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Ehlers transformation and accelerating spacetimes with a gravomagnetic monopole

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A chi nella vita ha coraggio

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Introduction

In 1915 Albert Einstein presented what would have been in the next 100 years one of the most fundamentals and elegant physical theories, currently at the core of our understanding of nature: General Relativity. Continuously validated by experimental evidences, General Relativity is at the moment the theory of gravitation which best describes gravitational interactions and the geometric aspects of spacetime. In fact, gravity is interpreted as the physical manifestation of the spacetime's curvature, which is in turn determined by the mass-energy distribution. Within the theory, spacetime is modeled by a differentiable 4-dimensional manifold equipped with a lorentzian metric $g_{\mu\nu}$, whose curvature is determined according to Einstein's equations:

$$G_{\mu\nu} = 8\pi T_{\mu\nu},$$

where $G_{\mu\nu}$ is Einstein's tensor, which is defined by the metric tensor $g_{\mu\nu}$, and $T_{\mu\nu}$ is the stress-energy tensor, which describes the distribution of energy and matter within the spacetime. Despite the elegant form in which they are presented, Einstein's equations are in fact a coupled set of non-linear second order PDE's for the metric components $g_{\mu\nu}$, for which no general methods of resolution are known. Nonetheless, if we restrict the search of solutions to spacetimes which possesses peculiar symmetries, this complexity can be overcome and solutions of physical interest may be found, like the one given from Schwarzschild in 1916, which led to the prediction of black holes; the less symmetries the spacetime has the harder it gets finding new solutions. However, since finding solutions to Einstein's equations represents a crucial point in order to make predictions, generating techniques were developed along the years in order to produce new solutions within certain classes of spacetimes. In a series of paper, [1] and [2], Ernst introduced one of these techniques for stationary and axisymmetric spacetimes with no cosmological constant, in both vacuum and electrovacuum case. In particular, Ernst showed that in this particular class of spacetimes, Einstein's equations could be further simplified introducing two complex potentials, currently known as "Ernst potentials", in terms of which the equations assumed a particularly simple form. From the analysis of the equation's symmetries, five transformations which leaves the equations unchanged can be found. Now, whenever such a transformation is found, it may be used to map a "seed" solution into a new solution of the same equations. The five transformations deriving from Ernst formalism are mostly Gauge transformations, which maps the seed solution into itself, just in some other coordinates system. However, two of them are non-trivial and they maps a seed solution in a physically different solution; they are usually known as "Ehlers transformation" and "Harrison transformation". In particular, the Ehlers transformation may be used to add the so called "NUT parameter", which is a sort of magnetic analogue for the gravitational field, to any seed metric we want in a straightforward way. Although nutty spacetimes have so many non-physical properties that Misner was driven in [3] to define them "a counterexample to almost everything", research is still active for this type of solutions. Indeed, very recently Podolský and Vrátný, in [4], presented a new and more convenient form of the accelerating NUT metric, first found by Chng, Mann and Stelea in [5]. Studying this metric they discovered it to be a type I solution, which is quite unusual for black holes metrics, since the majority of them is of type D and lives inside the Plebański-Demiański family of type D solutions. Now, aware of the fact that the Ehlers transformation is capable of adding NUT to any metric we want, it is natural to ask ourselves if applying the Ehlers to the C-metric, which represents an accelerating black hole, we will find the same type I metric presented by Podolský and Vrátný or a type D version of it which lives inside the Plebański-Demiański family of solutions. In this work we shows that what we obtain is the same type I solution studied by Podolský and Vrátný; moreover, we highlight the fact that this procedure gives us directly the accelerating NUT metric in the convenient form presented by Podolský and Vrátný, avoiding Chng, Mann and Stelea generating method (which involves higher dimensions) and all the work needed to put the metric in the spherical-type coordinates system presented by Podolský and Vrátný.

In the first chapter we introduce stationary and axisymmetric spacetimes, which are the class of solutions concerning Ernst's generating technique. Einstein-Maxwell equations are then casted in the simple form given by Ernst in his two papers and Ernst potentials are introduced. The symmetries of the effective action are then presented, including the Ehlers transformation. In the second chapter we introduce an algebraic way to classify spacetimes based on their Weyl tensor, which is commonly known as "Petrov classification"; when we were referring above to type I or type D solutions, we were indeed talking about the metric's Petrov type. In the third chapter, we introduce the NUT parameter with its most common physical interpretations and we give a first example of how the Ehlers transformation works when applied to a seed metric. In particular, we apply the Ehlers to the Reissner-Nordström metric, obtaining the Reissner-Nordström-NUT metric. In the fourth chapter, we introduce accelerating spacetimes, including the accelerating NUT metric. We shows how Chng, Mann and Stelea produced this metric for the first time and then we produce the same metric using the Ehlers transformation on the C-metric. It will be evident that the Ehlers transformation represents a straightforward and controlled tool capable of adding NUT to any metric we want. In the last chapter, we take one more step further and we apply the Ehlers transformation to the accelerating Reissner-Nordström metric, producing the accelerating Reissner-Nordström-NUT solution. This metric, which we could not find in literature, is a type I solution as the simpler accelerating NUT and like that it lives outside the Plebański-Demiański family of type D solutions.

Along this work we will use the natural units $G = c = 1$, the signature of the metric will always be $(-,+,+,+)$ and the order of the coordinates tetrad will be (t, r, θ, φ).

Chapter 1

Ernst's generating technique

In 1968 Ernst showed in a series of paper, [1] and [2], that for any axially symmetric stationary spacetime (with no cosmological constant) it is possible to reformulate Einstein-Maxwell equations (which are Einstein equations coupled with Maxwell's electromagnetism) in terms of two complex potentials. In this first chapter thus we show how the costraints given by the symmetries of a stationary and axisymmetric spacetime, according to [6] and [7], reduce the most general metric to the Lewys-Weyl-Papapetrou metric. Afterwards, following [8], we employ this metric to find a simpler version of Einstein-Maxwell (EM) equations. In particular, we show how defining Ernst potentials further simplify EM equations. In the end, we introduce the transformations under whose action the EM equations are left unchanged.

1.1 Stationary and axisymmetric space-time

1.1.1 The LWP metric

The context where Ernst's method works is the one of stationary and axisymmetric spacetimes. These solutions are not only interesting from a physical point of view, since they describe the spacetime outside a rotating object at equilibrium, but from a mathematical point of view too, since the high degree of symmetry which these solutions possess greatly simplify Einstein equations.
First thing first, a spacetime is said to be *stationary* if it has a timelike Killing vector ξ^μ whose orbits are complete. Similarly, a spacetime is said to be *axisymmetric* if it possess a spacelike Killing vector ψ^μ whose orbits are closed; the closeness of the orbits implies that the Killing vector is associated to an angular coordinate. We call a spacetime stationary and axisymmetric if it possesses both symmetries and if, in addition, the Killing vector fields ξ^μ and ψ^μ commute.
The existence of these two Killing vectors together with the request that these two vectors commute enable us to choose a temporal coordinate and and angular coordinate whose associated coordinate vector field correspond to the two Killing vector fields. Having $(x_0 = t, x_1 = \varphi, x_2, x_3)$, so that $\partial_t = \xi^\mu$ and $\partial_\varphi = \psi^\mu$, the metric components will be independent of coordinates t and φ.
Although we reduced the number of coordinates involved from four to only two, we can further simplify the components of the metric assuming that the Killing vectors satisfies the hypotheses of the following:

Theorem 1 *Let ξ^μ and ψ^μ be two commuting Killing fields such that:*
(i) $\xi_{[a}\psi_b\nabla_c\xi_{d]}$ and $\xi_{[a}\psi_b\nabla_c\psi_{d]}$ each vanishes at least in one point of the spacetime (it is sufficient that either ξ^μ or ψ^μ vanishes at one point)
(ii) $\xi^a R_a^{[b}\xi^c\psi^{d]} = \psi^a R_a^{[b}\xi^c\psi^{d]} = 0$.
Then the 2-planes orthogonal to ξ^μ and ψ^μ are integrable.

This theorem is a direct application of Frobenius's theorem, which gives the compatibility conditions under which the integral curves of n vector fields mesh into coordinate grids on an n-dimensional integrable manifold. In particular, condition (i) defines circular spacetimes, while property (ii) defines Ricci-circular spacetimes. Ricci-circularity is trivially satified if $R_{\mu\nu} = 0$ everywhere, but it is also satisfied if $T_{\mu\nu}$ is the stress-energy tensor of a stationary and axisymmetric electromagnetic field, which we will always assume to be true. This means that if we choose the remaining two coordinates of our coordinate system from this 2-dim surface, orthogonal to the tangent space spanned by the Killing vectors, we will be able to transport them along the integral curves of the Killing fields, covering the rest of the spacetime. In other words, we will choose as third and fourth coordinate the intersections between the 2-planes orthogonal to the tangent space spanned by the Killing vectors and the isometry's orbits associated with the Killing vectors. As a consequence, we will not have in the line element of the metric mixed terms between the coordinates associated with the symmetries and the other two, because of the orthogonality of the 2-dim surface with the tangent space; the metric will then look like this:

$$g_{\mu\nu} = \begin{pmatrix} -V & W & 0 & 0 \\ W & X & 0 & 0 \\ 0 & 0 & g_{22} & g_{23} \\ 0 & 0 & g_{23} & g_{33} \end{pmatrix},$$

where $\quad -V = -\xi^\mu\xi_\mu, \quad W = \xi^\mu\psi_\mu, \quad X = \psi^\mu\psi_\mu$.

We have now 6 unknown functions of the metric in 2 variables, rather than the 10 functions of 4 variables that we started with. However, we can do better by choosing carefully the remaining coordinates x_2 and x_3.
We define the scalar function ρ as:

$$\rho^2 = VX + W^2, \tag{1.1}$$

which is minus times the sub-determinant of the $t - \varphi$ part of the metric. Assuming $\nabla_\mu\rho \neq 0$ we can take ρ as the x_2 coordinate. The fourth and last coordinate x_3 is the scalar function z, which is built so that $\nabla_\mu z \perp \nabla_\mu\rho$ holds, condition which cancel the g_{23} element of the metric. This is possible setting z constant along the integral curves of $\nabla^\mu\rho$.

Substituting X as function of ρ into the metric, thanks to (1.1), we obtain the following line element:

$$ds^2 = -V(dt - \omega d\varphi)^2 + \Omega^2(d\rho^2 + \Lambda dz^2) + V^{-1}\rho^2 d\varphi^2, \tag{1.2}$$

where $\omega = W/V$. This is the more general metric for a stationary and axisymmetric spacetime, whose Killing vectors satisfies the conditions of the theorem above.

One last improvement is obtained for vacuum spacetimes when we set $R_{\mu\nu} = 0$. In particular, the Ricci flat condition yields:

$$D^\mu D_\mu \rho = 0. \tag{1.3}$$

This condition implies two things:

- If $\rho \neq$ constant then $\nabla_\mu \rho = 0$ only at isolated points. Because our coordinate system brake down when $\nabla_\mu \rho = 0$, our coordinate system can fail only at isolated points.
- Λ in (1.2) is a function of z only. Moreover, z was defined up to a transformation which we can now use to set $\Lambda = 1$.[1]

Eventually, introducing $f = V$ and $\gamma = \frac{1}{2} \ln (V\Omega^2)$, we obtain the Lewis-Weyl-Papapetrou metric (LWP metric):

$$ds^2 = -f(dt - \omega d\varphi)^2 + f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2]. \tag{1.4}$$

Actually, this is the most general metric not only for a vacuum stationary and axisymmetric spacetime, but also if an electromagnetic field, for which (1.3) still holds, is present. In case we were dealing with a $T_{\mu\nu}$ for which (1.3) did not hold, we would have not been allowed to use the LWP, since the most general metric would have been (1.2). Whenever this is the case, it is appropriate to use, in (1.1), $\alpha(\rho)$ instead of ρ. The particular $T_{\mu\nu}$ will impose that $\alpha(\rho)$ need to satisfy a certain equation, that in our case is exactly the "wave equation" $\nabla^2\alpha(\rho) = 0$, for which ρ is a solution. That is the reason why we may call the sub-determinant ρ and everything works out fine. On the other hand, this argument is also the reason why Ernst's generating technique does not work with a cosmological constant. Indeed, the cosmological constant introduce a new term in the wave equation written above, so that ρ is not a solution anymore.

Lastly, we notice one of the advantages of Ernst's generating technique. Since all the functions depend at most from both z and ρ, whose differentials appear inside the metric in the same form as they do in flat spacetime, we can use flat differential operators as if we were in a flat spacetime.

1.1.2 Einstein-Maxwell equations

Every stationary and axisymmetric spacetime can be casted in the LWP metric, specifying f, ω and $e^{2\gamma}$. However, not all metrics can be interpreted as belonging to a realistic spacetime, because not all metrics are solutions of Einstein-Maxwell's equations. These equations then will put some costraints on the missing functions of the LWP metric, imposing them to be solution of some other differential equations. Eventually, finding the metric of the spacetime, i.e. solving Einstein-Maxwell equations, will consist in solving the differential equations found for the LWP metric's functions. This task will be much easier than solving Einstein-Maxwell equations from just the stress-energy tensor. In order to find these simple equations we will follow the same procedure described in [8].

In order to find these differential equations for f, ω and γ, the most straight-forward method is to build from the LWP metric all the quantities involved in Einstein-Maxwell equations and throw these inside of:

[1] $z \to z' = \int \Lambda^{1/2} dz$

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{1.5}$$

with

$$T_{\mu\nu} = \frac{1}{4\pi}(F_{\mu}^{\ \alpha} F_{\nu\alpha} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta}) \tag{1.6}$$

in the electrovacuum case.

Just like not every metric is solution of (1.5), not every electro-magnetic field can be used to build $T_{\mu\nu}$. The electro-magnetic field has to satisfy Maxwell (homogeneous) equations, which can be expressed concisely in a covariant form using the Faraday tensor:

$$\nabla_{\mu} F^{\mu\nu} = 0, \tag{1.7}$$

with

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}, \tag{1.8}$$

$A = A_t(\rho, z)dt + A_{\rho}(\rho, z)d\rho$ being the most general 1-form vector potential compatible with the symmetries of the spacetime.

Now, computing $\rho^{-1}\partial_{\mu}(\sqrt{-g}F^{\mu t}) = 0$ and $-\rho^{-1}\partial_{\mu}(\sqrt{-g}F^{\mu\varphi}) = 0$, with $g = -e^{4\gamma}\rho^2 f^{-2}$ being the determinant of the LWP metric (1.4), we found the simplest form of Maxwell equations:

$$\boldsymbol{\nabla} \cdot [\rho^{-2} f(\boldsymbol{\nabla} A_{\varphi} + \omega \boldsymbol{\nabla} A_t)] = 0 \tag{1.9}$$

$$\boldsymbol{\nabla} \cdot [f^{-1} \boldsymbol{\nabla} A_t - \rho^{-2} f \omega (\boldsymbol{\nabla} A_{\varphi} + \omega \boldsymbol{\nabla} A_t)] = 0. \tag{1.10}$$

We can do the same for Einstein-Maxwell equations too, although is a bit more cumbersome. First of all we notice that the stress-energy tensor is traceless, so taking the trace of (1.5) we find that $R = 0$. Then, we can rewrite (1.5) as:

$$H_{\mu\nu} := R_{\mu\nu} - 8\pi T_{\mu\nu} = 0. \tag{1.11}$$

Now taking H_{tt}, removing the multiplicative factors and dividing by ρ^2, gives us the first relevant equation:

$$f\nabla^2 f = (\boldsymbol{\nabla} f)^2 - \rho^{-2} f^4 (\boldsymbol{\nabla}\omega)^2 + 2f[(\boldsymbol{\nabla} A_t)^2 + \rho^{-2} f^2 (\boldsymbol{\nabla} A_{\varphi} + \omega \boldsymbol{\nabla} A_t)^2]. \tag{1.12}$$

To obtain the last equation that we need, we have to combine H_{tt} and $H_{\varphi\varphi}$ in a particular way and add to this combination equation (1.9) rescaled by a factor. To be more precise, we define:

$$h_1 := \frac{\omega H_{tt} + H_{t\varphi}}{\rho^4}.$$

Then, computing $h_1 - 4A_t$ we obtain the last equation:

$$\nabla \cdot [\rho^{-2} f^2 \nabla\omega + 4\rho^{-2} f A_t(\nabla A_\varphi + \omega \nabla A_t)] = 0. \tag{1.13}$$

We note right away that (1.13) and (1.12) concerns only the functions ω and f. The equations regarding the function γ, which can be computed from $H_{\rho z}$ and $H_{\rho\rho} - H_{zz}$, are the following:

$$H_{\rho z} = \frac{\partial_z \gamma}{\rho} + \frac{2\partial_z A_t \partial_\rho A_t}{f} - \frac{2f(\omega\partial_z A_t + \partial_z A_\varphi)(\omega\partial_\rho A_t + \partial_\rho A_\varphi}{\rho^2} + \\ - \frac{\partial_z f \partial_\rho f}{2f^2} + \frac{f^2 \partial_z \omega \partial_\rho \omega}{2\rho^2} = 0 \tag{1.14}$$

$$\frac{1}{2}(H_{\rho\rho} - H_{zz}) = \frac{\partial_\rho \gamma}{\rho} + \frac{(\partial_\rho A_t)^2 - (\partial_z A_t)^2}{f} - \frac{(\partial_\rho f)^2 - (\partial_z f)^2}{4f^2} + \\ + \frac{f^2}{4\rho^2}(\partial_\rho \omega)^2 - (\partial_z \omega)^2 + \frac{f}{\rho^2}[(\partial_z A_\varphi)^2 - (\partial_\rho A_\varphi)^2 + \omega^2[(\partial_z A_t)^2 \\ - (\partial_\rho A_t)^2 + 2\omega(\partial_z A_t \partial_z A_\varphi - \partial_\rho A_t \partial_\rho A_\varphi)] = 0. \tag{1.15}$$

Finally, we can stop to appreciate what we obtained thanks to the symmetries of the spacetime. We reduced the problem of solving 10 equations (partial differential equations, coupled and non-linear) to a set of 4 equations coupled in ω and f, but decoupled from the last two equations in γ. In the next section we will show how Ernst noticed that these partial differential equations could be simplified even more introducing what are now known as Ernst potentials, showing an incredible simple form regarding the equations for ω and f.

One last thing before moving on is that Ernst, in his paper [2], found out that these equations could be easily retrieved from the following Lagrangian density [2]:

$$\mathcal{L} = -\frac{1}{2}\rho f^{-2} \nabla f \cdot \nabla f + \frac{1}{2}\rho^{-1} f^2 \nabla\omega \cdot \nabla\omega + \\ + 2\rho f^{-1} A_t \nabla A_t \cdot \nabla A_t - 2\rho^{-1} f(\nabla A_\varphi + \omega\nabla A_t) \cdot (\nabla A_\varphi + \omega \nabla A_t). \tag{1.16}$$

1.2 Ernst potentials

In the same series of paper mentioned above, Ernst reformulated equations (1.12), (1.13), (1.10) and (1.9), in terms of two complex potentials. These potentials lead to an incredible simple form of Einstein-Maxwell equations.

First thing to notice is that eq. (1.9) can be seen as an integrability condition. Here we exploit the fact that the divergence of the cross product between the versor in the azimuthal direction and the gradient of a function which is independent from the azimuthal coordinate is always zero. The new expression is written in terms of another function $\tilde{A}_\varphi$, independent from the azimuthal coordinate

[2]Be aware of the fact that Ernst in [2] made all his calculations starting from the LWP metric with $+\omega$, instead of $-\omega$ as he says. A change of sign in front of every ω is required in his equations.

φ, which is introduced and defined here as:

$$\hat{\varphi} \times \boldsymbol{\nabla}\tilde{A}_{\varphi} := \rho^{-1} f(\boldsymbol{\nabla} A_{\varphi} + \omega \boldsymbol{\nabla} A_t), \tag{1.17}$$

with $\hat{\varphi}$ versor in the azimuth direction. It may be verified that:

$$\boldsymbol{\nabla} \cdot [\rho^{-1}(\hat{\varphi} \times \boldsymbol{\nabla}\tilde{A}_{\varphi})] = 0. \tag{1.18}$$

Now, if we use (1.17) to express (1.10) in terms of the new potential $\tilde{A}_{\varphi}$ we obtain:

$$\boldsymbol{\nabla} \cdot [f^{-1}\boldsymbol{\nabla} A_t - \rho^{-1}\omega(\hat{\varphi} \times \boldsymbol{\nabla}\tilde{A}_{\varphi})] = 0. \tag{1.19}$$

The latter inspire us to manipulate (1.17) in order to obtain a similar expression. This can be achieved applying $\boldsymbol{\nabla} \cdot (\hat{\varphi}\times)$ to both side of (1.17) and then taking the divergence. Dividing everything by f we obtain [3]:

$$\boldsymbol{\nabla} \cdot [f^{-1}\boldsymbol{\nabla}\tilde{A}_{\varphi} + \rho^{-1}\omega(\hat{\varphi} \times \boldsymbol{\nabla} A_t)] = \boldsymbol{\nabla} \cdot [-\rho^{-1}(\hat{\varphi} \times \boldsymbol{\nabla} A_{\varphi})] = 0, \tag{1.20}$$

where in the last equality we used (1.18).

Looking at (1.19) and (1.20) we see that they are the same equation, where the role of A_t and $\tilde{A}_{\varphi}$ has been swapped between each other. This fact suggest to introduce the complex electro-magnetic potential:

$$\boldsymbol{\Phi} := A_t + i\tilde{A}_{\varphi}. \tag{1.21}$$

This potential let us merge the two equations in the following unique expression:

$$\boldsymbol{\nabla} \cdot (f^{-1}\boldsymbol{\nabla\Phi} + i\rho^{-1}\omega\hat{\varphi} \times \boldsymbol{\nabla\Phi}) = 0. \tag{1.22}$$

We can now move to the Einstein-Maxwell equations (1.13) and (1.12) with the same procedure in mind. We start from (1.13), applying to it (1.17) we have:

$$\boldsymbol{\nabla} \cdot [\rho^{-2} f^2 \boldsymbol{\nabla}\omega + 4\rho^{-1}\hat{\varphi} \times A_t \boldsymbol{\nabla}\tilde{A}_{\varphi}] = 0. \tag{1.23}$$

Now, the divergence of the second term can be rewritten using the following identities [4]:

$$\boldsymbol{\nabla} \cdot [\rho^{-1}\hat{\varphi} \times A_t \boldsymbol{\nabla}\tilde{A}_{\varphi}] = -\boldsymbol{\nabla} A_t \times \boldsymbol{\nabla}\tilde{A}_{\varphi} = \boldsymbol{\nabla}\tilde{A}_{\varphi} \times \boldsymbol{\nabla} A_t = -\boldsymbol{\nabla} \cdot [\rho^{-1}\hat{\varphi} \times \tilde{A}_{\varphi} \boldsymbol{\nabla} A_t],$$

(1.23) then becomes:

$$\boldsymbol{\nabla} \cdot [\rho^{-2} f^2 \boldsymbol{\nabla}\omega + 2\rho^{-1}\hat{\varphi} \times \mathrm{Im}(\boldsymbol{\Phi}^* \boldsymbol{\nabla\Phi})] = 0. \tag{1.24}$$

As before (1.24) may be regarded as the integrability condition for the existence of a new potential

[3] The following expression can be useful $\hat{\varphi} \times \hat{\varphi} \times \boldsymbol{\nabla}\tilde{A}_{\varphi} = -\boldsymbol{\nabla}\tilde{A}_{\varphi}$

[4] Two properties are used here:

1. $\boldsymbol{\nabla} \cdot (\boldsymbol{A} \times \boldsymbol{B}) = (\boldsymbol{\nabla} \times \boldsymbol{A}) \cdot \boldsymbol{B} - \boldsymbol{A} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A})$
2. $\boldsymbol{\nabla} \times (\psi \boldsymbol{A}) = \psi \boldsymbol{\nabla} \times \boldsymbol{A} + \boldsymbol{\nabla}\psi \times \boldsymbol{A}$

χ. We define the latter such that [5]:

$$\hat{\varphi} \times \boldsymbol{\nabla}\chi := -\rho^{-1} f^2 \boldsymbol{\nabla}\omega - 2\hat{\varphi} \times \mathrm{Im}(\boldsymbol{\Phi}^* \boldsymbol{\nabla}\boldsymbol{\Phi}). \tag{1.25}$$

As usual the condition:

$$\boldsymbol{\nabla} \cdot [\rho^{-1}(\hat{\varphi} \times \boldsymbol{\nabla}\chi)] = 0 \tag{1.26}$$

is verified.

Again we may apply $\boldsymbol{\nabla} \cdot (\hat{\varphi}\times)$ to both sides of (1.25) and exploiting the fact that the divergence of the term in ω is zero, we find the final form of (1.13):

$$\boldsymbol{\nabla} \cdot [f^{-2}(\boldsymbol{\nabla}\chi + 2\,\mathrm{Im}(\boldsymbol{\Phi}^* \boldsymbol{\nabla}\boldsymbol{\Phi}))] = 0. \tag{1.27}$$

Finally, we express (1.12) in terms of χ and $\boldsymbol{\Phi}$. The second term of the right side may be obtained squaring (1.25), which gives us:

$$\rho^{-2} f^4 (\boldsymbol{\nabla}\omega)^2 = [\boldsymbol{\nabla}\chi + 2\,\mathrm{Im}(\boldsymbol{\Phi}^* \boldsymbol{\nabla}\boldsymbol{\Phi})]^2, \tag{1.28}$$

while the third term of the right side is obtained thanks to the definition of $\boldsymbol{\Phi}$:

$$2f \boldsymbol{\nabla}\boldsymbol{\Phi} \cdot \boldsymbol{\nabla}\boldsymbol{\Phi}^* = 2f[(\boldsymbol{\nabla}A_t)^2 + \rho^{-2} f^2 (\boldsymbol{\nabla}A_\varphi + \omega \boldsymbol{\nabla}A_t)^2]. \tag{1.29}$$

Substituting in (1.12) we obtain:

$$f\nabla^2 f = (\boldsymbol{\nabla}f)^2 - [\boldsymbol{\nabla}\chi + 2\,\mathrm{Im}(\boldsymbol{\Phi}^* \boldsymbol{\nabla}\boldsymbol{\Phi})]^2 + 2f \boldsymbol{\nabla}\boldsymbol{\Phi} \cdot \boldsymbol{\nabla}\boldsymbol{\Phi}^*. \tag{1.30}$$

If we define a new complex "gravitational" potential:

$$\boldsymbol{\mathcal{E}} := f - |\boldsymbol{\Phi}|^2 + i\chi, \tag{1.31}$$

(1.30) and (1.27) combine to give the remarkable form:

$$(\mathrm{Re}(\boldsymbol{\mathcal{E}}) + \boldsymbol{\Phi}\boldsymbol{\Phi}^*) \boldsymbol{\nabla}^2 \boldsymbol{\mathcal{E}} = \boldsymbol{\nabla}\boldsymbol{\mathcal{E}} \cdot (\boldsymbol{\nabla}\boldsymbol{\mathcal{E}} + 2\boldsymbol{\Phi}^* \boldsymbol{\nabla}\boldsymbol{\Phi}). \tag{1.32}$$

On the other hand, (1.22) takes the similar form:

$$(\mathrm{Re}(\boldsymbol{\mathcal{E}}) + \boldsymbol{\Phi}\boldsymbol{\Phi}^*) \boldsymbol{\nabla}^2 \boldsymbol{\Phi} = \boldsymbol{\nabla}\boldsymbol{\Phi} \cdot (\boldsymbol{\nabla}\boldsymbol{\mathcal{E}} + 2\boldsymbol{\Phi}^* \boldsymbol{\nabla}\boldsymbol{\Phi}). \tag{1.33}$$

These are called "Ernst equations", i.e. Einstein-Maxwell equations and Maxwell equations reformulated in terms of Ernst potentials. However, these are not sufficient to determine a particular metric field, because they contains no information regarding the γ function inside the LWP metric. The equations ruling γ were (1.15) and (1.14), which expressed in terms of Ernst potentials take the following form:

[5] We set the sign of the integrability condition as we did because otherwise the generating technique fails, i.e. this is the right sign for the new potential definition.

$$\partial_\rho\gamma(\rho,z) = \frac{\rho}{4(\mathrm{Re}(\boldsymbol{\mathcal{E}})+\boldsymbol{\Phi}\boldsymbol{\Phi}^*)^2}[(\partial_\rho\boldsymbol{\mathcal{E}}+2\boldsymbol{\Phi}^*\partial_\rho\boldsymbol{\Phi})(\partial_\rho\boldsymbol{\mathcal{E}}^*+2\boldsymbol{\Phi}\partial_\rho\boldsymbol{\Phi}^*)+ \\ -(\partial_z\boldsymbol{\mathcal{E}}+2\boldsymbol{\Phi}^*\partial_z\boldsymbol{\Phi})(\partial_z\boldsymbol{\mathcal{E}}^*+2\boldsymbol{\Phi}\partial_z\boldsymbol{\Phi}^*] - \frac{\rho}{\mathrm{Re}(\boldsymbol{\mathcal{E}})+\boldsymbol{\Phi}\boldsymbol{\Phi}^*}(\partial_\rho\boldsymbol{\Phi}\partial_\rho\boldsymbol{\Phi}^*-\partial_z\boldsymbol{\Phi}\partial_z\boldsymbol{\Phi}^*) \quad (1.34)$$

$$\partial_z\gamma(\rho,z) = \frac{\rho}{4(\mathrm{Re}(\boldsymbol{\mathcal{E}})+\boldsymbol{\Phi}\boldsymbol{\Phi}^*)^2}[(\partial_\rho\boldsymbol{\mathcal{E}}+2\boldsymbol{\Phi}^*\partial_\rho\boldsymbol{\Phi})(\partial_z\boldsymbol{\mathcal{E}}^*+2\boldsymbol{\Phi}\partial_z\boldsymbol{\Phi}^*)+ \\ -(\partial_z\boldsymbol{\mathcal{E}}+2\boldsymbol{\Phi}^*\partial_z\boldsymbol{\Phi})(\partial_\rho\boldsymbol{\mathcal{E}}^*+2\boldsymbol{\Phi}\partial_\rho\boldsymbol{\Phi}^*] - \frac{\rho}{\mathrm{Re}(\boldsymbol{\mathcal{E}})+\boldsymbol{\Phi}\boldsymbol{\Phi}^*}(\partial_\rho\boldsymbol{\Phi}\partial_z\boldsymbol{\Phi}^*-\partial_z\boldsymbol{\Phi}\partial_\rho\boldsymbol{\Phi}^*). \quad (1.35)$$

1.3 Effective action and symmetries

Ernst's generating technique uses the symmetries of Ernst equations, (1.32) and (1.33), to produce new solutions once a "seed" solution to start with is given. The idea behind it is that once we find a transformation under whose action the equations of motions are left unchanged, the same transformation may be applied to a solution of the equations themselves, giving us a new solution. However, symmetries are often not so trivial to be spotted at first glance and for this reason methods to find the finite transformations from the infinitesimal generators of the symmetries were developed.

Without getting into details about how to find these symmetries, it is sufficient to know for the purposes of this thesis that the following effective action for the complex field couple ($\boldsymbol{\mathcal{E}}$,$\boldsymbol{\Phi}$):

$$S[\boldsymbol{\mathcal{E}},\boldsymbol{\Phi}] := \int \rho d\rho dz \left[\frac{(\boldsymbol{\nabla}\boldsymbol{\mathcal{E}}+2\boldsymbol{\Phi}^*\boldsymbol{\nabla}\boldsymbol{\Phi})\cdot(\boldsymbol{\nabla}\boldsymbol{\mathcal{E}}^*+2\boldsymbol{\Phi}\boldsymbol{\nabla}\boldsymbol{\Phi}^*)}{(\boldsymbol{\mathcal{E}}+\boldsymbol{\mathcal{E}}^*+2\boldsymbol{\Phi}\boldsymbol{\Phi}^*)^2} - \frac{2\boldsymbol{\nabla}\boldsymbol{\Phi}\cdot\boldsymbol{\nabla}\boldsymbol{\Phi}^*}{\boldsymbol{\mathcal{E}}+\boldsymbol{\mathcal{E}}^*+2\boldsymbol{\Phi}\boldsymbol{\Phi}^*}\right], \quad (1.36)$$

from which Ernst equations can be derived, is left unchanged (and so are its equations of motion) under the action of five independent transformations:

$$\begin{array}{rlll}
I) & \boldsymbol{\mathcal{E}} \longrightarrow \boldsymbol{\mathcal{E}}' = \lambda\lambda^*\boldsymbol{\mathcal{E}} & \boldsymbol{\Phi} \longrightarrow \boldsymbol{\Phi}' = \lambda\boldsymbol{\Phi} & \\
II) & \boldsymbol{\mathcal{E}} \longrightarrow \boldsymbol{\mathcal{E}}' = \boldsymbol{\mathcal{E}}+ib & \boldsymbol{\Phi} \longrightarrow \boldsymbol{\Phi}' = \boldsymbol{\Phi} & \\
III) & \boldsymbol{\mathcal{E}} \longrightarrow \boldsymbol{\mathcal{E}}' = \dfrac{\boldsymbol{\mathcal{E}}}{1+ic\boldsymbol{\mathcal{E}}} & \boldsymbol{\Phi} \longrightarrow \boldsymbol{\Phi}' = \dfrac{\boldsymbol{\Phi}}{1+ic\boldsymbol{\mathcal{E}}} & (1.37) \\
IV) & \boldsymbol{\mathcal{E}} \longrightarrow \boldsymbol{\mathcal{E}}' = \boldsymbol{\mathcal{E}}-2\beta^*\boldsymbol{\Phi}-\beta\beta^* & \boldsymbol{\Phi} \longrightarrow \boldsymbol{\Phi}' = \boldsymbol{\Phi}+\beta & \\
V) & \boldsymbol{\mathcal{E}} \longrightarrow \boldsymbol{\mathcal{E}}' = \dfrac{\boldsymbol{\mathcal{E}}}{1-2\alpha^*\boldsymbol{\Phi}-\alpha\alpha^*\boldsymbol{\mathcal{E}}} & \boldsymbol{\Phi} \longrightarrow \boldsymbol{\Phi}' = \dfrac{\alpha\boldsymbol{\mathcal{E}}+\boldsymbol{\Phi}}{1-2\alpha^*\boldsymbol{\Phi}-\alpha\alpha^*\boldsymbol{\mathcal{E}}}, &
\end{array}$$

where λ,α and β are complex constants, while b and c are real ones.
These are the symmetries that can be used to map a solution into a new solution. Be aware of the fact that not all of the above transformations will give you a physically new metric, but just the same metric in different coordinates. Indeed, transformations (I), (II) and (IV) are just gauge transformations, i.e. once the new metric is produced a change of coordinates can take us back to the seed metric. On the other hand, transformations (III) and (V) are less trivial, because they produce metrics with different physical features than the ones of the seed. In particular, (III) is called Ehlers

transformation and it is known for adding a new parameter to the metric, called NUT parameter, which we are going to discuss later on. On the other hand, (V) is called Harrison transformation and it adds an electromagnetic field to the seed metric. For example, applying to the Schwarzschild metric the Harrison transformation we would obtain the Reissner–Nordström metric.

Finally, one could argue what was the need of introducing Ernst potentials. Was it not possible to deduce the symmetries right from the beginning, using the simple form equations from where Ernst started? Actually yes, but no. In fact, if one tries to deduce the symmetries from the equations, without introducing Ernst potentials, he would find just gauge transformations, then trivial. The remarkable feature of Ernst generating technique then is that, introducing Ernst potentials, it somehow captures the new physical information which leads to non trivial transformations, like the Ehlers and the Harrison.

Chapter 2

Petrov classification

Petrov classification is an algebraic way to classify metrics based on their Weyl tensors. In particular, Petrov classification is an important tool that can be employed in order to distinguish spacetimes, e.g. two metrics very different at first glance could be just the same metric written in two different coordinate systems. One hint that this could be the case would be to check that both metric have the same Petrov type.

There are multiple ways of approaching Petrov classification, all equivalents in a 4 dimensions Lorentzian manifold. However, here we are going to propose only two ways of classifying the Weyl, the second of which will be the definition employed to classify the metrics obtained along our work.

In the first approach we classify the Weyl upon the Segre characteristic of its Jordan form. The Weyl tensor, with its components W_{abcd}, due to its symmetries may be written in the "block index" form W_{AB}, where each capital index can assume a value between 1 and 6 and represents a pair of skew indeces according to the following scheme:

$$
\begin{aligned}
1 &\longrightarrow (2,3) \qquad & 2 &\longrightarrow (3,1)\\
3 &\longrightarrow (1,2) \qquad & 4 &\longrightarrow (1,4)\\
5 &\longrightarrow (2,4) \qquad & 6 &\longrightarrow (3,4).
\end{aligned}
$$

The Weyl is thus mapped in a 6×6 matrix. Moreover, the Weyl symmetry $W^a{}_{bac} = 0$ allow us to write the matrix with block indices W_{AB} like this:

$$
W_{AB} = \begin{pmatrix} A & B \\ B^T & -A \end{pmatrix},
$$

where A and B are 3×3 trace-free matrices, with A symmetric. The matrix above can be converted in a 3×3 complex matrix $D = A + iB$, which is symmetric and trace free too. Because of the trace free condition, for the matrix D there are just the following three Jordan forms:

Petrov type I

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix},$$

Petrov type II

$$\begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & -2\alpha \end{pmatrix},$$

Petrov type III

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

where subcases of Petrov type I and II are:

Petrov type D Subcase of type **I** when $\alpha = \beta$
Petrov type O Subcase of type **I** when $\alpha = \beta = \gamma = 0 \iff W = 0$
Petrov type N Subcase of type **II** when $\alpha = 0$

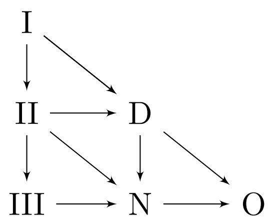

Figure 2.1: Schematic structure of the possible Petrov types. In the scheme we may see that the most general one is the Petrov type I, while all other Petrov type may be retrieved as sub-cases of it.

The second approach, which is the one we are going to use during this thesis, is based on what is commonly known as Newman-Penrose (NP) formalism, which is a “reformulation” of General Relativity in terms of some spinorial quantities. For our purposes, it is sufficient to know that NP formalism is a subcase of the more general tetrad formalism.
Assuming that we are dealing with a 4 dimensional Lorentzian manifold that can be interpreted as a spacetime, an orthonormal tetrad, also known as frame field, is a set of 4 point-wise orthogonal vector fields, one timelike and three spacelike, from which every tensorial quantity on the manifold can be built. A frame field carry the connection between the generalized coordinates of the spacetime and the local cartesian coordinates that can be defined in every point of the manifold. NP formalism take as tetrad four null vectors $(\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \bar{\boldsymbol{m}})$, which must satisfy the following conditions:

$$l^a k_a = -1, \qquad m^a \bar{m}_a = 1. \tag{2.1}$$

The 10 indipendent componenents of the Weyl tensors can be encoded in 5 complex scalars $\{\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4\}$, known as Weyl scalars. These scalars are defined as:

$$
\begin{aligned}
\Psi_0 &= W_{\kappa\lambda\mu\nu}\, k^\kappa\, m^\lambda\, k^\mu\, m^\nu \\
\Psi_1 &= W_{\kappa\lambda\mu\nu}\, k^\kappa\, l^\lambda\, k^\mu\, m^\nu \\
\Psi_2 &= W_{\kappa\lambda\mu\nu}\, k^\kappa\, m^\lambda\, \bar{m}^\mu\, l^\nu \\
\Psi_3 &= W_{\kappa\lambda\mu\nu}\, l^\kappa\, k^\lambda\, l^\mu\, \bar{m}^\nu \\
\Psi_4 &= W_{\kappa\lambda\mu\nu}\, l^\kappa\, \bar{m}^\lambda\, l^\mu\, \bar{m}^\nu .
\end{aligned}
\tag{2.2}
$$

The quantities above are relevant in terms of Petrov classification because, once we know which Weyl scalar is null, we know the type thanks to the following relations:

$$
\begin{aligned}
&\text{type } \mathbf{I} : \Psi_0 = 0 && \text{type } \mathbf{N} : \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0 \\
&\text{type } \mathbf{II} : \Psi_0 = \Psi_1 = 0 && \text{type } \mathbf{D} : \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0 \\
&\text{type } \mathbf{III} : \Psi_0 = \Psi_1 = \Psi_2 = 0 && \text{type } \mathbf{O} : \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0.
\end{aligned}
\tag{2.3}
$$

In the above relations it can easily be seen that Ψ_0 is always zero. However, it can happen that, chosen a null tetrad, when computing the Weyl scalars, Ψ_0 has a non zero value, since the value assumed by the Weyl scalars depends on the choice of the basis. So (2.3) just tell us that exist a specific null tetrad (maybe more than one) in which those scalars assume a zero value, forcing us to change null tetrad. Our null tetrad can be changed in the following ways:

(i) Null rotation around k:

$$
\boldsymbol{k}' = \boldsymbol{k} \qquad \boldsymbol{l}' = \boldsymbol{l} + L\bar{\boldsymbol{m}} + \bar{L}\boldsymbol{m} + L\bar{L}\boldsymbol{k} \qquad \boldsymbol{m}' = \boldsymbol{m} + L\boldsymbol{k} \tag{2.4}
$$

(ii) Null rotation around l:

$$
\boldsymbol{k}' = \boldsymbol{k} + K\bar{\boldsymbol{m}} + \bar{K}\boldsymbol{m} + K\bar{K}\boldsymbol{l} \qquad \boldsymbol{l}' = \boldsymbol{l} \qquad \boldsymbol{m}' = \boldsymbol{m} + K\boldsymbol{l} \tag{2.5}
$$

(iii) Lorentz boost:

$$
\boldsymbol{k}' = B\boldsymbol{k} \qquad \boldsymbol{l}' = B^{-1}\boldsymbol{l} \qquad \boldsymbol{m}' = e^{i\Phi}\boldsymbol{m} \tag{2.6}
$$

where L and K are complex, while B and Φ are real parameters. Together these three operations represent the six-parameter group of Lorentz transformations. We may rotate our null tetrad using (2.5) and see that Ψ_0 with the rotated tetrad becomes:

$$
\Psi_0' = \Psi_0 - 4K\Psi_1 + 6K^2\Psi_2 - 4K^3\Psi_3 + K^4\Psi_4. \tag{2.7}
$$

Imposing $\Psi_0' = 0$ and solving the quartic equation we get in K, we can actually find a tetrad where the Weyl scalar vanish. Thus, imposing $\Psi_0' = 0$ we obtain the quartic equation in K (for each K we have a new null vector $\boldsymbol{k}$):

$$
\Psi_0 - 4K\Psi_1 + 6K^2\Psi_2 - 4K^3\Psi_3 + K^4\Psi_4 = 0. \tag{2.8}
$$

Since this is a quartic expression in K, there are 4 complex roots, not necessarily distinct. Each of the null vectors $\boldsymbol{k'}$, obtained by rotation around $\boldsymbol{l}$ using as parameter the roots K_i of (2.8), are called *principal null directions* (*pnd* s).

These principal null directions can be used to identify the Petrov type of a spacetime:

type **I**: four distinct *pnd*s
type **II**: one *pnd* of multiplicity 2, the others are distinct
type **D**: two distinct *pnd*s of multiplicity 2
type **III**: one *pnd* of multiplicity 3, the others are distinct
type **N**: one *pnd* of multiplicity 4
type **O**: conformally flat.

However, our initial goal was to find a null tetrad where (2.3) would be valid. The null tetrad is the one having the null vector $\boldsymbol{k}$ aligned with the principal null direction which has the highest multiplicity. In the case of type **D** both null vectors $\boldsymbol{k}$ and $\boldsymbol{l}$ has to be aligned with the two *pnd*s doubly degenerate.

Petrov types and *pnd*s can be interpreted from a physical point of view too. For example, type **I** and type **D** describes stationary spacetimes, where no gravitational radiation is allowed. On the other hand, repeated *pnd*s of type **II**, **III** and **N** can be interpreted as the directions of gravitational radiation.

Be aware of the fact that the Petrov type is a point-wise property of the spacetime, i.e. we can have regions of different algebraic types in the same spacetime. However, remarkably most of known solutions have the same Petrov type in each point of the spacetime.

Lastly, when the computation of the Weyl scalars becomes heavy, it is useful to know that the following condition holds just for type I solutions, i.e. algebraically general spacetimes:

$$I^3 \neq 27J^2, \tag{2.9}$$

where

$$I = \Psi_0\Psi_4 - 4\Psi_1\Psi_3 + 3{\Psi_2}^2 \qquad J = \begin{vmatrix} \Psi_0 & \Psi_1 & \Psi_2 \\ \Psi_1 & \Psi_2 & \Psi_3 \\ \Psi_2 & \Psi_3 & \Psi_4 \end{vmatrix}.$$

Chapter 3

Example: Reissner-Nordström-NUT black hole

In this chapter we show how the Ehlers transformation can be used to produce, from the Reissner–Nordström metric, the Reissner-Nordström-NUT metric. The procedure we will follow to produce a new solution is quite general and does not depend in a relevant way on which metric we take as seed, although the more parameters the seed has the more complicated the computation will be. However before doing it, we will give a brief introduction about the NUT parameter, including some of its critical issues and its consequences on a black hole when present.

3.1 The NUT parameter

The NUT parameter is a continuous parameter which appears in the Taub-NUT solution (and its generalisations), which is often quoted in the form:

$$ds^2 = -f(r)(dt - 2l\cos\theta\, d\varphi)^2 + \frac{dr^2}{f(r)} + (r^2 + l^2)(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{3.1}$$

where

$$f(r) = \frac{r^2 - 2mr - l^2}{r^2 + l^2} \tag{3.2}$$

and l is the "NUT parameter" or "NUT charge". While the physical meaning of the parameter m, at least in the limit where $l \to 0$, is the mass of the source, the meaning of the NUT parameter depends on the interpretation we choose, since there is no unique interpretation for it. Moreover, whichever interpretation we choose for the NUT parameter, the Taub-NUT solution yields multiple nonphysical features which led Misner to define it "a counterexample to almost everything", like the fact that in the stationary region (called NUT region) the spacetime containes closed timelike curves. On the other hand, one of the most surprising features is the fact that there is no curvature singularity inside the black hole.

Since there is no curvature singularity in $r = 0$, is not clear where the source of this gravitational field should be located nor what could possibly generate this spacetime. However, the Taub-NUT solution possesses a torsion singularity which can be restricted to half of its symmetry axis and which can be interpreted in different ways, according to the interpretation we choose for the NUT parameter.

3.1.1 The gravomagnetic interpretation

The NUT parameter can be interpreted as the analogue of the magnetic monopole for the gravitational field.
The idea derive, says Bičák in [9], from the work done by Lynden-Bell and Nouri-Zonoz (1998) but it somehow can be traced back to Newton. In one of his scholia Newton discusses motion under the standard central force of gravitation plus a force which is perpendicular to the instantaneous plane of motion. Indeed, as described by Bičák, it is interesting to consider the motion of a particle according to the equation of motion:

$$m_1 \ddot{\boldsymbol{r}} = -V'(r)\, \hat{e}_r + \frac{m_0}{c} \dot{\boldsymbol{r}} \times \boldsymbol{B}, \tag{3.3}$$

where

$$\boldsymbol{B} = -\frac{Q}{r^2}\, \hat{e}_r \qquad \text{and} \qquad Q = \tilde{Q} \frac{c}{m_0}. \tag{3.4}$$

$\boldsymbol{B}$ can be though as the field of a “gravomagnetic” monopole of charge Q. Such a motion could correspond to an elliptic orbit in a plane with a wedge missing, and the removal of the wedge would result in an advance of the perihelion in successive orbits. This is actually what happens to test particles in the stationary regions of the Taub-NUT solution when l is present. Though, the source of the magnetic-like field has to be connected to the NUT parameter l, leading to the gravomagnetic interpretation. This interpretation of the NUT parameter as dual charge of the mass is supported by the fact that whenever it is present some components of the Weyl tensor which would be otherwise turned off are instead turned on, in analogy with the components of the Faraday tensor. Moreover, Lynden-Bell and Nouri-Zonoz also proved that all geodesics of NUT space lie on spatial cones. This has interesting consequences for the gravitational lensing properties of the spacetime, i.e. light rays are not just bent as they pass near the origin, but they are also twisted [1]. The differential twisting produces a characteristic spiral shear, peculiar to such gravomagnetic monopoles.

In the gravomagnetic interpretation of the NUT parameter the torsion singularity is justified by the presence of a Misner string, which is the analogue of a Dirac string for the gravitational field. Indeed, in the theory of magnetic monopoles, two of these objects are always connected to each other with a Dirac string and when we have just one of this monopoles we justifies its presence sending the other end of the string at infinity. In fact, the Misner string affects the spacetime, while Dirac strings have no effect at all. The Misner string seems more like an artifice to justify the presence of the gravomagnetic monopole than a real physical object and it could probably be eliminated by some kind of interaction, like Ernst did for the cosmic string, which we are going to see later. On the other hand, recently Clément, Gal'tsov and Guenouche in [10], showed that the presence of the Misner

[1] Because of this twist, l may be regarded as a twist parameter

string is not as problematic as the majority think it is, since the string is transparent to geodesics, with the significant consequence that the spacetime is geodesically complete. Moreover, despite the presence of closed timelike curves surrounding the string, for certain values of the parameter used to fix the location of the string, these closed timelike curves are not causal geodesics, so that causality is not violated.

3.1.2 Alternative interpretations

Another interpretation of the singular half axis is the one given by Bonnor (1969). Bonnor's interpretation of the singularity is that of a semi-infinite massless source of angular momentum, i.e. a semi infinite spinning rod. This is justified by two arguments. The first is that in the limit where l is small and $m = 0$ the metric perturbation correspond to that of a semi-infinite spike of angular momentum. The second one, is that (3.3) can actually be written in the form:

$$m_1 \ddot{\boldsymbol{r}} = -V'(r)\, \hat{e}_r - 2m_1 \boldsymbol{\Omega} \times \dot{\boldsymbol{r}}, \tag{3.5}$$

in which the term $2m_1 \boldsymbol{\Omega} \times \dot{\boldsymbol{r}}$ is the Coriolis force associated with the local frame of reference. In this interpretation the presence of a singularity is not unexpected, since a rotating field cannot be distributed over the surface of a sphere without the occurrence of at least one singularity. This spinning rod, although unlikely, seems more plausible than the Misner string and more importantly, seems more suitable to be replaced by some kind of interaction.

An alternative interpretation of the Taub-NUT solution was given by Misner (1963), where the axis of symmetry is forced to be regular. This is done by taking the metric just in the range $0 < \theta < \pi/2$ and joining it in $\theta = \pi/2$ with the other hemisphere $\pi/2 < \theta < \pi$, where the transformation $t \longrightarrow t' = t - 4l\varphi$ has been applied to the metric. Altough this procedure removes the singularity from the axis, in order to be consistent the coordinate t have to be periodic with a periodicity of $8\pi l$. This implies that all particles have to move along closed timelike curves.

3.2 Adding NUT to Reissner-Nordström

First of all, we present the Reissner-Nordström metric. The metric represents a static, spherically symmetric, asymptotically flat spacetime and it is commonly written in the following spherical-type coordinates:

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{3.6}$$

where m and e are the parameters representing respectively the mass of the source and his charge, assuming that $t \in (-\infty, +\infty)$, $r \in (0, \infty)$, $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi)$. The metric is accompanied by an electromagnetic field $F = dA$, with the vector potential given by:

$$A = -\frac{e}{r} dt. \tag{3.7}$$

The solution can be interpreted as the field generated by a charged black hole and the metric reduces to the Schwarzschild solution when $e = 0$. Besides, it has been proven that this is the unique[2] spherically symmetric, asymptotically flat solution of Einstein-Maxwell equation, being in addition static by Birkhoff's theorem. Thus, the exterior region of any bounded, spherically symmetric distribution of charged mass has to be described by the Reissner-Nordström metric.

Now applying the Ehlers transformation means changing the Ernst potentials of the seed metric according to (1.3), but in order to do that we first need to know what does $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\Phi}$ looks like for the Reissner-Nordström metric. These potentials are built from the vector potential components and by the functions f and ω of the LWP metric, which can be read comparing the LWP with the Reissner-Nordström metric, which is just the LWP metric where f, ω and $e^{2\gamma}$ had been written out explicitly. Since the two metrics are written in different coordinates systems, the first task will be to find out which is the change of coordinate we need in order to pass from the Weyl (cylindrical like) coordinates of the LWP to the spherycal-type coordinates used in (3.6), which is not trivial. The easiest part of the task is to find ρ as a function of r and θ, because we can read it directly by comparing the $d\varphi$ terms of both metrics, finding out that:

$$\rho(r, \theta) = \sin\theta \sqrt{r^2 - 2mr + e^2}. \tag{3.8}$$

A bit more complicated is finding z, because comparing directly the $d\rho$ and dz terms of LWP with the dr and $d\theta$ terms of Reissner-Nordström will give rise to an annoying system, containing the $e^{2\gamma}$ function too, which we still don't know. Nonetheless, the whole problem is simplified when we exploit the fact that in the Reissner-Nordström metric there is no mixed term $drd\theta$, which implies:

$$\frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta} = -\frac{\partial \rho}{\partial r}\frac{\partial \rho}{\partial \theta}.$$

Surprisingly, the stronger condition holds:

$$\frac{\partial z}{\partial r} = \frac{\partial \rho}{\partial \theta} \qquad \frac{\partial z}{\partial \theta} = -\frac{\partial \rho}{\partial r}. \tag{3.9}$$

Now it is easy to find that:

$$z(r, \theta) = \cos\theta \, (r - m). \tag{3.10}$$

We see that the coordinate change is a generalisation of the usual change from cylindrical to spherical coordinates used in a flat spacetime. Parameters like the mass and the electric charge modify the geometry and so they play a role in the change.

At this point, it would be natural to write the $d\rho^2 + dz^2$ term of LWP in terms of r and θ, in order to find the functions f, ω and $e^{2\gamma}$ by comparison with (3.6). However, it comes into play the fact that (1.34) and (1.35) are invariant under the action of the Ehlers transformation, i.e. γ does not change under the transformation. Because of this additional symmetry, we can read the whole $e^{2\gamma}(d\rho^2 + dz^2)$ in the Reissner-Nordström metric, as well as the other functions:

[2] The uniqueness is referred to the dyonic generalisation of the Reissner-Nordström spacetime, which essentially, being a duality transformation away from the solution we are currently using, represents the same solution.

$$f(r) = \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) \tag{3.11}$$

$$e^{2\gamma}(d\rho^2 + dz^2) = dr^2 + f(r)r^2 d\theta^2 \tag{3.12}$$

$$\omega = 0.$$

Before writing down the Ernst potentials, we note that since we passed from Weyl coordinates to a spherical-type coordinate system, the gradient operator, which before was the usual flat gradient in cylindrical coordinates, now has changed form in a non trivial way, according to (3.8) and (3.10). It assumes the following form:

$$\boldsymbol{\nabla} f(r,\theta) = \frac{1}{\sqrt{(r-m)^2 - (m^2 - e^2)\cos^2\theta}} \left(\frac{\partial f(r,\theta)}{\partial r} \sqrt{r^2 - 2mr + e^2}\; \hat{e}_r + \frac{\partial f(r,\theta)}{\partial \theta}\; \hat{e}_\theta \right).$$

Now, using (1.17) and (1.25)[3] we find that $\tilde{A}_\varphi$ and χ are constants which can be set to zero by the gauge freedom. Then, Ernst potential are simply:

$$\boldsymbol{\Phi} = -\frac{e}{r} \qquad \boldsymbol{\mathcal{E}} = 1 - \frac{2m}{r}. \tag{3.13}$$

We can now transform Ernst potentials according to (1.3), obtaining:

$$\boldsymbol{\Phi}' = \frac{\boldsymbol{\Phi}}{1 + ic\boldsymbol{\mathcal{E}}} = A'_t + i\tilde{A}'_\varphi \tag{3.14}$$

$$\boldsymbol{\mathcal{E}}' = \frac{\boldsymbol{\mathcal{E}}}{1 + ic\boldsymbol{\mathcal{E}}} = f' - |\boldsymbol{\Phi}'|^2 + i\chi' \tag{3.15}$$

and use (3.13) to get the following functions:

$$A'_t(r) = -\frac{er}{r^2 + c^2(r-2m)^2} \tag{3.16}$$

$$\tilde{A}'_\varphi(r) = \frac{ec(r-2m)}{r^2 + c^2(r-2m)^2} \tag{3.17}$$

$$f'(r) = \frac{r^2 - 2mr + e^2}{r^2 + c^2(r-2m)^2} \tag{3.18}$$

$$\chi'(r) = -\frac{c(r-2m)^2}{r^2 + c^2(r-2m)^2}. \tag{3.19}$$

We see that the Ehlers have introduced a non zero $\tilde{A}'_\varphi$ and χ', to which will be connected a non zero A'_φ and ω'. Indeed, using again (1.17) and (1.25) together with the functions written above, we find:

[3] The two definitions need to be manipulated in order to have the gradient of $\tilde{A}_\varphi$ and χ and not the cross product of their gradient

$$A'_\varphi(r,\theta) = \frac{4mcer\cos\theta}{r^2 + c^2(r-2m)^2} - ce\cos\theta \tag{3.20}$$

$$\omega'(\theta) = 4mc\cos\theta. \tag{3.21}$$

If we now use the primed functions to write the LWP metric, leaving (3.12) unchanged, we obtain the following new metric:

$$ds^2 = -\frac{Q_0(r)}{R_0^2}(dt - 4mc\cos\theta\, d\varphi)^2 + \frac{R_0^2}{Q_0(r)}dr^2 + R_0^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{3.22}$$

where

$$Q_0(r) = r^2 - 2mr + e^2$$
$$R_0^2 = r^2 + c^2(r-2m)^2.$$

First of all we verify that both the metric and the vector potential found satisfy respectively Einstein-Maxwell equations and Maxwell equations. We can identify this metric as the Reissner-Nordström-NUT metric with some arguments, like the absence of a curvature singularity, the fact that the metric is not asymptotically flat and that it is a type D metric. However, the confirm is given by the fact that under the following transformation of coordinates:

$$r \longrightarrow \sqrt{\frac{r_+}{r_+ - r_-}}(r - r_-), \qquad t \longrightarrow \sqrt{\frac{r_+}{r_+ - r_-}}\, t \tag{3.23}$$

and parameters:

$$2m \longrightarrow \sqrt{\frac{r_+}{r_+ - r_-}}(r_+ - r_-), \qquad e \longrightarrow \sqrt{\frac{r_+}{r_+ - r_-}}\, e, \qquad c \longrightarrow \frac{l}{r_+}, \tag{3.24}$$

where l is NUT parameter, $r_+ = m + \sqrt{m^2 + l^2}$ and $r_- = m - \sqrt{m^2 + l^2}$, we can actually retrieve the Reissner-Nordström-NUT metric in its canonical form:

$$ds^2 = -\frac{Q(r)}{R^2}(dt - 2l\cos\theta\, d\varphi)^2 + \frac{R^2}{Q(r)}dr^2 + R^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{3.25}$$

with

$$Q(r) = r^2 - 2mr - l^2 + e^2,$$
$$R^2 = r^2 + l^2.$$

It is noteworthy the fact that quantities such as r_+ and r_- play a significant role in the coordinates transformation, although we are not considering a pure Taub-NUT spacetime. When we apply the Ehlers transformation to the Schwarzschild spacetime, in order to get a Taub-NUT spacetime, the use of these quantities seems completely natural since they correspond to the radial positions of the horizons in the Taub-NUT solution. However, using others seed metrics with more parameters, such as the Reissner-Nordström metric, we would probably expect the relevant quantities involved in the transformation to be the radial positions of the new metric's horizons, i.e. in our current case $r_\pm = m \pm \sqrt{m^2 + l^2 - e^2}$. The reason could stand behind the fact that usually the mass m has to be mapped in $\sqrt{m^2 + l^2}$ and for this task the horizon's positions of the Taub-NUT solution seems more suitable. Moreover, a series of important identities make this quantities very easy to work with:

$$\begin{aligned} r_+ + r_- &= 2m \\ r_+ - r_- &= 2\sqrt{m^2 + l^2} = 2M \geqslant 0 \\ r_+ r_- &= -l^2 \\ r_+(r_+ - r_-) &= r_+^2 + l^2. \end{aligned} \tag{3.26}$$

The transformation used above to retrieve the canonical form of the metric is not sufficient to retrieve the vector potential usually associated to the metric. The reason why the transformation is not sufficient is that the Ehlers transformation, besides rotating in some sense part of the mass into the NUT parameter, rotates the electromagnetic field too. So, to retrieve the vector potential we need to realign the electromagnetic field through a duality rotation applied to the electromagnetic Ernst potential:

$$\mathbf{\Phi} \longrightarrow \bar{\mathbf{\Phi}} = \mathbf{\Phi}\, e^{i\beta}. \tag{3.27}$$

This is actually a special unitary sub-case of the more general transformation (I)-(1.3), when $\lambda = e^{i\beta}$. Performing the rotation and using (1.17), we find the new components of the vector potential in terms of the rotation's parameter:

$$\bar{A}_t(r) = -\frac{e\,[r\cos\beta + c(r-2m)\sin\beta]}{r^2 + c^2(r-2m)^2} \tag{3.28}$$

$$\bar{A}_\varphi(r,\theta) = -c\, e\cos\theta\cos\beta + e\cos\theta\sin\beta - \omega'(\theta)\bar{A}_t(r). \tag{3.29}$$

We have performed the rotation before transforming coordinates and parameters. Now, if we choose:

$$\cos\beta = \sqrt{\frac{r_+}{r_+ - r_-}} \tag{3.30}$$

and then transform coordinates and parameters according to (3.23) and (3.24), we retrieve the vector potential in its usual form:

$$A = -\frac{er}{r^2 + l^2}(dt - 2l\cos\theta d\varphi). \tag{3.31}$$

Lastly, looking at the canonical form of the vector potential we may see why we chose that specific value for $\cos\beta$. The reason is that the right choice of β is the one which cancel the first two terms on the right side of (3.29), leaving us with $\bar{A}_\varphi = -\omega'\bar{A}_t$.

Chapter 4

Generating accelerated NUT black holes

In this chapter we introduce the C-metric, i.e. a non-asymptotically flat solution which represents a couple of accelerating causally separated black holes, and we investigate some current explanations of the black hole's acceleration. Moreover, we introduce the Plebański-Demiański family of type D solutions and we show why the accelerating NUT solution does not belong to it. Eventually, we focus our attention on the accelerating NUT metric. We explain how the metric was found for the first time by Chng, Mann and Stelea in [5] (2006) and then reformulated in a spherical-type coordinates system by Podolský and Vrátný in [4] (2020). At the end, the same metric is produced using the Ehlers transformation with the C-metric as seed, showing the convenience of Ernst generating technique.

4.1 C-metric

The C-metric, with its analytic continuation, describes a pair of causally separated black holes which accelerate away from each other. It may be written using spherical-type coordinates, as presented in [11], in the following form:

$$ds^2 = \frac{1}{(1 - \alpha\, r\cos\theta)^2}\left(-\frac{Q(r)}{r^2}\, dt^2 + \frac{r^2\, dr^2}{Q(r)} + \frac{r^2\, d\theta^2}{P(\theta)} + P(\theta)\, r^2 \sin^2\theta\, d\varphi^2\right), \tag{4.1}$$

with

$$P(\theta) = 1 - 2\alpha m\cos\theta, \qquad Q(r) = \left(r^2 - 2mr\right)(1 - \alpha^2 r^2). \tag{4.2}$$

The spacetime, considering the region where $r > 0$, may be divided in three regions according to its Killing horizons, which are the typical event horizon of the black hole and an acceleration horizon. The only static region of these, where r is a spatial coordinate and t is a timelike coordinate, is the one between the two horizons. Thus, we will always assume later on that $2m < r < 1/\alpha$, which are respectively the radial positions of the two horizons.

Moreover, we notice that setting $\alpha = 0$ we recover the Schwarzschild solution, so it's natural to continue assuming in the C-metric that m is the mass of the source and r is the radial coordinate[1]. On the other hand, we still have to argue that the parameter α represents the acceleration of the black holes.

4.1.1 The Minkowski limit

In order to understand the physical meaning of the parameter α, we look at the metric when $m = 0$:

$$ds^2 = \frac{1}{(1-\alpha\, r\cos\theta)^2}\left(-(1-\alpha^2 r^2)\, dt^2 + \frac{dr^2}{(1-\alpha^2 r^2)} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right). \tag{4.3}$$

If we compute the curvature tensor through the Weyl scalars, using the following null tetrad:

$$\begin{aligned} \boldsymbol{k} &\equiv \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{-g_{tt}}}\partial_t + \frac{1}{\sqrt{g_{rr}}}\partial_r\right) \\ \boldsymbol{l} &\equiv \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{-g_{tt}}}\partial_t - \frac{1}{\sqrt{g_{rr}}}\partial_r\right) \\ \boldsymbol{m} &\equiv \frac{1}{\sqrt{2}}\left(\sqrt{\frac{g_{tt}}{D}}\partial_\varphi + \frac{g_{t\varphi}}{\sqrt{D}\, g_{tt}}\partial_t - \frac{i}{\sqrt{g_{xx}}}\partial_x\right), \end{aligned} \tag{4.4}$$

where $D = g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2 < 0$ is the subdeterminant of the C-metric and $x := \cos\theta$, we find out that the only non zero Weyl scalar is:

$$\Psi_2 = -m\left(\frac{1}{r} - \alpha\cos\theta\right)^3. \tag{4.5}$$

Besides noting that in $r = 0$ there is the usual Schwarzschild curvature singularity, we see that when $m = 0$ the spacetime is conformally flat, i.e. (4.3) must represent at least a part of the Minkowski spacetime. Indeed, we may apply the following transformation:

$$\hat{\zeta} = \frac{\sqrt{|1-\alpha^2 r^2|}}{\alpha(1-\alpha r\cos\theta)}, \qquad \hat{\rho} = \frac{r\sin\theta}{1-\alpha r\cos\theta}, \tag{4.6}$$

with $\tau = \alpha t$, without changing φ. Under this transformation the metric (4.3) in its static region with $r < 1/\alpha$, takes the usual form of the Rindler spacetime:

$$ds^2 = -\hat{\zeta}^2 d\tau^2 + d\hat{\zeta}^2 + d\hat{\rho}^2 + \hat{\rho}^2 d\varphi^2. \tag{4.7}$$

This can be put in the standard form of the Minkowski spacetime by the transformation $\hat{T} = \pm\hat{\zeta}\sinh\tau$, $\hat{Z} = \pm\hat{\zeta}\cosh\tau$, where $\hat{\zeta} \in [0,\infty)$ and $\tau \in (-\infty,\infty)$, with the two signs giving two copies of the same region. We can do the same with the region of C-metric where $r > 1/\alpha$, using the final transformation $\hat{T} = \pm\hat{\zeta}\cosh\tau$, $\hat{Z} = \pm\hat{\zeta}\sinh\tau$. Then, these four regions forms the complete Minkowski spacetime and they are separated by the null hypersurfaces $\hat{T} = \pm\hat{Z}$, where $r = 1/\alpha$; the null hypersurfaces are acceleration horizons. As a matter of fact, we may actually compute the timelike wordline $x^\mu(\lambda)$ of an observer moving with $r = \tilde{r} = costant$, $\theta = 0$ and $\varphi = 0$, with λ being

[1]Actually r may be considered as the conformal radial coordinate, since the canonical radial coordinate is usually identified as the term that multiplies the sphere element, which in our current discussion would be $r/(1-\alpha r\cos\theta)$

the proper time of the observer. The wordline is found by the condition $u_\mu u^\mu = -1$ of the 4-velocity u^μ, defined by $u^\mu = dx^\mu(\lambda)/d\lambda$, and reads as follows:

$$x^\mu(\lambda) = \left(\frac{1 - \alpha\, \tilde{r} \cos\theta}{\sqrt{1 - \alpha^2\, \tilde{r}^2}}\, \lambda,\ \tilde{r},\ 0,\ 0 \right). \tag{4.8}$$

The 4-acceleration of the observer is defined as $a^\mu = (\nabla_\nu u^\mu) u^\nu$ and computing its magnitude for $\tilde{r} = 0$ we find that:

$$|a| = \sqrt{a_\mu a^\mu}|_{\tilde{r}=0} = \alpha \tag{4.9}$$

Thus, we see that a test particle located in $r = 0$ has an acceleration exactly equal to α. We can conclude then the origin of the metric (4.3) is being accelerated with uniform acceleration α, i.e. α is the parameter which represents, at least in the weak field limit, the acceleration of the black holes.

4.1.2 The reason of the acceleration: cosmic strings

To understand where the source of the acceleration come from, we look at the ratio between the length of the circunference and the radius of a small circle around the half axis $\theta = 0$, with t and r constants. The computation gives:

$$\frac{\text{circunference}}{\text{radius}} = \lim_{\theta \to 0} \frac{2\pi P(\theta) \sin\theta}{\theta} = 2\pi(1 - 2m\alpha). \tag{4.10}$$

We can do the same thing around $\theta = \pi$, finding that:

$$\frac{\text{circunference}}{\text{radius}} = 2\pi(1 + 2m\alpha). \tag{4.11}$$

Now, the fact that the ratio between circumference and radius is less than the usual value of 2π can be explained in the following way: imagine the flat spacetime and cut out a slice of it, gluing the two edges of the spacetime back together. At the end of the procedure the spacetime will be affected by a topological defect called "conical singularity". The C-metric then on its "north" and "south" poles have two conical singularity with different conicity.

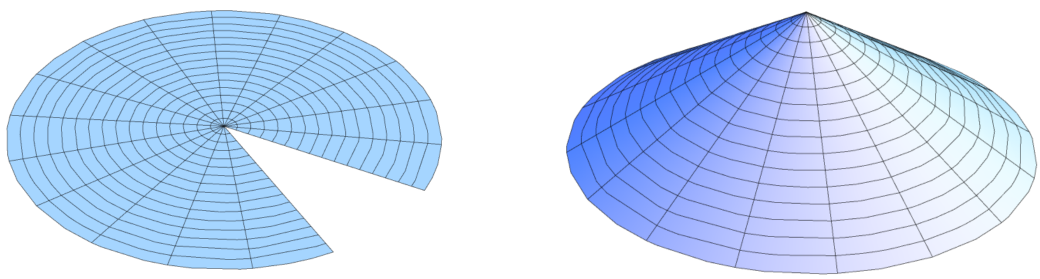

Figure 4.1: Removing a wedge from a flat disk, (which may represents all spatial sections of the Minkowski spacetime) and gluing back together the new edges produces a cone. We see then that redefining the range of the azimuthal angular coordinate produces a conical singularity, which may be interpreted as a model for a cosmic string.

The conical singularity, in flat spacetime, has a global effect on geodesics which pass either side of it. This mimics the behaviour of an attractive gravitational source even tough the curvature does not seem to diverge near the singularity.

In general, as explained in [11], when a conical singularity is present, a distributional component occurs in the Ricci tensor and this corresponds to a non zero stress component in the energy-momentum tensor. This is the reason way we usually imagine the conical singularity to be generated by what is commonly known as "cosmic string", i.e. an infinite line source under constant tension. This interpretation is not in contrast with the fact that we are considering an electro-vacuum space-time without matter, because this cosmic string may be represented by a δ-like stress-energy tensor. A δ-like $T_{\mu\nu}$ does not change the metric nor the equations of motions, since the metric is not well defined on the axis, although it causes scalar quantities to diverge along the axis. In the C-metric this infinite line source called cosmic string is taken as the source of the black hole's acceleration, once the angular excess at $\theta = \pi$ has been removed with the extension of the angular coordinate φ to the range $[0, 2\pi C)$, with $C = (1 + 2\alpha m)^{-1}$.

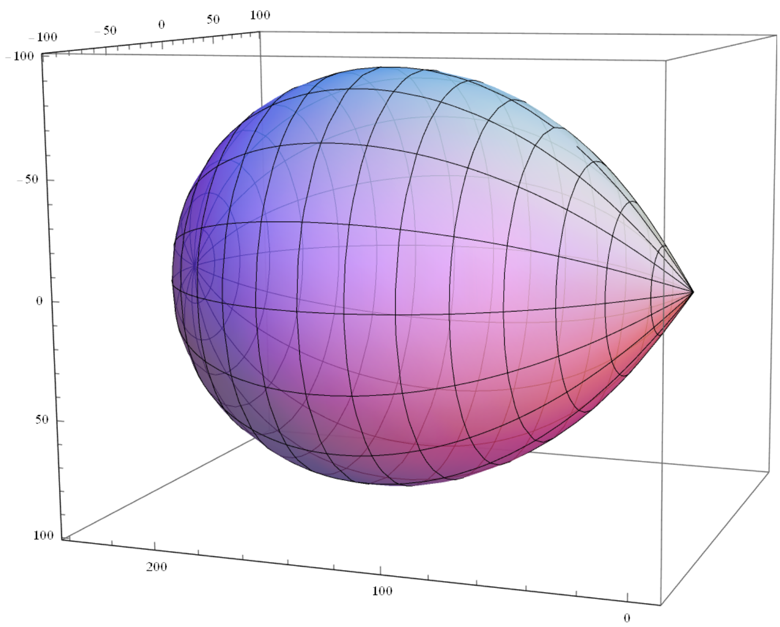

Figure 4.2: The event horizon of the regularized version of the C-metric (4.13), embedded in $\mathbb{R}^3$. The sharp edge is located at $\theta = 0$, while the conical singularity in $\theta = \pi$ has been removed.

This procedure is reasonable, since when a topological defect with an angular excess is present, its justification is given by the presence of an infinite strut, which "push" the black hole, instead of "pulling" it, as the cosmic string does. However, the matter should have a repulsive behaviour in order to build a strut, which seems then just a nonphysical object. Indeed, as explained in the appendix of [12], the non-vanishing components of the stress-energy tensor deriving from Einstein-Maxwell equations takes the following form:

$$T_t^t = T_\varphi^\varphi = 2\pi\mu\delta(x, y), \tag{4.12}$$

where

$$\mu = \frac{1-C}{4C}$$

As already mentioned, C represents the angular excess/deficit of the conical singularity, while μ represents the tension of the filament source of matter. We see from (4.12) that when we have $C < 1$, i.e. an angular deficit (cosmic string), the energy density associated to the stress-energy tensor is positive, although we still have diverging components. Conversely, when we are dealing with a strut, i.e. $C > 1$, the density of energy associated to the stress-energy tensor is negative, which means that the strut is composed of anti-gravitational matter. As we mentioned above, the strut may be cancelled rescaling the angular coordinate φ by a specific constant C, mapping then φ into $\tilde{\varphi} = C\varphi$; choosing $C = (1+2m\alpha)^{-1}$, we may cancel the angular excess computed in (4.11). The strut-less version of the C-metric takes then the following form:

$$ds^2 = \frac{1}{(1-\alpha\, r\cos\theta)^2}\left(-\frac{Q(r)}{r^2}\, dt^2 + \frac{r^2 dr^2}{Q(r)} + \frac{r^2 d\theta^2}{P(\theta)} + \frac{P(\theta)\, r^2 \sin^2\theta\, d\varphi^2}{(1+2\alpha m)^2}\right). \tag{4.13}$$

4.1.3 A more physical attempt to explain the acceleration

The reality of cosmic strings and struts seems quite unlikely from a phenomenological point of view. In fact, already Ernst in [13] (1976) thought that these nodal singularities were just a consequence of the neglect interaction that caused the black hole's acceleration. In his paper, Ernst was able to regularize both conical singularities of a charged C-metric, through the embedding of the latter in the Melvin universe and the tuning of the interaction between the intrinsic charge of the black hole and the background's electromagnetic field; see [14] for the dyonic charged generalisation, which is rotating. A similar embedding is possible thanks to transformation (V) of (1.3), also called "Harrison transformation". After two years in [15] (1978), Ernst suggested that the same thing could be done with the uncharged version of the C-metric, eliminating the conical singularity through a purely gravitational interaction between the black hole and the background.

Very recently, this idea was further developed in [16] (2022). In that paper a magnetic version of the Ehlers transformation is used to embed an accelerating Kerr black hole into a swirling universe (a sort of gravitational whirlpool as background) as shown in [17] (2022). Then, it is shown that it is possible to regularize the conical singularities of the metric thanks to a fine tuning of the parameter C mentioned above and the parameter introduced by the Ehlers transformation, which in its magnetic version represents the angular velocity of the background. The acceleration of the black hole thus can be justified by the "force" exerted by the spin-spin interaction between the angular momentum (for unit mass) of the black hole and the rotation of the gravitational background, without the necessity of strings or struts.

Eventually, imposing the regularisation of both conical singularities and that the acceleration parameter and the Ehlers parameter assume small values, we may retrieve a first order approximation for the Newtonian force responsible for the acceleration, which is:

$$mA \approx 4ja,$$

where A is the acceleration, a is the angular momentum (per unit of mass) of the black hole and j is the parameter which regulates the angular velocity of the background.

4.2 The Plebański-Demiański family of type D solutions

Plebański-Demiański spacetimes are a large class of type D black holes, which includes all the well known solutions, such as Schwarzschild, Kerr, Taub-NUT, Kerr-Newman, Reissner-Nordström and the C-metric. Recently, a new form of the Plebański-Demiański family was given by Podoslký and Vrátný in [18] (2021). The main advantage is that the new metric depends only on the typical parameters of all the solutions mentioned above. Thus, it is possible to retrieve all the relevant type D black holes from one single metric, just turning off the parameters that does not concern the particular metric. However, the metric itself, with all parameters turned on can be interpreted as a type D accelerating, rotating, NUT dyonic black hole. The new form presented by Podoslký and Vrátný employs spherical-type coordinates and takes the following form:

$$ds^2 = \frac{1}{\Omega^2}\Bigg(-\frac{Q}{\rho^2}\left[dt - \left(a\sin^2\theta + 4l\sin^2\frac{\theta}{2}\right)d\varphi\right]^2 + \frac{\rho^2}{Q(r)}dr^2 \\ + \frac{\rho^2}{P}d\theta^2 + \frac{P}{\rho^2}\sin^2\theta[adt - (r^2 + (a+l)^2)d\varphi]^2\Bigg), \quad (4.14)$$

where

$$\Omega = 1 - \frac{\alpha\, a}{a^2 + l^2} r(l + a\cos\theta) \quad (4.15)$$

$$\rho^2 = r^2 + (l + a\cos\theta)^2 \quad (4.16)$$

$$P(\theta) = \left(1 - \frac{\alpha\, a}{a^2 + l^2}\tilde{r}_+(l + a\cos\theta)\right)\left(1 - \frac{\alpha\, a}{a^2 + l^2}\tilde{r}_-(l + a\cos\theta)\right) \quad (4.17)$$

$$Q(r) = (r - \tilde{r}_+)(r - \tilde{r}_-)\left(1 + \alpha a\frac{a - l}{a^2 + l^2}r\right)\left(1 - \alpha a\frac{a + l}{a^2 + l^2}r\right), \quad (4.18)$$

being m, α, e, a, l and g, respectively the mass, acceleration, electric charge, Kerr-like rotation, NUT parameter and magnetic charge. Besides, $\tilde{r}_+$ and $\tilde{r}_-$ represents the two black hole horizons and they are located at:

$$\tilde{r}_\pm = m \pm \sqrt{m^2 + l^2 - a^2 - e^2 - g^2}.$$

Since we are interested in the accelerating (and non rotating) NUT metric, it is surprising to notice that this is the only solution that does not belong to the Plebański-Demiański family. Indeed, if we try to turn off simultaneously e, g and a we see that the parameter α disappear too, since it always appears in the metric multiplied by the parameter a. The reason why the accelerated NUT solution lives outside the Plebański-Demiański family is that this metric is not a type D solution, but rather a type I one, so algebraically special. This is argued by Podolský and Vrátný in [4] and will be confirmed by this work.

In addition, it can be easily verified that all the seed metrics used in this thesis belongs to the Plebański-Demiański family. Indeed, turning off l, a and g inside (4.14), we retrieve the accelerated Reissner-Nordström metric that we are going to use as seed in the last chapter; if we then turn off α too, we get the Reissner-Nordström metric, while turning off e (instead of α) we retrieve the uncharged C-metric.

4.3 Chng, Mann and Stelea accelerating NUT solution

In 2006 Chng, Mann and Stelea presented the metric of an accelerating NUT black hole, which takes the following form:

$$d\bar{s}^2 = -\frac{(y^2-1)F(y)}{\alpha^2(x-y)^2}\frac{C^2\delta}{\bar{H}(x,y)}\left[d\bar{t} + \frac{1}{C}\left(\frac{(1-x^2)F(x)}{\alpha^2(x-y)^2} + \frac{2Mx}{\alpha}\right)d\varphi\right]^2 + \frac{\bar{H}(x,y)}{\alpha^2(x-y)^2}\left[(1-x^2)F(x)d\varphi^2 + \frac{dx^2}{(1-x^2)F(x)} + \frac{dy^2}{(y^2-1)F(y)}\right], \quad (4.19)$$

where

$$F(x) = 1 + 2\alpha Mx$$
$$F(y) = 1 + 2\alpha My$$
$$\bar{H}(x,y) = \frac{1}{2} + \frac{\delta}{2}\left(\frac{(y^2-1)F(y)}{\alpha^2(x-y)^2}\right)^2,$$

and the parameter δ is the one related to the NUT parameter. Now, the thing we are most interested about is to highlight how much work was put into the generation of this metric, in order to show how the application of the Ehlers transformation represents a more efficient and more controlled approach.

4.3.1 Chng, Mann and Stelea generating method

The procedure followed by Chng, Mann and Stelea in order to produce the accelerating NUT metric is in some sense analogous to the Ehlers transformation. In particular, we both use the symmetries of Einstein-Maxwell equations as transformations employed to generate new solutions. In fact, they exploit a combination of these transformation in order to produce the metric given above.

Initially, they start from the ansatz of the Lewis-Weyl-Papapetrou metric written in the following form:

$$ds_4^2 = -e^{-\psi}dt^2 + e^{\psi}[e^{2\mu}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] \quad (4.20)$$

and they take its reduction in three dimensions along the timelike direction. Considering the three dimensional reduction they take the reduced Lagrangian from which the equations of motion can be derived, writing it in the following form:

$$\mathcal{L}_3 = \sqrt{g}R - \frac{1}{2}\sqrt{g}\Delta\psi. \quad (4.21)$$

From this lagrangian two of its symmetries are employed. The first one maps the pair of functions (ψ, μ) into the pair $(\gamma\psi, \gamma^2\mu)$, where γ is a real parameter. This “scaling” transformation, if applied to the Schwarzschild metric, maps the latter into the Zipoy-Voorhees metric, which can be though as an axisymmetric generalization of the Schwarzschild solution describing the gravitational field of a rod with generic mass and length.

The second symmetry employed is taken from the $SL(2,R)$ group of symmetries of the reduced lagrangian and it is an alternative version of the Harrison transformation - transformation (V) of (1.3). Indeed, this $SL(2,R)$ symmetry can be used to retrieve the Reissner-Nordström solution from the Schwarzschild solution taken as a seed metric. We will not get into the details of how this symmetry is able to charge the seed metric (see [5]), but we will just limit ourselves to follow the combination of transformations necessary to build the new solution.

The seed metric and the first step

The seed metric taken as starting point is an accelerated version of the Zipoy-Voorhees solution, presented by Teo in [19] (2006), which referring to the (4.20) functions reads:

$$e^{-\psi} = \frac{(y^2-1)F(y)}{A^2(x-y)^2}\left(\frac{F(y)}{F(x)}\right)^{\alpha-1}, \qquad e^{2\mu} = \frac{(y^2-1)F(y)}{f(x,y)G(x,y)}\frac{F(y)^{\alpha^2-1}F(x)^{(\alpha-1)^2}}{G(x,y)^{\alpha^2-1}}, \tag{4.22}$$

where

$$\begin{aligned} F(\xi) &= 1+2mA\xi \\ f(x,y) &= (y^2-1)F(x)+(1-x^2)F(y) \\ G(x,y) &= [1+mA(x+y)^2]^2 - m^2A^2(1-xy)^2, \end{aligned}$$

while the canonical Weyl coordinates ρ and z in terms of the pair of coordinates (x,y) are defined as:

$$\begin{aligned} \rho^2 &= \frac{(y^2-1)(1-x^2)F(x)F(y)}{A^4(x-y)^4}, \qquad z = \frac{(1-xy)[1+mA(x+y)]}{A^2(x-y)^2}, \\ d\rho^2+dz^2 &= \frac{f(x,y)G(x,y)}{A^4(x-y)^4}\left(\frac{dy^2}{(y^2-1)F(y)}+\frac{dx^2}{(1-x^2)F(x)}\right). \end{aligned}$$

The parameter A in (4.23) represents the acceleration, while the parameter α represents the Zipoy-Voorhees parameter, which is connected to the linear density of the rod.

This metric is then used as seed for the scaling symmetry mentioned above, which give us a further generalization of the Zipoy-Voorhees solution, indexed by two real parameters, where the relevant functions of the LWP metric takes the following form:

$$\begin{aligned} e^{-\psi} &= \left[\frac{(y^2-1)F(y)}{A^2(x-y)^2}\left(\frac{F(y)}{F(x)}\right)^{\alpha-1}\right]^{\gamma} \\ e^{2\mu} &= \left[\frac{(y^2-1)F(y)}{f(x,y)G(x,y)}\frac{F(y)^{\alpha^2-1}F(x)^{(\alpha-1)^2}}{G(x,y)^{\alpha^2-1}}\right]^{\gamma^2}. \end{aligned} \tag{4.23}$$

Charging the metric

The next step is to charge (4.23) with the $SL(2,R)$ transformation mentioned before. The transformation consist in replacing the e^{ψ} function with e^{λ}, which is:

$$e^{\lambda} = e^{\psi}\frac{(1-\delta e^{-\psi})^2}{4C^2\delta},$$

with the new constants δ and C that can be expressed in terms of the matrix elements involved in the $SL(2,R)$ symmetry. In addition, a vector potential appears after the transformation, assuming the form:

$$A_1 = \chi dt \qquad \text{with} \qquad \chi = \frac{4C\delta}{e^{\psi}-\delta}.$$

Thus, applying this charging transformation to (4.23) we find the following metric:

$$ds^2 = -e^{-\psi}\frac{1}{H_\gamma(x,y)}dt^2 + e^{\psi}H_\gamma(x,y)[e^{2\mu}(d\rho^2+dz^2)+\rho^2 d\varphi^2], \tag{4.24}$$

with

$$A_1 = \frac{4C\delta}{\left(\dfrac{A^2(x-y)^2}{(y^2-1)F(y)}\left(\dfrac{F(x)}{F(y)}\right)^{\alpha-1}\right)^{\gamma}-\delta}dt \tag{4.25}$$

$$H_\gamma(x,y) = \frac{\left(1-\delta\left(\dfrac{(y^2-1)F(y)}{A^2(x-y)^2}\left(\dfrac{F(x)}{F(y)}\right)^{\alpha-1}\right)^{\gamma}\right)^2}{4C^2\delta}.$$

Dualisation of the electromagnetic field

Once we have obtained the charged solution, the next step consist in performing a dualisation of the electromagnetic field, in order to obtain the magnetic charged version of (4.24). This can be accomplished, as explained above, using a special sub case of transformation (I)-(1.3). However, the authors compute the dual electromagnetic potential in the reduced three dimensional theory, using the reduced Lagrangian (we will not get into the details of this charging procedure either). This dualisation procedure, applied to (4.25), give us the following magnetic potential:

$$A_\varphi = \frac{\gamma}{C}\frac{(1-x^2)(\alpha F(x)+(1-\alpha)F(y))}{A^2(x-y)^2} + \frac{2m\gamma\alpha x}{AC}, \tag{4.26}$$

while the metric remains unchanged.

Last step

Finally, our magnetic solution of Einstein-Maxwell equations can be mapped into a solution of Einstein-Maxwell-Dilaton theory, with a specific value of the dilaton coupling, $a=-\sqrt{3}$. However, it is sufficient to know that the magnetic solution can be mapped into another solution (the one of the EMD theory), which may be seen as the dimensional reduction of a five-dimensional metric that satisfies Einstein vacuum equations. From this five-dimensional solution, since it may be seen as the product of a four-dimensional Euclidean metric with a time direction, a four-dimensional metric that solves Einstein vacuum equations can be extracted. So, if:

$$ds_4^2 = -e^{-\lambda}dt^2 + e^{\lambda}[e^{2\mu}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] \qquad \text{with} \qquad A_1 = A\varphi d\varphi \tag{4.27}$$

is our magnetic solution, this can be mapped into the following solution:

$$ds_4^2 = -e^{-\frac{\lambda}{2}}\left(dz + \frac{A_\varphi}{2}d\varphi\right)^2 + e^{\frac{\lambda}{2}}\left[(e^{2\mu})^{\frac{1}{4}}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right]. \tag{4.28}$$

Thus, we apply this procedure to (4.24) with (4.26), taking $\gamma = 2$. The result is the following metric:

$$\begin{aligned} ds^2 = & \frac{(y^2-1)F(x)}{A^2(x-y)^2}\left(\frac{F(y)}{F(x)}\right)^\alpha \frac{C^2\delta}{H(x,y)}(dt + A_\varphi d\varphi)^2 \\ & + \frac{H(x,y)}{A^2(x-y)^2}\left[\left(\frac{F(x)}{F(y)}\right)^\alpha (1-x^2)F(y)d\varphi^2 \right. \\ & \left. + \frac{(F(x)F(y))^{\alpha(\alpha-1)}}{G(x,y)^{\alpha^2-1}}\left(\frac{dy^2}{(y^2-1)F(y)} + \frac{dx^2}{(1-x^2)F(x)}\right)\right]. \end{aligned} \tag{4.29}$$

Eventually, taking $\alpha = 1$, we retrieve the metric (4.19), given at the beginning of this section. Note that setting $\alpha = 1$ in the usual Zipoy-Voorhees solution give us back the Schwarzschild solution, so setting $\alpha = 1$ in (4.29) cancel the characteristic "deformation" of Zipoy-Voorhees solution, giving us a generalization of the Schwarzschild metric. The final metric is actually the accelerating NUT solution.

4.3.2 Podoslký and Vrátný improved form of the metric

The metric produced by Chng, Mann and Stelea was recently put in a new improved form by Podolský and Vrátný in [4], which is:

$$ds^2 = \frac{1}{\Omega^2}\left[-\frac{\mathcal{Q}}{\mathcal{R}^2}\left(dt - 2l(\cos\theta - \alpha\mathcal{T}\sin^2\theta)d\varphi\right)^2 + \frac{\mathcal{R}^2}{\mathcal{Q}}dr^2 + \mathcal{R}^2\left(\frac{d\theta^2}{\mathcal{P}} + \mathcal{P}\sin^2\theta d\varphi^2\right)\right], \tag{4.30}$$

where

$$\begin{aligned} \Omega(r,\theta) &= 1 - \alpha(r - r_-)\cos\theta \\ \mathcal{P}(\theta) &= 1 - \alpha(r_+ - r_-)\cos\theta \\ \mathcal{Q}(r) &= (r - r_+)(r - r_-)(1 - \alpha(r - r_-))(1 + \alpha(r - r_-)) \\ \mathcal{T}(r,\theta) &= \frac{(r - r_-)^2\mathcal{P}}{(r_+ - r_-)\Omega^2} \\ \mathcal{R}^2(r,\theta) &= \frac{1}{r_+^2 + l^2}\left(r_+^2(r - r_-)^2 + l^2(r - r_+)^2\frac{[1 - \alpha^2(r - r_-)^2]^2}{[1 - \alpha(r - r_-)\cos\theta]^4}\right). \end{aligned} \tag{4.31}$$

In the new form the metric explicitly contains 3 parameters m, α and l, associated respectively with the mass, the acceleration and the NUT parameter of the black hole. Setting to 0 one or more of these parameters let us retrieve important type D solutions, which are subclasses of metric (4.30). Although all subclasses of the accelerating NUT are solutions of type D, the accelerating NUT metric is a solution of type I, which sits outside the Plebański-Demiański type D family.

4.4 Adding NUT to the C metric

We know that the Ehlers transformation add the NUT parameter to the seed metric once it is applied to it, see [20] and [21]. Then a natural question arises: what will we find applying the Ehlers transformation to the C-metric? Will we find an accelerating NUT solution of type D, that sits inside the Plebański-Demiański family or a type I solution that can be traced back to Podolský and Vrátný's accelerating NUT metric? The answer, as already anticipated many times above, is that we will find Podoslký and Vrátný's solution (4.30). We will go over again the procedure followed in section 3.2, with the simplification that here we have no electromagnetic field; thus, we will always assume that $\mathbf{\Phi} = 0$.

The seed metric is the C-metric in spherical-type coordinates, as written in (4.1), i.e. with both conical singularities at its poles.

The coordinates change from the Weyl coordinates (t, ρ, z, φ) to the spherical-type one (t, r, θ, φ) is found exploiting condition (3.9) and reads as follows:

$$\rho(r, \theta) = \frac{\sin\theta\sqrt{Q(r)P(\theta)}}{(1 - \alpha r\cos\theta)^2} \tag{4.32}$$

$$z(r, \theta) = \frac{(\cos\theta - \alpha r)(r - m - \alpha m r\cos\theta)}{(1 - \alpha r\cos\theta)^2}, \tag{4.33}$$

where $Q(r)$ and $P(\theta)$ were defined in (4.2). Besides, this coordinates change is clearly the accelerating generalisation of (3.8) and (3.10). Moreover, from this coordinates change it may be computed the gradient operator in spherical-type coordinates, that will be used later on. In fact, the only thing we need to know about the gradient operator are the coefficients that multiplies each derivative, since all the equations involved in the generating technique have gradient operators in both sides, cancelling any common term to which the gradient is multiplied for. So, the only thing we need to know about the gradient operator is that:

$$\boldsymbol{\nabla} f(r, \theta) \propto \sqrt{Q(r)}\,\partial_r f\,\hat{e}_r + \sqrt{P(\theta)}\,\partial_\theta f\,\hat{e}_\theta. \tag{4.34}$$

Once we know the coordinates change, it is easy to read out the relevant functions of the LWP metric in the case of the C-metric:

$$f(r, \theta) = \frac{Q(r)}{r^2(1 - \alpha r\cos\theta)^2}$$

$$e^{2\gamma}(d\rho^2 + dz^2) = \frac{dr^2}{(1 - \alpha r\cos\theta)^4} + \frac{Q(r)}{P(\theta)(1 - \alpha r\cos\theta)^4}\,d\theta^2 \tag{4.35}$$

$$\omega = 0.$$

Since we do not have an electromagnetic field, the C-metric will have just $\boldsymbol{\mathcal{E}} = f$ as Ernst potential, while $\boldsymbol{\Phi} = 0$. Then using the Ehlers transformation according to (III) of (1.3), it is possible to find the Ernst potential $\boldsymbol{\mathcal{E}'}$ of the accelerating NUT metric. Writing it as in (3.15), the new primed functions follows:

$$f'(r,\theta) = \frac{Q(r)(1-\alpha r\cos\theta)^2}{r^2(1-\alpha r\cos\theta)^4 + c^2(r-2m)^2(1-\alpha^2 r^2)^2} \tag{4.36}$$

$$\chi'(r,\theta) = -\frac{c\,(r-2m)^2(1-\alpha^2 r^2)^2}{r^2(1-\alpha r\cos\theta)^4 + c^2(r-2m)^2(1-\alpha^2 r^2)^2}. \tag{4.37}$$

From (4.37), using (1.25), the ω function of the accelerating NUT metric can be computed:

$$\omega'(r,\theta) = -\frac{2c(r^2\sin^2\theta\alpha - 2m\cos\theta(1-2r\alpha\cos\theta + r^2\alpha^2))}{(1-r\alpha\cos\theta)^2}. \tag{4.38}$$

Now, writing (4.35), (4.36) and (4.38) inside the LWP metric, i.e. (1.4), we obtain the accelerating NUT solution. Remarkably, the metric we found is a Petrov type I, although we started using as seed a type D metric; this is not trivial since type I metric are algebraically more general than type D metrics, being the latters just a subcase of the formers. We computed the Petrov type using the null tetrad defined in (4.4) and the definitions given in (2.2), from which we built the Weyl scalars for our solution. Due to the complexity of these scalars it was not possible to verify globally if the solution is a type I. However, we verified it using different sets of values for the coordinates and the parameters, strongly suggesting that the metric has Petrov type I. In particular, what we did was checking that condition (2.9) does hold for our metric and the reason why we did not looked at which one of the Weyl scalars was zero, according to (2.3), is that in the chosen null tetrad no one of the scalars annihilates.

Since our metric is a type I, this suggests that our metric is probably the same metric presented by Podolský and Vrátný, just written in a different coordinates system. This is confirmed by the fact that collecting the conformal factor $1/(1-r\alpha\cos\theta)^2$, our metric assumes the following form:

$$ds^2 = \frac{1}{\Omega_0^2}\left[-\frac{Q}{\mathcal{R}_0^2}\left(dt-\omega' d\varphi\right)^2 + \frac{\mathcal{R}_0^2}{Q}dr^2 + \mathcal{R}_0^2\left(\frac{d\theta^2}{P} + P\sin^2\theta d\varphi^2\right)\right], \tag{4.39}$$

with

$$\Omega_0(r,\theta) = 1 - r\alpha\cos\theta \tag{4.40}$$

$$\mathcal{R}_0^2(r,\theta) = r^2 + \frac{c^2(r-2m)^2(1-\alpha^2 r^2)^2}{(1-r\alpha\cos\theta)^4}, \tag{4.41}$$

which is the same structure assumed by Podolský's metric, even though its functions are different from the ones given in (4.31). In fact, we can retrieve Podolský and Vrátný's exact metric applying transformations (3.23) and (3.24), which we already used in section 3.2. These transformations will map our functions into the ones given in (4.31). The parameter α, which was not present when we applied the Ehlers to the Reissner-Nordström metric, has to be scaled in the following way:

$$\alpha \longrightarrow \alpha \sqrt{\frac{r_+ - r_-}{r_+}}. \tag{4.42}$$

Finally, it is worth noting the efficiency of the Ehlers transformation. Not only we produced without any particular difficulty the accelerating NUT solution, but we obtained it already in the form (4.30), avoiding all the work that Podoslký and Vrátný did in order to simplify Chng, Mann and Stelea's accelerating NUT metric.

Chapter 5

The accelerating Reissner-Nordström-NUT solution

In this last chapter we apply the Ehlers transformation to the accelerating version of the Reissner-Nordström solution, obtaining eventually the accelerating Reissner-Nordström-NUT solution. This is a type I solution which represents a generalization to the Plebański-Demiański family when the Kerr-like parameter a is set to zero.

The charged version of the C-metric is taken as seed and it may be written in the same form in which the C-metric was given in (4.1), with an appropriate generalisation of the functions $Q(r)$ and $P(\theta)$:

$$P(\theta) = 1 - 2\alpha m \cos\theta + \alpha^2 e^2 \cos^2\theta \qquad Q(r) = \left(r^2 - 2mr + e^2\right)(1 - \alpha^2 r^2), \tag{5.1}$$

and the same 1-form vector potential assumed in (3.7).

As usual, $\rho(r, \theta)$ can be deduced easily by the comparison of the seed metric with the LWP metric, while $z(r, \theta)$ is found exploiting condition (3.9). The transformation between Weyl coordinates and spherical-type coordinates reads as follows:

$$\rho(r, \theta) = \frac{\sin\theta\sqrt{Q(r)P(\theta)}}{(1 - \alpha r\cos\theta)^2} \tag{5.2}$$

$$z(r, \theta) = \frac{(\cos\theta - \alpha r)(r - m - \alpha\cos\theta(mr - e^2))}{(1 - \alpha r\cos\theta)^2}. \tag{5.3}$$

From this change the gradient operator, ignoring common terms to all derivatives, follows; it takes the same form given in (4.34), with the only exception that now $Q(r)$ and $P(\theta)$ are the ones defined in (5.1).

Moreover, the relevant functions of the LWP metric f and ω, together with the invariant piece $e^{2\gamma}(d\rho^2 + dz^2)$, take the same form given in (4.35) in the case of the C-metric. Ernst potentials $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\Phi}$ are then built from these functions, according to their definitions given in (1.31) and (1.21), and transformed under the action of the Ehlers transformation. The new Ernst potentials are then expanded according to (3.14) and (3.15), giving us the primed functions we need to build our solution:

$$A'_t(r,\theta) = -\frac{er^3(1-r\alpha\cos\theta)^4}{r^4(1-r\alpha\cos\theta)^4 + c^2(Q(r)-e^2(1-r\alpha\cos\theta)^2)^2} \tag{5.4}$$

$$\tilde{A}'_\varphi(r,\theta) = \frac{ecr(1-r\alpha\cos\theta)^2(Q(r)-e^2(1-r\alpha\cos\theta)^2)}{r^4(1-r\alpha\cos\theta)^4 + c^2(Q(r)-e^2(1-r\alpha\cos\theta)^2)^2} \tag{5.5}$$

$$f'(r,\theta) = \frac{r^2(1-r\alpha\cos\theta)^2 Q(r)}{r^4(1-r\alpha\cos\theta)^4 + c^2(Q(r)-e^2(1-r\alpha\cos\theta)^2)^2} \tag{5.6}$$

$$\chi'(r,\theta) = -\frac{c(Q(r)-e^2(1-r\alpha\cos\theta)^2)^2}{r^4(1-r\alpha\cos\theta)^4 + c^2(Q(r)-e^2(1-r\alpha\cos\theta)^2)^2}. \tag{5.7}$$

From χ' we can now compute ω' using (1.25):

$$\omega'(r,\theta) = \frac{2c\left[2m\cos\theta(1+r\alpha(r\alpha-2\cos\theta)) - \alpha\sin^2\theta\,(r^2-e^2(1-2r\alpha\cos\theta))\right]}{(1-r\alpha\cos\theta)^2} \tag{5.8}$$

and we may also compute, using (1.17), A'_φ from $\tilde{A}'_\varphi$, writing it in the following convenient form:

$$A'_\varphi(r,\theta) = \frac{ce[2(r-m\sin^2\theta)\alpha - \cos\theta(1+(r^2-e^2\sin^2\theta)\alpha^2)]}{(1-r\alpha\cos\theta)^2} - \omega'(r,\theta)A'_t(r,\theta). \tag{5.9}$$

Eventually, writing down f' and ω' inside the LWP metric, we obtain the metric of an accelerating Reissner-Nordström-NUT black hole, together with the vector potential $A = A'_t dt + A'_\varphi d\varphi$. Like we did in the case of the simpler accelerating NUT black hole, we compute the Weyl scalars from the null tetrad defined in (4.4) and we verify, for differents values of the coordinates and the parameters, that condition (2.9) does not holds, i.e. the metric is a type I one. We verify also, using the same sets of numerical values, that turning off the electric charge parameter the metric is still of type I, retrieving the accelerating NUT metric, while turning off the acceleration, the metric becomes a type D one, which is coherent with the fact that doing so the metric reduces to the Reissner-Nordström-NUT solution.

Now, even though we found the metric which we were looking for at the beginning of this chapter, we have seen in the cases of both Reissner-Nordström-NUT and accelerating NUT that a transformation of the coordinates r and t, together with a rescaling of all the parameters, is usually required in order to put the metric in a simpler form. Moreover, since we do not know our metric in its "canonical" right from the start, we cannot find the transformation we need from the comparison between our metric and the same one put in canonical form. However, it is plausible that the transformation required is still (3.23), since it worked for both the Reissner-Nordström-NUT solution and the accelerating NUT solution, and that the rescaling of the parameters remains (3.24), together with (4.42). As we did in the case of the accellerating NUT metric, before applying the transformation it is convenient to collect the conformal factor $1/(1-r\alpha\cos\theta)^2$ and put the metric in the same form given by (4.39), where P and Q are the ones defined in the current section, while the generalized radial coordinate $\mathcal{R}_0^2$ reads as follows:

$$\mathcal{R}_0^2 = r^2 + \frac{c^2(Q(r)-e^2(1-r\alpha\cos\theta)^2)^2}{r^2(1-r\alpha\cos\theta)^4}. \tag{5.10}$$

Then, applying the mentioned transformation we obtain a generalisation of Podoslký and Vrátný's accelerating NUT metric. Once we simplified the scaling constants introduced, the relevant functions of the metric are:

$$
\begin{aligned}
\Omega(r,\theta) &= 1-\alpha(r-r_-)\cos\theta \\
\mathcal{P}(\theta) &= 1-\alpha(r_+-r_-)\cos\theta + e^2\alpha^2\cos^2\theta \\
\mathcal{Q}(r) &= (r^2-2mr-l^2+e^2)(1-\alpha^2(r-r_-)^2) \\
\mathcal{T}(r,\theta) &= \frac{(r-r_-)^2\mathcal{P}-e^2(1-2(r-r_-)\alpha\cos\theta)}{(r_+-r_-)\Omega^2} \\
\mathcal{R}^2(r,\theta) &= \frac{1}{r_+^2+l^2}\left(r_+^2(r-r_-)^2+l^2\frac{(\mathcal{Q}-e^2\Omega^2)^2}{(r-r_-)^2\Omega^4}\right).
\end{aligned}
\tag{5.11}
$$

These functions give us the metric of an accelerating Reissner-Nordström-NUT solution, the only thing left to find is the associated vector potential. However, from section 3.2 we are aware of the fact that the Ehlers transformation rotates the electromagnetic field when it is applied to a charged seed metric. Performing the rotation of the vector potential and using (1.17), as we did at the end of section 3.2, we obtain the new components:

$$
\bar{A}_t(r,\theta) = -\frac{er\,\Omega_0^2\,[r^2\Omega_0^2\cos\beta + c(Q-e^2\Omega_0^2)\sin\beta]}{r^4\Omega_0^4+c^2(Q-e^2\Omega_0^2)^2}
\tag{5.12}
$$

$$
\begin{aligned}
\bar{A}_\varphi(r,\theta) = &-\frac{ce(\cos\theta-2(r-m\sin^2\theta)\alpha+\cos\theta(r^2-e^2\sin^2\theta)\alpha^2)\cos\beta}{\Omega_0^2} \\
&+e\cos\theta\sin\beta-\omega'(r,\theta)\bar{A}_t(r,\theta),
\end{aligned}
\tag{5.13}
$$

where Ω_0 is the one defined in (4.40).

Actually, this last step could have been avoided using the enhanced version of the Ehlers transformation, introduced by [21]. The enhanced Ehlers transformation has the advantage to only add NUT to the seed metric, without rotating the electromagnetic field, since the duality rotation is included in the new transformation.

As we did at the end of section 3.2, the criteria to follow in order to find the right choice of $\cos\beta$ is imposing that the first two terms in the right side of (5.13) cancel. It's clear that the current case is a bit more complicated than the previous one and that $\cos\beta$ is not going to have the same value as the one given in (3.30). Indeed, imposing that $\bar{A}_\varphi = -\omega'\bar{A}_t$ we find that the right choice is:

$$
\cos\beta = \frac{\cos\theta\,\Omega^2}{\sqrt{c^2(\cos\theta-2(r-m\sin^2\theta)\alpha+\cos\theta(r^2-e^2\sin^2\theta)\alpha^2)^2+\cos^2\theta\,\Omega^2}},
\tag{5.14}
$$

which in limit where $\alpha\to 0$ reduces to the value given in (3.30). Eventually, using (5.14) and applying the usual change of coordinates and parameters we used ever since, we get the following t-component of the vector potential:

$$
\bar{A}_t(r,\theta) = -\frac{e}{\mathcal{R}^2}\frac{(r_+^2(r-r_-)\cos\theta+l^2A(r,\theta)B(r,\theta))}{\sqrt{r_+^2\cos^2\theta+l^2B^2(r,\theta)}}\frac{1}{\sqrt{r_+^2+l^2}},
\tag{5.15}
$$

where we defined the functions A and B as follows:

$$A(r,\theta) = \frac{(\mathcal{Q} - e^2\Omega^2)}{(r-r_-)\Omega^2} \tag{5.16}$$

$$B(r,\theta) = \frac{\cos\theta(1 + ((r-r_-)^2 - e^2(1+x^2)\alpha^2) - (2r - 2m + (r_+ - r_-)\cos^2\theta)\alpha}{\Omega^2}. \tag{5.17}$$

Turning off α or e gives us back - in order - the Reissner-Nordström-NUT metric or the accelerating NUT solution. The hierarchy of all solutions involving mass, electric charge and NUT parameter, mentioned until now, is represented in figure 5.1:

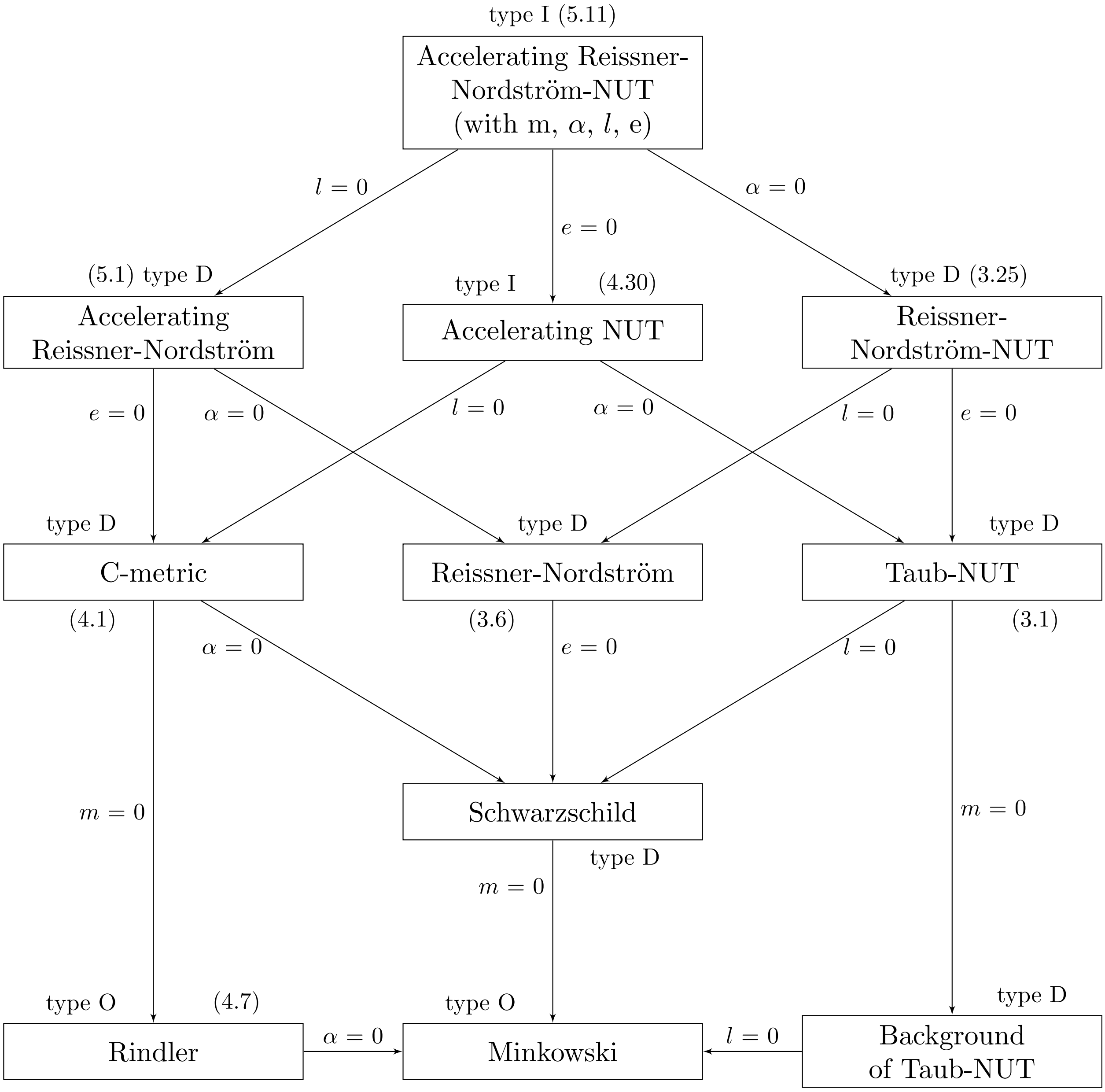

Figure 5.1: Structure of the type I family of solutions with no Kerr-like rotation. From the accelerating Reissner-Nordström-NUT metric, all other knowns accelerating solutions may be retrieved as subcases of the first one, turning off some of its paramters. When e is set to zero the solution remains algebraically general, while any other case leads to a double degenerate type D solution. Turning off all parameters the usual Minkowski spacetime is retrieved too

Conclusions

This work was devoted to show how the Ehlers transformation, and more in general Ernst's generating technique, represents an efficient and easy-to-use tool in order to produce new solutions of Einstein-Maxwell equations.

In the first chapter we derived the Lewis-Weyl-Papapetrou metric, i.e. the most general metric representing the class of electro-vacuum spacetimes involved in Ernst's generating technique, which are stationary and axisymmetric spacetimes with no cosmological constant. This class of spacetimes, although quite general, let us cast Einstein-Maxwell equations in a simpler form, where the introduction of two complex potentials $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\Phi}$ not only seems natural, but it also further simplify the equations. Eventually, what we obtain is a simple and symmetric set of two complex equations, decoupled from the other two equations in γ, called "Ernst equations". From an analysis of the equation's symmetries, that we did not carry out in this work, 5 transformations which maps solutions into solutions can be found, two of which are non-trivial. This work focused its attention on one of these non-trivial transformations, called the "Ehlers transformation", which may be employed to add the NUT parameter to any seed metric. Because of this in chapter 3, after having introduced the NUT parameter with some of its physical difficulties, we showed, on a simple metric such as the Reissner-Nordström one, how to use the Ehlers in order to add NUT. Once we understood how to use the Ehlers, in chapter 4 we finally applied it to the C-metric, in order to see if it was possible to retrieve Podoslký and Vrátný accelerating NUT metric. After having introduced the basic accelerating black hole, represented by the C-metric, and having explained how the accelerating NUT metric was generated for the first time by Chng, Mann and Stelea, we found out that we can easily retrieve Podoslký and Vrátný accelerating NUT metric just applying the Ehlers to the C-metric. In the last chapter, we applied the Ehlers to the charged version of the C-metric, obtaining the charged generalization of Podoslký and Vrátný accelerating NUT metric. Besides, an important role was played by the Petrov classification, which we introduced in the second chapter, since it represents a way to recognise the same metric when written in two different coordinates systems.

As conclusive comments, the Ehlers transformation does not just represents a useful tool to add the NUT charge, but it may be used as a useful tool to remove it, when undesired, from a particular solution. Applying the Ehlers to a solution which already possesses a NUT parameter and fine-tuning the transformation's parameter, we may indeed eliminate the initial NUT parameter, removing the possibly pathological features that comes with the NUT charge. This idea opens a question about what would happen if we tried to eliminate in such a way the NUT parameter from the Plebański-Demiański family. The question is not trivial since, as we have seen when talking about the Plebański-Demiański family in section 4.2, the parameter l that appears there is not the same NUT parameter of the accelerating NUT metric for example. The reason why they are not the same parameter is that the accelerating NUT metric does not belong to the Plebański-Demiański family, although the latter technically contains all the parameters we need to build the solution, which are α, m and l.

On the other hand, we have seen how the Ehlers transformation, when applied to an accelerating solution, is incline to give as result a type I metric. Moreover, we have also seen that there are black holes solutions, such as the accelerating NUT metric, that lives outside the Plebański-Demiański family of type D solutions. However, we saw that gradual extensions of the Plebański-Demiański family to type I solutions are possible when the Ehlers transformation is applied to seed metric with increasingly more parameters turned on. An example of this extensions was accomplished in the last chapter, when we found the accelerating Reissner-Nordström-NUT metric. Indeed, this metric may be seen as the type I extension of the Plebański-Demiański family, where the acceleration parameter is now decoupled from the Kerr-like rotation parameter, contrary to what happened in the Plebański-Demiański family as presented in section 4.2. This suggests that applying the Ehlers transformation to the accelerating Kerr-Newman would probably give us the complete extension of the Plebański-Demiański family to type I solutions, where all the precedent type D solutions could be retrieved turning off the different parameters. Theoretically, what we would have to do is nothing more than the same procedure that we followed in this work, although adding a rotation to the seed metric will remarkably increase the computational amount of work, making it more difficult to condensate the results in synthetic expressions. Instead, what we could do is starting with the accelerating Kerr metric and see into which metric is mapped under the action of the Ehlers. This would be interesting to compute, since the accelerating Kerr-NUT metric is present inside the Plebański-Demiański family of type D solutions. We could find a type I solution, which would confirm that the NUT parameter of the Plebański-Demiański family is not the same added by the Ehlers or we could found the same type D metric already present in the family of solutions.

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