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*Article*

# $f(\mathcal{R}, \mathcal{T})$-Gravity Model with Perfect Fluid Admitting Einstein Solitons

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**Abstract:** $f(\mathcal{R}, T)$-gravity is a generalization of Einstein's field equations $(EFE_s)$ and $f(\mathcal{R})$-gravity. In this research article, we demonstrate the virtues of the $f(\mathcal{R}, T)$-gravity model with Einstein solitons $(ES)$ and gradient Einstein solitons $(GES)$. We acquire the equation of state of $f(\mathcal{R}, T)$-gravity, provided the matter of $f(\mathcal{R}, T)$-gravity is perfect fluid. In this series, we give a clue to determine pressure and density in radiation and phantom barrier era, respectively. It is proved that if a $f(\mathcal{R}, T)$-gravity filled with perfect fluid admits an Einstein soliton $(g, \rho, \lambda)$ and the Einstein soliton vector field $\rho$ of $(g, \rho, \lambda)$ is Killing, then the scalar curvature is constant and the Ricci tensor is proportional to the metric tensor. We also establish the Liouville's equation in the $f(\mathcal{R}, T)$-gravity model. Next, we prove that if a $f(\mathcal{R}, T)$-gravity filled with perfect fluid admits a gradient Einstein soliton, then the potential function of gradient Einstein soliton satisfies Poisson equation. We also establish some physical properties of the $f(\mathcal{R}, T)$-gravity model together with gradient Einstein soliton.

**Keywords:** Einstein solitons; gradient Einstein solitons; perfect fluid spacetime; $f(\mathcal{R}, \mathcal{T})$-gravity; lorentzian manifolds

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## 1. Introduction

The general theory of relativity $(GR)$ says that the gravitation is a geometric property as symmetric curvature of spacetime. The physical matter symmetry is specially relating to the spacetime geometry. More specifically, the space time symmetries are used in the study of exact solutions of Einstein's field equations of general relativity. An important symmetry is a soliton that connects to geometrical flow of spacetime geometry. In fact, the Einstein flow is used to understand the idea of kinematics.

Hamilton suggested to use the evolution equation, known as Ricci flow, in order to establish Thurston's geometrization hypothesis in a three-dimensional manifold. In 1982, he [1] proposed the idea of the Ricci soliton $(RS)$ on a Riemannian manifold $M$ and noted that it moves under the Ricci flow $(\frac{\partial g}{\partial t} + 2\mathcal{R}ic(g) = 0)$ simply via diffeomorphism of the initial metric, where $g$, $\mathcal{R}ic$ and $t$ indicate the Riemannian metric, Ricci tensor and time, respectively. An $RS$ $(g, \mathcal{V}, \lambda)$ on $M$ takes the form

$$\mathcal{R}ic + \lambda g = -\frac{1}{2}\mathcal{L}_{\mathcal{V}} g, \tag{1}$$

where $\mathcal{L}_{\mathcal{V}}$ denotes the Lie derivative operator along $\mathcal{V}$ (termed as the soliton vector field) on $M$ and $\lambda \in \mathbb{R}$ (set of real numbers). An $RS$ will be shrinking, steady or expanding if

1. $\lambda < 0$,
2. $\lambda = 0$ and
3. $\lambda > 0$, respectively.

Shrinking *RS* yields an ancient self-similar solution to the Ricci flow with fixed annihilated time [2]. In addition, if $\mathcal{V}$ in $(g, \mathcal{V}, \lambda)$ is expressed as $\mathcal{V} = D\psi$, where $\psi : M \to \mathbb{R}$ and $D$ represents the gradient operator of $g$, then $g$ is referred as a gradient *RS* (in short, *GRS*), and Expression (1) leads to

$$\mathcal{R}ic + Hess(\psi) = -\lambda g. \tag{2}$$

Here, *Hess* symbolizes the Hessian operator. If $\psi =$ constant, then *GRS* is nothing more than an Einstein manifold. In a similar manner, the *RS* provides self-similar solutions to Ricci flows. Moreover, substantial attention has been given in recent days to the categorization of solutions that are self-similar to geometric flows. Catino and Mazzieri [2] defined the idea of Einstein soliton $(ES)$ in 2016, which creates some solutions that are self-similar to Einstein flow:

$$\frac{\partial}{\partial t} g + 2\left(\mathcal{R}ic - \frac{\mathcal{R}}{2} g\right) = 0, \tag{3}$$

where $\mathcal{R}$ denotes the scalar curvature of the manifold. Consider the equation

$$\mathcal{L}_{\mathcal{V}} g + 2\mathcal{R}ic + (2\lambda - \mathcal{R})g = 0. \tag{4}$$

If the data $(g, \mathcal{V}, \lambda - \frac{\mathcal{R}}{2})$ satisfy (4), then it is termed as *ES* [3] on $M$. Here, $\mathcal{V}$ is the soliton vector (known as Einstein soliton vector field). Recall that in a manifold $M$ with $\mathcal{R} =$ constant, the *ES* simplifies to an *RS*. Gradient vector field plays a crucial role to study the Morse–Smale theory. The gradient Einstein soliton $(GES)$ on a semi-Riemannian manifold $M$ is an *ES* with $\mathcal{V} = \mathcal{D}\psi$. As a result, Equation (4) may be reduced to the following form

$$Hess(\psi) + \mathcal{R}ic + \left(\lambda - \frac{\mathcal{R}}{2}\right) g = 0. \tag{5}$$

The smooth function $\psi$ is referred as the potential function of the *GES* in this context. Einstein solitons are generalization of Einstein metrics. A trivial *GES* is a *GES* with a constant potential function $\psi$. Einstein solitons are important in Einstein flow because they relate to solutions that are self-similar and frequently appear as singularity theories. In physics, quasi-Einstein is a smooth metric space $(M, g, e^{\psi}, dvol)$ with $Ric\,\psi = \lambda$g. On the other hand, the universe is all space and time and their contents, including stars, galaxies, planets and other forms of matter and energy. The *GR* is the most effective theoretical method for studying the large-scale structure of the universe. It is observed that *GR*, without taking into account the dark energy, cannot describe the acceleration of the early and late Universe. *GR* does not explain precisely gravity and it is quite reasonable to modify in order to obtain theories that admit inflation and imitate the *Dark Energy* $(DE)$. The conventional method to analyzing known cosmic dynamic is provided by Einstein's modification of gravitational field equations [4,5]. Einstein's field equations give the greatest approximation to the observable data, with the addition of a hypothetical element of the universe, described as *Dark Matter* [6].

In addition, the universe contains a strange component known as *DE*, which is thought to be the primary cause of the universe acceleration, extension and regulates the matter-energy ratio. This circumstance led various mathematicians and physicists to construct more advanced gravity theories, which emerged as a result of the Einstein–Hilbert action and the use of modified gravity theories such as $f(\mathcal{R})$-gravity [7], Gauss–Bonnet $f(G)$-gravity [8] and $f(\mathcal{T})$ theory [9], etc. These theories differ from Einstein's conventional gravity theory and might also give a good approximation to quantum gravity [10].

*GR* may be extended to the $f(\mathcal{R})$ gravity by Einstein–Hilbert Lagrangian density to a function $f(\mathcal{R})$, where $\mathcal{R}$ is the Ricci scalar. Higher order curvature solves the issue of huge neutron stars in the equations of motion of $f(\mathcal{R})$ gravity, for examples, see [11–13]. However, the $f(\mathcal{R})$ gravity has certain limits in terms of stability with the solar system and also fails to support the involvement of different cosmic models, such as stable stellar configuration (for more information, see [14,15]), raising concerns about its applicability. Harko et al. [16] presented a more extended gravity model by considering that the Lagrangian is an arbitrary function of $\mathcal{T}$ and $\mathcal{R}$, and named as $f(\mathcal{R},\mathcal{T})$-gravity theory. Here, $\mathcal{T}$ denotes the trace of energy–momentum tensor $T$. This idea was effectively employed to describe the universe late-time rapid expansion.

A spacetime can be characterized as a four-dimensional time orientated Lorentzian manifold $M$ which is a type of semi-Riemannian manifold with the Lorentzian metric $g$. The basic vectors characterization in the Lorentzian manifold were the starting point to study the properties of Lorentzian manifold geometry. As a reason, Lorentzian manifold $M$ is the finest choice for studying cosmological models. The material substance of the cosmos is known to behave like a perfect fluid spacetime $(PFST)$ in standard cosmological models. In $PFST$, we write the expression of $T$ as:

$$T(\mathcal{E},\mathcal{F}) = pg(\mathcal{E},\mathcal{F}) + (p+\sigma)\eta(\mathcal{E})\eta(\mathcal{F}), \tag{6}$$

where $\mathcal{E},\mathcal{F} \in \mathfrak{X}(M)$, $p$ and $\sigma$ indicate isotropic pressure and the energy density, respectively, of the perfect fluid ([17,18]). Here, $\mathfrak{X}(M)$ contains all smooth vector fields of $M$ and $\eta$ is the 1-form associated with the unit timelike velocity vector field $\rho$ of $PFST$ by the relation $\eta(\mathcal{E}) = g(\mathcal{E},\rho)$. In modern cosmology, the acceleration of universe expansion is assumed as a dark energy source.

Scalar fields are thought to play a vital character in the physics and cosmology of $f(\mathcal{R},\mathcal{T})$-gravity theory. However, adopting scalar fields as a source to build more generic gravitational models may lead to a clearer perspective of the general features of the gravitational field. Singh and Singh recreated the flat scalar and exponential models of $f(\mathcal{R},\mathcal{T})$-gravity in scalar field cosmology in [19]. Chaubey [20] investigated $f(\mathcal{R},\mathcal{T})$-gravity and demonstrated some findings. In [21,22], Capozziello et al. examined the characteristics of cosmological perfect fluid in $f(\mathcal{R})$ gravity. Many researchers also analyzed perfect fluid spacetime with different solitons. For further information, read [23–29].

The above studies inspire us enough to study the physical and geometrical property of $f(\mathcal{R},\mathcal{T})$-gravity with perfect fluid admitting $ES$ and $GES$. We use $\Theta^{\diamond}$ to denote $f(\mathcal{R},\mathcal{T})$-gravity model with perfect fluid. As a conclusion, it is essential to investigate the geometry and classification of $GES$.

## 2. Perfect Fluid Spacetime in $f(\mathcal{R},\mathcal{T})$-Gravity Theory

The physical property of matter plays a major role in studying $\Theta^{\diamond}$, and therefore we can obtain many hypothetical models for different choices of $\mathcal{R}$ and $\mathcal{T}$ [16]. For instance, we choose

$$\frac{\mathcal{R}}{2} = \frac{1}{2}f(\mathcal{R},\mathcal{T}) - f(\mathcal{T}), \tag{7}$$

where $f(\mathcal{T})$ represents a function of $\mathcal{T}$ only. Remark that the gravitational interaction between space matter and curvature are modified by the term $2f(\mathcal{T})$ appeared in the gravitational action. We assume the modified Einstein–Hilbert action term as:

$$\mathcal{H}_E = \frac{1}{16\pi}\int [f(\mathcal{R},\mathcal{T}) + L_m]\sqrt{(-g)}\, d^4x, \tag{8}$$

where $L_m$ is the matter Lagrangian of the scalar field. The stress energy tensor of the matter is given by

$$T_{rs} = \frac{-2\delta(\sqrt{-g}L_m)}{\sqrt{-g}\,\delta g^{rs}}. \tag{9}$$

Let us consider that $L_m$ depends only on $g_{rs}$ and not on its derivatives. The variation of action with respect to the $g_{rs}$ infers

$$f_{\mathcal{R}}(\mathcal{R},\mathcal{T})Ric_{rs} - \frac{1}{2}f(\mathcal{R},\mathcal{T})g_{rs} + (g_{rs}\nabla_c\nabla^c - \nabla_r\nabla_s)f_{\mathcal{R}}(\mathcal{R},\mathcal{T}) \tag{10}$$
$$= 8\pi\mathcal{T}_{rs} - f_{\mathcal{T}}(\mathcal{R},\mathcal{T})\mathcal{T}_{rs} - f_{\mathcal{T}}(\mathcal{R},\mathcal{T})h_{rs},$$

where $f_{\mathcal{R}} = \frac{\partial f(\mathcal{R},\mathcal{T})}{\partial \mathcal{R}}$ and $f_{\mathcal{T}} = \frac{\partial f(\mathcal{R},\mathcal{T})}{\partial \mathcal{T}}$. As per usual notation, $\nabla_r$ and $\Box \equiv \nabla_t \nabla^t$ stand for covariant derivative and d'Alembert operator, respectively. Moreover, we have

$$h_{rs} = -2T_{rs} + g_{rs}\mathcal{L}_m - 2g^{lk}\frac{\partial^2 L_m}{\partial g^{rs}\partial g^{lk}}. \tag{11}$$

Equations (8) and (9) with $f(\mathcal{R},\mathcal{T}) = f(\mathcal{R})$ provide the field equations of $f(\mathcal{R})$-gravity.

Let the matter be a perfect fluid with $p$, $\sigma$ and velocity vector $\eta^{\alpha}$. We have privilege in selection of $L_m$. Therefore, we fix $L_m = p$. Equation (6) can be rewritten as

$$T_{rs} = pg_{rs} + (p+\sigma)\eta_r\eta_s, \tag{12}$$

where

$$\eta^r \nabla_s \eta_r = 0, \quad \eta_r \cdot \eta^r = -1. \tag{13}$$

We have from (11) and (12)

$$h_{rs} = pg_{rs} - 2T_{rs}. \tag{14}$$

After adopting (7) and (10) we obtain

$$\mathcal{R}ic_{rs} = \frac{\mathcal{R}}{2}g_{rs} - 2f^{'}(\mathcal{T})T_{rs} - 2f^{'}(\mathcal{T})h_{rs} + f(\mathcal{T})g_{rs} + 8\pi T_{rs}. \tag{15}$$

In view of (12)–(14), Equation (15) becomes

$$\mathcal{R}ic_{rs} = \left\{f(\mathcal{T}) + \frac{\mathcal{R}}{2} + 8p\pi\right\}g_{rs} + \left\{(p+\sigma)(2f^{'}(\mathcal{T}) + 8\pi)\right\}\eta_r\eta_s, \tag{16}$$

reduces to

$$(\sigma + p)f^{'}(\mathcal{T}) - 2f(\mathcal{T}) - 4\pi(3p - \sigma) - \frac{\mathcal{R}}{2} = 0. \tag{17}$$

Thus, for *PFST* in $f(\mathcal{R},\mathcal{T})$-gravity, the Ricci tensor assumes the form

$$\mathcal{R}ic_{rs} = ag_{rs} + b\eta_r\eta_s, \tag{18}$$

where

$$a = \frac{1}{2}\mathcal{R} + f(\mathcal{T}) + 8p\pi \quad \text{and} \quad b = 2(\sigma + p)(4\pi + f^{'}(\mathcal{T})). \tag{19}$$

Throughout the manuscript, we suppose that $a$ and $b$ are not simultaneously zero. The similar processor has been followed in [20] to find the expression of Ricci tensor, but for more clarity we also gave the proof here. Thus, we have the following conclusion:

**Theorem 1.** *The Ricci tensor of* $\Theta^{\diamond}$ *is*

$$\mathcal{R}ic_{rs} = \left\{\frac{1}{2}\mathcal{R} + f(\mathcal{T}) + 8p\pi\right\}g_{rs} + \left\{(\sigma + p)(8\pi + 2f^{'}(\mathcal{T}))\right\}\eta_r\eta_s.$$

**Corollary 1.** *The scalar curvature tensor of* $\Theta^{\diamond}$ *is given by*

$$\frac{\mathcal{R}}{2} = (\sigma + p)f^{'}(\mathcal{T}) - 2f(\mathcal{T}) - 4\pi(3p - \sigma). \tag{20}$$

Now, Equation (16) can be written in index free notation as

$$b\eta \otimes \eta = \mathcal{R}ic - ag, \tag{21}$$

equivalently

$$Q\mathcal{E} = a\mathcal{E} + b\eta(\mathcal{E})\rho, \quad \forall \quad \mathcal{E} \in \mathfrak{X}(M). \tag{22}$$

Now, in the light of Equation (20), we have

$$p + \sigma = \frac{\frac{\mathcal{R}}{2} + 2f(\mathcal{T}) + 4\pi(3p - \sigma)}{f'(\mathcal{T})}, \tag{23}$$

provided $f^{'}(\mathcal{T}) \neq 0$. In [30], author proved that the equation of state ($\mathfrak{EOS}$) for dark energy is given by $p = -\sigma + f(a)$, where $f(a)$ is a smooth function of the scale factor "$a$" and $t$ being the cosmic time. He also showed that the equation $\omega = \frac{p}{\sigma} = -1$ gives phantom barrier, whereas $\omega > -1$ and $\omega < -1$ reflect a transition from non-phantom to phantom. This turns up the following:

**Proposition 1.** *If the matter of $f(\mathcal{R}, \mathcal{T})$-gravity is perfect fluid, then $\mathfrak{EOS}$ is given by (23).*

Next, we suppose that the source is of radiation type, then $\mathfrak{EOS}$ is $\omega = \frac{1}{3}$. This fact together with Equation (23) gives

$$p = \frac{1}{8}\left(\frac{\mathcal{R} + 4f(\mathcal{T})}{f'(\mathcal{T})}\right) \text{ and } \sigma = \frac{3}{8}\left(\frac{\mathcal{R} + 4f(\mathcal{T})}{f'(\mathcal{T})}\right), \tag{24}$$

where $f^{'}(T) \neq 0$.

**Corollary 2.** *Let the source of $f(\mathcal{R}, \mathcal{T})$-gravity be a radiation type. Then the pressure and density are governed by (24).*

In case of phantom barrier, $\sigma = -p = \frac{\mathcal{R}+4f(\mathcal{T})}{32\pi}$. Thus, we conclude

**Corollary 3.** *If the source of matter in $f(\mathcal{R}, \mathcal{T})$-gravity is phantom barrier type, then the pressure and energy density are evaluated as $\sigma = -p = \frac{\mathcal{R}+4f(\mathcal{T})}{32\pi}$.*

## 3. Einstein Solitons on Perfect Fluid Spacetime in $f(\mathcal{R}, \mathcal{T})$-Gravity

Consider Equation (4) and $\mathcal{V} = \rho$, we find

$$\mathcal{R}ic(\mathcal{E}, \mathcal{F}) = -\frac{1}{2}(\mathcal{L}_\rho g)(\mathcal{E}, \mathcal{F}) - \left(\lambda - \frac{\mathcal{R}}{2}\right)g(\mathcal{E}, \mathcal{F}). \tag{25}$$

Using explicit form of Lie derivative in (25) gives

$$\mathcal{R}ic(\mathcal{E}, \mathcal{F}) = -\left(\lambda - \frac{\mathcal{R}}{2}\right)g(\mathcal{E}, \mathcal{F}) - \frac{1}{2}[g(\nabla_{\mathcal{E}}\rho, \mathcal{F}) + g(\mathcal{E}, \nabla_{\mathcal{F}}\rho)]. \tag{26}$$

Contracting (26), we obtain

$$\mathcal{R} = 4\lambda + div\rho. \tag{27}$$

In view of (13), (21) and (27), we turn up

$$4a - b = 4\lambda + div\rho. \tag{28}$$

Putting $\mathcal{E} = \mathcal{F} = \rho$ in (21) and (25), respectively, we obtain $Ric(\rho, \rho) = -a + b$ and $Ric(\rho, \rho) = \lambda - \frac{\mathcal{R}}{2}$, since $(\mathcal{L}_\rho g)(\rho, \rho) = 0$. These relations together with $\mathcal{R} = 4a - b$ and (13) give

$$2a + b = 2\lambda. \tag{29}$$

The last two equations give

$$a = \lambda + \frac{div\rho}{6}. \tag{30}$$

In consequence of Equations (19), (27) and (30), we find

$$\lambda = 24\pi\left(p + \frac{\mathcal{R} + f(\mathcal{T})}{24\pi}\right). \tag{31}$$

Thus, we can state:

**Theorem 2.** *Let $\Theta^{\diamond}$ admit an ES $(g, \lambda, \rho)$. Then $(g, \lambda, \rho)$ is expanding for $p > -\frac{\mathcal{R}+f(\mathcal{T})}{24\pi}$, shrinking for $p < -\frac{\mathcal{R}+f(\mathcal{T})}{24\pi}$ and steady for $p = -\frac{\mathcal{R}+f(\mathcal{T})}{24\pi}$.*

We suppose that $\rho$ is Killing ($\mathcal{L}_{\rho} g = 0$), which implies that $div\rho = 0$ and hence $\lambda = a$ and $b = 0$, where Equations (28)–(30) are used. Now, from Equation (25), we have $\mathcal{R} = 4\lambda$. The fact is that the scalar curvature is constant and the Ricci tensor is proportional to the metric tensor. Thus, we conclude our finding as:

**Theorem 3.** *Let $\Theta^{\diamond}$ admit an ES $(g, \rho, \lambda)$. If the ES vector field $\rho$ is Killing, then the scalar curvature is constant and the Ricci tensor is proportional to the metric tensor.*

Let $\Theta^{\diamond}$ admit a non-steady type $ES$ $(g, \rho, \lambda)$. Particularly, we suppose that $f(\mathcal{T}) = 0$ and then $f(\mathcal{R}, \mathcal{T}) = \mathcal{R} + 2f(\mathcal{T}) = \mathcal{R}$, that is, the $f(\mathcal{R}, \mathcal{T})$-gravity reduces to $EFE_s$. Equations (27) and (31) give $p = -\frac{3\mathcal{R}+div\rho}{96\pi}$ and $\lambda = \frac{1}{4}(\mathcal{R} - div\rho) \neq 0$. Again, Equation (19) and $\mathcal{R} = 4a - b$ together reflect that $\sigma = \frac{\mathcal{R}-div\rho}{32\pi}$. Thus, we have

$$\frac{p}{\sigma} = \frac{1}{3} - \frac{4}{3}\frac{\mathcal{R}}{\mathcal{R} - div\rho}. \tag{32}$$

Now, we write

**Corollary 4.** *If a PFST satisfying the $EFE_s$ admits a non-steady type ES, then $\mathfrak{EOS}$ is given by (32). Moreover, the pressure and energy density of PFST are $p = -\frac{3\mathcal{R}+div\rho}{96\pi}$ and $\sigma = -\frac{\mathcal{R}-div\rho}{32\pi}$.*

Additionally, if we fix $\mathcal{R} = 0$ in Equation (32), then we have $\omega = \frac{p}{\sigma} = \frac{1}{3}$, which shows the radiation era. We state:

**Corollary 5.** *Let the PFST satisfying the $EFE_s$ admit a non-steady type ES and $\mathcal{R} = 0$. Then $\mathfrak{EOS}$ ($\omega = \frac{p}{\sigma} = \frac{1}{3}$) represents the radiation era.*

Next, we have the following remark.

**Remark 1.** *For $\Psi \in C^{\infty}(M)$ and the vector field $\rho$, a straight forward calculation gives*

$$div(\Psi\rho) = \rho(d\Psi) + \Psi div\rho. \tag{33}$$

*The function $\Psi \in C^{\infty}(M)$ is a last multiplier of vector field $\rho$ with respect to $g$ if $div(\Psi\rho) = 0$. The corresponding equation*

$$\rho(d\, ln\Psi) = -div(\rho) \tag{34}$$

*is called the* **Liouville's equation** *of the vector field $\rho$ with respect to $g$ (for more details, see [31]).*

Now, infer the above remark and Equation (28), we state:

**Theorem 4.** *Let $\Theta^{\diamond}$ admit an ES and the velocity vector field $\rho$ of the ES is of gradient type, then the Liouville equation of $f(\mathcal{R}, \mathcal{T})$-gravity satisfying by $\Psi$ and $\rho$ is,*

$$\rho(d\, ln\Psi) = f(\mathcal{T}) + 3(\lambda + 8p\pi). \tag{35}$$

Again, using the fact that, if $f(\mathcal{T}) = 0$ then $f(\mathcal{R}, \mathcal{T})$-gravity recover $f(\mathcal{R})$-gravity. Thus, we have the following corollary.

**Corollary 6.** *Let $\Theta^{\diamond}$ admit an ES and the velocity vector field $\rho$ of the ES is of gradient type, then the Liouville equation of $f(\mathcal{R})$-gravity satisfying by $\Psi$ and $\rho$ is,*

$$\rho(d\, ln\Psi) = 2[8p\pi - \lambda]. \tag{36}$$

## 4. Gradient Einstein Solitons in $f(\mathcal{R}, \mathcal{T})$-Gravity

In this segment, we focus on a specific condition when the soliton vector field $\mathcal{V}$ of *ES* $(g, \mathcal{V}, \lambda)$ is of gradient type, $\mathcal{V} = \mathcal{D}\psi$, in a $f(\mathcal{R}, \mathcal{T})$-gravity filled with perfect fluid.

Let $\mathcal{V} = \mathcal{D}\psi$, where $\psi$ is a smooth function and $\mathcal{D}$ stands for gradient operator of $g$. Then, we have from Equation (5)

$$\nabla \mathcal{D}\psi = -(\lambda - \frac{\mathcal{R}}{2})I - Q, \tag{37}$$

where $I$ and $Q$ denote the identity transformation and Ricci operator. The contraction of Equation (37) gives $\Delta\psi = \Upsilon$, where $\Delta$ represents the Laplace operator and $\Upsilon = \mathcal{R} - \lambda$. Thus, in light of (17), we can conclude the following results as:

**Theorem 5.** *Let $\Theta^{\diamond}$ admit a GES, then the potential function $\psi$ of GES satisfies the Poisson's equation*

$$\Delta\psi = 2[(p + \sigma)f'(\mathcal{T}) - 2f(\mathcal{T}) - 4\pi(3p - \sigma)] - \lambda. \tag{38}$$

Suppose $\mathcal{R} = \lambda$. Then $\Delta\psi = 0$, represents the Laplace equation. Hence, we state:

**Corollary 7.** *Assume that $\Theta^{\diamond}$ admits a GES. If the soliton constant $\lambda$ of the GES coincides with the scalar curvature of $\Theta^{\diamond}$, then the potential function of GES satisfies the Laplace equation.*

Remark that the $\mathfrak{EOS}$: $p = -\sigma$, $p = \sigma$, $p = \frac{\sigma}{3}$ and $p = 0$ represent, respectively, the dark matter era, stiff matter era, radiation era and dust matter era [32,33]. Now, we conclude our results as:

**Corollary 8.** *If $\Theta^{\diamond}$ admits a GES, then we have*

| *$f(\mathcal{R}, T)$-gravity represents* | $\mathfrak{EOS}$ | *Poisson's equation* |
|---|---|---|
| *Dark matter era* | $p = -\sigma$ | $\Delta\psi = -[\lambda + 4\{f(\mathcal{T}) + 8\pi p\}]$ |
| *Stiff matter era* | $p = \sigma$ | $\Delta\psi = 4[p\{f'(\mathcal{T}) - 4\pi\} - f(\mathcal{T})] - \lambda$ |
| *Radiation era* | $p = \frac{\sigma}{3}$ | $\Delta\psi = 4[2pf'(\mathcal{T}) - f(\mathcal{T})] - \lambda$ |
| *Dust matter era* | $p = 0$ | $\Delta\psi = 2[\sigma f'(\mathcal{T}) - 2f(\mathcal{T}) + 4\pi\sigma] - \lambda$ |

Fix $f(\mathcal{T}) = 0$, then $f(\mathcal{R}, \mathcal{T}) = \mathcal{R}$, that is, the $f(\mathcal{R}, \mathcal{T})$-gravity reduces to the $EFE_s$ in general relativity. Thus, from Theorem 5, we state:

**Proposition 2.** *Let the PFST obeying the $EFE_s$ admit a GES $(g, grad\psi, \lambda)$. Then $\psi$ satisfies the Poisson equation $\Delta\psi = 8\pi(\sigma - 3p) - \lambda$.*

We also state the following corollary with the help of Proposition 2 as:

**Corollary 9.** *Let the PFST M obey the EFE$_s$. If M admits a GES with the potential function $\psi$. Then, we have*

| *PFST represents* | $\mathfrak{EOS}$ | *Poisson's equation* |
|---|---|---|
| *Dark matter era* | $p = -\sigma$ | $\Delta\psi = 32\pi\sigma - \lambda$ |
| *Stiff matter era* | $p = \sigma$ | $\Delta\psi = -(\lambda + 16\pi\sigma)$ |
| *Radiation era* | $p = \frac{\sigma}{3}$ | $\Delta\psi = -\lambda$ |
| *Dust matter era* | $p = 0$ | $\Delta\psi = 8\pi\sigma - \lambda$ |

From (4) and (22), we can write

$$\nabla_{\mathcal{E}}\mathcal{D}\psi + Q\mathcal{E} + \left(\lambda - \frac{\mathcal{R}}{2}\right)\mathcal{E} = 0 \tag{39}$$

for all $\mathcal{E} \in \mathfrak{X}(M)$. From (39), we infer

$$\nabla_{\mathcal{F}}\nabla_{\mathcal{E}}\mathcal{D}\psi = -(\nabla_{\mathcal{F}}Q)(\mathcal{E}) - Q(\nabla_{\mathcal{F}}\mathcal{E}) - \left(\lambda - \frac{\mathcal{R}}{2}\right)\nabla_{\mathcal{F}}\mathcal{E} + \frac{\mathcal{F}(\mathcal{R})}{2}\mathcal{E}. \tag{40}$$

Using Equations (39) and (40) in the curvature identity

$$R(\mathcal{E},\mathcal{F})\mathcal{D}\psi = \nabla_{\mathcal{E}}\nabla_{\mathcal{F}}\mathcal{D}\psi - \nabla_{\mathcal{F}}\nabla_{\mathcal{E}}\mathcal{D}\psi - \nabla_{[\mathcal{E},\mathcal{F}]}\mathcal{D}\psi, \tag{41}$$

we lead to

$$R(\mathcal{E},\mathcal{F})\mathcal{D}\psi = (\nabla_{\mathcal{F}}Q)\mathcal{E} - (\nabla_{\mathcal{E}}Q)\mathcal{F} + \frac{1}{2}\{\mathcal{E}(\mathcal{R})\mathcal{F} - \mathcal{F}(\mathcal{R})\mathcal{E}\}, \tag{42}$$

where $R$ denotes the curvature tensor with respect to the Levi-Civita connection. In view of (22), we lead to

$$(\nabla_{\mathcal{E}}Q)(\mathcal{F}) = \mathcal{E}(a)\mathcal{F} + \mathcal{E}(b)\eta(\mathcal{F})\rho + b(\nabla_{\mathcal{E}}\eta)(\mathcal{F})\rho + b\eta(\mathcal{F})\nabla_{\mathcal{E}}\rho. \tag{43}$$

Using (43) in (42), we obtain

$$\begin{aligned} R(\mathcal{E},\mathcal{F})\mathcal{D}\psi = {} & \mathcal{F}(a)\mathcal{E} - \mathcal{E}(a)\mathcal{F} + [\mathcal{F}(b)\eta(\mathcal{E}) - \mathcal{E}(b)\eta(\mathcal{F})]\rho \\ & + b\{(\nabla_{\mathcal{F}}\eta)(\mathcal{E}) - (\nabla_{\mathcal{E}}\eta)(\mathcal{F})\}\rho + b[\eta(\mathcal{E})\nabla_{\mathcal{F}}\rho - \eta(\mathcal{F})\nabla_{\mathcal{E}}\rho] \\ & + \frac{1}{2}\{\mathcal{E}(\mathcal{R})\mathcal{F} - \mathcal{F}(\mathcal{R})\mathcal{E}\}. \end{aligned} \tag{44}$$

Taking a set of orthonormal frame field and contracting (44) over $\mathcal{E}$, we lead

$$\mathcal{R}ic(\mathcal{F},\mathcal{D}\psi) = \frac{1}{2}\mathcal{F}(b) - 3\mathcal{F}(a) - \rho(b)\eta(\mathcal{F}) - b[(\nabla_{\rho}\eta)(\mathcal{F}) + \eta(\mathcal{F})div\rho], \tag{45}$$

since $g(\rho,\rho) = -1$ and $g(\nabla_{\mathcal{E}}\rho,\rho) = 0$. Again, from (18) we have

$$\mathcal{R}ic(\mathcal{F},\mathcal{D}\psi) = a\mathcal{F}(\psi) + b\eta(\mathcal{F})\rho(\psi). \tag{46}$$

Setting $\mathcal{F} = \rho$ in (45) and (46), respectively, we obtain

$$\mathcal{R}ic(\rho,\mathcal{D}\psi) = \frac{3}{2}\rho(b) - 3\rho(a) + bdiv\rho$$

and

$$\mathcal{R}ic(\rho,\mathcal{D}\psi) = (a-b)\rho(\Psi).$$

The last two equations infer that

$$2(a-b)\rho(\psi) = 3\rho(b) - 6\rho(a) + 2bdiv\rho. \tag{47}$$

Let $\rho$ be Killing, that is, $\mathcal{L}_\rho g = 0$ and the scalars $a$ and $b$ remain invariant under $\rho$, that is, $\rho(b) = \rho(a) = 0$. Then, we obtain $div\rho = 0$. Thus, from Equation (47), we infer that either $a = b$ or $\rho(\psi) = 0$. Next, we split our study as:

**Case I.** We consider that $a = b$ and $\rho(\psi) \neq 0$. Then, from (19), we conclude that

$$\frac{p}{\sigma} = -\frac{f(\mathcal{T}) + \sigma(4\pi + f'(\mathcal{T}))}{(12\pi + f'(\mathcal{T}))\sigma}. \tag{48}$$

This gives the $\mathfrak{EOS}$ for $\Theta^\diamond$.

**Case II.** Let $\rho(\psi) = 0$ and $a \neq b$. The covariant derivative of $g(\rho, \mathcal{D}\psi) = 0$ along $\mathcal{E}$ gives

$$g(\nabla_{\mathcal{E}}\rho, \mathcal{D}\psi) = -[(\lambda - \frac{\mathcal{R}}{2} + (a-b)]\eta(\mathcal{E}), \tag{49}$$

where (21) and (39) are used. Since $\rho$ is Killing in $\Theta^\diamond$, that is, $g(\nabla_{\mathcal{E}}\rho, \mathcal{F}) + g(\mathcal{E}, \nabla_{\mathcal{F}}\rho) = 0$. Putting $\mathcal{F} = \rho$ in this equation, we obtain $g(\mathcal{E}, \nabla_\rho \rho) = 0$ because $g(\nabla_{\mathcal{E}}\rho, \rho) = 0$. Thus, we conclude that $\nabla_\rho\rho = 0$. Changing $U$ with $\rho$ in Equation (49) and using the last equation, we infer that

$$\lambda = b - a + \frac{\mathcal{R}}{2} \tag{50}$$

$$\Longleftrightarrow \lambda = 2(p+\sigma)f^{'}(\mathcal{T}) + 8\pi p - f(\mathcal{T}). \tag{51}$$

This reflects the following conclusions:

**Theorem 6.** *Let $\Theta^\diamond$ admit a GES. If the velocity vector field $\rho$ of the perfect fluid is Killing and the scalars a and b are invariant along $\rho$, then the GES is expanding, steady or shrinking if*

1. $2(p+\sigma)f^{'}(\mathcal{T}) + 8\pi p > f(\mathcal{T})$,
2. $2(p+\sigma)f^{'}(\mathcal{T}) + 8\pi p < f(\mathcal{T})$,
3. $2(p+\sigma)f^{'}(\mathcal{T}) + 8\pi p = f(\mathcal{T})$, *respectively.*

From Theorem 6, we can further state:

**Corollary 10.** *Let the metric of $\Theta^\diamond$ be a GES. If $\rho$ is Killing and the scalars a and b are invariant along $\rho$, then*

| *$\Theta^\diamond$ represents* | $\mathfrak{EOS}$ | *Gradient Einstein soliton is expanding, steady and shrinking accordingly* |
|---|---|---|
| *Dark matter era* | $p = -\sigma$ | $\sigma \gtreqless \frac{f(\mathcal{T})}{8\pi}$ |
| *Stiff matter era* | $p = \sigma$ | $\sigma \gtreqless \frac{f(\mathcal{T})}{4(f'(\mathcal{T})+2\pi)}$, $f'(\mathcal{T}) + 2\pi \neq 0$ |
| *Radiation era* | $p = \frac{\sigma}{3}$ | $p \gtreqless \frac{f(\mathcal{T})}{8(3\pi+f'(\mathcal{T}))}$, $3\pi + f'(\mathcal{T}) \neq 0$ |
| *Dust matter era* | $p = 0$ | $\sigma \gtreqless \frac{f(\mathcal{T})}{2(f'(\mathcal{T})+4\pi)}$, $f'(\mathcal{T}) + 4\pi \neq 0$ |

If we fix $f(\mathcal{T}) = 0$, then $f(\mathcal{R}, \mathcal{T}) = \mathcal{R}$ and hence $f(\mathcal{R}, \mathcal{T})$-gravity reduces to the $EFE_s$. Thus, we state:

**Corollary 11.** *Let the PFST obeying $EFE_s$ admit a GES. If $\rho$ is Killing and the scalars a and b are invariant along $\rho$, then the GES is expanding, steady or shrinking if $\sigma \gtreqless 0$.*

Next, Equations (45) and (46) together with the hypothesis take the form

$$\mathcal{R}ic(\mathcal{F}, \mathcal{D}\psi) = a\mathcal{F}(\psi) \tag{52}$$

and

$$\mathcal{R}ic(\mathcal{F}, \mathcal{D}\psi) = -3\mathcal{F}(a) + \frac{1}{2}\mathcal{F}(b) - \rho(b)\eta(\mathcal{F}). \tag{53}$$

In view of (50)–(53) and $\mathcal{R} = 4a - b$, we conclude

$$a\mathcal{F}(\psi) + 2\mathcal{F}(a) = 0 \Longleftrightarrow a\mathcal{D}\psi + 2\mathcal{D}a = 0. \tag{54}$$

Considering a set of orthonormal frame and contracting Equation (42) along vector field $U$ and using the fact that $trace\Big\{\mathcal{F} \longrightarrow \frac{1}{2}(\nabla_{\mathcal{F}}Q)\mathcal{E}\Big\} = \frac{1}{2}\nabla_{\mathcal{E}}\mathcal{R}$, we lead

$$\mathcal{R}ic(\mathcal{F}, \mathcal{D}\psi) = -\mathcal{F}(\mathcal{R}) = a\mathcal{F}(\psi), \tag{55}$$

where (52) has been used. Again, from (22) and (50), we infer that

$$\mathcal{F}(\mathcal{R}) = 2\mathcal{F}(a) = \mathcal{F}(b). \tag{56}$$

In consequence of Equations (50) and $\mathcal{R} = 4a - b$, we conclude that the associated scalars $a$, $b$ and scalar curvature $\mathcal{R}$ are constant. Now, using these facts in (54), we have

$$a\mathcal{F}(\psi) = 0, \tag{57}$$

which implies that either $a = 0$ or $\mathcal{F}(\psi) = 0 \Longleftrightarrow \psi = \text{constant}$. If $a = 0$ and $\psi$ is a non-zero smooth function on $\Theta^{\diamond}$, then from (44), we have

$$\mathcal{R}ic = -\mathcal{R}\eta \otimes \eta, \tag{58}$$

where $\mathcal{R} = -b \neq 0$. From (58), we observe that $\Theta^{\diamond}$ is *Ricci simple* [34]. Next, we consider that $a \neq 0$ and $\mathcal{D}\psi = 0 \Longleftrightarrow \psi = \text{constant}$. Thus, the *GES* on $\Theta^{\diamond}$ is trivial. Now, we conclude our results as:

**Theorem 7.** *Let $\Theta^{\diamond}$ admit a GES. If $\rho$ is Killing and the associated scalars $a$, $b$ are invariant along $\rho$. Then either*
*$(i)$ the $\mathfrak{EOS}$ of $\Theta^{\diamond}$ is governed by (48) and the soliton is expanding, shrinking or steady accordingly as $\mathcal{R}$ is positive, negative or zero, respectively, or*
*$(ii)$ $\Theta^{\diamond}$ is Ricci simple or the GES is trivial.*

Let $a \neq b$ on $\Theta^{\diamond}$. If the metric of $\Theta^{\diamond}$ is a *GES*, $\rho$ is Killing and $a$, $b$ are invariant along $\rho$, then from Theorem 7 we notice:

**Corollary 12.** *Let $\Theta^{\diamond}$ admit a GES. If $\rho$ is Killing and $a$, $b$ are invariant along $\rho$, then the scalar curvature of M is constant.*

**Corollary 13.** *Let a $f(\mathcal{R}, \mathcal{T})(= \mathcal{R})$-gravity model with perfect fluid admit a GES. If $\rho$ is Killing and $a$, $b$ are invariant along $\rho$, then either*
*$(i)$ the $\mathfrak{EOS}$ is $\frac{p}{\sigma} = \frac{1}{3}$, represents the radiation era, or*
*$(ii)$ $\mathcal{R}ic = -\mathcal{R}\eta \otimes \eta$ or the GES is trivial.*

As a consequence of Theorem 7 and Equation (48), we have following observation.

**Theorem 8.** *Let $\Theta^\diamond$ admit a non-trivial GES. Suppose $\rho$ is Killing and $a$, $b$ are invariant along $\rho$, then evolution of the universe is given in the following table through $\mathfrak{EOS}$ of $\Theta^\diamond$ as:*

| $\mathfrak{EOS}$ ($\frac{p}{\sigma} = \omega$) | *Restrictions on $f^{'}(T)$ and $f(T)$* | *Evolution of the universe* |
|---|---|---|
| $\omega = 1$ | $f(\mathcal{T}) = -2\sigma(8\pi + f'(\mathcal{T}))$ | *Ultra relativistic era* |
| $\omega > -1$ | $f(\mathcal{T}) < 8\pi\sigma$ | *Quintessence era* |
| $\omega < -1$ | $f(\mathcal{T}) > 8\pi\sigma$ | *Phantom era* |
| $\omega = 0$ | $f(\mathcal{T}) = -\sigma(8\pi + f'(\mathcal{T}))$ | *Dust era* |

A smooth function $h : M \longrightarrow \mathbb{R}$ is said to be harmonic if $\Delta h = 0$, where $\Delta$ is the Laplacian operator on $M$ [35], we turn up the following conclusions:

**Theorem 9.** *Let $\Theta^\diamond$ admit a GES $(g, D\psi, \lambda)$ with harmonic function $\psi$ on $M$, then $(g, D\psi, \lambda)$ is expanding, steady and shrinking accordingly as*

1. $(p + \sigma)f'(\mathcal{T}) > 2f(\mathcal{T}) + 4\pi(3p - \sigma)$,
2. $(p + \sigma)f'(\mathcal{T}) < 2f(\mathcal{T}) + 4\pi(3p - \sigma)$, *and*
3. $(p + \sigma)f'(\mathcal{T}) = 2f(\mathcal{T}) + 4\pi(3p - \sigma)$.

**Proof.** From Equation (38), we can easily obtain the desired result. □

Let us choose $f(\mathcal{R}, \mathcal{T}) = \mathcal{R}$ in $\Theta^\diamond$, then we obtain $EFE_s$. In this case, Equation (38) reduces to $\lambda = -8\pi(3p - \sigma)$. Thus, we state:

**Corollary 14.** *Let the PFST obeying the $EFE_s$ without cosmological constant admit a GES with the harmonic potential function $\psi$. Then, the GES is shrinking, expanding or steady if $\frac{p}{\sigma} > \frac{1}{3}$, $\frac{p}{\sigma} < \frac{1}{3}$, or $\frac{p}{\sigma} = \frac{1}{3}$, respectively.*

A smooth function $\psi$ on a semi-Riemannian manifold $M$ is said to be harmonic, subharmonic and superharmonic if $\Delta\psi = 0$, $\Delta\psi \geq 0$ and $\Delta\psi \leq 0$, respectively. These definitions together with Equation (38) state the following:

**Theorem 10.** *Let $\Theta^\diamond$ admit a GES with the potential function $\psi$. Then $\psi$ is harmonic, subharmonic and superharmonic if $2[(p + \sigma)f'(\mathcal{T}) - 2f(\mathcal{T}) - 4\pi(3p - \sigma)] = \lambda$, $2[(p + \sigma)f'(\mathcal{T}) - 2f(\mathcal{T}) - 4\pi(3p - \sigma)] \geq \lambda$ and $2[(p + \sigma)f'(\mathcal{T}) - 2f(\mathcal{T}) - 4\pi(3p - \sigma)] \leq \lambda$, respectively.*

In view of Theorem 10, we state:

**Corollary 15.** *If a PFST satisfying $EFE_s$ admits a GES, then the potential function of GES is harmonic, subharmonic and superharmonic if $8\pi(\sigma - 3p) = \lambda$, $8\pi(\sigma - 3p) \geq \lambda$ and $8\pi(\sigma - 3p) \leq \lambda$, respectively.*

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## Acronyms

EFEs: Einstein's Field Equations, ES: Einstein soliton, GES: gradient Einstein soliton, GR: General Theory of Relativity, RS: Ricci soliton, DE: Dark Energy, PFST: Perfect Fluid Space Time, $\mathfrak{EOS}$: Equation of State.

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