas in particular in the case where $\Phi_\pm$ are real fields. We shall denote them with the symbol

$$S_Y = S_{YL} + S_{YR} + h.c. \tag{346}$$

13.7.1. Symmetry Properties of Yukawa Couplings
Local Lorentz Invariance of Yukawa Terms

An infinitesimal local Lorentz transformation of ψ is defined by

$$\delta_L\psi = \Lambda_{ab}\Sigma^{ab}\psi, \qquad \Sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b] \tag{347}$$

It follows that

$$\delta_L\psi^c = \Lambda_{ab}\Sigma^{ab}\psi^c, \qquad \delta_L\overline{\psi}^c = -\overline{\psi}\Lambda_{ab}\Sigma^{ab}, \tag{348}$$

therefore, (336) is unchanged.

Gauge Invariance of Yukawa Terms

Suppose ψ is a (reducible) multiplet that transforms under a local $U(1)$ group with Hermitean generator Y:

$$\psi \to e^{i\widehat{\alpha}Y}\psi, \qquad \widehat{\alpha} = \alpha + \gamma_5\beta \tag{349}$$

Then

$$\psi^c \to e^{i\overline{\widehat{\alpha}}Y}\psi^c, \qquad \overline{\psi} \to \overline{\psi}e^{-i\overline{\widehat{\alpha}}Y}. \tag{350}$$

In order for (336) to be invariant under this transformation, $\widehat{\Phi}$ must be invariant too

$$\widehat{\Phi} \to \widehat{\Phi} \tag{351}$$

which holds in the case of the AE scalar field case we shall need below. The same holds in the case of $SU(2)$ or $SU(3)$ gauge transformations. Let the Hermitean generators be T^a. A local gauge transformation on ψ is

$$\psi \to e^{i\alpha_a T^a}\psi \tag{352}$$

which implies

$$\psi^c \to e^{i\overline{\alpha}_a T^a}\psi^c, \qquad \overline{\psi} \to \overline{\psi}e^{-i\overline{\alpha}_a T^a} \tag{353}$$

Thus, if $\widehat{\Phi}$ is invariant, (336) does not change.

Discrete Symmetry Properties of Yukawa Terms

The above Yukawa couplings are symmetric under AE diffeomorphisms, local Lorentz transformations and also under AE Weyl transformations. We have not yet examined their properties under C, P and CP transformations. Using the formulas of Appendix A, we see that they are not invariant under C and P, but this is in line with all the terms quadratic in the spinors, which have the same lack of C and P invariance, see (297), but are CP invariant. On the contrary the above Yukawa's may not be CP invariant. If we disregard the phases of the P and C transformations the sum $S'_{YL} + \overline{S}'_{YL}$ is neither parity nor charge conjugation invariant, and changes sign under CP. Analogously, the sum $S'_{YR} + \overline{S}'_{YR}$ is neither parity nor charge conjugation invariant and reverses its sign under CP. Of course, we can arrange the transformation phases $\widehat{\beta}_p, \widehat{\beta}_c$ of ψ in such a way as to absorb this minus sign and restore CP invariance, for instance, setting $\beta_p + \beta_c = \pi$. This choice does not affect the fermion kinetic terms or the Yukawa couplings of the next subsection. With this choice the total action (346) is parity invariant. However, the above Yukawa couplings do not reproduce the SM Yukawa terms. For this reason, we resort to other Yukawa couplings, which are CP invariant but oblige us to sacrifice in part the overall symmetry of the theory, see next subsection.

Mass Terms

Let us finally add a comment that will be useful later on. Yukawa couplings of scalar and fermions are, in particular, designed to produce fermion masses via the Higgs mechanism. Here we consider the one illustrated in Section 14.2 below. Let us consider as an example the mass terms produced by the vev v of the field Φ_+ in $S'_{YR} + h.c.$, (345). We take y_+ real, and we limit ourselves to the doublet $\begin{pmatrix} d_R \\ u_R \end{pmatrix}$ and the singlets $(d_L)^c, (u_L)^c$. Plugging them in $S'_{YR} + h.c.$ the integrand turns out to be proportional to (disregarding $\sqrt{g_+}$)

$$\frac{vy_+}{2}\left(\overline{d_L}(d_L)^c + \overline{(d_L)^c}d_L + \overline{d_R}(d_R)^c + \overline{(d_R)^c}d_R \right) \tag{354}$$

and the same expression for u. It is well-known that the kinetic terms in $S_f^{(+)}$, Equation (305), constitute the kinetic terms of a massless Dirac spinor $\psi'_d = d_L + d_R$ and $\psi'_u = u_L + u_R$.

The terms (354) form the Majorana mass term of that spinor, so the complete eom is the Majorana equation, which in the free case takes the form

$$\partial\!\!\!/\,\psi'_{d/u} + m\psi'^c_{d/u} = 0 \tag{355}$$

13.8. SM Yukawa Couplings

To reproduce the SM couplings (at least in the left part of our theory), we must use

$$S_{YdL} = \frac{y^-_{H_d}}{2} \int d^4\widehat{x} \sqrt{g_-} \left(\overline{\psi_{dL}} H_{d-} \chi_{sR} \right) + h.c. \tag{356}$$

where ψ_{dL} is a left-handed $SU(2)$ doublet, $\widehat{H}_{d-}$ is also an $SU(2)$ doublet, conjugate to the ψ_{dL} one in the inner product of the $SU(2)$ doublet representation space, while χ_{sR} is a right-handed singlet, all of them belonging to $\mathcal{T}_L$. Similarly, for $\mathcal{T}_R$,

$$S_{YdR} = \frac{y^+_{H_d}}{2} \int d^4\widehat{x} \sqrt{g_+} \left(\overline{\chi'_{dR}} H_{d+} \psi'_{sL} \right) + h.c. \tag{357}$$

Equations (357) and (356) are (separately) invariant under diffeomorphisms parametrized by $\xi^\mu_\pm$ and conformal transformations parametrized by $\omega_\pm$, respectively. They are invariant under $SU(2)$ gauge transformations and the other SM gauge transformations and are *CP* invariant. Again, it is not possible to write them in a compact form in terms of $\widehat{H}$ and $\widehat{g}_{\mu\nu}$ as in the previous actions, that is, by means of the full-fledged hypercomplex formalism. We stress that this does not spoil the L-R symmetry.

We shall denote the set of Yukawa couplings introduced in this subsection by

$$S_{Yd} = S_{YdL} + S_{YdR} \tag{358}$$

and its Hermitean conjugate by $\overline{S}_{Yd}$. An example of Yukawa coupling of S_{YdL} is

$$\frac{y^-_{H_d}}{2} \int d^4\widehat{x} \sqrt{g_-} \left(\overline{Q_L} H_{d-} \chi_{sR} \right) \tag{359}$$

where $Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$, $H_d = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ and $\chi_{sR} = u_R$ or d_R. Due to the breakdown of the electroweak symmetry the term (359) gives rise to a mass term proportional to the vev of H_d and to $\frac{y^-_{H_d}}{2}$, see Section 14.3. Together with the kinetic terms in $S_f^{(+)}$, Equation (305), it forms a Dirac fermion action, for instance for $\psi_d = d_L + d_R$, whose free eom is the ordinary massive Dirac equation:

$$\partial\!\!\!/\,\psi_d + m\psi_d = 0 \tag{360}$$

13.9. The Overall $\mathcal{T}$ Model

The action of the chirally symmetric model is

$$\widehat{S}_{ch-sym} = \widehat{S}_f + \widehat{S}_g + \widehat{S}_{aeg} + (S_Y + S_{Yd} + h.c.) + \widehat{S}_{aes} + \widehat{S}_{aed} + \widehat{S}_{EH} \tag{361}$$

where $\widehat{S}_g$ has gauge group $SU(2)$, while $\widehat{S}_{aeg}$ has gauge group $SU(3)_L \times SU(3)_R \times U(1)_L \times U(1)_R$. Concerning the discrete symmetries, with the choice made in Section Discrete Symmetry Properties of Yukawa Terms, $\mathcal{T}$ is P, C and so CP and T invariant. This theory is chirally symmetric and non-anomalous. It is, in particular, a *bimetric theory* [For a review see [96]. In general, a bimetric theory is introduced in order to realize a ghost-free massive spin 2 particle, see [97], a motivation different from the present one]. Since the fermion

spectrum is L-R symmetric by construction, it is completely free of type O anomalies. Excluding the term $S_{Yd} + h.c.$, the action (361) is invariant under AE diffeomorphisms. $S_{YdL} + h.c.$ and $S_{YdR} + h.c.$ are separately invariant under ξ_-^μ and ξ_+^μ. Concerning the Weyl transfomations: $\widehat{S}_f + \widehat{S}_g + \widehat{S}_{aeg} + (S_Y + h.c)$ are invariant under AE transformations. $S_{YdL} + h.c.$ and $S_{YdR} + h.c.$ are invariant under Weyl transformations with parameters ω_- and ω_+, respectively.

But the scalar plus gravity action, $\widehat{S}_{aes} + \widehat{S}_{aed} + \widehat{S}_{EH}$ is not invariant under AE Weyl transformations, only the remaining part is. We should add that, in addition, $\widehat{S}_{EH}$ is not renormalizable. We need an enhancement of symmetry in this part of the action, a formulation that preserves AE Weyl symmetry and, hopefully, achieves renormalizability. This is indeed possible by embedding the model in the so-called Weyl geometry, [98]. All this constitutes a very well-known subject. We are moving in a considerably explored ground and we do not have anything new to say about it. We limit ourselves to a short summary of a rather vast literature, see [99–104], and apply it to our context. The resulting model will be called $\mathcal{TW}$. Refs.: [96,98,101,103,105].

14. The Weyl Geometry and Gravity Embedding

In an ordinary gravitational background geometry, the Weyl transformation is given by

$$g_{\mu\nu} \to e^{2\omega} g_{\mu\nu} \tag{362}$$

The Christoffel symbols transform as

$$\Gamma_{\mu\nu}^\lambda \to \Gamma_{\mu\nu}^\lambda + \delta_\mu^\lambda \partial_\nu \omega + \delta_\nu^\lambda \partial_\mu \omega - g_{\mu\nu} g^{\lambda\rho} \partial_\rho \omega \tag{363}$$

We can construct Weyl invariant Christoffel symbols as follows

$$\widetilde{\Gamma}_{\mu\nu}^{(1)\lambda} = \Gamma_{\mu\nu}^\lambda - \left(\delta_\mu^\lambda \partial_\nu \varphi + \delta_\nu^\lambda \partial_\mu \varphi - g_{\mu\nu} g^{\lambda\rho} \partial_\rho \varphi \right) \tag{364}$$

where the field φ (a dilaton) under Weyl transforms as

$$\varphi \to \varphi + \omega \tag{365}$$

The field φ is dimensionless.

Alternatively, other Weyl invariant Christoffel symbols are

$$\widetilde{\Gamma}_{\mu\nu}^{(2)\lambda} = \Gamma_{\mu\nu}^\lambda - q \left(\delta_\mu^\lambda C_\nu + \delta_\nu^\lambda C_\mu - g_{\mu\nu} g^{\lambda\rho} C_\rho \right) \tag{366}$$

where the vector field C_μ transforms as

$$C_\mu \to C_\mu + \frac{1}{q} \partial_\mu \omega, \tag{367}$$

and q is a dimensionless parameter. Still, another set of equivalent Christoffel symbols is

$$
\begin{aligned}
\widetilde{\Gamma}_{\mu\nu}^\lambda &= \Gamma_{\mu\nu}^\lambda - \alpha \left(\delta_\mu^\lambda \partial_\nu \varphi + \delta_\nu^\lambda \partial_\mu \varphi - g_{\mu\nu} g^{\lambda\rho} \partial_\rho \varphi \right) - q(1-\alpha) \left(\delta_\mu^\lambda C_\nu + \delta_\nu^\lambda C_\mu - g_{\mu\nu} g^{\lambda\rho} C_\rho \right) \\
&= \Gamma_{\mu\nu}^\lambda - \alpha \left(\delta_\mu^\lambda \mathsf{D}_\nu \varphi + \delta_\nu^\lambda \mathsf{D}_\mu \varphi - g_{\mu\nu} g^{\lambda\rho} \mathsf{D}_\rho \varphi \right) - q \left(\delta_\mu^\lambda C_\nu + \delta_\nu^\lambda C_\mu - g_{\mu\nu} g^{\lambda\rho} C_\rho \right)
\end{aligned} \tag{368}
$$

where α is a dimensionless parameter and we have introduced the Weyl invariant derivative

$$\mathsf{D}_\mu \varphi = \partial_\mu \varphi - q C_\mu \tag{369}$$

We can use these Christoffel symbols to build the Riemann and Ricci tensors. For instance,

$$\widetilde{R}_{\mu\nu} = R_{\mu\nu} + 3D_\nu S_\mu - D_\mu S_\nu + g_{\mu\nu} D \cdot S + 2S_\mu S_\nu - 2g_{\mu\nu} S \cdot S \tag{370}$$

and

$$\widetilde{R} = R + 6(D \cdot S - S \cdot S), \tag{371}$$

where

$$S_\mu = \alpha\partial_\mu\varphi + (1-\alpha)qC_\mu \tag{372}$$

The conformal Weyl formalism also allows another decomposition of $\widetilde{R}_{\mu\nu}$. Summing $\widetilde{R}$ with $\alpha = 1$ multiplied by ϵ to $\widetilde{R}$ with $\alpha = 0$ multiplied by $1 - \epsilon$, one can write

$$\widetilde{R} = R + 6\epsilon(D \cdot \partial\varphi - \partial\varphi \cdot \partial\varphi) + 6(1-\epsilon)\left(qD \cdot C - q^2 C \cdot C\right), \tag{373}$$

where ϵ is a new dimensionless parameter (not to be confused with α).

Notice that, from (370),

$$\widetilde{R}_{[\mu,\nu]} = \frac{1}{2}\left(\widetilde{R}_{\mu\nu} - \widetilde{R}_{\nu\mu}\right) = 2(D_\nu S_\mu - D_\mu S_\nu) = 2(\partial_\nu S_\mu - \partial_\mu S_\nu) = 2q(\partial_\nu C_\mu - \partial_\mu C_\nu) \tag{374}$$

By construction, $\widetilde{R}_{\mu\nu}$ is conformal invariant, but

$$\widetilde{R} = g^{\mu\nu}\widetilde{R}_{\mu\nu} \to e^{-2\omega}\widetilde{R} \tag{375}$$

Thus we can construct a Weyl invariant gravitational action as follows. The first piece is the conformally enhanced EH action

$$S_{EH}^{(c)} = \frac{1}{2\kappa}\int d^4x \sqrt{g}\, e^{-2\varphi}\left(\widetilde{R} + \mathfrak{c}e^{-2\varphi}\right) \tag{376}$$

To it we can add

$$S_C^{(c)} = -\frac{1}{4}\int d^4x \sqrt{g}\, C_{\mu\nu}C^{\mu\nu}, \qquad C_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu \tag{377}$$

Finally there is also the following

$$S_W^{(c)} = -\frac{1}{\eta}\int d^4x \sqrt{g}\, C_{\mu\nu\lambda\rho}C^{\mu\nu\lambda\rho} \tag{378}$$

where $C_{\mu\nu\lambda\rho}$ is the Weyl tensor, which also is Weyl invariant, and η is a dimensionless constant. If we can disregard total derivatives in the action, (378) can be replaced by

$$S_W^{(c')} = -\frac{2}{\eta}\int d^4x \sqrt{g}\left(-R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2\right) \tag{379}$$

These have to be understood as interacting terms, while the kinetic terms are contained in (376) and (377).

We can easily embed also matter in Weyl geometry. For instance, consider a real scalar field Φ: the action

$$S_s^{(c)} = \frac{1}{2}\int d^4x \sqrt{g}\left[g^{\mu\nu}D_\mu\Phi D_\nu\Phi - m^2 e^{-2\varphi}\Phi^2 - \frac{\lambda}{4}\Phi^4\right] \tag{380}$$

is Weyl invariant provided Φ transform as

$$\Phi \to e^{-\omega}\Phi \tag{381}$$

In general, D_μ is the Weyl covariant derivative of the scalar Φ

$$\mathrm{D}_\mu\Phi = \left(\partial_\mu + \alpha\partial_\mu\varphi + q(1-\alpha)C_\mu\right)\Phi \tag{382}$$

Putting together (376) and (380), we can write down the conformal invariant scalar-gravity action

$$
\begin{aligned}
S^{(c)}_{EH+s+C} \;=\;& \frac{1}{2\kappa}\int d^4x\sqrt{g}\left(e^{-2\varphi} + \zeta\Phi^2\right)\left(\widetilde{R} + \mathfrak{c}e^{-2\varphi}\right) \\
&+ \frac{1}{2}\int d^4x\sqrt{g}\left[g^{\mu\nu}\mathrm{D}_\mu\Phi\mathrm{D}_\nu\Phi - m^2 e^{-2\varphi}\Phi^2 - \frac{\lambda}{4}\Phi^4\right] \\
&- \frac{1}{4}\int d^4x\sqrt{g}C_{\mu\nu}C^{\mu\nu} - \frac{1}{\eta}\int d^4x\sqrt{g}C_{\mu\nu\lambda\rho}C^{\mu\nu\lambda\rho},
\end{aligned}
\tag{383}
$$

where ζ is a constant that measures the non-minimal coupling of the scalars to gravity, and $\widetilde{R}$ is given by (373), that is

$$\widetilde{R} = R + 6\epsilon g^{\mu\nu}\left(D_\mu\partial_\nu\varphi - \partial_\mu\varphi\partial_\nu\varphi\right) + 6(1-\epsilon)g^{\mu\nu}\left(qD_\mu C_\nu - q^2 C_\mu C_\nu\right), \tag{384}$$

with ϵ a constant to be determined.

It should be clear that ζ, κ^{-1} and $\mathfrak{c}$ all have the dimension of a mass square, while λ and η are dimensionless.

14.1. Weyl Geometry: Other Field Theory Formulations

Another formulation of Weyl geometry in field theory is possible in terms of a scalar field χ of physical dimension 1. Instead of (376), the basic action is

$$S'^c_{EH} = \frac{1}{2\bar{\xi}}\int d^4x\sqrt{g}\chi^2\left(\widetilde{R} + \mathfrak{c}\chi^2\right) \tag{385}$$

where $\bar{\xi}$ is a dimensionless constant, while $\mathfrak{c}$ is as before (Of course χ cannot be identified with $e^{q\varphi}$ because they have different physical dimension, 1 and 0, respectively). χ transforms as

$$\chi \to e^{\omega}\chi \tag{386}$$

The relation between χ and φ can be written

$$\varphi = \log(\ell\chi) \tag{387}$$

where ℓ has the dimension of a length. Using this, one can translate all the above formulas in terms of χ (C_μ does not undergo any change). In the sequel we will use the formulation in terms of φ. It should be noted that we need a dimensional scale to relate χ to the Weyl geometry.

Returning to the main subject of the embedding in a Weyl geometry, the embedding of fermions is automatic because the corresponding kinetic term is Weyl invariant. The corresponding action term does not need the introduction of either φ or C_μ. As a consequence the anomalies remain the same as before the Weyl geometry embedding. No new anomalies are generated due to fermions.

The embedding of the gauge kinetic terms and of the Yukawa couplings does not require any modifications because they are Weyl invariant.

Remark 9. *Concerning the (local) Weyl symmetry, there are two possible attitudes. The first is to consider it as an ordinary gauge symmetry, characterized not only by a local parameter $\omega(x)$ but also by an associated (independent) gauge field C_μ with relevant curvature and covariant derivatives, as we have done above (in this case, we understand $\alpha = \epsilon = 0$ in all the previous formulas). Another attitude is to drop C_μ and consider the resulting theory as an unusual gauge theory symmetric under a local parameter but without a corresponding gauge field. This means we set $C_\mu = 0, \alpha = \epsilon = 1$ [It is actually not necessary to set $C_\mu = 0$, it suffices to require $C_\mu \sim \partial_\mu \varphi$, which defines an integrable Weyl structure]. Both attitudes are legitimate. But it is worth remarking that, contrary to what occurs in ordinary gauge theories, it is not necessary to introduce a vector gauge potential C_μ in order to guarantee the gauge invariance of the action; $\partial_\mu \varphi$ can do the same job. But, in such a case, we do not introduce new degrees of fredom (say C_μ); the field φ has the same number of degrees of freedom as the gauge parameter.*

The most frequent approach in the literature consists in introducing C_μ and treating it in the usual way as in gauge theories with a Higgs mechanism, which makes it a massive vector field by absorbing a scalar. It is worth recalling the latter mechanism, although it is extremely well known because it plays a prominent role in the sequel. We consider the two cases that will appear below: a real singlet and a doublet Higgs mechanism with complex scalars.

Refs. [106–111], for early applications of Weyl geometry to the SM, see [112–121].

14.2. The Real Singlet Higgs Mechanism

In a theory like (383), the Higgs mechanism is realized as follows. We single out the action

$$S_s = \frac{1}{2} \int d^4 \widehat{x} \sqrt{g} \left(g^{\mu\nu} D_\mu \Phi D_\nu \Phi + m^2 \Phi^2 - \frac{\lambda}{4} \Phi^4 \right) \tag{388}$$

where

$$D_\mu \Phi = \left(\partial_\mu + q C_\mu \right) \Phi \tag{389}$$

and the term m^2 is produced, for instance, by the gauge choice $\varphi = 0$, i.e., $m^2 = \frac{\zeta \mathfrak{c}}{\kappa}$.

Setting $\Phi = v + \rho$, where ρ is a real field and $v = \sqrt{\frac{2m^2}{\lambda}}$, a real number, denotes the minimum of the potential, we have

$$D_\mu \Phi D_\mu \Phi = \left(\partial_\mu + q C_\mu \right)(v + \rho)\left(\partial^\mu + q C^\mu \right)(v + \rho) \tag{390}$$

The quadratic terms in $\frac{1}{2} D_\mu \Phi D_\mu \Phi - V$ (V is the potential) are

$$\frac{1}{2} \partial_\mu \rho \partial^\mu \rho + \frac{1}{2} q^2 C_\mu C^\mu v^2 - \frac{1}{2} \lambda v^2 \rho^2 \tag{391}$$

The result is a massive scalar ρ as well as a massive vector field C_μ.

Including this case among the Higgs mechanisms is somewhat of an abuse of language because there is no gauge field eating a scalar, but we can consider it as a sort of limiting case. A massive C_μ produced by a Higgs mechanism, is often considered in the literature, see [99,100,102].

Finally, the Yukawa coupling (345), for Φ real, when $\langle \Phi \rangle = v$, produce mass terms proportional to v such as those in Equation (354).

Remark 10. *If ψ denotes a singlet or a multiplet, it transforms under a gauge transformation*

$$\psi \to e^{i\alpha_a T^a}\psi, \qquad (T^a)^\dagger = T^a, \tag{392}$$

then

$$\psi^c \to e^{i\alpha_a T^a}\psi^c, \qquad \overline{\psi} \to \overline{\psi}e^{-i\alpha_a T^a} \tag{393}$$

Therefore a mass term like (354) remains gauge invariant after the breaking.

14.3. The Doublet Higgs Mechanism

Suppose Φ is a complex doublet field transforming as

$$\Phi \to e^{i\alpha_a T^a}\Phi, \qquad \Phi^\dagger \to \Phi^\dagger e^{-i\alpha_a T^a} \tag{394}$$

under a local $SU(2)$ transformation, $a = 1, 2, 3$. The relevant terms of the action are

$$S_d^{(c)} = \int d^4x \sqrt{g}\left[g^{\mu\nu}D_\mu\Phi^\dagger D_\nu\Phi + M^2\Phi^\dagger\Phi - \frac{\lambda}{4}\left(\Phi^\dagger\Phi\right)^2\right] \tag{395}$$

where

$$D_\mu\Phi = \left(\partial_\mu - igV_\mu + qC_\mu\right)\Phi \tag{396}$$

The symmetry breaking is determined by the decomposition

$$\Phi = \begin{pmatrix} \phi \\ v + \frac{h+i\chi}{\sqrt{2}} \end{pmatrix} \qquad \langle\Phi\rangle = \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{397}$$

where ϕ is a complex scalar field, h and χ are real scalar fields. ϕ and χ are Goldstone bosons destined to become components of the three $SU(2)$ gauge fields while h is bound to become the massive Higgs field. The three Goldstone degrees of freedom represented by ϕ and χ can be absorbed by an $SU(2)$ gauge transformation. So with a suitable choice of the parameters α_a, Φ can be represented as

$$\Phi = e^{i\alpha_a T^a}\Phi_h, \qquad \Phi_h = \begin{pmatrix} 0 \\ v + \frac{h}{\sqrt{2}} \end{pmatrix} \tag{398}$$

We recall that if M^2 is the mass of Φ and λ, the strength of the quartic coupling in the action, v is given by

$$v^2 = \frac{2M^2}{\lambda} \tag{399}$$

Under this $SU(2)$ gauge transformation, the covariant derivative of Φ,

$$D_\mu\Phi = \left(\partial_\mu - igW_\mu + qC_\mu\right)\Phi, \qquad W_\mu = W_\mu^a T^a \tag{400}$$

becomes

$$D_\mu\Phi \to e^{i\alpha_a T^a}D_\mu'\Phi_h \tag{401}$$

where, in the covariant derivative, igW_μ is replaced by

$$igX_\mu = e^{-i\alpha_a T^a}\left(-\partial_\mu + igW_\mu\right)e^{i\alpha_a T^a} \tag{402}$$

In this way χ and the two components of ϕ are absorbed into X_μ. Disregarding for simplicity C_μ, we have

$$D_\mu \Phi^\dagger D^\mu \Phi \to \frac{1}{2} \partial_\mu h \partial^\mu h + \frac{g^2}{2} X_\mu^a X^{a\mu} \left(v + \frac{h}{\sqrt{2}} \right)^2 \tag{403}$$

summed over a. Since the kinetic term in the action is gauge invariant, this assigns a mass square $v^2 g^2$ to the three gauge bosons X_μ^a.

On the other hand

$$\Phi^\dagger \Phi \to \Phi_h^\dagger \Phi_h = \left(v + \frac{h}{\sqrt{2}} \right)^2 \tag{404}$$

The scalar potential V becomes

$$V = -\frac{1}{2} M^2 v^2 + M^2 h^2 + \mathcal{O}(h^3) \tag{405}$$

This gives the mass of the Higgs field h. Via the Yukawa term S_{Yd} also the fermions coupled to Φ acquire a mass proportional to v.

In the sequel, we shall indeed consider two types of scalar field actions. In one, Φ is a real scalar field in $\mathcal{T}$ denoted $\widehat{\Phi}$. In the second case, Φ is a complex $SU(2)$ multiplet, which in the AE case will be denoted $\widehat{H}$. The first action will be denoted $\widehat{S}_{aes}$ and the second $\widehat{S}_{aed}$.

The next step consists of embedding $\mathcal{T}$ in the AE Weyl geometry, see [99,101–103].

15. Embedding in AE Weyl Geometry

The embedding in axially extended (AE) Weyl geometry is straightforward. The embedding of the fermion kinetic terms and of Yukawa couplings does not require modifications. We can write

$$\widehat{S}_f^{(c)} = \widehat{S}_f, \qquad \widehat{S}_g^{(c)} = \widehat{S}_g, \qquad \widehat{S}_{aeg}^{(c)} = \widehat{S}_{aeg}, \qquad S_Y^{(c)} = S_Y, \qquad S_{Yd}^{(c)} = S_{Yd} \tag{406}$$

For the gravitational part we can write

$$\widehat{S}_{EH}^{(c)} = \frac{1}{4} \int d^4 \hat{x} \, \mathrm{tr} \left[\frac{1}{\widehat{\kappa}} \sqrt{\widehat{g}} e^{-2\widehat{\varphi}} \left(\widetilde{\widehat{R}} + \widehat{\mathfrak{c}} e^{-2\widehat{\varphi}} \right) \right] \tag{407}$$

where $\widehat{\varphi} = \varphi + \gamma_5 \omega$, $\widehat{\kappa} = \kappa_+ P_+ + \kappa_- P_-$, and

$$\widetilde{\widehat{R}} = \widehat{R} + 6 \left(\widehat{D} \cdot \widehat{S} - \widehat{S} \cdot \widehat{S} \right) \tag{408}$$

where [Here and below, we treat the two $\widehat{S}_\mu$ as separate cases. One could just as well consider a combination of the two with coefficients α and $1 - \alpha$, respectively, as was done before.]

$$\widehat{S}_\mu = \partial_\mu \widehat{\varphi}, \qquad \text{or} \qquad \widehat{S}_\mu = q \widehat{C}_\mu, \qquad \widehat{C}_\mu = C_\mu + \gamma_5 E_\mu \tag{409}$$

We can, of course, easily write down the splitting in $+$ and $-$ part of the actions. To start with

$$\widehat{S}_{EH}^{(c)} = S_{EH}^{(c+)} + S_{EH}^{(c-)} \tag{410}$$

where

$$S_{EH}^{(c\pm)} = \frac{1}{2\kappa_\pm} \int d^4x \sqrt{g_\pm} e^{-2\varphi_\pm} \left(\widetilde{R}^{(\pm)} + \widehat{\mathfrak{c}} e^{-2\varphi_\pm} \right) \tag{411}$$

which is invariant both under diffeomorphisms with parameters $\xi_\pm$ and under Weyl transformations with parameters $\omega_\pm$. Here

$$\widetilde{R}^{(\pm)} = R^{(\pm)} + 6 \left(D^{(\pm)} \cdot S^{(\pm)} - S^{(\pm)} \cdot S^{(\pm)} \right) \tag{412}$$

where

$$S_\mu^{(\pm)} = q C_\mu^{(\pm)} \qquad \text{or} \qquad S_\mu^{(\pm)} = \partial_\mu \varphi^{(\pm)} \tag{413}$$

and

$$\varphi_\pm = \varphi \pm \omega, \qquad C_\mu^{(\pm)} = C_\mu \pm E_\mu \tag{414}$$

Similarly,

$$\widehat{S}_C^{(c)} = -\frac{1}{8} \int d^4\widehat{x}\,\mathrm{tr}\left[\sqrt{\widehat{g}}\widehat{g}^{\mu\mu'}\widehat{g}^{\nu\nu'}\widehat{C}_{\mu\nu}\widehat{C}_{\mu'\nu'} \right] = S_C^{(c+)} + S_C^{(c-)} \tag{415}$$

where $\widehat{C}_{\mu\nu} = \widehat{\partial}_\mu\widehat{C}_\nu - \widehat{\partial}_\nu\widehat{C}_\mu$, and

$$S_C^{(\pm)} = -\frac{1}{4} \int d^4x \sqrt{g_\pm} g_\pm^{\mu\mu'} g_\pm^{\nu\nu'} C_{\mu\nu}^{(\pm)} C_{\mu'\nu'}^{(\pm)}, \tag{416}$$

For a real AE scalar $\widehat{\Phi}$:

$$\widehat{S}_{aes}^{(c)} = \frac{1}{2} \int d^4\widehat{x}\,\mathrm{tr}\left\{ \sqrt{\widehat{g}}\left[\widehat{g}^{\mu\nu}\widehat{D}_\mu\widehat{\Phi}\widehat{D}_\nu\widehat{\Phi} + \widehat{m}^2 e^{-2\widehat{\varphi}}\widehat{\Phi}^2 - \frac{\widehat{\lambda}}{4}\widehat{\Phi}^4 \right] \right\} = \widehat{S}_{aes}^{(c+)} + \widehat{S}_{aes}^{(c-)} \tag{417}$$

where $\widehat{D}_\mu = \partial_\mu + q\widehat{C}_\mu$ or $\widehat{D}_\mu = \partial_\mu + \partial_\mu\widehat{\varphi}$, $\widehat{\Phi} = \phi + \gamma_5\pi$ and $\widehat{m}^2 = m_+^2 P_+ + m_-^2 P_-$. Finally

$$S_{aes}^{(c\pm)} = \frac{1}{2} \int d^4x \sqrt{g_\pm} \left[g_\pm^{\mu\nu} \left(D_\mu^\pm\Phi_\pm \right) \left(D_\nu^\pm\Phi_\pm \right) + m_\pm^2 e^{2\varphi_\pm}\Phi_\pm^2 - \frac{\lambda_\pm}{4}\Phi_\pm^4 \right] \tag{418}$$

with $D_\mu^\pm = \partial_\mu + q C_\mu^{(\pm)}$ or $D_\mu^\pm = \partial_\mu + \partial_\mu\varphi^{(\pm)}$ and $\Phi_\pm = \phi \pm \pi$.

Finally, for an $SU(2)$ doublet $\widehat{H}$:

$$\widehat{S}_{aed}^{(c)} = \int d^4\widehat{x}\,\mathrm{tr}\left\{ \sqrt{\widehat{g}}\left[\widehat{g}^{\mu\nu} \left(\widehat{D}_\mu\widehat{H} \right)^\dagger \left(\widehat{D}_\nu\widehat{H} \right) + \widehat{M}^2\widehat{H}^\dagger\widehat{H} - \frac{\widehat{\lambda}_h}{4} \left(\widehat{H}^\dagger\widehat{H} \right)^2 \right] \right\} = \widehat{S}_{aed}^{(c+)} + \widehat{S}_{aed}^{(c-)} \tag{419}$$

where $\widehat{D}_\mu = \partial_\mu + q\widehat{C}_\mu - igW_\mu$ or $\widehat{D}_\mu = \partial_\mu + \partial_\mu\widehat{\varphi} - igW_\mu$, W_μ being a gauge field valued in the $SU(2)$ Lie algebra representation to which $\widehat{H}$ belongs (beware, W_μ is not split).

$$S_{sm}^{(c\pm)} = \int d^4x \sqrt{g_\pm} \left[g_\pm^{\mu\nu} (D_\mu H_\pm)^\dagger (D_\nu H_\pm) + M_\pm^2 H_\pm^\dagger H_\pm - \frac{\lambda_{h\pm}}{4} \left(H_\pm^\dagger H_\pm \right) \right] \tag{420}$$

with $D_\mu^\pm = \partial_\mu + q C_\mu^{(\pm)} - igW_\mu$ or $D_\mu^\pm = \partial_\mu + \partial_\mu\varphi^{(\pm)} - igW_\mu$.

Putting everything together, we can write down the most general AE scalar plus gravity action

$$\widehat{S}^{(c)}_{EH+aes+aed+C} = \frac{1}{2}\int d^4\widehat{x}\,\mathrm{tr}\left[\frac{1}{\widehat{\kappa}}\sqrt{\widehat{g}}\left(e^{-2\widehat{\varphi}}+\widehat{\zeta}_h\widehat{H}^\dagger\widehat{H}+\widehat{\zeta}\widehat{\Phi}^2\right)\left(\widetilde{\widehat{R}}+\widehat{c}e^{-2\widehat{\varphi}}\right)\right]$$

$$+\int d^4\widehat{x}\,\mathrm{tr}\left\{\sqrt{\widehat{g}}\left[\widehat{g}^{\mu\nu}\left(\widehat{D}_\mu\widehat{H}\right)^\dagger\left(\widehat{D}_\nu\widehat{H}\right)+\widehat{M}^2\widehat{H}^\dagger\widehat{H}-\frac{\widehat{\lambda}_h}{4}\left(\widehat{H}^\dagger\widehat{H}\right)^2\right]\right\}$$

$$+\frac{1}{2}\int d^4\widehat{x}\,\mathrm{tr}\left\{\sqrt{\widehat{g}}\left[\widehat{g}^{\mu\nu}\widehat{D}_\mu\widehat{\Phi}\widehat{D}_\nu\widehat{\Phi}+\widehat{m}^2e^{-2\widehat{\varphi}}\widehat{\Phi}^2-\frac{\widehat{\lambda}}{4}\widehat{\Phi}^4\right]\right\}$$

$$-\frac{1}{4}\int d^4\widehat{x}\,\mathrm{tr}\left[\sqrt{\widehat{g}}\widehat{C}_{\mu\nu}\widehat{C}^{\mu\nu}\right]-\int d^4x\,\mathrm{tr}\left[\frac{1}{\widehat{\eta}}\sqrt{\widehat{g}}\widehat{C}_{\mu\nu\lambda\rho}\widehat{C}^{\mu\nu\lambda\rho}\right], \tag{421}$$

where

$$\widetilde{\widehat{R}} = \widehat{R}+6\epsilon\widehat{g}^{\mu\nu}\left(\widehat{D}_\mu\widehat{\partial}_\nu\widehat{\varphi}-\widehat{\partial}_\mu\widehat{\varphi}\widehat{\partial}_\nu\widehat{\varphi}\right)+6(1-\epsilon)\widehat{g}^{\mu\nu}\left(q\widehat{D}_\mu\widehat{C}_\nu-q^2\widehat{C}_\mu\widehat{C}_\nu\right), \tag{422}$$

and

$$\widehat{C}_{\mu\nu}\widehat{C}^{\mu\nu} = \widehat{g}^{\mu\mu'}\widehat{g}^{\nu\nu'}\widehat{C}_{\mu\nu}\widehat{C}_{\mu'\nu'}, \qquad \text{etc.}$$

We have also added the non-minimal coupling strenths $\widehat{\zeta}_h$ and $\widehat{\zeta}$, with $\widehat{\zeta}_h = \zeta_{h+}P_+ + \zeta_{h-}P_-$, $\widehat{\zeta} = \zeta_+P_+ + \zeta_-P_-$, etc.

It is also possible to replace $\widehat{\Phi}$ with a complex field. The action has the same terms as the field $\widehat{H}$, by replacing it with $\widehat{\Phi}$ and W_μ with a $U(1)$ gauge field V_μ.

In conclusion, the overall conformal invariant model $\mathcal{TW}$ is defined by the action

$$\widehat{S}^{(c)}_{ch-sym} = \widehat{S}_f + \widehat{S}_g + \widehat{S}_{aeg} + (S_Y + S_{Yd} + h.c.) + \widehat{S}^{(c)}_{EH+aes+aed+C} \tag{423}$$

It splits into

$$S^{(c-)}_{ch-sym} = S^{(-)}_f + S^{(-)}_g + S^{(-)}_{aeg} + (S_{YL} + S_{YdL} + h.c.) + S^{(c-)}_{EH+aes+aed+C} \tag{424}$$

and

$$S^{(c+)}_{ch-sym} = S^{(+)}_f + S^{(+)}_g + S^{(+)}_{aeg} + (S_{YR} + S_{YdR} + h.c.) + S^{(c+)}_{EH+aes+aed+C} \tag{425}$$

The label $(-/+)$ stands for left/right.

To summarize, $\mathcal{TW}$ is a theory of massless fields, which is not only classically symmetric under the already mentioned symmetries, i.e., under diffeomorphisms, local Lorentz and Weyl transformations, as well as the $SU(3)_L \times SU(3)_R \times SU(2) \times U(1)_L \times U(1)_R$ gauge transformations, but also has the right interaction terms to be classified as renormalizable. However, we have no guarantee about its unitarity. This is due, in the perturbative approach, to the possible appearance in the gravity propagators of ghosts (wrong sign of the quadratic kinetic operator).

Therefore let us proceed to illustrate some consequences of the quantization of $\mathcal{TW}$.

16. $\mathcal{TW}$ as a Quantum Theory

The $\mathcal{TW}$ theory has several local symmetries: diffeomorphism symmetries, local Lorentz invariance, Weyl invariance, besides the symmetries under $SU(3)_L \times SU(3)_R \times SU(2) \times U(1)_L \times U(1)_R$ gauge transformations. Quantization requires gauge fixing. The reason is that the gauge degrees of freedom are unphysical and have to be eliminated. But there is also another reason: if the gauge is not fixed, the propagators of the gauge fields and of the metric are not defined (the corresponding kinetic operators are not invertible),

and, therefore, quantization is impossible. The two things are connected: gauge invariance means that, in the space of field configurations, besides the physical degrees of freedom, we have a set of unphysical configurations of infinite volume (in the path integral sense). This infinity is the origin of the non-invertibility of the relevant kinetic operator, being related to the (infinite volume) space of its zero modes. An educated guess for gauge fixing, accompanied by the introduction of FP ghosts in order to fix the mismatch between the unphysical degrees of freedom and the dofs eliminated by the gauge fixing, can lead to the solution of both problems. If we now look at the Weyl invariance we see a difference: in Section 14 we have already noticed that the introduction of a vector gauge field C_μ is not required in order to implement a Weyl symmetry, the scalar field φ alone can do the job. In $\mathcal{TW}$, we would like to exploit this possibility and explore the formulation where the vector gauge potential $\widehat{C}_\mu$ is absent. In such a case, the propagator of $\widehat{\varphi}$ exists even when the gauge has not been fixed and there is a perfect matching between the unphysical dofs and those eliminated, for instance, by the gauge choice $\widehat{\varphi} = 0$ (thus, no FP ghosts are necessary). This seems to suggest that the nature of Weyl symmetry is different from that of ordinary gauge symmetries. In the sequel, therefore, we would like to explore an unusual possibility: we will not fix this gauge but keep it unfixed (in particular, therefore, there will be neither a ghost nor a BRST symmetry corresponding to it), but much like we do for rigid symmetries we wish to preserve this local symmetry throughout the process of quantization. In the literature, the attitude is generally different, and the Weyl symmetry is treated as an ordinary gauge symmetry and quantized with gauge fixing, ghost and corresponding BRST symmetry; see notably [122–124]. In the last reference the authors signal the difficulty of producing compatible BRST Weyl, diffeomorphism and Stückelberg gauge symmetry fixings, see Section 18 for a further comment. In our approach, this problem is avoided, and it should be recalled that we are not violating orthodoxy: solving a gauge theory without fixing the gauge is not in itself so heretical, it is simply extremely complicated and perhaps even impossible when gauge invariance implies the lack of a propagator; but, we repeat, in our case the relevant φ propagator exists and, so, it makes sense to preserve Weyl gauge invariance through (the perturbative) quantization.

Our choice to ignore the field $\widehat{C}_\mu$ is tantamount to setting $\widehat{C}_\mu = \partial_\mu \widehat{\varphi}$, which, in the literature, defines an *integrable Weyl structure*. This structure, among other things, has the virtue of avoiding the SCE (the second clock effect), which revives an objection raised by Einstein against non-metricity, [105]: the parallel transport of a unit vector along curves with the same initial and final point would lead to different results, involving both space and time intervals. In particular the time difference (SCE) would be given by the integral of $C_{\mu\nu}$ over the surface comprised between the two curves, which, of course, in the integrable Weyl structure vanishes. For more details and references, see [100,125–127].

Summarizing, the quantum $\mathcal{TW}$ theory is dealt with by fixing the gauge for diffeomorphism and gauge invariance, introducing the corresponding FP ghosts and recovering the appropriate BRST symmetries, while trying to preserve Weyl symmetry. But quantization gives rise to a new problem involving precisely the preservation of Weyl symmetry due to even trace anomalies.

17. The Problem of Even Trace Anomalies

The object of this section is the problem of even trace anomalies in $\mathcal{TW}$ in its symmetric phase (i.e., before the *primitive chiral symmetry breaking*, see below).

Trace anomalies have been discussed above at various stages, and especially in Section 6. The analysis was limited there to theories of free fermions coupled to background potentials (metric or gauge fields) and mostly to odd parity anomalies. Now, we

have to make do with a fully interacting theory, not only for matter fields but also for the gauge potentials, the metrics and the dilatons.

In general, trace anomalies for matter fields interacting with background potentials arise as a response to the effective action to a Weyl variation of the metric $g_{\mu\nu} \to e^{2\omega} g_{\mu\nu}$ for a local parameter ω. The classical definition of the e.m. tensor for matter fields interacting with a background metric is

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}} \tag{426}$$

S being the classical action. If the latter is Weyl invariant, the e.m. tensor is traceless. This follows from the classical WI

$$\delta_\omega S = \int d^{\mathsf{d}} x \left(\frac{\delta S}{\delta g^{\mu\nu}} \delta_\omega g^{\mu\nu} + \sum_i \frac{\delta S}{\delta f_i} \delta_\omega f_i \right) = 0 \tag{427}$$

where f_i denote generic matter fields, and, for infinitesimal ω, $\delta_\omega g^{\mu\nu} = -2\omega g^{\mu\nu}, \delta_\omega f_i = -2y_i \omega f_i$ (where y_i is 0 for gauge fields, $\frac{\mathsf{d}-2}{2}$ for scalars and $\frac{\mathsf{d}-1}{4}$ for fermions, etc.). If the matter fields are on shell, i.e., $\frac{\delta S}{\delta f_i} = 0$, it follows that $T_{\mu\nu} g^{\mu\nu} = 0$ due to the arbitrariness of ω.

When the metric is dynamical, the interpretation of (426) changes for $\frac{\delta S}{\delta g^{\mu\nu}}$ does not represent the e.m. tensor of the metric field, but the LHS of its equation of motion. It is well known that in the EH gravity, this leads to the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu} \tag{428}$$

where Λ is the cosmological constant and $T_{\mu\nu}$ is the e.m. tensor of the matter fields.

Therefore, in an interacting theory like $\mathcal{TW}$ [In this section, we refer to a general abstract theory with the same general features as $\mathcal{TW}$. For this reason, we use a generic notation for the fields and couplings, dropping, in particular, the $_\pm$ labels, but the following discussion applies as well to $\mathcal{TW}$, and to $\mathcal{TW}_L$ or $\mathcal{TW}_R$ separately or in conjunction, in the symmetric phase] the WI for Weyl transformations has the same form as (427), but the interpretation of e.m. tensor of Equation (426), while it still holds for matter fields, is not valid for the metric itself. If we refer, for instance, to (376) we recall that, since we have dropped C_μ, the tilded Ricci scalar is

$$\widetilde{R} = R + 6\Box\varphi - 6\partial\varphi \cdot \partial\varphi \tag{429}$$

Therefore the equation of motion for the metric is

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left(R + \mathfrak{c} e^{-2\varphi} \right) = -\kappa e^{2\varphi} T_{\mu\nu}^{(m)} \tag{430}$$

where $T_{\mu\nu}^{(m)}$ denotes the e.m. tensor of all the matter fields (including the dilaton) coupled to the metric. Let us recall that this equation is a Weyl covariant.

Taking the trace of (430), we get that, when all the other fields are on shell, the equation

$$R + 2\mathfrak{c} e^{-2\varphi} = \kappa e^{2\varphi} T^{(m)}, \qquad T^{(m)} = g^{\mu\nu} T_{\mu\nu}^{(m)} \tag{431}$$

should hold as a consequence of (427) (since ω is arbitrary).

It must be remarked that, as required by (427), we obtain a dynamical equation for $g_{\mu\nu}$ that does not automatically imply the vanishing of the trace of $T_{\mu\nu}^{(m)}$. That is, in order to preserve Weyl invariance, it is, in general, not necessary for the matter e.m. tensor to be

traceless. There may exist situations of this type, but in view of the discussion that follows, they fall in the realm of non-perturbative solutions. For simplicity, we exclude them from our analysis.

Now, the matter e.m. tensor $T_{\mu\nu}^{(m)}$ contains the contributions from the e.m. tensors of all the fields, including φ, except the metric. However, in general, $T_{\mu\nu}^{(m)}$ is not simply the direct sum of the e.m. tensors of the various fields coupled only to the metrics (like, for instance, in (158)), because there are also interaction terms that mix different fields beside the metrics. This means that the problem we face with the trace of the RHS of (430) is not simply the sum of the distinct problems of computing the trace of each field species separately [For the same reason we cannot factor the overall path integral of the theory as a product of separate path integrals, each of them involving the quadratic action of each field species interacting with other fields treated as external sources]. When dealing with a free field interacting with external potentials, we have the possibility to use either a perturbative or a non-perturbative approach. The latter consists in resorting to a one-loop quantum energy–momentum tensor $\langle\langle T_{\mu\nu}\rangle\rangle$, which, for matter fields, is defined by

$$\langle\langle T_{\mu\nu}\rangle\rangle = \frac{2}{\sqrt{g}}\frac{\delta W}{\delta g^{\mu\nu}} \tag{432}$$

where W is the effective action, and then using the WI

$$\delta_\omega W = -\int d^{\mathsf{d}}x\,\sqrt{g}\,\omega\,\langle\langle T_{\mu\nu}\rangle\rangle g^{\mu\nu} \tag{433}$$

in order to identify a possible trace anomaly.

But this approach is not available when dealing with a fully interacting theory like $\mathcal{TW}$. We must turn to the perturbative method, which consists in general in fixing the gauges and introducing the corresponding ghosts, in determining the Feynman rules and, in particular, writing down propagators and vertices, and, finally, after introducing an appropriate external source for each elementary field, writing the appropriate Slavnov–Taylor identity for each BRST symmetry (this will be commented further on in Section 18). However, rewriting (427) as a Slavnov–Taylor identity would obscure the role of the e.m. tensors, which are composite operators (not elementary fields). In order to highlight this role, we will proceed in another way. Concerning Weyl anomalies, the available quantum results at present are (perturbative and/or non-perturbative) local expressions of trace anomalies obtained by integrating out in each separate path integral various matter fields, for instance, fermions, scalars, etc., which feature quadratically in expressions where they interact with other fields considered as background. The question is whether these results are of any use when we consider a fully interacting theory such as $\mathcal{TW}$. In the sequel, we would like to discuss how they can, in fact, be brought to bear in a perturbative approach.

The $\mathcal{TW}$ action contains terms coming from the quadratic kinetic terms of the various dynamical fields coupled to the metric, the dilaton and the gauge potentials, but also several interaction terms: non-Abelian gauge fields interacting among themselves, scalar fields interacting with fermions, gauge fields and the dilaton, gravitons interacting with the other fields and among themselves. The perturbative approach singles out kinetic terms, each being quadratic in one species of fields, from which the propagators are extracted, and deals with all the other terms as interaction terms from which the vertices are derived. It is important to recall that in the perturbative approach, the fields we consider are fluctuations around a background; for most fields (fermions, vectors and also scalars in the symmetric phase) the fluctuating fields are usually identified with the fields themselves, while for the metric the fluctuating field is $h_{\mu\nu}$ with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ [The dilaton could be expanded

around a non-vanishing value φ_0, but in the present discussion we set $\varphi_0 = 0$]. In the following discussion, we set, for the moment, $\mathfrak{c} = 0$.

Next, we proceed as follows. We expand both sides of (431) in series of $h_{\mu\nu}$ and forget the interaction terms of all the other fields. The 0-th order term on the LHS vanishes [Because its lowest order is linear in $h_{\mu\nu}$, for, at the lowest order, $R = \partial_\mu \partial_\nu h^{\mu\nu} - \Box h^\mu_\mu$]. Therefore, at the lowest order, the RHS, which is the sum of the traces of the e.m. tensors of the various species of fields, must vanish too. But since the e.m. tensors of the various species are non connected by interaction terms, this implies that they must be separately traceless. This is all classical.

To quantize (430) and (431), the perturbative approach allows us to start from the classical zeroth order expression of the matter e.m. tensor or its trace, which is the sum of the free e.m. tensors for each distinct field species and treat all the other terms as interactions. To extract explicit formulas we expand the full action in terms of the fluctuating fields and forget their interactions, the action becomes a free field action (and the equations of motion are the free ones). From it, we can extract the lowest order (on shell conserved and traceless) e.m. tensor for each species via the canonical formulas. Alternatively we can use (426) by coupling every single species with the metric $g_{\mu\nu}$ in such a way as to realize Weyl invariance, using (426) and eventually setting the metric equal to the flat one. Said another way, we expand the full action in series of $h_{\mu\nu}$: the 0-th order term is made of the free actions while the coefficient of $h^{\mu\nu}$ in the first order term identifies the on-shell conserved and traceless e.m. tensors.

The zeroth order e.m. tensors obtained in this way are

$$T^{(f)}_{\mu\nu} = \frac{i}{4}\overline{\psi}\gamma_\mu \overset{\leftrightarrow}{\partial_\nu}\psi + (\mu \leftrightarrow \nu) - \eta_{\mu\nu}\frac{i}{2}\overline{\psi}\gamma^\lambda \overset{\leftrightarrow}{\partial_\lambda}\psi, \tag{434}$$

for fermions (which, in particular, may be chiral), and

$$T^{(g)}_{\mu\nu} = -\frac{1}{g^2}\left(\partial_\mu A_\lambda \partial_\nu A^\lambda + \partial_\lambda A_\mu \partial^\lambda A_\nu - \frac{1}{2}\eta_{\mu\nu}\partial_\lambda A_\rho \partial^\lambda A^\rho\right) \tag{435}$$

for Abelian gauge fields. This is obtained after fixing the Lorenz gauge. The non-Abelian formula differs from it only for the trace symbol in front of the RHS, the trace being applied to the gauge Lie algebra generators.

The above e.m. tensors are (on shell of the free equations of motion) conserved and traceless. At variance with these examples, the derivation of the e.m. tensors for scalar fields via Formula (426) applied to the scalar fields in (421) does not automatically lead to conservation and tracelessness. However, if we now return to Formula (431) in the spirit of perturbation theory, we have seen that at the lowest order, the LHS vanishes. Therefore, as already pointed out, the matter e.m. tensor must indeed be traceless at this order. It follows that also each e.m. tensor for scalars must be separately traceless.

Therefore, for the complex scalar field Φ, we expect

$$T^{(s)}_{\mu\nu} = \partial_\mu \Phi^\dagger \partial_\nu \Phi + \partial_\nu \Phi^\dagger \partial_\mu \Phi - \eta_{\mu\nu}\partial_\lambda \Phi^\dagger \partial^\lambda \Phi + \frac{1}{3}\left(\eta_{\mu\nu}\Box - \partial_\mu \partial_\nu\right)\Phi^\dagger \Phi \tag{436}$$

and an analogous formula for H. In analogy, we expect a similar expression for the dilaton

$$T^{(\varphi)}_{\mu\nu} = \frac{6}{\kappa}\left(\partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2}\eta_{\mu\nu}\partial\varphi \cdot \partial\varphi + \frac{1}{6}\left(\eta_{\mu\nu}\Box - \partial_\mu \partial_\nu\right)\varphi^2\right) \tag{437}$$

These expressions are known as improved e.m. tensors. But what do they have to do with the action of $\mathcal{TW}$ and in particular with (421)?

Consider a free scalar field ϕ in dimension d. Its improved (on-shell conserved and traceless) e.m. tensor is

$$T_{\mu\nu}^{(s)} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial_\lambda\phi\partial^\lambda\phi + \frac{d-2}{4(d-1)}\left(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu\right)\phi^2, \tag{438}$$

assuming the free equation of motion is $\Box\phi = 0$. It can be derived with the improved canonical formula. But it can also be derived from the action

$$S_\phi = \frac{1}{2}\int d^d x\sqrt{g}\left(\partial_\mu\phi\partial^\mu\phi + \frac{d-2}{4(d-1)}R\phi^2\right) \tag{439}$$

where R is the Ricci scalar. This action is conformal invariant. Applying (426) to it, integrating by parts and setting $g_{\mu\nu} = \eta_{\mu\nu}$ one obtains (438). Now, what we do is the following: to the free action of ϕ, we add and subtract a term $\int d^4x\sqrt{g}R\phi^2$ (with the appropriate coefficient). The expansion in $h_{\mu\nu}$ gives

$$2\int d^4x h^{\mu\nu}\left(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu\right)\phi^2 + \mathcal{O}(h^2) \tag{440}$$

from which one can derive (438). Adding and subtracting the same term of course does not change anything, but while the lowest order approximant (440) allows us to write (438), the term with the opposite sign will remain and contribute an interaction term, which was not present in the original action, and must be taken into account when computing the trace anomaly. All the terms of higher order in (440) cancel one another. Said another way, we expand the metric in the RHS of (439): the 0-th order term is the free action of ϕ and is left unchanged by the addition of $\int d^4x\sqrt{g}R\phi^2$, the coefficient of $h^{\mu\nu}$ in the first order term identifies the on-shell conserved and traceless e.m. tensor (438); the first order term of $-\int d^4x\sqrt{g}R\phi^2$ is accounted for as an interaction term. All the higher-order terms of $\int d^4x\sqrt{g}R\phi^2$ disappear. On the other hand, if in the original action, there are mass terms like in ((388) and (395)) and ((420) and (419)), these terms are to be accounted for among the interaction terms.

This procedure can be adopted for the scalar fields φ, Φ and H.

Finally we have to consider also the contribution to the total trace anomaly due to the Faddeev-Popov ghosts. There are two types of these ghosts: those coming from the gauge fixing of non-Abelian gauge symmetries, denoted c^a and $\bar{c}^a$ and those derived from fixing the diffeomorphism symmetry, denoted by ξ^μ and $\bar{\xi}^\mu$.

A gauge fixed and FP ghost action for the former can be found in Appendix C below. The ghost action to the 0-th and first order in $h_{\mu\nu}$, excluding all the other interaction terms, can be extracted from

$$S^{c,\bar{c}} = \int d^4x\sqrt{g}g^{\mu\nu}\partial_\mu\bar{c}^a\partial_\nu c^a \tag{441}$$

summed over a. But the e.m. we obtain in this way

$$T_{0\mu\nu}^{(c,\bar{c})} = \partial_\mu\bar{c}^a\partial_\nu c^a + \partial_\nu\bar{c}^a\partial_\mu c^a - \eta_{\mu\nu}\partial_\lambda\bar{c}^a\partial^\lambda c^a \tag{442}$$

is neither conserved nor traceless, in other words we face the same problem as with the scalar field when trying to extract a conserved and traceless e.m. tensor ((441) is not Weyl invariant). Therefore we proceed as in the scalar field case and add to the Lagrangian term in (441) the term $\frac{1}{3}\sqrt{g}R\bar{c}^a c^a$. In this way, the corresponding lowest order e.m. tensor is

$$T_{\mu\nu}^{(c,\bar{c})} = \partial_\mu\bar{c}^a\partial_\nu c^a + \partial_\nu\bar{c}^a\partial_\mu c^a - \eta_{\mu\nu}\partial_\lambda\bar{c}^a\partial^\lambda c^a + \frac{1}{3}\left(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu\right)\bar{c}^a c^a \tag{443}$$

which is conserved and traceless.

For diffeomorphisms we use the DeDonder gauge, whose lowest order form is as follows: $2\partial_\mu h^\mu_\lambda - \partial_\lambda h^\mu_\mu = 0$. Proceeding as in the gauge case we find a ghost action (see Appendix C, below),

$$S^{(\xi,\bar\xi)} = -\int d^4x \sqrt{g} g^{\mu\nu} \bar\xi_\lambda \partial_\mu \partial_\nu \xi^\lambda \tag{444}$$

Equation (444) is not Weyl invariant. Adding, as in the scalar field case, a term $\int d^4x \sqrt{g} R \bar\xi_\lambda \xi^\lambda$ to this action, at the lowest order, we find the traceless e.m. tensor

$$T^{(\xi,\bar\xi)}_{\mu\nu} = \partial_\mu \bar\xi_\lambda \partial_\nu \xi^\lambda + \partial_\nu \bar\xi_\lambda \partial_\mu \xi^\lambda - \eta_{\mu\nu} \partial_\lambda \bar\xi_\rho \partial^\lambda \xi^\rho + \frac{1}{3}\left(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu\right)\bar\xi_\lambda \xi^\lambda \tag{445}$$

Comment. Other backgrounds

In the perturbative approach described so far, we have started from the background $g_{\mu\nu} = \eta_{\mu\nu}, \mathfrak{c} = 0$ and $\varphi = 0$. This is the simplest background. Other more general backgrounds are possible, $g_{\mu\nu} = g_{0\mu\nu} \neq \eta_{\mu\nu}, \mathfrak{c} \neq 0$ and $\varphi = \varphi_0 \neq 0$ and even $V_\mu = V_{0\mu} \neq 0, W_\mu = W_{0\mu} \neq 0$. They are supposed to satisfy the background Equation (430)

$$R_{0\mu\nu} - \frac{1}{2} g_{0\mu\nu} \left(R_0 + \mathfrak{c}e^{-2\varphi_0} \right) = -\kappa e^{2\varphi_0} T^{(m)}_{0\mu\nu} \tag{446}$$

whose trace is

$$R_0 + 2\mathfrak{c}e^{-2\varphi_0} = \kappa e^{-2\varphi_0} T^{(m)}_0 \tag{447}$$

The new fluctuating fields are $h_{\mu\nu} = g_{\mu\nu} - g_{0\mu\nu}$ and $\varphi - \varphi_0$, $V_\mu - V_{0\mu}$ and $W_\mu - W_{0\mu}$. Now, we expand (430) in terms of the fluctuating fields around the classical solution of (446), in the same way we have done before around the trivial flat solution. As before, the 0-th order term of the expansion of the full action in $h_{\mu\nu}$ provides the free field equations of motion and the first order one the free e.m. tensors. The quantization is carried out as before, except that e.m. tensors and vertices will contain insertions of the background fields. We will not pursue this here. We only notice that since the calculation of the trace anomalies must be carried out according to the Formula (161), the two sides of Equation (447), will drop out of the final result. Refs.: [128,129].

17.1. Even Trace Anomalies: Further Examples

The previous argument allows us to conclude that the first perturbative level of the RHS of Equation (431) can be computed for each species separately. To this end, we have to single out the propagators and vertices. The propagators are well-known; some have already been written down. We summarize them for the reader's convenience, they are as follows: $\frac{i}{\not p}$ for fermions, $-\frac{i}{p^2}\eta_{\mu\nu}\delta^{ab}$ for gluons (in the Lorenz gauge), $\frac{i}{p^2}$ for real or complex scalars, $\frac{i}{p^2}\delta^{ab}$ for FP gauge ghost. The propagator between $h_{\mu\nu}$ and $h_{\alpha\beta}$ has a complicated form depending on the classical action we start from and on the gauge fixing conditions. For the present purposes, enough is to know that the generic form of this propagator in the UV limit is $\frac{i}{2\kappa p^2} t_{\mu\nu\alpha\beta}$, where the constant tensor t is constructed out of the flat Lorentz metric $\eta_{\mu\nu}$. This simplified form is sufficient to recognize at a glance the order of divergence of a Feynman diagram. The propagator for the dilaton φ is $\frac{i\kappa}{6}\frac{1}{p^2}$.

The relevant vertices for fermions have been introduced earlier, for instance, the vertex V_{ffh}, of two fermions and one $h_{\mu\nu}$ gravity fluctuation field in Equation (165). The fermion–

fermion–gluon vertex is $V_{ffg} : ig\gamma_\mu T^a$. A fermion–fermion–scalar vertex generated by a Yukawa term with coupling y takes the simple form

$$V_{ffs} \quad : \quad iy \tag{448}$$

Similarly the scalar–scalar–$h_{\mu\nu}$-graviton vertex is given by

$$V_{ssh} \quad : \quad -\frac{i}{2}\left(\eta_{\mu\nu}p\cdot p' - \left(p_\mu p'_\nu + p_\nu p'_\mu\right)\right) \tag{449}$$

where p' is an entering scalar particle momentum and p an exiting one. The subtraction of the term (440) entails the addition of another scalar–scalar–graviton vertex

$$V'_{ssh} \quad : \quad \frac{i}{6}\left(\eta_{\mu\nu}p\cdot p' - \left(p_\mu p'_\nu + p_\nu p'_\mu\right)\right) \tag{450}$$

The complex scalar–complex scalar–gluon vertex V_{ssg} is

$$V_{ssg} \quad : \quad iT^a \mathrm{g}\left(p_\mu - p'_\mu\right) \tag{451}$$

where the gluon leg is represented by $V_\mu^a T^a$ and g is the relevant gauge coupling. The gluon–gluon–graviton vertex is

$$V_{ggh} \quad : \quad \frac{i}{2}\delta^{\alpha\beta}\left[(p_\alpha p'_\beta + p_\beta p'_\alpha)\eta_{\mu\nu} - p\cdot p'\left(\eta_{\alpha\mu}\eta_{\beta\nu} + \eta_{\alpha\nu}\eta_{\beta\mu} + \frac{1}{2}\eta_{\alpha\beta}\eta_{\mu\nu}\right)\right] \tag{452}$$

referring to the legs $h_{\alpha\beta}$, $V_\mu^a T^a$, $V_\nu^b T^b$ with p entering, p' exiting for the last two. Here, we have assumed the normalization $\mathrm{Tr}(T^a T^b) = \delta^{ab}$. The triple gluon coupling, V_{ggg}, can be found in any gauge field theory textbook.

The ghost–ghost–V_μ-gluon vertex is

$$V_{c\bar{c}g} \quad : \quad -\mathrm{g}f^{abc}p_\mu \tag{453}$$

The ghost–ghost–$h_{\mu\nu}$-graviton vertex is

$$V_{c\bar{c}h} \quad : \quad -\frac{i}{2}(p_\mu p'_\nu + p_\nu p'_\mu - \eta_{\mu\nu}p\cdot p') \tag{454}$$

The vertex $V_{\xi\bar{\xi}h}$ is

$$V_{\xi\bar{\xi}h} \quad : \quad -\frac{i}{2}(p_\mu p'_\nu + p_\nu p'_\mu - \eta_{\mu\nu}p\cdot p')\eta_{\lambda\rho} \tag{455}$$

where the legs are $h_{\mu\nu}, \bar{\xi}^\lambda, \xi^\rho$ with momenta q, p, p' respectively $(q + p + p' = 0)$.

As for the dilaton φ, $V_{\varphi\varphi h}$ is the same as V_{ssh} multiplied by $\frac{3}{\kappa}$. Then

$$V_{H^\dagger H\varphi} \quad : \quad ip\cdot p' \tag{456}$$

where p is the momentum of $H^\dagger$ and p' is the momentum of φ. The vertex $V_{\Phi^\dagger\Phi\varphi}$ is the same. The mass terms of scalar fields such as those in (388), (420) and (419) originate interaction terms proportional to the square mass.

There are also vertices due to terms like (440) and higher order vertices. But the ones shown above are sufficient to illustrate the main point.

Thus, at least at the lowest loop order, we can compute the overall anomaly of $\mathcal{TW}$ using the results and methods presented in the first part of this article. As for the odd parity

(i.e., type O) trace anomalies in $\mathcal{TW}$ this confirms what has already been said above: they cancel out by construction. But in $\mathcal{TW}$ there are even (type NO) trace anomalies. It is worth recalling the difference between the two types of anomalies we are talking about here.

Odd trace anomalies are produced in models of chiral fermions coupled to a background metric or gauge field. They are a consequence of the lack of the corresponding fermion propagators. With this kind of anomaly, a theory simply does not make sense. Even trace anomalies, on the contrary, do not correspond to a lack of propagator, and the corresponding theory is well-defined. It is true that, for instance, in the calculation of the even trace anomaly in a theory of a Maxwell field coupled to a metric, the propagator is ill-defined because the kinetic operator is not invertible due to gauge invariance, but this fact is remedied by suitably fixing a gauge (and taking into account the contribution of the appropriate FP ghosts). Once the relevant propagator is defined the calculation is not unlike the odd parity case we have seen above. The results are anomalies taking the form of Equation (147). The most well-known have density $T[g]$ given by (148) and (149). In perturbation theory, their lowest order terms are found from the triangle diagram contribution of the one-loop e.m. tensor $\langle\!\langle T^{(f)}_{\mu\nu}\rangle\!\rangle$, according to the prescription (161). But there are others, perhaps less well-known or plainly ignored, anomalies. They are characterized by the fact that their densities are covariant pieces of actions. An example is represented by Equation (151). Its explicit computation can be found in [5]. Here, we summarize the main facts. It shows up, for instance, in a theory of Dirac fermions coupled both to an external metric and to an external Maxwell potential. At the lowest level, it is again born out of $\langle\!\langle T^{(f)}_{\mu\nu}\rangle\!\rangle$, but from a triangle diagram amplitude formed by joining a lowest level chiral e.m. tensor to two V_{ffg} vertices and joining each other by means of three fermion propagators, with the addition of the crossed amplitude. These amplitudes are divergent, and by extracting the finite part, one finds

$$\mathcal{A}^{(g)}_\omega = -\frac{1}{12\pi^2}\int d^4x\,\omega\left(-\partial_\nu V_\lambda\partial^\nu V^\lambda + \partial_\nu V_\lambda\partial^\lambda V^\nu\right) = \frac{1}{24\pi^2}\int d^4x\,\omega F_{\nu\lambda}F^{\nu\lambda} \tag{457}$$

This is the lowest order result. The all-order expression invariant under both diffeomorphisms and gauge transformations is

$$\mathcal{A}^{(g)}_\omega = \frac{1}{24\pi^2}\int d^4x\,\sqrt{g}\,\omega F_{\nu\lambda}F^{\nu\lambda}, \tag{458}$$

Its non-Abelian extension is

$$\mathcal{A}^{(g)}_\omega = \frac{1}{24\pi^2}\int d^4x\,\sqrt{g}\,\omega\,\mathrm{tr}\left(F_{\nu\lambda}F^{\nu\lambda}\right). \tag{459}$$

From this example it is clear what has to be done in order to identify the trace anomalies of this type. We must construct all possible divergent triangle diagrams formed by joining a zeroth order e.m. tensor with two vertices and the latter with each other by means of three suitable propagators; the e.m. tensor carries an incoming momentum, and the other two outgoing lines represent particles created by the same field (two fermions, two scalars, etc.). Then, we must regularize the associated integral and extract its finite part. It is easy to see that the triangle diagrams with two outgoing fermions always vanish [This is true at any loop order, therefore is an exact result. It is due to the fact that all vertices involve either two fermions or none. Therefore, in order to form a triangle diagram with one bosonic and two fermionic external legs, we need either an even number of fermion vertices and an odd number of fermion propagators or vice versa. In both cases the number of γ matrices involved in the diagram is odd; thus, the gamma matrix trace vanishes.] Therefore, we cannot have an even trace anomaly of the above type whose density $T[g]$

is the Lagrangian density of the fermion kinetic term. Analogous considerations hold for anomalies with density $T[g]$ corresponding to the Lagrangian density of ghosts, although the vanishing mechanism is different: it is easy to see that there are potentially divergent triangle diagrams with two outgoing $\bar{c}^a$ and c^b legs, but the corresponding amplitude vanishes when summed with the crossed amplitude due to anticommutativity and the lack of an appropriate tensor antisymmetric in $a \leftrightarrow b$. The same holds for diffeomorphism ghosts. However, this is not the case for scalars.

Consider, for instance, the triangle diagram very similar to the one that gives rise to the familiar ABJ anomaly, except that the two outgoing legs represent real scalars, say Φ. It is formed by joining the zeroth order fermion e.m. tensor $T_{\mu\nu}^{(f)}$ to two fermion-fermion-scalar vertices $V_{ffs}^{(s)}$, and these to each other, by means of three fermion propagators. The only difference with the ABJ calculation is the absence of a γ_5 matrix and the trace of four γ matrices instead of six. The integral is divergent; it must be regularized, and the trace anomaly must be calculated according to (161). The result is

$$\mathcal{A}_{\omega}^{(s)} = \frac{1}{48\pi^2} \int d^4x \omega \left(8\Phi\Box\Phi - \partial_\mu\Phi\partial^\mu\Phi\right) \tag{460}$$

This is the triangle diagram result. It does not satisfy the WZ consistency conditions. The all-order expression is either [In the perturbative context of this analysis, the expressions of trace anomalies we need involve only the orders of approximation relevant for the perturbative order we are considering. However, the identification of all-order expressions is both a simple way to check the WZ consistency conditions and a practical way to find all the orders of approximation]

$$\Delta_{\omega}^{1s)} \sim \int d^4x \sqrt{g}\,\omega g^{\mu\nu}D_\mu\Phi D_\nu\Phi \tag{461}$$

or

$$\Delta_{\omega}^{(2s)} \sim \int d^4x \sqrt{g}\,\omega \left(\partial_\mu\Phi\partial^\mu\Phi + \frac{1}{6}R\Phi^2\right) \tag{462}$$

or

$$\Delta_{\omega}^{(3s)} \sim \int d^4x \sqrt{g}\,\omega \left(\Phi\Box\Phi - \frac{1}{6}R\Phi^2\right) \tag{463}$$

They all satisfy the WZ consistency conditions. What combination of the three is the appropriate one in any specific case can come only from the explicit calculation. In the case of trace anomaly due to the scalar Φ, which involves interacting with a Dirac fermion, the final result is

$$\mathcal{A}_{\omega}^{(s)} = \frac{1}{48\pi^2} \int d^4x \omega \left(8\Phi\Box\Phi - \partial_\mu\Phi\partial^\mu\Phi - 9R\Phi^2\right) \tag{464}$$

The difference between this last expression and Equation (462) is due to higher-order diagram contributions, see Appendix D.

In the round brackets of Equations (461) and (462) we can add also a term Φ^4, which form a consistent cocycle by itself and can appear at higher orders of the calculation, but for our present discussion the above formulas are sufficient.

Similar expressions also hold with the obvious replacements for a complex doublet H instead of the real scalar Φ.

As for the dilaton, we can have

$$\mathcal{A}_\omega^{(\varphi)} \sim \int d^4x \sqrt{g}\, \omega \left(\partial_\mu \varphi \partial^\mu \varphi + \frac{1}{3} R \varphi^2 \right) \quad \text{or} \quad \sim \int d^4x \sqrt{g}\, \omega \left(\varphi \Box \varphi + \frac{1}{6} R \left(\varphi^2 - 3\varphi \right) \right) \tag{465}$$

Both terms satisfy the WZ consistency conditions. Other consistent expressions are possible.

A similar result is obtained from the triangle diagram constructed by joining $T_{\mu\nu}^{(g)}$, by means of two gluon propagators, to two vertices V_{ssg} and the latter by means of a scalar propagator, the two outgoing legs being scalars.

There are many other similar examples. The anomalies (148) and (149) may come not only from the triangle diagrams of $\langle\!\langle T_{\mu\nu}^{(f)} \rangle\!\rangle$ but also from those of $\langle\!\langle T_{\mu\nu}^{(g)} \rangle\!\rangle$ and $\langle\!\langle T_{\mu\nu}^{(s)} \rangle\!\rangle$ both for Φ and H, and $\langle\!\langle T_{\mu\nu}^{(\varphi)} \rangle\!\rangle$, as well as in $\langle\!\langle T_{\mu\nu}^{(c,\bar{c})} \rangle\!\rangle$ and $\langle\!\langle T_{\mu\nu}^{(\zeta,\bar{\zeta})} \rangle\!\rangle$, provided we construct the triangle with the right vertices. Each of the corresponding anomalies will have its own coefficient.

The anomalies of the type (458) can be produced not only in the triangle diagram of $\langle\!\langle T_{\mu\nu}^{(f)} \rangle\!\rangle$ but also in $\langle\!\langle T_{\mu\nu}^{(g)} \rangle\!\rangle$, $\langle\!\langle T_{\mu\nu}^{(c,\bar{c})} \rangle\!\rangle$ and $\langle\!\langle T_{\mu\nu}^{(s)} \rangle\!\rangle$, both for Φ and H.

Anomalies of the type (461)–(463) can be produced in triangle diagrams of $\langle\!\langle T_{\mu\nu}^{(f)} \rangle\!\rangle$, $\langle\!\langle T_{\mu\nu}^{(s)} \rangle\!\rangle$ and $\langle\!\langle T_{\mu\nu}^{(g)} \rangle\!\rangle$. And so on.

It is out of the scope of this article to explicitly calculate these potential anomalies. They are not really challenging but rather lengthy calculations, where for completeness, one should also consider other diagrams besides the ones mentioned before. For instance, in the case of $\langle\!\langle T_{\mu\nu}^{(f)} \rangle\!\rangle$, one should also consider bubble diagrams involving two fermion propagators and a 2-fermion–2-graviton vertex. Although there are no known explicit examples of nontrivial contributions to anomalies of this type, it is still necessary to examine them.

The examples already shown should, however, be enough to illustrate the problem. We have plenty of trace anomalies that break Weyl invariance. Considering the wide randomness of their rational coefficients, a mutual cancelation, like in the case of the odd trace anomalies (mode (B)), looks impossible in any reasonable theory. There is, however, another method for that, and we would like now to focus on it: the breaking of conformal invariance by even trace anomalies can be repaired by adding to the effective action suitable WZ terms. Let us briefly discuss them. Refs. [3,5,51,63,64,130–139].

17.2. WZ Terms

Any trace anomaly can be written in the form

$$\mathcal{A}_\omega[g, f] = \int d^{\mathsf{d}} x \sqrt{g}\, \omega F[g, f] \tag{466}$$

where $g = \{g_{\mu\nu}\}$ is the metric, ω is the Weyl transformation parameter $\delta_\omega g_{\mu\nu} = 2\omega g_{\mu\nu}$, f denotes any other field and F is a local function of g and f. Assuming ω to be an anticommuting Abelian field, the anomaly must satisfy the consistency condition

$$\delta_\omega \mathcal{A}_\omega = 0 \tag{467}$$

which expresses simply the fact that two subsequent Weyl transformations made in reverse order yield the same result. This is, in fact, an integrability condition. It means that, with the help of an auxiliary field σ, which transforms as $\delta_\omega \sigma = -\omega$, we can construct a local functional $\mathcal{W}_{WZ}[\sigma, g, f]$ such that

$$\delta_\omega \mathcal{W}_{WZ}[\sigma, g, f] = -\mathcal{A}_\omega[g, f] \tag{468}$$

The construction parallels the analogous one for chiral anomalies [5]. Introduce the parameter t, $0 \leq t \leq 1$ and the interpolating metric

$$g_{\mu\nu}(t) = e^{2\sigma t}g_{\mu\nu} \tag{469}$$

such that

$$\delta_\omega g_{\mu\nu}(t) = 2(1-t)\omega g_{\mu\nu}(t), \tag{470}$$

as well as an interpolating field

$$f(t) = e^{-yt\sigma}f, \qquad \delta_\omega f(t) = -y(1-t)\omega f(t) \tag{471}$$

where $y = 0$ for a gauge field, $y = \frac{d-2}{2}$ for a scalar field. We have also

$$\frac{d}{dt}g_{\mu\nu}(t) = 2\sigma g_{\mu\nu}(t), \qquad \frac{d}{dt}f(t) = -y\sigma f(t) \tag{472}$$

For the field φ we put $f(t) \equiv \varphi(t) = \varphi + \sigma t$, thus

$$\delta_\omega \varphi(t) = \omega(1-t), \qquad \frac{d}{dt}\varphi(t) = \sigma \tag{473}$$

Then, we construct the functional

$$
\begin{aligned}
\mathcal{W}_{WZ}[\sigma, g, f] &= \frac{1}{8}\int_0^1 dt \int d^d x \sqrt{g(t)}F[g(t), f(t)]g^{-1}(t)\frac{d}{dt}g(t) \\
&= \int_0^1 dt \int d^d x \sqrt{g(t)}F[g(t), f(t)]\sigma
\end{aligned}
\tag{474}
$$

Now we notice that the t derivatives of the interpolating fields coincides with their Weyl variation except for the replacement of σ with $(1-t)\omega$.

A simple evaluation yields

$$
\begin{aligned}
\delta_\omega \mathcal{W}_{WZ}[\sigma, g, f] &= \int_0^1 dt \frac{d}{dt}\left(\int d^d x \sqrt{g(t)}F[g(t), f(t)]\right)(1-t)\omega \\
&\quad - \int_0^1 dt \int d^d x \sqrt{g(t)}F[g(t), f(t)]\omega = -\mathcal{A}_\omega[g, f]
\end{aligned}
\tag{475}
$$

The WZ terms have the same formal expression as the anomaly with ω replaced by σ, $g_{\mu\nu}$ replaced by $g_{\mu\nu}(t)$ and f with $f(t)$. For instance, the WZ term for the anomaly with density $\sim F_{\mu\nu}F^{\mu\nu}$ takes the extremely simple form

$$\mathcal{W}_{WZ}[\sigma, g, V] \sim \int d^4 x \sqrt{g}\sigma F_{\mu\nu}F^{\mu\nu} \tag{476}$$

This is the case also for the anomaly (461) and for the square Weyl tensor anomaly (148). Refs. [5,21,62].

Application of WZ Terms to $\mathcal{TW}$

From the previous subsection, we can see all the even trace anomalies present in $\mathcal{TW}$ can be canceled by adding to the effective action the corresponding WZ term constructed by replacing σ in the above formulas with φ_+ or φ_- according to whether we refer to $\mathcal{TW}_+$ or $\mathcal{TW}_-$. For instance, the WZ term (476) takes the form

$$\mathcal{W}_{WZ}^{(\pm)}[\varphi_\pm, g_\pm, V] \sim \int d^4 x \sqrt{g_\pm}\varphi_\pm \mathrm{tr}\left(F_{\mu\nu}F_{\mu'\nu'}\right)g_\pm^{\mu\mu'}g_\pm^{\nu\nu'} \tag{477}$$

to cancel the even trace anomaly induced by the (common to both sides) SU(2) gauge field. These are new interaction terms in the quantum action.

A remark is in order should one use, instead of $\varphi_\pm$, the field $\chi_\pm$, see Equation (385). The WZ term would be

$$\mathcal{W}_{WZ}^{(\pm)}[\chi_\pm, g_\pm, V] \sim \int d^4x \sqrt{g_\pm} \log(\ell\chi_\pm) \mathrm{tr}\left(F_{\mu\nu}F_{\mu'\nu'}\right) g_\pm^{\mu\nu'} g_\pm^{\nu\nu'} \tag{478}$$

To write it down one must introduce a scale of length ℓ, which may create problems for renormalization (see a discussion in Appendix E).

17.3. Conformal Symmetry Restoration

As stressed before, our way to deal with conformal symmetry is not the traditional way of treating gauge symmetries. As pointed out several times, in $\mathcal{TW}$ there are several other local symmetries, notably the two diffeomorphism symmetries and the SM gauge symmetries. Our approach to quantization consists in gauge fixing the latter symmetries while leaving free the conformal symmetry during quantization. This allows us to apply the perturbative approach as outlined above. This, in turn, requires introducing the FP ghosts for the former symmetries in order to vindicate the existence of relevant metric and gauge boson propagators and recover the corresponding BRST symmetries, an indispensable implement for unitarity and renormalization. It is important to highlight the different standing of local conformal symmetry on one side and diffeomorphism and/or gauge symmetry on the other. The latter are absolutely needed for renormalization; they are the fundamental symmetries that one has to restore in the form of (mutually compatible) BRST symmetries at every step of the perturbative renormalization of the relevant theory. The former does not have the same status—it is not strictly required to fix the gauge (the relevant propagator exists anyhow). It may, however, be violated by even trace anomalies without compromising the consistency of the theory. It is nevertheless possible to restore conformal invariance by adding suitable WZ terms.

The way to proceed with (one loop) renormalization of $\mathcal{TW}$ is, therefore, the following. In $\mathcal{TW}$ at one-loop, we meet divergent integrals (in diagrams with a low enough number of legs), and one has to regularize them and carefully define their infinite parts. Let us suppose that this has been done without meeting obstacles. The next step consists of absorbing such infinities by means of field and coupling constant renormalizations. The one-loop trace calculations of the various matter, e.m. tensors, are not affected by this procedure, but, as we have seen, they also give rise to infinities whose finite parts generate trace anomalies. To recover the conformal invariance of the one-loop action, we have to add the relevant WZ terms. The price is the addition to the quantum action of new interaction terms that involve the dilaton, such as (476) and, in general, (475).

At this stage, we would like to stress an important point. As has been noted, see for instance [140], any classical local field theory is implicitly conformal invariant. To make the invariance explicit, it is enough to use a dilaton field and introduce suitable expressions of it in the local terms of the action, which are not already conformal invariant. This is the procedure we have adopted above to immerse $\mathcal{T}$ in the Weyl geometry. To this end, we have introduced dilaton fields, represented by the symbols $\varphi, \widehat{\varphi}, \varphi_+, \varphi_-$. What we are considering now is different, we are checking if Weyl symmetry can survive quantization. Once the effective action has been one-loop renormalized, the Weyl invariance turns out to be broken by several trace anomalies. However, as we have seen, such trace anomalies can be canceled by suitable WZ terms, which restore in this way Weyl invariance. It should not be forgotten that this cancelation cannot be carried out for arbitrary local field expressions. It can be done for trace anomalies because the latter satisfies the WZ consistency conditions. Now, introducing WZ terms entails introducing new interaction

terms in the action. Considering $\mathcal{TW}_+$ and $\mathcal{TW}_-$, this operation can be carried out by identifying σ with φ_+ and φ_-, respectively, in the WZ terms of the previous section. In this way, we end up with two one-loop conformal invariant theories, $\mathcal{TW}_+^{(1)}$ and $\mathcal{TW}_-^{(1)}$.

The previous one may hardly be considered a grand result—WZ terms are more than fifty years old—although it is remarkable that the overall cost is the use of a unique AE field $\widehat{\varphi}$, which is already there in the classical theory. However things become more interesting if we can proceed with the two-loop quantization. At two loops, there are many more interaction terms and diagrams to be considered than at one loop. Due to the number of fields and interactions, and so of diagrams involved, a direct computational approach is simply daunting. What saves our day are the WZ consistency conditions. For we know that the trace of the matter, e.m. tensor, must satisfy these conditions at every perturbative order. Now, given the fields at hand, the quantum numbers and the dimensions, there exist only certain local expressions of the fields that satisfy the WZ conditions. In our specific case, they are the same cocycles that appear at one loop, i.e., the Gauss–Bonnet and square Weyl tensor anomalies and those in Equations (458), (459), (461), (462) and (465). None else; in particular, there are no anomalies whose density $T[g]$ is the fermion or ghost part of the action. The only change at two loops can lie in the renormalization of their coefficients. But, altogether, the result will be again conformal invariant. And this argument can be repeated at higher orders. So, we expect the quantum action to be Weyl invariant. Of course, it remains to examine unitarity, and for this, see the next section and Appendix E.

Finally, let us return to Equation (431). Its validity guarantees conformal invariance. We have seen that expanding in $h_{\mu\nu}$, the 0-th order term vanishes on the LHS and implies that also the RHS must vanish. On the other hand, quantizing the RHS we have argued that, by using WZ terms, it is possible to preserve Weyl invariance, which implies a vanishing quantum RHS. We, therefore, expect that the quantum version of the LHS should also be vanishing. Proceeding as in the RHS, the result should be obvious because the starting operator to be quantized is simply zero. This seems to be confirmed by the Formula (161), because the LHS corresponds to $g^{\mu\nu}\langle\langle R_{\mu\nu}\rangle\rangle - \langle\langle R\rangle\rangle$. However, we interpret this expression; it does not seem to give rise to ambiguities such as those that generate known anomalies. But no doubt a direct check is necessary in terms of explicit calculations. See refs. [101,103,112,113,140–143], for other examples of conformal symmetry restoration.

18. Renormalization and Unitarity—A Lightning Review

Like in any theory also involving gravity in $\mathcal{T}$, one cannot avoid the problem of renormalizability and unitarity. In this paper, we do not intend to tackle these huge problems. We do not have anything original to add to what can be found in the literature. But we would like to present anyhow a quick summary to put our proposed theory in perspective against the backdrop of current and past research.

In $\mathcal{T}$, we have two basic ingredients, the SM and gravity. As for the first, an Abelian and non-Abelian gauge theory coupled to matter, the problem of renormalizability and unitarity has been solved long ago in various stages and by various people [24–26,144]. A detailed proof of renormalization for the GWS model of electroweak interactions can be found in [145], where one can also find the relevant literature. Instead, the problem is still open for gravity. As compared with an enormous amount of literature on the classical aspects of gravity, the one concerning its quantum aspects is, perhaps understandably, rather limited and lacking undisputed conclusions. In fact, it became clear from the very beginning that gravity in the form of the EH action is not renormalizable due to the presence of a dimensionful coupling constant. However, it was realized that this initial obstacle can be circumvented by introducing in the action other terms quadratic in the curvature [146,147]. This renders the theory formally renormalizable but raises another

problem: the gravitational propagator contains a massive component with the wrong overall sign (corresponding to a negative norm state) that renders it a physical ghost.

Such physical ghosts are the thorn in the side of any gravitational theory as long as one cannot exclude that they do not contribute to the internal lines of a Feynman diagram, thus possibly compromising unitarity. We cannot illustrate here all the proposals presented to overcome this difficulty. We choose to illustrate two of them, which are particularly fitting to our case. The first [148,149], studies the renormalization of a gravitational model where two more square curvature terms have been added to the EH action, each with its own coupling constant. The authors choose as variable the metric fluctuation $h_{\mu\nu}$, fix the DeDonder gauge, introduce the FP ghosts and determine the BRST symmetry of the model. Then, they write down the corresponding ST (Slavnov–Taylor) identities and apply the BPHZL (Bogoliubov–Parasiuk–Hepp–Zimmermann–Lowenstein) renormalization scheme. They analyze the double pole in the propagator of $h_{\mu\nu}$, which splits into a massless simple pole and a massive unphysical one, as usual. This is the situation at the tree level. However the authors' claim is that in higher loops the massive component of $h_{\mu\nu}$ does not develop poles, therefore it does not correspond to particles. In this way, the S-matrix unitarity turns out to be violated only at the tree level. One of the authors [150], has pushed the analysis further by coupling the previous model to a scalar field with quartic interaction, arriving at the same conclusion: the negative norm state shows up only at the tree level.

The second proposal is due to Oda and collaborators [122–124,151–153]. In [123] the author focuses on a theory defined by the classical action (439) in d = 4, which, as we noted, is Weyl invariant and is called Weyl invariant scalar-tensor gravity. They fix both the diffeomorphism and the Weyl gauge and work out the corresponding BRST symmetries. Unlike the previous example, the quantization is carried out in the canonical way. The author determines the equal time (anti)commutation relations among all the fields and their conjugates. On this basis, applying a formalism originally developed by Nakanishi [154,155], he proves the existence of a global symmetry the 'choral' symmetry, which is spontaneously broken at the quantum level. Its Nambu-Goldstone bosons are the graviton and the dilaton, which are consequently massless. The author is also able to analyze the physical S-matrix using the method introduced by Kugo and Ojima [156,157] and prove that it is unitary.

The authors of ref. [151] consider a somewhat related conformal invariant model (the third proposal). They deal with the Weyl symmetry as a gauge symmetry and introduce the corresponding gauge field (which we called C_μ above). Simultaneously, they add to the action (439) the kinetic term for C_μ, and three more terms, each with its own independent coupling: a Weyl covariant kinetic term, a quartic coupling for ϕ and a term $\phi^2 \widetilde{R}^2$, where $\widetilde{R}$ is the same as in (373) with $\epsilon = 0$. They fix both gauges in a mutually compatible way and develop a parallel analysis to [123]. They show that the S-matrix in this model is unitary, and its physical spectrum consists of the two components of a massless graviton and the three components of a massive Proca field, which comes from C_μ having absorbed ϕ via the Higgs mechanism.

The other side of these positive results is that they do not seem to be unitary. Once we declare that they are Weyl invariant, renormalization demands to be introduced in the action of all terms with the right dimensions that are Weyl invariant, hence the term (378). It is known that this term is likely to carry in the spectrum a massive ghost. This is precisely what is analyzed in ref. [124], where the term (378) is added to the action (439). In order to apply the canonical quantization the presence of four derivatives requires the introduction of an auxiliary (Stückelberg) vector field, which entails an additional (Stückelberg) symmetry. There are, therefore, three symmetries, and it turns out to be impossible to find mutually compatible (i.e., mutually anticommuting) BRST transformations for all three.

Therefore, the authors choose two BRST operators, which correspond to diffeomorphisms and to a mixture of the other two. Then, they proceed to the canonical quantization to extract the equal time commutators and to analyze the asymptotic states. The particle spectrum is made of a massless graviton and a massive ghost with negative norm, together with zero norm states, which form a quartet. Since all these states satisfy the Kugo–Ojima criterion for physicality, this means that unitarity is explicitly violated.

Let us return now to the $\mathcal{TW}$ theory. We can see that several of its features are contained in the previous examples. Therefore, we can try to apply some of the previous results to it, well conscious, however, that an *ad hoc* careful analysis is still missing. Nevertheless, we believe it sensible to expect that this theory is, at least formally, renormalizable, but with the presence in its particle spectrum of negative norm states, unless we exclude by hand a term like (378), which renders the theory non-unitary (see also the discussion in Appendix E). Squeezed between the almost unreasonable perfection of the SM, which is renormalizable and unitary, and the promised land of superstring theories that lure us with the ultimate goal of UV finiteness, we can only recognize that $\mathcal{TW}$ is far away from both; it looks like a simple, effective field theory, defective from the point of view of UV completeness and unitarity.

To be more precise, looking at the second option above, one would be induced to give up Weyl invariance in favor of renormalizability and unitarity. In this perspective, Weyl invariance would be an accidental symmetry that manifests itself in approximate form only at the classical level. If, instead, we consider the first and third options, we would be led to conclude that Weyl symmetry imposes too strong conditions for them to be compatible with unitarity in $\mathcal{TW}$. From this point of view, therefore, there is no happy news. On the other hand, the contrary would be quite surprising. It is a rather common belief, and there are arguments in support, that a consistent quantum field theory, including quantum gravity, cannot be constructed with a finite number of fields [158–160].

On the other hand, the right attitude is perhaps to consider these two alternatives not as excluding each other but in physical sequence: they would refer to different ranges of energy, the first to lower and the second to higher energy, the lack of unitarity of the second denoting a UV incompleteness, that is signaling missing degrees of freedom in the spectrum of $\mathcal{TW}$. In this regard, a reference to [140] is not out of context. In that paper, the authors argue that in the space of field configurations encircling a conformal field theory, there exists a trajectory of the renormalization group, which is entirely made of conformal theories from the UV to the IR. At first glance, this seems to contradict our previous conclusions. But this may not be the case. The theories along a renormalization group trajectory do not necessarily have always the same form (same fields, same couplings, ...), but may undergo transitions to differet forms with different elementary degrees of freedom.

Leaving this issue unprejudiced, we believe it makes sense to explore the physical implications of $\mathcal{TW}$. After all, it is constructed on the basis of coherence between the SM and gravity, and it is perhaps not unreasonable to hope that the incompleteness that affects it can be disregarded at least in a limited region of energy. Although very conjectural, the following does not seem to us totally unreasonable. See also [161–163].

19. Reduction and Gauge Fixing

The action (423) is a rather general (even though not the most general) chirally symmetric and Weyl invariant action for $\mathcal{TW}$. In the sequel, we somewhat simplify it in order to limit ourselves to a contained, though hopefully still meaningful, discussion. The first simplification is to set $H_+ = H_- = H$. That is, not only the $SU(2)$ gauge bosons are common to the left and right sectors, but also the Higgs field. All the formulas introduced

above for $H_\pm$ are still valid with the simple substitution of symbols. Next, we remark that (423) brings in new fields besides the ones of gravity and SM, i.e., $\widehat{C}_\mu$ and $\widehat{\varphi}$. As we have already remarked, there is no compelling reason to introduce $\widehat{C}_\mu$, thus, as before, in order to somewhat simplify the discussion, we will drop it, i.e., we set $\widehat{C}_\mu = 0, \widehat{C}_{\mu\nu} = 0$ and $\alpha = 1, \epsilon = 1$. The field $\widehat{\varphi}$ is sufficient to implement AE Weyl invariance. The degrees of freedom contained in it are the same as the dofs in $\widehat{\omega}$. There is no mismatch, unlike with gauge fixing in the SM, for instance. Consequently, fixing the gauge $\widehat{\varphi} = 0$ does not require any FP ghosts. Choosing the gauge $\widehat{\varphi} = 0$ produces a mass term for $H = H_\pm$ and $\Phi_\pm$ given by

$$M_h^2 = M_{h\pm}^2 = M_\pm^2 + \frac{\mathfrak{c}_\pm \zeta_{h\pm}}{\kappa_\pm} \qquad \text{and} \qquad M_{\Phi\pm}^2 = m_\pm^2 + \frac{\mathfrak{c}_\pm \zeta_\pm}{\kappa_\pm}, \tag{479}$$

We will do this operation in two steps. First, we consider a partial breaking of the AE Weyl symmetry by setting

$$\varphi_+ = 0 \tag{480}$$

but leaving φ_- free. This will leave $\mathcal{TW}_L$ unchanged and symmetric under the Weyl transformations parametrized by ω_-, while the Weyl symmetry of $\mathcal{TW}_R$ is broken. This operation produces masses only for H and Φ_+,

$$M_h^2 = M_\pm^2 + \frac{\mathfrak{c}_\pm \zeta_{h\pm}}{\kappa_\pm} \qquad \text{and} \qquad M_{\Phi+}^2 = m_+^2 + \frac{\mathfrak{c}_+ \zeta_+}{\kappa_+}, \tag{481}$$

respectively.

From now on the two theories $\mathcal{TW}_L$ and $\mathcal{TW}_R$, split. After the partial Weyl gauge fixing they are not anymore AE conformal invariant, so that we call them $\mathcal{TW}_L$ and $\mathcal{T}_R$, respectively. What is more important, they have two different evolutions.

19.1. The Fate of the Two Halves

The Right-Handed $\mathcal{T}_R$ Model and Its Breaking

We postulate an early (i.e., high energy) breakdown of the chiral symmetry in the right theory, according to the pattern described in Section 14.2 with the scalar field identified with Φ_+[Actually, in order to determine the minimum of the action we should analyze the global potential involving all the scalar fields. But since the Φ_+, Φ_- and H do not have mixed interaction terms and their minima are supposed to differ by several order of magnitude], we think we can analyze them separately.

Setting $\Phi_+ = v_+ + \rho_+$, where ρ_+ is a real field and $v_+ = \sqrt{\frac{2M_+^2}{\lambda_+}}$, a real number, denotes the minimum of the potential, we have

$$D_\mu \Phi_+ D_\mu \Phi_+ \quad = \quad \partial_\mu (v_+ + \rho_+) \partial^\mu (v_+ + \rho_+) \tag{482}$$

The quadratic terms in $\frac{1}{2} D_\mu \Phi D_\mu \Phi - \mathsf{V}$ (V is the potential) are

$$\frac{1}{2} \partial_\mu \rho_+ \partial^\mu \rho_+ - \frac{1}{2} \lambda v_+^2 \rho_+^2 \tag{483}$$

The result is a massive scalar ρ_+.

Finally, the Yukawa couplings introduced in Section 13.7, see Equation (345), $\sqrt{g_+} \frac{y_+}{2} \overline{\psi}_R' \Phi_+ (\psi_R')^c + h.c.$, when $\langle \Phi_+ \rangle = v_+$, produces a mass term proportional to v_+, see Equation (354). Here ψ_R' is a concise notation for the multiplet (296). As we have explained in Section 13.7, see Equation (354), ψ_R' are fermion that satisfy the Majorana

equation, see (355). Since all these masses, as well as the mass of ρ_+, are proportional to v_+, they can be made as large as we wish. This allows us to suppose that the energy at which this Higgs mechanism takes place can be very high, between the Planck and the Grand Unification scales. We can call it the *primitive spontaneous chiral symmetry breaking scale*. We recall that the just illustrated breaking does not affect the other gauge symmetries. As for the symmetry under the AE diffeomorphisms $\widehat{\xi}^\mu$, it is broken by the Yukawa terms and by the gauge fixing, but a symmetry remains with respect to the diffeomorphisms parametrized by ξ_-^μ on the left and ξ_+^μ on the right. The $SU(2)$ gauge fields W_μ (valued in the Lie algebra $\mathfrak{su}(2)$) are in common to both theories and, up to this stage, untouched by the gauge fixing (the $SU(2)$ gauge symmetry survives the previous breaking). The actions $S_{aeg}^{(+)}$ and $S_{aeg}^{(-)}$ are certainly invariant under the diffeomorphisms parametrized by ξ_+^μ and ξ_-^μ, respectively, *as long as we consider them separate theories*; but, since W_μ belong to both, only the invariance under the ordinary diffeomorphisms ξ^μ survives.

The AE Weyl symmetry is also limited to the transformations parametrized by ω, it is broken by the gauge fixing in $\mathcal{TW}_R$, but a Weyl symmetry under the ω transformations survives in the $\mathcal{TW}_L$, whose fields remain massless.

At this stage, all the gauge bosons are massless. But there is room for a breakdown of the electroweak symmetry $SU(2) \times U(1)_L$, which generates masses for the three $SU(2)$ gauge bosons and for H. We recall that the masses of the gauge bosons and the Higgs field are, of course, the same on both sides $M_W^2 = \mathsf{g}_+^2 v_+^2 = \mathsf{g}_-^2 v_-^2$ ($\mathsf{g}_\pm$ are the $SU(2)$ gauge couplings in the two sides) and $M_h^2 = M_\pm^2 + \frac{\mathfrak{c}_\pm \zeta_{h\pm}}{\kappa_\pm}$. In this process, both the $SU(3)$ and the $U(1)_R$ symmetries remain unbroken.

Let us summarize the situation. Before the chiral symmetry breaking, we have a theory, $\mathcal{TW}$, split into two halves, with a different particle spectrum and each with its own gravity (mediated by $g_{+\mu\nu}$ and $g_{-\mu\nu}$, respectively) but having in common the $SU(2)$ vector bosons and H scalars. They communicate through the exchange of these particles: the relative terms in the action that produces this interaction are the fermion-fermion-boson vertices on the two sides and the cubic and quartic self-interaction terms for the vector bosons, as well as the potential interaction terms involving the H scalars. Other direct interactions between the two halves are not permitted by the ubiquitous presence of the projectors $P_\pm$. Of course, there remains the possibility of indirect interactions: for instance, the metric $g_{+\mu\nu}$ can interact with an $SU(2)$ vector boson, the latter with another $SU(2)$ vector boson, and this with $g_{-\mu\nu}$. These and similar interactions are likely to be extremely weak, except at extremely high energies, if grand unification of forces is allowed in this theory.

If this is so, then we may suppose that the right and left-handed theories evolve almost independently because the links between them are represented by the exchange of weak interacting $SU(2)$ bosons and scalars. After the spontaneous symmetry breaking, the right-handed fermions become very massive, and the interaction with the left-handed model is even less important. Needless to say, an analogy with what we know about SM does not make sense under these different boundary conditions, and an ab initio analysis is necessary for the right-handed model. Perhaps, however, some general features can be guessed; for instance, since the right-handed quarks and leptons are very heavy, any nucleation is unlikely, leaving us with a gas of these massive particles. We could say that while $\mathcal{T}_R$ becomes almost 'fossil' theory, $\mathcal{TW}_L$ is in full evolution, except for the exchange of weak interacting $SU(2)$ bosons, which can be treated as a perturbation. This 'misconceived' $\mathcal{T}_R$ theory, however, has many properties (heavy particles, weak interaction with luminous matter, coupled to its own gravity) that can help describe the dark matter. Refs.: [164–166]

19.2. The Left-Handed $\mathcal{TW}_L$ Model and the SM

After the spontaneous chiral symmetry breaking, the left side theory $\mathcal{TW}_L$ almost decouples from the right side one and evolves almost autonomously. Its particle spectrum is that of the MSM plus a right-handed neutrino and a massive Φ_- scalar with a quartic potential, coupled to a gravity described by the $g_{-\mu\nu}$ metric. We recall that it is described by the action

$$S_{ch-sym}^{(c-)} = S_f^{(-)} + S_g^{(-)} + S_{aeg}^{(-)} + (S_{YdL} + h.c.) + S_{EH+aes+aed+C}^{(c-)} \tag{484}$$

It is invariant under diffeomorphisms spanned by ξ_-^μ and under $SU(3)_L \times SU(2) \times U(1)_L$ gauge transformations. It is also invariant under Weyl transformations spanned by ω_-. As long as we can treat the interaction with the right-handed half as a disturbance, it would seem that we cannot repeat straightaway what we said for the entire $\mathcal{TW}$ model, for $\mathcal{TW}_L$ in isolation has an overall nontrivial odd parity trace anomaly. However remember that this anomaly is a topological fact, it is related to the family's index theorem. From the theory and recent experiments of entangled systems we learn that topological features may survive separation. It is, therefore, likely that the cancelation of these anomalies in $\mathcal{TW}$ is not suppressed by the splitting into $\mathcal{TW}_+$ and $\mathcal{TW}_-$ due to the chiral breaking. We take it as a plausible hypothesis.

At this stage, we can fix the gauge $\varphi_- = 0$, which exhibits the SM coupled to EH gravity carried by the metric $g_{-\mu\nu}$. In the course of this evolution, it is to be expected that at a certain stage, the gravity quanta become irrelevant and the gravitational field starts behaving entirely classically. From that stage on the theory is a renormalizable one on a classical gravitational background (odd trace anomalies have become irrelevant). After a while (in the cosmological evolution), the scale of electroweak breaking is reached, the $SU(2) \times U(1)_L$ symmetry breaks down to the $U(1)$ of electromagnetism. A story we already know, apart from the fate of the real Φ_- scalar field, is still to be explored. Needless to say, it is a natural candidate for the inflaton field. To allow such an interpretation, however, we have to arrange the parameters in such a way that $M_{\Phi_-}^2$ is negative, see Equation (479), and in combination with λ_-, this satisfies the conditions of slow-roll inflation.

What has been said so far in this section is a conjectural, very qualitative picture of the evolution of the $\mathcal{TW}$ theory. This paper intends to highlight the role of anomalies in the bottom–up construction of a theory that enlarges the SM to include gravity. Now, starting from the above results and conjectures, explicit calculations and quantitative checks are necessary to confirm or refute them. An entirely new chapter opens up. But, perhaps a comment is necessary from the start concerning the presence in $\mathcal{TW}$ of two metrics. We have already remarked that our entire construction works even if we consider only a unique metric $g_{+\mu\nu} = g_{-\mu\nu}$. But we have maintained throughout the more general setting of two distinct metrics. In this more general case it would seem that we have two ways of measuring distances and times. In a sense, this is true, but not for us. We, as observers, see only one metric, the other is seen only by a right-handed observer, if it can exist. The interaction between the two metrics is extremely feeble, although one cannot exclude that it may lead to observable consequences.

20. Conclusions

In the construction of the SM, chiral gauge anomalies have played a prominent role. In a theory constructed with Weyl fermions, the absence of consistent chiral anomalies was one of the basic principles, and looking at how it has been realized, one cannot but feel admiration for its creators. But once we immerse this jewel in a background of gravity, it seems that the fortune that accompanied them in their endeavor has abandoned us. New

anomalies appear, and what we prove in this paper is that it is impossible to get rid of them with the spectrum of fields offered by the SM. We have to be more specific. The anomalies we are talking about, the dangerous ones, which we have named type O (O stands for obstruction), are not only the traditional consistent chiral gauge anomalies, which vanish in the MSM, but also those generated by the diffeomorphisms and the mixed ones, which fortunately also vanish in the MSM. But among the type O anomalies, there is another kind of anomalies, which have apparently been disregarded in the literature. They are the chiral trace anomalies, which appear in general in models of Weyl fermions coupled to a metric or a gauge potential. Like all the type O anomalies, they signal the fundamental drawback of these models; the corresponding Weyl propagators are missing. This is due to a topological obstruction (a non-perturbative fact) guaranteed by the family's index theorem.

Once the appropriate anomaly hygiene is applied to the SM and its extensions, which are minimally coupled to a background gravitational field, we found out that none of them are free of these anomalies. In fact, the more 'resistant' ones are the chiral trace anomalies induced by the $SU(2)$ gauge fields of the SM. These are the results of the first part of our paper. In the second part, we have proposed a scheme for overcoming these difficulties. We have noticed first that a left–right symmetric model is fit for taming this problem. If it is not absolutely the only way, it is certainly the simplest and most elegant. Therefore, we proposed, as an example, an L-R model, which is free of all type O anomalies and can be, in fact, enlarged to an entire family of similar models; this is the reason why we prefer to call it a scheme. This model has a left–right symmetry, which we named primitive chiral symmetry. It almost splits into two models, one with left-handed fermions, the other with right-handed ones, each of which almost a copy of an enlarged MSM. By almost, we mean that the $SU(2)$ gauge potentials have commonalities with both of them. Each of them is coupled to a different gravity theory, which is characterized by its own metric. At this stage, we came across a dilemma: to be or not to be conformal.

As it is well known by now, it is rather simple to make a local field theory Weyl invariant by simply adding a new field, a dilaton, to the spectrum. Then, the problem is twofold: Is this enlargement required by physics? Is it compatible with quantization? We do not have definite answers to either question. Tentatively, we are oriented to think that it would be quite natural (and even experimentally supported by the Bjorken scaling limit) that at very high energies, such as those of the very early universe, the physics would be insensitive to a dimensionful scale. This is what an approach, à la Wilson, would suggest, if we are allowed to be inspired by the analogy with low energy statistical model: the high energy limit of the universe regressing toward the beginning should be a UV fixed point, most likely a conformal fixed point. On the other hand, it should be in accord with quantization. Here, we have found a real difficulty: while it is possible to figure out conformal models that are similar to ours and are renormalizable, there are some indications that unitarity is not respected, which suggests a likely incompleteness of the spectrum of fields.

Notwithstanding these difficulties, we believe that our scheme, constructed on the basis of mathematical consistency, should be able to describe at least a limited stretch of the evolution of the universe. We have envisaged an early breaking of the primitive chiral symmetry, which should have split the history of the two L and R models. The latter describes the physics of a gas of very massive quarks and leptons; the former evolves toward a model of ordinary classical gravity coupled to the ordinary SM; and the two keep in touch, so to speak, via the weak interacting $SU(2)$ gauge fields.

Several aspects of the above scheme are still too crudely described or incomplete. For instance, we do not have an explanation for the mechanism that can trigger the early chiral symmetry breaking, but from this perspective, we are in good company with many

phenomena, which we can describe but not explain. In this paper, we have not considered the consequences of our model involving the cosmological constants as well as other cosmological aspects. Neither have we analyzed the evolution of the right model. Needless to say, many explicit checks and calculations are needed. At the same time, though, this scheme is not only mathematically consistent (it is anomaly-free), it may describe the 'strange' chiral asymmetry of the SM by exhibiting it as the existence of the other face, the dark matter world, and leads naturally at low enough energy to the SM, which is coupled to classical gravity equipped with inflation. Refs.: [2,167–172].

Appendix A. Gamma Matrices in Flat Spacetime

In the absence of gravity, we refer to theories on a flat Minkowski spacetime with flat metric $\eta_{\mu\nu}$ with mostly $(-)$ signature. The gamma matrices satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ and

$$\gamma_\mu^\dagger = \gamma_0 \gamma_\mu \gamma_0.$$

The charge conjugation operator C is defined to satisfy

$$\gamma_\mu^T = -C^{-1}\gamma_\mu C, \qquad CC^* = -1, \qquad CC^\dagger = 1. \tag{A1}$$

The chiral matrix $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ has the properties

$$\gamma_5^\dagger = \gamma_5, \qquad (\gamma_5)^2 = 1, \qquad C^{-1}\gamma_5 C = \gamma_5^T.$$

and

$$\mathrm{tr}(\gamma_5 \gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\rho) = -4i\varepsilon_{\mu\nu\lambda\rho} \tag{A2}$$

The chiral projectors

$$P_L \equiv P_- = \frac{1-\gamma_5}{2}, \qquad P_R \equiv P_+ = \frac{1+\gamma_5}{2}, \qquad \text{with} \qquad P_+ + P_- = 1, \qquad P_\pm^2 = P_\pm. \tag{A3}$$

In terms of a Dirac fermion ψ Weyl fermions are defined by $\psi_{L,R} = P_{L,R}\psi$.

The generators of the Lorentz group in the 'anti-hermitean' version are $\Sigma_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu]$.

Under parity, a Dirac spinor transforms as

$$\mathcal{P}\psi(\vec{x}, t)\mathcal{P}^{-1} = e^{i\beta_p}\gamma_0\psi(-\vec{x}, t) \tag{A4}$$

Under charge conjugation, it transforms as

$$\mathcal{C}\psi(\vec{x}, t)\mathcal{C}^{-1} = e^{i\beta_c}\gamma_0 C\psi^*(\vec{x}, t) \tag{A5}$$

and for a complex scalar field, we have

$$\mathcal{P}\phi(\vec{x},t)\mathcal{P}^{-1} = e^{i\alpha_p}\phi(-\vec{x},t), \qquad \mathcal{C}\phi(\vec{x},t)\mathcal{C}^{-1} = e^{i\alpha_c}\phi^\dagger(\vec{x},t) \tag{A6}$$

where $\alpha_p, \alpha_c, \beta_p, \beta_c$ are arbitrary phases, which we usually set equal to zero.

A vector field V_μ transform as

$$\mathcal{P}V_\mu(\vec{x},t)\mathcal{P}^{-1} = V^\mu(-\vec{x},t), \qquad \mathcal{C}V_\mu(\vec{x},t)\mathcal{C}^{-1} = -V_\mu(\vec{x},t) \tag{A7}$$

and for a pseudovector field A_μ

$$\mathcal{P}A_\mu(\vec{x},t)\mathcal{P}^{-1} = -A^\mu(-\vec{x},t), \qquad \mathcal{C}A_\mu(\vec{x},t)\mathcal{C}^{-1} = A_\mu(\vec{x},t) \tag{A8}$$

A metric $g_{\mu\nu}$ transforms as

$$\mathcal{P}g_{\mu\nu}(\vec{x},t)\mathcal{P}^{-1} = g^{\mu\nu}(-\vec{x},t), \qquad \mathcal{C}g_{\mu\nu}(\vec{x},t)\mathcal{C}^{-1} = g_{\mu\nu}(\vec{x},t) \tag{A9}$$

while

$$\mathcal{P}f_{\mu\nu}(\vec{x},t)\mathcal{P}^{-1} = -f^{\mu\nu}(-\vec{x},t), \qquad \mathcal{C}f_{\mu\nu}(\vec{x},t)\mathcal{C}^{-1} = -f_{\mu\nu}(\vec{x},t) \tag{A10}$$

Appendix B. Euclidean Field Theory Calculations

Most classical and quantum field theories, such a the SM, are formulated in a Minkowski metric (both flat and non-flat). However, true calculations with Minkowski metrics are often not available and in order to extract concrete results, theoretical physicists have been obliged to sell their souls to the Euclidean devil. This is done via Wick rotations. At the end of the calculations, the results are turned Minkowski via inverse Wick rotations.

A Wick rotation means: $x^0 \to \tilde{x}^0 = ix^0, k^0 \to \tilde{k}^0 = ik^0$ and $\gamma^0 \to \tilde{\gamma}^0 = i\gamma^0$, while x^i, k_i, γ^i remain unchanged. A tilde on top of an object represents its Wick-rotated correspondent. We have, in particular,

$$\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = 2\tilde{\eta}^{\mu\nu} \equiv -2\delta^{\mu\nu} \tag{A11}$$

and

$$\tilde{\gamma}_\mu^\dagger = -\tilde{\gamma}_\mu \tag{A12}$$

The chiral matrix γ_5 changes sign under a Wick rotation, $\tilde{\gamma}_5 = -\gamma_5$.

In this article, Wick rotations are used for perturbative and non-perturbative calculations (via the SDW method) in the intermediate steps. A particular caution has to be employed with Weyl fermions. It is not possible to write down a full Euclidean action for Weyl fermions using the Osterwalder-Schrader prescription [173–177]. The attempt to Wick rotate a Lagrangian like $i\overline{\psi}\gamma^\mu\partial_\mu\psi$, where ψ is Dirac, ends up in a doubling of fermionic degrees of freedom: $\overline{\psi}$ can only be a spinor independent of ψ. More elaborated approaches where a Wick rotation of a Weyl spinor ψ is accompanied by a rotation of the same cannot comply with the hermiticity, see [5]. In summary, it is nonsense to make a Euclidean calculation for Weyl fermions starting from an action formulated in terms of Euclidean Weyl fermions proper, because this would change the nature of the problem. The best we can do is use the Wick rotation to make sense of the Feynman and path integrals wherever they are met in the course of the calculations. In perturbative calculations we set up the Feynman diagram calculations in Minkowski terms and switch to Euclidean when evaluating the Feynman integrals. In the SDW method, we Wick-rotate the square Dirac

operator to transform it into a self-adjoint operator and make sense of the *s*-integrals. In both cases, we use Wick rotations only in such junctures, without any attempt to identify the Euclidean field theories underlying them, which is impossible.

Finally, in the third approach we use, which is based on the family's index theorem, we start at the very beginning from the Euclidean formulation of the Dirac operator because a proof of that theorem in a Lorentzian metric is not available. Anyhow, this does not meet the just mentioned difficulties because what we analyzed is the invertibility of the Dirac operator in isolation, without reference to a particular action it is inserted in. We assume that the results of the theorem are true also for the Minkowski version.

Appendix C. Gauge Fixing and Ghosts

The gauge fixing and ghost action for a non-Abelian YM theory coupled to a metric in the Lorenz gauge is

$$S_{g.f.}^{c,\bar{c}} = \int d^4 x \sqrt{g} \left(b^a \widetilde{\nabla}_\mu V_\nu^a g^{\mu\nu} + \frac{\alpha}{2} b^a b^a - g^{\mu\nu} \bar{c}^a \widetilde{\nabla}_\mu (D_\nu c)^a \right) \tag{A13}$$

summed over a, where $\widetilde{\nabla}$ denotes the metric covariant derivative with Christoffel symbols $\widetilde{\Gamma}$, (368) with $\alpha = 1$, and $D_\mu c = \partial_\mu c + [V_\mu, c]$. This action is invariant under diffeomorphisms and also under Weyl transformations, provided we assign Weyl weight -2 to b^a and $\bar{c}^a$, and 0 to c^a. Equation (A13) is, of course, invariant under the gauge BRST transformations (31) and $\delta_c \bar{c}^a = b^a$, $\delta_c b^a = 0$.

A full gauge fixing and ghost action for diffeomorphisms in the DeDonder gauge is

$$S_{g.f.}^{(\xi,\bar{\xi})} = \int d^4 x \sqrt{g} \left(b_\lambda \widetilde{\Gamma}_{\mu\nu}^\lambda g^{\mu\nu} + \frac{\alpha}{2} b_\lambda g^{\lambda\rho} b_\rho e^{2\varphi} - \bar{\xi}_\lambda g^{\mu\nu} \partial_\mu \partial_\nu \xi^\lambda \right) \tag{A14}$$

which is BRST invariant. For the $g_{\mu\nu}$ and ξ^μ BRST transformations see (123) and (127), for $\bar{\xi}_\lambda$: $\delta_\xi \bar{\xi}_\lambda = -b_\lambda + \xi \cdot \partial \bar{\xi}_\lambda + \partial_\lambda \xi^\rho \bar{\xi}_\rho$, while b_λ transforms as an ordinary covariant vector. Equation (A14) is also Weyl invariant provided we assign Weyl weight -2 to b_λ and $\bar{\xi}_\lambda$ and weight 0 to ξ^μ.

Appendix D. Consistency of Scalar Cocycles

Let us consider, for instance, the cocycle (462). To understand its origin from the second term in (460), which comes from an explicit triangle diagram calculation, and its relation with other diagrams, we must abandon the identification of Φ as the fluctuating field and write instead $\Phi = \Phi_0 + \phi$, where Φ_0 is a classical constant background and ϕ is the fluctuating field. We split the cocycle (462) in a series expansion based on the number i of fluctuating (infinitesimal) fields:

$$\Delta_\omega = \sum_{i=0}^{\infty} \Delta_\omega^{(i)} \tag{A15}$$

where

$$\Delta_\omega^{(0)} = 0, \qquad \Delta_\omega^{(1)} = \frac{1}{3} \int d^4 x \omega \left(\partial_\mu \partial_\nu h^{\mu\nu} - \Box h_\lambda^\lambda \right) \Phi_0^2,$$

$$\Delta_\omega^{(2)} = \int d^4 x \omega \left(\eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{2}{3} \partial_\mu \partial_\nu h^{\mu\nu} \Phi_0 \phi - \frac{2}{3} \Box h_\lambda^\lambda \Phi_0 \phi \right), \quad \text{etc.} \tag{A16}$$

In parallel, we must change the Weyl transformation in accordance with this series expansion

$$\delta_\omega^{(0)} h_{\mu\nu} = \omega \eta_{\mu\nu}, \quad \delta_\omega^{(1)} h_{\mu\nu} = 2\omega h_{\mu\nu}, \quad \delta_\omega^{(2)} h_{\mu\nu} = 0, \dots \tag{A17}$$

$$\delta_\omega^{(0)} \phi = -\omega \Phi_0, \quad \delta_\omega^{(1)} \phi = -\omega \phi, \quad \delta_\omega^{(2)} \phi = 0, \quad \dots \tag{A18}$$

and expand the WZ consistency condition $\delta_\omega \Delta_\omega = 0$. The first two non-trivial terms are

$$\delta_\omega^{(0)} \Delta_\omega^{(1)} + \delta_\omega^{(1)} \Delta_\omega^{(0)} = 0$$
$$\delta_\omega^{(1)} \Delta_\omega^{(1)} + \delta_\omega^{(0)} \Delta_\omega^{(2)} = 0 \tag{A19}$$

It is not hard to prove that they hold true. The series goes on with

$$\delta_\omega^{(1)} \Delta_\omega^{(2)} + \delta_\omega^{(0)} \Delta_\omega^{(3)} = 0, \quad \dots \tag{A20}$$

but in this case we must introduce new approximation terms for Δ_ω, which we dispense with here. It is interesting to notice that, apart from the first terms of $\Delta_\omega^{(2)}$, all the other terms do not come from the triangle diagram calculation mentioned before Equation (460). They are, in fact, higher order terms, which can be obtained from the fourth point diagrams where the fermion e.m. tensor is attached to three vertices ($V_{ffh}, V_{ff\phi}, V_{ff\Phi_0}$) by means of four fermion propagators. This is another example of a trace calculation, which is undecidable at the lowest perturbative order.

Appendix E. Dilaton and Dimensions

The field φ (the dilaton), introduced in Section 14, is not the only possible choice. We could have set $\varphi = \ell\phi$, where ℓ is a constant with the dimension of a length and ϕ is a dimension one field. In this case ϕ becomes an ordinary real scalar field with a canonical kinetic term (after normalization). The problem with this choice is its identification with the field σ in the WZ terms: $\sigma = \ell\phi$. It transforms the WZ term into a five-dimensional operator expression multiplied by ℓ. In the framework of renormalization theory such a term renders the theory formally non-renormalizable.

A similar problem arises also with the identification $\sigma = \log(\ell\chi)$, as already pointed out in Section Application of WZ Terms to $\mathcal{TW}$. The choice $\sigma = \varphi$ seems the most logical, but it is itself not exempt from problems in subsequent developments. Once we have introduced a dimensionless field φ in a theory, we must, upon renormalization, allow for the appearance of all the terms with the right dimensions and symmetries, which also involve φ. In our case, we can construct, for instance, the expression

$$Q = \Box\varphi - \partial_\mu\varphi\partial^\mu\varphi + \frac{1}{6}R \tag{A21}$$

Under a Weyl transformation it transforms as $Q \to e^{-2\omega}Q$. Therefore

$$\int d^4x \sqrt{g} Q^2 \tag{A22}$$

is Weyl invariant, has the right dimensions and, thus, it can arise in the renormalization process. Unfortunately, it contains a quartic kinetic term, which, when added to the quadratic kinetic term, splits the propagator into a massless plus a massive one, the latter representing either a tachyon or a ghost, thus possibly affecting unitarity.

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