



Thermodynamics of a sufficient small singly spinning Kerr-AdS black hole

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Abstract

In this paper, we will analyze the thermodynamics of a small singly spinning Kerr-AdS black hole. As the black hole will be sufficient small, its temperature will be large and so we can not neglect the effects of thermal fluctuations. We will demonstrate that these thermal fluctuations correct the entropy of singly spinning Kerr-AdS black hole by a logarithmic correction term. We will analyze the implications of the logarithmic correction on other thermodynamic properties of this black hole, and analyze the stability of such a black hole. We will observe that this form of correction becomes important when the size of the black hole is sufficient small. We will also analyze the effect of these thermal fluctuations on the critical phenomena for such a black hole.

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1. Introduction

Black holes are maximum entropy objects, and so a black hole has more entropy than any other object with the same volume [1–5]. It is important to associate a maximum entropy with a black

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hole as finite entropy objects can cross the horizon of a black hole. This would spontaneously reduce the entropy of the universe, and thus the second law of thermodynamics can get violated if a maximum entropy is not associated with a black hole. This maximum entropy associated with a black hole scales with the area of the horizon. In fact, the entropy of the black hole can be expressed in terms of the area of the horizon as $s = A/4$. The observation that the maximum entropy of a region of space scales with its area has motivated the holographic principle [6,7]. This principle states that the number of degrees of freedom in any region of space is equal to the number of degrees of freedom on the boundary of that region.

The holographic principle has found various applications in many different areas of physics. However, it is expected that the holographic principle will get modified near Planck scale [8,9]. This is because the area–entropy law of black holes gets modified due to the quantum gravitational effects. In fact, almost all approaches to quantum gravity predict the same functional form for these quantum corrections to the area–entropy relation i.e., the area–entropy law gets corrected by a logarithmic correction term. However, the coefficient of this logarithmic correction term depends on the specific approach chosen, and is different for different approaches to quantum gravity. Such logarithmic correction term has been obtained using the non-perturbative quantum general relativity [10]. This was done by using the relation between the density of states of a black hole and the conformal blocks of a well defined conformal field theory. The Cardy formula has been used for obtaining such corrections terms to the area–entropy relation [11]. The correction for a BTZ black hole has been calculated, and it has been demonstrated that these are logarithmic corrections [11].

The effect of matter fields surrounding a black hole has been studied [12–14]. This analysis has also been used to obtain corrections to the area–entropy relation, and it was observed that this correction term is logarithmic. The string theoretical corrections to the entropy of a black hole have been calculated, and it has been found that the entropy of a black hole gets corrected by logarithmic term generated from string theoretical effects [15–18]. The logarithmic correction to the entropy of a dilatonic black hole has been obtained [19]. The partition function of a black hole has been used to obtain the logarithmic correction to the area–entropy law of a black hole [20]. The corrections obtained using the generalized uncertainty principle are also logarithmic [21,22]. The thermodynamics and statistics of Gödel black hole with logarithmic correction has been studied [23]. Furthermore, $P - V$ criticality of dyonic charged AdS black hole with a logarithmic correction has been also been analyzed [24].

It may be noted that in the Jacobson formalism, the Einstein's equation can be derived from the first law of thermodynamics [25,26]. In this formalism, it is required that the Clausius relation holds for all the local Rindler causal horizons through each space–time point, and this gives rise to the Einstein's equation. As there exists a relation between the geometry and thermodynamics of a black hole, we expect that thermal fluctuations in the thermodynamics will give rise to the quantum fluctuations in the metric. In fact, it has been demonstrated that the entropy of the black hole gets corrected by logarithmic terms due to these thermal fluctuations [27,28]. As such corrections are expected to occur from most approaches to quantum gravity, we will study the effect of such term on our system. Furthermore, as the coefficient of such terms depends on the approach to the quantum gravity, we will not fix the coefficient of such correction term, and hence see the effect of such correction term on the system, which would be generated from different approaches to quantum gravity.

Already, the effect of thermal fluctuations on the thermodynamics quantities of an AdS charged black hole has been analyzed [29]. It was demonstrated that the thermal fluctuations decrease the certain thermodynamics potentials associated with the system, for example, the free

energy reduced due to these fluctuations. The modification to the thermodynamics of a black Saturn because of the thermal fluctuations has also been studied [30], and it was observed that logarithmic corrections do not affect stability of the black Saturn. The logarithmic corrections to entropy of a modified Hayward black hole has been analyzed [31], and it was observed that the value of the pressure and internal energy reduced due to such corrections. It is also demonstrated that the first law of thermodynamics is satisfied for the this black hole in the presence of thermal fluctuations. The correction to the thermodynamics of a charged dilatonic black Saturn has been studied [32], and it has been demonstrated that for this system the corrections obtained from a conformal field theory are the same as the corrections obtained from the fluctuations in the energy. It may be noted that such corrections where studied by analyzing the thermal fluctuations very close to the equilibrium, and this approximation is expected to breakdown near the Planck scale. This is because near the Planck scale, the thermal fluctuations will become so large that the equilibrium thermodynamics cannot be used to describe the system. However, as long as the system remains close to the equilibrium, it is possible to analyze the effect of thermal fluctuations as a perturbation around the equilibrium state [33,34].

In this paper, we will analyze the effects of such thermal fluctuation on a higher dimensional singly spinning Kerr-AdS black hole [35]. It may be noted that the thermodynamics of such a black hole has already been studied [5,36], and we shall analyze the corrections to the thermodynamics by thermal fluctuations. We will show that thermal fluctuations have an important effect on the thermodynamics and critical points of singly spinning Kerr-AdS black holes in higher dimensions. We will study the effect of logarithmic correction on the partition function, and also investigate the special case of an ordinary Kerr-AdS black hole in four dimensions.

2. Singly spinning Kerr-AdS black hole

In this section, we will discuss thermodynamic properties of a higher dimensional singly spinning Kerr-AdS black hole. The metric for a d -dimensional Kerr-AdS black hole in Boyer–Lindquist coordinates can be written as [37],

$$\begin{aligned}
 ds^2 = & -W \left(1 + \frac{r^2}{l^2} \right) d\tau^2 + \frac{2m}{U} \left(W d\tau - \sum_{i=1}^N \frac{a_i \mu_i^2 d\phi_i}{\Xi_i} \right)^2 \\
 & + \sum_{i=1}^N \frac{r^2 + a_i^2}{\Xi_i} \mu_i^2 d\phi_i^2 + \frac{U dr^2}{\mathcal{F} - 2m} + \sum_{i=1}^{N+\epsilon} \frac{r^2 + a_i^2}{\Xi_i} d\mu_i^2 \\
 & - \frac{1}{W(r^2 + l^2)} \left(\sum_{i=1}^{N+\epsilon} \frac{r^2 + a_i^2}{\Xi_i} \mu_i d\mu_i \right)^2,
 \end{aligned} \tag{1}$$

where

$$\begin{aligned}
 W &= \sum_{i=1}^{N+\epsilon} \frac{\mu_i^2}{\Xi_i}, \\
 U &= r^\epsilon \sum_{i=1}^{N+\epsilon} \frac{\mu_i^2}{r^2 + a_i^2} \prod_j (r^2 + a_j^2),
 \end{aligned}$$

$$\mathcal{F} = r^{\varepsilon-2} \left(1 + \frac{r^2}{l^2} \right) \prod_j^N (r^2 + a_j^2),$$

$$\Xi_i = 1 - \frac{a_i^2}{l^2}. \tag{2}$$

Here, m and a_i are mass and rotation parameters, respectively. The coordinates μ_i satisfy the following constraint

$$\sum_{i=1}^{N+\varepsilon} \mu_i^2 = 1, \tag{3}$$

where $\varepsilon = 0$ for odd d or $\varepsilon = 1$ for even d . In the case of $d = 4$, the above space–time reduces to the four-dimensional Kerr-AdS metric. Thermodynamics of this system has already been studied [5]. The mass of this black hole (as well as enthalpy) is given by,

$$M = \frac{m\omega_{d-2}}{4\pi(\prod_j \Xi_j)} \left(\sum_{i=1}^N \frac{1}{\Xi_i} - \frac{1-\varepsilon}{2} \right), \tag{4}$$

where ω_{d-2} is the volume of the unit $(d - 2)$ -sphere which is given by,

$$\omega_{d-2} = \frac{2\pi^{\left(\frac{d-1}{2}\right)}}{\Gamma\left(\frac{d-1}{2}\right)}. \tag{5}$$

The angular momenta is given by,

$$J_i = \frac{a_i m \omega_{d-2}}{4\pi \Xi_i (\prod_j \Xi_j)}. \tag{6}$$

The angular velocities of the horizon are given by,

$$\Omega_i = \frac{a_i (r_+^2 + l^2)}{l^2 (r_+^2 + a_i^2)}. \tag{7}$$

Furthermore, the temperature T can be expressed as

$$T = \frac{1}{2\pi} \left[r_+ \left(1 + \frac{r_+^2}{l^2} \right) \sum_{i=1}^N \frac{1}{r_+^2 + a_i^2} - \frac{1}{r_+} \left(\frac{1}{2} - \frac{r_+^2}{2l^2} \right)^\varepsilon \right]. \tag{8}$$

The entropy of this black hole is given by $s = \frac{A}{4}$, where

$$A = \frac{\omega_{d-2}}{r_+^{1-\varepsilon}} \prod_{i=1}^N \frac{r_+^2 + a_i^2}{\Xi_i}. \tag{9}$$

Here, the horizon radius r_+ is the largest root of $\mathcal{F} - 2m = 0$. The thermodynamic volume is given by,

$$V = \frac{Ar_+}{d-1} + \frac{8\pi}{(d-1)(d-2)} \sum_i a_i J_i. \tag{10}$$

In this paper, we will study the logarithmic corrections to the thermodynamics of singly spinning Kerr-AdS black hole. A singly spinning Kerr-AdS black holes can be described using one

non-zero rotation parameter $a_1 = a$ (while other a_i are zero), and so the metric (1) takes the following form,

$$\begin{aligned}
 ds^2 = & -\frac{\Delta}{\rho^2} \left(dt - \frac{a}{\Xi} \sin^2 \theta d\varphi \right)^2 + \frac{\rho^2}{\Delta} dr^2 + \frac{\rho^2}{\Sigma} d\theta^2 \\
 & + \frac{\Sigma \sin^2 \theta}{\rho^2} \left[a dt - \frac{r^2 + a^2}{\Xi} d\varphi \right]^2 + r^2 \cos^2 \theta d\Omega_{d-4}^2,
 \end{aligned} \tag{11}$$

where

$$\begin{aligned}
 \Delta &= (r^2 + a^2) \left(1 + \frac{r^2}{l^2} \right) - 2mr^{5-d}, \\
 \Sigma &= 1 - \frac{a^2}{l^2} \cos^2 \theta, \\
 \rho^2 &= r^2 + a^2 \cos^2 \theta, \\
 \Xi_i &= 1 - \frac{a^2}{l^2}.
 \end{aligned} \tag{12}$$

Thermodynamics quantities associated with this metric have been studied [38]. The temperature and entropy of this black hole, are given by

$$T = \frac{1}{2\pi} \left[r_+ \left(1 + \frac{r_+^2}{l^2} \right) \left(\frac{1}{r_+^2 + a^2} + \frac{d-3}{2r_+^2} \right) - \frac{1}{r_+} \right], \tag{13}$$

and

$$s = \frac{\omega_{d-2}}{4} \frac{(r_+^2 + a^2) r_+^{d-4}}{\Xi_i}, \tag{14}$$

where r_+ is the largest positive real root of $\Delta = 0$, which has been obtained using from the first equation in (12).

Plots of the Fig. 1 show that there is at least one positive root. There are special choices of (l, m, a) parameters such as $(1.3, 1, 0.5)$, $(1.5, 1.2, 0.8)$, $(1.3, 1.2, 0.7)$, ... where $r_+ = 1$ for all d . We see small differences between Figs. 1(a), (b) and (c) where $r_+ = 1$, while Fig. 1(d) has separated r_+ for different dimensions. Figs. 1(e) and (f) show variation of event horizon with m for $d = 4$ and $d = 10$, respectively. We can observe that for the fixed l and a , r_+ varies with m . It will be important to study behavior of temperature with r_+ . Also, from the last equation of (12), we should set $a^2 < l^2$ to have positive entropy.

In the Fig. 2 we can see behavior of the temperature with r_+ . As for the fixed l and a , r_+ varies with m , hence, T varies with r_+ . We can see different behavior for $d \geq 6$ and $d \leq 5$. In the cases of $d = 4$ and $d = 5$ (solid and dashed lines of Fig. 2), temperature is totally increasing function of r_+ . On the other hand, for $d \geq 6$, temperature has a minimum. So, it decreases with small r_+ , and increases with large r_+ . Minimum temperature occurs at the critical value r_{+c} ($d \geq 6$), which is root of the following equation,

$$\begin{aligned}
 (d-1)r_+^6 + (2a^2d - l^2(d-3))r_+^4 \\
 + a^2((d-3)a^2 - 2l^2(d-6))r_+^2 - a^4l^2(d-5) = 0.
 \end{aligned} \tag{15}$$

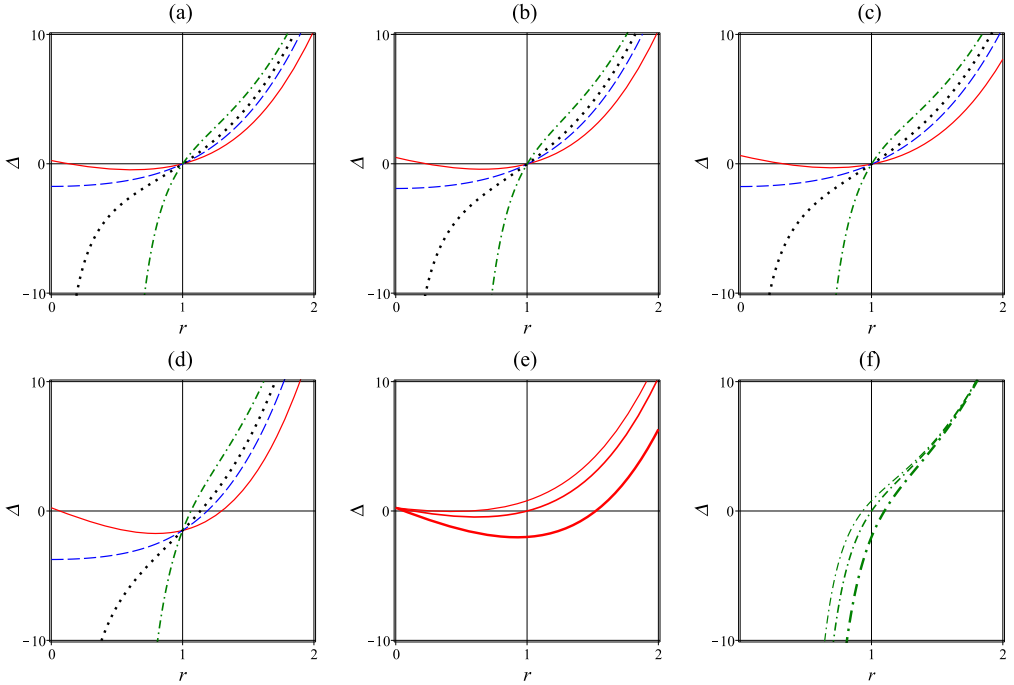


Fig. 1. Δ in terms of r . (a) $l = 1.3$, $m = 1$, and $a = 0.5$. (b) $l = 1.3$, $m = 1.2$, and $a = 0.7$. (c) $l = 1.5$, $m = 1.2$, and $a = 0.8$. (d) $l = 1$, $m = 2$, and $a = 0.5$. (e) $l = 1.3$, $a = 0.5$, $m = 0.6$ (upper thin), $m = 1$ (middle), and $m = 2$ (lower thick). (f) $l = 1.3$, $a = 0.5$, $m = 0.6$ (upper thin), $m = 1$ (middle), and $m = 2$ (lower thick). $d = 4$ (solid red), $d = 5$ (dashed blue), $d = 6$ (dotted black), $d = 10$ (dash dotted green). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

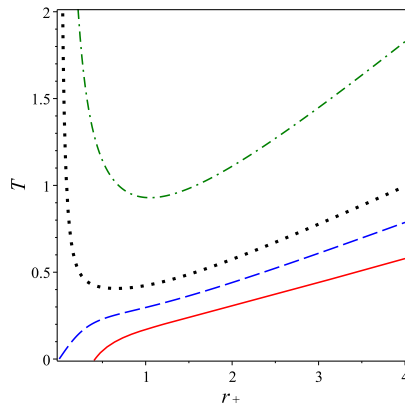


Fig. 2. Temperature in terms of r_+ for $l = 1.3$, and $a = 0.5$. $d = 4$ (solid red), $d = 5$ (dashed blue), $d = 6$ (dotted black), $d = 10$ (dash dotted green). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

We can find critical value of event horizon radius by variation of m . So, we observe that $r_{+c} \approx 1$ corresponding to $d = 10$ and $d = 4$ obtained for $m \approx 1$ and $m \approx 0.1$, respectively (for $a = 0.5$ and $l = 1.3$).

Other thermodynamics quantities like mass, angular momenta, angular velocity and volume can be expressed as

$$M = \frac{m\omega_{d-2}}{4\pi\Xi^2} \left(1 + \frac{(d-4)\Xi}{2} \right), \quad (16)$$

$$J = \frac{ma\omega_{d-2}}{4\pi\Xi^2}, \quad (17)$$

$$\Omega = \frac{a}{l^2} \frac{r_+^2 + l^2}{r_+^2 + a^2}, \quad (18)$$

$$V = \frac{Ar_+}{d-1} \left[1 + \frac{a^2(r_+^2 + l^2)}{l^2\Xi(d-2)r_+^2} \right], \quad (19)$$

where $A = 4s$, as given by the relation (14). It is clear that, in the special case of $d = 4$ and $a = 0$, we get $V = \frac{4}{3}\pi r_+^3$ as expected. Hence, the effect of a is that it increases the volume in four dimension. Situation is similar for the higher dimensional case. Finally, Gibbs free energy is given by

$$G = \frac{\omega_{d-2}r_+^{d-5}}{16\pi\Xi^2} \left(3a^2 + r_+^2 - \frac{(r_+^2 - a^2)^2}{l^2} + \frac{3a^2r_+^4 + a^4r_+^2}{l^4} \right). \quad (20)$$

Behavior of G and critical points are discussed. We will discuss other thermodynamics potentials like Helmholtz free energy, PV diagram, critical point and stability of system in the next section, and also analyze the effect of thermal fluctuations on the thermodynamics of this system.

3. Thermal fluctuations

The entropy of a black hole will be corrected by a logarithmic term due to the thermal fluctuations. It may be noted that we will be analyzing this system very close to the equilibrium, and so we will analyze the thermal fluctuations as perturbations around the equilibrium. This approximation will be valid as long as the correction due to the thermal fluctuations is small compared to the original quantity, i.e., as long as $S - s/s \ll 1$, where S is the corrected entropy and s is the original entropy of the system. It should also be noted that at very high temperatures, i.e., near the Planck scale, the thermal fluctuations will be very large, and at this stage the system cannot be analyzed as a perturbation around equilibrium temperature. However, for such temperatures, where we can analyze this system as a perturbation around equilibrium, we can write the corrected entropy as [33,34,27],

$$S = s - \frac{\ln s''}{2}, \quad (21)$$

here prime denotes derivative with respect to T^{-1} , where T is the equilibrium temperature. Furthermore, for such systems, the second derivative of the entropy can be expressed in terms of fluctuations of the energy, and so the corrected entropy can be written as [27]

$$S = s - \frac{\alpha}{2} \ln(sT^2), \quad (22)$$

where s is original entropy given by the Eq. (14). The parameter α is added by hand to analyze effect of thermal fluctuations. Furthermore, as the logarithmic corrections to the entropy are generated from almost all approaches to quantum gravity, but the coefficient of such corrections

varies between different approaches, we will keep such a coefficient as a variable in this paper. Thus, we can effectively discuss the effect of the corrections to the thermodynamics from various different approaches to quantum gravity. It may be noted that the limiting cases, $\alpha = 1$ is valid for the entropy corrected by thermal fluctuations, and $\alpha = 0$ is valid for the original entropy of this system.

As we will be analyzing a very general form of the corrections to the entropy, we will generalize this result. We can generalize this result by looking at the ideal gas entropy, written as,

$$S = \frac{5}{2} Nk_B - Nk_B \ln\left(\frac{N}{V} \frac{h^3}{(2\pi mk_B T)^{\frac{3}{2}}}\right), \tag{23}$$

where k_B is Boltzmann constant and N is the total particle number. Motivated by the equation (23), we propose the following logarithmic entropy,

$$S(T) = s + \alpha \ln(sf(T)), \tag{24}$$

where $f(T)$ is specific function of temperature, and α is free parameter which is used to incorporate the effect of logarithmic corrections. If we assume $s = \frac{5}{2} Nk_B$, $\alpha = -Nk_B$ and $f(T) = \frac{2h}{5V} \left(\frac{2\pi}{m}\right)^{\frac{3}{2}} T^{-\frac{3}{2}}$, then the ideal gas entropy (23) reproduced. Presence of \hbar in the $f(T)$ shows that thermal fluctuations are indeed quantum effect and will be tested for several physical system.

Furthermore, if we assume $f(T) = T^2$ and $\alpha = -\frac{1}{2}$, then the corrections to the entropy due to thermal fluctuations are reproduced.

In the canonical ensemble, one can relate entropy $S(T)$ to the partition function Z ,

$$S(T) = k_B \ln Z + k_B T \left(\frac{\partial \ln Z}{\partial T}\right). \tag{25}$$

So, for the any physical system with ordinary entropy s , one can obtain,

$$\ln Z = \frac{1}{T} \int \frac{s + \alpha \ln(sf(T))}{k_B} dT. \tag{26}$$

Using the equation (26), one can obtain other thermodynamics quantities like internal energy,

$$E = k_B T^2 \frac{d \ln Z}{dT} = T(s + \alpha \ln(sf(T))) - \int (s + \alpha \ln(sf(T))) dT, \tag{27}$$

Helmholtz free energy,

$$F = -k_B T \ln Z = - \int (s + \alpha \ln(sf(T))) dT, \tag{28}$$

specific heat at constant volume,

$$C_V = T \left[\frac{d}{dT} (s + \alpha \ln(sf(T))) \right]_V, \tag{29}$$

specific heat at constant pressure,

$$C_P = C_V + \left[P + \left(\frac{\partial E}{\partial V}\right)_T \right] \left(\frac{\partial V}{\partial T}\right)_P, \tag{30}$$

and pressure,

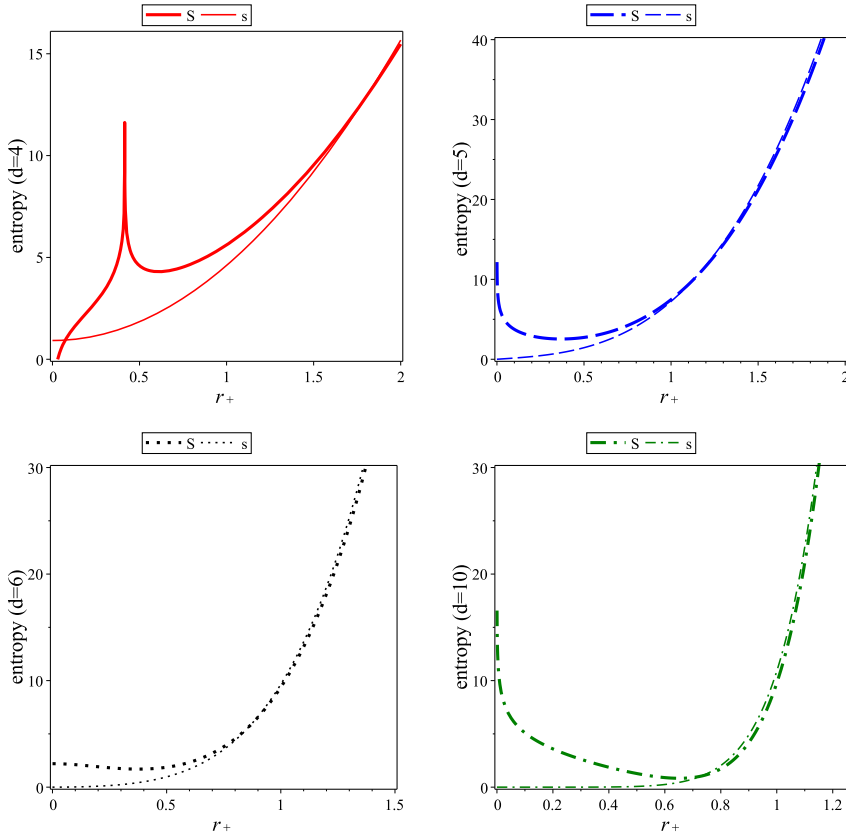


Fig. 3. Entropy in terms of r_+ for $l = 1.3$, and $a = 0.5$. Thin curves show ordinary entropy ($\alpha = 0$) given by the equation (14), while thick curves show logarithmic corrected entropy ($\alpha = 1$) given by the equation (22). $d = 4$ (solid red), $d = 5$ (dashed blue), $d = 6$ (dotted black), $d = 10$ (dash dotted green). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$P = k_B T \left(\frac{d \ln Z}{dV} \right)_T. \tag{31}$$

Then, we can obtain enthalpy and Gibbs free energy as

$$H = E + PV, \tag{32}$$

and

$$G = F + PV = H - T(s + \alpha \ln(sf(T))). \tag{33}$$

Now we will assume $f(T) = T^2$, and investigate thermodynamics properties of this system.

In the plots of the Fig. 3, we can see the behavior of the entropy for various dimensions (we focus on $d = 4, 5, 6, 10$). It is expected that logarithmic correction is important for small black hole, so we see that for the larger r_+ both S and s coincide. On the other hand, for the small r_+ , corrected entropy has completely different behavior, so a minimum entropy is available for $d \geq 5$. In the special case of $d = 4$, there is a maximum for the entropy, which has a singular behavior. However, it is possible that this approximation breaks down near this point, as the perturbations can not be used to analyze such points.

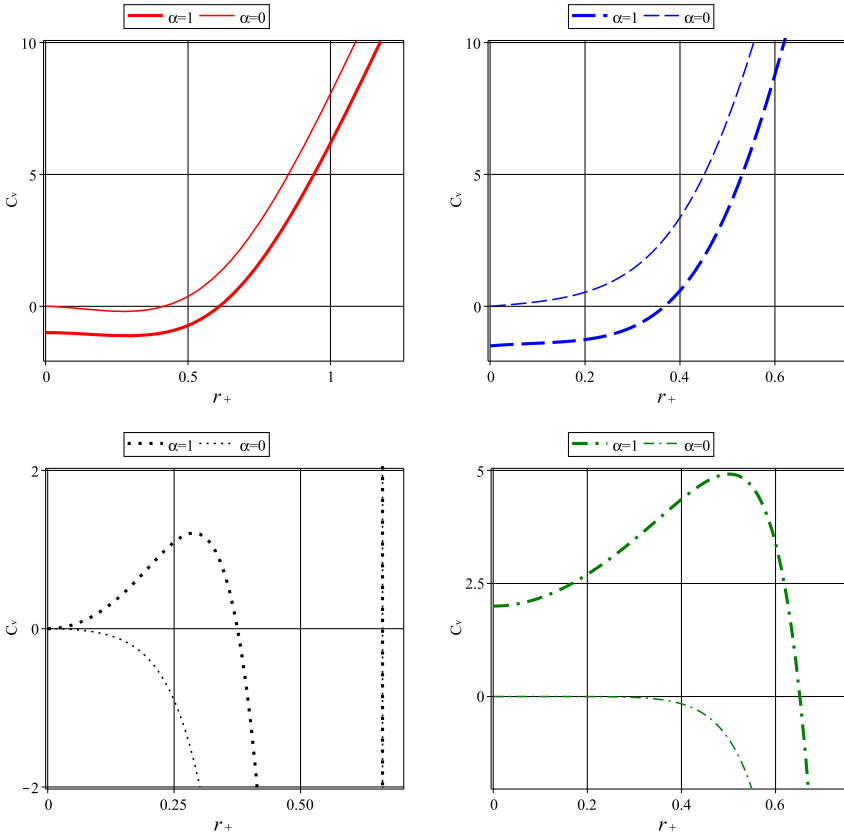


Fig. 4. Specific heat in terms of r_+ for $l = 1.3$, and $a = 0.5$. Thin curves show ordinary case ($\alpha = 0$), while thick curves show logarithmic corrected case ($\alpha = 1$). $d = 4$ (solid red), $d = 5$ (dashed blue), $d = 6$ (dotted black), $d = 10$ (dash dotted green). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Now, we can use the specific heat given by the equation (29) to study stability of the black hole. Plots of the Fig. 4 show behavior of specific heat in terms of r_+ for various dimensions. In absence of extra dimensions ($d = 4$), we can see unstable region for the small black hole. It means that there is a minimum size for the black hole. So, the thermal fluctuations can cause a phase transition in the black hole, (as one can see from the second, third and last plots of Fig. 4), and so without the logarithmic corrections there is no phase transition, except in four dimensions. We can see that the black hole is completely stable in $d = 5$, without logarithmic correction. However, thermal fluctuations make the black hole unstable. Situation is completely different for the cases of $d \geq 6$. In this cases the black hole is completely unstable. It is an interesting result that the logarithmic corrections are needed to make the black holes stable in presence of at least two extra dimensions ($d \geq 6$).

In order to obtain Helmholtz free energy we need to calculate internal energy,

$$E = \int C dT = E_1 + \alpha E_2, \tag{34}$$

where E_1 and E_2 for the case of $d = 4$ are given by,

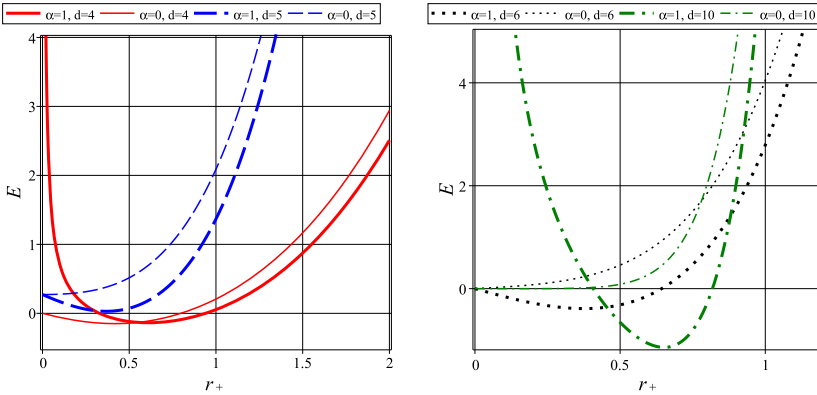


Fig. 5. Internal energy in terms of r_+ for $l = 1.3$, and $a = 0.5$. Thin curves show ordinary case ($\alpha = 0$), while thick curves show logarithmic corrected case ($\alpha = 1$). $d = 4$ (solid red), $d = 5$ (dashed blue), $d = 6$ (dotted black), $d = 10$ (dash dotted green). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$E_1 = \frac{2a(l^2 - a^2) \tan^{-1}\left(\frac{r_+}{a}\right) + r_+(2a^2 - l^2 - r_+^2)}{2(a^2 - l^2)}, \tag{35}$$

and

$$E_2 = \frac{ar_+(a^2 + r_+^2) \tan^{-1}\left(\frac{r_+}{a}\right) + \frac{a^2}{4}(l^2 - 5r_+^2) - \frac{3}{2}r_+^4}{\pi l^2 r_+(a^2 + r_+^2)}. \tag{36}$$

In presence of extra dimension, we can obtain approximately similar results which are illustrated by Fig. 5. In the case of $d = 4$, we can see important effect for the small black hole, which have infinite energy because of thermal fluctuations. In the cases of $d = 6$ and $d = 10$, we obtain a minimum for internal energy, due to the thermal fluctuations.

Now, we can calculate Helmholtz free energy given by the equation (28). In the plots of the Fig. 6, we can see behavior of the Helmholtz free energy in terms of r_+ for various dimensions. In the cases of $d \geq 5$, we can see that logarithmic correction decreases value of F . In the special case of $d = 4$, there is a critical horizon radius r_c , where $F(\alpha = 1) = F(\alpha = 0)$. If $r_+ > r_c$, then the effect of logarithmic correction is that it decreases the Helmholtz free energy, while if $r_+ < r_c$, then the effect of logarithmic correction is that it increases the Helmholtz free energy to a maximum value. Now, $F(\alpha = 0) \rightarrow +\infty$, and $F(\alpha = 1) \rightarrow -\infty$ at $r_+ \rightarrow 0$. Interesting point is that value of corrected F at $r_+ \approx 1$ is the same for $d = 4$ and $d = 5$. In higher dimensions, Helmholtz free energy is zero for small black hole without thermal fluctuations, while it has negative infinite value in presence of thermal fluctuations.

Gibbs potential obtained using enthalpy (which interpreted as mass given by the equation (16)), temperature and entropy which is given by the equation (33). We can see that value of G is depend on m , a and l . So, at the fixed m , a and l , value of r_+ is fixed. In the Table 1, we can see effect of logarithmic correction on the Gibbs potential. We can conclude that thermal fluctuations decrease value of G in the $d = 4$ and $d = 5$ dimensions. On the other hand, for the cases of $d \geq 6$, thermal fluctuation increases value of the Gibbs potential. For the cases of $d = 9$ and $d = 10$, Gibbs free energy is negative in absence of logarithmic correction, but it is positive in presence of thermal fluctuations.

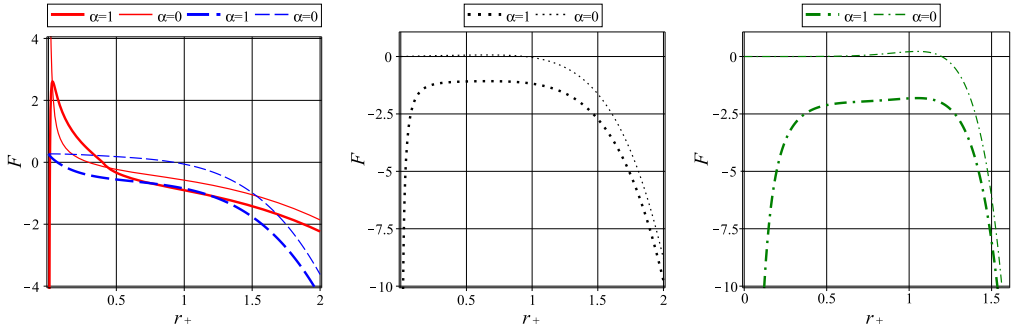


Fig. 6. Helmholtz free energy in terms of r_+ for $l = 1.3$, and $a = 0.5$. Thin curves show ordinary case ($\alpha = 0$), while thick curves show logarithmic corrected case ($\alpha = 1$). $d = 4$ (solid red), $d = 5$ (dashed blue), $d = 6$ (dotted black), $d = 10$ (dash dotted green). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Table 1
Value of the Gibbs potential for $l = 1.3$, $m = 1$ and $a = 0.5$ for various dimensions.

$\alpha = 0$	$\alpha = 1$	d
0.38932	0.21797	4
0.47997	0.4132	5
0.46452	0.5807	6
0.34055	0.6809	7
0.14279	0.7235	8
-0.076259	0.7434	9
-0.26761	0.7751	10

Now, using the equations (19), (28) and

$$P = - \left(\frac{\partial F}{\partial V} \right), \tag{37}$$

we can study behavior of pressure. In order to find PV diagram, we can find r_+ in terms of V from the Eq. (19), then remove r_+ in the Eq. (37), to obtain behavior of P in terms of V . In the case of $d = 5$, we can obtain,

$$r_+^2 = \frac{\pi l a^2 (2l^2 - a^2) - (a^2 - l^2) \sqrt{\pi^2 a^4 l^2 + 6(3l^2 - 2a^2)V}}{\pi l (2a^2 - 3l^2)}. \tag{38}$$

Using the relation (38) in the Eq. (37) for $d = 5$, we can obtain behavior of P as illustrated in the Fig. 7. We can see critical point shifted due to the thermal fluctuations. It is clear that the interested case is when $l = 1.3$ and, $a = 0.5$, and for this we obtain a completely stable black hole. We should note that, there is some negative regions in the Fig. 7 which are unphysical since it would require analytically continuing $l \rightarrow il$, and so we cannot use the values below $P = 0$ to analyze the van der Waals behavior. These unphysical situations are obtained for small a and large l , so one can remove such negative region by fixing a and l . It produces a maximum value for l and minimum value for a , with an uncertainty like equation. Fitting the curves of Fig. 7, suggests the following Virial expansion form of pressure,

$$P = \frac{A}{V} + \frac{B}{V^2} + \frac{C}{V^3} + \dots \tag{39}$$

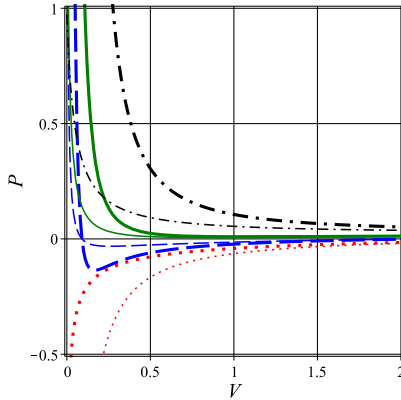


Fig. 7. Pressure in terms of V for $l = 1.3$ and $d = 5$. Thin curves show ordinary case ($\alpha = 0$), while thick curves show logarithmic corrected case ($\alpha = 1$). $a = 0$ (dotted red), $a = 0.25$ (dashed black), $a = 0.333$ (solid green), $a = 0.5$ (dash dotted black). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

and hence we can write,

$$\frac{PV}{T} = A(T) + \frac{B(T)}{V} + \frac{C(T)}{V^2} + \dots \tag{40}$$

It means that singly spinning Kerr-AdS black hole in five dimension behave as van der Waals fluid. The best fitted values of a stable case for Virial coefficients $A(T)$, $B(T)$ and $C(T)$ yields to the following relation,

$$\frac{PV}{T} = 0.3 + \frac{0.7}{V^2}, \tag{41}$$

so $B(T) = 0$, and a black hole is in Boyle temperature ($B(T) = 0$).

Situation is complicated for other dimensions where analytic expression of r_+ in terms of V is not available. For example, using the relation (19), with $d = 10$, we have the following equation,

$$(8l^2 - 7a^2)r^9 + (9a^2l^2 - 7a^4)r^7 + l^2a^4r^5 - \frac{945(a^2 - l^2)^2}{l^2\pi^4}V = 0. \tag{42}$$

However, we can assume small a and do numerical analysis to obtain pressure in terms of volume, and this is illustrated by the Fig. 8. We can see that the thermal fluctuations are necessary to have critical point, and obtain expected curves. Right plot of the Fig. 8, shows that a and l should be small to have the expected curves.

4. Four dimensional Kerr-AdS black hole

In this section, we will discuss about the special case of a rotating black hole in four dimensions, and also analyze the effects of thermal fluctuation on such a black hole. The corresponding metric is obtained by putting $d = 4$ in the solution given by (11) and (12). Thermodynamics quantities of Kerr-AdS black hole are listed below. The black hole mass and angular momenta are given by

$$M = \frac{m}{\Xi^2}, \tag{43}$$

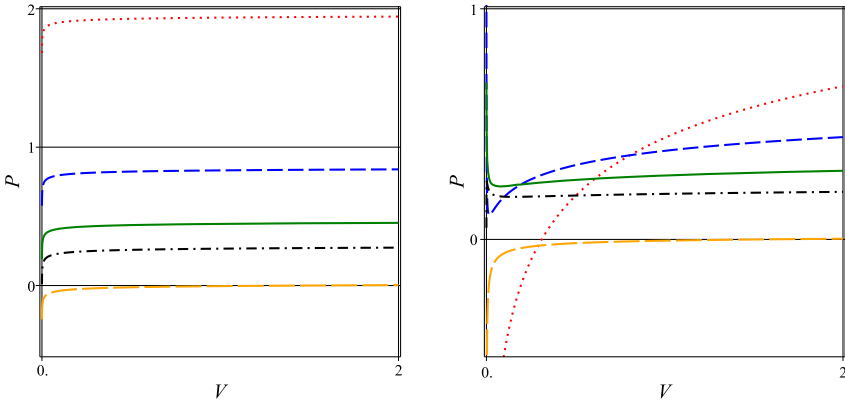


Fig. 8. Pressure in terms of V for $a = 0.01$ and $d = 10$. Left plot shows ordinary case ($\alpha = 0$), while right plot shows logarithmic corrected case ($\alpha = 1$). $l = 0.2$ (dotted red), $l = 0.3$ (dashed blue), $l = 0.4$ (solid green), $l = 0.5$ (dash dotted black), $l = 1.3$ (long dashed orange). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

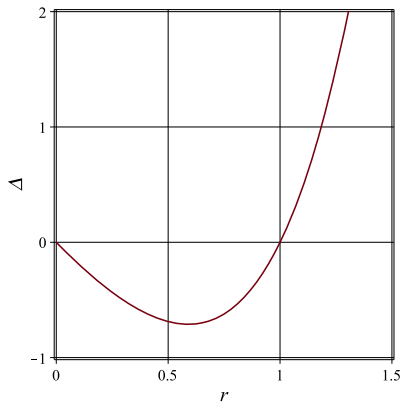


Fig. 9. Typical behavior of Δ in terms of r with $l = 1$, $m = 1$, and $a = 0$.

and

$$J = \frac{ma}{\Xi^2}, \tag{44}$$

while angular velocity of the horizon is given by

$$\Omega = \frac{a(r_+^2 + l^2)}{l^2(r_+^2 + a^2)}, \tag{45}$$

where r_+ is the largest root of $\Delta = 0$. By appropriate choice of l , m and a , there are two real positive roots, r_{\pm} , which illustrated by the Fig. 9. In the Table 2, we can see value of r_+ , with some possible values of free black hole parameters.

Furthermore, the temperature T is given by,

$$T = \frac{1}{2\pi r_+} \left[\frac{(a^2 + 3r_+^2)(r_+^2 + l^2)}{2l^2(a^2 + r_+^2)} - 1 \right]. \tag{46}$$

Table 2
Value of r_+ .

r_+	a	m	l
1	0	1	1
0.5 (extremal)	0.742	1	1
0.5	0.2	0.367	1
1	0.4	0.728	2
1	1	2	1

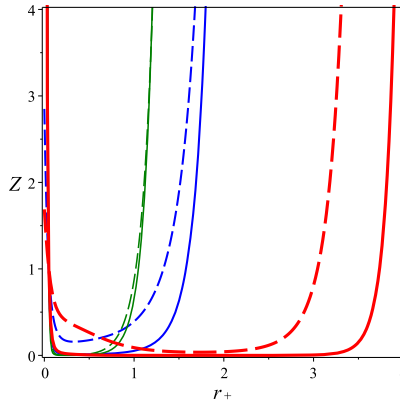


Fig. 10. Partition function in terms of r_+ with $l = 1$, and $a = 0.5$ (blue); $l = 1$, and $a = 0.742$ (green); $l = 2$, and $a = 0.4$ (red). Solid and dashed lines represent $\alpha = 1$ and $\alpha = 0$ respectively. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

The ordinary entropy of the black hole is given by $s = \frac{A}{4}$, and hence

$$s = \pi \frac{r_+^2 + a^2}{\Xi}. \tag{47}$$

The thermodynamic volume is given by,

$$V = \frac{4\pi r_+ r_+^2 + a^2}{3 \Xi} \left(1 + \frac{(r_+^2 + l^2)a^2}{2l^2 r_+^2 \Xi} \right). \tag{48}$$

We will now study logarithmic corrections to the thermodynamics of this Kerr-AdS black hole. In the Fig. 10, we can see behavior of the partition function for various values of a and l . We can see that logarithmic correction reduces value of the partition function. We can reproduce behavior of Z , using the following function at the region of $r \leq 2$,

$$Z \approx c_1 + \frac{(c_2 - c_3)(r_+ - b)^4}{r_+} + \frac{c_4(r_+ - b)}{(r_+ - d)^2}, \tag{49}$$

where c_1, c_2, c_3 ($c_2 > c_3$), c_4, b and d are constants which can fit above function corresponding to the curves of the Fig. 2. Here $c_1 = 0$ corresponds to the corrected partition function ($\alpha \neq 0$), while $c_1 \neq 0$ and $c_3 = 0$ corresponds to the ordinary entropy ($\alpha = 0$).

In the Fig. 11 we can see typical behavior of free energy F as a function of r_+ . It is illustrated that the effect of thermal fluctuations increases as r_+ becomes smaller. It is clear that logarithmic correction increases Helmholtz free energy.

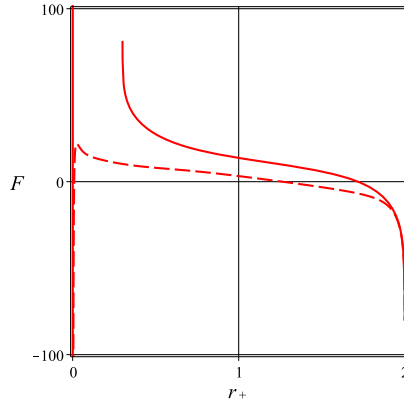


Fig. 11. Helmholtz free energy in terms of r_+ with $l = 1$, and $a = 0.2$. Solid and dashed lines represent logarithmic corrected and ordinary free energy respectively.

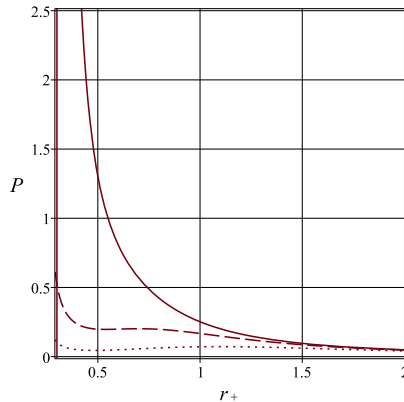


Fig. 12. Pressure in terms of r_+ with $l = 1$, and $a = 0.5$. Solid ($P > P_C$), dashed ($P = P_C$), dotted ($P < P_C$).

Now, by using the equation (31) one can discuss about pressure, and critical point, which are obtained by solving $\partial_V P = \partial_{V'} P = 0$. In the Fig. 12, we can see behavior of pressure compared it with the critical pressure P_C .

5. Conclusions and discussion

In this paper, we analyzed the effects of thermal fluctuations on the thermodynamics of a small singly spinning AdS-Kerr black hole in higher dimensional space–time. It was observed that the entropy of this black hole gets corrected by a logarithmic correction term due to the thermal fluctuations, and this has important consequences for the properties of such a black hole. We also calculated the corrections to the various thermodynamical quantities due to the thermal fluctuations. Then, we analyzed the stability of this black hole in higher dimensions, and the effect of thermal fluctuations on the stability of this black hole. We also studied the critical phenomena for such black hole, and the effect of thermal fluctuations on critical phenomena.

It was observed that the effect of these thermal fluctuations can be neglected for a large black hole, but it becomes important for a sufficiently small black hole. This was expected, as the correction to the entropy–area relation occurs due to quantum fluctuations in the geometry, and these fluctuations become important only at small scale.

This was done studying the phase transition in canonical ensemble. The curve of specific heat for this system was also analyzed, and it was observed that they have two divergent points (see for example the third plot of the Fig. 4 corresponding to $d = 6$). Furthermore, it was also observed that in four dimensions both the large radius region and the small radius region are thermodynamically stable, with positive specific heat. However, the medium radius region is unstable with negative specific heat. For the case of $d \leq 5$ it is clear that when the size of black hole is small, logarithmic correction makes the black hole unstable. However in higher dimension situation is the inverse.

It may be noted that there are several interesting asymptomatic geometries, and it is possible to study the correction to the thermodynamics of such geometries by thermal fluctuations. It would also be interesting to analyze the effect of thermal fluctuations on black holes in modified theories of gravity. It is expected that these thermal fluctuations will also correct the entropy of modified theories of gravity. Such correction to the entropy of modified theories of gravity, will also produce correction terms for other thermodynamical quantities. This can also affect the critical phenomena which can be studied for AdS black holes in modified theories of gravity. It may be noted that the critical phenomena has been studied for AdS black holes in $f(R)$ theories of gravity [39].

The topological black hole solutions of third order Lovelock gravity couple with two classes of Born–Infeld type nonlinear electrodynamics have been also been studied [40]. In this analysis, the geometric and thermodynamics properties of AdS black hole solutions were discussed, and it was observed that the first law of thermodynamics holds for such solutions. The heat capacity and determinant of Hessian matrix for these black holes were used to evaluate thermal stability in both canonical and grand canonical ensembles. It would be interesting to analyze the effect of thermal fluctuations on the thermodynamics and stability of such black in Lovelock gravity couple with two classes of Born–Infeld type nonlinear electrodynamics.

It may be noted that recently the correction to the thermodynamics of black holes has also been obtained from gravity’s rainbow [41–45]. It has been argued that the correction from gravity’s rainbow can help to resolve the black hole information paradox [46–48]. It would be interesting to analyze the effects of gravity’s rainbow on the thermodynamics of singly spinning AdS–Kerr black hole. In fact, it is possible to use the results of this paper to obtain certain constraints on the rainbow functions. This is because as the logarithmic correction is so universally produced by almost all approaches to quantum gravity, we expect that such corrections should also be produced by gravity’s rainbow. This can be used to constrain the parameters used in the rainbow function. It would also be interesting to perform such an analysis for other kind of black holes, and to analyze the relation between correction obtained from thermal fluctuations and corrections obtained from gravity’s rainbow. It may be noted that the thermodynamics of black holes has also been studied in massive gravity [49–52]. It would also be interesting to analyze the effects of thermal fluctuations on the thermodynamics of black holes in massive gravity. Finally, it may be interesting to study logarithmic correction effect on the 3D hairy black hole, as such systems have also been studied [53–56]. It would be interesting to analyze the effect that logarithmic corrections can have on such black holes.

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