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One-point functions in ABJM theory and integrability

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Tiivistelmä — Referat — Abstract

Correlation functions in a superconformal field theory are strictly constrained by conformal symmetry. Notably, one-point functions of conformal operators always vanish. However, when a defect is inserted into the spacetime of the field theory, certain one-point functions become non-zero due to the broken conformal symmetry, highlighting the special properties of the defect. One interesting type of defect is the domain wall, which separates spacetime into two regions with distinct vacua. The domain wall version of $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory has been extensively studied in recent years. In this context, the supersymmetric domain wall preserves integrability, allowing one to evaluate one-point functions in the defect field theory using integrability techniques.

As an analogous study of the domain wall version of $\mathcal{N} = 4$ SYM theory, this thesis focuses on the ABJM theory with a 1/2-BPS domain wall, meaning that the domain wall preserves half the original supersymmetry. We first review integrability methods, e.g. the Coordinate Bethe ansatz and the Algebraic Bethe ansatz for $\mathfrak{su}(2)$ Heisenberg spin chain. The spectrum of the spin chain can be determined by solving sets of the Bethe equations. Moreover, the Rational Q-system is examined, which solves the Bethe equations efficiently and eliminates all nonphysical solutions automatically.

On the field theory side, we first review the original ABJM theory and its spectral integrability following J. A. Minahan's work in 2009. There exists an underlying quantum $\mathfrak{su}(4)$ spin chain with alternating even and odd sites, whose Hamiltonian can be identified with the two-loop dilation operator of ABJM theory in the planar limit. This correspondence allows us to find the spectrum of ABJM theory using the Bethe ansatz. We study the $\mathfrak{su}(4)$ alternating spin chain and demonstrate the procedure for constructing eigenstates of ABJM theory.

Finally, we study the tree-level one-point functions in the domain wall version of ABJM theory. We derive the classical solutions for the scalar fields that describe a domain wall and explicitly demonstrate how the domain wall preserves half of the supersymmetry. With these classical solutions, we define a domain wall version of ABJM theory. Then, we introduce the so-called Matrix Product State, which is a boundary state in the spin chain's Hilbert space. The domain wall can be identified with an integrable matrix product state, leading to a compact determinant formula for the one-point functions in spin chain language. Consequently, we can evaluate one-point functions explicitly using the Bethe ansatz and boundary integrability.

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Contents

1. Introduction

The Bethe ansatz provides a way to find exact solutions for the eigenvalues and eigenstates of the Hamiltonian in integrable models. It was first introduced by Hans Bethe in 1931 to solve the simplest $\mathfrak{su}(2)$ Heisenberg spin chain [1], known as the Coordinate Bethe Ansatz (CBA). Later, developed by the Leningrad (now Saint Petersburg) School in the mid-1970s, the Quantum Inverse Scattering Method (QISM) or the Algebraic Bethe Ansatz (ABA) plays a central role in solving integrable models [2]. The method relies on the underlying algebraic structure of the model, where the integrability can be proven rigorously. Therefore, the Bethe ansatz can be regarded as the starting point of quantum integrability.

Integrability techniques have found successful applications in gauge theories. The spectral problem in the planar limit of supersymmetric Yang-Mills theory can be mapped to solving one-dimensional integrable spin chains, which was proposed by J. Minahan and K. Zarembo in 2003 [3]. Specifically, they demonstrated that the one-loop dilation operator acting on a sub-sector of scalars in $\mathcal{N} = 4$ SYM can be identified with the Hamiltonian of an integrable Heisenberg spin chain. This observation can be generalized to the full scalar sector, which corresponds to an integrable $\mathfrak{so}(6)$ spin chain. Consequently, the spectrum of anomalous dimensions of scalar composite operators in planar $\mathcal{N} = 4$ SYM is solvable with the Bethe ansatz method, and hence the $\mathcal{N} = 4$ SYM theory is integrable in this sense.

Inspired by the spectrum integrability in $\mathcal{N} = 4$ SYM, another gauge theory proposed by Aharony, Bergman, Jafferis and Maldacena, known as the ABJM theory [4], also turns out to be integrable in the planar limit [5]. Analogous to the $\mathcal{N} = 4$ SYM, the planar two-loop dilation operator in the scalar sector of ABJM theory is identified with the Hamiltonian of an integrable spin chain with sites alternating between the fundamental and the anti-fundamental representations of $\mathfrak{su}(4)$. Besides, ABJM theory has a high degree of superconformal symmetry in 3-dimensions and seems to be an effective theory describing the world-volume theory of a stack of N M2 branes moving on the orbifold $\mathbb{C}^4/\mathbb{Z}_k$. This is another example of the AdS/CFT correspondence conjectured by Juan Maldacena in 1998 [6], referred to as $\mathrm{AdS}_4/\mathrm{CFT}_3$. This correspondence relates two rather different types of models and provides an intuitive

understanding of the origin of integrability in CFT_3; it translates to the integrability of the corresponding string sigma model, for which integrability is a common phenomenon [7, 8]. Consequently, integrability has become an important tool to perform exact calculations on both sides of AdS/CFT.

This holographic duality can be deformed by introducing defects (probe branes) on the field theory (string theory) side, known as the AdS/dCFT correspondence, which leads to novel features on pairs of models. For instance, certain one-point functions become non-zero due to the broken conformal symmetry, highlighting the special properties of the defect. The one-point functions in a domain wall version of $\mathcal{N} = 4$ SYM can be expressed as an overlap between the Bethe state of $\mathfrak{su}(2)$ spin chain and the boundary state corresponds to the defect. The domain wall preserves part of the supersymmetry, leading to boundary integrability and significantly simplifying the overlap [9, 10]. Inspired by the study of domain wall set-ups in $\mathcal{N} = 4$ SYM, one can expect there exists supersymmetric domain wall defects in ABJM theory [11]. With the classical solutions of scalar fields, a 1/2-BPS domain wall version of ABJM theory can be defined. By studying the corresponding boundary state in spin chain picture, it turns out that this domain wall preserves integrability as well. Thus, the overlap between the Bethe state and the integrable boundary state simplifies to a determinant formula. Furthermore, the half-BPS domain wall in ABJM theory is holographic dual to a D4 probe brane embedding in the Type IIA background $\mathrm{AdS}_4 \times \mathbb{CP}^3$, resulting in a D2-D4 probe brane system. Integrability of the Green-Schwarz sigma model in the dual string description was checked in [12], which suggests the ABJM domain wall is integrable to all loop orders and for any value of the bond dimension.

The thesis is organized as follows. We first review integrability methods, e.g. the Coordinate Bethe ansatz and the Algebraic Bethe ansatz for $\mathfrak{su}(2)$ Heisenberg spin chain in Chapter 2. Then, the Rational Q-system is examined in Chapter 3, which solves the Bethe equations efficiently and eliminates all nonphysical solution automatically. In Chapter 4, we review the original ABJM theory and its spectrum integrability. We study the $\mathfrak{su}(4)$ alternating spin chain and demonstrate the procedure for constructing the eigenvectors of ABJM theory. We also introduce the basic concepts of the duality at last. In Chapter 6, we study the tree-level one-point functions in the domain wall version of ABJM theory . We derive the classical solutions for the scalar fields that describe a domain wall and explicitly demonstrate that the domain wall preserves half of the supersymmetry. We define a domain wall version of ABJM theory and briefly introduce its string theory description. Next, we discuss how boundary integrability leads to a compact determinant formula for the one-point functions. This overlap formula allows us to evaluate one-points in the spin chain picture. Finally, Chapter 7 contains our conclusion and outlook.

2. Coordinate Bethe ansatz

In this chapter, we begin with the famous Heisenberg spin chain, a model proposed by Werner Heisenberg in 1928 to study magnetism. We will briefly review how the coordinate Bethe ansatz constructs exact solutions for the eigenvalues and eigenstates of the Heisenberg spin chain. The key idea of the coordinate Bethe ansatz is to express the wave function of the model as a superposition of plane waves.

2.1 Heisenberg spin chain

The Hamiltonian of the spin chain is given by

$$H = -\sum_{n=1}^{L} \left(J_x S_n^x S_{n+1}^x + J_y S_n^y S_{n+1}^y + J_z S_n^z S_{n+1}^z \right), \tag{2.1}$$

where J_x, J_y, J_z are coupling constants in each spacial direction. We consider the simplest case, $J_x = J_y = J_z = J \neq 0$ and spin operators satisfy the commutation relation of $\mathfrak{su}(2)$ algebra, which is known as the $\mathfrak{su}(2)$ Heisenberg XXX spin-chain with Hamiltonian

$$\begin{aligned} H_{\mathrm{XXX}} &= -J \sum_{n=1}^{L} (S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + S_n^z S_{n+1}^z) \\ &= -\frac{J}{2} \sum_{n=1}^{L} (S_n^- S_{n+1}^+ + S_n^+ S_{n+1}^- + 2 S_n^z S_{n+1}^z), \end{aligned} \tag{2.2}$$

where $S^{\pm} \equiv S^x \pm i S^y$ and $S_n^i = \frac{1}{2}\sigma_n^i$, $i = x, y, z$ (in fundamental representation). Following naturally, we should find the spectrum of the H_{XXX} under the periodic boundary condition $S_{L+1}^i \equiv S_1^i$. The Hilbert space of the spin chain is the tensor product of the spins of each site:

$$\mathcal{V} = V_1 \otimes V_2 \otimes \cdots \otimes V_L. \tag{2.3}$$

It is of dimension 2^L since each spin is $V = \mathbb{C}^2$. So the size of the Hamiltonian (2.2) is $2^L \times 2^L$ and hence for large systems it is hard or impossible to diagonalize it directly. Probably the spectrum has to be determined with some other methods.

2.2 General discussion

Without loss of generality, let $J = 1$. For a length-L spin chain, consider spin-ups as vacuum and spin-downs as magnetic excitation (magnons). This decomposes the Hilbert space into several subspaces, the vacuum sector, one magnon sector, two magnon sector, etc., since the Hamiltonian H_{XXX} does not change the number of up spins and down spins (total spin is preserved).

Pseudo-vacuum state with all spin-ups is defined as

$$|\Omega\rangle = |\uparrow\rangle_1 \otimes |\uparrow\rangle_2 \otimes \cdots \otimes |\uparrow\rangle_L \equiv |\uparrow\uparrow\cdots\uparrow\rangle = |\uparrow^L\rangle. \tag{2.4}$$

For simplicity, we omit the symbol of tensor product here. With $S^+|\uparrow\rangle = 0$, $S^z|\uparrow\rangle = \frac{1}{2}|\uparrow\rangle$, we can see

$$H_{\mathrm{XXX}}|\Omega\rangle = -\sum_{n=1}^{L} S_n^z S_{n+1}^z |\Omega\rangle = -\frac{L}{4}|\Omega\rangle \equiv E_0|\Omega\rangle. \tag{2.5}$$

Thus, $|\Omega\rangle$ is indeed the eigenstate of the Hamiltonian and the vacuum energy is $E_0 = -\frac{L}{4}$.

The basis vectors in the N-magnon sector, generated by acting with N lowering operators on the vacuum, are denoted by

$$|n_1, n_2, \ldots, n_N\rangle = S_{n_1}^- S_{n_2}^- \cdots S_{n_N}^- |\Omega\rangle, \tag{2.6}$$

where $n_1, n_2, \ldots, n_N$ are the positions of down spins. Any energy eigenstate with N flipped spins can be expressed as the linear combination of these basis vectors

$$|\psi\rangle = \sum_{1\le n_1<\ldots<n_N\le L} a\left(n_1, n_2 \ldots, n_N\right) |n_1, n_2, \ldots, n_N\rangle\,, \tag{2.7}$$

with some unknown coefficients $a\left(n_1, n_2 \ldots, n_N\right)$ satisfying periodic boundary condition

$$a(n_2, \ldots, n_N, n_1 + L) = a(n_1, \ldots, n_N). \tag{2.8}$$

Hans Bethe proposed a formula for the coefficients in 1931, known as the Coordinate Bethe ansatz (CBA). The inspiration comes from that the Hamiltonian (2.2) is invariant under the one-site shift of the spin chain , since it is homogeneous. Next, let us figure out the CBA from the specific case to the general.

2.3 One magnon sector

The basis vectors in one magnon sector are $|n\rangle$. For any operator X_n acting on site n, the one-site shift operation relocates the operator from site n to site $n + 1$, namely

$U^{-1}X_nU = X_{n+1}$. Easy to see the Hamiltonian (2.2) is invariant under the one-site shift . Thus it commutes with the shift operator U and hence they have simultaneous eigenstates. The one-site shift is just a lattice version of space translation, which implies the energy eigenstate takes the discrete version of a plane-wave

$$|\Psi(p)\rangle = \sum_{n=1}^{L} e^{ipn}|n\rangle. \tag{2.9}$$

Namely, we propose that wave function in coordinate representation is $a(n) = e^{ipn}$ and here parameter p can be regarded as the momentum of magnon. With the following relations

$$\begin{aligned}
\sum_{l=1}^{L} S_l^+ S_{l+1}^-|n\rangle &= |n+1\rangle, \\
\sum_{l=1}^{L} S_l^- S_{l+1}^+|n\rangle &= |n-1\rangle, \\
\sum_{l=1}^{L} S_l^z S_{l+1}^z|n\rangle &= \frac{L-4}{4}|n\rangle,
\end{aligned} \tag{2.10}$$

it is easy to find

$$\begin{aligned}
H_{\mathrm{XXX}}|\Psi(p)\rangle &= \sum_{n=1}^{L} e^{ipn} H_{\mathrm{XXX}}|n\rangle \\
&= -\frac{1}{2}\sum_{n=1}^{L} e^{ipn}\left(|n+1\rangle + |n-1\rangle + \frac{L-4}{2}|n\rangle\right) \\
&= -\frac{1}{2}\left(e^{ip} + e^{-ip} + \frac{L-4}{2}\right)\sum_{n=1}^{L} e^{ipn}|n\rangle \\
&= -\left(\cos(p) + \frac{L-4}{4}\right)|\Psi(p)\rangle \equiv E_1(p)|\Psi(p)\rangle.
\end{aligned} \tag{2.11}$$

The energy of an excitation, one magnon, is

$$\varepsilon(p) = E_1(p) - E_0 = 1 - \cos(p) \geq 0. \tag{2.12}$$

From periodic boundary condition (2.8), we get the momentum quantization condition $e^{ipL} = 1$. Hence, the momentum of magnon only takes values

$$p = \frac{2\pi m}{L}, \quad m = 1, 2, \ldots, L. \tag{2.13}$$

There are L values of p totally, corresponding to L eigenstates. This is consistent with the dimension of the Hilbert space of one magnon sector. (L basis vectors in this sector and hence should have L eigenstates)

2.4 Two magnon sector

In this sector, we propose the following ansatz for the energy eigenstate

$$|\Psi(p_1,p_2)\rangle = \sum_{1\le n_1<n_2\le L} a(\boldsymbol{p}|\boldsymbol{n})|n_1,n_2\rangle, \tag{2.14a}$$

$$a(\boldsymbol{p}|\boldsymbol{n}) = A(p_1,p_2)e^{ip_1n_1+ip_2n_2} + A(p_2,p_1)e^{ip_2n_1+ip_1n_2}, \tag{2.14b}$$

where parameters p_1, p_2 are the momenta of the two magnons. The wave function includes two parts: approximately two single-particle states that are nearly free, and the other states resulting from the exchange of two momenta. The exact expression for the amplitude is not important. We only need the ratio of the amplitude, given by the scattering matrix

$$S(p_1,p_2) = \frac{A(p_2,p_1)}{A(p_1,p_2)}. \tag{2.15}$$

Now we must have

$$H_{\mathrm{XXX}}|\Psi(p_1,p_2)\rangle = E_2(p_1,p_2)|\Psi(p_1,p_2)\rangle. \tag{2.16}$$

With the following relations,

$$\begin{aligned}
&\sum_{l=1}^{L} S_l^+ S_{l+1}^- |n_1,n_2\rangle = |n_1+1,n_2\rangle + |n_1,n_2+1\rangle,\\
&\sum_{l=1}^{L} S_l^- S_{l+1}^+ |n_1,n_2\rangle = |n_1-1,n_2\rangle + |n_1,n_2-1\rangle\,,\\
&\sum_{l=1}^{L} S_l^z S_{l+1}^z |n_1,n_2\rangle = \frac{L-8}{4}|n_1,n_2\rangle,
\end{aligned} \tag{2.17}$$

we can solve the energy $E_2(p_1,p_2)$ and the S-matrix by substituting (2.14a) and (2.14b) into (2.16). Actually, there is a summation over the positions of the down-spins in (2.16), which means (2.16) holds for $|n_1-n_2|>1$ terms and $n_2=n_1+1$ terms separately. We first work in the case that two spin-downs are not neighbors, i.e. $|n_1-n_2|>1$, since (2.17) is only valid in this case.

It is straightforward to get

$$\begin{aligned}
H_{\mathrm{XXX}}\sum_{n_1,n_2} a(\boldsymbol{p}|\boldsymbol{n})|n_1,n_2\rangle =& -\frac{1}{2}\sum_{n_1,n_2} a(\boldsymbol{p}|\boldsymbol{n})\Big(|n_1-1,n_2\rangle + |n_1+1,n_2\rangle\\
&+\frac{L-8}{2}|n_1,n_2\rangle + |n_1,n_2-1\rangle + |n_1,n_2+1\rangle\Big)\\
=& -\sum_{n_1,n_2}\left(\cos(p_1)+\cos(p_2)+\frac{L-8}{4}\right)a(\boldsymbol{p}|\boldsymbol{n})|n_1,n_2\rangle\\
=& E_2(p_1,p_2)\sum_{n_1,n_2} a(\boldsymbol{p}|\boldsymbol{n})|n_1,n_2\rangle.
\end{aligned} \tag{2.18}$$

Thus, the energy derived from the region $|n_1 - n_2| > 1$ is

$$E_2(p_1, p_2) - E_0 = 2 - \cos(p_1) - \cos(p_2) = \varepsilon(p_1) + \varepsilon(p_2). \tag{2.19}$$

This is just the sum of the energy of two magnons. For the $n_2 = n_1 + 1$ case, we claim that the energy $E_2(p_1, p_2)$ still satisfies (2.19) because it is in front of the summation over n_1, n_2 in (2.16). Then with modified relations

$$\begin{aligned}
\sum_{l=1}^{L} S_l^+ S_{l+1}^- |n_1, n_1 + 1\rangle &= |n_1, n_1 + 2\rangle, \\
\sum_{l=1}^{L} S_l^- S_{l+1}^+ |n_1, n_1 + 1\rangle &= |n_1 - 1, n_1 + 1\rangle\,, \\
\sum_{l=1}^{L} S_l^z S_{l+1}^z |n_1, n_1 + 1\rangle &= \frac{L-4}{4} |n_1, n_1 + 1\rangle,
\end{aligned} \tag{2.20}$$

the Schrödinger equation (2.16) gives

$$\left(E_2 + \frac{L-4}{4}\right) a(\boldsymbol{p}|n_1, n_1 + 1) = -\frac{1}{2}\left(a(\boldsymbol{p}|n_1 - 1, n_1 + 1) + a(\boldsymbol{p}|n_1, n_1 + 2)\right). \tag{2.21}$$

The scattering matrix is obtained by substituting explicit form of $a(\boldsymbol{p}|\boldsymbol{n})$ and E_2 into the above equation. Finally we get

$$S(p_1, p_2) = -\frac{1 - 2e^{ip_2} + e^{i(p_1+p_2)}}{1 - 2e^{ip_1} + e^{i(p_1+p_2)}} = \frac{\frac{1}{2}\cot\frac{p_1}{2} - \frac{1}{2}\cot\frac{p_2}{2} - i}{\frac{1}{2}\cot\frac{p_1}{2} - \frac{1}{2}\cot\frac{p_2}{2} + i}. \tag{2.22}$$

The quantization condition for the momenta is derived from the periodic boundary condition $a(\boldsymbol{p}|n_1, n_2) = a(\boldsymbol{p}|n_2, n_1 + L)$. Also, notice that the relation $S(p_1, p_2)S(p_2, p_1) = 1$ clearly follows from (2.22) and is consistent with the definition (2.15) of S-matrix. Then, we get the set of quantization conditions, which is so-called Bethe ansatz equations (BAE)

$$e^{ip_1 L} S(p_1, p_2) = 1, \qquad e^{ip_2 L} S(p_2, p_1) = 1. \tag{2.23}$$

2.5 N-magnon sector

Inspired by the previous discussion, we can propose a similar form for the wave function. An energy eigenstate in N-magnon sector takes the following form

$$|\Psi(\boldsymbol{p})\rangle = \sum_{1 \le n_1 < \ldots < n_N \le L} a(\boldsymbol{p}|\boldsymbol{n}) |n_1, \ldots, n_N\rangle, \tag{2.24a}$$

$$a(\boldsymbol{p}|\boldsymbol{n}) = \sum_{\sigma \in S_N} A(\boldsymbol{p}_\sigma) e^{i\left(p_{\sigma(1)} n_1 + p_{\sigma(2)} n_2 + \ldots + p_{\sigma(N)} n_N\right)}, \tag{2.24b}$$

here we denote the set of all possible permutations of $\{1, ..., N\}$ by S_N. For each σ, the corresponding permutation of momenta $\{p_1, ..., p_N\}$ is denoted by $\boldsymbol{p}_\sigma$. Thus there are $N!$ terms in the sum in (2.24b) while the unknown quantities we need to determine are the eigenvalue E_N and $(N! - 1)$ ratios $A(\boldsymbol{p}_\sigma)/A(\boldsymbol{p})$.

For the total energy E_N, based on the results in one and two magnon sector, we conjecture that the energy of each magnon takes a simple form

$$\varepsilon(p_k) = 1 - \cos(p_k). \tag{2.25}$$

Then, the eigenvalue in N-magnon sector is

$$E_N(\boldsymbol{p}) = E_0 + \sum_{k=1}^{N} \varepsilon(p_k). \tag{2.26}$$

The crucial conjecture for any amplitude is given by

$$A(\cdots, p_j, p_k, \cdots) = S(p_k, p_j) A(\cdots, p_k, p_j, \cdots), \tag{2.27}$$

which comes from the physical interpretation of the two body S-matrix. Then, the ratios $A(\boldsymbol{p}_\sigma)/A(\boldsymbol{p})$ factorize into a sequence of two-body S-matrices since any $A(\boldsymbol{p}_\sigma)$ can be brought to $A(\boldsymbol{p})$ by a sequence of exchanges of adjacent particles. With this conjecture, the periodic boundary condition (2.8) leads to the following set of equations

$$e^{ip_k L} = \prod_{j \neq k}^{N} s(p_j, p_k), \tag{2.28}$$

which is the quantization condition for momenta $\{p_1, ..., p_N\}$, namely N coupled Bethe ansatz equations. Then, let us make a change of variables from momenta to the so-called rapidity,

$$e^{ip_k} = \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}, \qquad u_k = \frac{1}{2} \cot \frac{p_k}{2}, \tag{2.29}$$

resulting in a simpler form of the BAEs

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \right)^L = \prod_{j \neq k}^{N} \frac{u_k - u_j + i}{u_k - u_j - i} \qquad k = 1, 2, ..., N. \tag{2.30}$$

And the dispersion relation for each magnon also becomes simpler

$$\varepsilon(p_k) = 1 - \cos(p_k) = \frac{2}{4u_k^2 + 1}. \tag{2.31}$$

3. Algebraic Bethe ansatz

While the coordinate Bethe ansatz provides a physical picture that magnons propagating in the vacuum and interact with each other in an integrable way, it does not prove the integrability of the system. Let us review another method for solving integrable models, called Quantum Inverse Scattering Method (QISM) or Algebraic Bethe Ansatz (ABA), which was mainly developed by the Leningrad School. In this method, the underlying algebraic structure of the model plays a role and the integrability can be proven rigorously.

3.1 Building blocks

Instead of constructing the eigenstates immediately, we first define the building blocks for the ABA. The starting point is the so-called Lax operator L_{an} defined on the local Hilbert space of lattice site n and the auxiliary space labeled by an abstract index a, namely L_{an} acts on $\mathbb{C}_a \otimes V_n$. It is parameterized by the complex spectral parameter u and generally takes the form:

$$L_{an}(u) = (u - \frac{i}{2})I_{an} + iP_{an}, \tag{3.1}$$

where P_{an} is the permutation operator. For the $\mathfrak{su}(2)$ spin chain, the auxiliary space can be taken as $\mathbb{C}_a^2$ and hence $P_{an} = \frac{1}{2}(I_a \otimes I_n) + \sum_{\alpha=1}^{3} \sigma_a^\alpha \otimes S_n^\alpha$. Then the Lax operator can be rewritten as

$$L_{an}(u) = uI_{an} + i \sum_{\alpha=x,y,z} \sigma_a^\alpha \otimes S_n^\alpha = \begin{pmatrix} u + iS_n^z & iS_n^- \\ iS_n^+ & u - iS_n^z \end{pmatrix}_a, \tag{3.2}$$

which is a 2×2 matrix in $\mathbb{C}_a^2$ with local spin operators serving as entries. And it is familiar that these spin operators satisfy the standard $\mathfrak{su}(2)$ algebra.

The most important property of the Lax operator defined in (3.2) is that it satisfies the RLL-relation:

$$R_{ab}(u - v)L_{an}(u)L_{bn}(v) = L_{bn}(v)L_{an}(u)R_{ab}(u - v), \tag{3.3}$$

where a, b denote two distinct two-dimensional auxiliary spaces and R_{ab} is given by

$$R_{ab}(u-v) = (u-v)I_{ab} + iP_{ab} = \begin{pmatrix} u-v+i & 0 & 0 & 0 \\ 0 & u-v & i & 0 \\ 0 & i & u-v & 0 \\ 0 & 0 & 0 & u-v+i \end{pmatrix}_{ab} . \tag{3.4}$$

The relation (3.3) can be verified explicitly by rewriting L_{an} and L_{bn} as the 4×4 matrices in the tensor product space $\mathbb{C}^2_a \otimes \mathbb{C}^2_b$. It is easy to see that the RLL-relation relies on the underlying $\mathfrak{su}(2)$ algebra.

Now we can build the central object for the ABA, which is known as the Monodromy matrix and defined as the ordered product of Lax operators:

$$M_a(u) = L_{a1}(u)L_{a2}(u)\dots L_{aL}(u) \equiv \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}_a , \tag{3.5}$$

where $A(u), B(u), C(u), D(u)$ are complicated operators consisting of local spin operators. However, for most of our purpose it is unnecessary to figure out the explicit form of these four operators. The point is the algebra they satisfy, which can be extracted from the RMM-relation

$$R_{ab}(u-v)M_a(u)M_b(v) = M_b(v)M_a(u)R_{ab}(u-v). \tag{3.6}$$

This relation directly follows from the RLL-relation (3.3). To find the algebra among the operators A, B, C, D, we similarly rewrite the monodromy as the 4×4 matrix:

$$M_a(u) = M_a(u) \otimes \mathbb{I}_b = \begin{pmatrix} A(u) & 0 & B(u) & 0 \\ 0 & A(u) & 0 & B(u) \\ C(u) & 0 & D(u) & 0 \\ 0 & C(u) & 0 & D(u) \end{pmatrix}_{ab} , \tag{3.7}$$

and

$$M_b(v) = \mathbb{I}_a \otimes M_b(v) = \begin{pmatrix} A(v) & B(v) & 0 & 0 \\ C(v) & D(v) & 0 & 0 \\ 0 & 0 & A(v) & B(v) \\ 0 & 0 & C(v) & D(v) \end{pmatrix}_{ab} . \tag{3.8}$$

Then substituting (3.7), (3.8) and (3.4) into (3.6) yields 16 relations among A, B, C, D. The following algebra is our concern:

$$\begin{aligned} [A(u), A(v)] = [B(u), B(v)] = [C(u), C(v)] = [D(u), D(v)] = 0, \\ A(u)B(v) = f(v-u)B(v)A(u) + g(u-v)B(u)A(v), \\ D(u)B(v) = f(u-v)B(v)D(u) + g(v-u)B(u)D(v), \\ [C(u), B(v)] = \frac{1}{g(u-v)} \left(A(v)D(u) - A(u)D(v)\right), \end{aligned} \tag{3.9}$$

where $f(u) = \frac{u+i}{u}$, $g(u) = \frac{i}{u}$.

The last important quantity we need to define is the Transfer matrix, which is the trace of the monodromy matrix in the auxiliary space

$$T(u) = \mathrm{tr}_a M_a(u) = A(u) + D(u). \tag{3.10}$$

It is easy to show

$$[T(u), T(v)] = 0, \tag{3.11}$$

through the algebra in (3.9). Because the Lax operator is linear in the spectral parameter u, the transfer matrix $T(u)$ is a polynomial of order L in u whose coefficients are operators acting on the quantum space. As a result of (3.11), all of these operators commute with each other and hence we find a set of L mutually commuting operators

$$[Q_i, Q_j] = 0, \quad i, j = 1, \ldots, L, \tag{3.12}$$

where Q_l can be organized as the local conserved charge involving interactions between l neighboring spins. In this sense, $T(u)$ plays the role of generating function of these L conserved charges and therefore the $\mathfrak{su}(2)$ spin chain constitutes an integrable system. We will see later that the Hamiltonian belongs to this set of conserved charges and acts as Q_2.

3.2 Yang-Baxter equation

So far we have not discussed how we determine the matrix R_{ab} in (3.4). If we define another kind of Lax operator with a different u dependence, we would have to use a different R-matrix and finally would find another algebra instead of (3.9). However, it turns out that the R-matrix is highly constrained and cannot be taken arbitrarily if we want our algebra to be compatible. The consistency relation constraining the R-matrix is known as the Yang-Baxter equation [13,14], which was discovered in different contexts in the study of integrable models.

In order to derive the Yang-Baxter equation, let us reorder the product of three monodromy as follows

$$M_a(u_1) M_b(u_2) M_c(u_3) \to M_c(u_3) M_b(u_2) M_a(u_1), \tag{3.13}$$

using the RMM-relation (3.6). This can be done in two different ways:

$$\begin{aligned} M_c M_b M_a &= R_{ab} \left(M_c M_a M_b\right) R_{ab}^{-1} \\ &= R_{ab} R_{ac} \left(M_a M_c M_b\right) R_{ac}^{-1} R_{ab}^{-1} \\ &= R_{ab} R_{ac} R_{bc} \left(M_a M_b M_c\right) R_{bc}^{-1} R_{ac}^{-1} R_{ab}^{-1}, \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} M_c M_b M_a &= R_{bc}\left(M_b M_c M_a\right) R_{bc}^{-1} \\ &= R_{bc} R_{ac}\left(M_b M_a M_c\right) R_{ac}^{-1} R_{bc}^{-1} \\ &= R_{bc} R_{ac} R_{ab}\left(M_a M_b M_c\right) R_{ab}^{-1} R_{ac}^{-1} R_{bc}^{-1}. \end{aligned} \tag{3.15}$$

These two ways of reordering must be consistent. Therefore, we find the Yang-Baxter equation that R-matrix must satisfy

$$R_{ab}(u_1,u_2)R_{ac}(u_1,u_3)R_{bc}(u_2,u_3) = R_{bc}(u_2,u_3)R_{ac}(u_1,u_3)R_{ab}(u_1,u_2). \tag{3.16}$$

Solving above equation is a highly non-trivial and mathematical problem. If we assume that $R(u_1,u_2) = R(u_1-u_2)$ and certain non-degeneracy conditions, it can be shown that there are three types of solutions for $R(u)$: rational, trigonometric and elliptic, corresponding to the R-matrix in the XXX, XXZ and XYZ spin chain models respectively.

3.3 Construct the conserved charges

Using the building blocks introduced before, we can figure out the ABA for the Heisenberg spin chain with spin-$\frac{1}{2}$ representation of $SU(2)$. In this representation, the permutation operator acting on $\mathbb{C}_a^2 \otimes \mathbb{C}_n^2$ takes the form

$$\mathrm{P}_{an} = \frac{1}{2}\Big(\mathrm{I}_a \otimes \mathrm{I}_n + \sum_{\alpha} \sigma_a^{\alpha} \otimes \sigma_n^{\alpha}\Big), \quad \alpha = x,y,z. \tag{3.17}$$

It is called the permutation operator since we have

$$\mathrm{P}_{an}\left(|x\rangle_a \otimes |y\rangle_n\right) = |y\rangle_a \otimes |x\rangle_n, \tag{3.18}$$

for any vector $|x\rangle_a$ and $|y\rangle_n$ in the two spaces. The relation (3.18) can be checked explicitly by writing P_{an} in the matrix form. Besides, we have the following identities

$$\mathrm{P}_{n,a}\mathrm{P}_{n,b} = \mathrm{P}_{a,b}\mathrm{P}_{n,a} = \mathrm{P}_{n,b}\mathrm{P}_{b,a}, \qquad \mathrm{P}_{a,b} = \mathrm{P}_{b,a}. \tag{3.19}$$

The point is that the Hamiltonian (2.2) of the Heisenberg spin chain can be written in terms of permutation operators

$$H_{\mathrm{XXX}} = -\sum_{n=1}^{L} \vec{S}_n \cdot \vec{S}_{n+1} = -\frac{1}{4}\sum_{n=1}^{L} \vec{\sigma}_n \cdot \vec{\sigma}_{n+1} = \frac{L}{4} - \frac{1}{2}\sum_{n=1}^{L} \mathrm{P}_{n,n+1}. \tag{3.20}$$

We see that the Hamiltonian H_{XXX} is essentially a sum of permutation operators up to a constant shift. Recall that the Lax operator (3.1) is also expressed in terms of permutation operator and we have the following relations

$$\begin{aligned} L_{an}(i/2) &= i\mathrm{P}_{an}, \\ \frac{\mathrm{d}}{\mathrm{d}u} L_{an}(u) &= \mathrm{I}_{an}. \end{aligned} \tag{3.21}$$

Using the above relations we can extract the momentum operator and the Hamiltonian from the transfer matrix. The shift operator is simply given by

$$\begin{aligned} U &= i^{-L}T(i/2) = i^{-L}\mathrm{tr}_a M_a(i/2) \\ &= \mathrm{tr}_a \mathrm{P}_{a,1}\mathrm{P}_{a,2}\ldots\mathrm{P}_{a,L} = (\mathrm{tr}_a \mathrm{P}_{a,L})\,\mathrm{P}_{L,L-1}\ldots\mathrm{P}_{2,3}\mathrm{P}_{1,2} \\ &= \mathrm{P}_{L,L-1}\ldots\mathrm{P}_{3,2}\mathrm{P}_{2,1}. \end{aligned} \tag{3.22}$$

For any operator X_n at site-n, the operator U shifts all spins by one site

$$U^{-1}X_nU = X_{n+1}, \tag{3.23}$$

where we have used the fact $\mathrm{P}_{nm}X_m\mathrm{P}_{nm} = X_n$. Since the shift operator can be seen as $U = e^{i\hat{P}}$, we have found the momentum operator $\hat{P}$ acting as the conserved charge Q_1

$$\hat{P} = \frac{1}{i}\log(i^{-L}T(\frac{i}{2})). \tag{3.24}$$

Next, let us reconstruct the Hamiltonian (3.20) through the transfer matrix. In order to expand the transfer matrix around $u = \frac{i}{2}$, we first consider the derivative of monodromy

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}u}M_a(u)\bigg|_{u=i/2} &= \frac{\mathrm{d}}{\mathrm{d}u}(L_{a1}(u)\ldots L_{aL}(u))\bigg|_{u=i/2} \\ &= i^{L-1}\sum_{n=1}^{L}\mathrm{P}_{a,1}\ldots\widehat{\mathrm{P}}_{a,n}\ldots\mathrm{P}_{a,L} \\ &= i^{L-1}\sum_{n=1}^{L}\mathrm{P}_{a,L}\mathrm{P}_{L,1}\ldots\widehat{\mathrm{P}}_{L,n}\ldots\mathrm{P}_{L,L-1}, \end{aligned} \tag{3.25}$$

where $\widehat{\mathrm{P}}_{L,n}$ means the permutation operator $\mathrm{P}_{L,n}$ is missing from the string of operators. Then, by taking the trace over the auxiliary space and employing a similar trick as when deriving the shift operator, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}u}T(u)\bigg|_{u=i/2} = i^{L-1}\sum_{n=1}^{L}\mathrm{P}_{L,L-1}\ldots\mathrm{P}_{n+2,n+1}\mathrm{P}_{n+1,n-1}\mathrm{P}_{n-1,n-2}\ldots\mathrm{P}_{2,1}. \tag{3.26}$$

With the relation (3.22), we find that

$$\left(\frac{\mathrm{d}}{\mathrm{d}u}T(u)\right)T(u)^{-1}\bigg|_{u=i/2} = \frac{1}{i}\sum_{n=1}^{L}\mathrm{P}_{n,n+1}. \tag{3.27}$$

The left-hand side of (3.27) can be expressed as the logarithm derivative of $T(u)$. Comparing with (3.20), the Hamiltonian can be written in terms of the transfer matrix

$$H_{\mathrm{XXX}} = \frac{L}{4} - \frac{i}{2}\frac{\mathrm{d}}{\mathrm{d}u}\log T(u)\bigg|_{u=i/2}. \tag{3.28}$$

We have seen that the transfer matrix generates the Hamiltonian, which serves as the conserved charge Q_2.

The third charge Q_3 involves interactions between 3 neighboring spins, which can be chosen as

$$Q_3 = \sum_{i=1}^{L} [H_{i,i+1}, H_{i+1,i+2}]. \tag{3.29}$$

For the higher conserved charges, a specific boosting procedure is utilized for their construction [15, 16].

3.4 Eigenvectors

We have demonstrated the significance of the transfer matrix in the integrable systems; it plays the role of generating a set of conserved charges. In this section, we consider diagonalizing the transfer matrix, which help us find the eigenvectors of the conserved charges, particularly for the Hamiltonian (3.28).

The pseudo-vacuum is defined in the same way as in the CBA. On each lattice site, we equivalently have an upper triangular matrix acting on the state

$$L_{an}(u)|\uparrow\rangle_n = \begin{pmatrix} u + iS_n^z & iS_n^- \\ iS_n^+ & u - iS_n^z \end{pmatrix} |\uparrow\rangle_n = \begin{pmatrix} u + \frac{i}{2} & iS_n^- \\ 0 & u - \frac{i}{2} \end{pmatrix} |\uparrow\rangle_n. \tag{3.30}$$

Since the product of a sequence of upper triangular matrices is still an upper triangular matrix, the action of the monodromy matrix on the pseudo-vacuum is

$$\begin{aligned} M_a(u)|\uparrow^L\rangle &= \begin{pmatrix} u + \frac{i}{2} & iS_1^- \\ 0 & u - \frac{i}{2} \end{pmatrix} \begin{pmatrix} u + \frac{i}{2} & iS_2^- \\ 0 & u - \frac{i}{2} \end{pmatrix} \cdots \begin{pmatrix} u + \frac{i}{2} & iS_L^- \\ 0 & u - \frac{i}{2} \end{pmatrix} |\uparrow^L\rangle \\ &= \begin{pmatrix} (u + \frac{i}{2})^L & \star \\ 0 & (u - \frac{i}{2})^L \end{pmatrix} |\uparrow^L\rangle \\ &= \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} |\uparrow^L\rangle, \end{aligned} \tag{3.31}$$

here $A(u), B(u), C(u), D(u)$ are the same operators introduced in (3.5). From the general composition of these four operators, $A(u), D(u)$ do not change the number of up-spins and down-spins, while $B(u)$ flips one up-spin and $C(u)$ flips one down-spin. Thus, $B(u)$ and $C(u)$ play the role of lowering operator and raising operator respectively. We can read off the following eigenvalue equations from (3.31)

$$A(u)|\uparrow^L\rangle = a(u)|\uparrow^L\rangle, \quad D(u)|\uparrow^L\rangle = d(u)|\uparrow^L\rangle, \quad C(u)|\uparrow^L\rangle = 0, \tag{3.32}$$

with eigenvalues $a(u) = \left(u + \frac{i}{2}\right)^L$ and $d(u) = \left(u - \frac{i}{2}\right)^L$. The pseudo-vacuum state is an eigenstate of $A(u), D(u)$ and annihilated by $C(u)$, as expected. Such a state is

known as a highest weight state or reference state. The existence of a reference state is a nontrivial property of the system and is a necessary condition that the system can be solved by Bethe ansatz.

We have known the action of $A(u), D(u)$ and $C(u)$ on the reference state, while the action of $B(u)$ remains complicated. We claim that the following state

$$|\mathbf{u}_N\rangle = B(u_1)B(u_2)\dots B(u_N)|\uparrow^L\rangle \tag{3.33}$$

is an eigenstate of $T(u) = A(u) + D(u)$ if the spectral parameters $\mathbf{u}_N = \{u_1, \dots, u_N\}$ satisfy certain conditions. Since $B(u)$ acts as the lowering operator, $|\mathbf{u}_N\rangle$ is a state with N down-spins, analogous to the N-magnon state in the CBA. Let us figure out the action of $A(u)$ and $D(u)$ on the state $|\mathbf{u}_N\rangle$ by using algebraic relations in (3.9).

For $A(u)$, We know the state $|\uparrow^L\rangle$ diagonalizes it and hence we want to move $A(u)$ through a string of B operators. From the algebra, when $A(u)$ passes through one B operator, their spectral parameters might be swapped. Thus, We arrive at the following final result

$$\begin{aligned} A(u)B(u_1)\dots B(u_N)|\uparrow^L\rangle = a(u)\prod_{k=1}^{N} f(u_k-u)B(u_1)\dots B(u_N)|\uparrow^L\rangle \\ + \sum_{k=1}^{N} M_k(u|\mathbf{u}_N)B(u_1)\dots\widehat{B}(u_k)\dots B(u_N)B(u)|\uparrow^L\rangle, \end{aligned} \tag{3.34}$$

where $\widehat{B}(u_k)$ denotes that the operator $B(u_k)$ is absent. The first term on the right-hand side is called the 'wanted term' since it appears in the form of an eigenstate of $A(u)$. The rest terms on the right-hand side are labeled as 'unwanted terms' because we aim for them to cancel out in some manner. The coefficients $M_k(u|\mathbf{u}_N)$ can be determined as follows: firstly, it is straightforward to find the coefficient of $B(u_2)B(u_3)\dots B(u_N)B(u)|\uparrow^L\rangle$ by using the second term of the algebra once, and then use the first term for the rest of the commutation relations

$$\begin{aligned} & g(u-u_1)B(u)A(u_1)B(u_2)\dots B(u_N)|\uparrow^L\rangle \\ =& g(u-u_1)f(u_2-u_1)B(u)B(u_2)A(u_1)B(u_3)\dots B(u_N)|\uparrow^L\rangle \\ =& g(u-u_1)\prod_{k=2}^{N} f(u_k-u_1)B(u)B(u_2)\dots B(u_N)A(u_1)|\uparrow^L\rangle \\ =& g(u-u_1)a(u_1)\prod_{k=2}^{N} f(u_k-u_1)B(u_2)\dots B(u_N)B(u)|\uparrow^L\rangle. \end{aligned} \tag{3.35}$$

Therefore, we find

$$M_1(u|\mathbf{u}_N) = g(u-u_1)a(u_1)\prod_{k=2}^{N} f(u_k-u_1). \tag{3.36}$$

Secondly, since all B operators commute, we can obtain $M_k(u|\mathbf{u}_N)$ simply from $M_1(u|\mathbf{u}_N)$ by replacing u_1 with u_k

$$M_k(u|\mathbf{u}_N) = g(u-u_k)a(u_k)\prod_{j\neq k}^{N} f(u_j-u_k). \tag{3.37}$$

Similarly, using the algebra we obtain the following result for $D(u)$

$$\begin{aligned} D(u)B(u_1)\dots B(u_N)|\uparrow^L\rangle = d(u)\prod_{k=1}^{N} f(u-u_k)B(u_1)\dots B(u_N)|\uparrow^L\rangle \\ + \sum_{k=1}^{N} N_k(u|\mathbf{u}_N)B(u_1)\dots\widehat{B}(u_k)\dots B(u_N)B(u)|\uparrow^L\rangle, \end{aligned} \tag{3.38}$$

where the coefficients can be determined in the same way

$$N_k(u|\mathbf{u}_N) = g(u_k-u)d(u_k)\prod_{j\neq k}^{N} f(u_k-u_j). \tag{3.39}$$

The unwanted terms cancel under the condition $M_k + N_k = 0$, which leads to

$$\begin{aligned} g(u-u_k)a(u_k)\prod_{j\neq k}^{N} f(u_j-u_k) &= -g(u_k-u)d(u_k)\prod_{j\neq k}^{N} f(u_k-u_j), \\ a(u_k)\prod_{j\neq k}^{N} f(u_k-u_j) &= d(u_k)\prod_{j\neq k}^{N} h(u_k-u_j) \quad k=1,2,\dots,N. \end{aligned} \tag{3.40}$$

Or written more explicitly

$$\left(\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}}\right)^L \prod_{j\neq k}^{N}\frac{u_k-u_j-i}{u_k-u_j+i} = 1, \qquad k=1,2,\dots,N. \tag{3.41}$$

This is noting but the Bethe ansatz equations for the Heisenberg spin chain, exactly the same as (2.30) that we derived by the CBA. Under the condition (3.41), the state $|\mathbf{u}_N\rangle$ is an eigenstate of the transfer matrix $T(u)$

$$(A(u)+D(u))|\mathbf{u}_N\rangle = \tau(u|\mathbf{u}_N)|\mathbf{u}_N\rangle, \tag{3.42}$$

with the eigenvalue

$$\begin{aligned} \tau(u|\mathbf{u}_N) &= a(u)\prod_{k=1}^{N} f(u_k-u) + d(u)\prod_{k=1}^{N} f(u-u_k) \\ &= a(u)\prod_{k=1}^{N}\frac{u-u_k-i}{u-u_k} + d(u)\prod_{k=1}^{N}\frac{u-u_k+i}{u-u_k}. \end{aligned} \tag{3.43}$$

The eigenvalue of all conserved charges can be obtained from $\tau(u|\mathbf{u}_N)$ due to the fact that the transfer matrix generates all conserved charges. For instance, the eigenvalue equation of the momentum operator (3.24) on a Bethe state is

$$\hat{P}|\mathbf{u}_N\rangle = \frac{1}{i^{1+L}}\log T(\frac{i}{2})|\mathbf{u}_N\rangle = \sum_{k=1}^{N} p(u_k)|\mathbf{u}_N\rangle, \tag{3.44}$$

where

$$p(u_k) = \frac{1}{i}\log\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}}. \tag{3.45}$$

And for the energy of $|\mathbf{u}_N\rangle$, using the Hamiltonian (3.28) we find

$$\begin{aligned} H_{\text{XXX}}|\mathbf{u}_N\rangle &= \left(\frac{L}{4} - \frac{i}{2}\frac{\mathrm{d}}{\mathrm{d}u}\log T(u)\Big|_{u=\frac{i}{2}}\right)|\mathbf{u}_N\rangle \\ &= \left(\frac{L}{4} - \frac{L}{2} + \sum_{k=1}^{N}\frac{2}{1+4u_k^2}\right)|\mathbf{u}_N\rangle \\ &= \left(\sum_{k=1}^{N}\varepsilon(u_k) - \frac{L}{4}\right)|\mathbf{u}_N\rangle, \end{aligned} \tag{3.46}$$

where $\varepsilon(u_k)$ can be regarded as the energy of each magnon. The result (3.46) is consistent with the energy of N-magnon state (2.26) derived by the CBA. Thus, we have seen that the spin chain Hamiltonian can be exactly solved by the ABA. We can derive the eigenvalue of higher conserved charges in a similar way.

3.5 Generalization

So far, algebraically, We have constructed the Bethe state (3.33) and derived the Bethe ansatz equations (3.41) for the Heisenberg spin chain, where the spin operators satisfy the $\mathfrak{su}(2)$ algebra and lie in the spin-1/2 representation. It is natural to consider the more general spin chain with higher spin representation or higher-rank symmetry. It is now clear that the algebraic structure underlying integrable models is the RMM-relation (3.6) where the R-matrix is analogous to the structure constant in a Lie algebra and the monodromy plays the role of generators. Thus, essentially we can construct different monodromy and R-matrix to generalize the Heisenberg spin chain.

3.5.1 Higher spin representation

Consider the spin-s representation of the local spin operators. The Lax operator takes the same form, while the spin operators can not be written as Pauli matrices anymore. The monodromy and the transfer matrix can be constructed in exactly the same way. It is straightforward to find

$$A(u)|\Omega_s\rangle = (u+is)^L|\Omega_s\rangle, \qquad D(u) = (u-is)^L|\Omega_s\rangle, \tag{3.47}$$

where $|\Omega_s\rangle$ is the pseudo-vacuum state in the spin-s representation. And the transfer matrix is diagonalized by the Bethe state with the corresponding eigenvalue

$$\tau(u|\mathbf{u}_N) = a(u)\prod_{k=1}^{N}\frac{u-u_k-i}{u-u_k} + d(u)\prod_{k=1}^{N}\frac{u-u_k+i}{u-u_k}, \tag{3.48}$$

here $a(u) = (u + is)^L$ and $d(u) = (u - is)^L$. The Bethe equations for the XXX_s spin chain read as follows:

$$\left(\frac{u_k + is}{u_k - is}\right)^L \prod_{j\neq k}^{N} \frac{u_k - u_j - i}{u_k - u_j + i} = 1, \qquad k = 1, 2, \ldots, N. \tag{3.49}$$

Thus, we observe that the spin representation is labeled by the eigenvalues of the operators $A(u)$ and $D(u)$. Interestingly, this observation arises in other integrable models as well. Different physical models can serve as representations of the same algebra (3.6), characterized by $a(u)$ and $d(u)$. In this sense, the algebraic Bethe ansatz is a more universal approach for integrable models.

3.5.2 Higher rank symmetry

Another generalization of the Heisenberg spin chain is achieved by selecting a symmetry algebra with rank $r > 1$, indicating that the system enjoys a higher-rank symmetry instead of $\mathfrak{su}(2)$. For instance, the $\mathfrak{su}(N)$ spin chain has the symmetry algebra with rank $r = N - 1$, where the nested Bethe Ansatz [17] must be applied to solve the spectrum. The basic idea of the nested Bethe Ansatz is to reduce the rank of the symmetry algebra successively until it reaches rank 1, namely the well-known $\mathfrak{su}(2)$ case. For each reduction, we obtain a set of Bethe equations. Adding the last set of equations from the rank 1 case, totally there should be $(N - 1)$ sets of nested Bethe equations.

The Hamiltonian of the $\mathfrak{su}(N)$ fundamental spin chain takes the following form:

$$H_{\mathrm{XXX}} = \sum_{n=1}^{L} (\mathrm{I}_{n,n+1} - \mathrm{P}_{n,n+1}), \tag{3.50}$$

where permutation opeartor $\mathrm{P}_{n,n+1}$ acts on $\mathbb{C}^N_n \otimes \mathbb{C}^N_{n+1}$ and is defined by standard basis matrix $(E_{ij})_{lk} = \delta_{il}\delta_{jk}$:

$$\mathrm{P}_{ab} = \sum_{i,j=1}^{N} E^a_{ij} \otimes E^b_{ji}. \tag{3.51}$$

For the lattice site-n, local Hilbert space is $V_n = \mathbb{C}^N_n$. We designate $|1\rangle$ as the vacuum state for each site, and $|2\rangle, |3\rangle, \ldots, |N\rangle$ as distinct excitations. The coefficients in front of the identity operator and permutation operator were fixed by acting the Hamiltonian (3.50) on the vacuum $|1^L\rangle$.

From the definition (3.51), the permutation operator in components is $\mathrm{P}^{kl}_{ij} = \delta_{il}\delta_{jk}$. And for the identity operator, we have $\mathrm{I}^{kl}_{ij} = \delta_{ik}\delta_{jl}$, where the indices take the value

$$(ij) = (11), (12), \ldots, (1N), (21), \ldots, (NN). \tag{3.52}$$

which implies the Hamiltonian of $\mathfrak{su}(N)$ fundamental spin chain is a $N^2 \times N^2$ matrix. This Hamiltonian can be expressed in terms of generators of $\mathfrak{su}(N)$. Using the properties of tensor product, we find

$$(t_n^a \otimes t_{n+1}^a)_{i(N-1)+j,k(N-1)+l} = t_{ik}^a \times t_{jl}^a = \frac{1}{2}\left(\delta_{il}\delta_{kj} - \frac{1}{N}\delta_{ik}\delta_{jl}\right)_{n,n+1}, \tag{3.53}$$

here we have used the Fierz identity satisfied by the fundamental generators of $\mathfrak{su}(N)$. Then the tensor product of generators can be rewritten as the following

$$(t_n^a \otimes t_{n+1}^a)_{ij}^{kl} = \frac{1}{2}\left(\mathrm{P}_{ij}^{kl} - \frac{1}{N}\mathrm{I}_{ij}^{kl}\right)_{n,n+1}. \tag{3.54}$$

Now the Hamiltonian (3.50) can be recast using (3.54)

$$H_{\mathrm{XXX}}^{\mathfrak{su}(N)} = \sum_{n=1}^{L} \mathcal{H}_{n,n+1} = \sum_{n=1}^{L}\left[(1 - \frac{1}{N})\mathrm{I}_{n,n+1} - 2t_n^a \otimes t_{n+1}^a\right]. \tag{3.55}$$

It is easy to show that this Hamiltonian is $\mathfrak{su}(N)$ invariant, i.e.

$$\left[H_{\mathrm{XXX}}^{\mathfrak{su}(N)}, T^a\right] = 0, \qquad a = 1, 2, 3, \ldots, N^2 - 1. \tag{3.56}$$

where

$$T^a = \frac{1}{2}\sum_{n=1}^{L}\left(t_n^a \otimes \mathrm{I}_{n+1} + \mathrm{I}_n \otimes t_{n+1}^a\right). \tag{3.57}$$

are analogous to the global spin operators in Heisenberg spin chain. With the commutation relations of $\mathfrak{su}(N)$ Lie algebra $\left[t_n^a, t_m^b\right] = if^{abc}\delta_{nm}t^c$, one can show that

$$\sum_{n,m}\left[t_n^a \otimes \mathrm{I}_{n+1} + \mathrm{I}_n \otimes t_{n+1}^a, t_m^b \otimes t_{m+1}^b\right] = 0, \tag{3.58}$$

which leads to (3.56). The R-matrix of the $\mathfrak{su}(N)$ fundamental spin chain has the form

$$R(u-v) = \mathrm{I} + g(u-v)\mathrm{P}, \qquad g(u-v) = \frac{i}{u-v}. \tag{3.59}$$

And other building blocks of Algebraic Bethe Ansatz are defined in a similar manner

$$\begin{aligned} L_{an}(u) &= R_{an}(u - \frac{i}{2}), \\ M_a(u) &= R_{a1}(u - \frac{i}{2})R_{a2}(u - \frac{i}{2})\ldots R_{aL}(u - \frac{i}{2}), \\ T(u) &= \mathrm{Tr}_a M_a(u) = \sum_{k=1}^{N} M_{kk}(u). \end{aligned} \tag{3.60}$$

The RMM-relation (3.6) still holds and leads to the algebra among the monodromy matrix entries

$$[M_{ij}(u), M_{kl}(v)] = g(u-v)\left(M_{kj}(v)M_{il}(u) - M_{kj}(u)M_{il}(v)\right), \tag{3.61}$$

which is the famous Yangian algebra Y[$\mathfrak{su}(N)$] [18]. We expect the representation of Yangian algebra is highest weight if

$$\begin{aligned} M_{ii}\left|\Omega\right\rangle &= \lambda_i(u)\left|\Omega\right\rangle, \qquad i = 1, \ldots, N. \\ M_{ji}\left|\Omega\right\rangle &= 0, \qquad\qquad 1 \leq i < j \leq N. \end{aligned} \tag{3.62}$$

where $\lambda_i(u)$ are the vacuum eigenvalues characterizing the representation. The monodromy is a $N \times N$ matrix in the auxiliary space $\mathbb{C}_a^N$, we can decompose it into:

$$M_a(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}_a, \tag{3.63}$$

with entries $B(u) = \{B_1(u), B_2(u), \ldots B_{N-1}(u)\}$ as a row vector and $C(u)$ as a column vector. The $(N-1) \times (N-1)$ matrix $D(u)$ can be regarded as a sub-monodromy satisfying the RMM-relation of $\mathfrak{su}(N-1)$. Thus, the transfer matrix becomes $T(u) = A(u) + \mathrm{Tr}_a D(u)$. Monomial Bethe state (3.33) generally is not invariant under the action of $T(u)$. Therefore, we claim that eigenvectors of the transfer matrix take a polynomial form

$$\left|\Psi\right\rangle = \sum_{\beta_1, \ldots, \beta_{N_1}} B_{\beta_1}(u_1) B_{\beta_2}(u_2) \ldots B_{\beta_{N_1}}(u_{N_1}) F_{\beta_1, \ldots, \beta_{N_1}} \left|\Omega\right\rangle, \tag{3.64}$$

where $\beta_i = 1, 2, \ldots, N-1$. To determine the function $F_{\beta_1, \ldots, \beta_{N_1}}$, we can map it to the energy eigenstate of inhomogeneous $\mathfrak{su}(N-1)$ fundamental spin chain with length N_1

$$F_{\beta_1, \ldots, \beta_{N_1}} \to \left|\beta_1, \ldots, \beta_{N_1}\right\rangle. \tag{3.65}$$

We identify $F_{\beta_1, \ldots, \beta_{N_1}}$ as the wavefunction of the spin chain eigenstate, allowing us to determine it by solving the $\mathfrak{su}(N-1)$ spin chain, for which the rank of the algebra is reduced by 1. Repeating this process, we can eventually return to the $\mathfrak{su}(2)$ spin chain, which has been extensively studied before.

After the reduction, a total of $N-1$ Bethe states will be constructed, with the last Bethe state having the same form as (3.33). Acting the corresponding transfer matrix on each Bethe state, to ensure the unwanted terms vanish, one set of Bethe equations should be imposed. Therefore there are $N-1$ sets of Bethe equations totally. Here we present the general Bethe equations for the $\mathfrak{su}(N)$ fundamental spin chain without

going into the detailed derivation [19]

$$\begin{aligned}
\left(\frac{u_{1,k}+\frac{i}{2}}{u_{1,k}-\frac{i}{2}}\right)^L &= \prod_{j\neq k}^{K_1} S(u_{1,k},u_{1,j}) \prod_{j=1}^{K_2} \tilde{S}(u_{1,k},u_{2,j}),\\
1 &= \prod_{j\neq k}^{K_2} S(u_{2,k},u_{2,j}) \prod_{j=1}^{K_1} \tilde{S}(u_{2,k},u_{1,j}) \prod_{j=1}^{K_3} \tilde{S}(u_{2,k},u_{3,j}),\\
1 &= \prod_{j\neq k}^{K_3} S(u_{3,k},u_{3,j}) \prod_{j=1}^{K_2} \tilde{S}(u_{3,k},u_{2,j}) \prod_{j=1}^{K_4} \tilde{S}(u_{3,k},u_{4,j}),\\
&\cdots\cdots\\
1 &= \prod_{j\neq k}^{K_{N-1}} S(u_{N-1,k},u_{N-1,j}) \prod_{j=1}^{K_{N-2}} \tilde{S}(u_{N-1,k},u_{N-2,j}),
\end{aligned} \tag{3.66}$$

with

$$S(u,v)=\frac{u-v+i}{u-v-i} \quad \tilde{S}(u,v)=\frac{u-v-\frac{i}{2}}{u-v+\frac{i}{2}}. \tag{3.67}$$

There are $N-1$ sets of Bethe roots $\{\mathbf{u_1},\mathbf{u_2},\ldots,\mathbf{u_{N-1}}\}$ with each set having $K_1,K_2,\ldots,K_{N-1}$ roots, respectively. After finding these Bethe roots, we can determine the energy eigenvalues and construct the corresponding eigenstates of the Hamiltonian. The energy is given by

$$E=\sum_{k=1}^{K_1}\frac{1}{u_{1,k}^2+\frac{1}{4}}. \tag{3.68}$$

The Bethe roots $\mathbf{u_1}$ are special because they contribute alone to the momentum and energy, hence they are called momentum carrying roots, while the other Bethe roots are referred to as auxiliary roots.

From the representation theory, we know that any irreducible representation of $\mathfrak{gl}(n)$ is also an irreducible representation of $\mathfrak{su}(N)$ and vice versa. Consequently, the Hamiltonian of $\mathfrak{gl}(N)$ fundamental spin chain is exactly the same as (3.55), and the Bethe ansatz for the $\mathfrak{gl}(N)$ spin chain remains unchanged. We have discussed the fundamental spin chain, while for other representations, the Hamiltonian of the spin chain would be more complicated, requiring some adjustments to the Bethe ansatz.

4. Rational Q-system

The Bethe ansatz allows us to determine the spectrum of integrable models by solving sets of algebraic equations. However, finding solutions to these non-linear coupled equations is by no means a simple task, especially for generalized spin chains where dealing with nested equation sets is highly complicated. Moreover, regarding the completeness of the Bethe ansatz, solving the Bethe equations directly may yield too many solutions, including unwanted nonphysical ones. The Rational Q-system, proposed by Marboe and Volin in 2016 [20], offers a powerful approach for analytically finding Bethe roots, automatically eliminating all nonphysical solutions. We work with the Heisenberg XXX spin chain for illustrative purpose.

4.1 Completeness of Bethe ansatz

For the $\mathfrak{su}(2)$ XXX_s spin chain with length L and magnon number N, we first count the number of expected solutions to the Bethe equations. Denoting the irreducible spin-s representation by $\mathcal{D}^{(s)}$, the length-L Hilbert space decomposes as:

$$\left[\mathcal{D}^{(s)}\right]^{\otimes L} = \bigoplus_{J=J_{\min}}^{sL} d_s(L,J)\mathcal{D}^{(J)}, \tag{4.1}$$

where $J_{\min} = 0$ for even L and $J_{\min} = s$ for odd L. $d_s(L,J)$ is the number of spin-J representations. The decomposition (4.1) can be computed by repeatedly applying the Clebsch-Gordan decomposition for any two representations

$$\mathcal{D}^{(\ell)} \otimes \mathcal{D}^{(\ell')} = \mathcal{D}^{|\ell'-\ell|} \oplus \mathcal{D}^{|\ell'-\ell|+1} \oplus \cdots \oplus \mathcal{D}^{(\ell'+\ell)}. \tag{4.2}$$

For a given magnetization m, the number of coupled basis states $b_s(L,m)$ can be computed by expanding the left-hand side of the following equation

$$(z^{-s} + z^{-s+1} + \ldots + z^{s})^L = \sum_{m=-sL}^{sL} b_s(L,m) z^m. \tag{4.3}$$

On the other hand, each spin-J representation with $J \geq |m|$ contributes one magnetization-m state to $b_s(L,m)$, hence

$$b_s(L,m) = d_s(L,|m|) + d_s(L,|m|+1) + \ldots + d_s(L,sL). \tag{4.4}$$

The relation (4.4) indicates that the number of spin-J representations can be evaluated by

$$d_s(L,J) = b_s(L,J) - b_s(L,J+1). \tag{4.5}$$

For the Heisenberg XXX spin chain, from (4.3) we can see that $b_{1/2}(L,m)$ is just the binomial coefficient

$$b_{1/2}(L,m) = b_{1/2}(L,-m) = \binom{L}{|L/2-m|} = \binom{L}{N}, \tag{4.6}$$

where we have rewritten m in terms of the magnon number by $m = \frac{L}{2} - N$. The Bethe state (3.33) is the highest weight state of the spin-$(\frac{L}{2}-N)$ representation, since

$$S^+|\mathbf{u}_N\rangle = 0, \qquad S^z|\mathbf{u}_N\rangle = \left(\frac{L}{2}-N\right)|\mathbf{u}_N\rangle. \tag{4.7}$$

In other words, $|\mathbf{u}_N\rangle$ corresponds to the primary state $|J,J\rangle$ in a spin-J representation. Therefore, we expect that the number of Bethe states equal to the number of spin-$(\frac{L}{2}-N)$ representations, namely the number of physical solutions $\mathcal{N}(L,N)$ is expected to be

$$\mathcal{N}(L,N) = d_{1/2}(L,\frac{L}{2}-N) = \binom{L}{N} - \binom{L}{N-1}. \tag{4.8}$$

We restrict $N \leq \frac{L}{2}$ because $J \geq 0$. The dimension of a spin-J representation is $2J+1$, and the total dimension of the Hilbert space of the spin chain is the sum of the dimension of each subspace

$$\begin{aligned} &\sum_{J=J_{\min}}^{\frac{L}{2}} d_{\frac{1}{2}}(L,J)\times(2J+1) \\ =&\sum_{N=0}^{\left[\frac{L}{2}\right]} d_{\frac{1}{2}}(L,\frac{L}{2}-N)\times(L-2N+1) \\ =&2^L, \end{aligned} \tag{4.9}$$

which is consistent with the dimension of the tensor product space $\mathcal{V} = (\mathbb{C}^2)^{\otimes L}$.

If we directly solve the Bethe equations (2.30), for example, for $L=4, N=2$, we find six solutions, whereas from (4.8) we only expect two solutions. This suggests that some of the solutions are not physical, meaning that although they satisfy (2.30), the corresponding Bethe state is no longer an eigenstate of the Hamiltonian or transfer matrix. There are two kinds of such solutions that require more careful analysis: coinciding roots and singular solutions that introduce poles in the dispersion relation (2.31).

In general, for the solution $\{u_0,u_0,\ldots,u_0,u_1,u_2,\ldots,u_N\}$, where the first K roots are coinciding, there are $N+1$ Bethe equations and $K-1$ additional constraints to

ensure the solution is physical. However, with only $N+1$ variables, such a system is over-determined and often does not have solutions. For the Heisenberg spin chain, there is good evidence that we do not have physical solutions with repeated roots, while for the $\mathfrak{su}(2)$ XXX_s spin chain with $s \geq 1$, the physical solutions with coinciding roots are allowed [21]. Besides, for the general singular solution $\{\frac{i}{2}, -\frac{i}{2}, u_1, \ldots, u_N\}$ the singular roots $\pm\frac{i}{2}$ need to be regularized carefully. After proper regularization, we see that some of the singular solutions are physical while others are not. The physical singular solutions need to satisfy an additional condition.

Regrading the completeness of Bethe ansatz for the Heisenberg spin chain, the actual number of physical solutions should match the expected number of solutions $\mathcal{N}(L,N)$ in (4.8). Denoting the number of non-singular and physical-singular solutions without repeated roots by $\mathcal{N}_1(L,N)$ and $\mathcal{N}_2(L,N)$, respectively. The completeness conjecture [21] proposed by Hao, Nepomechie and Sommese indicates that

$$\mathcal{N}_1(L,N) + \mathcal{N}_2(L,N) = \binom{L}{N} - \binom{L}{N-1}. \tag{4.10}$$

where we find that the right-hand side of (4.10) is precisely the number of solutions that we expect in (4.8). Thus, if the conjecture (4.10) holds true, the Bethe ansatz for Heisenberg XXX spin chain is complete. In fact, this conjecture has been tested quite non-trivially up to $L = 14$ [21].

So far, we see that there are in general too many solutions when solving Bethe equations directly and we have to discard solutions with coinciding roots and non-physical singular solutions. To find physical solutions efficiently, other alternative formulations of the Bethe equations should be considered, such as Baxter's TQ-relation and Rational Q-system. We focus on the Rational Q-system since it is more efficient to find Bethe roots and can be generalized to solve generic Bethe ansatz equation [22].

4.2 The formalism

Firstly, we need a Young tableaux on which the Q-system is defined. For the Heisenberg spin chain with length L and magnon number N, a Young tableaux with two rows $(L-N, N)$ is shown in figure 4.1. At each lattice point (a,s), we associate a polynomial Q-function $Q_{a,s}(u)$ whose degree is equal to the number of boxes to the upper right of the point. Secondly, we impose the boundary conditions by fixing the Q-functions on the upper and left boundary:

$$Q_{2,s}(u) = 1, \qquad Q_{0,0}(u) = u^L, \qquad Q_{1,0}(u) = \prod_{k=1}^{N}(u - u_k), \tag{4.11}$$

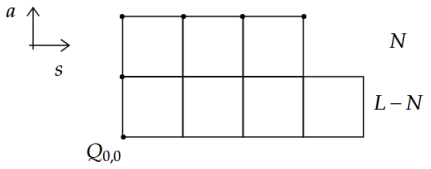

Figure 4.1: Young tableaux for the Bethe equations (2.30)

where $Q_{1,0}(u)$ is just the Baxter Q-function whose zeros are Bethe roots $\mathbf{u_k}$. For convenience, we parameterize $Q_{1,0}(u)$ as follows

$$Q_{1,0}(u) = u^N + \sum_{k=0}^{N-1} c_k u^k, \tag{4.12}$$

where the coefficients c_k encoding the Bethe roots are to be determined. We want to derive a set of algebraic equations for $\{c_k\}$ from the Q-system. Finally, we impose the QQ-relation to relate the four Q-functions at the four corners of each box

$$Q_{a+1,s}(u)Q_{a,s+1}(u) = Q^+_{a+1,s+1}(u)Q^-_{a,s}(u) - Q^-_{a+1,s+1}(u)Q^+_{a,s}(u), \tag{4.13}$$

with $Q^{\pm}(u) \equiv Q(u \pm \frac{i}{2})$. Thus, we can find all the Q-functions on the Young tableaux using the QQ-relation and the boundary conditions. The point is that we require all the Q-functions should be polynomial, which yields the zero remainder condition. We will see that this requirement is non-trivial and leads to a set of algebraic equations for $\{c_k\}$.

4.3 Solve the Q-system

Before solving the Q-system, we first prove the equivalence between the QQ-relations (4.13) and the Bethe ansatz equations (2.30). Considering the first column of the Young tableaux in figure 4.1, we have the following QQ-relations

$$Q_{1,1}(u) = Q^+_{1,0}(u) - Q^-_{1,0}(u), \tag{4.14a}$$

$$Q_{1,0}(u)Q_{0,1}(u) = Q^+_{1,1}(u)Q^-_{0,0}(u) - Q^-_{1,1}(u)Q^+_{0,0}(u). \tag{4.14b}$$

Evaluating the equation (4.14b) at $u = u_k$, we notice that $Q_{1,0}(u_k) = 0$, which yields:

$$Q^+_{1,1}(u_k)Q^-_{0,0}(u_k) - Q^-_{1,1}(u_k)Q^+_{0,0}(u_k) = 0. \tag{4.15}$$

To determine $Q^{\pm}_{1,1}(u_k)$, we can evaluate the equation (4.14a) at $u = u_k \pm \frac{i}{2}$, leading to

$$Q^+_{1,1}(u_k) = Q^{++}_{1,0}(u_k), \qquad Q^-_{1,1}(u_k) = -Q^{--}_{1,0}(u_k). \tag{4.16}$$

Substituting them into (4.15), we obtain

$$Q^{+}_{0,0}(u_k)Q^{--}_{1,0}(u_k) + Q^{-}_{0,0}(u_k)Q^{++}_{1,0}(u_k) = 0, \tag{4.17}$$

or equivalently,

$$\frac{Q^{+}_{0,0}(u_k)Q^{--}_{1,0}(u_k)}{Q^{-}_{0,0}(u_k)Q^{++}_{1,0}(u_k)} = -1. \tag{4.18}$$

Upon plugging in (4.11), we find that the equation (4.18) is equivalent to the Bethe ansatz equations (2.30). Therefore, we can find Bethe roots by solving $Q_{1,0}(u)$ through QQ-relations.

The basic strategy for solving the Q-system is to determine the Q-functions using (4.13), row by row, from top to bottom. We begin with the row $a = 1$, the QQ-relation reads

$$Q_{1,s+1}(u) = Q^{-}_{1,s}(u) - Q^{+}_{1,s}(u). \tag{4.19}$$

The solution to this difference equation is

$$Q_{1,s}(u) = D^s Q_{1,0}(u), \qquad Df(u) \equiv f(u - \frac{i}{2}) - f(u + \frac{i}{2}). \tag{4.20}$$

We then consider the next row with $a = 0$. The QQ-relation reads

$$Q_{0,s+1}(u)Q_{1,s}(u) = Q^{+}_{1,s+1}(u)Q^{-}_{0,s}(u) - Q^{-}_{1,s+1}(u)Q^{+}_{0,s}(u), \tag{4.21}$$

where functions $Q_{1,s}$ are already determined by (4.20). This equation can be used to determine all $Q_{0,s}$ by writing it as

$$Q_{0,s+1}(u) = \frac{Q^{+}_{1,s+1}(u)Q^{-}_{0,s}(u) - Q^{-}_{1,s+1}(u)Q^{+}_{0,s}(u)}{Q_{1,s}(u)}. \tag{4.22}$$

Generally, the right-hand side of the above equation is a rational function of u, instead of a polynomial. Thus, we need to impose constraints to ensure all the Q-functions are polynomial in u, meaning that the remainder of the right-hand side of (4.22) must be zero. It turns out that the zero remainder condition leads to a set of equations for coefficients $\{c_k\}$ in (4.12). Then we can solve these equations and determine the Baxter Q-function $Q_{1,0}(u)$ whose zeros are Bethe roots.

To see how rational Q-system works, we present an example for $(L, N) = (4, 2)$. The Young tableaux is shown in figure 4.2. The boundary conditions are given by (4.11) where the Baxter Q-function becomes

$$Q_{1,0}(u) = u^2 + c_1 u + c_0. \tag{4.23}$$

We first consider the row $a = 1$, the QQ-relations for $s = 0, 1$ are given by

$$\begin{aligned} Q_{1,1}(u) &= Q^{-}_{1,0} - Q^{+}_{1,0} = DQ_{1,0}(u) = -i(2u + c_1), \\ Q_{1,2}(u) &= DQ_{1,1}(u) = -2, \end{aligned} \tag{4.24}$$

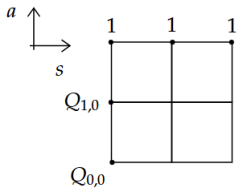

Figure 4.2: Young tableaux for $L = 4$, $N = 2$

where the operator D is defined in (4.20). For the next row $a = 0$, we have the following QQ-relation relations

$$\begin{aligned} Q_{0,1}(u) &= \frac{Q^+_{1,1}Q^-_{0,0} - Q^-_{1,1}Q^+_{0,0}}{Q_{1,0}(u)}, \\ Q_{0,2}(u) &= \frac{Q^+_{1,2}Q^-_{0,1} - Q^-_{1,2}Q^+_{0,1}}{Q_{1,1}(u)}, \end{aligned} \tag{4.25}$$

where the explicit form of $Q_{0,1}(u)$ is

$$Q_{0,1}(u) = i\frac{\left[2(u-\frac{i}{2})+c_1\right](u+\frac{i}{2})^4 - \left[2(u+\frac{i}{2})+c_1\right](u-\frac{i}{2})^4}{u^2+c_1u+c_0}. \tag{4.26}$$

Since the right-hand side of the above equation should be a polynomial in u, we find

$$\begin{aligned} Q_{0,1}(u) &= -6u^2 + 2c_1u - 2c_1^2 + 6c_0 - 1, \\ R_{0,1}(u) &= (2c_1^3 - 8c_1c_0 + 2c_1)u + (2c_0c_1^2 - 6c_0^2 + c_0 + \frac{1}{8}), \end{aligned} \tag{4.27}$$

where $R_{0,1}(u)$ denotes the remainder of the right-hand side of (4.26). Now we can evaluate $Q_{0,2}(u)$ and the corresponding remainder $R_{0,2}(u)$ in the same way

$$Q_{0,2}(u) = 12, \qquad R_{0,2}(u) = 16ic_1. \tag{4.28}$$

We require the remainder $R_{0,1}(u)$ and $R_{0,2}(u)$ vanish, leading to a set of equations for $\{c_1, c_0\}$

$$2c_1^3 - 8c_1c_0 + 2c_1 = 0, \qquad 2c_0c_1^2 - 6c_0^2 + c_0 + \frac{1}{8} = 0, \qquad 16ic_1 = 0. \tag{4.29}$$

Solving for $\{c_1, c_0\}$ is straightforward, and then the Bethe roots are derived from the zeros of Baxter Q-function (4.23):

$$\left\{\frac{i}{2}, -\frac{i}{2}\right\}, \qquad \left\{\frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}\right\}. \tag{4.30}$$

We can see there are only two physical solutions to the Bethe equations with $(L, N) = (4, 2)$, which is consistent with the conjecture (4.10). Compared with solving the Bethe equations directly, which yields a total of 6 solutions, the crucial advantage of solving Q-system is that it eliminates all non-physical solutions and preserves the completeness

conjecture. Although deriving the algebraic equations for $\{c_k\}$ requires some effort, it is definitely worthwhile because the zero remainder conditions are much more efficient to handle, particularly when (L, N) are large.

It is important to notice that in practice, not all QQ-relations are non-trivial. For example, when $(L, N) = (6, 2)$, although there are 8 unknown Q-functions on the Young tableaux, only 4 of them are non-vanishing. As a result, the QQ-relations remain the same as the case $(L, N) = (4, 2)$ but with slightly different boundary conditions. Furthermore, not all zero remainder conditions are necessary. For $(L, N) = (6, 3)$, we have 3 zero remainder conditions. However, it is not necessary to use all of them to solve for $\{c_k\}$. By appropriately choosing a subset of these conditions, we can find the desired physical solutions as expected. Indeed, there exists a set of minimal choices of zero remainder conditions [23].

4.4 Generalization

The rational Q-system can be generalized to solve other types of Bethe equations corresponding to various quantum integrable spin chains, such as inhomogeneous spin chains where the vacuum on each site carries a weight, XXZ-type spin chains with open or twisted boundary conditions [24–26], and more. For our concern, we want to generalize the formulation of rational Q-system for solving Bethe equations corresponding to the spin chains with higher spin representation and higher rank symmetry algebra.

The Bethe ansatz equations of $\mathfrak{su}(N)$ fundamental spin chain are given by (3.66), which can be denoted graphically by the Dynkin diagram with $N-1$ nodes as shown in Figure 4.3. here the solid circle is termed as momentum carrying node while the others

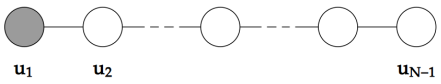

Figure 4.3: Dynkin diagram for $\mathfrak{su}(N)$ fundamental spin chain

are considered auxiliary nodes. Each node in the Dynkin diagram is associated with a set of Bethe roots $\boldsymbol{u_a}$, which corresponds to the a-th set of Bethe equations involving the self-interactions of roots $\boldsymbol{u_a}$ and their interactions with neighboring roots $\boldsymbol{u_{a\pm 1}}$. Furthermore, the momentum carrying nodes have an additional term contributing to the Bethe equations

$$\left(\frac{\boldsymbol{u_a} + \frac{i}{2}}{\boldsymbol{u_a} - \frac{i}{2}}\right)^L . \tag{4.31}$$

It turns out that the conserved charges such as momentum and energy of the Bethe state only explicitly depend on the Bethe roots associated with the momentum carrying

nodes. The rational Q-system for Bethe equations (3.66) can be readily extended from the case of Heisenberg spin chain. Now, the Young tableaux has N rows $\vec{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ and the number of boxes in each row is given by

$$\begin{aligned} \lambda_1 &= L - K_1, \\ \lambda_2 &= K_1 - K_2, \\ \lambda_3 &= K_2 - K_3, \\ &\cdots \\ \lambda_N &= K_{N-1}, \end{aligned} \tag{4.32}$$

where $\{K_1, K_2, \ldots, K_{N-1}\}$ are the number of Bethe roots $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_{N-1}}\}$, respectively. The boundary conditions become

$$\begin{aligned} &\mathbb{Q}_{0,0}(u) = u^L, \qquad\qquad \mathbb{Q}_{N,s}(u) = 1, \\ &\mathbb{Q}_{a,0}(u) = Q_a(u) = u^{K_a} + \sum_{k=0}^{K_a-1} c_k^{(a)} u^k, \end{aligned} \tag{4.33}$$

where $a = 1, 2, \ldots, N-1$ and $Q_a(u)$ are Baxter Q-functions solving for the $N-1$ sets of Bethe roots $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_{N-1}}\}$. The QQ-relation remains the same as (4.13). All the Q-functions should be polynomial and the zero remainder conditions would lead to systems of equations for $\{c_k^{(a)}\}$.

Particularly, we would like to figure out the rational Q-systems for $\mathfrak{so}(6)$ and $\mathfrak{su}(4)$ alternating spin chains which are useful in various contexts. The Dynkin diagrams of the Bethe equations corresponding to these two types of spin chains are illustrated in Figure 4.4.

(a) $\mathfrak{so}(6)$ fundamental spin chain **(b)** $\mathfrak{su}(4)$ alternating spin chain

Figure 4.4: Dynkin diagrams for $\mathfrak{so}(6)$ and $\mathfrak{su}(4)$ spin chains

The $\mathfrak{so}(6)$ spin chain plays an important role in the context of integrability in planar $\mathcal{N} = 4$ super-symmetric Yang-Mills theory (SYM). The one-loop dilation operator in the scalar sector was calculated by Minahan and Zarembo in 2002 [3]. They proposed that this dilation operator can be identified with the Hamiltonian of the $\mathfrak{so}(6)$ spin chain

$$H_{\mathrm{SO}(6)} = \sum_{n=1}^{L} (\mathrm{K}_{n,n+1} + 2\mathrm{I}_{n,n+1} - 2\mathrm{P}_{n,n+1}). \tag{4.34}$$

At each site of the spin chain, there are 6 possible polarizations corresponding to the six real scalar fields in $\mathcal{N} = 4$ SYM. Since the Hamiltonian (4.34) can be diagonalized

by Bethe ansatz, meaning that it is integrable, the problem of finding the one-loop anomalous dimensions of composite operators in $\mathcal{N}=4$ SYM simplifies to solving sets of Bethe equations. The Bethe ansatz equations for $\mathfrak{so}(6)$ spin chain can be read out from the Dynkin diagram:

$$\begin{aligned}
1 &= \prod_{j\neq k}^{K_1}\frac{u_{1,k}-u_{1,j}+i}{u_{1,k}-u_{1,j}-i}\prod_{l=1}^{K_2}\frac{u_{1,k}-u_{2,l}-\frac{i}{2}}{u_{1,k}-u_{2,l}+\frac{i}{2}},\\
\left(\frac{u_{2,k}+\frac{i}{2}}{u_{2,k}-\frac{i}{2}}\right)^L &= \prod_{l=1}^{K_1}\frac{u_{2,k}-u_{1,l}-\frac{i}{2}}{u_{2,k}-u_{1,l}+\frac{i}{2}}\prod_{j\neq k}^{K_2}\frac{u_{k,2}-u_{j,2}+i}{u_{k,2}-u_{j,2}-i}\prod_{l=1}^{K_3}\frac{u_{k,2}-u_{l,3}-\frac{i}{2}}{u_{k,2}-u_{l,3}+\frac{i}{2}},\\
1 &= \prod_{j\neq k}^{K_3}\frac{u_{3,k}-u_{3,j}+i}{u_{3,k}-u_{3,j}-i}\prod_{l=1}^{K_2}\frac{u_{3,k}-u_{2,l}-\frac{i}{2}}{u_{3,k}-u_{2,l}+\frac{i}{2}}.
\end{aligned}\tag{4.35}$$

The rational Q-system for Bethe equations (4.35) includes a Young tableaux with four rows $\vec{\lambda}=(L-K_1, L+K_1-K_2, K_2-K_3, K_3)$, the QQ-relation (4.13) and the boundary conditions:

$$\begin{aligned}
&\mathbb{Q}_{0,0}(u)=(u-\frac{i}{2})^L(u+\frac{i}{2})^L, && \mathbb{Q}_{4,s}(u)=1,\\
&\mathbb{Q}_{1,0}(u)=u^L Q_1(u), && \mathbb{Q}_{2,0}(u)=Q_2(u),\\
&\mathbb{Q}_{3,0}(u)=Q_3(u), && Q_a(u)=u^{K_a}+\sum_{k=0}^{K_a-1}c_k^{(a)}u^k.
\end{aligned}\tag{4.36}$$

The $\mathfrak{su}(4)$ alternating spin chain serves a crucial role in the study of integrability in ABJM theory, which constitutes one of the key points throughout the thesis. In the next chapter, we will delve into the $\mathfrak{su}(4)$ alternating spin chain in detail. Here, we just present the Bethe equations and the corresponding rational Q-system, which helps solve the spectrum of ABJM theory later on. From the graphical representation of the Bethe equations, we can read off

$$\begin{aligned}
\left(\frac{u_{1,k}+\frac{i}{2}}{u_{1,k}-\frac{i}{2}}\right)^L &= \prod_{j\neq k}^{K_1}\frac{u_{1,k}-u_{1,j}+i}{u_{1,k}-u_{1,j}-i}\prod_{l=1}^{K_2}\frac{u_{1,k}-u_{2,l}-\frac{i}{2}}{u_{1,k}-u_{2,l}+\frac{i}{2}},\\
1 &= \prod_{l=1}^{K_1}\frac{u_{2,k}-u_{1,l}-\frac{i}{2}}{u_{2,k}-u_{1,l}+\frac{i}{2}}\prod_{j\neq k}^{K_2}\frac{u_{k,2}-u_{j,2}+i}{u_{k,2}-u_{j,2}-i}\prod_{l=1}^{K_3}\frac{u_{k,2}-u_{l,3}-\frac{i}{2}}{u_{k,2}-u_{l,3}+\frac{i}{2}},\\
\left(\frac{u_{3,k}+\frac{i}{2}}{u_{3,k}-\frac{i}{2}}\right)^L &= \prod_{j\neq k}^{K_3}\frac{u_{3,k}-u_{3,j}+i}{u_{3,k}-u_{3,j}-i}\prod_{l=1}^{K_2}\frac{u_{3,k}-u_{2,l}-\frac{i}{2}}{u_{3,k}-u_{2,l}+\frac{i}{2}}.
\end{aligned}\tag{4.37}$$

The Young tableaux has four rows $\vec{\lambda}=(2L-K_1, L+K_1-K_2, L+K_2-K_3, K_3)$, with the following conditions on the upper and left boundary:

$$\begin{aligned}
&\mathbb{Q}_{0,0}(u)=(u-i)^L u^{2L}(u+i)^L, && \mathbb{Q}_{1,0}(u)=(u-\frac{i}{2})^L(u+\frac{i}{2})^L Q_1(u),\\
&\mathbb{Q}_{2,0}(u)=u^L Q_2(u), && \mathbb{Q}_{3,0}(u)=Q_3(u), && \mathbb{Q}_{4,s}(u)=1.
\end{aligned}\tag{4.38}$$

5. ABJM theory and integrability

Inspired by the spectrum integrability in $\mathcal{N}=4$ SYM [3], another gauge theory proposed by Aharony, Bergman, Jafferis and Maldacena, known as the ABJM theory [4], also turns out to be integrable in the planar limit [5]. In this chapter, we review the setup of ABJM theory and investigate its spectrum integrability through the underlying $\mathfrak{su}(4)$ alternating spin chain. We exhibit the procedure for constructing the eigenvectors in the scalar sector of ABJM theory and briefly introduce the duality at last.

5.1 Field theory setup

The ABJM theory is a Chern-Simons-matter theory in three dimensions with gauge group $\mathrm{U}(N)_k \times \hat{\mathrm{U}}(N)_{-k}$ at Chern-Simons levels $\pm k$. The action of the ABJM theory in the $\mathrm{SU}(4)_R$ invariant component form is given by [27]

$$\begin{aligned} S =& \frac{k}{4\pi}\int d^3x \Bigg[\epsilon^{\mu\nu\lambda}\,\mathrm{Tr}\left(A_\mu\partial_\nu A_\lambda + \frac{2i}{3}A_\mu A_\nu A_\lambda - \hat{A}_\mu\partial_\nu\hat{A}_\lambda - \frac{2i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda\right) \\ &+ \mathrm{Tr}\left(D_\mu Y_A^\dagger D^\mu Y^A + i\psi^{\dagger A}\gamma^\mu D_\mu\psi_A\right) - V_{\mathrm{ferm}} - V_{\mathrm{bos}}\Bigg], \end{aligned} \tag{5.1}$$

with the potential terms

$$\begin{aligned} V_{\mathrm{ferm}} =& \frac{i}{2}\,\mathrm{Tr}\Big(Y_A^\dagger Y^A\psi^{\dagger B}\psi_B - Y^A Y_A^\dagger\psi_B\psi^{\dagger B} + 2Y^A Y_B^\dagger\psi_A\psi^{\dagger B} \\ &- 2Y_A^\dagger Y^B\psi^{\dagger A}\psi_B + \epsilon^{ABCD}Y_A^\dagger\psi_B Y_C^\dagger\psi_D - \epsilon_{ABCD}Y^A\psi^{\dagger B}Y^C\psi^{\dagger D}\Big), \end{aligned} \tag{5.2a}$$

$$\begin{aligned} V_{\mathrm{bos}} =& -\frac{1}{12}\,\mathrm{Tr}\Big(Y_A^\dagger Y^A Y_B^\dagger Y^B Y_C^\dagger Y^C + Y^A Y_A^\dagger Y^B Y_B^\dagger Y^C Y_C^\dagger \\ &+ 4Y_A^\dagger Y^B Y_C^\dagger Y^A Y_B^\dagger Y^C - 6Y^A Y_B^\dagger Y^B Y_A^\dagger Y^C Y_C^\dagger\Big), \end{aligned} \tag{5.2b}$$

where the indices $A, B, C = 1, 2, 3, 4$. The covariant derivatives are defined as

$$D_\mu Y^A = \partial_\mu Y^A + A_\mu Y^A - Y^A\hat{A}_\mu, \qquad D_\mu Y_A^\dagger = \partial_\mu Y_A^\dagger + \hat{A}_\mu Y_A^\dagger - Y_A^\dagger A_\mu. \tag{5.3}$$

The field contents in the action (5.1) include gauge fields A_μ and $\hat{A}_\mu$ transforming in the adjoint representation of the groups $\mathrm{U}(N)_k$ and $\hat{\mathrm{U}}(N)_{-k}$ respectively, along with

four complex scalar fields Y^A and Weyl spinors ψ_A, both in the bi-fundamental representation of the gauge group $\mathrm{U}(N)_k \times \hat{\mathrm{U}}(N)_{-k}$. The gauge structure of the ABJM theory can be represented by the quiver diagram shown in Figure 5.1, where the arrows indicate the gauge group representations of the various fields, pointing from a fundamental to an anti-fundamental representation.

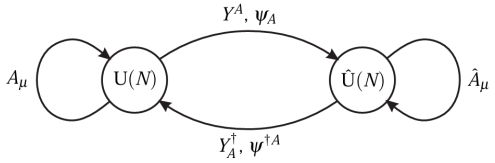

Figure 5.1: Quiver diagram of ABJM theory. The arrows indicate the representations of the fields under the gauge groups. The arrows are drawn from a fundamental to an anti-fundamental representation.

There are two parameters in ABJM theory, the Chern-Simons level k and the rank of the gauge group N. Since the level k is an overall factor in the action, $g^2_{\mathrm{CS}} \equiv \frac{1}{k}$ plays the role of coupling constant. Besides, the level k must be an integer, as required by the gauge symmetry [28]. For $k \gg 1$, the ABJM theory is weakly coupled and can be treated using perturbation theory. Particularly, the ABJM theory becomes integrable in the 't Hooft (or planar) limit [5, 29, 30], realized by a small coupling and a large number of colors:

$$N \to \infty,\ k \to \infty, \qquad \text{with } \frac{N}{k} \text{ fixed}, \tag{5.4}$$

which introduces the effective coupling $\lambda \equiv g^2_{\mathrm{CS}} N$, referred to as the 't Hooft coupling. For infinite N, k and finite λ, essentially λ is continuous.

5.2 Global symmetries

As mentioned above, the action of the ABJM theory possesses a global $\mathrm{SO}(6)_R \cong \mathrm{SU}(4)_R$ symmetry, known as the R-symmetry, which is a symmetry of the superalgebra itself. The fields $Y^A, \psi^\dagger_A$ and $Y^\dagger_A, \psi_A$ transform as the $\mathbf{4}$ and $\bar{\mathbf{4}}$ representations of the $\mathrm{SU}(4)_R$ respectively, ensuring the scalar and fermionic terms in the action (5.1) are invariant under both gauge and R-symmetry transformations. This symmetry is essential for the supersymmetric nature of the theory and affects the supersymmetry transformations and the form of the action.

For the spacetime symmetry, the ABJM theory is invariant under Lorentz transformations and spacetime translations, which constitute the PoincarÃ© symmetry.

Furthermore, incorporating the symmetries under scale transformations, the ABJM theory exhibits the 3-dimensional conformal symmetry $\mathrm{Sp}(4) \cong \mathrm{SO}(2,3)$.

The action (5.1) is also invariant under the $\mathcal{N}=6$ supersymmetry (SUSY) transformations:

$$\begin{aligned}
\delta Y^A =& i\omega^{AB}\psi_B, \qquad\qquad \delta Y_A^\dagger = i\psi^{\dagger B}\omega_{AB},\\
\delta\psi_A =& -\gamma^\mu\omega_{AB}D_\mu Y^B + \frac{2\pi}{k}\left[-\omega_{AB}(Y^C Y_C^\dagger Y^B - Y^B Y_C^\dagger Y^C) + 2\omega_{CD}Y^C Y_A^\dagger Y^D\right],\\
\delta\psi^{A\dagger} =& D_\mu Y_B^\dagger \omega^{AB}\gamma^\mu + \frac{2\pi}{k}\left[-(Y_B^\dagger Y^C Y_C^\dagger - Y_C^\dagger Y^C Y_B^\dagger)\omega^{AB} + 2Y_D^\dagger Y^A Y_C^\dagger \omega^{CD}\right],\\
\delta A_\mu =& \frac{\pi}{k}(-Y^A\psi^{B\dagger}\gamma_\mu\omega_{AB} + \omega^{AB}\gamma_\mu\psi_A Y_B^\dagger),\\
\delta\hat{A}_\mu =& \frac{\pi}{k}(-\psi^{A\dagger}Y^B\gamma_\mu\omega_{AB} + \omega^{AB}\gamma_\mu Y_A^\dagger\psi_B),
\end{aligned} \tag{5.5}$$

where ω^{AB} and ω_{AB} are SUSY transformation parameters and anti-symmetric in spinor indices AB, given by

$$\omega_{AB} = \varepsilon_i(\Gamma^i)_{AB}, \qquad \omega^{AB} = \varepsilon_i((\Gamma^i)^*)^{AB}. \tag{5.6}$$

Six $(2+1)$-dimensional majorana spinors ε_i are generators of $\mathcal{N}=6$ SUSY. And we have the following relations for Γ matrices:

$$\begin{aligned}
&\left\{\Gamma^i, \Gamma^{j\dagger}\right\} = 2\delta_{ij}, \qquad (\Gamma^i)_{AB} = -(\Gamma^i)_{AB},\\
&\frac{1}{2}\varepsilon^{ABCD}\Gamma^i_{CD} = -(\Gamma^{i\dagger})^{AB} = ((\Gamma^i)^*)^{AB}.
\end{aligned} \tag{5.7}$$

Hence the SUSY parameters satisfy

$$(\omega^{AB})_\alpha = (\omega^*_{AB})_\alpha, \qquad \omega^{AB} = \frac{1}{2}\epsilon^{ABCD}\omega_{CD}. \tag{5.8}$$

Together with the R-symmetry, the SUSY transformations (5.5) significantly restricts the possible interactions in the theory. Moreover, the SUSY plays a crucial role in the cancellation of certain types of anomalies and quantum corrections.

The representations of the ABJM field contents under the gauge and global symmetries are collected in table 5.1. The global symmetries, including $\mathrm{SU}(4)_R$ R-symmetry, $\mathrm{SO}(2,3)$ conformal symmetry, and $\mathcal{N}=6$ supersymmetry generators ε_i, combine into a larger symmetry group, the ortho-symplectic supergroup $\mathrm{OSp}(6|4)$ [31–33]. In this sense, the ABJM theory is a $\mathcal{N}=6$ superconformal theory.

	U(N)	$\hat{\text{U}}$(N)	SU(4)$_R$	SO(2,1)	U(1)$_\Delta$	U(1)$_b$
Y^A	$\mathbf{N}$	$\overline{\mathbf{N}}$	$\mathbf{4}$	$\mathbf{1}$	$\frac{\mathbf{1}}{\mathbf{2}}$	$\mathbf{1}$
$Y^{\dagger A}$	$\overline{\mathbf{N}}$	$\mathbf{N}$	$\overline{\mathbf{4}}$	$\mathbf{1}$	$\frac{\mathbf{1}}{\mathbf{2}}$	$\mathbf{-1}$
ψ_A	$\mathbf{N}$	$\overline{\mathbf{N}}$	$\overline{\mathbf{4}}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$
$\psi^{\dagger A}$	$\overline{\mathbf{N}}$	$\mathbf{N}$	$\mathbf{4}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{-1}$
A^μ	adj	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{0}$
$\hat{A}^\mu$	$\mathbf{1}$	adj	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{0}$

Table 5.1: Field contents of ABJM theory and representations carried by the fields under the gauge symmetry, the R-symmtry, the Lorentz group SO(2,1). As well as the conformal dimension Δ and the baryonic charge U(1)$_b$.

5.3 Correlation functions

The presence of conformal symmetry in the ABJM theory strictly constrains the form of correlation functions. The one-point function of a scalar operator $\langle\mathcal{O}(x)\rangle$ should be constant due to the translations invariance. Furthermore, the operator $\mathcal{O}(x)$ must transform appropriately under the dilation:

$$x \to \lambda x, \qquad \mathcal{O}(x) \to \lambda^\Delta \mathcal{O}(\lambda x), \tag{5.9}$$

where Δ is the scaling dimension of the operator $\mathcal{O}(x)$. For the one-point function, the scaling invariance implies

$$\langle\mathcal{O}(x)\rangle \to \langle\mathcal{O}(\lambda x)\rangle = \lambda^{-\Delta}\langle\mathcal{O}(x)\rangle. \tag{5.10}$$

To satisfy (5.10), the one-point function must vanish

$$\langle\mathcal{O}(x)\rangle = 0. \tag{5.11}$$

Similarly, the two-point functions are completely fixed by the conformal symmetry. The translation and rotation invariance imply that the two-point functions can only depend on the distance between the two points, namely

$$\langle\mathcal{O}_1(x)\mathcal{O}_2^\dagger(y)\rangle = f(|x-y|) \equiv f(r), \tag{5.12}$$

where $f(r)$ is an arbitrary function. The dilation invariance requires

$$f(\lambda r) = \lambda^{-\Delta_1-\Delta_2} f(r) \tag{5.13}$$

holds for any λ. Thus, we have

$$f(r) = \frac{C}{r^{\Delta_1+\Delta_2}}, \tag{5.14}$$

where C is a constant. Additionally, the two-point functions should transform correctly under special conformal transformations, leading to the non-vanishing condition $\Delta_1 = \Delta_2$. Finally, we see that the two-point functions are fixed up to a constant

$$\langle \mathcal{O}_1(x)\mathcal{O}_2^\dagger(y)\rangle = \frac{\delta_{12}C}{|x-y|^{\Delta_1+\Delta_2}}. \tag{5.15}$$

Actually, all correlation functions in a conformal field theory are fixed in terms of the so-called conformal data (Δ, λ). Here, Δ represents the conformal or scaling dimension of the operator, as previously mentioned. And λ (distinct from the parameter in the dilation) is the structure constant that appears in the operator product expansion (OPE):

$$\mathcal{O}_i(x)\mathcal{O}_j(y) = \frac{\delta_{ij}C}{|x-y|^{\Delta_i+\Delta_j}} + \sum_k \frac{\lambda_{ij}{}^k}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}} d(x-y,\partial_y)\mathcal{O}_k(y), \tag{5.16}$$

where the sum runs over conformal primary operators and the differential operator d accounts for the presence of conformal descendants. Following from the OPE (5.16), the three-point functions are fixed up to the structure constant λ_{ijk}. Moreover, all higher-point functions can be determined by repeatedly using the OPE (5.16), which leads to their factorization into a sum of two-point functions.

5.4 Gauge invariant operators

Since the gauge symmetry of ABJM theory is $\mathrm{U}(N) \times \hat{\mathrm{U}}(N)$, the scalars and gauge fields transform as

$$\begin{aligned} Y^A &\to U Y^A \hat{U}^\dagger, & A_\mu &\to U A_\mu U^\dagger - iU\partial_\mu U^\dagger, \\ Y_A^\dagger &\to \hat{U} Y_A^\dagger U^\dagger, & \hat{A}_\mu &\to \hat{U}\hat{A}_\mu\hat{U}^\dagger - i\hat{U}\partial_\mu\hat{U}^\dagger, \end{aligned} \tag{5.17}$$

where $(U, \hat{U}) \in \mathrm{U}(N) \times \hat{\mathrm{U}}(N)$. For spinors, there are similar gauge transformations. We aim to construct gauge invariant operators because only the correlation functions of these operators are physically observable. There are several classes of gauge invariant operators. The simplest examples of such operators are single trace operators, taking the following form

$$\mathcal{O} = \mathrm{Tr}\left(X_{i_1} X_{i_2}^\dagger X_{i_3} X_{i_4}^\dagger \dots X_{i_{n-1}} X_{i_n}^\dagger\right), \tag{5.18}$$

where X_{2k-1} are any fields that transform in the bi-fundamental representations $(\mathrm{N}, \bar{\mathrm{N}})$ of the gauge group, while X_{2k} represents fields in the anti-bi-fundamental representations $(\bar{\mathrm{N}}, \mathrm{N})$. Moreover, we can build gauge invariant multi-trace operators by taking products of such single trace operators (5.18). We observe that maintaining gauge symmetry requires an alternating sequence of fields within the trace. Therefore, at

odd sites we can have any of the $4_B + 8_F$ fields $Y^A, \psi_{A\alpha}$, while at even sites any of the $4_B + 8_F$ fields $Y^\dagger_A, \psi^{\dagger A}_\alpha$ can reside. Additionally, any number of covariant derivatives $D_{\alpha\beta}$ can be introduced to act on the fields.

We will work in the scalar sector of ABJM theory, and gauge invariant composite operators can be built with these scalars in the form

$$\mathcal{O} = \Psi^{B_1\dots B_L}_{A_1\dots A_L} \mathrm{tr}(Y^{A_1} Y^\dagger_{B_1} Y^{A_2} Y^\dagger_{B_2} \dots Y^{A_L} Y^\dagger_{B_L}), \tag{5.19}$$

where the coefficient $\Psi^{B_1\dots B_L}_{A_1\dots A_L}$ is invariant under cyclic permutations of the sequence $\{A_1, B_1, \dots, A_L, B_L\}$ due to cyclic property of trace. The operator $\mathcal{O}$ can be seen as a SU(4) alternating tensor in $(\mathbf{4} \otimes \bar{\mathbf{4}})^{\otimes L}$.

5.5 Spectral problem

The spectral problem in a conformal field theory involves finding the spectrum of conformal dimensions. The classical or bare dimension Δ_0 of a composite operator $\mathcal{O}$ can be obtained by standard power counting. However, conformal dimension Δ can receive quantum corrections, implying that Δ depends on the coupling constant in general. The corrections $\gamma = \Delta - \Delta_0$ is called the anomalous dimension by historical reasons. The dilation operator D measures the dimension Δ of a composite operator $\mathcal{O}$ by the eigenvalue equation

$$D\mathcal{O}(x) = \Delta\mathcal{O}(x). \tag{5.20}$$

It plays an important role in the spectral problem. The eigenvectors of D are good conformal operators with definite conformal dimensions. We will see that the anomalous dimension of the operator $\mathcal{O}$ arises in the perturbative expansion of its two-point correlation function.

The two-point function at quantum level, by the conformal symmetry, still takes the form (5.15), but with Δ expressed as a power series in the coupling constant λ

$$\Delta(\lambda) = \sum_{n=0}^{\infty} \lambda^{2n} \Delta^{(n)}. \tag{5.21}$$

For small coupling, $\gamma \ll \Delta_0$, expanding $\Delta(\lambda)$ in the two-point function, we have

$$\langle \mathcal{O}(x)\mathcal{O}^\dagger(y)\rangle = \frac{1}{|x-y|^{2\Delta_0}} \left(1 - \lambda^2 \Delta^{(1)} \log(\Lambda^2|x-y|) + O(\lambda^4)\right), \tag{5.22}$$

where $\Delta^{(1)}$ is the anomalous dimension at leading order. Since the interactions in ABJM theory are of types Y^6 and $Y^2\psi^2$, the lowest quantum corrections to the scaling dimension come at two-loop order. Here, Λ is a renormalization scale with dimension of mass, introduced to ensure that the logarithm is dimensionless. From (5.22), we see

that quantum corrections to the two-point functions of local composite operators are manifested through the coupling dependence of scaling dimensions.

Let us investigate the spectral problem in the scalar sector of ABJM theory. For a general operator composed of scalar fields

$$\mathcal{O}^{A_1,\ldots,A_L}_{B_1,\ldots,B_L} = \mathrm{tr}(Y^{A_1}Y^\dagger_{B_1}Y^{A_2}Y^\dagger_{B_2}\ldots Y^{A_L}Y^\dagger_{B_L}), \tag{5.23}$$

the two-point function $\langle \mathcal{O}_I(x)\mathcal{O}^\dagger_J(y)\rangle$ at tree level does not receive quantum corrections, meaning that it only depends on Δ_0. There are three types of tree-level Feynman diagrams contributing to the two-point function, as shown in Figure 5.2. We can

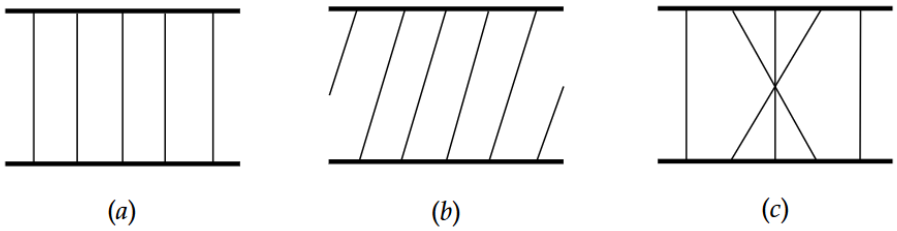

Figure 5.2: Tree-level diagrams contributing to the two-point function

evaluate these diagrams by contracting scalar fields with the propagator

$$\left\langle (Y^A(x))^a{}_b (Y^\dagger_B(y))^c{}_d \right\rangle = \frac{1}{k}\frac{\delta^A{}_B\delta^a{}_d\delta^c{}_b}{|x-y|}, \tag{5.24}$$

where $a, b, c, d = 1, 2, \ldots, N$ are color indices and A, B are flavor indices. The contributions from diagrams a and b are proportional to N^{2L} since every two propagators give a factor $\delta^a_d\delta^c_b\delta^d_a\delta^b_c = N^2$. While for the diagram c, such contractions swap the order of two of the propagators, making it proportional to $(N^2)^{L-1}$. Hence, the contributions from non-planar diagrams such as c are suppressed in the 't Hooft limit. It is straightforward to calculate the contributions from diagram a and c. The result is

$$\left\langle \mathcal{O}^{A_1,\ldots,A_L}_{B_1,\ldots,B_L}\mathcal{O}^{\dagger\,B'_1,\ldots,B'_L}_{A'_1,\ldots,A'_L} \right\rangle = \frac{\lambda^{2L}}{|x-y|^{2L}}\left(\delta^{A_1}_{A'_1}\ldots\delta^{A_L}_{A'_L}\delta^{B'_1}_{B_1}\ldots\delta^{B'_L}_{B_L} + \text{cyclic perm.}\right). \tag{5.25}$$

We see that the classical dimension of composite operator (5.23) is $\Delta_0 = L$, consistent with the power counting.

To find the anomalous dimension, we can first calculate the two-point function at loop order and then extract the logarithmically divergent pieces from it. In the $\mathcal{N} = 4$ SYM theory, the dilation operator can be read off from the two-point function under a specific renormalization scheme [34]. Alternatively, in ABJM theory, we can obtain the dilation operator by studying the one-point functions of the composite operators (5.19). The divergences due to quantum corrections require the renormalization

$$\mathcal{O}^{\mathrm{ren}}_a = \mathcal{Z}_a{}^b(\lambda,\Lambda)\mathcal{O}^{\mathrm{bare}}_b, \tag{5.26}$$

where matrix $\mathcal{Z}_a^{\ b}$ cancels the appearing divergences and implies there is a mixing of different bare operators. The operator mixing matrix Γ describes how the basis (5.23) transform under the renormalization and can be viewed as the matrix representation of the dilation operator, which is given by

$$\Gamma = \mathcal{Z}^{-1}\frac{d\mathcal{Z}}{d\ln\Lambda}. \tag{5.27}$$

The two-loop renormalization in the basis (5.23) was studied in [5,30] and the dilation operator in the planar limit is given by

$$\Gamma_{\text{2-loop}} = \frac{\lambda^2}{2}\sum_{l=1}^{2L}\left(2 - 2P_{l,l+2} + P_{l,l+2}K_{l,l+1} + K_{l,l+1}P_{l,l+2}\right). \tag{5.28}$$

Now the spectral problem of anomalous dimension translates to diagonalizing the dilation operator. The set of basis (5.23) forms a tensor product space $V_1 \otimes V_2 \otimes \ldots \otimes V_L$ with $V_n = \mathbf{4} \otimes \bar{\mathbf{4}}$, on which the dilation operator acts. And the eigenvectors of the dilation operator generally are linear combinations of single trace operators, taking the form (5.19). The dilation operator in this basis is a matrix of size 4^{2L}. Thus, for large L it is hard or impossible to diagonalize it directly. Fortunately, the dilation operator (5.28) can be identified with the Hamiltonian of the SU(4) alternating spin chain, where the spectral problem can be solved using integrability technique, e.g. Bethe ansatz.

5.6 SU(4) alternating spin chain

The Hamiltonian of the SU(4) alternating spin chain is exactly same as the dilation operator (5.28), where permutation operator P and trace operator K acting on $\mathbb{C}^4 \otimes \mathbb{C}^4$ are defined by

$$\mathrm{P}_{ij}|i\rangle \otimes |j\rangle = |j\rangle \otimes |i\rangle, \qquad \mathrm{K}_{ij}|i\rangle \otimes |j\rangle = \delta_{ij}\sum_{k=1}^{4}|k\rangle \otimes |k\rangle. \tag{5.29}$$

The spin chain shown in Figure 5.3 has two types of lattice sites, where spins at odd sites and even sites sit in the $\mathbf{4}$ and $\bar{\mathbf{4}}$ representations of SU(4) group respectively.

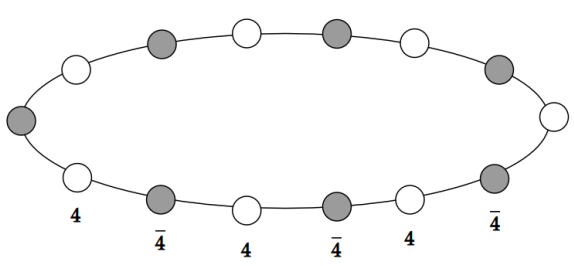

Figure 5.3: $\mathfrak{su}(4)$ alternating spin chain

Since the scalar fields $Y^A, Y_A^\dagger$ in ABJM theory transform as the fundamental and anti-fundamental representations of SU(4) R-symmetry group, we can map these fields to the spins

$$Y^A \mapsto |A\rangle, \qquad\qquad Y_B^\dagger \mapsto |\bar{B}\rangle, \tag{5.30}$$

where bars are used to distinguish odd and even sites. Then, the composite operator (5.23) can be mapped to a state in the Hilbert space of the spin chain

$$\mathrm{tr}\left(Y^{A_1}Y_{B_1}^\dagger Y^{A_2}Y_{B_2}^\dagger\cdots\right) \quad \mapsto \quad |A_1\bar{B}_1A_2\bar{B}_2\cdots\rangle. \tag{5.31}$$

Therefore, we can diagonalize the dilation operator (5.28) within the spin chain framework by utilizing the Bethe ansatz, which was detailed in the previous chapter.

5.6.1 Eigenvalue

The Bethe ansatz equations for the SU(4) alternating spin chain are given by (4.37), and we rewrite it as

$$\begin{aligned}
1 = e^{i\phi_{u_k}} &= \left(\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}}\right)^L \prod_{\substack{j=1\\ j\neq k}}^{K_{\mathbf{u}}} S(u_k,u_j)\prod_{j=1}^{K_{\mathbf{w}}}\tilde{S}(u_k,w_j),\\
1 = e^{i\phi_{w_k}} &= \prod_{\substack{j=1\\ j\neq k}}^{K_{\mathbf{w}}} S(w_k,w_j)\prod_{j=1}^{K_{\mathbf{u}}}\tilde{S}(w_k,u_j)\prod_{j=1}^{K_{\mathbf{v}}}\tilde{S}(w_k,v_j),\\
1 = e^{i\phi_{v_k}} &= \left(\frac{v_k+\frac{i}{2}}{v_k-\frac{i}{2}}\right)^L \prod_{\substack{j=1\\ j\neq k}}^{K_{\mathbf{v}}} S(v_k,v_j)\prod_{j=1}^{K_{\mathbf{w}}}\tilde{S}(v_k,w_j),
\end{aligned} \tag{5.32}$$

where the S-matrices are the same as those in (3.67), and we introduce the functions ϕ for later use. Through the rational-Q system (4.38), we can solve these coupled equations efficiently. The two-loop anomalous dimension of the conformal operator (5.19) equals to the energy eigenvalue

$$\Delta - L = \lambda^2\left(\sum_{k=1}^{K_{\mathbf{u}}}\frac{1}{u_k^2+\frac{1}{4}} + \sum_{k=1}^{K_{\mathbf{v}}}\frac{1}{v_k^2+\frac{1}{4}}\right). \tag{5.33}$$

5.6.2 Eigenvector

The eigenvectors of the dilation operator are identified with the energy eigenstate of SU(4) alternating spin chain, which can be constructed by the nested coordinate Bethe ansatz (CBA). There are 4 possible states at each site of the spin chain while the odd and even sites should be distinguished. Each scalar field in ABJM theory is mapped

to a specific combination of the Bethe roots as follows:

$$Y^1 \mapsto |1\rangle = |\bullet\rangle, \qquad Y^2 \mapsto |2\rangle = |\overset{u}{\bullet}\rangle, \qquad Y^3 \mapsto |3\rangle = |\overset{w}{\underset{u}{\bullet}}\rangle, \qquad Y^4 \mapsto |4\rangle = |\overset{w}{\underset{uv}{\bullet}}\rangle,$$
$$Y_1^\dagger \mapsto |\bar{1}\rangle = |\overset{w}{\underset{uv}{\circ}}\rangle, \qquad Y_2^\dagger \mapsto |\bar{2}\rangle = |\overset{w}{\underset{v}{\circ}}\rangle, \qquad Y_3^\dagger \mapsto |\bar{3}\rangle = |\overset{v}{\circ}\rangle, \qquad Y_4^\dagger \mapsto |\bar{4}\rangle = |\circ\rangle. \tag{5.34}$$

Here, we choose $|\Omega\rangle = |1\bar{4}\rangle^L$ as the vacuum state, corresponding to a chiral primary operator $\mathcal{O}_{gs} = \mathrm{tr}[(Y^1 Y_4^\dagger)^L]$. It is easy to see $\mathcal{O}_{gs}$ is symmetric under any permutation $\mathrm{P}_{l,l+2}$ and vanishing under any trace $\mathrm{K}_{l,l+1}$. Thus, $\mathcal{O}_{gs}$ is the ground state of the dilation operator (5.28) with zero anomalous dimension. Moreover, the chiral primaries in ABJM theory are those in (5.19) with symmetric and traceless $\Psi^{B_1,\ldots,B_L}_{A_1,\ldots,A_L}$. Their conformal dimension are protected from quantum corrections by the supersymmetry.

To construct eigenvectors, we initially distribute the Bethe roots $\{\mathbf{u}, \mathbf{w}, \mathbf{v}\}$ to the sites of the spin chain in a specific manner:

1. Distribute the momentum carrying roots $\{u_1, \ldots, u_{K_\mathbf{u}}\}$ and $\{v_1, \ldots, v_{K_\mathbf{v}}\}$ to the vacuum state $|\bullet_1, \circ_1, \cdots \bullet_L, \circ_L\rangle$. Either a single root u_k or a pair of roots $u_k v_j$ can be placed at odd sites. Likewise, a single v-type root or two roots of uv-type can be assigned to even sites.

2. Distribute the auxiliary roots $\{w_1, \ldots, w_{K_\mathbf{w}}\}$ to the sites that contain roots u or v. Each root w_k can be allocated to the top of either u, v or uv.

3. Distributions with configurations not listed in (5.34), such as $|\overset{uv}{\bullet}\rangle$ which does not correspond to any of the fields, are set to zero.

Following this procedure, the possible configurations at site-n are given by

$$\mathrm{s}_{2n-1} \in \{\bullet, \overset{u_k}{\bullet}, \overset{w_i}{\underset{u_k}{\bullet}}, \overset{w_i}{\underset{u_k v_j}{\bullet}}\}, \qquad \mathrm{s}_{2n} \in \{\circ, \overset{v_j}{\circ}, \overset{w_i}{\underset{v_j}{\circ}}, \overset{w_i}{\underset{u_k v_j}{\circ}}\}. \tag{5.35}$$

Any possible distribution is denoted by

$$\vec{\mathbf{s}} \equiv \{\mathrm{s}_1, \mathrm{s}_2, \ldots, \mathrm{s}_{2L-1}, \mathrm{s}_{2L}\}. \tag{5.36}$$

All the distributions $\vec{\mathbf{s}}$ form a set of basis vectors $|\vec{\mathbf{s}}\rangle$. Any eigenvector is the linear combination of these basis vectors

$$|\mathbf{u}, \mathbf{w}, \mathbf{v}\rangle = \sum_{\vec{\mathbf{s}}} \Psi_{\vec{\mathbf{s}}}(\mathbf{u}, \mathbf{w}, \mathbf{v}) |\vec{\mathbf{s}}\rangle, \tag{5.37}$$

where Ψ represents the wavefunction of each distribution. It takes the following general form:

$$\Psi_{\vec{\mathbf{s}}}(\mathbf{u}, \mathbf{w}, \mathbf{v}) = \mathbb{S} \times \prod_{n=1}^{2L} \Phi(\mathrm{s}_n). \tag{5.38}$$

Single-site wave functions $\Phi(\mathrm{s}_n)$ with at most one Bethe root are given by

$$\Phi(\bullet_n) = \Phi(\circ_n) = 1, \qquad \Phi(\overset{u_k}{\bullet}_n) = \left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^n, \qquad \Phi(\overset{v_k}{\circ}_n) = \left(\frac{v_k + \frac{i}{2}}{v_k - \frac{i}{2}}\right)^n, \tag{5.39}$$

and $\Phi(\mathrm{s}_n)$ with two Bethe roots are

$$\begin{aligned} \Phi(\overset{w_i}{\underset{u_k}{\bullet}}_n) &= \left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^n \times \psi(w_i|z_j) \times \frac{-1}{w_i - u_k - \frac{i}{2}}, \\ \Phi(\overset{w_i}{\underset{v_k}{\circ}}_n) &= \left(\frac{v_k + \frac{i}{2}}{v_k - \frac{i}{2}}\right)^n \times \psi(w_i|z_j) \times \frac{1}{w_i - v_k - \frac{i}{2}}, \end{aligned} \tag{5.40}$$

where $\psi(w_i|z_j)$ indicates that wavefunctions (5.40) depend on the Bethe roots at other sites. The explicit form of $\psi(w|z_j)$ is

$$\psi(w|z_j) = \prod_j \frac{w - z_j + \frac{i}{2}}{w - z_j - \frac{i}{2}}, \tag{5.41}$$

and z_j denotes all the momentum carrying roots located at sites to the left of the site containing w. Similarly, $\Phi(\mathrm{s}_n)$ with three Bethe roots also depends on z_j, given by

$$\begin{aligned} \Phi(\overset{w_i}{\underset{u_k v_j}{\bullet}}_n) &= \left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^n \left(\frac{v_j + \frac{i}{2}}{v_j - \frac{i}{2}}\right)^n \times \psi(w_i|z_l) \times \frac{-(v_j - \frac{i}{2})}{(w_i - u_k - \frac{i}{2})(w_i - v_j - \frac{i}{2})}, \\ \Phi(\overset{w_i}{\underset{u_k v_j}{\circ}}_n) &= \left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^n \left(\frac{v_j + \frac{i}{2}}{v_j - \frac{i}{2}}\right)^n \times \psi(w_i|z_l) \times \frac{+(u_k + \frac{i}{2})}{(w_i - u_k - \frac{i}{2})(w_i - v_j - \frac{i}{2})}. \end{aligned} \tag{5.42}$$

Now, the single-site wavefunction $\Phi(\mathrm{s}_n)$ for all eight types of configurations in (5.35) has been determined. The remaining task is to determine the factor $\mathbb{S}$ in (5.38), which is a product of S-matrices (3.67). The factor $\mathbb{S}$ accounts for the reordering of Bethe roots into the canonical order, namely the same order as $\{u_1, u_2, \dots u_k\}$, $\{v_1, v_2, \dots v_k\}$ and $\{w_1, w_2, \dots w_k\}$.

5.6.3 Norms and Gaudin determinant

The normalization problem involves the norm of Bethe states. The Gaudin conjecture points out that for any integrable spin chains, the norm of Bethe states is proportional to the determinant of the Gaudin matrix [35]. The Gaudin matrix is defined by

$$(G_{ab})_{jk} = \frac{\partial \phi_{u_{b,k}}}{\partial u_{a,j}}, \tag{5.43}$$

where the function ϕ is the logarithm of Bethe equations, as introduced in (5.32). And $u_{a,j}$ is an element in the a-th set of Bethe roots. Thus, the Gaudin matrix in general is a block matrix.

For the SU(4) alternating spin chain, the Gaudin matrix is a 3×3 block matrix with a total size of $(K_u + K_w + K_v) \times (K_u + K_w + K_v)$:

$$G = \begin{pmatrix} \partial_{u_i}\phi_{u_j} & \partial_{u_i}\phi_{w_j} & \partial_{u_i}\phi_{v_j} \\ \partial_{w_i}\phi_{u_j} & \partial_{w_i}\phi_{w_j} & \partial_{w_i}\phi_{v_j} \\ \partial_{v_i}\phi_{u_j} & \partial_{v_i}\phi_{w_j} & \partial_{v_i}\phi_{v_j} \end{pmatrix}. \tag{5.44}$$

We see that the Gaudin matrix would be large even for small (K_u, K_w, K_v). However, if the Bethe roots $\{\mathbf{u}, \mathbf{w}, \mathbf{v}\}$ are parity-symmetric, the determinant of the Gaudin matrix (5.44) factorizes into the determinants of two sub-matrices. This factorization has already been observed in the SU(2) Heisenberg spin chain when studying the integrability in $\mathcal{N} = 4$ SYM [36]. We will see this factorization in ABJM theory later. The norm of the Bethe states (5.37), as conjectured in [37], is given by

$$\begin{aligned} \langle \mathbf{u}, \mathbf{w}, \mathbf{v} | \mathbf{u}, \mathbf{w}, \mathbf{v} \rangle = & \left(\prod_{i<j} \frac{S(u_i, u_j)}{S(u_i^*, u_j^*)} \right)^{\frac{1}{2}} \left(\prod_{i<j} \frac{S(v_i, v_j)}{S(v_i^*, v_j^*)} \right)^{\frac{1}{2}} \left(\prod_{i<j} \frac{S(w_i, w_j)}{S(w_i^*, w_j^*)} \right)^{\frac{1}{2}} \\ & \times \left(\prod_j \frac{1}{\partial_u p(u_j)} \right) \left(\prod_k \frac{1}{\partial_v p(v_k)} \right) \det G, \end{aligned} \tag{5.45}$$

with

$$S(u, v) = \frac{u - v - i}{u - v + i}, \qquad p(u) = \frac{1}{i} \log \frac{u + \frac{i}{2}}{u - \frac{i}{2}}, \qquad \partial_u p(u) = \frac{-1}{u^2 + \frac{1}{4}}. \tag{5.46}$$

The conjecture for the norm (5.45) can be tested in the specific case. For the parity-symmetric Bethe states which satisfy $\mathcal{P}|\mathbf{u}, \mathbf{w}, \mathbf{v}\rangle = |\mathbf{u}, \mathbf{w}, \mathbf{v}\rangle$, the norm formula simplifies to

$$\langle \mathbf{u}, \mathbf{w}, \mathbf{v} | \mathbf{u}, \mathbf{w}, \mathbf{v} \rangle = \prod_{k=1}^{\left[\frac{K_\mathbf{w}}{2}\right]} \frac{(w_k - \frac{i}{2})(w_k^* + \frac{i}{2})}{(w_k + \frac{i}{2})(w_k^* - \frac{i}{2})} \prod_{j=1}^{K_\mathbf{u}} \left(u_j^2 + \frac{1}{4} \right)^2 \det G. \tag{5.47}$$

5.7 Duality

The AdS/CFT correspondence proposed by Maldacena in [6], indicates that the boundary of an anti-de Sitter space can act as the spacetime for a d-dimensional conformal field theory. And this CFT_d is exactly equivalent to the gravitational theory in the bulk of the AdS_{d+1} space, suggesting a correspondence where every entity in the CFT has a counterpart on the gravity side (and vice versa). There are several examples of the AdS/CFT correspondence; for instance, $\mathcal{N} = 4$ SYM theory in four dimensions is equivalent type IIB string theory on the product space $\text{AdS}_5 \times S^5$, referred to as $\text{AdS}_5/\text{CFT}_4$. The integrability in planar $\mathcal{N} = 4$ SYM theory is manifested through the classically integrable string world-sheet model on the string theory side [38, 39].

Specifically, the dilation operator in the CFT can be mapped to a Hamiltonian of a spin chain, whose integrability mirrors that of the string sigma model in AdS space. Thus, integrability aids in computing these spectra exactly on both sides, providing a powerful check of the duality.

Another example of the AdS/CFT correspondence is $\mathrm{AdS}_4/\mathrm{CFT}_3$ [4]: the ABJM theory is gravitational dual to M-theory on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$. In other words, ABJM theory describes the world-volume theory of a stack of N M2 branes moving on the orbifold $\mathbb{C}^4/\mathbb{Z}_k$. In the 't Hooft limit (planar limit), where k and N grow large with equal powers, we have $k^5 \gg N$ and hence M-theory is approximated by weekly coupled IIA string theory on $\mathrm{AdS}_4 \times \mathbb{C}P^3$ background. The string coupling constant and tension are given by:

$$g_s \sim \left(\frac{N}{k^5}\right)^{1/4} = \frac{\lambda^{5/4}}{N}, \qquad \frac{R^2}{\alpha'} = 4\pi\sqrt{2\lambda}, \tag{5.48}$$

where R is the radius of $\mathbb{C}P^3$ and twice the radius of AdS_4. We see that both the string theory and the field theory are controlled by only two parameters k and N. As discussed in [7,8], planar integrability was established for the dual string sigma model, which provides an intuitive understanding of the origin of integrability in CFT_3.

6. One-point functions in defect ABJM theory

As previously discussed, correlation functions in a conformal field theory (CFT) are significantly constrained by conformal symmetry. In particular, one-point functions must vanish due to symmetry arguments, see Eq.(5.11). However, we can introduce a defect in the spacetime of a CFT, which leads to several novel effects. The presence of defects can break conformal symmetry and affect the correlation functions. We consider a flat co-dimension 1 defect that acts as an interface, as depicted in Figure 6.1. This interface allows some of the physical modes to propagate while blocking others. Furthermore, it reduces the conformal symmetry by one dimension and necessarily breaks some of the supersymmetry as well.

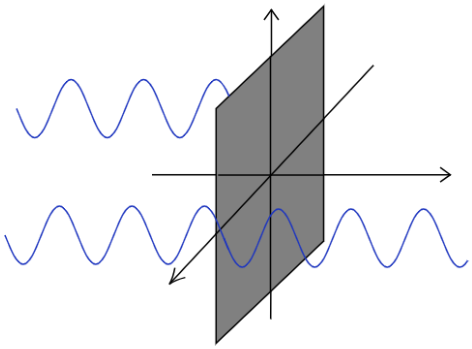

Figure 6.1: A co-dimension-1 defect in the spacetime of a CFT

In such a defect CFT, one-point functions and two-point functions of operators with unequal conformal dimensions are less restricted and hence become non-vanishing. Specifically, one-point functions are now fixed up to a constant C [40]

$$\langle \mathcal{O}_I(x) \rangle = \frac{C_I}{z^{\Delta_I}}, \tag{6.1}$$

where z denotes the distance from x to the interface. And two-point functions are no

longer proportional to a delta function:

$$\langle \mathcal{O}_I(x)\mathcal{O}_J(y)\rangle = \frac{f_{IJ}(\xi)}{z_x^{\Delta_I} z_y^{\Delta_J}}, \qquad \xi = \frac{|x-y|^2}{4z_x z_y}, \tag{6.2}$$

with the conformal ratio ξ. Additionally, correlation functions involving bulk and boundary fields emerge.

A defect version of ABJM theory with a co-dimension 1 defect can be built by introducing a domain wall on which Nahm pole boundary conditions are applied to certain field components. This defect ABJM theory is holographic dual to a string theory with D2-D4 probe brane system, referred to as the AdS/dCFT correspondence. The defect, as well as its holographic dual description, can be identified with an integrable boundary state in the Hilbert space of underlying spin chain. Consequently, overlaps between these integrable boundary states and Bethe eigenstates encode the one-point functions, demonstrating that integrability is preserved in this setup.

6.1 Supersymmetric defect in ABJM theory

The domain wall in a field theory separates regions of different vacuum configurations. It is described by the classical scalar-field profile and can be viewed as an interface where scalar fields transition from one vacuum configuration to another in this interface. There exists super-symmetric domain walls with certain boundary conditions, which preserve a fraction of the original supersymmetry. In ABJM theory, the domain wall described by Nahm pole boundary conditions preserves half of the supersymmetry and hence is named the 1/2-BPS domain wall. The Nahm pole boundary conditions are solutions to the Nahm's equations, which can be derived by rewriting the BPS equations.

We first derive the BPS equations by minimizing the energy functional and then use supersymmetry arguments to demonstrate that they indeed preserve half of the supersymmetry in ABJM theory. The bosonic part of the Lagrangian of ABJM theory follows from (5.1):

$$\begin{aligned}\mathcal{L} =& \frac{k}{4\pi}\mathrm{tr}\Big[\varepsilon^{\mu\nu\lambda}\Big(A_\mu\partial_\nu A_\lambda + \frac{2}{3}A_\mu A_\nu A_\lambda - \hat{A}_\mu\partial_\nu\hat{A}_\lambda - \frac{2}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda\Big)\\ &+ D_\mu Y_A^\dagger D^\mu Y^A + \frac{1}{12}Y^A Y_A^\dagger Y^B Y_B^\dagger Y^C Y_C^\dagger + \frac{1}{12}Y^A Y_B^\dagger Y^B Y_C^\dagger Y^C Y_A^\dagger\\ &- \frac{1}{2}Y^A Y_A^\dagger Y^B Y_C^\dagger Y^C Y_B^\dagger + \frac{1}{3}Y^A Y_B^\dagger Y^C Y_A^\dagger Y^B Y_C^\dagger + \text{fermions}\Big].\end{aligned} \tag{6.3}$$

And the energy functional of scalars reads

$$E = \frac{k}{4\pi}\int dx\,\mathrm{tr}\left(D_\mu Y_A^\dagger D^\mu Y^A + V_{\text{bos}}\right), \tag{6.4}$$

with the potential energy V_{bos} give by (5.2b). The energy of a BPS configuration takes the absolute minimum value within its topological sector. Thus, we aim to re-express the energy functional (6.4) as a total square plus a total derivative and then require the total square to vanish. Although the potential V_{bos} is not a total square in general, it turns out that V_{bos} can become such if we consider a configuration with only two of the scalar fields Y^A being non-zero. We choose Y^1, Y^4 as the non-zero components for later convenience, collectively denoted by Y^α.

Contracting $Y^\dagger_\alpha Y^\delta Y^\dagger_\beta Y^\varepsilon Y^\dagger_\gamma Y^\omega$ with the Levi-Civita tensor $\varepsilon^{\alpha\beta\gamma}\varepsilon_{\delta\varepsilon\omega}$, which is always zero for 2-dimensional indices $\alpha, \beta, \gamma, \delta, \varepsilon, \omega = 1, 4$, we establish the following identity:

$$\begin{aligned}\operatorname{tr} Y^\dagger_\alpha Y^\beta Y^\dagger_\gamma Y^\alpha Y^\dagger_\beta Y^\gamma =& 3\operatorname{tr} Y^\dagger_\alpha Y^\beta Y^\dagger_\beta Y^\alpha Y^\dagger_\gamma Y^\gamma - \operatorname{tr} Y^\dagger_\alpha Y^\beta Y^\dagger_\beta Y^\gamma Y^\dagger_\gamma Y^\alpha \\ &- \operatorname{tr} Y^\dagger_\alpha Y^\alpha Y^\dagger_\beta Y^\beta Y^\dagger_\gamma Y^\gamma,\end{aligned} \tag{6.5}$$

which implies that there are only three kinds of independent tensor structures in the potential term

$$V_{\text{bos}} = \frac{1}{4} Y^\alpha Y^\dagger_\alpha Y^\beta Y^\dagger_\beta Y^\gamma Y^\dagger_\gamma + \frac{1}{4} Y^\alpha Y^\dagger_\beta Y^\beta Y^\dagger_\gamma Y^\gamma Y^\dagger_\alpha - \frac{1}{2} Y^\alpha Y^\dagger_\alpha Y^\beta Y^\dagger_\gamma Y^\gamma Y^\dagger_\beta. \tag{6.6}$$

Along with

$$\begin{aligned}\operatorname{tr}\left[D_\mu (Y^\dagger_\alpha Y^\alpha Y^\dagger_\beta Y^\beta)\right] &= \operatorname{tr}\left[D_\mu (Y^\dagger_\alpha Y^\alpha) Y^\dagger_\beta Y^\beta + Y^\dagger_\alpha Y^\alpha D_\mu (Y^\dagger_\beta Y^\beta)\right] \\ &= 2\operatorname{tr}\left[D_\mu (Y^\dagger_\alpha Y^\alpha) Y^\dagger_\beta Y^\beta\right].\end{aligned} \tag{6.7}$$

we can rewrite the energy functional as

$$\begin{aligned}E = \frac{k}{4\pi}\int dx \operatorname{tr}\Bigg[& D_\mu Y^\dagger_\alpha D^\mu Y^\alpha - \frac{1}{2} D_\mu (Y^\dagger_\alpha Y^\alpha) Y^\dagger_\beta Y^\beta + \frac{1}{2} Y^\alpha Y^\dagger_\alpha D_\mu (Y^\beta Y^\dagger_\beta) \\ &+ V_{\text{bos}} + \frac{1}{4} D_\mu (Y^\dagger_\alpha Y^\alpha Y^\dagger_\beta Y^\beta) - \frac{1}{4} D_\mu (Y^\alpha Y^\dagger_\alpha Y^\beta Y^\dagger_\beta)\Bigg].\end{aligned} \tag{6.8}$$

Then, it can be expressed in a modulus-squared form

$$\begin{aligned}E =& \frac{k}{4\pi}\int dx \operatorname{tr}\left(\frac{dY^\dagger_\alpha}{dx} - \frac{1}{2} Y^\dagger_\beta Y^\beta Y^\dagger_\alpha + \frac{1}{2} Y^\dagger_\alpha Y^\beta Y^\dagger_\beta\right) \\ &\times \left(\frac{dY^\alpha}{dx} - \frac{1}{2} Y^\alpha Y^\dagger_\beta Y^\beta + \frac{1}{2} Y^\beta Y^\dagger_\beta Y^\alpha\right) + \text{total derivatives}.\end{aligned} \tag{6.9}$$

where covariant derivative D_μ reduces to ∂_μ by a specific gauge choice and $x \equiv x_2$ because we assume $Y^\alpha{=}Y^\alpha(x_2)$. To ensure that the energy functional takes the smallest value, we set the total squared term to zero, which leads to the BPS equations

$$\frac{dY^\alpha}{dx} - \frac{1}{2} Y^\alpha Y^\dagger_\beta Y^\beta + \frac{1}{2} Y^\beta Y^\dagger_\beta Y^\alpha = 0. \tag{6.10}$$

The solutions to the BPS equations are consistent with the equations of motion of the scalars. Hence, these solutions describe a BPS domain wall.

To explicitly see that the BPS equations (6.10) preserve half of the supersymmetry, we can re-derive them from the SUSY transformations (5.5). Here we still assume that $Y^\alpha = Y^\alpha(x_2), Y^3 = Y^4 = 0$, along with

$$\gamma^2\omega_{12} = \omega_{12}, \quad \gamma^2\omega_{34} = \omega_{34}, \quad \gamma^2\omega_{a\dot{b}} = -\omega_{a\dot{b}}, \quad \gamma^2\omega_{\dot{b}a} = -\omega_{\dot{b}a}, \tag{6.11}$$

where ω_{AB} are SUSY transformation parameters defined in (5.6) with $a = 1, 2$ and $\dot{b} = 3, 4$. The condition (6.11) implies, for instance, $\omega_{12} = \omega_{34}^*$. Thus, there are only three independent ω_{AB}, resulting in the SUSY transformation (5.5) coinciding with the usual $\mathcal{N} = 3$ SUSY transformation. The BPS condition reads $\delta\psi_A = 0$, whose solutions will preserve 6 supercharges, namely half of the original supersymmetry. Under the choice (6.11), the SUSY transformation for ψ becomes

$$\begin{aligned} &\frac{dY^1}{dx_2} + \frac{2\pi}{k}\left(Y^2Y_2^\dagger Y^1 - Y^1Y_2^\dagger Y^2\right) = 0, \\ &\frac{dY^2}{dx_2} + \frac{2\pi}{k}\left(Y^1Y_1^\dagger Y^2 - Y^2Y_1^\dagger Y^1\right) = 0, \end{aligned} \tag{6.12}$$

or collectively

$$\frac{dY^\alpha}{dx_2} = -\frac{2\pi}{k}(Y^\beta Y_\beta^\dagger Y^\alpha - Y^\alpha Y_\beta^\dagger Y^\beta), \tag{6.13}$$

which will be exactly same as the BPS equations (6.10) after re-scaling the fields. Now we see that the solutions to (6.10) indeed correspond to a 1/2-BPS domain wall.

6.2 1/2-BPS solutions and Nahm's equations

Let us find the solutions to the BPS equations (6.10). The solution for the half-space $x > 0$ takes the form

$$Y^\alpha(x) = \frac{S^\alpha}{\sqrt{x}}, \qquad x > 0, \tag{6.14}$$

where S^α sit in the bi-fundamental representation $(\mathbf{N}, \bar{\mathbf{N}})$ of the gauge group and satisfy the following algebra

$$S^1 = S^2S^{2\dagger}S^1 - S^1S^{2\dagger}S^2, \tag{6.15a}$$

$$S^2 = S^1S^{1\dagger}S^2 - S^2S^{1\dagger}S^1. \tag{6.15b}$$

With the $\mathrm{U}(N) \times \hat{\mathrm{U}}(N)$ gauge symmetry, we can always diagonalize S^2 by the gauge transformation (5.18), and hence we take

$$(S^2)_{ij} = \alpha_i\delta_{ij}, \tag{6.16}$$

where α_i is a real and non-negative number. Without loss of generality, we can assume $\alpha_{i+1} \leq \alpha_i$, facilitating the formulation for diagonal S^2. Then, equation (6.15a) gives

$$(S^1)_{ij} = \left(\alpha_i^2 - \alpha_j^2\right)(S^1)_{ij}, \tag{6.17}$$

which implies $(S^1)_{ij} = 0$ if $\alpha_i^2 - \alpha_j^2 \neq 1$. Thus, S^1 is block diagonalized if $\alpha_i^2 - \alpha_{i+1}^2 = 1$ and $\alpha_i^2 - \alpha_j^2 \neq 1$ when $j \neq i+1$, where we can assume

$$(S^1)_{ij} = \beta_i \delta_{i,j-1}, \qquad i,j = 1, \cdots, N. \tag{6.18}$$

With $(S^1(S^1)^\dagger)_{ij} = \delta_{ij}(\beta_i)^2$ and $((S^1)^\dagger S^1)_{ij} = \delta_{ij}(\beta_{i-1})^2$, equation (6.15b) reads

$$\left(\beta_i^2 - \beta_{i-1}^2\right)\alpha_i = \alpha_i, \qquad i = 1, \ldots, N. \tag{6.19}$$

For convenience, we set $\alpha_N = 0$ and $\beta_0 = 0$. Then, equation (6.19) implies $\beta_1 = 1$ when $i = 1$. Therefore, we conclude that

$$(S^1)_{ij} = \delta_{i,j-1}\sqrt{i}, \qquad (S^2)_{ij} = \delta_{ij}\sqrt{N-i}, \qquad i,j = 1, \ldots, N. \tag{6.20}$$

These solutions, combined with (6.14), is the BPS solution for $x > 0$.

We consider a configuration where we take $i = 1, \ldots, q-1$ and $j = 1, \ldots q$ in (6.20), assuming $q \leq N$. This configuration still satisfies 1/2-BPS equations and serves as the classical solution for the scalar fields, given by

$$Y_{\rm cl}^{\alpha}(x) = \frac{1}{\sqrt{x}} \begin{pmatrix} S^{\alpha}_{(q-1)\times q} & 0_{(q-1)\times(N-q)} \\ 0_{(N-q+1)\times q} & 0_{(N-q+1)\times(N-q)} \end{pmatrix}, \qquad x > 0. \tag{6.21}$$

And all other fields, i.e. Y^3, Y^4, gauge fields and fermions vanish classically. In other words, for $x > 0$, the gauge symmetry breaks down to $\mathrm{U}(N-q+1) \times \hat{\mathrm{U}}(N-q)$ due to the non-zero part of $Y_{\rm cl}^{\alpha}$ and the symmetry will restored to $\mathrm{U}(N) \times \hat{\mathrm{U}}(N)$ as $x \to \infty$. In the other half-space $x < 0$, to maintain consistency, the gauge symmetry is taken to be $\mathrm{U}(N-q+1) \times \hat{\mathrm{U}}(N-q)$. All fields in this region transform under the representations of this symmetry group and possess vanishing classical solutions. Consequently, this defect field theory setup define a domain wall version of ABJM theory, as shown in Figure 6.2, where the domain wall breaks the SU(4) symmetry of scalar fields to $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$.

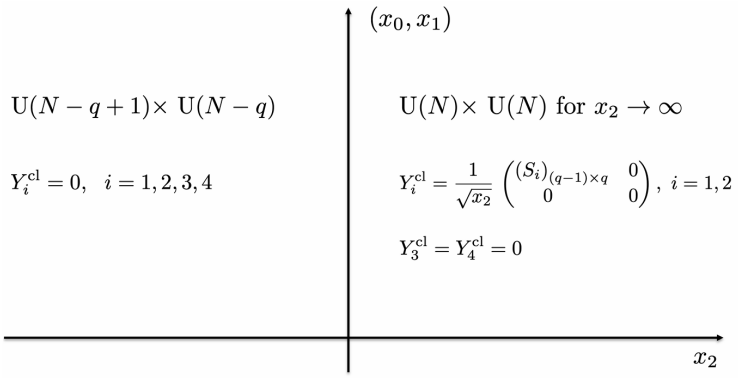

Figure 6.2: The domain wall version of ABJM theory [41]

Equivalently, the domain wall can also be described by Nahm pole boundary conditions [42], which solve Nahm's equations [43]. In fact, Nahm's equations describe super-symmetric domain walls in 4D [44] and play a significant role in the parallel studies of defect $\mathcal{N} = 4$ SYM [45]. We aim to demonstrate that the BPS equations (6.10) are equivalent to Nahm's equations, in the sense that they represent a square root of Nahm's equation.

Defining the composite field with $(q-1) \times q$ dimensional scalar fields Y^α

$$\Phi^\alpha{}_\beta = Y^\alpha Y^\dagger_\beta, \tag{6.22}$$

we can combine the BPS equation (6.10) and its conjugate into a single equation

$$\frac{d\Phi^\alpha{}_\beta}{dx} = \Phi^\alpha{}_\gamma \Phi^\gamma{}_\beta - \frac{1}{2}\left\{\Phi^\gamma{}_\gamma, \Phi^\alpha{}_\beta\right\}. \tag{6.23}$$

Furthermore, expanding the composite field $\Phi^\alpha{}_\beta$ in the $(\boldsymbol{\sigma}, \mathbb{I})$ basis,

$$\Phi^\alpha{}_\beta = \Phi^i \sigma^\alpha_{i\beta} + \Phi \delta^\alpha_\beta, \tag{6.24}$$

the equation (6.23) simplifies to

$$\begin{aligned} \frac{d\Phi^i}{dx} &= \frac{i}{2}\varepsilon^{ijk}\left[\Phi^j, \Phi^k\right], \\ \frac{d\Phi}{dx} &= \Phi^i \Phi^i - \Phi^2, \end{aligned} \tag{6.25}$$

where the first equation is the Nahm equation. It is easy to find the simplest Nahm-pole solution:

$$\Phi^i = \frac{t^i}{x}, \tag{6.26}$$

with t^i satisfying the $\mathfrak{su}(2)$ algebra. Since Φ^i is of size $(q-1) \times (q-1)$ by the definition (6.22), t^i form a $(q-1)$-dimensional representation of $\mathfrak{su}(2)$. Hence, the Casimir operator $\Phi^i \Phi^i = \frac{q(q-2)}{4}\mathbb{I}$. Then, we find the solution for the singlet component

$$\Phi = \frac{q\mathbb{I}}{2x}. \tag{6.27}$$

The same arguments apply to the dual bi-linear

$$\hat{\Phi}_\alpha{}^\beta = Y_\alpha^\dagger Y^\beta \equiv \hat{\Phi}_i \sigma_\alpha^{i\beta} + \hat{\Phi}\delta_\alpha{}^\beta. \tag{6.28}$$

For which, the BPS equation can be reformulated as Nahm's equation:

$$\begin{aligned} \frac{d\hat{\Phi}_i}{dx} &= -\frac{i}{2}\varepsilon_{ijk}[\hat{\Phi}_j, \hat{\Phi}_k], \\ \frac{d\hat{\Phi}}{dx} &= -\hat{\Phi}_i\hat{\Phi}_i + \hat{\Phi}^2. \end{aligned} \tag{6.29}$$

Similarly, it is straightforward to find the simplest Nahm pole solution

$$\begin{aligned} \hat{\Phi}_i &= -\frac{\hat{t}_i}{x}, \\ \hat{\Phi} &= \frac{(q-1)\mathbb{I}}{2x}. \end{aligned} \tag{6.30}$$

Notice that in this case $\hat{t}_i$ form a q-dimensional representation of $\mathfrak{su}(2)$, instead of $(q-1)$-dimensional, because the dual bi-linear $\hat{\Phi}_\alpha{}^\beta$ is of size $(q \times q)$. We can check the Nahm pole solutions (6.26), (6.27) and (6.30) are consistent with the BPS solution (6.21), implying that they do correspond to a 1/2-BPS domain wall. The simplest Nahm pole solution arises when $q = 2$, given by

$$\begin{aligned} \Phi^\alpha{}_\beta &= \frac{1}{x}\delta^\alpha{}_\beta, \\ \hat{\Phi}_\alpha{}^\beta &= \frac{\mathbb{I}_{2\times 2}\delta_\alpha{}^\beta - \sigma_3\sigma_i\sigma_3\sigma_\alpha^{i\beta}}{2x}, \end{aligned} \tag{6.31}$$

here we choose the representation $\hat{t}_i = \sigma_3\sigma_i\sigma_3$ for consistency. With the classical solution (6.21) for $q = 2$, we explicitly find

$$Y^\alpha Y_\beta^\dagger = \frac{1}{x}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad Y_\alpha^\dagger Y^\beta = \frac{1}{x}\begin{bmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{bmatrix}, \tag{6.32}$$

which is exactly consistent with the Nahm pole solution (6.31).

6.3 String theory description

As mentioned before, planar ABJM theory describes the world-volume theory of a stack of N D2 branes in type IIA superstring theory. This holographic duality can be deformed by introducing defects (probe branes) on the field theory (string theory) side. The defect version of ABJM theory with a 1/2-BPS domain wall, constructed in

the previous section, has a dual string theory description where a D4 probe brane is embedded in the type IIA background $AdS_4 \times \mathbb{CP}^3$. The resulting D2-D4 probe brane system consists of N coinciding D2-branes and one single probe D4-brane inserted among them. The relative orientation of the branes is shown in Table 6.1.

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
D2	•	•	•							
D4	•		•	•		•	•			

Table 6.1: D2-D4 probe brane system in ten-dimensional string theory background

The probe brane system has the geometry $AdS_3 \times \mathbb{CP}^1$ and carries q units of world volume gauge field flux on the $\mathbb{CP}^1$. It preserves half of the supersymmetry in the string theory [46], namely it is 1/2-BPS, which is consistent with the domain wall in ABJM theory. In fact, integrability of the string boundary conditions on the probe D4-brane within the Green-Schwarz sigma model was checked in [12], which suggests the ABJM domain wall is integrable to all loop orders and for any value of the bond dimension.

This is not surprising, since similar results are obtained in AdS_5/CFT_4. The D3-D5 probe brane setup describing $\mathcal{N} = 4$ SYM with a supersymmetric domain wall was studied in [9,10,36], where the investigation of tree-level one-point functions has uncovered notable signs of integrability and enabled a test of the AdS/dCFT correspondence at classical level [47]. Furthermore, at quantum level the study of one-loop one-point functions has provided a positive test of the AdS/dCFT correspondence [48,49]. What is more, there exists a probe brane setup that breaks all supersymmetry of $\mathcal{N} = 4$ SYM, known as the D3-D7 brane setup, which comes in two variants. In one variant, the D7-brane has the geometry $AdS_4 \times S^4$ [50]; in the other, it is $AdS_4 \times S^2 \times S^2$ [51]. The former setup exhibits SO(5) symmetry, while the latter one displays SU(2)×SU(2) symmetry. The one-point functions were evaluated for the SO(5) D3-D7 setup in [52], including those of non-protected operators. And the boundary integrability in this dCFT was proved in [53]. However, so far, for the SU(2)×SU(2) D3-D7 setup, only tree-level one-point functions of chiral primaries have been calculated, and these calculations were consistent with string-theory predictions [54].

Finally, it is worth mentioning that integrability in the context of the AdS/CFT correspondence has been studied for other duality configurations involving D7-branes. These include a D7-D3 probe setup with the geometry $AdS_5 \times S^3$ [55–57], and a D7-O7-D3 setup which involves an orientifold plane [58–61].

6.4 Integrable boundary states

Now we focus on the one-point functions in the 1/2-BPS domain wall version of ABJM theory. At tree level, the one-point functions of composite operator (5.19) can be evaluated by inserting the classical solution of scalar fields (6.21):

$$\langle \mathcal{O}(x) \rangle_{\text{tree}} = \frac{1}{x^L} \Psi^{B_1 \dots B_L}_{A_1 \dots A_L} \operatorname{tr}(S^{A_1} S^\dagger_{B_1} S^{A_2} S^\dagger_{B_2} \dots S^{A_L} S^\dagger_{B_L}). \tag{6.33}$$

The coefficients $\Psi^{B_1 \dots B_L}_{A_1 \dots A_L}$ ensure that the operator $\mathcal{O}(x)$ is an eigenstate of the dilation operator. With the underlying integrable quantum spin chain in ABJM theory, $\Psi^{B_1 \dots B_L}_{A_1 \dots A_L}$ can be determined by Bethe ansatz method as analysed in section 5.6.

Alternatively, we can calculate the one-point functions exploiting the integrability of the domain wall. Since the domain wall preserves half of the supersymmetry, it turns out that such super-symmetric defects possess integrability, which is manifested through the corresponding integrable boundary state in spin chain picture.

6.4.1 Definition

There is no rigours definition for the integrable boundary state. One possible definition of the integrable boundary state was proposed by Piroli, Pozsgay, Vernier in 2017 [62], which is inspired from the definition of boundary states in quantum field theories [63]. They suggested that an boundary state $|\Psi_0\rangle$ is integrable if it is annihilated by all local conserved charges of the model that are odd under space reflection:

$$Q_{2k+1} |\Psi_0\rangle = 0, \qquad k = 1, 2, \dots. \tag{6.34}$$

The integrability condition (6.34) is equivalent to another integrability condition using the transfer matrix $T(u)$ that generates the set of conserved charges

$$\Pi T(u) \Pi |\Psi_0\rangle = T(u) |\Psi_0\rangle, \tag{6.35}$$

where Π is the parity operator, acting on the all sites of spin chain as

$$\Pi |i_1, i_2, \cdots, i_L\rangle = |i_L, i_{L-1}, \cdots, i_1\rangle. \tag{6.36}$$

The parity transformation $\Pi T(u) \Pi$ in the condition (6.35) are model-dependent, as the structure of the transfer matrix varies between different models. For instance, the transfer matrix of $\mathfrak{su}(2)$ spin chain constructed in section 3.1, transforms as [64]

$$\Pi T(u) \Pi = T(-u). \tag{6.37}$$

We will discuss the significance of the integrability conditions when studying the overlap between boundary states and Bethe states.

6.4.2 Matrix product state

We consider a specific type of boundary state represented by the so-called matrix product state. The matrix product state (MPS) [65] is a representation of quantum many-body states, especially useful for one-dimensional systems, such as spin chains. It expresses the quantum state of a system in terms of a product of matrices. A generic (periodic) MPS can be defined as

$$|\text{MPS}\rangle = \sum_{i_1,\ldots,i_L=1}^{d} \text{tr}\left[A_1^{(i_1)} A_2^{(i_2)} \ldots A_L^{(i_L)}\right] |i_1, i_2, \ldots, i_L\rangle, \tag{6.38}$$

where d is the dimension of the physical space and $A_n^{(i_n)}$ are $d_{n-1} \times d_n$ dimensional matrices with d_n being arbitrary positive integer numbers, referred to as bond dimensions. The $|\text{MPS}\rangle$ that corresponds to the domain wall can be constructed by replacing matrices $A_n^{(i_n)}$ with the classical solution of scalar fields. In studies of the integrability of defect $\mathcal{N} = 4$ SYM theory, the constructed MPS is indeed integrable [9, 10, 66], which implies that the defect preserves integrability and there exists a closed formula for one-point functions. We will see this later in ABJM theory.

In the presence of a 1/2-BPS domain wall in ABJM theory, the boundary state $|\text{MPS}\rangle$ in the spin chain's Hilbert space, takes the following form

$$|\text{MPS}\rangle = \sum_{A_k,B_k=1}^{2} \text{tr}\left[S^{A_1} S^\dagger_{B_1} S^{A_2} S^\dagger_{B_2} \ldots S^{A_L} S^\dagger_{B_L}\right] |A_1, B_1, \ldots, A_L, B_L\rangle. \tag{6.39}$$

Combining odd and even sites into $\Phi^\alpha{}_\beta$ and inserting Nahm pole solutions (6.26) and (6.27), we find the wavefunction of $|\text{MPS}\rangle$

$$\begin{aligned} \text{MPS} &= \text{tr}_a\, M_{1\bar{1}} \ldots M_{L\bar{L}}, \\ M_{n\bar{n}} &= \frac{q}{2}\mathbb{I}_a \otimes \mathbb{I}_n + t_a^i \otimes \sigma_n^i, \end{aligned} \tag{6.40}$$

where $M_{n\bar{n}}$ are matrices acting on the $(q-1)$-dimensional auxiliary space V_a and quantum spaces of two neighboring sites $V_n = |A_n\rangle \otimes |B_n\rangle$. Equivalently, using dual bi-linear solutions $\hat{\Phi}_\alpha{}^\beta$ in (6.30), another representation of the boundary state $|\text{MPS}\rangle$ can be obtained

$$\begin{aligned} \text{MPS} &= \text{tr}_b\, \widehat{M}_{\bar{1}2} \widehat{M}_{\bar{2}3} \ldots \widehat{M}_{\bar{L}1}, \\ \widehat{M}_{\bar{n},n+1} &= \frac{q-1}{2}\mathbb{I}_b \otimes \mathbb{I}_n - \hat{t}_b^i \otimes \sigma_n^i, \end{aligned} \tag{6.41}$$

here the auxiliary space $V_b = \mathbb{C}^q$ and quantum space $V_n = |B_n\rangle \otimes |A_{n+1}\rangle$. When the bond dimension parameter $q = 2$, the 1-dimensional representation of $\mathfrak{su}(2)$ is trivial $t^i = 0$ and there is no auxiliary space to be traced. Thus, the boundary state given in (6.40) becomes a product of two-site states:

$$\text{MPS}_{q=2} = \delta^{A_1}_{B_1} \delta^{A_2}_{B_2} \ldots \delta^{A_L}_{B_L}. \tag{6.42}$$

Such a boundary state is named the valence bond State (VBS), defined as

$$\langle \mathrm{MPS}_{q=2}| = \langle K|^{\otimes L} \equiv \langle \mathrm{VBS}|, \tag{6.43}$$

where $\langle K|$ is a two-site state with components $K^A_B = \delta^A_B$.

6.5 Overlap formula

The one-point functions (6.33) can be re-expressed in spin chain picture. Since the dilation operator (5.28) is identified with the Hamiltonian of $\mathfrak{su}(4)$ alternating spin chain, the conformal operator $\mathcal{O}(x)$ is proportional to energy eigenstate of spin chain. In general, the energy eigenstate of $\mathfrak{su}(4)$ spin chain can be written as

$$|\mathbf{u}\rangle = \Psi^{B_1\dots B_L}_{A_1\dots A_L}|A_1, B_1, \dots, A_L, B_L\rangle, \tag{6.44}$$

with the same coefficients $\Psi^{B_1\dots B_L}_{A_1\dots A_L}$ as in equation (6.33). The overlap between the eigenstate $|\mathbf{u}\rangle$ and the boundary state $|\mathrm{MPS}\rangle$ in (6.39) is

$$\langle \mathrm{MPS}|\mathbf{u}\rangle = \Psi^{B_1\dots B_L}_{A_1\dots A_L}\,\mathrm{tr}\left(S^{A_1}S^\dagger_{B_1}S^{A_2}S^\dagger_{B_2}\dots S^{A_L}S^\dagger_{B_L}\right). \tag{6.45}$$

We observe that the overlap is equal to the one-point functions in (6.33), up to a normalization factor. It turns out that the one-point functions at tree level have the following exact form in terms of the overlap

$$\langle \mathcal{O}(x)\rangle_{\mathrm{tree}} = \frac{1}{x^L}\frac{1}{\lambda^L L^{\frac{1}{2}}}\frac{\langle \mathrm{MPS}_q|\mathbf{u}\rangle}{\langle \mathbf{u}|\mathbf{u}\rangle^{\frac{1}{2}}}, \qquad x > 0, \tag{6.46}$$

which allows to evaluate one-point functions using integrability technique in spin chain language. Specifically, the eigenstate $|\mathbf{u}\rangle$ and its norm can be found using the Bethe ansatz as discussed in section 5.6, and the overlap $\langle \mathrm{MPS}|\mathbf{u}\rangle$ simplifies greatly under the integrability condition (6.35) for boundary states.

6.5.1 Selection rules

Before we discuss the overlap formula, we first need to examine the conditions under which the one-point functions are non-vanishing. These conditions are referred to as selection rules in this context. From the field theory perspective, the non-vanishing one-point functions (6.33) are determined by the classical solutions $Y^A Y^\dagger_B$ with $A, B = 1, 4$, which serve as the building blocks of the single trace operator. It is easy to see that the one-point function of the chiral primary vanishes, since

$$\langle \mathcal{O}_{gs}\rangle = \mathrm{Tr}(Y^1Y^\dagger_4\dots Y^1Y^\dagger_4), \qquad Y^1Y^\dagger_4 \equiv \Phi^1{}_2 = \frac{t^i\sigma^1_i{}_2}{x} = \frac{2t^-}{x}, \tag{6.47}$$

where t^- is a $(q-1)$-dimensional lower triangular matrix with the same structure as the lowering operator, implying that the trace of any product of $Y^1Y_4^\dagger$ vanish. The other three types of field combinations in (6.33) are

$$Y^4Y_1^\dagger = \frac{2t^+}{x}, \qquad Y^1Y_1^\dagger = \frac{q+2t^3}{2x}, \qquad Y^4Y_4^\dagger = \frac{q-2t^3}{2x}, \tag{6.48}$$

where t^+ serves as the raising operator, and t^3 is a diagonal matrix. Thus, we find that for non-vanishing one-point functions an arbitrary number of field combinations of the two types $Y^1Y_1^\dagger$ and $Y^4Y_4^\dagger$ are allowed within the trace, whereas a combination $Y^1Y_4^\dagger$ needs another combination $Y^4Y_1^\dagger$ to form a pair.

By this observation, in order to have the non-vanishing one-point functions of an operator that is excited from the vacuum $\mathcal{O}_{gs}$ based on the correspondence in (5.34), we require $K_u = K_v = K_w$ so that only $Y^1, Y_1^\dagger, Y^4, Y_4^\dagger$ arise in the excited operator. Furthermore, we need an equal number of combinations $Y^4Y_1^\dagger$ and $Y^1Y_4^\dagger$ to ensure they form a pair, which yields

$$K_u = K_v = K_w = L, \tag{6.49}$$

where L is half the length of the operator or corresponding spin chain. Besides, the selection rule (6.49) is consistent with the perspective of the spin chain. The boundary state $|\text{MPS}\rangle$ constructed in (6.39) only contains states $|1\rangle, |4\rangle, |\bar{1}\rangle, |\bar{4}\rangle$. For the overlap $\langle\text{MPS}|\mathbf{u}\rangle$ non-vanishing, the Bethe state $|\mathbf{u}\rangle$ should only involve these four states, leading to the condition $K_u = K_v = K_w$. Note that the selection rule (6.49) is just a consequence of the global symmetry $\text{SU}(2) \times \text{SU}(2) \times \text{U}(1)$ in the domain wall version of ABJM theory, and it is not due to the integrability of the matrix product state.

A Bethe state with (K_u, K_w, K_v) roots belongs to the $\mathfrak{su}(4)$ representation with the Dynkin labels

$$\left[L - 2K_u + K_w, K_u - 2K_w + K_v, L - 2K_v + K_w\right], \tag{6.50}$$

and the excitation numbers must satisfy

$$2K_u \leqslant L + K_w, \qquad 2K_v \leqslant L + K_w, \qquad 2K_w \leqslant K_u + K_v. \tag{6.51}$$

Thus, the selection rule (6.49) requires the Bethe state $|\mathbf{u}\rangle$ is a singlet $[0,0,0]$ of $\mathfrak{su}(4)$.

Next, let us discuss the implications of the boundary integrability, which will also lead to selection rules. Assuming that the boundary state $\langle\text{MPS}|$ is integrable, we have

$$\langle\text{MPS}|\Pi t(u)\Pi = \langle\text{MPS}|t(u), \tag{6.52}$$

where $t(u)$ is one of the transfer matrices of $\mathfrak{su}(4)$ alternating spin chain. These transfer matrices can be constructed with the R-matrix $R_{ab}(u) = u\mathbb{I}_{ab} - \text{P}_{ab}$

$$\begin{aligned} t(u) &= \text{Tr}_a R_{a1}(u)\bar{R}_{a2}(u)\ldots R_{a,2L-1}(u)\bar{R}_{a,2L}(u), \\ \bar{t}(u) &= \text{Tr}_a \bar{R}_{a1}(u)R_{a2}(u)\ldots \bar{R}_{a,2L-1}(u)R_{a,2L}(u). \end{aligned} \tag{6.53}$$

The R-matrix R_{an} acts on the fundamental representation $V_a = \mathbb{C}^4$ of $\mathfrak{su}(4)$ and the quantum space V_n, while $\bar{R}_{an}$ carries the anti-fundamental representation $\bar{\mathbf{4}}$ as the auxiliary space. They are related by crossing symmetry

$$\bar{R}_{an}(u) = R^{T_a}_{an}(2-u) = R^{T_n}_{an}(2-u), \tag{6.54}$$

with T_a representing transposition in space V_a. Then, the parity transformation on $t(u)$ can be related to the other transfer matrix $\bar{t}(u)$ by

$$\begin{aligned}\Pi t(u)\Pi &= \mathrm{Tr}_a \bar{R}_{a,2L}(u) R_{a,2L-1}(u) \dots \bar{R}_{a2}(u) R_{a1}(u) \\ &= \mathrm{Tr}_a \bar{R}_{a1}(2-u) R_{a2}(2-u) \dots \bar{R}_{a,2L-1}(2-u) R_{a,2L}(2-u) \\ &= \bar{t}(2-u).\end{aligned} \tag{6.55}$$

Thus, the integrability condition (6.52) implies

$$t(u) = \bar{t}(2-u). \tag{6.56}$$

Applying this equation to the Bethe state we find that

$$\begin{aligned}\left\langle \mathrm{MPS} \middle| t(u) \middle| \mathbf{u} \right\rangle &= \left\langle \mathrm{MPS} \middle| \bar{t}(2-u) \middle| \mathbf{u} \right\rangle, \\ \tau(u) \left\langle \mathrm{MPS} \middle| \mathbf{u} \right\rangle &= \bar{\tau}(2-u) \left\langle \mathrm{MPS} \middle| \mathbf{u} \right\rangle,\end{aligned} \tag{6.57}$$

where $\tau(u)$ and $\bar{\tau}(u)$ are eigenvalues of the two transfer matrices respectively and they have the same structure. Thus, the non-vanishing overlap $\left\langle \mathrm{MPS} \middle| \mathbf{u} \right\rangle$ yields $\tau(u) = \bar{\tau}(2-u)$. With the explicit form of $\tau(u)$ and $\bar{\tau}(u)$ given in [5, 37], the boundary integrability (6.52) ultimately leads to the following selection rules on sets of Bethe roots:

1. The set of roots $\mathbf{w}$ must have the following paired structure:

$$\mathbf{w} = \begin{cases} (w_1, -w_1, \dots, w_{\frac{K_\mathbf{w}}{2}}, -w_{\frac{K_\mathbf{w}}{2}}), & K_w : \text{even}, \\ (w_1, -w_1, w_2, -w_2, \dots, 0), & K_w : \text{odd}. \end{cases} \tag{6.58}$$

   In other words, $\mathbf{w} = -\mathbf{w}$.

2. The set of sets of roots $\{\mathbf{u}, \mathbf{w}, \mathbf{v}\}$ is invariant under the parity reflection, namely

$$\{\mathbf{u}, \mathbf{w}, \mathbf{v}\} = \{-\mathbf{v}, -\mathbf{w}, -\mathbf{u}\} = \{-\mathbf{v}, \mathbf{w}, -\mathbf{u}\}, \tag{6.59}$$

   here we use the parity-symmetric property of $\mathbf{w}$. This condition implies that $\mathbf{v} = -\mathbf{u}$.

6.5.2 Determinant formula for the overlap

So far, we have determined the selection rules (6.49), (6.58) and (6.59) for the non-vanishing overlap between the Bethe state and the integrable matrix product state. Now let us discuss the overlap formula for the one-point functions (6.46). We will see that the one-point functions can be expressed as a determinant formula, which includes a universal part for overlaps involving integrable boundary states.

The three sets of Bethe roots under these selection rules take the form:

$$\begin{aligned} &\mathbf{u} = (u_1, u_2, \dots, u_L), \\ &\mathbf{v} = -\mathbf{u} = (-u_1, -u_2, \dots, -u_L), \\ &\mathbf{w} = (w_1, -w_1, \dots, w_{\frac{L}{2}}, -w_{\frac{L}{2}}), \end{aligned} \tag{6.60}$$

here we assume L is even without loss of generality. For convenience, we divide $\mathbf{w}$ into $\mathbf{w}^+ = (w_1, w_2, \dots, w_{\frac{L}{2}})$ and $\mathbf{w}^- = (-w_1, \dots, -w_{\frac{L}{2}})$. Then, the $3L$-dimensional Gaudin matrix defined in (5.44), can be written as

$$G = \begin{pmatrix} U_u & U_+ & U_- & U_v \\ W_{+u} & W_{++} & W_{+-} & W_{+v} \\ W_{-u} & W_{-+} & W_{--} & W_{-v} \\ V_u & V_+ & V_- & V_v \end{pmatrix}, \tag{6.61}$$

with sub-blocks given by

$$\begin{aligned} &[U_u]_{ij} \equiv \partial_{u_i}\phi_{u_j}, && [U_\pm]_{ij} \equiv \partial_{u_i}\phi_{w_j^\pm}, && [U_v]_{ij} \equiv \partial_{u_i}\phi_{v_j}, \\ &[W_{\pm u}]_{ij} \equiv \partial_{w_i^\pm}\phi_{u_j}, && [W_{\pm v}]_{ij} \equiv \partial_{w_i^\pm}\phi_{v_j}, && [W_{\pm\pm}]_{ij} \equiv \partial_{w_i^\pm}\phi_{w_j^\pm}, \\ &[V_u]_{ij} \equiv \partial_{v_i}\phi_{u_j}, && [V_\pm]_{ij} \equiv \partial_{v_i}\phi_{w_j^\pm}, && [V_v]_{ij} \equiv \partial_{v_i}\phi_{v_j}. \end{aligned} \tag{6.62}$$

The setup (6.60) implies the following relations among these sub-blocks of the Gaudin matrix:

$$\begin{aligned} &U_u = V_v, && U_v = V_u, && W_{\pm u} = W_{\mp v}, \\ &U_\pm = V_\mp, && W_{+\pm} = W_{-\mp}. \end{aligned} \tag{6.63}$$

The determinant of Gaudin matrix (6.61) is invariant under swapping rows and columns:

$$\det G = \det \begin{pmatrix} U_u & U_+ & U_v & U_- \\ W_{+u} & W_{++} & W_{+v} & W_{+-} \\ V_u & V_+ & V_v & V_- \\ W_{-u} & W_{-+} & W_{-v} & W_{--} \end{pmatrix}. \tag{6.64}$$

With the relations (6.63), the determinant $\det G$ simplifies further

$$\begin{aligned}\det G =& \det \begin{pmatrix} U_u & U_+ & U_v & U_- \\ W_{+u} & W_{++} & W_{+v} & W_{+-} \\ U_v & U_- & U_u & U_+ \\ W_{+v} & W_{+-} & W_{+u} & W_{++} \end{pmatrix} \\ =& \det \begin{pmatrix} U_u + U_v & U_+ + U_- & 0 & 0 \\ W_{+u} + W_{+v} & W_{++} + W_{+-} & 0 & 0 \\ U_v & U_- & U_u - U_v & U_+ - U_- \\ W_{+v} & W_{+-} & W_{+u} - W_{+v} & W_{++} - W_{+-} \end{pmatrix} \\ =& \det G^+ \det G^-, \end{aligned} \tag{6.65}$$

where we introduce G^+ and G^- defined as:

$$G^+ = \begin{pmatrix} U_u + U_v & U_+ + U_- \\ W_{+u} + W_{+v} & W_{++} + W_{+-} \end{pmatrix}, \quad G^- = \begin{pmatrix} U_u - U_v & U_+ - U_- \\ W_{+u} - W_{+v} & W_{++} - W_{+-} \end{pmatrix}. \tag{6.66}$$

We see that, under the selection rules, the determinant of the Gaudin matrix (6.61) factorizes into two determinants of sub-matrices.

Normalized overlaps between integrable boundary states and Bethe states are described by a remarkably compact formula [67]:

$$\frac{|\langle \Psi_0 | \mathbf{u} \rangle|^2}{\langle \mathbf{u} | \mathbf{u} \rangle} = \underbrace{\prod_{j,\nu} \mathcal{F}_\nu(u_j^\nu)}_{\text{boundary dependent}} \times \underbrace{\frac{\det G^+}{\det G^-}}_{\text{universal}}, \tag{6.67}$$

where the universal term is a quotient of Gaudin-like determinants, just like the matrices we defined in (6.66). The universal term is independent of the boundary state $|\Psi_0\rangle$ and depends only on the specific spin chain and the corresponding Bethe roots. Systematic derivations of this formula exist for the XXX and XXZ spin chains [68–70], while overlaps for models with higher-rank symmetry groups or higher spin representations are mainly based on conjecture. For our purpose, as analyzed in [67], the overlap formula involving the valence bond state is explicitly given by

$$\frac{\langle \text{VBS} | \mathbf{u} \rangle}{\langle \mathbf{u} | \mathbf{u} \rangle^{\frac{1}{2}}} = 2^{-L} Q_2(i) \sqrt{\frac{\text{Sdet} G}{Q_2(0) Q_2\left(\frac{i}{2}\right)}}, \tag{6.68}$$

where $\langle \text{VBS}|$ defined in (6.43) is a special case of $\langle \text{MPS}|$ when $q = 2$. And Q_2 is the Baxter Q-function for Bethe roots $\mathbf{w}$. The $\text{Sdet} G$ represents the quotient of $\det G^+$ and $\det G^-$:

$$\text{Sdet}\, G = \frac{\det G^+}{\det G^-}. \tag{6.69}$$

With this overlap formula (6.68), we can evaluate tree-level one-point functions (6.46) for $q = 2$. Therefore, in the domain wall version of ABJM theory, the one-point functions can be calculated totally by integrability methods. Once the set of Bethe roots $\{\mathbf{u}, \mathbf{w}, \mathbf{v}\}$ is solved, we can evaluate Sdet$G$ using (6.66) and then find the explicit expression of the one-point functions of specific composite operators. Moreover, the selection rules for non-trivial one-point functions strictly constrain the degrees of freedom of the input. The only parameter we should consider is the half-length of the operator or spin chain, when calculating one-point functions. Other input parameters such as (K_u, K_w, K_v) are fixed by selection rules.

On the other hand, the overlap formula (6.68) can be verified from the field theory side. The eigenvectors of the dilation operator can be determined either by the Bethe ansatz or by direct diagonalization. By substituting the classical solution (6.21) into specific composite operators, we obtain their tree-level one-point functions, which are expected to match those calculated by the overlap formula (6.68). The validity of the overlap formula indicates that the boundary state is integrable, which suggests the 1/2-BPS domain wall in ABJM theory preserves integrability as well.

7. Conclusion and outlook

In this thesis, we review the integrability techniques, e.g. Bethe ansatz and Rational Q-system for solving the spectrum of integrable models. The coordinate Bethe ansatz provides an intuitive physical picture of Heisenberg spin chain, namely magnons propagating on top of the vacuum and interact with each other in an integrable way, while the algebraic Bethe ansatz highlights the integrability through the algebraic nature of the method. Both of them transfer the spectral problem into solving sets of algebraic Bethe equations. For finding solutions to Bethe equations analytically and efficiently, we introduce the Rational Q-system which automatically eliminates all nonphysical solutions. These integrability techniques play an important role in the later discussion.

Next, the setup of ABJM theory is introduced and the spectrum of anomalous dimensions of scalar composite operators is discussed. With the underlying integrable $\mathfrak{su}(4)$ alternating spin chain, the spectral problem in the scalar sector of ABJM theory can be solved using integrability techniques.

The focus of this thesis is on the one-point functions in the defect version of ABJM theory, featuring a 1/2-BPS domain wall. We demonstrate that the 1/2-BPS domain wall preserves half the supersymmetry of ABJM theory. In spin chain language, the domain wall can be mapped to a Matrix Product State which is a fixed state in the Hilbert space. The one-point functions of scalar composite operators can be expressed as an overlap between the Bethe state and the matix product state. The boundary integrability implies the selection rules for non-vanishing one-point functions, and the overlap simplifies to a compact determinant formula that includes a universal term and a boundary-dependent term. The overlap formula for the matrix product state with a bond dimension of 1 is explicitly demonstrated, and one-point functions can be calculated using this formula together with Bethe ansatz.

It would be beneficial to check the integrability on the string theory side, as in the thesis there is only a brief introduction to the string theory description of ABJM theory and the domain wall version of ABJM theory. Furthermore, it is possible to evaluate one-point functions in the dual string theory by a variant of the GKPW prescription [71]. This allows us to compare the one-point functions on both sides and assess the credibility of the AdS/CFT correspondence.

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