

# Geometrization of Newtonian Dynamics

Yaakov Friedman and Tzvi Scarr

Jerusalem College of Technology  
Departments of Mathematics and Physics  
P.O.B. 16031 Jerusalem 91160, Israel

E-mail: [friedman@jct.ac.il](mailto:friedman@jct.ac.il), [scarr@g.jct.ac.il](mailto:scarr@g.jct.ac.il)

**Abstract.** Riemann's principle "force equals geometry" provided the basis for Einstein's General Relativity - the geometric theory of gravitation. In this paper, we follow this principle to derive the dynamics for any conservative force. We introduce the *relativity of spacetime*: an object lives in its own spacetime, whose geometry is determined by all of the forces affecting it. We also introduce the *Generalized Principle of Inertia* which unifies Newton's first and second laws and states that: *An inanimate object moves freely, that is, with zero acceleration, in its own spacetime.* We derive the metric of an object's spacetime in two ways. The first way uses conservation of energy to derive a Newtonian metric. We reveal a physical deficiency of this metric (responsible for the inability of Newtonian dynamics to account for relativistic behavior), and remove it. The dynamics defined by the corrected Newtonian metric leads to a new *Relativistic Newtonian Dynamics (RND)* for both massive objects and massless particles moving in any static, conservative force field, not necessarily gravitational. In the case of the gravitational field of a static, spherically symmetric mass distribution, this metric turns out to be the Schwarzschild metric. This dynamics reduces in the weak field, low velocity limit to classical Newtonian dynamics and also exactly reproduces the classical tests of General Relativity. In the second way, we obtain the *RND* metric directly, without first obtaining a Newtonian metric. Instead of conservation of energy, we use conservation of angular momentum, a carefully defined Newtonian limit and Tangherlini's condition. The non-static case is handled by applying Lorentz covariance to the static case.

## 1. Introduction

Bernhard Riemann, although best known as a mathematician, became interested in physics in his early twenties. His lifelong dream was to develop the mathematics to unify the laws of electricity, magnetism, light and gravitation. At an 1894 conference in Vienna, the mathematician Felix Klein said:

"I must mention, first of all, that Riemann devoted much time and thought to physical considerations. Grown up under the tradition which is represented by the combinations of the names of Gauss and Wilhelm Weber, influenced on the other hand by Herbart's philosophy, he endeavored again and again to find a general mathematical formulation for the laws underlying all natural phenomena .... The point to which I wish to call your attention is that these physical views are the mainspring of Riemann's purely mathematical investigations [1]."

Riemann's approach to physics was geometric. As pointed out in [2], "one of the main features of the local geometry conceived by Riemann is that it is well suited to the study of gravity and



more general fields in physics.” He believed that the forces at play in a system determine the geometry of the system. For Riemann, *force equals geometry*.

The application of Riemann’s mathematics to physics would have to wait for two more essential ideas. While Riemann considered how forces affect *space*, physics must be carried out in *spacetime*. One must consider trajectories in spacetime, not in space. For example, in flat spacetime, an object moves with constant velocity if and only if his trajectory in *spacetime* is a straight line. On the other hand, knowing that an object moves along a straight line in *space* tells one nothing about whether the object is accelerating. As Minkowski said, “Henceforth, space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality [3].” This led to the second idea. Riemann worked only with positive definite metrics, whereas Minkowski’s metric on spacetime is not positive definite. The relaxing of the requirement of positive-definiteness to non-degeneracy led to the development of *pseudo-Riemannian geometry*.

Fifty years after Riemann’s death, Einstein used pseudo-Riemannian geometry as the cornerstone of General Relativity (*GR*). Acknowledging his reliance on Riemann, Einstein said:

“But the physicists were still far removed from such a way of thinking; space was still, for them, a rigid, homogeneous something, incapable of changing or assuming various states. Only the genius of Riemann, solitary and uncomprehended, had already won its way to a new conception of space, in which space was deprived of its rigidity, and the possibility of its partaking in physical events was recognized. This intellectual achievement commands our admiration all the more for having preceded Faraday’s and Maxwell’s field theory of electricity [4].”

*GR* is a direct application of “force equals geometry.” In *GR*, the gravitational force curves spacetime. Since, by the Equivalence Principle, the acceleration of an object in a gravitational field is independent of its mass, curved spacetime can be considered a *stage* on which objects move. In other words, the geometry is the same for all objects. However, the Equivalence Principle holds only for gravitation. In this way, *GR* singles out the gravitational force from other forces which are not treated geometrically. For example, the potential of an electric force depends on the charge of the particle, and the particle’s acceleration depends on its charge-to-mass ratio. Thus, the electric field does not create a common stage on which all particles move. Indeed, a neutral particle does not feel any electric force at all. The way spacetime curves due to an electric potential depends on *both* the potential *and* intrinsic properties of the object. This was also recognized in the geometric approach of [5]. How, then, are we to apply Riemann’s principle of “force equals geometry” to other forces?

We answer this question here for any conservative force. One of the main new ideas is the *relativity of spacetime*. This means that spacetime is an object-dependent notion. An object lives in its own spacetime, its own geometric world, which is defined by the forces which affect it. For example, in the vicinity of an electric field, a charged particle and a neutral particle exist in different worlds, in different spacetimes. In fact, for the neutral particle, the electric field does not exist. Likewise, in the vicinity of a magnet, a piece of iron and a piece of plastic live in two different worlds.

The second new idea is the *Generalized Principle of Inertia*. It is based on the fact that an inanimate object has no internal mechanism with which to change its velocity. Hence, it has constant velocity, or zero acceleration, in *its own* world (spacetime). The *Generalized Principle of Inertia* unifies Newton’s first and second laws and states that: **An inanimate object moves freely, that is, with zero acceleration, in its own spacetime, whose geometry is determined by all of the forces affecting it.** This is a generalization, or more accurately, a relativization of Einstein’s idea.

In *GR*, a freely-falling object in a gravitational field moves along a geodesic determined by the metric of the spacetime. With respect to this metric, the object’s acceleration is zero. The

Generalized Principle of Inertia extends this idea so that *every* object moves along a geodesic in *its* spacetime. This geodesic is with respect to an appropriate metric, which we call the *metric of the object's spacetime*. The metric in *GR* is determined solely by the gravitational sources, while in our model, the metric of the object's spacetime depends on all of the forces affecting the object. In the case of static, conservative forces, the metric will depend only on the potentials of these forces, defined in some preferred inertial frame, which may be the rest frame of the Universe.

Since the motion is by geodesics, we use a variational principle and derive Euler-Lagrange type equations and the ensuing conservation laws.

“Many results in both classical and quantum physics can be expressed as variational principles, and it is often when expressed in this form that their physical meaning is most clearly understood. Moreover, once a physical phenomenon has been written as a variational principle, ... it is usually possible to identify conserved quantities, or symmetries of the system of interest, that otherwise might be found only with considerable effort [6].”

In the classification [7] of alternative gravitation theories, our theory is a preferred frame, Lagrangian-based metric theory.

We call our theory *Relativistic Newtonian Dynamics (RND)*. *RND* has the following features:

- (i) It is based on the classical, unmodified Newtonian potential.
- (ii) It avoids the complicated field equations of *GR*.
- (iii) It reveals the physical mechanism responsible for relativistic phenomena.
- (iv) For the gravitational field of a non-rotating, spherically symmetric mass distribution, the *RND* metric is the Schwarzschild metric
- (v) *RND* exactly reproduces the classical tests of *GR*.
- (vi) It is also valid for non-spherically symmetric fields.
- (vii) It does not rely on the Equivalence Principle, and so is applicable to *any combination* of conservative force fields whose potentials vanish at infinity.

We note that the *RND* metric was also derived in [8] using the escape trajectory and the influence of the potential on spacetime.

In the literature, there are other alternative approaches to reproducing the relativistic gravitational features of *GR*. One approach uses modified Newtonian-like potentials. This so-called “pseudo-Newtonian” approach, introduced in [9], is much simpler mathematically than *GR*, with no need for covariant differentiation and complicated tensorial equations. Numerous authors ([10, 11, 12, 13, 14, 15, 16]) have proposed various modified Newtonian-like potentials. However, none of these potentials are able to reproduce the tests of *GR*, even in the weak field regime. Moreover, as stated in [17], most of these modified potentials “are arbitrarily proposed in an ad hoc way” and, more fundamentally, are “not a physical analogue of local gravity and are not based on any robust physical theory and do not satisfy Poisson’s equation.”

More recently, the above shortcomings were addressed in [18]. Using a metric approach and hypothesizing a generic relativistic gravitational action and a corresponding Lagrangian, the authors derive a velocity-dependent relativistic potential which generalizes the classical Newtonian potential. For a static, spherically symmetric geometry, this potential exactly reproduces relativistic gravitational features, including the tests of *GR*. Even more recently, one finds a fundamental grounding to these velocity-dependent pseudo-Newtonian potentials in [19]. The authors generalize the pseudo-Newtonian approach to any stationary spacetime. They also include additional forces, such as the electromagnetic force.

## 2. A geometric approach to dynamics

We present the preferred frame, Lagrangian approach to geometrize the motion of an object of mass  $m$  in a force field. The results here are quite general and may be applied to any force field. Later, we will consider the particular case of a static, spherically symmetric, attractive force field.

Let

$$ds^2 = g_{\alpha\beta}(q)dq^\alpha dq^\beta \quad (1)$$

be the metric of the object's spacetime, where  $q^\alpha$ ,  $\alpha = 0, 1, 2, 3$  are the coordinates of an inertial observer far removed from the sources of the field. This is similar to Earth-based observations of motion within a distant galaxy and to observations within the solar system measured with respect to the far-removed stars.

In order to define the length of the trajectory, we introduce the following function  $L$  of the eight independent variables  $q^\alpha, \dot{q}^\alpha$ :

$$L(q, \dot{q}) = mc\sqrt{g_{\alpha\beta}(q)\dot{q}^\alpha\dot{q}^\beta}, \quad (2)$$

where the constant  $mc$  has been chosen to be compatible with the classical notion of momentum.

Let  $q : \sigma \rightarrow x, a \leq \sigma \leq b$  be a trajectory of an object of mass  $m$ , where  $\sigma$  is an arbitrary parameter. The length  $l(q)$  of the trajectory  $q$  is given by

$$l(q) = mc \int_a^b \frac{ds}{d\sigma} d\sigma = \int_a^b L\left(q, \frac{dq}{d\sigma}\right) d\sigma. \quad (3)$$

It is well known that the length of the trajectory does not depend on the parametrization.

From the Generalized Principle of Inertia, the length of the trajectory  $q(\sigma)$  is extremal. By a standard argument, it follows that  $q(\sigma)$  satisfies the Euler-Lagrange equations

$$\frac{\partial L}{\partial q^\mu} - \frac{d}{d\sigma} \frac{\partial L}{\partial \dot{q}^\mu} = 0, \quad (4)$$

where  $L$  is defined by (2) and  $\dot{q} = \frac{dq}{d\sigma}$ . In this case, the conjugate momentum  $p_\mu$  is

$$p_\mu = \left. \frac{\partial L}{\partial \dot{q}^\mu} \right|_{\dot{q} = \frac{dq}{d\sigma}} = \frac{mcg_{\mu\beta} \frac{dq^\beta}{d\sigma}}{ds/d\sigma} = mcg_{\mu\beta} \frac{dq^\beta}{ds}. \quad (5)$$

Note that the second term in equation (4) contains differentiation by two parameters on the curve. The first differentiation is by  $s$ , as seen in equation (5). The second differentiation is by  $\sigma$ . In order to obtain a differential equation with a single parameter, we will choose  $\sigma$  to be proportional to  $s$ . More precisely, we choose  $\sigma$  to be the parameter

$$\tau = c^{-1}s, \quad (6)$$

called *proper time*, which is proportional to  $s$  and reduces to the coordinate time  $t$  in the classical limit.

Using  $\tau$  will turn equation (4) into a second-order differential equation. We denote differentiation of  $q$  with respect to  $\tau$  by  $\dot{q}$ . From (4) and (5), the Euler-Lagrange equations become

$$\frac{\partial L}{\partial q^\mu} = \frac{dp_\mu}{d\tau}, \quad (7)$$

where

$$p_\mu = mg_{\mu\beta}\dot{q}^\beta. \quad (8)$$

Note that momentum is a covector, while velocity is a vector. Therefore, the metric is necessary in (8) in order to lower the index. It can be shown that (7) is equivalent to the geodesic equation.

The following proposition follows immediately from equation (7).

**Proposition 1** If the metric coefficients  $g_{\alpha\beta}$  do not depend on the coordinate  $q^\mu$ , then the  $\mu$  component  $p_\mu = mg_{\mu\beta}\dot{q}^\beta$  of the conjugate momentum is conserved on the trajectory.

### 3. Method I: Relativistic Metric from Newtonian metric

#### 3.1. Geometric Formulation of Newtonian Dynamics

We begin our derivation by applying the geometric approach to classical Newtonian dynamics. Based on the discussion in the previous section, we replace the coordinate time  $t$  (the classical evolution parameter) by the proper time  $\tau$ . In this modification, Newton's second law for a force with potential  $U$  becomes

$$m \frac{d^2 \mathbf{x}}{d\tau^2} = -\nabla U. \quad (9)$$

Taking the Euclidean dot product of both sides of (9) with  $\dot{\mathbf{x}}$  gives

$$m \ddot{\mathbf{x}} \cdot \dot{\mathbf{x}} = -\nabla U \cdot \dot{\mathbf{x}},$$

which, upon integration, yields

$$\frac{m \dot{\mathbf{x}}^2}{2} + U(\mathbf{x}) = E, \quad (10)$$

where the integral of motion  $E$  is the total energy on the object's worldline. The only difference between equation (10) and the classical energy conservation equation is the kinetic energy term, in which  $d\mathbf{x}/dt$  has been replaced by  $\dot{\mathbf{x}}$ . In the classical limit, we have  $dt = d\tau$ , and then equation (10) reduces to the classical energy conservation equation.

Assume now that the potential  $U(\mathbf{x}) \leq 0$  and vanishes at infinity. Introduce the *dimensionless potential*

$$u(\mathbf{x}) = \frac{-2U(\mathbf{x})}{mc^2}, \quad (11)$$

where  $c$  denotes the speed of light. With this definition, equation (10) yields the dimensionless energy conservation equation

$$\frac{\dot{\mathbf{x}}^2}{c^2} - u = \mathcal{E}, \quad (12)$$

where  $\mathcal{E}$  denotes the dimensionless total energy on the worldline. The total energy is a sum of kinetic energy, depending on the magnitude of the velocity, and potential energy, depending on position.

We turn now to the construction of the metric of the object's spacetime, for motion satisfying (9), where  $U$  is the potential of a static force. In our inertial lab frame, the metric is of the form

$$ds^2 = f(\mathbf{x})c^2 dt^2 - g(\mathbf{x})d\mathbf{x}^2, \quad (13)$$

where  $f(\mathbf{x})$  and  $g(\mathbf{x})$  depend solely on  $\mathbf{x}$ . Note that there are no anisotropic terms in the metric because in Newtonian dynamics, space is isotropic. Moreover, assuming Einstein synchrony, a straightforward argument shows that there are no time-space cross terms (see [20], page 187).

For  $u(\mathbf{x}) \ll 1$ , the worldlines are approximately straight, implying that this metric is asymptotically Minkowski. Hence,  $f(\mathbf{x}) \rightarrow 1$  and  $g(\mathbf{x}) \rightarrow 1$  as  $u(\mathbf{x}) \rightarrow 0$ .

Since the metric is static, Proposition 1 implies that the zero component of the conjugate momentum is conserved. Thus,

$$f(\mathbf{x})\dot{t} = k \quad , \quad \dot{t} = \frac{k}{f(\mathbf{x})}, \quad (14)$$

for some constant  $k$  related to the total energy on the worldline. The square of the norm with respect to (13) of the four-velocity  $(\dot{t}, \dot{\mathbf{x}})$  is

$$f(\mathbf{x})c^2\dot{t}^2 - g(\mathbf{x})\dot{\mathbf{x}}^2 = c^2,$$

which, by the use of (14), leads to

$$\frac{k^2}{f(\mathbf{x})} - g(\mathbf{x})\frac{\dot{\mathbf{x}}^2}{c^2} = 1. \quad (15)$$

This can be considered a conservation equation on the worldline.

We can now determine the metric coefficients  $f(\mathbf{x})$  and  $g(\mathbf{x})$  by comparing this conservation to the energy conservation. Adding 1 to both sides of equation (12) and dividing by  $-(1-u)$ , we obtain

$$\frac{1}{1-u} \left( \mathcal{E} + 1 - \frac{\dot{\mathbf{x}}^2}{c^2} \right) = 1. \quad (16)$$

Comparing (15) and (16), and using  $f(\mathbf{x}) \rightarrow 1$  as  $u \rightarrow 0$ , one obtains

$$g(\mathbf{x}) = \frac{1}{1-u} \quad , \quad f(\mathbf{x}) = 1-u \quad , \quad k = \sqrt{\mathcal{E} + 1}. \quad (17)$$

From (13) and (17), we obtain the *Newtonian metric*

$$ds^2 = (1-u(\mathbf{x}))c^2 dt^2 - \frac{1}{1-u(\mathbf{x})} d\mathbf{x}^2. \quad (18)$$

Reversing our argument shows that a trajectory which is minimal with respect to this metric satisfies Newton's second law (9).

In order to complete the spacetime description of the worldline, from (14) and (17) we obtain

$$\dot{t} = \frac{k}{1-u(\mathbf{x})} = \frac{\sqrt{\mathcal{E} + 1}}{1-u(\mathbf{x})}. \quad (19)$$

### 3.2. Preferability of our Newtonian metric

In the *GR* literature, we find several metrics, like (18), whose first-order approximation leads to Newtonian dynamics. For example, in terms of  $u$ , the metric in [20] is

$$ds^2 = (1-u(\mathbf{x}))c^2 dt^2 - d\mathbf{x}^2, \quad (20)$$

and in [21] and [22], the metric is

$$ds^2 = (1-u(\mathbf{x}))c^2 dt^2 - (1+u(\mathbf{x}))d\mathbf{x}^2. \quad (21)$$

Lagrangians  $L_1$  and  $L_2$  are said to be *equivalent* ( $L_1 \sim L_2$ ) if they lead to the same equation of motion. It is obvious that if  $L_2 = a + bL_1$ , for constants  $a$  and  $b$ , then  $L_1 \sim L_2$ . For the three metrics (18), (20) and (21), we have

$$L = \frac{ds}{dt} = c\sqrt{1-u-f(u)\beta^2},$$

where  $\beta^2 = v^2/c^2$  and  $f(u) = \frac{1}{1-u}$ ,  $1$  and  $1+u$ , respectively. In the low velocity ( $\beta^2 \ll 1$ ) and weak field ( $u \ll 1$ ) limit, the first-order approximation in  $u$  and  $\beta^2$  is

$$L = c \left( 1 - \frac{1}{2}u - \frac{1}{2}\beta^2 \right).$$

Dividing by  $c$ , subtracting 1, and then multiplying by  $-mc^2$ , yields the classical Lagrangian

$$L = \frac{1}{2}mv^2 - U.$$

Hence, in the first-order approximation, all three metrics lead to classical Newtonian dynamics. From this point of view, there is no reason to prefer any one of the metrics over the other two.

We thus compare the equations of motion for these metrics. The equation of motion for our metric (18) is equation (9), while for the metrics (20) and (21), the equations of motion are

$$m\ddot{\mathbf{x}} = -\nabla U \frac{1 + \frac{\dot{\mathbf{x}}^2}{c^2}}{1 + \frac{2U}{mc^2}},$$

and

$$m\ddot{\mathbf{x}} = -\nabla U \frac{1 - \frac{4U\dot{\mathbf{x}}^2}{mc^4}}{1 - \frac{4U^2}{m^2c^4}},$$

respectively. The relative simplicity of equation (9) makes our metric (18) preferable over the others.

### 3.3. The deficiency of the Newtonian metric and the corrected metric

The huge success of Newtonian dynamics implies that the Newtonian metric (18) is close to the one that governs the laws of Nature. Nevertheless, the observed astrophysical deviations from the predictions of this dynamics indicate that this metric has a deficiency and needs to be corrected.

It is natural to assume that time intervals are influenced by the potential and should be altered by a factor defined by this potential when translated to the inertial frame. This influence is handled by the coefficient  $1-u$  of  $c^2 dt^2$  in (18) and accurately predicts the known gravitational time dilation in a spherically symmetric gravitational field. It is also natural to assume that the space increments in the direction of the gradient  $\nabla U$  are influenced by the potential and should be altered by a factor defined by this potential when translated to the inertial frame. This influence is also present in (18).

However, the metric (18) is deficient in that it is isotropic - it alters the spatial increments equally in *all* spatial directions. The potential, on the other hand, influences only the direction of the gradient  $\nabla U$  and has no influence on the spatial increments in the directions transverse to the gradient. To remove this problem, we alter the metric so that the potential affects *only* the direction of the force, in the same way as is in (18), and leaves the transverse directions unaffected.

More precisely, introduce at each  $\mathbf{x}$  where  $\nabla U(\mathbf{x}) \neq \mathbf{0}$  a normalized vector

$$\mathbf{n}(\mathbf{x}) = \frac{\nabla U(\mathbf{x})}{|\nabla U(\mathbf{x})|} \quad (22)$$

in the direction of the gradient of  $U(\mathbf{x})$ , or the negative of the direction of the force. Let  $d\mathbf{x}_n = (d\mathbf{x} \cdot \mathbf{n})\mathbf{n}$  and  $d\mathbf{x}_{tr} = d\mathbf{x} - (d\mathbf{x} \cdot \mathbf{n})\mathbf{n}$ , respectively, denote the projections of the spatial increment  $d\mathbf{x}$  in the parallel and transverse directions to  $\mathbf{n}(\mathbf{x})$ . With this notation, the *Relativistic Newtonian Dynamics (RND) metric* is

$$ds^2 = (1 - u(\mathbf{x}))c^2 dt^2 - \frac{1}{1 - u(\mathbf{x})} d\mathbf{x}_n^2 - d\mathbf{x}_{tr}^2. \quad (23)$$

At the points  $\mathbf{x}_0 = x_0^j$  where  $\nabla U(\mathbf{x}_0) = \mathbf{0}$ , the normalized vector  $\mathbf{n}$  is not defined. We claim that on any smooth trajectory  $q(\sigma)$  in spacetime with  $q(\sigma_0) = (x_0^0, \mathbf{x}_0) = x_0$ , the metric (23) can

be extended continuously to the point  $x_0$ . The Taylor expansion of the potential  $U(\mathbf{x})$  at  $\mathbf{x}_0$  to second order is  $U(\mathbf{x}) \approx U(\mathbf{x}_0) + \frac{1}{2}U_{,jk}(\mathbf{x}_0)(x^j - x_0^j)(x^k - x_0^k)$ . The limit of  $\mathbf{n}(\sigma)$  as  $\Delta\sigma = \sigma - \sigma_0 \rightarrow 0$  on the trajectory  $q(\sigma)$  can be calculated by the limit along its tangent line  $x_0^\alpha + a^\alpha \Delta\sigma$ , where  $a$  is the tangent vector to the trajectory, and use of the second-order approximation of  $U(\mathbf{x})$  to calculate  $\nabla U(\mathbf{x})$ . This yields  $\nabla U(\mathbf{x}(\Delta\sigma))_k = \frac{1}{2}U_{,jk}(\mathbf{x}_0)(a^j \Delta\sigma) = \mathbf{b}\Delta\sigma$ . Hence,

$$\lim_{\Delta\sigma \rightarrow 0^+} \mathbf{n}(\Delta\sigma) = - \lim_{\Delta\sigma \rightarrow 0^-} \mathbf{n}(\Delta\sigma).$$

Since the metric is not affected by the sign of  $\mathbf{n}$ , this proves our claim. Moreover, since, in general, the measure of such points  $\mathbf{x}_0$  is zero, the length of the trajectory is not affected by the metric at these points.

In the case of the gravitational field of a non-rotating, spherically symmetric body of mass  $M$ , in spherical coordinates with origin at its center, the potential is  $U(r) = -GmM/r$ , and the dimensionless potential (11) is  $u(r) = 2GM/c^2 r = r_s/r$ , where  $r_s = 2GM/c^2$  is the Schwarzschild radius. In this case, the metric (23) is

$$ds^2 = \left(1 - \frac{r_s}{r}\right) c^2 dt^2 - \frac{1}{1 - r_s/r} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2, \quad (24)$$

which is the well-known Schwarzschild metric ([23]). It is shown in section 5 that *RND* passes the classical tests of *GR*.

#### 3.4. The Equations of Relativistic Newtonian Dynamics

We now obtain the dimensionless and dimensional *energy conservation equations* and *equations of motion* of *RND*. The derivation is similar to the reversal of the derivation in Section 3.1 for the metric (18). Since the metric (23) is static, Proposition 1 implies here, as in (14), that

$$\dot{t} = \frac{k}{1 - u(\mathbf{x})}. \quad (25)$$

The square of the norm with respect to (23) of the four-velocity  $\dot{x}$  is

$$c^2 \frac{k^2}{1 - u} - \frac{1}{1 - u} \dot{\mathbf{x}}_n^2 - \dot{\mathbf{x}}_{tr}^2 = c^2. \quad (26)$$

Multiplying by  $\frac{1-u}{c^2}$ , using  $\dot{\mathbf{x}}^2 = \dot{\mathbf{x}}_n^2 + \dot{\mathbf{x}}_{tr}^2$  and rearranging terms, we obtain the *dimensionless energy conservation equation*

$$\frac{\dot{\mathbf{x}}^2}{c^2} - u - u \frac{\dot{\mathbf{x}}_{tr}^2}{c^2} = k^2 - 1. \quad (27)$$

The corresponding *dimensional energy conservation equation* is

$$\frac{m\dot{\mathbf{x}}^2}{2} + U(\mathbf{x}) + U(\mathbf{x}) \frac{\dot{\mathbf{x}}_{tr}^2}{c^2} = E, \quad (28)$$

where the integral of motion  $E$  is the total energy on the worldline. As in the energy conservation equation (10) of modified Newtonian dynamics, equation (28) has a kinetic energy term and a potential energy term. But in (28), there is also a *mixed term* which depends on both the velocity of the object and the potential. This means that in order to reproduce relativistic effects, one can no longer distinguish between potential and kinetic energy, as in Newtonian dynamics. This also explains the need to include the velocity in the modified Newtonian potentials proposed in

[17, 18, 24, 25, 26]. The mixed term in (28) is approximately  $\beta^2 U(\mathbf{x})$  and is therefore only seen for high velocities or in high-precision experiments. Note that we can also write (28) as

$$\frac{m}{2} (\dot{\mathbf{x}}_n^2 + (1-u)\dot{\mathbf{x}}_{tr}^2) + U(\mathbf{x}) = E. \quad (29)$$

This is the usual “kinetic plus potential” energy from of energy conservation, only now the square of the velocity is computed with respect to the metric (23). Notice also that the force in *RND* is no longer conservative.

Let  $\phi = U/m$  denote the potential per unit mass. Differentiating equation (28) with respect to  $\tau$ , one obtains the *equation of motion* of *RND*

$$\ddot{\mathbf{x}} = -\nabla\phi - \nabla\phi \frac{\dot{\mathbf{x}}_{tr}^2}{c^2} + 2\frac{\phi(\mathbf{x})}{c^2}(\dot{\mathbf{x}} \cdot \dot{\mathbf{n}})\mathbf{n}, \quad (30)$$

which has now two additional terms not appearing in the corresponding classical equation (9). In the classical regime, both of these terms are small and have therefore gone unrecognized.

For potentials such as the gravitational potential, for which  $\phi$  is independent of  $m$ , equations (27) and (30) can be extended to massless particles as well by using the symbol  $\varepsilon$ , which equals 1 for *objects* with non-zero mass and 0 for massless *particles*. However, for massless particles, the proper time  $\tau$  is not defined. Instead, we will use an affine parameter (see, for example, [23]). There is no need here to specify this parameter, because to test our theory, we obtain parameter-free equations.

For massless particles, the norm of the four-velocity  $\dot{x} = (\dot{t}, \dot{\mathbf{x}})$  is 0. Replacing  $c^2$  by 0 on the right-hand side of equation (26), we obtain

$$\frac{\dot{\mathbf{x}}_n^2}{c^2} + (1-u) \left( \frac{\dot{\mathbf{x}}_{tr}^2}{c^2} + \varepsilon \right) = k^2. \quad (31)$$

Differentiating with respect to  $\tau$  yields, in turn,

$$\ddot{\mathbf{x}} = -\varepsilon\nabla\phi - \nabla\phi \frac{\dot{\mathbf{x}}_{tr}^2}{c^2} + 2\frac{\phi(\mathbf{x})}{c^2}(\dot{\mathbf{x}} \cdot \dot{\mathbf{n}})\mathbf{n}, \quad (32)$$

which is valid everywhere except on the Schwarzschild horizon. Equations (31) and (32) are, respectively, the *dimensionless energy conservation equation* and the *dimensionless equation of motion for objects/particles* in *RND*. Equations (25) and (32) provide a complete description of a worldline in *RND*.

Note that even though the classical force is zero for a massless particle, the second and third terms in the equation of motion (32) nevertheless remain and account for phenomena such as gravitational lensing and the Shapiro time delay. In the classical regime, both of these terms are small and have therefore gone unrecognized. Details of the derivation of (30) can be found in [27] and [28].

It is clear from these equations that *RND* reduces in the low velocity, weak field limit to classical Newtonian dynamics.

#### 4. Method II: Relativistic metric from Tangherlini's condition

In this section, we derive the *RND* metric directly, that is, without first obtaining a Newtonian metric. The derivation here is based on a carefully defined Newtonian limit and Tangherlini's condition, which states that the metric coefficients must satisfy  $g_{tt}g_{rr} = -1$  [29].

#### 4.1. The metric of an object moving in a static, spherically symmetric, attractive central force field

Here, we derive the metric of the spacetime of an object of mass  $m$  moving in a static, spherically symmetric, attractive central force field. Let  $K_\infty$  be an inertial frame whose origin is at infinity. Let  $K$  be the same frame but with the origin shifted to the center of symmetry of the field. We use standard spherical coordinates  $ct, r, \theta, \varphi$  in  $K$ . Let  $U(r)$  denote the potential of this field. We assume that  $U \leq 0$  and vanishes at infinity.

#### 4.2. The implications of spherical symmetry

We turn now to the construction of the metric of the object's spacetime. From spherical symmetry, the metric in  $K$  is of the form

$$ds^2 = f(r)c^2 dt^2 - g(r)dr^2 - h(r)r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (33)$$

Since the force is static, the metric coefficients do not depend on  $t$ . Moreover, there are no time-space cross terms (see [20], page 187). By spherical symmetry, the functions  $f, g$  and  $h$  cannot depend on  $\varphi$ . For  $u(r) \ll 1$ , the worldlines are approximately straight, implying that this metric is asymptotically the Minkowski metric

$$\eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1). \quad (34)$$

Hence,

$$\lim_{r \rightarrow \infty} f(r) = \lim_{r \rightarrow \infty} g(r) = \lim_{r \rightarrow \infty} h(r) = 1. \quad (35)$$

Without loss of generality, we may assume that the motion is in the plane  $\theta = \pi/2$ . Since the metric coefficients do not depend on  $\varphi$ , Proposition 1 implies that the  $\varphi$  component of the conjugate momentum is conserved on the trajectory of the object. Thus,

$$h(r)r^2\dot{\varphi} = J, \quad (36)$$

for some constant  $J$ . Since the force is central, the acceleration  $\ddot{\mathbf{r}}$  is parallel to  $\mathbf{r}$  implying that the angular momentum per unit mass  $\mathbf{r} \times \dot{\mathbf{r}} = r^2\dot{\varphi}$  is also conserved on the trajectory. Thus  $h(r)$  remain constant for any values  $r$  on the trajectory of the object. Since around any value  $r > 0$  there can be a trajectory of the object, this imply that  $h(r)$  is constant for all  $r$  and from (35)

$$h(r) = 1. \quad (37)$$

#### 4.3. Newtonian approximation for radial motion

Since angular momentum conservation does not provide any information about radial motion ( $\varphi = \text{constant}$ ), we now consider radial motion. The function  $L(x, \dot{x})$  defined by (2) in this case becomes

$$L(x, \dot{x}) = mc\sqrt{f(r)c^2\dot{t}^2 - g(r)\dot{r}^2}. \quad (38)$$

From (8), the  $r$ -momentum is  $p_r = -mg(r)\dot{r}$ , and the  $r$  component of the Euler-Lagrange equation (7) is

$$\frac{m}{2} (f'(r)c^2\dot{t}^2 - g'(r)\dot{r}^2) + mg'(r)\dot{r}^2 + mg\ddot{r} = 0 \quad (39)$$

Dividing (33) by  $d\tau^2$ , we obtain the norm equation

$$f(r)c^2\dot{t}^2 - g(r)\dot{r}^2 = c^2, \quad (40)$$

implying that  $c^2 \dot{t}^2 = \frac{c^2}{f(r)} + \frac{g(r)}{f(r)} \dot{r}^2$ . Substituting this into (39), we obtain

$$\ddot{r} = -\frac{1}{2} \left( \left( \frac{f'(r)}{f(r)} + \frac{g'(r)}{g(r)} \right) \dot{r}^2 + \frac{c^2 f'(r)}{f(r)g(r)} \right). \quad (41)$$

In Newtonian dynamics, the acceleration of an object depends only on the forces acting on it and on the object's mass, but is independent of its velocity. As explained in the previous section, in the geometric approach to dynamics, the acceleration should be  $\ddot{r} = \frac{d^2 r}{d\tau^2}$ . Can we define the metric so that this acceleration for radial motion will be independent on the velocity of the object? From equation (41), it follows that this can be achieved if we set

$$\frac{f'(r)}{f(r)} + \frac{g'(r)}{g(r)} = 0, \quad (42)$$

which implies that  $\ln(f(r)g(r)) = \text{const}$  and  $f(r)g(r) = \text{const}$ . Using (35), this implies that

$$f(r)g(r) = 1. \quad (43)$$

The condition (43) was also used in [29]. For some interesting conditions equivalent to (43), see [30] and [31]. Under this condition, (41) becomes

$$\ddot{r} = -\frac{c^2 f'(r)}{2}. \quad (44)$$

The function  $f(r)$  will be defined from the classical limit. Let  $r_0$  be an arbitrary value of  $r$ . Consider the radial motion of an object whose velocity at  $r_0$  is  $v(r_0) = 0$ . We now connect this object's acceleration  $\ddot{r}$  with the classical acceleration  $\frac{d^2 r}{dt^2}$  at  $r_0$ . Since near the point  $r_0$ , we have  $d\tau = \sqrt{f(r)} dt$ , the acceleration  $\ddot{r}$  at  $r_0$  is

$$\ddot{r} = \frac{d^2 r}{d\tau^2} = \frac{d}{d\tau} \left( \frac{1}{\sqrt{f}} \frac{dr}{dt} \right) = \left( -\frac{1}{2} f^{-3/2} f' \left( \frac{dr}{dt} \right)^2 + \frac{1}{\sqrt{f}} \frac{d^2 r}{dt^2} \right) f^{-1/2}.$$

Since  $v(r_0) = 0$ , we have

$$\ddot{r}(r_0) = \frac{1}{f(r_0)} \frac{d^2 r}{dt^2} \Big|_{r_0}. \quad (45)$$

The Newtonian radial acceleration in tensorial form is  $\frac{d^2 r}{dt^2} = m^{-1} \eta^{1\beta} U_{,\beta}$  (see [32]). Since the object's spacetime is not flat, this formula should be replaced by  $\frac{d^2 r}{dt^2} = m^{-1} g^{1\beta} U_{,\beta}$ . For radial motion with metric (33) satisfying (43), this acceleration is  $\frac{d^2 r}{dt^2} = -\frac{1}{mg(r)} \frac{dU}{dr} = -\frac{f(r)}{m} \frac{dU}{dr}$ . Hence, the acceleration  $\ddot{r}$  at  $r_0$  is

$$\ddot{r}(r_0) = \frac{1}{f(r_0)} \frac{d^2 r}{dt^2} \Big|_{r_0} = -\frac{1}{f(r_0)} \frac{f(r_0)}{m} \frac{dU}{dr} \Big|_{r_0} = -\frac{1}{m} \frac{dU}{dr} \Big|_{r_0}. \quad (46)$$

This is the analog of Newton's second law which, for geometric dynamics, holds only in the case in which the velocity is in the direction of the acceleration. Equation (46) holds for any radial motion, not just motion with has zero velocity at some point, since the acceleration  $\ddot{r}$  which satisfies (44) is independent of the velocity of the object. Moreover, since  $r_0$  was arbitrary, the radial acceleration is

$$\ddot{r} = -\frac{1}{m} \frac{dU}{dr}. \quad (47)$$

For radial motion, the geometrization of Newton's second law involves only replacing the time parameter with the proper time parameter. Our derivation assumes only that  $\dot{r}$  is independent of the velocity, that Newton's second law holds for motion with initial zero velocity and that the raising the index of the covector  $\frac{dU}{dr}$  to the acceleration vector is done via the metric of the object's spacetime.

From (44), we obtain  $\frac{c^2 f'(r)}{2} = \frac{dU}{dr}$ . Thus,  $f(r) = \frac{2U}{mc^2} + \text{const}$ , and from (35),  $f(r) = 1 + \frac{2U}{mc^2}$ . Introducing the *dimensionless potential*

$$u(r) = \frac{-2U(r)}{mc^2}, \quad (48)$$

we have

$$f(r) = 1 - u(r), \quad (49)$$

and from (43),

$$g(r) = \frac{1}{1 - u(r)}. \quad (50)$$

Thus, using  $u(r)$  defined by (48), the metric (33) is

$$ds^2 = (1 - u(r))c^2 dt^2 - \frac{1}{1 - u(r)} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\varphi^2), \quad (51)$$

which in the case of the gravitational field of a spherically symmetric mass distribution is the Schwarzschild metric. This metric was also derived without using Einstein's equation in [33], from assumptions different from those used here.

The metric (51) can be generalized to motion under any conservative force with potential  $U(\mathbf{x})$ , as follows. As in the previous section, introduce at each  $\mathbf{x}$  where  $\nabla U(\mathbf{x}) \neq \mathbf{0}$  a normalized vector

$$\mathbf{n}(\mathbf{x}) = \frac{\nabla U(\mathbf{x})}{|\nabla U(\mathbf{x})|} \quad (52)$$

in the direction of the gradient of  $U(\mathbf{x})$ , or the negative of the direction of the force. Let  $d\mathbf{x}_n = (d\mathbf{x} \cdot \mathbf{n})\mathbf{n}$  and  $d\mathbf{x}_{tr} = d\mathbf{x} - (d\mathbf{x} \cdot \mathbf{n})\mathbf{n}$ , respectively, denote the projections of the spatial increment  $d\mathbf{x}$  in the parallel and transverse directions to  $\mathbf{n}(\mathbf{x})$ . With this notation, our metric (51) becomes

$$ds^2 = (1 - u(\mathbf{x}))c^2 dt^2 - \frac{1}{1 - u(\mathbf{x})} d\mathbf{x}_n^2 - d\mathbf{x}_{tr}^2, \quad (53)$$

which is the *RND* metric.

## 5. The Classical Tests of GR

In this section, we show that *RND* passes the following classical tests of *GR*:

- (1) the anomalous precession of Mercury
- (2) the periastron advance of a binary star
- (3) gravitational lensing
- (4) the Shapiro time delay

For the gravitational field of a non-rotating, spherically symmetric body of mass  $M$ , the unit vector  $\mathbf{n}$ , defined by (52), is the radial direction, so  $\dot{\mathbf{x}}_n = \dot{r}$ . If the initial position and velocity of the object/particle are in the plane  $\theta = \pi/2$ , then they will remain in this plane throughout the motion ([34]). Thus, one may chose the coordinate system so that  $\theta = \pi/2$ .

Moreover, the metric coefficients (24) are independent of  $\varphi$ . Hence, by Proposition 1, we have

$$r^2\dot{\varphi} = J, \quad (54)$$

where, for objects with non-zero mass, the constant  $J$  has the meaning of angular momentum per unit mass. This implies that  $\dot{\mathbf{x}}_{tr}^2 = r^2\dot{\varphi}^2 = \frac{J^2}{r^2}$ , and one can rewrite equation (31) as

$$\frac{\dot{r}^2}{c^2} + (1 - u(r)) \left( \frac{J^2}{c^2 r^2} + \varepsilon \right) = k^2. \quad (55)$$

This, together with the definition of  $J$ , leads to the *path equation for a central force*

$$\left( \frac{J}{cr^2} \frac{dr}{d\varphi} \right)^2 + (1 - u(r)) \left( \frac{J^2}{c^2 r^2} + \varepsilon \right) = k^2, \quad (56)$$

which depends on the two integrals of motion  $k$  and  $J$  and coincides with the geodesic equation of the Schwarzschild metric ([23]).

Furthermore, from (25) and (54), the *time dependence equation for a central force* on this path is

$$\frac{dt}{d\varphi} = \frac{kr^2}{J(1 - u(r))}. \quad (57)$$

For a non-rotating, spherically symmetric object of mass  $M$ , the dimensionless gravitational potential defined by (11) is  $u(r) = \frac{r_s}{r}$ , where  $r_s = \frac{2GM}{c^2}$  is its Schwarzschild radius.

**1.** To describe the trajectory of Mercury ( $\varepsilon = 1$ ) in the gravitational field of the Sun, we rewrite (56) in terms of the dimensionless potential energy  $u$  by substituting  $r = r_s/u$ . Defining the orbit parameter  $\mu = \frac{1}{2} \left( \frac{cr_s}{J} \right)^2$ , we obtain the *RND equation for the planetary orbit*

$$\left( \frac{du}{d\varphi} \right)^2 = u^3 - u^2 + 2\mu u + 2\mu(k^2 - 1), \quad (58)$$

which is similar to the corresponding equation in *GR* (see, for example, [35]).

The corresponding classical Newtonian equation for this orbit is

$$\left( \frac{du}{d\varphi} \right)^2 = -u^2 + 2\mu u + 2\mu(k^2 - 1). \quad (59)$$

For a bounded orbit, the maximum and minimum values of  $u$  are the roots  $u_p, u_a$  of the quadratic on the right-hand side of this equation, corresponding to the perigee and apogee, respectively. This equation has the obvious classical solution  $u_{cl}(\varphi) = \mu(1 + e \cos(\varphi - \varphi_0))$ , where  $\varphi_0$  is the polar angle of the perigee and  $e$  is the eccentricity of the orbit. Here,  $u_p = \mu(1 + e)$  and  $u_a = \mu(1 - e)$ . Then  $\mu = (u_p + u_a)/2$  is the average energy on the trajectory. In polar coordinates, we have

$$r_{cl}(\varphi) = \frac{r_s/\mu}{1 + e \cos(\varphi - \varphi_0)}, \quad (60)$$

which is a non-precessing ellipse. The reason there is no precession is that the radial and angular periodicities are both equal to  $2\pi$ . This is no longer the case in *RND* dynamics, due to the anisotropy of the metric (24).

The *RND* solution (the solution of (58)) is of the form  $u(\varphi) = \mu(1 + e \cos \alpha(\varphi))$ , where the angle  $\alpha$  satisfies  $r_{cl}(\alpha) = r(\varphi)$ . As in the classical case,  $u_p, u_a$  are again roots of the cubic on the

right-hand side of equation (58), but this cubic has an additional root  $u_3 = 1 - (u_p + u_a) = 1 - 2\mu$ . Substituting this into (58), one obtains  $d\alpha/d\varphi = \sqrt{1 - 3\mu - \mu e \cos \alpha}$  and the explicit dependence

$$\varphi(\alpha) = \varphi_0 + \int_0^\alpha (1 - 3\mu - \mu e \cos \tilde{\alpha})^{-1/2} d\tilde{\alpha}, \quad (61)$$

which eventually yields the known perihelion precession formula ([23])

$$\varphi(2\pi) - \varphi(0) - 2\pi \approx 3\pi\mu \frac{rad}{rev}. \quad (62)$$

Substituting the value of  $\mu$  for Mercury, we obtain its observed anomalous precession.

**2.** Since, in *RND*, we use the unmodified Newtonian potential, the potential of a binary star is the same as the potential of a classical two-body problem. Therefore, we can reduce the problem to a one-body problem in the gravitational field of an object with mass  $M$ , the combined mass of the binary, located at the center of mass of the binary. Hence, the *RND* treatment of the binary is the same as for Mercury and will once again produce precessing elliptic orbits for each component of the binary, with precession given by (62). As shown in [36], this leads to a periastron advance

$$\dot{\omega} = 3 \frac{(GM)^{2/3}}{c^2(1-e^2)} \left( \frac{P_b}{2\pi} \right)^{-5/3}, \quad (63)$$

where  $P_b$  is the orbital period of the binary and  $\omega$  is the angular position of the periastron. This formula is identical to the post-Keplerian formula for the relativistic advance of the periastron found in [37].

**3.** Gravitational lensing and the Shapiro time delay (or gravitational time delay) describe the deflection of a light ray and the slowing of a light pulse ( $\varepsilon = 0$ ) as it moves from a point  $A$  to a point  $B$  in the gravitational potential of a spherically symmetric massive object of mass  $M$ . Denote by  $r_0$  the distance from the point  $P$  on the trajectory closest to the massive object. Since  $\frac{dr}{d\varphi} = 0$  at the point  $P$ , it follows from (56) that

$$\frac{J^2}{c^2 k^2} = -\frac{r_0^2}{1 - r_s/r_0}. \quad (64)$$

To obtain the formula for gravitational lensing, substitute (64) into (56), which yields

$$\left( \frac{r_0}{r^2} \frac{dr}{d\varphi} \right)^2 + \left( 1 - \frac{r_s}{r} \right) \frac{r_0^2}{r^2} = 1 - \frac{r_s}{r_0}. \quad (65)$$

For any angle  $\varphi$  on the trajectory, one may associate an angle  $\alpha(\varphi)$  for which  $r(\varphi) = \bar{r}(\alpha)$ , where  $\bar{r}(\alpha) = \frac{r_0}{\sin \alpha}$  is the straight-line approximation of the trajectory at the point  $P$ , chosen to be the  $x$  direction. This suggests the substitution  $r = \frac{r_0}{\sin \alpha}$ , which implies  $\frac{dr}{d\varphi} = -\cos \alpha \frac{r_0^2}{r_0} \frac{d\alpha}{d\varphi}$  and

$$\frac{d\varphi}{d\alpha} = \left( 1 - \frac{r_s}{r_0} \left( \sin \alpha + \frac{1}{1 + \sin \alpha} \right) \right)^{-1/2}. \quad (66)$$

Hence, the deflection angle of a light ray moving from  $A$  to  $B$  is

$$\delta\phi = \int_{\alpha_A}^{\alpha_B} \left( 1 - \frac{r_s}{r_0} \left( \sin \alpha + \frac{1}{1 + \sin \alpha} \right) \right)^{-1/2} d\alpha - \pi, \quad (67)$$

where  $\alpha_A, \alpha_B$  are the  $\alpha$  values of the points  $A$  and  $B$ , respectively. Assuming that these points are very remote from the massive body ( $\alpha_A \approx \pi, \alpha_B \approx 0$ ) and that  $r_s/r_0 \ll 1$ , the weak deflection angle becomes

$$\delta\phi \approx \frac{2r_s}{r_0} = \frac{4GM}{c^2 r_0}, \quad (68)$$

which is identical to the angle given by Einstein's formula for weak gravitational lensing using *GR* ([23, 38]).

4. To obtain the formula for the Shapiro time delay, one substitutes the value of  $J/ck$  from (64) into the *RND* time dependence equation for a central force (57). Hence, the time of passage of light from the point  $P$  to  $B$  is given by

$$c(T_B - T_P) = \int_{\varphi_B}^{\pi/2} \frac{r^2 \sqrt{1 - r_s/r_0}}{(1 - r_s/r)r_0} d\varphi. \quad (69)$$

For the common case  $r_s/r_0 \ll 1$ , we work in first order in  $r_s/r_0$ . Then, using (66) and the same substitution  $r = \frac{r_0}{\sin \alpha}$  as above, the time propagation between  $P$  and  $B$  is

$$c(T_B - T_P) \approx x_B + r_s \ln \frac{r_B + x_B}{r_0}, \quad (70)$$

where  $x_B$  denotes the  $x$  coordinate of  $B$ . Using this approximation, the Shapiro time delay for a signal traveling from  $A$  to  $B$  and back is

$$r_s \ln \frac{4x_B |x_A|}{r_0^2}, \quad (71)$$

which is the known formula for the Shapiro time delay ([23, 38]), confirmed by several experiments.

## 6. The *RND* metric of several objects moving in an inertial frame

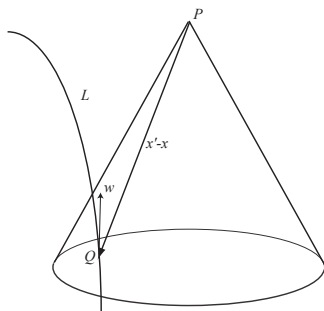
If the inertial frame  $K$  is the rest frame of the Universe, and the rest frame  $K'$  of the Sun moves uniformly with respect to  $K$ , then as pointed out in [39], the theory will pass the tests of *GR* only if it is Lorentz covariant. Therefore, we now extend the previous results by imposing Lorentz covariance. In electromagnetism, the potential of a moving charge can be defined from the potential of a stationary charge by applying Lorentz covariance. This leads to the Liénard-Wiechert retarded four-potential. Our approach here is similar.

Consider a spherically symmetric object of rest mass  $M$  moving uniformly in an inertial frame  $K$ . Denote by  $P = x^\mu$  the spacetime point in  $K$  at which we want to define the four-potential. Let  $L : (x')^\mu(\tau)$  denote the worldline of the object generating the gravitational field. Let the point  $Q = x'(\tau(x))$  be the unique point of intersection of the past light cone at  $P$  with the worldline  $x'(\tau)$ , see Figure 1. The time on the worldline of the object corresponding to this intersection is uniquely determined by the point  $P$ . It is called the *retarded time* and will be denoted by  $\tau(x)$ . Note that radiation emitted at  $Q = x'(\tau(x))$  at the retarded time will reach  $P$  at time  $t = (x^0 - (x')^0(\tau(x)))/c$ .

The four-potential per unit mass at  $P$ , in an inertial frame, is given by

$$A_\alpha(P) = \frac{-GMw_\alpha}{(x'(\tau(x)) - x) \cdot w}, \quad (72)$$

where  $x'(\tau(x)) - x$  is the relative position of  $P$  and the position of the object at the retarded time, and  $w$  is its four-velocity at that time. Formula (72) certainly holds in an inertial frame



**Figure 1.** The four-vectors  $x' - x, w$  associated with an observer and a moving object.

in which  $M$  is at rest. By Lorentz invariance, this formula holds in any inertial frame. The derivation parallels that of the Liénard-Wiechert potentials of the field of an arbitrarily moving charge (see, for example, [40], pp. 174-6).

Now consider a gravitational field generated by several objects of masses  $M_j$  with worldlines  $x'_j(\tau)$ . The total four-potential of the sources is the sum of the individual four-potentials (see [42]). To calculate the total four-potential, we have to define first their retarded times  $\tau_j(x)$ , their relative positions  $x'_j(\tau_j(x)) - x$ , and their four-velocities  $w_j$  at the retarded time. Then, the total gravitational four-potential at  $P$  is

$$A_\alpha(P) = \sum_j \frac{-GM_j(w_j)_\alpha}{(x'_j(\tau_j(x)) - x) \cdot w_j}. \quad (73)$$

Note that this four-potential tends to 0 when  $P$  is far removed from the sources.

Denote the time and spatial components of the four potential by

$$A_\alpha = (\Phi, -\mathbf{A}), \quad (74)$$

where the spatial part has a minus sign because it is a co-vector. Since each individual four-potential is a timelike vector, the total four-potential is also timelike. Therefore, we can apply a Lorentz transformation  $\Lambda$  from  $K$  to an inertial frame  $K'$  such that  $\Lambda(A) = (U, 0, 0, 0)$ . In  $K'$ , the four-potential behaves like a “static” potential since it has zero spatial component. Therefore, in  $K'$ , the metric is defined by formula (23), with  $u$  defined by (11) and  $\mathbf{n}$  defined by (52). To obtain the metric in  $K$ , apply the inverse Lorentz transformation  $\Lambda^{-1}$  to the metric in  $K'$ .

## 7. Discussion

Riemann’s approach to unify the laws of electricity, magnetism, light and gravitation was geometric. He believed that the forces at play in a system determine the geometry of the system. Put simply, for Riemann, *force equals geometry*. His quest failed, unfortunately, because he considered how forces affect *space*, not *spacetime*. Nevertheless, his geometric approach led to the development of *pseudo-Riemannian geometry* which fifty years later provided the cornerstone for Einstein’s *GR*. However, *GR* singles out the gravitational force from other forces which are not treated geometrically.

In this paper we introduced the relativity of spacetime in order to apply Riemann’s principle of “force equals geometry” to the dynamics under any conservative force. We accomplished

this by describing the geometry of the spacetime of a moving object via a metric derived from the potential of the force field acting on the object. Since an inanimate object has no internal mechanism with which to change its velocity, it has constant velocity in *its* own world. This led us to formulate our new *Generalized Principle of Inertia*, which states that: **An inanimate object moves freely, that is, with zero acceleration, in *its* own spacetime, whose geometry is determined by all of the forces affecting it.**

This is a generalization, or more accurately, a relativization of what Einstein accomplished. In *GR*, an object freely falling in a gravitational field is in free motion. Along a geodesic, the acceleration is zero. The Generalized Principle of Inertia states that *every* object is in free motion in *its* own world, determined by the forces which affect it. Thus, we assumed the existence of a metric with respect to which the length of the object's trajectory is extremal, enabling us to use a variational principle and conserved quantities to calculate trajectories.

Specifically, we began by treating classical Newtonian dynamics within this framework. Using second Newton's law and energy conservation, we derived a *Newtonian metric*. Nevertheless, this Newtonian metric is still deficient, since it fails to reproduce the tests of *GR*. The deficiency lies in the fact that this metric is isotropic, while the potential influences only the direction of the force and has no influence on directions transverse to the force. We removed this deficiency and obtained a corrected Newtonian metric (23). The dynamics built on the corrected metric is called *Relativistic Newtonian Dynamics (RND)*. We derived the *dimensionless energy conservation equation* (31) and the *dimensionless equation of motion* (32) of *RND*, for both massless particles and objects with non-zero mass. For a gravitational field of a spherically symmetric massive body, the *RND* metric is the Schwarzschild metric.

It is clear from these equations that this dynamics reduces in the low velocity, weak field limit to classical Newtonian dynamics. Moreover, as a partial validation of our approach, we have shown in section 5 that for a gravitational potential, *RND* exactly reproduces the classical tests of *GR*. The derivation of our metric is much simpler than in *GR* and uses potentials defined by the sources via Poisson's equation for the static case. We expect *RND* to be useful for studying relativistic gravitational astrophysical (or other) phenomena.

The first approach reveals why Newtonian dynamics cannot produce relativistic behavior and what modification is needed to produce such behavior. However, this approach is based on energy conservation which relies on a flat metric and not on the Generalized Principle of Inertia. This led us to the second approach which uses only the metric-free parts of Newtonian dynamics.

In our second approach, we took advantage of the symmetry of the problem and angular momentum conservation, which helped to derive the form of the metric in the direction transverse to the force. The remaining part of the metric was obtained from considering radial motion. We defined the metric so that for radial motion, the proper acceleration is independent of the velocity, as in the classical case. This property, which is equivalent to Tangherlini's condition, cannot be kept for arbitrary motion, only for motion in the direction of the force. We used the classical limit to complete the derivation of the metric.

In classical physics, the total energy has two mutually exclusive contributions: the kinetic energy depending only on the magnitude of the velocity of the object, and the potential energy depending on the field and the object's position. This leads to the isotropy of the spatial part of the Newtonian metric (18). In *RND*, on the other hand, the metric is not spatially isotropic because space is not isotropic. The direction of the force is a preferred direction for the given object. This is reflected in the additional term

$$U(\mathbf{x}) \frac{\dot{\mathbf{x}}_{tr}^2}{c^2}$$

of the energy conservation equation (28), which depends on both the position and the velocity of the moving object. This implies that in order to reproduce relativistic effects, one can no longer

separate these contributions. Indeed, some authors [17, 18, 24, 25, 26] have defined velocity-dependent “potentials” in order to reproduce relativistic effects. However, these potentials are not potentials in the true sense. They are “Newtonian analogous potentials” (see [18]).

The non-static case is handled as follows. First, we consider a single source which is moving in our inertial frame. Using Lorentz invariance, the potential of this source at a given spacetime point becomes a four-potential, which depends on the position of the source at the retarded time with respect to the given spacetime point. For several sources, the metric of the object’s spacetime depends on the sum of the four-potential of the sources which affect our object. The formula for the four-potentials is similar to the that of the Liénard-Wiechert potentials. Since the total four-potential is timelike, there is an inertial frame in which it behaves like a scalar potential. In this frame, the metric is defined as in the static case. The metric in the original frame is then computed via a Lorentz transformation. A similar approach can be found in [42].

At this stage, the *RND* model does not describe modern tests of *GR* beyond the classical tests. In particular, since the potential of a collapsing binary is not static, the model in its current form does not provide a mechanism for the recently discovered gravitational waves by the LIGO team. We hope to extend the model to handle these modern tests as well.

Our methodology here differs from the standard approach in the way we define the radial coordinate  $r$ . The standard approach (see [20, 44], for example) is to *define*  $r$  so that a sphere of radius  $r$  has surface area  $4\pi r^2$ . Our approach, on the other hand, is to measure  $r$  in the inertial frame at infinity. This is a more natural way to define the radial coordinate, since, in practice, this is what is measured.

## References

- [1] Klein, F.: Riemann und seine Bedeutung für die Entwicklung der modernen Mathematik. In: Klein, F. Gesammelte mathematische Abhandlungen. Springer-Verlag, Berlin (1923) (reprinted 1973) 3: 482-497. English translation: Riemann and his significance for the development of modern mathematics. Bull. Amer. Math. Soc. 1 No. 5, 165-180 (1895)
- [2] Papadopoulos, A.: Physics in Riemann’s Mathematical Papers. In: Ji, L., Papadopoulos A., Yamada S. (eds.) From Riemann to Differential Geometry and Relativity, pp. 151-207. Springer, Cham, Switzerland (2017)
- [3] Minkowski, H.: Raum und Zeit (Space and Time). Phys Z 10, 7588 (1908/1909)
- [4] Einstein, A.: Ideas and opinions, based on C. Seeling (ed.) Mein Weltbild. New translations and revisions by S. Bargmann. Crown Publishers, Inc., New York (1954)
- [5] Duarte, C.: The classical geometrization of the electromagnetism. International Journal of Geometric Methods in Modern Physics 12 (2015)
- [6] Riley, K.F., Hobson, M.P., Bence, S.J.: Mathematical Methods for Physics and Engineering, Third Edition. Cambridge University Press, New York (2006) p. 787
- [7] Will, C.M., Nordtvedt, K., Jr.: Astrophys. J. 177, 757-774 (1972)
- [8] Friedman, Y., Steiner, J. M.: Predicting Mercury’s precession using simple relativistic Newtonian dynamics. Europhys Lett. 113, 39001 (2016)
- [9] Paczyński, B., Wiita, P.J.: Thick accretion disks and supercritical luminosities. Astron Astrophys. 88, 23-31 (1980)
- [10] Abramowicz, M.A., Czerny, B., Lasota, J.P., Szuszkiewicz, E.: Slim accretion disks. Astrophys J. 332, 646-658 (1988)
- [11] Chakrabarti, S., Titarchuk, L.G.: Spectral Properties of Accretion Disks around Galactic and Extragalactic Black Holes. Astrophys J. 455, 623 (1995)
- [12] Hawley, J.F., Balbus, S.A.: The Dynamical Structure of Nonradiative Black Hole Accretion Flows. Astrophys J. 573, 738-748 (2002)
- [13] Lee, W.H., Ramírez-Ruiz, E.: Accretion Modes in Collapsars: Prospects for Gamma-Ray Burst Production. Astrophys J. 641, 961-971 (2006)
- [14] MacFadyen, A.I., Woosley, S.E.: Collapsars: Gamma-Ray Bursts and Explosions in “Failed Supernovae”. Astrophys J. 524, 262-289 (1999)
- [15] Matsumoto, R., Kato, S., Fukue, J., Okazaki, A.T.: Viscous transonic flow around the inner edge of geometrically thin accretion disks. Astron Soc Japan 36, 71-85 (1984)

- [16] Rosswog, S., Ramírez-Ruiz, E., Hix, W.R.: Tidal Disruption and Ignition of White Dwarfs by Moderately Massive Black Holes. *Astrophys J.* 695, 404-419 (2009)
- [17] Ghosh, S., Sarkar, T., Bhadra, A.: Newtonian analogue of corresponding spacetime dynamics of rotating black holes: implication for black hole accretion. *Mon Not R Astron Soc.* 445, 4463-4479 (2014)
- [18] Ghosh, S., Sarkar, T., Bhadra, A.: Exact Relativistic Newtonian Representation of Gravitational Static Spacetime Geometries. *Astrophys J.* 828, 6-9 (2016)
- [19] Witzany, V., Lämmerzahl, C.: Pseudo-Newtonian Equations for Evolution of Particles and Fluids in Stationary Space-times. *Astrophys J.* 841, 105-118 (2017)
- [20] Rindler, W.: *Relativity, Special, General and Cosmological.* Oxford, New York (2001)
- [21] Bartelmann, M.: *General Relativity.* Heidelberg: Institut für Theoretische Astrophysik Universität Heidelberg, [www.ita.uni-heidelberg.de/research/bartelmann/files/relativity.pdf](http://www.ita.uni-heidelberg.de/research/bartelmann/files/relativity.pdf), 1970
- [22] Carroll, S.: *Spacetime and Geometry: An Introduction to General Relativity.* Pearson, London (2003)
- [23] Misner, C., Thorne, K., Wheeler, J.: *Gravitation.* Freeman, San Francisco (1973)
- [24] Tejada, E., Rosswog, S.: An accurate Newtonian description of particle motion around a Schwarzschild black hole. *Mon Not R Astron Soc.* 433, 1930-1940 (2013)
- [25] Sarkar, T., Ghosh, S., Bhadra, A.: Newtonian analogue of Schwarzschild–de Sitter spacetime: Influence on the local kinematics in galaxies. *Phys Rev D* 90, 063008-063027 (2014)
- [26] Ghosh, S., Sarkar, T., Bhadra, A.: Newtonian analogue of static general relativistic spacetimes: An extension to naked singularities. *Phys Rev D* 92, 083010-083032 (2015)
- [27] Friedman, Y.: Relativistic Newtonian Dynamics for objects and particles. *Europhys. Lett.* 117, 49003 (2017)
- [28] Friedman, Y., Steiner, J. Relativistic Newtonian Dynamics. *J. Phys. Conf. Ser.* 845 012028 (2017)
- [29] Tangherlini, F.R.: Postulational Approach to Schwarzschild’s Exterior Solution with Application to a Class of Interior Solutions. *Il Nuovo Cimento* 25 (5), 10811105 (1962)
- [30] Guendelman, E., Rabinowitz, A.: Linearity, non self-interacting spherically symmetric gravitational fields, the “sphereland equivalence principle” and Hamiltonian bubbles. *Gen. Rel. Grav.* 28(2), 117 (1996)
- [31] Jacobson, T.: When is  $g_{tt}g_{rr} = -1$ ? *Class. Quant. Grav.* 24, 5717 (2007)
- [32] Itin, Y.: Premetric representation of mechanics, electromagnetism and gravity. *International Journal of Geometric Methods in Modern Physics* 15 (2018) 1840002
- [33] Guendelman, E., Rabinowitz, A., Bank, A.: Finding Schwarzschild Metric Component  $g_{r,r}$  and FLRW’s  $k$  without Solving the Einstein Equation, Rather by a Synergistic Matching between Geometric Results Enfranchised by Newtonian Gravity. *Bulg. J. Phys.* 45, 180 (2018)
- [34] Grøn, Ø.: *Lecture Notes on the General Theory of Relativity.* Springer, New York (2009)
- [35] Brown, K.: *Reflections on Relativity.* lulu.com (2015)
- [36] Friedman, Y., Livshitz, S., Steiner, J.M.: Predicting the relativistic periastron advance of a binary without curving spacetime. *Europhys Lett.* 116, 59001-59006 (2016)
- [37] Damour, T., Deruelle, N.: General Relativistic Celestial Mechanics of binary systems I. The post-Newtonian motion. *Ann Inst Henri Poincaré A* 43, 107-132 (1985)
- [38] Will, C.M.: The Confrontation between General Relativity and Experiment. *Liv Rev Rel* 17, 4 (2014)
- [39] Nordtvedt, K., Jr., Will, C.M.: *Astrophys. J.* 177, 775-792 (1972)
- [40] Landau, L., Lifshitz, E.: *The Classical Theory of Fields.* Addison-Wesley (1951)
- [41] Friedman, Y., with the assistance of Scarr, T.: *Physical Applications of Homogenous Balls.* Birkhauser(2005)
- [42] Mashhoon, B.: Gravitoelectromagnetism: A Brief Review in Pascual-Sanchez, J.-F., Floria, L., San Miguel, A., Vicente, F. eds. *Reference Frames and Gravitomagnetism.* World Scientific (2001) pp. 121-32
- [43] Friedman, Y., Scarr, T., Steiner, J.M.: *Int. J. Geom. Meth. Mod. Phys* to appear
- [44] Rindler, W.: Counterexample to the Lenz-Schiff Argument. *Am J Phys* 36 (6), 540-544 (1968)