Lorentzian wormholes without exotic matter in Eddington-inspired Born-Infeld gravity

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Within the framework of Eddington-inspired Born-Infeld (EiBI) gravity, we show that it is possible to construct a wide class of exact Lorentzian wormholes without violating the weak or null energy condition. The wormholes exist in a certain region of the parameter space spanned by the Eddington-Born-Infeld theory parameter and the parameters related to mass and energy density. Below the critical value of a parameter defined in our work, we have wormholes. Above the critical value, an event horizon is formed around the wormhole throat resulting in a regular black hole geometry. The traversability constraints on the wormholes, which restrict the tidal acceleration at the throat to values below one Earth gravity (g), lead to lower limits on the theory parameter and the throat radius. As a special case of our solution, we retrieve the wormhole supported by an electric field for a charge-to-mass ratio greater than the critical value 1.144.

1. Introduction

It is well known that the violation of energy conditions and hence the requirement of exotic matter are generic features of a wormhole in general relativity (GR) [1, 2]. However, this may not be true in modified or alternative gravity theories. A simple study on Raychaudhuri equation for a bundle of light rays passing through a wormhole, point out that, in general, the violation of the null convergence condition is a generic feature of a wormhole, not the violations of the energy conditions. In GR, the violation of null convergence condition is translated to the violation of null energy condition via Einstein field equation. This, in turn, leads to the violations of all other energy conditions (weak, strong, dominant, etc.). But, because of the modified field equations in some modified or alternative theories of gravity, a violation of null convergence condition may not lead to a violation of the energy conditions. Therefore, in such theories, we may have wormhole supported by non-exotic matter.

In this article, we review one such wormhole solution supported by non-exotic matter and obtained in [3] in the context of Eddington-inspired Born-Infeld gravity (EiBI) [4]. Born-Infeld type of gravitational action was first suggested by Deser and Gibbons [5], inspired by the earlier work of Eddington [6] and the nonlinear electrodynamics of Born and Infeld [7]. They considered metric formulation of the action. However, we shall focus on the Palatini formulation of the action with the matter coupling considered by Banados and Ferreira [4]. Banados and Ferreira's formulation of Born-Infeld gravity is commonly cited as Eddington-inspired Born-Infeld gravity (EiBI). Various aspects of EiBI gravity such as spherically symmetric solutions, cosmology, and astrophysical aspects, have been studied by many authors in the recent past [8, 9, 10, 11, 12]. Many authors have attempted to obtain wormhole solution in this theory. The matter supporting the wormhole solution obtained in [8] in three dimensional EiBI gravity, satisfies the energy condition for a negative cosmological constant. However, the energy conditions are violated for non-negative cosmological constant. The anisotropic fluid supporting the wormhole solution obtained in [9] in four dimension, has negative energy density and hence violates the energy conditions. The author in [10] obtained a wormhole solution supported by Maxwell electric field. However, we retrieve this wormhole solution as a special case of the general wormhole solution obtained in [3].

2. Wormholes, Raychaudhuri equations and energy conditions

The spacetime representing a static, spherically symmetric wormhole geometry is generically written as [1]

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1-\frac{b(r)}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

where $\Phi(r)$ and $b(r)$ are, respectively, the redshift function and the wormhole shape function. The wormhole throat is at $r = r_0$ such that $b(r_0) = r_0$. One of the necessary conditions to construct a traversable wormhole is that $\Phi(r)$

must be finite everywhere (no horizon condition). The spatial shape of the wormhole is visualized by embedding the t =constant, $\theta = \pi/2$ spatial section of the wormhole spacetime in background cylindrical coordinates (z, r, ϕ) system using an embedding function $z(r)$. Therefore, the line element on the embedding surface can be written as

$$ds_2^2 = dz(r)^2 + dr^2 + r^2 d\phi^2 = \left[1 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + r^2 d\phi^2.$$

Matching this with the t = constant, $\theta = \pi/2$ section of the metric in (1), one obtains the embedding equation $\frac{dz}{dr} = \pm\sqrt{\frac{b/r}{1-b/r}}$. The embedding diagram is obtained by taking the surface of revolution of the curve $z = z(r)$ by varying ϕ from 0 to 2π. The embedding diagram of a wormhole for a typical shape function $b(r)$ is shown in Fig. 1. The inverse embedding function $r = r(z)$ has a minimum at the throat. This is known as the minimality of the

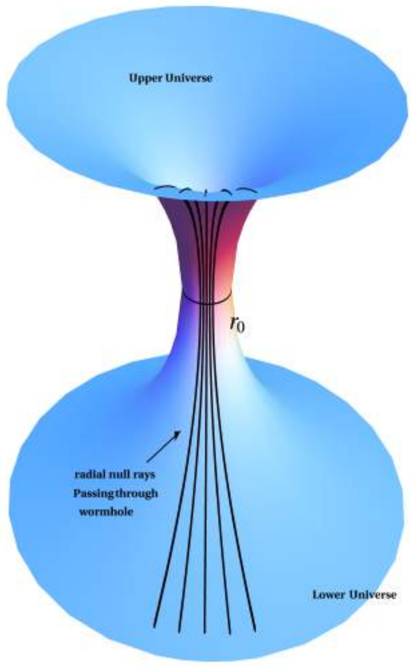

Figure 1. Embedding diagram of a wormhole.

wormhole throat. This leads to the well-known flare-out condition at the throat

$$\frac{d}{dz}\left(\frac{dr}{dz}\right) = \frac{b - b'r}{2b^2} > 0. \tag{2}$$

The minimality of the throat can be reinterpreted as divergent null rays passing through the throat. This requires the violation of the null convergence condition, from the Raychaudhuri equation. The Raychaudhuri equation for a bundle of light rays is given by

$$\frac{d\hat{\theta}}{d\lambda} + \frac{1}{2}\hat{\theta}^2 + \hat{\sigma}^2 - \hat{\omega}^2 + R_{\alpha\beta}\hat{u}^\alpha \hat{u}^\beta = 0,$$

where $\hat{\sigma}^2 = \hat{\sigma}_{\alpha\beta}\hat{\sigma}^{\alpha\beta}$, $\hat{\omega}^2 = \hat{\omega}_{\alpha\beta}\hat{\omega}^{\alpha\beta}$and $\hat{u}^\alpha$ is the four velocity of the light ray. There are also evolution equation for the shear ($\hat{\sigma}_{\alpha\beta}$) and rotation ($\hat{\omega}_{\alpha\beta}$) tensors [13]. For a radial null ray passing through the wormhole (see Fig. 1) in the equatorial plane ($\theta = \pi/2$), $\hat{u}^t = e^{-2\Phi(r)}$ and $\hat{u}^r = \pm e^{-\Phi(r)}\sqrt{1 - \frac{b(r)}{r}}$. Note that the family is ingoing at one side and outgoing at the other side of the throat. Therefore, the expansion for this family becomes $\hat{\theta} = \nabla_\alpha \hat{u}^\alpha = \pm\frac{2}{r}e^{-\Phi}\sqrt{1 - \frac{b(r)}{r}}$, where upper and lower signs are for outgoing and ingoing null rays, respectively. Note that the null expansion vanishes at the throat, i.e., $\hat{\theta}(r_0) = 0$. In the neighbourhood of the throat, the null expansion is positive at one side and negative at the other side of the throat. Along the radial null rays, null expansion goes from negative to zero at the throat and then becomes positive at the other side. Therefore, in the neighbourhood of the throat, $\frac{d\hat{\theta}}{d\lambda}$ is positive along this family. It can be shown that the rotation tensor $\hat{\omega}_{\alpha\beta} = (\partial_\beta \hat{u}_\alpha - \partial_\alpha \hat{u}_\beta)$ identically vanishes for this family. Also, $\hat{\sigma}^2 \geq 0$ since $\hat{\sigma}_{\alpha\beta}$ is spatial [13]. Therefore, in order to satisfy the Raychaudhuri equation for the family of radial null rays passing through the wormhole, we must have $R_{\alpha\beta}\hat{u}^\alpha \hat{u}^\beta < 0$. Note that, for a timelike velocity u^α, $R_{\alpha\beta}u^\alpha u^\beta \geq 0$ is known as timelike convergence condition. Therefore, **irrespective of the gravitational theory, the null convergence condition must be violated at the wormhole throat**.

In general relativity, after using the Einstein field equation $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$, one can show that the violation of null convergence condition, i.e., $R_{\alpha\beta}\hat{u}^\alpha \hat{u}^\beta < 0$ implies violation of null energy condition, i.e., $T_{\alpha\beta}\hat{u}^\alpha \hat{u}^\beta < 0$ which, in turn, implies the violation of other energy conditions. The matter violating energy conditions are termed as exotic matter. Therefore, exotic matter is needed to support a wormhole in general relativity. It should be noted that the null convergence condition and null energy condition are same in general relativity. However, in some alternative or modified theory of gravity, these two conditions are different in general. This is because of the modified field equation in these theories. Therefore, in such theories, we may have violation of the null convergence condition without violating the energy condition, i.e., we may have wormholes without exotic matter. Therefore, **in general, the violation of the null convergence condition is a generic feature of a wormhole, not the violation of the energy conditions**. In the subsequent sections, we show that such a wormhole solution, without exotic matter, is possible in Eddington-inspired Born-Infeld (EiBI) gravity.

3. Eddington-inspired Born-Infeld (EiBI) gravity

The action in EiBI gravity is given by [4]

$$S_{BI}[g, \Gamma, \Psi] = \frac{c^4}{8\pi G\kappa} \int d^4x \left[\sqrt{-|g_{\mu\nu} + \kappa R_{\mu\nu}(\Gamma)|} - \lambda\sqrt{-g}\right] + S_M(g, \Psi),$$

where c is the speed of light, G is Newton's gravitational constant, $\lambda = 1 + \kappa\Lambda$, $R_{\mu\nu}(\Gamma)$ is the symmetric part of the Ricci tensor built with the connection Γ and $S_M(g, \Psi)$ is the action for the matter field. Λ is the cosmological constant. Variations of this action with respect to the metric tensor $g_{\mu\nu}$ and the connection Γ yield, respectively [4, 10, 11],

$$\sqrt{-q}q^{\mu\nu} = \lambda\sqrt{-g}g^{\mu\nu} - \bar{\kappa}\sqrt{-g}T^{\mu\nu}, \qquad \nabla^\Gamma_\alpha \left(\sqrt{-q}q^{\mu\nu}\right) = 0, \tag{3}$$

where $\bar{\kappa} = \frac{8\pi G\kappa}{c^4}$, ∇^Γ denotes the covariant derivative defined by the connection Γ and $q^{\mu\nu}$ is the inverse of the auxiliary metric $q_{\mu\nu}$ defined by $q_{\mu\nu} = g_{\mu\nu} + \kappa R_{\mu\nu}(\Gamma)$. To obtain these equations, it is assumed that both the connection Γ and the Ricci tensor $R_{\mu\nu}(\Gamma)$ are symmetric, i.e., $\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}$ and $R_{\mu\nu}(\Gamma) = R_{\nu\mu}(\Gamma)$. Equation (3) gives the metric compatibility equation which yields

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}q^{\mu\sigma}\left(q_{\nu\sigma,\rho} + q_{\rho\sigma,\nu} - q_{\nu\rho,\sigma}\right).$$

Therefore, the connection $\Gamma^\mu_{\nu\rho}$ is the Levi-Civita connection of the auxiliary metric $q_{\mu\nu}$.

4. Wormhole solution in EiBI gravity

To obtain wormhole solution, we consider following metric ansatze for the physical and auxiliary metric:

$$ds_g^2 = -\psi^2(r)f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{4}$$

$$ds_q^2 = -G^2(r)F(r)dt^2 + \frac{dr^2}{F(r)} + H^2(r)(d\theta^2 + \sin^2\theta d\phi^2). \tag{5}$$

For the matter part, we consider an anisotropic fluid having an energy-momentum tensor of the form,

$$T^\mu_\nu = diag(-\rho, p_r, p_\theta, p_\theta) = diag(-\rho, -\rho, \alpha\rho, \alpha\rho).$$

It has been shown that the above $T^{\mu\nu}$ can be obtained from non-linear electrodynamics having action of the form [14],

$$S_M = \frac{1}{8\pi}\int d^4x\sqrt{-g}\left(-\frac{1}{2}F_{\mu\nu}F^{\mu\nu}\right)^{\frac{1+\alpha}{2\alpha}}. \tag{6}$$

For $\alpha = 1$, it reduces to the Maxwell electrodynamics action. The energy conservation equation can be integrated to obtain $\rho = \frac{C_0}{r^{2(\alpha+1)}}$, where C_0 is an integration constant. To satisfy the energy conditions, we must have $C_0 > 0$ and $0 \le \alpha \le 1$. The full solution for the physical metric is given by [3]

$$\psi(r) = \left[1 \mp \frac{r_0^{2(\alpha+1)}}{r^{2(\alpha+1)}}\right]^{-\frac{1}{2}}, \tag{7}$$

$$f(r) = \frac{1 \mp \frac{r_0^{2(\alpha+1)}}{r^{2(\alpha+1)}}}{1 \pm \alpha\frac{r_0^{2(\alpha+1)}}{r^{2(\alpha+1)}}}\left[1 - \frac{r_0^{(2\alpha+1)}}{3|\kappa| r^{2\alpha}} - \frac{2}{r\sqrt{1 \mp \frac{r_0^{2(\alpha+1)}}{r^{2(\alpha+1)}}}}\left(\bar{M} + \frac{(\alpha+1)r_0^{2(\alpha+1)}}{3|\kappa|}I(r)\right)\right], \tag{8}$$

$$I(r) = \int \frac{dr}{r^{2\alpha}\sqrt{1 \mp \frac{r_0^{2(\alpha+1)}}{r^{2(\alpha+1)}}}} = \begin{cases} \frac{2}{3}\log\left[\left(\frac{r}{r_0}\right)^{\frac{3}{2}} + \sqrt{\left(\frac{r}{r_0}\right)^3 \mp 1}\right] & : \alpha = \frac{1}{2} \\ \frac{r^{1-2\alpha}}{1-2\alpha}\,{}_2F_1\left[\frac{1}{2}, \frac{2\alpha-1}{2\alpha+2}, \frac{4\alpha+1}{2\alpha+2}; \pm\left(\frac{r_0}{r}\right)^{2\alpha+2}\right] & : \alpha \neq \frac{1}{2}, \end{cases} \tag{9}$$

where $r_0 = (|\kappa|C_0)^{\frac{1}{2(\alpha+1)}}$ and $\bar{M} = \frac{GM}{c^2}$, M being related to the mass. Here, upper and lower signs are for $\kappa < 0$ and $\kappa > 0$, respectively. For radial null rays

$$\left|\hat{\theta}\right| = \frac{2}{r}\sqrt{1 \mp \frac{r_0^{2(\alpha+1)}}{r^{2(\alpha+1)}}}, \qquad R_{\alpha\beta}\hat{u}^\alpha\hat{u}^\beta = \mp 2(\alpha+1)\frac{r_0^{2(\alpha+1)}}{r^{2(\alpha+2)}}. \tag{10}$$

We have seen that $\hat{\theta} = 0$ and $R_{\alpha\beta}\hat{u}^\alpha\hat{u}^\beta < 0$ at the wormhole throat. Therefore, from the above expression, it is clear that we must have $\kappa < 0$ to have wormhole solution and $r = r_0$ represents the wormhole throat. The traversability and no-horizon conditions demand $e^{2\Phi(r)} = \psi^2(r)f(r)$ to be non-zero, positive, and finite, in the range $r_0 \le r < \infty$. But, for $\kappa < 0$, $\psi^2 f$ diverges as $r \to r_0$. However, this divergence can be removed by taking the following relation between κ, M and r_0 [3]:

$$\bar{M} = -\frac{(\alpha+1)r_0^{2(\alpha+1)}}{3|\kappa|}I(r_0) = \begin{cases} 0 & : \alpha = \frac{1}{2} \\ \frac{(\alpha+1)r_0^3}{3(2\alpha-1)|\kappa|}\,{}_2F_1\left[\frac{1}{2}, \frac{2\alpha-1}{2\alpha+2}, \frac{4\alpha+1}{2\alpha+2}; 1\right] & : \alpha \neq \frac{1}{2}. \end{cases} \tag{11}$$

It is clear that, we must have $\alpha \ge \frac{1}{2}$ to satisfy the above condition for non-negative mass M. It has been shown in [3] that the invariant scalars such as Ricci scalar and the Kretschmann scalar are finite at the throat. But, they diverge at the throat if the above condition is not satisfied. At the throat, we have

$$1 - \frac{b(r)}{r}\Big|_{r_0} = f(r)\big|_{r_0} = 0, \quad e^{2\phi(r)}\big|_{r_0} = \psi^2(r)f(r)\big|_{r_0} = \frac{1}{\alpha+1}(1-x)$$

$$\frac{b - b'r}{2b^2}\Big|_{r_0} = \frac{f'}{2(1-f)^2}\Big|_{r_0} = \frac{1}{r_0}(1-x), \qquad x = \frac{r_0^2}{|\kappa|}$$

Therefore, to satisfy the flare-out condition as well as $\psi^2 f > 0$ at the throat, we must have $x < 1$, i.e., $r_0 < |\kappa|^{1/2}$. Since, for $x < 1$, $f = 0$ and $f' > 0$ at the throat, f does not have any zeroes at $r > r_0$. But, for $x > 1$, it has zero (giving an event horizon) at $r = r_h > r_0$. Therefore, for $x > 1$, an event horizon is formed around the throat, thereby giving a regular black hole solution. The critical value $x_c = 1$ distinguishes the wormhole and black hole solutions. For $\alpha = 1$, we retrieve the electrically charged solution discussed in [10]. The energy density $\rho = \frac{1}{8\pi}\frac{Q^2}{r^4}$ gives $C_0 = \frac{Q^2}{8\pi}$, where Q is the charge. In this case, we obtain the critical charge-to-mass ratio $\left(\frac{Q}{M}\right)_c \approx 1.144$ from equation (11) and $x_c = 1$. Therefore, we have wormhole for $\frac{Q}{M} > \left(\frac{Q}{M}\right)_c$ and black hole for $\frac{Q}{M} < \left(\frac{Q}{M}\right)_c$, where we have taken $G = 1$ and $c = 1$. This is similar to the Reissner-Nordström metric where we have naked singularity for $\frac{Q}{M} > 1$ and black hole for $\frac{Q}{M} < 1$. For a traversable wormhole, the tidal acceleration between two parts of a traveller's body travelling through the wormhole must be within a tolerable limit. For the radial tidal acceleration to be below one Earth gravity g, the Eddington-Born-Infeld theory parameter κ and the wormhole throat must be greater than the minimum values $r_{0min} = \sqrt{\frac{c^2}{3g}} \simeq 8.64 R_E$ and $|\kappa|_{min} \simeq 3.0 \times 10^{15} m^2$ [3], where $R_E \simeq 6400$ km is the Earth radius. However, for the solar constraint $|\kappa| \lesssim 1.8 \times 10^{14} m^2$ obtained by Casanellas $et\ al.$ [15], the minimum radial acceleration is given by $17g$.

5. Conclusion

We have seen that, in general, the violation of the null convergence condition is a generic feature of a wormhole not the violations of the energy conditions. In GR, a violation of null convergence condition leads to a violation of the energy conditions, thereby requiring exotic matter to support a wormhole. But, this is not true in EiBI gravity, in general. In this gravity theory, we have obtained an exact wormhole solution supported by non-exotic matter. We have chosen a special relation between the mass M, wormhole throat radius r_0 and the theory parameter κ to remove the singularity appearing at the throat. We have also obtained the critical value $x_c = \left(\frac{r_0^2}{|\kappa|}\right)_c = 1$ that distinguishes between wormhole and black hole. As a special case, we retrieve the wormhole solution supported by electric field. For this special solution, we obtain the critical charge-to-mass ratio $\left(\frac{Q}{M}\right)_c \approx 1.144$. We have wormhole for $\frac{Q}{M} > \left(\frac{Q}{M}\right)_c$ and black hole for $\frac{Q}{M} < \left(\frac{Q}{M}\right)_c$.

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