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Entropic c–functions in $T\overline{T}$, $J\overline{T}$, $T\overline{J}$ deformations

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Abstract

We study the holographic entanglement entropy of an interval in a quantum field theory obtained by deforming a holographic two–dimensional conformal field theory via a general linear combination of irrelevant operators that are closely related to, but nonetheless distinct from, $T\overline{T}$, $J\overline{T}$ and $T\overline{J}$, and compute the Casin–Huerta entropic c–function. In the ultraviolet, for a particular combination of the deformation parameters, we find that the leading order dependence of the entanglement entropy on the length of the interval is given by a square root but not logarithmic term. Such power law dependence of the entanglement entropy on the interval length is quite peculiar and interesting. We also find that the entropic c–function is ultraviolet regulator independent, and along the renormalization group upflow towards the ultraviolet, it is non–decreasing. We show that in the ultraviolet the entropic c–function exhibits a power law divergence as the interval length approaches a minimum finite value determined in terms of the deformation parameters. This value sets the non–locality scale of the theory.

1. Introduction

String theory on the background $AdS_3 \times \mathcal{X}$ with Neveu–Schwartz two–form B field contains a sector of long strings that extend to the boundary [1–4]. The effective theory of N coincident long strings on $AdS_3 \times \mathcal{X}$ in the weak coupling regime is believed to be well described by the symmetric product theory [4,5]

$$\mathcal{M}^N/S_N, \tag{1.1}$$

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where the conformal field theory $\mathcal{M}$ is the dual theory of a single long string [3]. Note that the symmetric product theory (1.1) is not equal to the full dual boundary (spacetime) theory since $\mathcal{M}$ only describes the long string sector.

In [5] the authors considered deforming the Lagrangian of the worldsheet string theory on the background $AdS_3 \times \mathcal{X}$ by a general linear combination of truly marginal current–current operators

$$\lambda J^-_{\rm SL}\overline{J}^-_{\rm SL} + \epsilon_+ K\overline{J}^-_{\rm SL} + \epsilon_- \overline{K} J^-_{\rm SL}. \tag{1.2}$$

They showed that, in the long string sector, this deformation is equivalent to the deformation of the theory $\mathcal{M}$ (with spectral flow or winding number $w = 1$) by a general linear combination of the recently much studied irrelevant operators [6–9]

$$-tT\overline{T} - \mu_+ J\overline{T} - \mu_- \overline{J}T, \tag{1.3}$$

where J and $\overline{J}$ are the left and right moving $U(1)$ currents, respectively, and T and $\overline{T}$ are the left and right moving stress tensor components, respectively, of the conformal field theory $\mathcal{M}$.

String theory on the background $AdS_3 \times \mathcal{X}$ contains a class of (integrated) vertex operators $\overline{A}$, A and D constructed in [10], which depends on the coordinates $(x, \overline{x})$ of the dual conformal field theory. It is shown in [5,11,12] that the deformation (1.2) in the action amounts to adding a linear combination of the operators $\int d^2x\overline{A}$, $\int d^2xA$ and $\int d^2xD$. In the boundary (spacetime) theory the operators $\overline{A}$, A and D have the same scaling dimensions as $T\overline{J}$, $J\overline{T}$ and $T\overline{T}$, respectively, however, they are single trace operators, in the sense that, in the long string sector, they are interpreted as sum over the N copies of the operators $T\overline{J}$, $J\overline{T}$ and $T\overline{T}$, respectively, of the field theory $\mathcal{M}$.

The operators $J\overline{T}$ and $T\overline{J}$ have left and right scaling dimensions $(1, 2)$ and $(2, 1)$, respectively, and therefore, the deformation (1.3) results in a theory that breaks Lorentz invariance [9,13,12].

The spacetime couplings t, μ_+ and μ_- are related to the worldsheet dimensionless couplings λ, ϵ_+ and ϵ_- via the relations [5],

$$t = \pi\alpha'\lambda, \quad \mu_\pm = 2\sqrt{2\alpha'}\epsilon_\pm, \quad \alpha' = l_s^2, \tag{1.4}$$

where α' is the Regge slope, and l_s is the intrinsic string length.

The deformation (1.3) is irrelevant and therefore the couplings grow as we ascend the renormalization group. In general, under an irrelevant deformation of a quantum field theory, in the ultraviolet, the associated coupling is large and the description of the theory in terms of the original infrared degrees of freedom often breaks down. The theory also suffers from ambiguities and/or arbitrariness. It is also often the case that quantum corrections generate an infinite number of irrelevant operators. Under the deformation (1.3) it is shown, however, that the theory is solvable, in the sense that the spectrum on an infinite cylinder [7–9,14,15] and the partition function on a torus [16–21] can be computed exactly. The theory also does not acquire new couplings. On a torus modular invariance can be employed to constrain the theory [16–18,22]. In the case in which the coupling t is positive and $\mu_\pm = 0$ the theory involves no ambiguities [17,18]. We obtain in this paper the exact entanglement entropy and entropic c–function in the deformation (1.3). This increases the number of non–trivial quantities that one can exactly solve and study in this class of theories.

It is shown in [5] that in the space of couplings in which the combination

$$F = \frac{t}{\pi} - \frac{(\mu_+ + \mu_-)^2}{8}, \tag{1.5}$$

is positive, $F > 0$, the energies of states are real, and the density of states asymptotically exhibits Hagedorn growth. In the limit $F \to 0^+$ however the theory appears to be distinct. The density of states (in a fixed charge sector) asymptotically exhibits an intermediate growth between Cardy and Hagedorn growths. We also show later in the paper that in this limit the Von Neumann entanglement entropy at short distances exhibits a square root area law correction but not logarithmic correction. Such square root correction term is quite peculiar and interesting. In the case in which $F < 0$ the energies become complex above a scale fixed by the couplings and the corresponding bulk geometry is singular and/or it has either closed timelike curves or no timelike direction. The signature of the bulk metric switches signs beyond a finite radial distance where the singularity occurs. We will not consider this case in this paper as it is not clear how to consistently apply the Ryu and Takayanagi holographic prescription and its covariant generalization. We comment on this later in the discussion section.

We now briefly mention the holographic proposals for the closely related double trace deformations. In this class of deformations T and J are the left moving energy momentum tensor and current, respectively, of the full boundary theory. The right moving energy momentum tensor and current are denoted by $\overline{T}$ and $\overline{J}$ respectively. In these holographic proposals the coupling t is negative and therefore $F < 0$. For negative t with $\mu_{\pm} = 0$ the dual bulk spacetime of a $T\overline{T}$ deformed two dimensional holographic conformal field theory is proposed to be AdS_3 with a Dirichlet boundary at finite radial distance fixed by the coupling t [23]. For either sign of μ_+ (and $\mu_- = 0$ or vice versa) and $t = 0$ it is shown in [24] that the dual bulk spacetime of a $J\overline{T}$ (or $T\overline{J}$) deformed two dimensional holographic conformal field theory is AdS_3 with boundary conditions that mix the metric and a gauge field dual to the current J (or $\overline{J}$). In either of these cases in the field theory side we have states with complex energies.

In this paper we study the (Von Neumann) entanglement entropy for a spatial interval of length L in the deformed (full) spacetime theory with $F \geq 0$ from its bulk string theory description and compute the Casin–Huerta entropic c–function. We study the monotonicity property of the entropic c–function along the renormalization group upflow and its independence of regularization scheme that one introduces to regularize the ultraviolet divergence of entanglement entropy. This provides further support that the renormalization group upflow is under better control. It also gives important insight into the nature of the theory in the ultraviolet as it is not governed by an ultraviolet fixed point [25].

The rest of the paper is organized as follows. In section 2 we review the corresponding bulk string theory background obtained under the deformation (1.2). In section 3 we compute the entanglement entropy in the deformed dual spacetime theory using the holographic prescription. Following this, we discuss its large and small L limits. In section 4 we compute the entropic c–function and study its ultraviolet and infrared limits. In section 5 we discuss the main results and future research directions. We comment on the entropic c–function of a double trace deformed field theory. We also comment on the case where $F < 0$. In the appendix we collect some of the intermediate results that are required in section 3.

2. The bulk deformed string background

We begin with string theory on

$$AdS_3 \times S^1 \times \mathcal{N}, \tag{2.1}$$

with Neveu–Schwartz two–form B field. Where the component $\mathcal{N}$ is an internal six–dimensional compact manifold. Its presence is irrelevant in our discussion. The S^1 component gives rise in the boundary conformal field theory to a $U(1)$ current algebra generated by the spacetime currents J and $\overline{J}$ [26,10].

The bosonic part of the worldsheet theory on $AdS_3 \times S^1$ is described by the action

$$S = \frac{k}{2\pi} \int d^2z \left(\partial\phi\overline{\partial}\phi + e^{2\phi}\partial\overline{\gamma}\overline{\partial}\gamma + \frac{1}{k}\partial\psi\overline{\partial}\psi \right), \tag{2.2}$$

where $(\phi, \gamma, \overline{\gamma})$ are the coordinates on AdS_3, and $\psi \sim \psi + 2\pi$ is the coordinate on S^1. The boundary of AdS_3 is located at $\phi = +\infty$. The coordinates γ and $\overline{\gamma}$ are

$$l_s\gamma = t + x, \quad l_s\overline{\gamma} = -t + x. \tag{2.3}$$

The level k is given by

$$l^2 = l_s^2 k, \tag{2.4}$$

where l is the radius of curvature of AdS_3.

The action has an affine $SL(2,R)_L \times SL(2,R)_R \times U(1)_L \times U(1)_R$ symmetry with left mover worldsheet currents

$$J^-_{\rm SL} = e^{2\phi}\partial\overline{\gamma}, \quad J^+_{\rm SL} = -2\gamma\partial\phi - \partial\gamma + \gamma^2 e^{2\phi}\partial\overline{\gamma}, \quad J^3_{\rm SL} = \gamma e^{2\phi}\partial\overline{\gamma} - \partial\phi, \quad K = \partial\psi, \tag{2.5}$$

and similar expressions for the right movers $\overline{J}^-_{\rm SL}, \overline{J}^+_{\rm SL}, \overline{J}^3_{\rm SL}$ and $\overline{K}$.

Consider deforming the worldsheet theory (2.2) by adding to its Lagrangian the deformation (1.2). The deformation (1.2) is truly marginal, and therefore, it preserves the conformal symmetry. It breaks the affine $SL(2,R)_L \times SL(2,R)_R \times U(1)_L \times U(1)_R$ worldsheet symmetry down to $U(1)_L \times U(1)_R \times U(1)_L \times U(1)_R$ affine symmetry.

The deformation corresponds to a deformation of the metric g, dilaton Φ, and Neveu–Schwartz two–form B [5,27]. We shall refer to the deformed background as $\mathcal{M}_4$,

$$ds^2 = d\phi^2 + hd\gamma d\overline{\gamma} + \frac{2h\epsilon_+}{\sqrt{k}}d\psi d\overline{\gamma} + \frac{2h\epsilon_-}{\sqrt{k}}d\psi d\gamma + \frac{1}{k}hf^{-1}d\psi^2, \tag{2.6}$$

$$e^{2\Phi} = g_s^2 e^{-2\phi}h, \tag{2.7}$$

$$B_{\gamma\overline{\gamma}} = g_{\gamma\overline{\gamma}}, \quad B_{\gamma\psi} = g_{\gamma\psi}, \quad B_{\psi\overline{\gamma}} = g_{\overline{\gamma}\psi}, \tag{2.8}$$

where

$$h^{-1} = e^{-2\phi} + \lambda - 4\epsilon_+\epsilon_-, \quad f^{-1} = h^{-1} + 4\epsilon_+\epsilon_-. \tag{2.9}$$

For $\lambda = 0$, $\epsilon_\pm = 0$ $\mathcal{M}_4$ reduces to our starting background $AdS_3 \times S^1$. For $\lambda = 0$ $\mathcal{M}_4$ reduces to a warped $AdS_3 \times S^1$ background [13,12]. For $\epsilon_\pm = 0$ $\mathcal{M}_4$ reduces to a background that is asymptotically $AdS_3 \times S^1$ for large negative ϕ and $\mathbb{R}_\phi \times \mathbb{R}^{1,1} \times S^1$ for large positive ϕ [11].

It is shown in [5] that for a combination of the couplings

$$\Psi = \lambda - (\epsilon_+ + \epsilon_-)^2, \tag{2.10}$$

with $\Psi \geq 0$ the geometry is smooth and it has no closed timelike curves. This positivity condition on (2.10) is the dual analogue of the positivity condition on (1.5).

In this paper we consider the case in which

$$\epsilon_+ = \frac{\epsilon}{2}, \quad \epsilon_- = \frac{\epsilon}{2}, \quad \Psi \geq 0. \tag{2.11}$$

In this case the background $\mathcal{M}_4$ is

$$\begin{aligned} ds^2 &= \alpha' d\phi^2 - h dt^2 + h\left(dx + \epsilon\sqrt{\frac{\alpha'}{k}}d\psi\right)^2 + \frac{\alpha'}{k}d\psi^2, \\ e^{2\Phi} &= g_s^2 e^{-2\phi} h, \quad h^{-1} = e^{-2\phi} + \Psi. \end{aligned} \tag{2.12}$$

In what follows we work on this background to compute the entanglement entropy and the entropic c–function for an interval of length L in the deformed dual conformal field theory using the holographic prescription. We note that for $\lambda < 0$ or equivalently $\Psi < 0$ the function h changes sign beyond a finite radial distance and therefore the metric signature also changes signs. It is not clear in this case how to consistently apply the holographic prescription. We will not consider this case in this paper. We comment on this later in the discussion section.

3. Holographic entanglement entropy

In this section we compute the entanglement entropy in the deformed spacetime conformal field theory for a spatial interval of length L with endpoints at $x = -L/2$ and $x = +L/2$. It is defined as the Von Neumann entropy,

$$S = -\mathrm{Tr}\rho \log \rho, \tag{3.1}$$

corresponding to the reduced density matrix ρ of the subregion L. The reduced density matrix ρ is obtained by tracing the global density matrix over the states of the complement of the subregion L. The entanglement entropy measures the amount of entanglement between the subregion L and its complement. Due to short range correlations near the boundary of the subregion L the entropy is, however, divergent. To regulate this ultraviolet divergence we introduce a finite cutoff. The entropy is an intrinsic property of the subregion L. Thus, it is a useful tool to characterize a field theory along a renormalization group flow. An equally useful and well–defined quantity that we will discuss in detail in the next section is the entropic c–function which, in any local and Lorentz invariant quantum field theory, is ultraviolet regulator independent and finite.

In holographic field theories, entanglement entropy is encoded in certain geometrical quantities in the bulk geometry [28–35]. In the context of the AdS/CFT correspondence [36–38], entanglement entropy is given by the area of a co–dimension two minimal surface in the bulk geometry [28,29,33]. See [30] for a covariant generalization. In this paper we work under the assumption that the holographic entanglement prescription is also valid for quantum field theories that have dual gravity or string theory descriptions. In what follows we begin by briefly stating this prescription.

Suppose we have a d–dimensional holographic quantum field theory. Suppose also the dual string theory is on a background $\mathcal{M}_{d+1}$. We assume the background $\mathcal{M}_{d+1}$ is static. This is the case in the theory we are considering. To compute the entanglement entropy $S_{\mathcal{R}}$ of a given spatial region $\mathcal{R}$ in the boundary quantum field theory we first find a co–dimension two static surface $\mathcal{K}$ in the bulk geometry $\mathcal{M}_{d+1}$ that ends on the boundary of $\mathcal{R}$. The surface $\mathcal{K}$ is homologous to $\mathcal{R}$ and minimizes the area functional. The entanglement entropy $S_{\mathcal{R}}$ in the d–dimensional boundary quantum field theory is then given by

$$S_{\mathcal{R}} = \frac{\mathrm{Area}(\mathcal{K})}{4G_{\mathrm{N}}^{(d+1)}}, \tag{3.2}$$

where $G_{\mathrm{N}}^{(d+1)}$ is the $d+1$–dimensional Newton's constant of the $\mathcal{M}_{d+1}$ geometry.

Following the holographic prescription we now compute the entanglement entropy for the spatial interval of length L in the deformation dual to (1.2). The bulk string geometry (2.12) at a moment of time is

$$ds^2 = \alpha' d\phi^2 + h dy^2 + \frac{\alpha'}{k} d\psi^2, \quad h^{-1} = e^{-2\phi} + \Psi, \tag{3.3}$$

where $y = x + \epsilon\sqrt{\alpha'/k}\,\psi$.

We now look for a two–dimensional surface $\phi(y,\psi)$ in the geometry (3.3) (and wrapping the internal $\mathcal{N}$ space) that minimizes globally (in the space of functions) the area functional (3.2) which taking into account the dilaton [28,32] is[1]

$$S = \frac{\sqrt{k\alpha'}}{4G_N^{(4)}} \int_0^{2\pi} d\psi \int_{-\frac{L}{2}+\epsilon\sqrt{\alpha'/k}\,\psi}^{+\frac{L}{2}+\epsilon\sqrt{\alpha'/k}\,\psi} dy\, e^{2\phi} \sqrt{\frac{1}{h}\left(1 + \frac{\alpha'}{h}\left(\partial_y\phi\right)^2 + k\left(\partial_\psi\phi\right)^2\right)}, \tag{3.4}$$

with the boundary conditions

$$\phi(\pm L/2 + \epsilon\sqrt{\alpha'/k}\,\psi, \psi) = \infty, \quad \phi(y,0) = \phi(y + \epsilon\sqrt{\alpha'/k}\,2\pi, 2\pi), \tag{3.5}$$

where ψ is on S^1 that is $\psi \sim \psi + 2\pi$.

We note that under the following continuous and discrete spacetime transformations

$$\psi \to \psi + \delta, \quad y \to y + \epsilon\sqrt{\frac{\alpha'}{k}}\delta, \quad \text{and} \quad \psi \to -\psi + 2\pi, \quad y \to -y + \epsilon\sqrt{\frac{\alpha'}{k}}2\pi, \tag{3.6}$$

where δ is an arbitrary constant, the bulk background (3.3) and the boundary conditions (3.5) are invariant. The surface

$$\phi(y,\psi) = \phi(y - \epsilon\sqrt{\alpha'/k}\,\psi) = \phi(-y + \epsilon\sqrt{\alpha'/k}\,\psi), \tag{3.7}$$

is invariant under the above symmetry transformations (3.6) and thus we expect that it minimizes the area functional. The minimal surface (3.7) is generated by translating the curve, for example at $\psi = 0$, $\phi(y)$, along the line $y = \epsilon\sqrt{\alpha'/k}\,\psi$. This curve has the parity symmetry $y \to -y$. The entanglement entropy is then obtained using (3.4). We find

$$S = \frac{\sqrt{k}}{4G_N^{(3)}} \int_{-\frac{L}{2}}^{+\frac{L}{2}} dx \sqrt{H(U)}\sqrt{1 + \beta(U)(\partial_x U)^2}, \tag{3.8}$$

where $G_{\rm N}^{(3)} = G_{\rm N}^{(4)}/2\pi l_s$, and

$$U = e^{\phi}, \quad U^2 h^{-1} = 1 + U^2\left(\lambda - \epsilon^2\right), \quad U^{-2}H(U) = U^2 h^{-1}, \quad U^4\beta(U) = (1 + U^2\lambda)\alpha'. \tag{3.9}$$

The boundary conditions (3.5) now take the form

$$U(\pm L/2) = U_\infty, \tag{3.10}$$

where U_∞ is an ultraviolet cutoff.

[1] Here we rescaled the metric (3.3) by the level k.

We denote the value at which the curve U takes its minimum value by U_0. This value is related to the length of the interval L. This follows from the Euler's variational equation of the action (3.8) with the boundary conditions (3.10). One finds

$$L(U_0) = 2\sqrt{H(U_0)} \int_{U_0}^{U_\infty} dU \frac{\sqrt{\beta(U)}}{\sqrt{H(U) - H(U_0)}}. \tag{3.11}$$

The entropy using the Euler's equation of motion becomes

$$S = \frac{\sqrt{k}}{2G_N^{(3)}} \int_{U_0}^{U_\infty} dU \sqrt{\frac{\beta(U)}{H(U) - H(U_0)}} H(U). \tag{3.12}$$

We rewrite the expression (3.11) of the interval length L in terms of the minimum value U_0 as

$$L(U_0) = \frac{\sqrt{\alpha'}}{U_0} \int_1^{x_\infty} \frac{dx}{x} \sqrt{\frac{(1+\alpha x)(1+\alpha_-)}{x(x-1)(\alpha_- x + \alpha_- + 1)}}, \tag{3.13}$$

where

$$\alpha = \lambda U_0^2, \quad \alpha_- = \Psi U_0^2, \quad x_\infty = \frac{U_\infty^2}{U_0^2}. \tag{3.14}$$

The integral (3.13) is ultraviolet convergent and it solves to

$$\frac{L}{2\sqrt{\alpha'\lambda}} = \sqrt{\frac{1+\alpha}{\alpha}} E\left(\arcsin\sqrt{\frac{1+\alpha_-}{1+2\alpha_-}}, \sqrt{\frac{1+2\alpha_-}{(1+\alpha)(1+\alpha_-)}}\right), \tag{3.15}$$

where $E(\varphi, k)$ is the incomplete elliptic integral of the second kind,[2]

$$E(\varphi, k) = \int_0^{\varphi} d\theta \sqrt{1 - k^2 \sin^2\theta}. \tag{3.16}$$

In the limit in which $\Psi \to 0^+$ or, equivalently $\alpha_- \to 0^+$, we note that the interval L (3.15) takes the following simpler form

$$\frac{L}{2\sqrt{\alpha'\lambda}} = \sqrt{\frac{1+\alpha}{\alpha}} E\left(\sqrt{\frac{1}{1+\alpha}}\right), \tag{3.17}$$

where $E(k) = E(\pi/2, k)$ is the complete elliptic integral of the second kind. Sending $\alpha \to 0^+$ in (3.17) yields

$$\frac{L}{\sqrt{\alpha'}} = \frac{2}{U_0}. \tag{3.18}$$

[2] Here and in what follows note our notation of elliptic integrals. There are different notations of elliptic integrals in the literature.

We rewrite the entanglement entropy (3.12) as

$$S=\frac{\sqrt{k\alpha'}}{4G_{\mathrm{N}}^{(3)}}\int\limits_{1}^{x_\infty}dx\sqrt{\frac{\alpha x+1}{x(x-1)(\alpha_- x+\alpha_-+1)}}\cdot(\alpha_- x+1). \tag{3.19}$$

We note that for $\alpha_-\neq 0$ the entropy diverges as $S\sim x_\infty$, and in the limit in which we take $\alpha_-\to 0^+$ with $\alpha\neq 0$ it diverges as $S\sim\sqrt{x_\infty}$. We also note that in the case in which we take both $\alpha\to 0^+,\alpha_-\to 0^+$ it diverges as $S\sim\log(x_\infty)$. The integral (3.19) solves with the ultraviolet cutoff x_∞ to

$$\begin{aligned}S=&\frac{\sqrt{k\alpha'}}{2G_{\mathrm{N}}^{(3)}}\frac{1}{\sqrt{(\alpha+1)(\alpha_-+1)}}\Bigg\{\left(\alpha_-+\alpha-\alpha\alpha_-\frac{d}{d\xi}\right)\\&\left[\frac{1}{\xi+1}\cdot\Pi\left(\arcsin\sqrt{\frac{\alpha_-+1}{2\alpha_-+1}\cdot\left(1-\frac{1}{x_\infty}\right)},\frac{2\alpha_-+1}{(\xi+1)(\alpha_-+1)},\sqrt{\frac{2\alpha_-+1}{(\alpha+1)(\alpha_-+1)}}\right)\right]_{\xi=0}\\&+F\left(\arcsin\sqrt{\frac{\alpha_-+1}{2\alpha_-+1}\cdot\left(1-\frac{1}{x_\infty}\right)},\sqrt{\frac{2\alpha_-+1}{(\alpha+1)(\alpha_-+1)}}\right)\Bigg\},\end{aligned} \tag{3.20}$$

where $\Pi(\varphi,n,k)$ is the incomplete elliptic integral of the third kind, and $F(\varphi,k)=\Pi(\varphi,0,k)$ is the incomplete elliptic integral of the first kind,

$$\Pi(\varphi,n,k)=\int\limits_0^{\varphi}\frac{d\theta}{(1-n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}. \tag{3.21}$$

In the limit $\Psi\to 0^+$ or, equivalently $\alpha_-\to 0^+$, the entropy (3.20) gives

$$\begin{aligned}S=&\frac{\sqrt{k\alpha'}\sqrt{1+\alpha}}{2G_{\mathrm{N}}^{(3)}}\left[F\left(\arcsin\sqrt{1-\frac{1}{x_\infty}},\sqrt{\frac{1}{1+\alpha}}\right)-E\left(\arcsin\sqrt{1-\frac{1}{x_\infty}},\sqrt{\frac{1}{1+\alpha}}\right)\right]\\&+\frac{\sqrt{k\alpha'}}{2G_{\mathrm{N}}^{(3)}}\sqrt{(\alpha x_\infty+1)\cdot\left(1-\frac{1}{x_\infty}\right)}.\end{aligned} \tag{3.22}$$

Taking $\alpha\to 0^+$ in (3.22) gives

$$S=\frac{\sqrt{k\alpha'}}{2G_{\mathrm{N}}^{(3)}}\mathrm{Log}(2\sqrt{x_\infty}). \tag{3.23}$$

In the rest of the current section we study the above results in turns.

3.1. *Case* $\Psi=0:\lambda=0,\ \epsilon=0$

In this case we have for the interval length L and entanglement entropy S from (3.18) and (3.23)

$$\frac{L}{\sqrt{\alpha'}}=\frac{2}{U_0},\quad S=\frac{\sqrt{k\alpha'}}{2G_{\mathrm{N}}^{(3)}}\mathrm{Log}\left(2\frac{U_\infty}{U_0}\right). \tag{3.24}$$

We write the entropy as

$$S=\frac{c}{3}\mathrm{Log}\left(2\frac{L}{L_\Lambda}\right),\quad \frac{L_\Lambda}{\sqrt{\alpha'}}:=\frac{2}{U_\infty},\quad c=\frac{3\sqrt{k\alpha'}}{2G_N^{(3)}},\tag{3.25}$$

where L_Λ is an ultraviolet cutoff. This result is the well–known entanglement entropy for a two–dimensional holographic conformal field theory with (Brown–Henneaux) central charge c that is dual to pure AdS_3 [39,29,40].

3.2. Case $\Psi=0:\lambda=\epsilon^2\neq 0$

In this case we have from (3.17) and (3.22) that the interval length L and entanglement entropy S are given by

$$\frac{L}{2\sqrt{\alpha'\lambda}}=\sqrt{\frac{1+\alpha}{\alpha}}E\left(\sqrt{\frac{1}{1+\alpha}}\right),\tag{3.26}$$

$$\begin{aligned}S=&\frac{\sqrt{k\alpha'}\sqrt{1+\alpha}}{2G_N^{(3)}}\left[F\left(\arcsin\sqrt{1-\frac{1}{x_\infty}},\sqrt{\frac{1}{1+\alpha}}\right)-E\left(\arcsin\sqrt{1-\frac{1}{x_\infty}},\sqrt{\frac{1}{1+\alpha}}\right)\right]\\&+\frac{\sqrt{k\alpha'}}{2G_N^{(3)}}\sqrt{(\alpha x_\infty+1)\cdot\left(1-\frac{1}{x_\infty}\right)},\quad \alpha=\lambda U_0^2,\quad x_\infty=\frac{U_\infty^2}{U_0^2}.\end{aligned}\tag{3.27}$$

Note that this case corresponds taking $F=0$ in (1.5).

We find from (3.26) that in the large U_0 limit the interval length L asymptotes to a minimum value which we denote by L_0. It takes the value

$$L_0=\pi\sqrt{\alpha'\lambda}=\sqrt{\pi t}.\tag{3.28}$$

We find using (3.26) the following large U_0 expansion of the interval length L,

$$\frac{L}{L_0}=1+\frac{1}{4}\cdot\frac{1}{\alpha}-\frac{3}{64}\cdot\frac{1}{\alpha^2}+\mathcal{O}\left(\frac{1}{\alpha^3}\right),\quad \alpha=\lambda U_0^2.\tag{3.29}$$

Inverting the above equation one finds

$$\alpha=\frac{1}{4\xi}-\frac{3}{16}+\mathcal{O}(\xi),\quad \xi=\frac{L}{L_0}-1.\tag{3.30}$$

We will use this result momentarily to write the entropy in the ultraviolet in terms of the length L.

The small U_0 expansion of the interval length L is

$$\frac{L}{\sqrt{\alpha'}}=\frac{2}{U_0}\left(1-\frac{1}{4}\alpha\ln\alpha+\frac{1}{32}\alpha^2\ln\alpha+\mathcal{O}(\alpha^3)\right),\quad \alpha=\lambda U_0^2.\tag{3.31}$$

The leading term corresponds to the deep AdS_3 geometry (see (3.24)). Therefore, in the large L limit the surface is deep inside the bulk. We note also that the correction starts at order $\lambda=\epsilon^2$.

Inverting equation (3.31) we find

$$\alpha=\left(\frac{2}{\pi}\cdot\frac{L_0}{L}\right)^2\left[1-\left(\frac{2}{\pi}\cdot\frac{L_0}{L}\right)^2\mathrm{Log}\left(\frac{2}{\pi}\cdot\frac{L_0}{L}\right)+\mathcal{O}\left(\frac{2}{\pi}\cdot\frac{L_0}{L}\right)^4\right].\tag{3.32}$$

We can similarly study the large and small U_0 or, equivalently, the large and small L limits of the entanglement entropy (3.27). One finds in the large interval length L limit

$$S = \frac{c}{3}\left[\sqrt{\alpha x_\infty} - \frac{1}{2}\mathrm{Log}(\alpha) - \frac{\alpha}{8}\mathrm{Log}(\alpha) + \mathcal{O}\left(\alpha^2\right)\right], \quad \alpha = \lambda U_0^2, \quad x_\infty = \frac{U_\infty^2}{U_0^2}, \tag{3.33}$$

which upon using (3.32) gives

$$S = \frac{c}{3}\left[\frac{L_0}{L_\Lambda} - \mathrm{Log}\left(\frac{2}{\pi}\cdot\frac{L_0}{L}\right) + \frac{1}{4}\left(\frac{2}{\pi}\cdot\frac{L_0}{L}\right)^2 \mathrm{Log}\left(\frac{2}{\pi}\cdot\frac{L_0}{L}\right) + \mathcal{O}\left(\frac{2}{\pi}\cdot\frac{L_0}{L}\right)^4\right], \tag{3.34}$$

where

$$L_\Lambda := \frac{\pi\sqrt{\alpha'}}{U_\infty}, \tag{3.35}$$

and L_Λ is an ultraviolet cutoff. The leading logarithmic term is the contribution from the AdS_3 region found deep inside the bulk. The coefficient of this term is $-c/3$, as expected. We note that the logarithmic terms depend on the ratio L/L_0, and thence L_0 sets the non–locality scale of the theory. This will become even more evident in the next section.

As we approach L_0 we find

$$S = \frac{c}{3}\left[\sqrt{\alpha x_\infty} + \frac{\pi}{4}\frac{1}{\alpha^{1/2}} - \frac{\pi}{32}\frac{1}{\alpha^{3/2}} + \mathcal{O}\left(\frac{1}{\alpha^{5/2}}\right)\right], \quad \alpha = \lambda U_0^2, \quad x_\infty = \frac{U_\infty^2}{U_0^2}, \tag{3.36}$$

which simplifies using (3.30) to

$$S = \frac{c}{3}\left[\frac{L_0}{L_\Lambda} + \frac{\pi}{2}\left(\frac{L}{L_0} - 1\right)^{\frac{1}{2}} - \frac{\pi}{16}\left(\frac{L}{L_0} - 1\right)^{\frac{3}{2}} + \mathcal{O}\left(\left(\frac{L}{L_0} - 1\right)^{\frac{5}{2}}\right)\right]. \tag{3.37}$$

We note that there is no a logarithmic correction term. The entanglement entropy instead shows a square root area law correction at next–to–leading order. The entanglement entropy scales as the square root of the length of the interval. Such power law scaling is quite peculiar and interesting. In local and Lorentz–invariant even–dimensional quantum field theories, however, in general the presence of a logarithmic correction term is generic, and its coefficient is expected to be universal [41]. It would be interesting to understand this theory better. We discuss its entropic c–function in the next section.

3.3. Case $\Psi > 0 : \epsilon = 0$

This case is studied in [42]. Setting $\alpha = \alpha_-$ in (3.15) and (3.20) we find for the length L and entanglement entropy S

$$\frac{L}{2\sqrt{\alpha'\lambda}} = \sqrt{\frac{1+\alpha}{\alpha}}\,E\left(\arcsin\sqrt{\frac{1+\alpha}{1+2\alpha}}, \sqrt{\frac{1+2\alpha}{1+2\alpha+\alpha^2}}\right), \tag{3.38}$$

$$
\begin{aligned}
S =& \frac{\sqrt{k\alpha'}}{2G_{\mathrm{N}}^{(3)}}\frac{1}{\alpha+1}\left\{\left(2\alpha-\alpha^2\frac{d}{d\xi}\right)\right.\\
&\left[\frac{1}{\xi+1}\cdot\Pi\left(\arcsin\sqrt{\frac{\alpha+1}{2\alpha+1}\cdot\left(1-\frac{1}{x_\infty}\right)},\frac{2\alpha+1}{(\xi+1)(\alpha+1)},\sqrt{\frac{2\alpha+1}{\alpha^2+2\alpha+1}}\right)\right]_{\xi=0}\\
&\left.+F\left(\arcsin\sqrt{\frac{\alpha+1}{2\alpha+1}\cdot\left(1-\frac{1}{x_\infty}\right)},\sqrt{\frac{2\alpha+1}{\alpha^2+2\alpha+1}}\right)\right\},\quad \alpha=\lambda U_0^2,\quad x_\infty=\frac{U_\infty^2}{U_0^2}.
\end{aligned}
\tag{3.39}
$$

We find that in the large U_0 limit the interval length L asymptotes to a minimum value which we denote by L_0 (this should cause no ambiguity)

$$L_0=\frac{\pi\sqrt{\alpha'\lambda}}{2}=\frac{1}{2}\sqrt{\pi t}. \tag{3.40}$$

We note that there is a factor of 2 difference between (3.28) and (3.40). We find from (3.38) that the interval length L has the following large U_0 expansion

$$\frac{L}{L_0}=1+\frac{2}{\pi\alpha}+\frac{3\pi-16}{16\pi\alpha^2}+\mathcal{O}\left(\frac{1}{\alpha^3}\right),\quad \alpha=\lambda U_0^2. \tag{3.41}$$

Inverting the above equation one finds

$$\alpha=\frac{2}{\pi\xi}+\frac{3\pi-16}{32}+\mathcal{O}(\xi),\quad \xi=\frac{L}{L_0}-1. \tag{3.42}$$

The small U_0 expansion takes the form

$$\frac{L}{\sqrt{\alpha'}}=\frac{2}{U_0}\left[1-\frac{\alpha^2}{4}\mathrm{Log}(\alpha)+\frac{3}{8}\alpha^3\mathrm{Log}(\alpha)+\mathcal{O}\left(\alpha^4\right)\right],\quad \alpha=\lambda U_0^2. \tag{3.43}$$

We note that for a long interval the surface is deep inside the bulk in the AdS_3 region. We also note that the term linear in α is zero. Therefore, the correction starts, in this case, at order λ^2.

Inverting the above equation we find

$$\alpha=\left(\frac{4}{\pi}\frac{L_0}{L}\right)^2\left[1-\left(\frac{4}{\pi}\frac{L_0}{L}\right)^4\mathrm{Log}\left(\frac{4}{\pi}\frac{L_0}{L}\right)+\mathcal{O}\left(\left(\frac{4}{\pi}\frac{L_0}{L}\right)^6\right)\right]. \tag{3.44}$$

In the large L limit we find that the entanglement entropy S has the following series expansion

$$S=\frac{c}{3}\left[\frac{1}{2}\alpha x_\infty-\frac{1}{2}\mathrm{Log}(\alpha)-\frac{1}{4}\alpha+\mathcal{O}\left(\alpha^2\mathrm{Log}(\alpha)\right)\right],\quad \alpha=\lambda U_0^2,\quad x_\infty=\frac{U_\infty^2}{U_0^2}, \tag{3.45}$$

which using (3.44) gives

$$
\begin{aligned}
S=&\frac{c}{3}\left[\frac{1}{2}\cdot\frac{L_0^2}{L_\Lambda^2}\cdot\frac{16}{\pi^2}-\mathrm{Log}\left(\frac{L_0}{L}\cdot\frac{4}{\pi}\right)-\frac{1}{4}\left(\frac{L_0^2}{L^2}\cdot\frac{16}{\pi^2}\right)+\mathcal{O}\left(\frac{L_0^2}{L^2}\cdot\frac{16}{\pi^2}\right)^2\right],\\
U_\infty :=& \frac{2\sqrt{\alpha'}}{L_\Lambda}.
\end{aligned}
\tag{3.46}
$$

The (leading) logarithmic term is due to the deep AdS_3 region in the bulk. The coefficient of this term is $-c/3$, as expected. We also note that the arguments of the leading logarithmic terms in (3.34) and (3.46) are also equal despite the different L_0 values.

As we approach L_0 the entropy S takes the form

$$S = \frac{c}{3}\left[\frac{1}{2}\alpha x_\infty - \frac{1}{2}\text{Log}(\alpha) + \mathcal{O}\left(\frac{1}{\alpha}\right)\right], \quad \alpha = \lambda U_0^2, \quad x_\infty = \frac{U_\infty^2}{U_0^2}. \tag{3.47}$$

Using (3.42) this gives

$$S = \frac{c}{3}\left[\frac{1}{2}\cdot\frac{16}{\pi^2}\cdot\frac{L_0^2}{L_\Lambda^2} + \frac{1}{2}\text{Log}\left(\frac{\pi}{2}\left(\frac{L}{L_0}-1\right)\right) + \mathcal{O}\left(\frac{L}{L_0}-1\right)\right]. \tag{3.48}$$

In this limit the geometry is a linearly varying dilaton background. We note that in this case we have a logarithmically divergent term as opposed to the former $\Psi = 0$ case. The coefficient of this term is $c/6$.

3.4. *Case* $\Psi > 0 : \epsilon \neq 0$

In the large U_0 limit we find from (3.15) that the interval length approach a minimum value L_0

$$L_0 = \frac{\pi\sqrt{\alpha'\lambda}}{2} = \frac{1}{2}\sqrt{\pi t}, \tag{3.49}$$

that is determined solely by λ.

The length L has the following large U_0 expansion

$$\frac{L}{L_0} = 1 + \frac{2-\delta^2}{\pi(1-\delta^2)\alpha} + \frac{3\pi - 16 + 2(4-\pi)\delta^2 - \pi\delta^4}{16\pi(1-\delta^2)^2\alpha^2} + \mathcal{O}\left(\frac{1}{\alpha^3}\right), \quad \delta^2 = \frac{\epsilon^2}{\lambda}, \quad \alpha = \lambda U_0^2. \tag{3.50}$$

Inverting this we find

$$\alpha = \frac{2-\delta^2}{\pi(1-\delta^2)\xi} + \frac{3\pi - 16 + 2(4-\pi)\delta^2 - \pi\delta^4}{32 - 48\delta^2 + 16\delta^4} + \mathcal{O}(\xi), \quad \xi = \frac{L}{L_0} - 1. \tag{3.51}$$

The small U_0 expansion takes the form

$$\frac{L}{\sqrt{\alpha'}} = \frac{2}{U_0}\left[1 - \frac{\delta^2}{4}\alpha\text{Log}(\alpha) - \frac{\alpha^2}{4}(1+\delta^2 p)\text{Log}(\alpha) + \mathcal{O}(\alpha^3)\right], \quad \delta^2 = \frac{\epsilon^2}{\lambda}, \quad \alpha = \lambda U_0^2, \tag{3.52}$$

here p is a polynomial in δ^2. The leading term corresponds to the deep AdS_3 region. We note that in this case the correction starts at order ϵ^2.

Inverting the above equation one finds

$$\begin{aligned}\alpha = &\left(\frac{4}{\pi}\frac{L_0}{L}\right)^2\left(1 - \delta^2\left(\frac{4}{\pi}\frac{L_0}{L}\right)^2\text{Log}\left(\frac{4}{\pi}\frac{L_0}{L}\right)\right.\\ &\left.- \left(\frac{4}{\pi}\frac{L_0}{L}\right)^4(1+\delta^2 p)\text{Log}\left(\frac{4}{\pi}\frac{L_0}{L}\right) + \mathcal{O}\left(\left(\frac{4}{\pi}\frac{L_0}{L}\right)^6\right)\right).\end{aligned} \tag{3.53}$$

In the large L limit we find the following expansion for the entropy S

$$S = \frac{c}{3}\left[\frac{1}{2}\alpha x_\infty \sqrt{1-\delta^2} - \frac{1}{2}\mathrm{Log}(\alpha) - \frac{1}{4}\alpha\left(\frac{1}{(1-\delta^2)^2} + \frac{1}{2}\delta^2 \mathrm{Log}(\alpha)\right) + \mathcal{O}\left(\alpha^2\right)\right], \tag{3.54}$$

which upon using (3.53) gives

$$\begin{aligned} S = \frac{c}{3}\Bigg[& \left(\frac{4L_0}{\pi L_\Lambda}\right)^2 \sqrt{\frac{1-\delta^2}{4}} - \mathrm{Log}\left(\frac{4L_0}{\pi L}\right) \\ & - \frac{1}{4}\left(\frac{4L_0}{\pi L}\right)^2 \left[\frac{1}{(1-\delta^2)^2} - \delta^2 \mathrm{Log}\left(\frac{4L_0}{\pi L}\right)\right] + \mathcal{O}\left(\frac{L_0}{L}\right)^4 \Bigg], \end{aligned} \tag{3.55}$$

where L_Λ is defined in (3.46). The leading logarithmic term as in the previous cases corresponds to the AdS_3 region found deep inside the bulk. The coefficient of this term is $-c/3$, as expected.

As we approach L_0 we find

$$S = \frac{c}{3}\left[\frac{1}{2}\alpha x_\infty - \frac{2-\delta^2}{4\sqrt{1-\delta^2}}\mathrm{Log}(\alpha) + \mathcal{O}\left(\frac{1}{\alpha}\right)\right], \tag{3.56}$$

which using (3.51) gives

$$\begin{aligned} S = \frac{c}{3}\Bigg[& \frac{1}{2}\cdot\frac{16}{\pi^2}\cdot\frac{L_0^2}{L_\Lambda^2}\cdot\sqrt{1-\delta^2} + \frac{2-\delta^2}{4\sqrt{1-\delta^2}}\mathrm{Log}\left(\left(\frac{L}{L_0}-1\right)\cdot\frac{\pi(1-\delta^2)}{2-\delta^2}\right) \\ & + \mathcal{O}\left(\frac{L}{L_0}-1\right)\Bigg]. \end{aligned} \tag{3.57}$$

We note that the leading logarithmic term has a coefficient that depends on δ^2.

In the next section we study in the above cases the Casin–Huerta entropic c–function.

4. Casin–Huerta entropic *c*–function

In quantum field theory entanglement entropy is ultraviolet divergent. It requires an ultraviolet cutoff to regularize the divergence. However, in two–dimensional local and Lorentz–invariant quantum field theories the Casin–Huerta entropic c–function [43] which is derived from the entanglement entropy as

$$C := L\frac{\partial S}{\partial L}, \tag{4.1}$$

is independent of the ultraviolet cutoff and finite. The interval length L is interpreted as a renormalization group scale.

The entropic c–function is also a monotonic function of L, and at fixed points of renormalization group flow it is proportional to the corresponding central charges. For scale invariant theories the entropic c–function is constant; it is independent of L.

The entropic c–function is a useful tool to probe phase transitions. In this section we study the entropic c–function for the different cases we studied in the former section. We study in each case its monotonicity as a function of L and its independence of the ultraviolet regulator. We also examine its behavior in the ultraviolet regime.

4.1. *Case* $\Psi = 0 : \lambda = 0,\ \epsilon = 0$

In this case we have

$$C = \frac{c}{3}. \tag{4.2}$$

This is the result for a two–dimensional holographic conformal field theory with central charge c [43]. It is independent of the interval length L. The entropic c–function is non–negative and constant.

4.2. *Case* $\Psi = 0 : \lambda = \epsilon^2 \neq 0$

In this case we find that the entropic c–function $C(\alpha)$ is given by

$$C = \frac{c}{3}\sqrt{1+\alpha}\,E\left(\sqrt{\frac{1}{1+\alpha}}\right), \quad \alpha = \lambda U_0^2. \tag{4.3}$$

We study the small and large U_0 limit, or equivalently the large and small L limits of the entropic c–function (4.3).

In the large L limit we find

$$C = \frac{c}{3}\left(1 + \frac{2}{\pi^2\xi^2}\mathrm{Log}\left(\frac{2}{\pi}\cdot\xi\right) + \mathcal{O}\left(\frac{1}{\xi^4}\right)\right), \quad L := \xi L_0, \quad L_0 = \pi\sqrt{\alpha'\lambda} = \sqrt{\pi t}. \tag{4.4}$$

In the small L limit we find

$$C = \frac{c}{3}\left(\frac{\pi}{4}\cdot\frac{1}{\xi^{\frac{1}{2}}} + \frac{5\pi}{32}\cdot\xi^{\frac{1}{2}} + \mathcal{O}(\xi^{3/2})\right), \quad \xi = \frac{L}{L_0} - 1. \tag{4.5}$$

One can think of L as a renormalization group scale. We note that the entropic c–function increases as we ascend the renormalization group, and it diverges in the ultraviolet at L_0. At short distances it diverges as

$$C \sim \xi^{-\frac{1}{2}}, \quad \xi = \frac{L}{L_0} - 1. \tag{4.6}$$

This is the case since in the ultraviolet the theory is not governed by a fixed point. We also note that the entropic c–function is independent of the ultraviolet cutoff that we introduced to regularize the entanglement entropy. The variable α can be expressed using the result (3.26) in terms of the interval length L and the coupling t to write (4.3) as a function of only L and t.

4.3. *Case* $\Psi > 0 : \epsilon = 0$

This case is studied in [42].[3] In this case we find that the entropic c–function $C(\alpha)$ in closed–form is given by

$$C = \frac{c}{3}(1+\alpha)E\left(\arcsin\sqrt{\frac{1+\alpha}{1+2\alpha}}, \sqrt{\frac{1+2\alpha}{(1+\alpha)^2}}\right), \quad \alpha = \lambda U_0^2. \tag{4.7}$$

[3] It is also studied in closely related works [44,45].

Using the results (3.42) and (3.44) for U_0 we find in the large L limit

$$C = \frac{c}{3}\left(1 + \frac{8}{\pi^2\xi^2} + \mathcal{O}\left(\frac{1}{\pi\xi}\right)^4\right), \quad \xi := \frac{L}{L_0}, \quad L_0 = \frac{\pi\sqrt{\alpha'\lambda}}{2} = \frac{1}{2}\sqrt{\pi t}, \tag{4.8}$$

and in the small L limit

$$C = \frac{c}{3}\left(\frac{1}{2}\left(1 - \frac{1}{\xi}\right)^{-1} + \mathcal{O}\left(1 - \frac{1}{\xi}\right)\right), \quad \xi := \frac{L}{L_0}. \tag{4.9}$$

We also note in this case that the entropic c–function is non–negative and increasing. At short distances it diverges as

$$C \sim \xi^{-1}, \quad \xi = \frac{L}{L_0} - 1. \tag{4.10}$$

Our results (4.8) and (4.9) are in agreement with the corresponding analyses in [42]. Our result (4.7) gives a non–perturbative answer and it can be written as a function of the interval length L and the coupling t. We also note from (4.7) that the entropic c–function is independent of the ultraviolet cutoff.

4.4. *Case* $\Psi > 0 : \epsilon \neq 0$

In this case we find that the c–function $C(\alpha, \chi)$ is given by

$$C = \frac{c}{3}\sqrt{(1+\alpha)(1+\alpha\chi)}E\left(\arcsin\sqrt{\frac{1+\alpha\chi}{1+2\alpha\chi}}, \sqrt{\frac{1+2\alpha\chi}{(1+\alpha)(1+\alpha\chi)}}\right), \tag{4.11}$$

where

$$\alpha = \lambda U_0^2, \quad \Psi = \lambda\chi. \tag{4.12}$$

In this case the large L limit of the entropic c–function takes the form

$$C = \frac{c}{3}\left(1 + \frac{8}{\pi^2\xi^2}\left(\frac{1}{\chi^2} + (1-\chi)\mathrm{Log}\left(\frac{\pi\xi}{4}\right)\right) + \mathcal{O}\left(\frac{1}{\pi\xi}\right)^4\right), \tag{4.13}$$

where

$$\xi := \frac{L}{L_0}, \quad L_0 = \frac{\pi\sqrt{\alpha'\lambda}}{2} = \frac{1}{2}\sqrt{\pi t}. \tag{4.14}$$

In the small L limit we find

$$C = \frac{c}{3}\left(\frac{1+\chi}{4\sqrt{\chi}}\left(1 - \frac{1}{\xi}\right)^{-1} + \mathcal{O}\left(1 - \frac{1}{\xi}\right)\right), \quad \xi := \frac{L}{L_0}. \tag{4.15}$$

In this case also the entropic c–function is non–negative, ultraviolet cutoff independent and increasing. In the ultraviolet it diverges as

$$C \sim \chi^{-\frac{1}{2}} \cdot \xi^{-1}, \quad \xi = \frac{L}{L_0} - 1. \tag{4.16}$$

We note that setting $\chi = 1$ in (4.11) gives (4.7), and setting $\chi = 0$ gives (4.3). At $\alpha = 0$ it gives (4.2). The entropic c–function C (4.11) and the interval length L (3.15) satisfy the curious relation

$$\frac{C}{c_0} = \frac{L}{l_0} \cdot \sqrt{1+\alpha\chi}, \quad l_0 = \frac{2\sqrt{\alpha'}}{U_0}, \quad c_0 = \frac{c}{3}, \quad \alpha = \lambda U_0^2, \tag{4.17}$$

where l_0 and c_0 can be thought of as the interval length L and the entropic c–function C at $\lambda = 0$ or in the infrared, respectively.

5. Discussion

In this paper we computed the (Von Neumann) entanglement entropy and the entropic c–function for an interval of length L. We found that the entropic c–function is given by

$$C = \frac{c}{3}\sqrt{(1+\alpha)(1+\alpha\chi)}E\left(\arcsin\sqrt{\frac{1+\alpha\chi}{1+2\alpha\chi}}, \sqrt{\frac{1+2\alpha\chi}{(1+\alpha)(1+\alpha\chi)}}\right), \tag{5.1}$$

where[4]

$$\alpha = \lambda U_0^2, \quad \chi = 1-\delta^2, \quad \delta^2 = \frac{\epsilon^2}{\lambda} = \frac{\pi\mu^2}{8t}. \tag{5.2}$$

The variable α is related to the interval length L via

$$\frac{L}{2\sqrt{\alpha'\lambda}} = \sqrt{\frac{1+\alpha}{\alpha}}E\left(\arcsin\sqrt{\frac{1+\alpha\chi}{1+2\alpha\chi}}, \sqrt{\frac{1+2\alpha\chi}{(1+\alpha)(1+\alpha\chi)}}\right), \quad \alpha'\lambda = \frac{t}{\pi}. \tag{5.3}$$

We found that the entropic c–function is non–negative and ultraviolet cutoff independent. This is required for a theory that is internally consistent and it is a non–trivial consistency check. Therefore, this provides further evidence that the deformed theory is very sound and under control. We also found that along the renormalization group upflow towards the ultraviolet it is non–decreasing. At long distances it is proportional to the central charge of the original conformal field theory. At short distances it diverges. This is the case since in the ultraviolet the deformed theory is not governed by an ultraviolet fixed point. The minimum distance at which the entropic c–function diverges sets the non–locality scale of the theory.

In Fig. 1 we show plots of the entropic c–function C (5.1) as a function of the interval length L (5.3) for $\delta = 0$ and $\delta^2 = 1$. In the plots we normalized C with respect to $c_0 = \frac{c}{3}$ and L with respect to $L_0 = \frac{1}{2}\sqrt{\pi t}$.

In the case in which $F > 0$, the entropic c–function diverges in the ultraviolet as

$$C \sim \chi^{-\frac{1}{2}} \cdot \xi^{-1}, \quad \xi = \frac{L}{L_0} - 1, \quad L_0 := \frac{\pi\sqrt{\alpha'\lambda}}{2} = \frac{1}{2}\sqrt{\pi t}. \tag{5.4}$$

Note that L_0 is determined solely by λ; it is independent of ϵ.

In the case in which $F = 0$, the entropic c–function diverges at short distances as

$$C \sim \xi^{-\frac{1}{2}}, \quad \xi = \frac{L}{L_0} - 1, \quad L_0 := \pi\sqrt{\alpha'\lambda} = \sqrt{\pi t}. \tag{5.5}$$

4 In terms of the field theory side deformation parameters δ^2 is given by $\frac{\pi\mu^2}{8t}$, and $\mu = 2\mu_+ = 2\mu_-$.

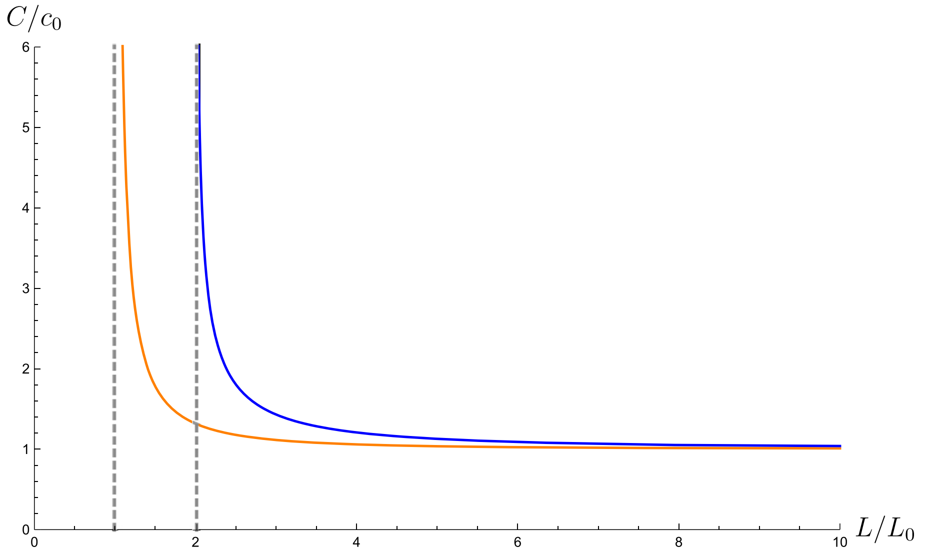

Fig. 1. Entropic c–function C per c_0 as a function of interval length L per L_0. Where $c_0 = \frac{c}{3}$ and $L_0 = \frac{1}{2}\sqrt{\pi t}$. The orange plot is for $\delta = 0$, and the blue plot is for $\delta^2 = 1$. The (normalized) entropic c–function for the case $\delta = 0$ diverges at L_0. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Note that the minimum interval length L_0 in this case is twice larger than the corresponding length in the former case. This is depicted in Fig. 1 with $\delta = 0$ and $\delta^2 = 1$.

We note that the exponent of ξ in the case in which $F > 0$ is -1, and in the case in which $F = 0$ it is $-1/2$. This is due to the fact that at short distances the entanglement entropy for the case in which $F > 0$ (3.57) contains a logarithmically divergent term (with a coefficient that depends on χ) whereas in the case in which $F = 0$ (3.37) it does not contain a logarithmically divergent term. In the latter case the entanglement entropy area law exhibits a square root correction (3.37). The entanglement entropy scales as the square root of the length of the interval,

$$S = \frac{c}{3}\left[\frac{L_0}{L_\Lambda} + \frac{\pi}{2}\left(\frac{L}{L_0} - 1\right)^{\frac{1}{2}} + \mathcal{O}\left(\left(\frac{L}{L_0} - 1\right)^{\frac{3}{2}}\right)\right], \tag{5.6}$$

where L_Λ is the ultraviolet cutoff. Such power law scaling is quite distinct and interesting.

It would be interesting to compute the entropic c–function directly in $J\overline{T}$, $T\overline{J}$, $T\overline{T}$ deformed field theory, for example in perturbation theory, and compare it with the bulk calculation result (5.1) (where α is given by (5.3)). It would be also nice to understand better the theory corresponding to the case in which $F = 0$. We leave these for future work.

In this paper we mainly focused on the case in which $F \geq 0$. As we mentioned in the introduction, in the case $F < 0$ or equivalently $\Psi < 0$, however, the bulk geometry is singular and/or it has either closed timelike curves or no timelike direction. In the field theory side this is related to the presence of states with complex energies. If we look, for example, the case where $\epsilon = 0$, the signature of the boundary (spacetime) metric changes signs as we turn on the coupling λ. The boundary (spacetime) spatial coordinate x becomes temporal at the outset of the deformation and it is not clear how to consistently apply the holographic prescription in this setting. That is, it is not clear as to whether or not we should consider the region beyond the finite radial distance at which the bulk metric signature changes signs. The holographic prescription may also have to be modified if we excise the region. It would be nice to understand the entropic c–function in this case and compare it with results obtained from the closely related double trace deformations. For example, in the holographic proposal with radial cutoff the entropic c–function for an interval of length L in the large c limit is shown to be given by [44][5]

$$C = \frac{c}{3} \cdot \frac{1}{\sqrt{1 - \frac{2\pi}{3} \cdot \frac{ct}{L^2}}}, \tag{5.7}$$

with ct/L^2 finite. It is also shown in [44][6] that this result agrees with a field theory calculation performed in a $T\overline{T}$ deformed conformal field theory. We also left out the case $F \geq 0$ and $\mu_+ \neq \mu_-$ which requires using the Hubeny, Rangamani, and Takayanagi prescription [30]. We leave these for future work.

A fairly similar analysis can be done at finite temperature by considering black hole in the string background (2.6), (2.7), (2.8). At finite temperature the case in which $\epsilon = 0$ is studied in [47,42]. It is shown that the inverse maximum (or Hagedorn) temperature which characterizes the theory in the ultraviolet is given by L_0 (up to numerical factor of order 0),

$$\beta_H = 2\sqrt{\pi t}. \tag{5.8}$$

It is interesting to understand how the maximum temperature depends on the couplings in the other remaining cases. We hope to address this and related questions in future work.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A

We collect the intermediate results required in section 3 to compute the interval length and the Von Neumann entanglement entropy, see [48].

We have (with $u > a > b > c > d$),

$$\int_a^u dx \sqrt{\frac{x-c}{(x-a)(x-b)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(b-c)F(\varphi, k) + (a-b)\Pi(\varphi, n, k)\right], \tag{A.1}$$

[5] See also [45] for a related result.

[6] See also [46] for a related result.

$$\int_a^u dx\sqrt{\frac{x-b}{(x-a)(x-c)(x-d)}}=\frac{2(a-b)}{\sqrt{(a-c)(b-d)}}\Pi(\varphi,n,k), \tag{A.2}$$

$$\int_a^u \frac{dx}{x-b}\sqrt{\frac{x-c}{(x-b)(x-a)(x-d)}}=\frac{2}{a-b}\sqrt{\frac{a-c}{b-d}}E(\varphi,k), \tag{A.3}$$

here

$$\varphi=\arcsin\sqrt{\frac{(b-d)(u-a)}{(a-d)(u-b)}},\quad n=\frac{a-d}{b-d},\quad k=\sqrt{\frac{(b-c)(a-d)}{(a-c)(b-d)}}. \tag{A.4}$$

We have (with $a>u\geq b>c,\ r\neq a$),

$$\int_u^a \frac{dx}{x-r}\frac{1}{\sqrt{(a-x)(x-b)(x-c)}}=\frac{2}{(a-r)\sqrt{a-c}}\Pi(\varphi,n,k), \tag{A.5}$$

here

$$\varphi=\arcsin\sqrt{\frac{a-u}{a-b}},\quad n=\frac{a-b}{a-r},\quad k=\sqrt{\frac{a-b}{a-c}}. \tag{A.6}$$

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