

# Durham E-Theses

---

## *Giant Magnons and Giant Gravitons*

CIAVARELLA, ANDREW, MICHAEL

---

### How to cite:

CIAVARELLA, ANDREW, MICHAEL (2011) *Giant Magnons and Giant Gravitons*, Durham theses, Durham University. Available at Durham E-Theses Online: <http://etheses.dur.ac.uk/836/>

---

### Use policy

The full-text may be used and/or reproduced, and given to third parties in any format or medium, without prior permission or charge, for personal research or study, educational, or not-for-profit purposes provided that:

- a full bibliographic reference is made to the original source
- a link is made to the metadata record in Durham E-Theses
- the full-text is not changed in any way

The full-text must not be sold in any format or medium without the formal permission of the copyright holders.

Please consult the [full Durham E-Theses policy](#) for further details.

# Giant Magnons and Giant Gravitons

**Andrew Ciavarella**

A Thesis presented for the degree of  
Doctor of Philosophy



Centre for Particle Theory  
Department of Mathematical Sciences  
University of Durham  
England

March 2011

*Dedicated to*

Mum and Dad

# Giant Magnons and Giant Gravitons

Andrew Ciavarella

Submitted for the degree of Doctor of Philosophy  
March 2011

## Abstract

In this thesis we shall present work concerning the description of the emergence of solitonic fundamental strings from stable, finite energy, compact D3-branes in a subspace of  $AdS_5 \times S^5$  and their subsequent interaction. We work in the planar limit and focus on states of large angular momentum  $J$  corresponding to large R-charge in the dual gauge theory.

We begin by constructing the full set of boundary giant magnons on  $\mathbb{R} \times S^2$  attached to the maximal  $Z = 0$  giant graviton by mapping from the general solution to static sine-Gordon theory on the interval. We then compute the values of the anomalous dimension,  $\Delta - J$ , of the dual gauge theory operators at finite  $J$ , examining the behaviour of the leading order corrections when  $J$  is large.

We then consider the Born-Infeld theory of the giant graviton itself coupled to the background 5-form flux. Constructing BIon spike solutions that correspond to the world volume description of the boundary giant magnons we find a limit amenable to analysis which returns the full range of behaviour exhibited at finite  $J$ .

Finally we produce the open strings on  $\mathbb{R} \times S^2$  that correspond to the solutions of integrable boundary sine-Gordon theory. Relating the boundary parameters in a way that ensures a given set of string boundary conditions we describe the scattering of giant magnons with non-maximal  $Y = 0$  giant gravitons and calculate the leading contribution to the associated magnon scattering phase. Our method necessarily describes all integrable scatterings of giant magnons with giant gravitons.

# Declaration

The work in this thesis is based on research carried out at the Centre for Particle Theory, the Department of Mathematical Sciences, Durham University, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

**Copyright © 2011 by Andrew Ciavarella.**

“The copyright of this thesis rests with the author. No quotations from it should be published without the author’s prior written consent and information derived from it should be acknowledged”.

# Acknowledgements

An advantage of taking a little longer to complete my doctorate has been the pleasure of getting to know so many other PhD students, with many of whom I have spent a great deal of time in *the* office and some of whom I have seen come *and* go. My thanks goes out to you all for making not just lunch times but all of the last half decade of my life so memorable. I wish you all happiness for the future.

My supervisor has always made time for me and has always provided insightful commentary on my thinking. Most importantly we collaborated on my first paper in mathematical physics! Thank you, Peter, for all of your time which has been invaluable.

Susan, you've made me happy while my work load has spiralled and my future hung in the balance; thank you for your patience and company.

Finally, none of this would have been possible without the love and support of my parents and the home they have provided. Thank you Mum and Dad, and thank you to all my family.

# Contents

<b>Abstract</b>	iii
<b>Declaration</b>	iv
<b>Acknowledgements</b>	v
<b>1 Background: <math>AdS/CFT</math> and integrability</b>	1
1.1 Introduction . . . . .	1
1.2 $AdS_5 \times S^5 / \mathcal{N} = 4$ SYM . . . . .	4
1.2.1 Type IIB theory and $AdS/CFT$ . . . . .	4
1.2.2 Global symmetries . . . . .	14
1.2.3 Spectrum and chiral primary operators . . . . .	18
1.3 Aspects of integrability . . . . .	20
1.3.1 The string sigma model . . . . .	21
1.3.2 Pohlmeyer reduction of the $O(3)$ model . . . . .	23
1.3.3 Sine-Gordon model and integrability . . . . .	26
1.3.4 BMN limit, spin chains and the magnon spectrum . . . . .	32
<b>2 Giant states: semi-classical strings and branes in <math>AdS/CFT</math></b>	35
2.1 Spinning strings to giant magnons . . . . .	36
2.1.1 Spinning strings . . . . .	37
2.1.2 The Giant Magnon . . . . .	38
2.1.3 Finite $J$ giant magnons . . . . .	46
2.1.4 Further generalisations of the giant magnon . . . . .	48
2.2 Born-Infeld effective theory and giant gravitons . . . . .	50

---

2.2.1	World volume solitons: BIon solutions	51
2.2.2	Giant gravitons from the effective action	53
2.2.3	Gauge theory dual to giant gravitons	55
2.2.4	Open strings on giant gravitons	57
<b>3</b>	<b>Boundary giant magnons</b>	<b>60</b>
3.1	Static Sine-Gordon Solutions on the Interval	60
3.1.1	$c > 1$ Solution	61
3.1.2	$ c  \leq 1$ Solution	62
3.1.3	Boundary Conditions	63
3.2	String Solutions and $\Delta - J$	66
3.2.1	$\dot{\phi} \geq 1$ solution	68
3.2.2	$0 \leq \dot{\phi} \leq 1$ solution	70
3.2.3	Behaviour of anomalous dimension: $\Delta - J$	71
3.2.4	Finite $J$ corrections	74
3.3	Conclusion	76
<b>4</b>	<b>World volume description</b>	<b>78</b>
4.1	D3-brane action and ansatz	79
4.1.1	D3-brane equations of motion	79
4.1.2	Classical string	82
4.2	$\dot{\phi} = 1$	84
4.3	$\dot{\phi} \neq 1$	89
4.4	Conclusion	97
<b>5</b>	<b>Giant magnons and non-maximal giant gravitons</b>	<b>98</b>
5.1	Boundary sine-Gordon theory	99
5.2	Open string on $\mathbb{R} \times S^2$	104
5.2.1	N-magnon string solution on $\mathbb{R} \times S^2$ with moduli	105
5.2.2	Method of images on $\mathbb{R} \times S^2$	110
5.3	Scattering giant magnons off giant gravitons and semi-classical phase shifts	114

---

5.3.1	Scattering with maximal $Z = 0$ and $Y = 0$ giant gravitons . . . . .	115
5.3.2	Scattering with non-maximal $Y = 0$ giant gravitons . . . . .	122
5.3.3	Other open strings with integrable sine-Gordon boundary conditions . . . . .	128
5.4	Conclusion . . . . .	131
<b>6</b>	<b>Discussion</b>	<b>133</b>
	<b>Bibliography</b>	<b>139</b>
	<b>Appendix</b>	<b>149</b>
<b>A</b>	<b>Elliptic integrals and functions</b>	<b>149</b>
A.1	Elliptic integrals . . . . .	149
A.2	Elliptic functions . . . . .	151
A.3	Series representations . . . . .	152
<b>B</b>	<b>N=3 solution on <math>\mathbb{R} \times S^2</math></b>	<b>153</b>

# List of Figures

2.1	The geometry of the giant magnon. . . . .	43
3.1	Sine-Gordon solution with $c > 1$ on the interval $0 \leq x \leq L$ . . . . .	65
3.2	Sine-Gordon solution with $ c  \leq 1$ on the interval $0 \leq x \leq L$ . . . . .	65
3.3	Open string solution with $\dot{\phi} > 1$ . . . . .	69
3.4	Open string solution with $0 \leq \dot{\phi} \leq 1$ . . . . .	71
3.5	Open string energy in units of $\sqrt{\lambda}/2\pi$ . . . . .	72
3.6	Open string angular momentum in units of $\sqrt{\lambda}/2\pi$ . . . . .	73
3.7	Open string $\Delta - J$ in units of $\sqrt{\lambda}/2\pi$ . . . . .	73
4.1	The $\dot{\phi} = 1$ BIon solution on $\mathbb{R} \times S^2$ . . . . .	85
4.2	The $\dot{\phi} \geq 1$ BIon solution on $\mathbb{R} \times S^2$ . . . . .	93
4.3	The $0 \leq \dot{\phi} \leq 1$ BIon solution on $\mathbb{R} \times S^2$ . . . . .	94
5.1	The time shift due to boundary scattering via the method of images. . . . .	102
5.2	An example of sine-Gordon boundary / soliton scattering using the method images. . . . .	103
5.3	A kink/anti-kink/kink 3-magnon solution at early time. . . . .	109
5.4	A kink/kink/kink 3-magnon solution at early time. . . . .	110
5.5	A giant magnon attached to a maximal $Z = 0$ giant graviton. . . . .	119
5.6	The scattering of a giant magnon with a maximal $Z = 0$ giant graviton. . . . .	121
5.7	A giant magnon attached to a non-maximal $Y = 0$ giant graviton. . . . .	123
5.8	The scattering of a giant magnon with a non-maximal $Y = 0$ giant graviton. . . . .	125

# Chapter 1

## Background: $AdS/CFT$ and integrability

### 1.1 Introduction

The  $AdS/CFT$  correspondence [1–4] is a conjectured duality between superstring or M theory [5,6] on backgrounds of the form  $AdS_d \times M_{D-d}$  and a conformally invariant quantum field theory on a flat  $d-1$  dimensional space interpreted as the boundary of  $AdS_d$ .  $AdS_d$  is Anti-de-Sitter space of dimension  $d$  while  $M_{D-d}$  is a compact space of dimension  $D-d$ , with  $D=10$  for superstring theory and  $D=11$  for M theory.

The importance of Maldacena’s conjecture is hard to exaggerate as it promises that

- a quantum-gravitational theory may be described by a field theory in flat space,
- a non-abelian gauge theory may be described non-perturbatively by supergravity,

and since the late 1990’s probing the conjecture has become an industry. Much of the work of the last decade has concerned solvable limits, or integrable subsectors, of the correspondence that have allowed analytical methods to venture beyond the perturbative regimes of the respective theories.

*AdS/CFT* makes concrete an idea that has long been entertained [7] that there may exist a string description of non-abelian field theory and it was of course in exactly this way that string theory itself was first obtained. Bound states of fundamental matter in Yang-Mills theory require a gauge invariant description that invokes the description of string-like objects, which are physically realised as flux tubes. *AdS/CFT* then sees the energy scale of processes as an additional dimension. Most astonishingly the additional symmetries and supersymmetries of the theory complement the higher dimensional theory with precisely the required fields and dimensions of type IIB string theory.

This thesis presents research conducted upon states of type IIB string theory on  $AdS_5 \times S^5$  that are dual to operators of  $\mathcal{N} = 4$  super-Yang-Mills (SYM). This particular instance of *AdS/CFT* has been most intensively studied due to the unique level of symmetry that both sides of the correspondence (necessarily) display. The product of anti-de-Sitter 5-space and the 5-sphere is one of only two curved backgrounds that preserve the same number of supersymmetries as flat space [8]. Under *AdS/CFT* duality the isometries of the compactification space determine the supersymmetries of the field theory at the boundary of *AdS* space and for  $\mathcal{M}_5 = S^5$  this gives the maximal number of global supersymmetries possible for a 3+1 dimensional field theory.

The level of symmetry of these two theories is at the heart of progress in their understanding and yet this symmetry must be broken to study more realistic examples, such as a non-supersymmetric  $SU(3) \times SU(2) \times U(1)$  theory. Even within  $\mathcal{N} = 4$   $SU(N)$  SYM however it is still not easy to probe beyond the perturbative level and it is as part of an effort to do so that this thesis fits into the wider picture. In particular we shall be concerned with semi-classical states of type IIB string theory named giant magnons [9] and giant gravitons [10] whose description is both non-trivial enough to be informative yet simple enough to be accessible.

In chapter 1 we shall introduce *AdS/CFT* with focus on the case in point,  $AdS_5 \times S^5 / \mathcal{N} = 4$  SYM, briskly moving on to those aspects of the theoretical framework that are relevant to the research presented in later chapters. In section 1.2.2 we

---

shall discuss the symmetries and quantum numbers of the states that here interest us and provide an understanding of the class of states that take central stage. In section 1.3 we move on to the integrable aspects of these states of which we shall make great use.

Chapter 2 reviews the specific material necessary to discuss the giant magnons and giant gravitons, the former being solitonic states of a fundamental string and the latter being stable, compact D-branes [11, 12] whose world volume theory can be seen to give rise to the fundamental strings which are deemed to end upon them. We shall discuss a variety of generalisations of the giant magnon along the way as well as the gauge theory interpretation of the giant gravitons [13].

Chapter 3 begins the presentation of original research. We will show how the description of open strings ending on giant gravitons may be seen in the non-interacting string limit from the perspective of a useful form of the world sheet theory and construct the corresponding target space string solutions. This will result in a family of solutions of which half are novel. We calculate the energies of these solutions for a range of parameters, commenting on the physical significance and making comparisons between them and similar string solutions.

In chapter 4 we turn to the world volume theory of D3-branes from which it should also be possible to describe fundamental strings. We successfully rederive the behaviour of the complete set of solutions, generalising similar existing work away from the security of the infinite charge simplifying limit.

Chapter 5 returns us to the world sheet picture where we make great use of integrable aspect of the world sheet theory to describe the scattering of giant magnons with giant gravitons. We will again take ourselves away from a simplifying limit, this time the maximal size of the giant graviton, to find that we may describe new interactions of giant magnons with giant gravitons and make non-trivial statements about gauge theory quantities along the way.

Finally, chapter 6 is the discussion. The research presented in this thesis can also be found in published form in [14, 15].

## 1.2 $AdS_5 \times S^5 / \mathcal{N} = 4$ SYM

The purpose of this section is to present a brief overview of the  $AdS/CFT$  hypothesis and the relevant concepts that arise therein. We will begin in subsection 1.2.1 with a discussion of the relevant theoretical framework and the idea of the duality itself. Subsections 1.2.2 and 1.2.3 develop some of these ideas.

### 1.2.1 Type IIB theory and $AdS/CFT$

The argument for  $AdS/CFT$  begins with Type IIB superstring theory in Minkowski space. The massless spectrum of this theory [6] is the product of two copies of the tree-level massless spectrum of an open superstring (with momentum shared between the left and right moving modes).

In both left- and right-moving sectors a tachyon free, spacetime supersymmetric spectrum of states satisfying the physical state condition lives in a representation of  $SO(8)$ , the little group of  $SO(1, 9)$ , of the form  $\mathbf{8}_v \oplus \mathbf{8}_s$  where the 'v' indicates a spacetime vector and the 's' a spacetime spinor. The GSO projection required we pick a particular chirality for states transforming as a spacetime spinor in the  $\mathbf{8}_s$ . The Type II spectrum is then obtained as the states living in a product of these representations where we must pick the chirality of the spacetime spinors to be the same or opposite. Type IIB takes the same chirality on both sides so that we have states in  $(\mathbf{8}_v \oplus \mathbf{8}_s) \otimes (\mathbf{8}_v \oplus \mathbf{8}_s)$ .

The fields arising from these representations are found by decomposition into irreducible representations. We will be interested in the spacetime bosonic fields which arise from the R-R and NS-NS sectors. In the NS-NS sector we have

$$\mathbf{8}_v \otimes \mathbf{8}_v = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35} \quad (1.1)$$

which may be identified with the dilaton  $\Phi$ , Kalb-Ramond 2-form  $B_{\mu\nu}$  and the metric  $G_{\mu\nu}$ . From the R-R sector we get

$$\mathbf{8}_s \otimes \mathbf{8}_s = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35} \quad (1.2)$$

which are the R-R pseudo-scalar, a 2-form and the self-dual 4-form potentials.<sup>1</sup> Each  $(p+1)$ -form potential will give rise to a  $(p+2)$ -form field strength.

The fundamental string is not a source for these  $(p+1)$ -form fields arising out of the R-R sector. Instead they are sourced by  $p$ -branes [11] which are required for a complete description of the string theory. In fact every odd value of  $p$  less than 10 is allowed in Type IIB sourcing their corresponding potentials and field strengths.

A low energy effective field theory in 10 dimensions may be written down for these fields which also includes the spacetime fermions coming from the R-NS and NS-R sectors. This has the form [16, 17]

$$S_s = -s \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{|g|} \left( e^{-2\Phi} (R + 4g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi) - \frac{1}{2} \sum_n \frac{1}{n!} F_n^2 + \dots \right) \quad (1.3)$$

where  $s$  is  $-1(+1)$  for Minkowski (Euclidean) signature and the metric is “mostly plus”. Dots represent fermionic terms as well as the NS-NS 3-form field strength.  $\Phi$  is the dilaton and the  $n$ -form field strengths  $F_n$  belong to the R-R sector. For Type IIB we have only  $n$  odd and if the the 5-form field strength appears it is a self-dual tensor. The above action is adequate to form the equations of motion but a modification is required to properly include the self-dual field strength [18].

Solutions to the equations arising from the above action may be found that are interpreted as  $Dp$ -branes acting as ‘electric’ and ‘magnetic’ sources for the corresponding field strengths while the geometry is divided into the translationally invariant  $p$  dimensional world volume of the brane and a  $10-p$  dimensional transverse space with black hole characteristics. On the other hand the physics of a  $Dp$ -brane is described by a world volume theory in  $p+1$  dimensions. This theory is also supersymmetric and has bosonic fields that are described by the Dirac-Born-Infeld action [19]

$$S_{\text{DBI}} = T_p \int d^{p+1}x e^{-\Phi} \sqrt{-\det(\tilde{G}_{ab} + \tilde{B}_{ab} + 2\pi l_s^2 F_{ab})}, \quad a, b = 0, \dots, p. \quad (1.4)$$

Here the determinant is referring to the  $a, b$  structures appearing.  $\tilde{G}_{ab}$  can be thought of as the metric induced on the world volume of the brane embedded into a higher

---

<sup>1</sup>Although the dimensions of the irreducible representations in these two sectors are the same they consist of distinct representations.

dimensional space, or the pullback in the language of differential geometry. Thus the term

$$\tilde{G}_{ab} = \frac{\partial x^\mu}{\partial x^a} \frac{\partial x^\nu}{\partial x^b} G_{\mu\nu}, \quad a, b = 0, \dots, p \quad \mu, \nu = p+1, \dots, 9 \quad (1.5)$$

is the kinetic term for the  $10 - (p+1)$  scalar fields  $x^\mu$  that live on the world volume of the brane and, analogously to the description of a fundamental string, describe the embedding of the brane into target space.  $\tilde{B}_{ab}$  is similarly the pullback of the anti-symmetric Kalb-Ramond field  $B_{\mu\nu}$ . Those directions longitudinal to the brane world volume result in gauge fields  $A^a$  whose field strength appears also under the square root.  $T_p$  is the  $Dp$ -brane tension.

$D$ -branes are  $\frac{1}{2}$ -BPS objects, which is to say that they preserve exactly half of the supersymmetry of the background, equal to that preserved by the open strings that are defined to end upon them. BPS objects of like-charge experience a cancellation of the attractive and repulsive forces acting between them so that the energy of  $N$  such objects placed at arbitrary locations with respect to one another is equal to the energy of  $N$  asymptotically separated objects, or indeed  $N$  such objects stacked one on top of the other. Upon stacking  $N$   $Dp$ -branes the gauge potentials on the world volume receive new ‘internal’ indices leading to a  $p+1$  dimensional non-Abelian gauge theory on the branes [5, 12] with the same supersymmetries as the single  $Dp$ -brane. The action (1.4) is modified to include a trace over the gauge group structure.

In the ‘low energy limit’  $\alpha' = l_s^2 \rightarrow 0$  and in the absence of any  $B_{ab}$  we may expand the square root of the DBI action (1.4) to obtain the action for a Yang-Mills theory. Using  $\det^{\frac{1}{2}}(1 + M) = \exp \left[ \frac{1}{2} \text{Tr} \left( M - \frac{1}{2} M^2 + \dots \right) \right]$  then for  $p = 3$  (1.4) becomes, using the expression for the  $Dp$ -brane tension  $T_p = (2\pi)(2\pi l_s)^{-(p+1)} g_s^{-1}$ ,

$$S_{\alpha' \rightarrow 0} = \frac{1}{4\pi g_s} \text{Tr} (F^{\mu\nu} F_{\mu\nu} + \dots) \quad (1.6)$$

where the dots represent longer products of the field strengths coming multiplied by positive powers of  $\alpha'$ , all of which has a trace over the (suppressed) gauge group indices. We may then make an identification between the string coupling  $g_s$  and Yang-Mills gauge coupling  $g_{\text{YM}}$ ,

$$g_{\text{YM}}^2 = 4\pi g_s. \quad (1.7)$$

In the case of a stack of  $N$  D3-branes this theory is an  $SU(N)$  Yang-Mills theory<sup>2</sup> with a 4-extended supersymmetry [20]. The theory lives in 3+1 spacetime dimensions and in addition to the standard Poincaré group of spacetime symmetries possesses conformal symmetry. The field content is uniquely fixed for such a theory to be the gauge field  $A^\mu$ , four complex Weyl fermions  $\lambda_{\alpha r}$  ( $\alpha = 1, 2$ ,  $r = 1, 2, 3, 4$ ) and six real scalars  $\phi^I$ , all of which live in the adjoint representation of the gauge group. The action of  $\mathcal{N} = 4$  superconformal Yang-Mills is of the form

$$S_{SYM} = \frac{1}{g_{YM}^2} \text{Tr} \int d^4x \left\{ -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + D_\mu \phi_I D^\mu \phi^I + [\phi_I, \phi_J] [\phi^I, \phi^J] + \text{fermions} \right\}. \quad (1.8)$$

The global symmetries of this theory, the superconformal symmetry and R-symmetries, will be discussed in section 1.2.2.  $\mathcal{N} = 4$  SYM is in fact the unique field theory in 4 dimensions of maximal global spacetime symmetry [20].

It is this dual description of D-branes that leads directly to the  $AdS/CFT$  correspondence [1–4]. In the ‘low energy limit’  $\alpha' \rightarrow 0$  in which we may neglect massive string modes ( $\text{mass}^2 \propto \alpha'^{-1}$ ) the excitations of the effective world volume theory and the 10 dimensional effective theory in the bulk decouple. Meanwhile the same limit viewed from the perspective of the D-brane as sourcing a background for type IIB string theory decouples low energy excitations of the flat space far from the black hole from the arbitrarily large energy but red-shifted excitations close to the event horizon. Both perspectives describe two decoupled systems where one is supergravity in flat 10 dimensional space. The  $AdS/CFT$  conjecture is then that the other systems in each case are identified, namely, the type IIB excitations in the near-horizon geometry of the black hole are identified with the gauge theory on the world volume of the brane.

The D3-brane / black hole solution [21] will be presented below together with a discussion of the near horizon geometry on which the string theory that is conjectured to be dual to  $\mathcal{N} = 4$  SYM is defined. That near horizon geometry includes  $AdS_5$  in the directions along the world volume of the brane and in the radial direction in the transverse six dimensions, plus a 5-sphere due to the symmetry of those

---

<sup>2</sup>The theory on the stack of branes is in fact  $U(N)$ , but a  $U(1)$  decouples giving  $SU(N)$ .

transverse directions. The radius  $R$  of both parts of this space are equal. All of type IIB string theory on this background (together with a 5-form flux) is dual to all of  $\mathcal{N} = 4$  SYM. [1]

On each side of the correspondence we have two dimensionless parameters: the string coupling and the ratio of the common radius of  $AdS_5 \times S^5$  to the fundamental string length on the string side and the Yang-Mills coupling and the size of the gauge group on the gauge theory side,

$$\begin{aligned} \text{Type IIB : } & g_s \quad \frac{R}{l_s} \\ \mathcal{N} = 4 \text{ SYM : } & g_{\text{YM}} \quad N. \end{aligned} \quad (1.9)$$

The combination  $\lambda \equiv g_{\text{YM}}^2 N$  is known as the 't Hooft coupling and is the effective gauge theory coupling constant. The  $AdS/CFT$  correspondence is conjectured to hold when

$$\lambda \equiv g_{\text{YM}}^2 N = \frac{R^4}{l_s^4}. \quad (1.10)$$

$g_{\text{YM}}$  is related to  $g_s$  via (1.7).

The flat 4 dimensional space on which the field theory lives is to be found at the boundary of  $AdS_5$  [2, 3]. While  $d$  dimensional anti-de-Sitter space is not compact it has a notion of a  $d - 1$  dimensional boundary that is scale invariant, as will be discussed below. The gauge theory is then thought of as defined on this boundary, on which the asymptotic values of the bulk fields appear as currents that must couple to operators in the appropriate fashion.

For this reason we shall associate string states with gauge theory operators. A central prediction of the conjecture is that the energies of string states on  $AdS_5 \times S^5$  are equal to the conformal dimension  $\Delta$  of such operators in  $\mathcal{N} = 4$  SYM.

The correspondence is particularly simple in the limit that we have a large number  $N$  of stacked  $D$ -branes. If  $N \rightarrow \infty$  at fixed  $\lambda$  then from (1.10) and (1.7)  $g_s \rightarrow 0$  so that the strings becomes free and we may use the classical theory, a feature that emerges from an earlier intuition. The possibility of a string description of non-abelian gauge theory in this limit has been known since the 1970's when it was observed by 't Hooft [7] that by power counting the factors of  $\lambda$  and  $N$  that appear in the vertices and propagators of an arbitrary perturbative expansion it appears as

a double expansion over the 't Hooft coupling and, most significantly, the genus  $g$  of a 2 dimensional surface suggestive of the world sheet of a string,

$$\sum_{g=0}^{\infty} N^{2-2g} f(\lambda). \quad (1.11)$$

This then implies that surfaces of the lowest genus dominate processes at large  $N$  and as  $N \rightarrow \infty$  only tree level string diagrams, or *planar* field theory diagrams contribute.

What is more, if we take  $\lambda$  to be large so that the curvature of the background is small then we may use the supergravity approximation to string theory. In such a way classical supergravity may be used to describe *strongly* interacting *quantum* field theory.

It is clear then that the duality is of the strong / weak type: in the perturbative gauge theory regime  $R$  is small compared to  $l_s$  so that the string theory lives on a strongly curved background, and when at strong coupling in gauge theory curvature is small in the string theory. While one theory is tractable the other is intractable and vice-versa. On the one hand this allows predictions to be made about regimes of each theory that have been otherwise inaccessible, on the other hand this makes testing the conjecture in general very difficult.

In gauge theory the conformal dimension of an operator will in general run with the coupling developing an anomalous dimension that at strong coupling (large  $\lambda$ ) may become incalculable via perturbative field theory but ultimately calculable as the energy of a corresponding string state. In section 1.2.3 we will discuss a special class of operator for which the problem of strong / weak duality is overcome by virtue of their BPS nature; the conformal dimension is equal to that in the free theory. Along with comparison of the global symmetries of each theory, to be examined in section 1.2.2, these operators / states provide for an easy first check of the correspondence.

It is nevertheless desirable to find operators / states for which  $\Delta$  evolves with  $\lambda$  and may be calculated across a large range of  $\lambda$ . It is the discovery of integrability [22–27] in the  $AdS_5 \times S^5 / \mathcal{N} = 4$  SYM correspondence that has allowed a diversity of such instances to be found which generically consist of deformations

away from more trivially solvable limits. In section 1.3 we will review a few of the aspects of integrability relevant to the work in this thesis and in chapter 2 go on to review the semiclassical states of particular interest.

Below we will briefly discuss anti-de-Sitter space together with its conformal boundary and a couple of useful coordinate systems. Then we discuss the appearance of  $AdS_5 \times S^5$  as the near horizon geometry of a solution to the equations of motion derived from (1.3) that possesses the correct symmetries, interpreted as a stack of  $N$  D3-branes embedded in Minkowski space.

The quantum superconformal field theory lives on a 4 dimensional ‘boundary’ of  $AdS$  space [28]. This conformal-boundary is now identified. Firstly, we consider embedding  $AdS_{n+1}$  into a pseudo-Euclidean  $n+2$  dimensional space with coordinates  $y^a = (y^0, y^1, \dots, y^{n+1})$  and metric  $\eta_{ab} = \text{diag}\{+, -, \dots, -, +\}$  so that the length squared denoted  $y^2 \equiv y^a y^b \eta_{ab}$  is preserved by the action of elements  $\Lambda^a_b$  of the isometry group  $SO(2, n)$ ,

$$y^a \rightarrow y'^a = \Lambda^a_b y^b, \quad \Lambda^a_b \in SO(2, n), \quad (1.12)$$

$$y'^2 = y^2 = -(y^0)^2 - (y^{n+1})^2 + \sum_{i=1}^n y^i y^i. \quad (1.13)$$

The surface

$$y^2 = R^2 \quad (1.14)$$

is then  $AdS_{n+1}$  with ‘radius’  $R$ . It is a hyperboloid embedded in  $\mathbb{R}^{2,n}$ .

A set of coordinates for the hyperboloid (1.14) that cover the whole of the surface is given by

$$y^0 = R \cosh(\rho) \cos(\tau), \quad y^{n+1} = R \cosh(\rho) \sin(\tau), \quad (1.15)$$

$$y^i = R \sinh(\rho) w^i, \quad w^i w^i = 1, \quad i = 1, \dots, n. \quad (1.16)$$

These are the global coordinates of  $AdS_{n+1}$  and they give a metric of

$$ds^2 = R^2 (-\cosh^2(\rho) d\tau^2 + d\rho^2 + \sinh^2(\rho) d\Omega_{n-1}^2). \quad (1.17)$$

In fact to cover the whole hyperboloid only once we must restrict to  $0 \leq \tau \leq 2\pi$  (and  $0 \leq \rho$ ).

Another commonly used set of coordinates on  $AdS_{n+1}$  that cover only half of the space are the Poincaré coordinates  $(t, u, x^i)$ ,  $0 < u$ , given by

$$y_0 = \frac{1}{2u}(1 + u^2(R^2 + x^i x^i - t^2)), \quad y_{n+1} = R u t, \quad (1.18)$$

$$y^i = R u x^i, \quad i = 1, \dots, n-1, \quad (1.19)$$

$$y^n = \frac{1}{2u}(1 - u^2(R^2 - x^i x^i + t^2)). \quad (1.20)$$

Now the metric on  $AdS_{n+1}$  becomes

$$ds^2 = R^2 \left( \frac{du^2}{u^2} + u^2(-dt^2 + (dx^i)^2) \right). \quad (1.21)$$

The conformal-boundary of  $AdS_{n+1}$  is then defined by the equivalence class of coordinates under the scaling

$$y^a = b \tilde{y}^a, \quad b \rightarrow \infty. \quad (1.22)$$

The distance squared is then  $\tilde{y}^2 = -(\tilde{y}^0)^2 - (\tilde{y}^{n+1})^2 + \tilde{y}^i \tilde{y}^i = \frac{R^2}{b^2} \rightarrow 0$  and so

$$\{y^a\} \sim t\{y^a\}, \quad \forall t \in \mathbb{R} \quad (1.23)$$

are a set of coordinates on the conformal boundary. Using the scaling this may be taken to be  $(\tilde{y}^0)^2 + (\tilde{y}^{n+1})^2 = \tilde{y}^i \tilde{y}^i = 1$  so that it is clear that the boundary is  $n$  dimensional. At the same time this makes apparent an  $SO(2) \times SO(n) \subset SO(2, n)$  symmetry of the space at the boundary. In the following section we shall see the generators of the group  $SO(2) \times SO(4)$  appearing as special combinations of the generators of the conformal group on  $\mathbb{R}^{1,3}$  which leads us to the observation that the isometry group  $SO(2, 4)$  of  $AdS_5$  acts on the boundary of  $AdS_5$  as the conformal group acting on Minkowski space  $\mathcal{M}_4$ . This is precisely the appearance of the conformal nature of the theory on the boundary of anti-de-Sitter spacetime from a theory in the bulk.

Now we discuss<sup>3</sup> the appearance of  $AdS_5 \times S^5$  as the near horizon geometry of a black hole solution interpreted as a stack of  $N$  D3-branes [21]. The action (1.3)

---

<sup>3</sup>For this discussion we follow the lecture notes [18].

is directly obtained as the low energy effective theory of type IIB strings. To move to the Einstein frame in which the Ricci curvature and scalar kinetic terms appear canonically we perform a Weyl rescaling of the metric

$$g_{\mu\nu} \rightarrow e^{2\sigma\Phi} g_{\mu\nu}, \quad (1.24)$$

$$\sqrt{|g|}e^{-2\Phi}R \rightarrow \sqrt{|g|}e^{-\Phi(\sigma(D-2)+2)} \left\{ R + 2\sigma(D-1) \frac{1}{\sqrt{|g|}} \partial_\mu(\sqrt{|g|}\partial^\mu\Phi) \right. \quad (1.25)$$

$$\left. - \sigma^2(D-1)(D-2)(\partial_\mu\Phi\partial^\mu\Phi) \right\}. \quad (1.26)$$

With the choice  $\sigma = -\frac{2}{D-2}$  this becomes

$$S_E = -s \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{|g|} \left\{ R - \frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2} \sum_n \frac{1}{n!} e^{a_n\Phi} F_n^2 + \dots \right\} \quad (1.27)$$

with  $a_n = -\frac{1}{2}(n-5)$ . If for simplicity we assume only one of the field strengths  $F_n$  to be non-zero then from this action (1.27) the equations of motion for the metric are

$$R^\mu_\nu = \frac{1}{2}\partial^\mu\Phi\partial_\nu\Phi + \frac{1}{2n!}e^{a_n\Phi} \left( nF^{\mu\xi_2\dots\xi_n}F_{\nu\xi_2\dots\xi_n} - \frac{n-1}{D-2}\delta_\nu^\mu F_n^2 \right), \quad (1.28)$$

for  $\Phi$  is

$$\nabla^2\Phi = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}\partial_\nu\Phi g^{\mu\nu}) = \frac{a_n}{2n!}F_n^2 \quad (1.29)$$

and for the  $(n-1)$ -form potential  $A^{\xi_1\dots\xi_{n-1}}$  are

$$\partial_\mu(\sqrt{g}e^{a_n\Phi}F^{\mu\xi_2\dots\xi_n}) = 0. \quad (1.30)$$

We want static solutions to equations (1.28)-(1.30) with translational symmetry in  $p$  directions that will correspond to flat, solitonic  $p$ -branes.

With coordinates

$$z^\mu = (t, x^i, y^a), \quad \mu = 0, \dots, D-1, \quad i = 1, \dots, p, \quad a = 1, \dots, d, \quad D = p+1+d \quad (1.31)$$

and

$$r^2 \equiv \sum_{a=1}^d (y^a)^2 \quad (1.32)$$

where  $r$  is a radial coordinate in the  $d$  dimensional space transverse to the  $p$  dimensional flat space then an ansatz for the metric that respects  $p$  dimensional translational symmetry and is isotropic in the transverse space is

$$ds^2 = g_{\mu\nu}dz^\mu dz^\nu = sB^2dt^2 + C^2 \sum_{i=1}^p (dx^i)^2 + F^2dr^2 + G^2r^2d\Omega_{d-1}^2. \quad (1.33)$$

Here the metric components  $B$ ,  $C$ ,  $F$  and  $G$  all depend on  $r$  and  $d\Omega_{d-1}^2$  is the metric on the unit  $d-1$ -sphere. We shall require that as  $r \rightarrow \infty$  the metric becomes flat.

A generic solution to these equations then takes the form of a higher dimensional black hole with a horizon at  $r = r_0$ ; these are the non-extremal solutions, given when

$$B = f^{\frac{1}{2}} H^{-\frac{d-2}{\Delta}}, \quad C = H^{-\frac{d-2}{\Delta}}, \quad F = f^{-\frac{1}{2}} H^{\frac{p+1}{\Delta}}, \quad e^\Phi = H^{a_n \frac{D-2}{\Delta}} \quad (1.34)$$

$$ds^2 = H^{-2\frac{d-2}{\Delta}} \left( s f dt^2 + \sum_{i=1}^p (dx^i)^2 \right) + H^{2\frac{p+1}{\Delta}} (f^{-1} dr^2 + r^2 d\Omega_{d-1}^2) \quad (1.35)$$

with

$$H = 1 + \left( \frac{h}{r} \right)^{d-2}, \quad f = 1 - \left( \frac{r_0}{r} \right)^{d-2}, \quad \Delta = (p+1)(d-2) + \frac{1}{2} a_n^2 (D-2), \quad (1.36)$$

and

$$h^{2(d-2)} + r_0^{d-2} h^{d-2} = \frac{\Delta Q}{2(d-2)(D-2)}. \quad (1.37)$$

Then as  $r \rightarrow \infty$  it is clear that the metric components  $\rightarrow 1$  so that we recover flat space at an infinite distance from the source of the curvature. The parameter  $Q$  appears as a charge for the electric ansatz for the  $(p+2)$ -form field strength

$$F_{p+2} \equiv F_{ti_1 \dots i_p r} = \epsilon_{i_1 \dots i_p} H^{-2} \frac{Q}{r^{d-1}}. \quad (1.38)$$

For type IIB theory with  $n = 5$  we should really take  $F_5 + *F_5$  [18],  $*F_5$  being the Hodge dual

$$(*F_5)_{\mu_{n+1}, \dots, \mu_D} \equiv \frac{1}{5!} \frac{s}{\sqrt{g}} \epsilon_{\mu_1 \dots \mu_D} F^{\mu_1 \dots \mu_n}. \quad (1.39)$$

The simplest case is the extremal case where the event horizon of the background shrinks to zero size and equivalently the D-brane is unexcited. We must take  $r_0 = 0$ . Inserting  $D = 10$ ,  $p = 3$ ,  $d = 6$ ,  $n = 5$  gives  $\Delta = 16$ ,  $h^4 = \frac{Q}{4}$  and  $f = 1$ , and then

$$H = 1 + \frac{Q}{4r^4}. \quad (1.40)$$

The solution for the background metric is then

$$ds^2 = H^{-\frac{1}{2}} (s dt^2 + \sum_{i=1}^p (dx^i)^2) + H^{\frac{1}{2}} (dr^2 + r^2 d\Omega_{d-1}^2). \quad (1.41)$$

From (1.29) it is clear that because  $a_5 = 0$  the field strength  $F_5$  does not source the dilaton, which may therefore be taken to be constant,  $\Phi = 0$ .  $Q$  can be related to the electric charge density on the brane by

$$\mu_3 \equiv \frac{1}{\sqrt{16\pi G_{10}}} \int_{S^5} \tilde{F}_5 = \frac{\Omega_5 Q}{\sqrt{16\pi G_{10}}} \quad (1.42)$$

where  $\tilde{F}_5 = e^{a\Phi(=0)} * F$ . Alternatively, using the  $p$ -brane tension

$$T_p = \frac{2\pi}{(2\pi l_s)^{p+1} g_s} \quad (1.43)$$

and  $16\pi G_{10} = \frac{(2\pi l_s)^8}{2\pi} g_s$  we may write

$$\mu_3 = T_3 \sqrt{16\pi G_{10}}. \quad (1.44)$$

It is then apparent that for a stack of  $N$  D3-branes we would have

$$\mu_3 = NT_3 \sqrt{16\pi G_{10}} \quad (1.45)$$

and therefore the solution is modified by taking

$$Q = Ng_s \frac{(2\pi l_s)^4}{\Omega_5}, \quad h_5^4 = Ng_s \frac{(2\pi l_s)^4}{4\Omega_5}. \quad (1.46)$$

Introducing the rescaled variable  $u = \frac{r}{l_s^2}$  we examine the  $r \rightarrow 0$  limit of the geometry while taking  $l_s \rightarrow 0$  such that  $u$  remains finite. Hence  $H \approx \frac{4\pi g_s N}{l_s^4 u^4}$  as  $u \rightarrow 0$  and the metric (1.41) becomes

$$ds^2 = R^2 \left\{ \frac{du^2}{u^2} + u^2 d\tilde{x}^2 + d\Omega_5^2 \right\} \quad (1.47)$$

where the coordinates on the stack of D3-branes are rescaled as  $\tilde{x}^i = \lambda^{-\frac{1}{4}} x^i$ . This may be compared with equation (1.21) defining the metric of  $AdS_{n+1}$  in the Poincaré coordinate system. The rescaled variable  $u$  has appeared as the radius-like coordinate on an  $AdS_5$  space with the other 4 coordinates  $\{\tilde{x}^\mu\} = \{t, \tilde{x}^i\}$  being the coordinates longitudinal to the world volume of the D3 brane. Together with the independent spherical part of the metric that shares the same radius as the  $AdS$  part this is the metric of  $AdS_5 \times S^5$ .

### 1.2.2 Global symmetries

A zeroth order check of the correspondence is that the global symmetries on both sides match. It is also of course these symmetries that determine the quantum numbers possessed by the states of the corresponding theories and we shall need to understand parts of the resulting spectrum to proceed. We therefore present a

discussion of the global symmetry algebra present on both sides, though here from the perspective of the gauge theory.

Firstly we have the Poincaré group of symmetries [30] arising from the flat 4 dimensional space upon which the gauge theory may be defined. The Poincaré algebra is

$$i[J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\sigma\mu} J^{\rho\nu} + \eta^{\sigma\nu} J^{\rho\mu}, \quad (1.48)$$

$$i[P^\mu, J^{\rho\sigma}] = \eta^{\mu\rho} P^\sigma - \eta^{\mu\sigma} P^\rho, \quad (1.49)$$

$$[P^\mu, P^\rho] = 0. \quad (1.50)$$

The  $P^\mu$  generate translations and the  $J^{\mu\nu}$  rotations and Lorentz boosts.  $\eta^{\mu\nu}$  is the inverse of the Minkowski metric.

We also have the set of conformal transformations that preserve the form of the spacetime metric up to an arbitrary local scaling

$$g_{\mu\nu}(x) \rightarrow \Omega^2(x) g_{\mu\nu}(x) \quad (1.51)$$

which includes the action of a global scaling of the coordinates

$$x^\mu \rightarrow \lambda x^\mu. \quad (1.52)$$

The conformal algebra [29] includes the Poincaré algebra as a subalgebra and introduces two new generators,  $K^\mu$  generating the special conformal transformations and  $D$  generating scale transformations. The extra commutation relations are

$$[P^\mu, D] = iP^\mu, \quad [K^\mu, D] = -iK^\mu, \quad (1.53)$$

$$[P^\mu, K^\nu] = 2i\eta^{\mu\nu}D + 2iJ^{\mu\nu}, \quad [K^\mu, K^\nu] = 0, \quad (1.54)$$

$$[J^{\rho\sigma}, K^\mu] = i\eta^{\mu\nu}K^\sigma - i\eta^{\mu\sigma}K^\rho, \quad [J^{\rho\sigma}, D] = 0. \quad (1.55)$$

The vacuum of a conformal theory is annihilated by each of these generators individually and in combination. The infinitesimal group element

$$U(1 + \omega, \epsilon, \lambda, \rho) = 1 + \frac{i}{2}J_{\mu\nu}\omega^{\mu\nu} + iP_\mu\epsilon^\mu + i\lambda D + iK_\mu\rho^\mu \quad (1.56)$$

induces an infinitesimal spacetime transformation

$$x^\mu \rightarrow x^\mu + \omega^{\mu\nu}x_\nu + \epsilon^\mu + \lambda x^\mu + \rho^\mu x^\nu x_\nu - 2x^\mu \rho^\nu x_\nu \quad (1.57)$$

which is the most general infinitesimal transformation of the coordinates that leaves the light-cone invariant.

The algebra is isomorphic to that of the group  $SO(2, 4)$  whose generators are obtained by repackaging the generators of the conformal group as a new set  $\tilde{J}_{ab}$ ,  $a, b = \{0, 1, \dots, 5\}$  according to

$$\tilde{J}_{\mu\nu} = J_{\mu\nu}, \quad \tilde{J}_{\mu 4} = \frac{1}{2}(K_\mu - P_\mu), \quad \tilde{J}_{\mu 5} = \frac{1}{2}(K_\mu + P_\mu), \quad \tilde{J}_{54} = D. \quad (1.58)$$

The generator  $\tilde{J}_{05}$  corresponds to the  $SO(2) \subset SO(2) \times SO(4) \subset SO(2, 4)$  that appeared as the isometry group of the conformal boundary of  $AdS_5$  in the previous section while the generators  $\tilde{J}_{\mu\nu}$  correspond to the  $SO(4)$  part.

Operators are classified according to their commutators with the generator of dilations  $D$  whereupon an operator  $\mathcal{O}$  is said to have scaling dimension  $\Delta$  if  $[\mathcal{O}, D] = i\Delta\mathcal{O}$  and a field  $\phi(x)$  resulting from using the operator on the vacuum will be an eigenfunction of  $D$ , transforming as

$$\phi(x) \rightarrow \lambda^\Delta \phi(\lambda x) \quad (1.59)$$

under the scaling (1.52).

The conformal algebra (1.53)-(1.55) implies that the various conformal generators possess scaling dimensions

$$P^\mu : \Delta = +1 \quad (1.60)$$

$$J^{\rho\sigma} : \Delta = 0 \quad (1.61)$$

$$K^\mu : \Delta = -1 \quad (1.62)$$

$$D : \Delta = 0. \quad (1.63)$$

It can be seen that the operators  $P^\mu$  and  $K^\mu$  raise and lower the scaling dimension respectively. Using (1.53) then for some operator  $\Phi$  of dimension  $\Delta$  we have

$$DP^\mu \Phi |0\rangle = (P^\mu D - iP^\mu) \Phi |0\rangle \quad (1.64)$$

$$= -i(\Delta + 1)P^\mu \Phi |0\rangle \quad (1.65)$$

and

$$DK^\mu \Phi |0\rangle = (K^\mu D + iP^\mu) \Phi |0\rangle \quad (1.66)$$

$$= -i(\Delta - 1)P^\mu \Phi |0\rangle. \quad (1.67)$$

Since in any unitary quantum field theory there exists a lower bound on the dimension of a field then in a given representation of the conformal group there exists some operator of lowest dimension that is annihilated by  $K^\mu$ . These are the *primary* operators of a conformal field theory. In a given representation of the conformal group, specified by a Lorentz representation and a  $\Delta$ , we may then move between the various fields by application of  $P_\mu$  to the primary field. Fields in non-primary representations are named descendants.

In addition to conformal symmetry  $\mathcal{N} = 4$  SYM has an extended supersymmetry [20]. The supersymmetry algebra of  $\mathcal{N} = 4$  contains 16 fermionic supercharges combined into 4 sets of complex Majoranas  $Q_{\alpha r}$ ,  $\alpha = \pm\frac{1}{2}$ ,  $r = 1, \dots, 4$ , and their complex conjugates  $\bar{Q}_{\dot{\alpha} s}$ . These obey the standard SUSY algebra

$$\{Q_{\alpha r}, \bar{Q}_{\dot{\beta} s}\} = 2\delta_{rs}\sigma_{\alpha\dot{\beta}}^\mu P_\mu, \quad (1.68)$$

$$\{Q_{\alpha r}, Q_{\beta s}\} = 0 \quad (1.69)$$

$$\{\bar{Q}_{\alpha r}, \bar{Q}_{\dot{\beta} s}\} = 0 \quad (1.70)$$

where  $\sigma_\mu$  are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (1.71)$$

In general the right hand side of (1.69) (and (1.70)) could be equal to  $\epsilon_{\alpha\beta}Z_{rs}$  (and  $\epsilon_{\alpha\beta}\bar{Z}_{rs}$ ) where the  $Z_{rs}$  are central charges but in  $\mathcal{N} = 4$  these are all zero. There exists an R-symmetry group  $SU(4)$  under which the  $Q_{\alpha r}$  transform in the fundamental **4** and the  $\bar{Q}_{\dot{\alpha} r}$  in the anti-fundamental **4̄**.

Commutation of  $Q_{\alpha r}$  with  $K^\mu$  results in a further 16 fermionic generators [31]  $S_{\alpha r}$ ,  $\bar{S}_{\alpha r}$ . Defining<sup>4</sup>

$$K_{\alpha\dot{\alpha}} \equiv K_\mu\sigma_{\alpha\dot{\alpha}}^\mu, \quad P_{\alpha\dot{\alpha}} \equiv P_\mu\sigma_{\alpha\dot{\alpha}}^\mu \quad (1.72)$$

and

$$\sigma_{\alpha\dot{\beta}}^\mu \sigma_{\gamma\dot{\delta}}^\nu J_{\mu\nu} \equiv J_{\alpha\gamma}\epsilon_{\dot{\beta}\dot{\delta}} + \bar{J}_{\dot{\beta}\dot{\delta}}\epsilon_{\alpha\gamma} \quad (1.73)$$

---

<sup>4</sup>We largely follow the conventions of [18].

where

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (1.74)$$

we have firstly

$$[K_{\alpha\dot{\beta}}, Q_{\gamma r}] = 2i\epsilon_{\alpha\gamma}\bar{S}_{\dot{\beta}r}, \quad (1.75)$$

and then the other relations involving the fermionic generators in commutation or anti-commutation with the generators of the conformal algebra are

$$[Q_{\alpha r}, D] = \frac{i}{2}Q_{\alpha r}, \quad [P_{\alpha\dot{\beta}}, \bar{S}_{\dot{\gamma}r}] = 2i\epsilon_{\dot{\beta}\dot{\gamma}}Q_{\alpha r}, \quad (1.76)$$

$$\{Q_{\alpha r}, \bar{S}_{\dot{\beta} s}\} = \delta_{rs}K_{\alpha\dot{\beta}}, \quad (1.77)$$

$$\{Q_{\alpha r}, S_{\beta s}\} = \delta_{rs}(\epsilon_{\alpha\beta}D + J_{\alpha\beta}) + i\epsilon_{\alpha\beta}T_{rs}, \quad (1.78)$$

$$[\bar{S}_{\dot{\alpha} r}, T_{st}] = 2\delta_{rs}\bar{S}_{\dot{\alpha} t} - \frac{1}{2}\delta_{st}\bar{S}_{\dot{\alpha} r}. \quad (1.79)$$

The  $T_{rs}$  appearing here are the R-symmetry generators [20, 31] that by  $AdS/CFT$  become the generators of rotations about  $S^5$ , the angular momenta. These relations, together with the Poincaré and the rest of the conformal algebra, determine the algebra of the Lie super-group  $SU(2, 2|4)$ .

This completes the specification of the global spacetime symmetries of the SYM theory.

This is to be compared with the IIB theory on  $AdS_5 \times S^5$  where the relevant symmetry groups are more immediately realised as the set of isometries of the background. The isometry group of  $S^5$  is  $SO(6)$  while that of  $AdS_5$  is  $SO(2, 4)$ , however involvement of spinors requires that we consider the covering group of these which are, respectively,  $SU(4)$  and  $SU(2, 2)$ , and in fact the exact supergroup is given as  $SU(2, 2|4)$ .

There is also a discrete  $SL(2, \mathbb{Z})$  symmetry shared by both theories that will not concern us.

### 1.2.3 Spectrum and chiral primary operators

The semi-classical giant states that are the object of study in this thesis correspond under the  $AdS/CFT$  duality to integrable deformations away from a simple set of operators which are named chiral primary operators. It is the object of this

subsection to define these operators and place them within the spectrum of the gauge theory.

The fields of  $\mathcal{N} = 4$  in  $\mathbb{R}^{1,3}$  live in a single vector multiplet, containing spins not greater than one [20]. This consists of a Lorentz-vector field  $A_\mu$ , four complex Weyl fermions  $\lambda_{\alpha r}$  and six real scalars  $\phi^i$  where, as above, the  $r$  index lives in the  $\bar{\mathbf{4}}$  of  $SU(4)$  while the  $i$  lives in the  $\mathbf{6}$  of  $SU(4) \sim SO(6)$ . All of these fields then live in the adjoint representation of the gauge group,  $SU(N)$ , and the spectrum of operators must be formed from gauge invariant combinations thereof [4].

Local operators, where each of the fields are evaluated at the same spacetime point and composed of a single trace over the gauge group play a special role in that all gauge invariant operators must be expressible as (linear combinations of) products of such single traces (of products) [4]. Furthermore, the large  $N$  't Hooft limit suppresses the contributions of multi-trace operators by powers of  $N$ . Therefore we will be concerned often with operators of the form

$$\mathcal{O} \sim \text{Tr}(\mathcal{O}^{I_1} \mathcal{O}^{I_2} \dots \mathcal{O}^{I_n}) \quad (1.80)$$

were the  $\mathcal{O}^{I_i}$  are any of the operators of  $\mathcal{N} = 4$  SYM.

Representations of superconformal symmetry naturally contain multiple representations of the conformal symmetry, whose algebra (1.53)-(1.55) is a subalgebra of the full superconformal algebra. The various conformal primaries are stepped through by acting on the lowest state, annihilated now by both  $K_\mu$  and  $S_{\alpha r}$ , with the operators  $Q_{\alpha r}$  which raise the helicity by  $+\frac{1}{2}$ .

Application with a number of the  $Q_\alpha$  on a state of helicity  $s$  will in the most general case lead to a state with maximum helicity of  $s+4$  or a minimum helicity of  $s-4$ , while acting with more than 8  $Q_\alpha$  of the same helicity  $\alpha$  leads to annihilation or the creation of a conformal descendent. Smaller representations of the superconformal algebra exist however and in the small representation the range of helicities is between  $s-2$  and  $s+2$ . Fields in these representations are therefore annihilated by half of (or some combinations of) the supercharges and are known as the *chiral primary* operators.

For chiral primary operators, which are annihilated by  $K$ ,  $S$  and half of the  $Q$ , the scaling dimension can be determined in terms of the R-charges and Lorentz charges through the anti-commutation relation (1.78). It is as such that the dimension of chiral primary operators is protected from quantum corrections. In particular a single trace operator of the form (1.80) where each of the  $n$  operators inside the trace is just one of the scalar fields satisfies

$$\Delta = n \quad (1.81)$$

which is the value of the operator's dimension in the free theory with each scalar operator carrying a single unit of R-charge. In chapters 3, 4 and 5 we will calculate departures from this relationship, i.e. the development of an *anomalous dimension*, by considering string and brane configurations dual to operators that are no longer chiral primary.

### 1.3 Aspects of integrability

Integrability is now a ubiquitous feature of the  $AdS_5 \times S^5/\mathcal{N} = 4$  SYM correspondence [32], residing mainly in the planar  $N \rightarrow \infty$ ,  $g_s \rightarrow 0$  limit where the string is non-interacting. In this regime we may ignore phenomena such as string splitting and if  $\lambda$  is large then we typically require only the supergravity approximation to full string theory.

Nevertheless it has been pointed out that for operators / string states of high bare dimension such as we are concerned with in this thesis then the inclusion of purely supergravity fields, whose origin are the massless modes of strings, may be inadequate [33, 34]. The class of operators whose string duals we shall study in this thesis are deformations away from the simpler chiral primaries whose conformal dimensions are protected from running with  $\lambda$ . Such operators are supposed to be dual to excited string states obeying  $m^2 \propto \frac{1}{\alpha'}$  so that these operators shall have dimensions growing as  $\Delta \gtrsim \lambda^{\frac{1}{4}}$ .

In fact the work of Gubser, Klebanov and Polyakov [34] showed that we must invoke the use of string solitons as duals to these operators. In particular for oper-

ators with large R-charge the scaling of their conformal dimension is greater than the  $\lambda^{\frac{1}{4}}$  lower bound; it is  $\Delta \propto \sqrt{\lambda}$ .

The proportionality of the string charges to  $\sqrt{\lambda}$  appears throughout the chapters of this thesis. First, later in this chapter we shall introduce the relation  $\Delta = \Delta(n, \sqrt{\lambda})$  that appears as the dispersion relation of the fundamental excitations of the dual theories around the  $n \rightarrow \infty$  ‘vacuum’ [22]. In chapter 2 where we meet the semi-classical objects playing the main parts we shall see how the constants of motion again violate the  $\lambda^{\frac{1}{4}}$  scaling to reproduce that of the operators of this chapter.

Classically, the integrability of the string model was first shown by Pohlmeyer via the relation between  $O(n)$  sigma-models<sup>5</sup> and sine-Gordon theories [35, 36] and it is the map between these models that provides much of the focus of the research we present. In section 1.3.2 below we shall therefore demonstrate this connection explicitly, deriving the sine-Gordon equation from the string sigma-model after setting up our notation and the sigma-model itself in section 1.3.1.

Subsequently in section 1.3.3 we shall describe the sine-Gordon model and demonstrate some of its integrable aspects, and hence make the argument for the integrability of our string model. Finally in section 1.3.4 we shall introduce the structures and language of the large R-charge ( $n \rightarrow \infty$ ) limit of the gauge theory to which our string solitons are relevant; in particular we shall encounter the *magnons*.

### 1.3.1 The string sigma model

To derive the action and equations of motion for the string  $O(n)$  sigma model we begin with the Polyakov string whose action, ignoring world sheet fermions and working in the planar limit with large  $\lambda$ , is given by [5]

$$S = -\frac{1}{4\pi\alpha'} \int_{r_-}^r d\tau d\sigma \, \gamma^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu g_{\mu\nu}. \quad (1.82)$$

The target space fields are  $x^\mu$ ,  $\mu = 0, \dots, 9$ , the target space metric is  $g_{\mu\nu}$  and  $\gamma^{\alpha\beta} \equiv \sqrt{-h} h^{\alpha\beta}$  is the Weyl-invariant combination of the world sheet metric  $h_{\alpha\beta}$  and its determinant  $h$ .

---

<sup>5</sup>The  $n$  of  $O(n)$  is not related to the quantity of R-charge just discussed.

The target space metric is that of  $AdS_5 \times S^5$  but we shall be interested in strings that remain at rest in all spatial  $AdS_5$  directions. Using global coordinates (1.15) and (1.16) on  $AdS_5$  we set  $\rho = 0$ ; the whole string sits at the centre of  $AdS$  space but the (global) time coordinate is of course still important. The string is thus confined to move on the space  $\mathbb{R}_t \times S^5$ .

Turning to the coordinates on  $S^5$ , it is often convenient to deal with a target space embedded into a higher dimensional flat space. We may embed  $S^5$  into a Euclidean space of six dimensions,  $S^5 \subset \mathbb{R}^6$ , taking our coordinates  $x^\mu \equiv RX^\mu$  to be given by

$$X^\mu = \{X^0, X^i\}, \quad i = 1, \dots, 6, \quad g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1, 1, 1) \quad (1.83)$$

together with the ‘sigma model constraint’

$$X^i X^i \equiv X^2 = 1. \quad (1.84)$$

Observing that we have taken an overall scale factor of  $R^2$  out of the Lagrangian density the Polyakov action becomes, using the relation (1.10) for  $\lambda$ ,

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int_{r-}^r d\tau d\sigma \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}. \quad (1.85)$$

To make contact with the  $O(n)$  sigma model we must gauge fix the action. We use the static (partial) gauge where  $\tau$  is taken to be the global time variable of  $AdS_5$  introduced in (1.15). We then adopt the conformal gauge  $\gamma^{\alpha\beta} = \text{diag}(-1, 1)$ . This will impose the so called Virasoro constraints which will become of great importance in the following section.

The interacting Lagrangian density of the resulting sigma model may be re-expressed by inserting a Lagrange multiplier  $\Lambda(\sigma, \tau)$  into the ‘free’ Lagrangian density  $\mathcal{L}_0$ . Defining

$$\mathcal{L}_0 \equiv \partial_\alpha X^i \partial^\alpha X_i, \quad g_{ij} = \delta_{ij} \quad (1.86)$$

we solve

$$\partial_\gamma \left( \frac{\mathcal{L}_0}{\partial(\partial_\gamma X^\sigma)} \right) - \frac{\partial \mathcal{L}_0}{\partial X^\sigma} + \Lambda \frac{\partial G}{\partial X^\sigma} = 0 \quad (1.87)$$

with  $G \equiv X^i X_i - 1$  to find  $\Lambda = \partial_\gamma X^i \partial^\gamma X_i$ . Hence the classical string on  $\mathbb{R} \times S^5$  can be described by the Lagrangian density

$$\mathcal{L} = \frac{\sqrt{\lambda}}{4\pi} [\partial_\gamma X^i \partial^\gamma X_i + \partial_\gamma X^i \partial^\gamma X_i (X^i X_i - 1)]. \quad (1.88)$$

for freely fluctuating fields  $X^i \in \mathbb{R}^6$ . We obtain the Euler-Lagrange equations of motion

$$\partial_\alpha \partial^\alpha X^i + (\partial_\beta X^j \partial^\beta X_j) X^i = 0. \quad (1.89)$$

These are the  $O(n)$  sigma model equations.

We note that an immediate solution of the  $O(n)$  sigma model is given by (any constant  $O(n)$  rotation of)

$$(X^i) = (\cos(\omega t), \sin(\omega t), 0, \dots, 0) \quad (1.90)$$

which represents a point moving at speed  $R\omega$  around a great circle of the sphere  $S^{n-1}$ , which is a string with only zero modes excited. This solution has a special role to play in the  $AdS_5 \times S^5/\mathcal{N} = 4$  SYM correspondence which shall become clear in section 1.3.4.

### 1.3.2 Pohlmeyer reduction of the $O(3)$ model

The Pohlmeyer reduction [35] is a procedure by which, having identified quantities constructed from the target space fields that are invariant under the isometries of the target space, we reduce the original model to a new one by finding the equations of motion obeyed by these new quantities. The case of the  $O(3)$  sigma model is particularly simple, possessing only one field that may be thus defined. This single real scalar field can be shown to obey the sine-Gordon equation.

For other spaces we obtain theories with more scalar fields. For the  $O(n)$  sigma model with  $n > 3$  the Pohlmeyer reduced versions are the symmetric-space sine-Gordon models. These were constructed originally [37, 38] as the equations obtained as the condition for a field  $\omega$  in the algebra of the symmetry group of a symmetric space (such as  $SO(n)/SO(n-1)$ ) to obey the flat connection condition  $d\omega - \omega \wedge \omega = 0$  but can also be obtained via a gauged Wess-Zumino-Witten model [39]. The sine-Gordon model itself is the simplest member of this set of models.

In that the sine-Gordon model is a (prototypically) integrable relativistic field theory this reduction procedure is sufficient to prove the classical integrability of the string  $O(3)$  sigma model. However the correspondence between the two models obtains at the level of the classical solutions and does not extend to the equality of their

respective Lagrangians. It follows that the Poisson structures of the two theories are different and that the ensuing quantisation of the models will be different.<sup>6</sup>

In this thesis we will make extensive use of the relation between the  $O(3)$  sigma model and sine-Gordon theory. In particular, the sine-Gordon model possesses soliton solutions by virtue of integrability and the corresponding string solitons will be a major focus. We will introduce the sine-Gordon model in the following subsection after discovering its part in describing strings on  $\mathbb{R} \times S^2$ .

We did not mention the boundary conditions associated with the points  $\sigma = \pm r$  implicit from the range of  $\sigma$  appearing in the action (1.85). The range of the spatial world sheet coordinate in the conformal gauge is set (as it typically is) by the total energy of the string. In fact the conformal gauge is that in which the energy density of the string is a constant. Hence we must have  $E \propto r$ .

As has been mentioned this thesis is often concerned with strings (and D-branes) possessing divergent energy and angular momentum. We shall therefore be interested in either *bulk* world sheet theories without boundaries, or world sheets for which we consider only a single end point giving a boundary theory. Both possess domains of infinite extent (at least in one direction) in which we may sensibly consider excitations to be asymptotically separated at early and late times. For now we consider  $r = \infty$  and rename our world sheet coordinate  $\sigma$  as  $x$  where  $-\infty \leq x \leq \infty$  and  $\tau$  as  $t$ .

Adopting then light-cone coordinates  $\zeta = \frac{t+x}{2}$ ,  $\eta = \frac{t-x}{2}$ , and temporarily dropping the target space indices of  $X^i \equiv X$ , we denote  $X_\zeta \equiv \partial_\zeta X$ ,  $X_\eta \equiv \partial_\eta X$ , so that the sigma model equations (1.89) become

$$X_{\zeta\eta} + (X_\zeta \cdot X_\eta)X = 0. \quad (1.91)$$

Our sigma model and Virasoro constraints become

$$X^2 \equiv X \cdot X = 1, \quad (X_\zeta)^2 = 1, \quad (X_\eta)^2 = 1, \quad (1.92)$$

---

<sup>6</sup>A hope was entertained that by Pohlmeyer reducing the full superstring theory on  $AdS_5 \times S^5$  [40, 41] the Pohlmeyer reduced theory and the superstring theory would turn out to be quantum equivalent nonetheless [42]. However, it appears this may not be the case after all [43].

and from the sigma model constraint we also automatically satisfy

$$X_\zeta \cdot X = X_\eta \cdot X = 0. \quad (1.93)$$

The equations of motion (1.91) imply two conservation laws

$$\{(X_\eta)^2\}_\zeta = 0, \quad \{(X_\zeta)^2\}_\eta = 0 \quad (1.94)$$

which in turn tell us that we could have had  $(X_\zeta)^2 = h(\zeta)^2$  and  $(X_\eta)^2 = k(\eta)^2$  for some arbitrary functions  $h(\zeta)$  and  $k(\eta)$ . The Virasoro constraints (1.92) have fixed this local scale invariance.

The sole  $O(3)$ -invariant we may define is

$$\cos(\varphi) \equiv X_\zeta \cdot X_\eta \quad (1.95)$$

where we will show that  $\varphi$  is the sine-Gordon field defined by

$$\varphi_{\zeta\eta} = -\sin(\varphi). \quad (1.96)$$

From (1.95) we may write

$$\varphi_{\zeta\eta} = \partial_\eta \left\{ -\frac{\partial_\zeta(X_\zeta \cdot X_\eta)}{\sqrt{1 - \cos^2(\varphi)}} \right\} \quad (1.97)$$

for which we will require the second derivatives  $X_{\zeta\zeta}$  and  $X_{\eta\eta}$  which must ultimately be expressed in terms of  $\varphi$ . Given the relations (1.92) and that  $X_{\zeta\eta} \propto X$  by (1.91) we see that the three vectors  $\{X, X_\zeta, X_\eta\}$  span  $\mathbb{R}^3$  and may be taken as a local basis in which to express the remaining second derivatives. Hence we write

$$X_{\zeta\zeta} = \alpha_1 X + \beta_1 X_\zeta + \gamma_1 X_\eta, \quad (1.98)$$

$$X_{\eta\eta} = \alpha_2 X + \beta_2 X_\zeta + \gamma_2 X_\eta, \quad (1.99)$$

where the coefficients will be functions of  $\varphi$  and its derivatives for which we must solve. For example, we have

$$X_{\zeta\zeta} \cdot X = \alpha_1 X^2, \quad \text{and} \quad X_{\zeta\zeta} \cdot X = \partial_\zeta[X_\zeta \cdot X] - (X_\zeta)^2 = 0 - 1, \quad (1.100)$$

which together imply that  $\alpha_1 = -1$ . Less trivially we obtain relations such as

$$X_{\zeta\zeta} \cdot X_\zeta : \quad 0 = \beta_1 + \gamma_1 \cos(\varphi), \quad (1.101)$$

$$X_{\zeta\zeta} \cdot X_\eta : \quad -\varphi_\zeta \sin(\varphi) = \beta_1 \cos(\varphi) + \gamma_1. \quad (1.102)$$

Altogether the second derivatives are given by (1.91) and

$$X_{\zeta\zeta} = -X + \varphi_\zeta \cot(\varphi) X_\zeta - \varphi_\zeta \operatorname{cosec}(\varphi) X_\eta, \quad (1.103)$$

$$X_{\eta\eta} = -X - \varphi_\eta \operatorname{cosec}(\varphi) X_\zeta + \varphi_\eta \cot(\varphi) X_\eta. \quad (1.104)$$

The equation (1.97) for  $\varphi_{\zeta\eta}$  becomes

$$\varphi_{\zeta\eta} = \varphi_\eta \frac{\cos(\varphi)}{\sin^2(\varphi)} X_{\zeta\zeta} \cdot X_\eta - \frac{1}{\sin(\varphi)} \{ \partial_\zeta (X_{\zeta\eta}) \cdot X_\eta + X_{\zeta\zeta} \cdot X_{\eta\eta} \} \quad (1.105)$$

which using the above becomes

$$\varphi_{\zeta\eta} = -\frac{1}{\sin(\varphi)} \{ \varphi_\zeta \varphi_\eta [-\cos(\varphi) + \cos(\varphi)] + \cos^2(\varphi) - 1 \} \quad (1.106)$$

$$\Rightarrow \varphi_{\zeta\eta} = -\sin(\varphi). \quad (1.107)$$

The world sheet physics of the classical string on  $\mathbb{R} \times S^2$  is therefore seen to be governed by a new relativistic (though not scale invariant) equation, the sine-Gordon equation.

### 1.3.3 Sine-Gordon model and integrability

As the sine-Gordon model is integrable we expect to be able to associate with the sine-Gordon field an infinite number of constants of motion, the hierarchy of conserved charges, and a Bäcklund transformation. The former tell us that the model is in some sense infinitely constrained, and quantum mechanically leads to the factorisation of the multi-particle S-matrix into two-particle factors [44]. The latter allows one to start with a known solution to the equations of motion and obtain another solution, typically adding or subtracting a soliton [35].

In this subsection we will first introduce the sine-Gordon model in its own right, following this by an examination of the Bäcklund transformation for the sine-Gordon field and relating this to a conservation law and thence to an infinite hierarchy of covariant conserved currents.

Sine-Gordon theory of a single scalar field  $\varphi(x, t) \in \mathbb{R}$  may be defined by the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \{ \dot{\varphi}^2 - \varphi'^2 \} - g (1 - \cos(\beta\varphi)). \quad (1.108)$$

By a field redefinition  $\beta\varphi \rightarrow \varphi$  we have only one parameter appearing, a coupling equal to  $g\beta^2$ . Classically we are free to pick  $g = \beta^{-2}$  so that we may instead consider

$$\mathcal{L} = \frac{1}{2} \{ \dot{\varphi}^2 - \varphi'^2 \} - (1 - \cos(\varphi)). \quad (1.109)$$

The Euler-Lagrange equation of motion is then

$$\ddot{\varphi} - \varphi'' = -\sin(\varphi) \quad (1.110)$$

which is the equation  $\varphi_{\zeta\eta} = -\sin(\varphi)$  we have seen above in light-cone coordinates.

The potential appearing in (1.109) is

$$V(\varphi) = 1 - \cos(\varphi) \quad (1.111)$$

which has an infinite number of equally spaced minima at  $\varphi = 2n\pi$ ,  $n \in \mathbb{Z}$ . If we allow the field  $\varphi$  to have enough energy then it is possible to traverse one or more of the maxima of  $V(\varphi)$  so that the boundary conditions at  $x \rightarrow \pm\infty$  are different and the field configuration interpolates between different vacua of the theory. These are the soliton solutions of sine-Gordon theory. Field configurations interpolating between different vacua are not homotopically equivalent, being classified by the integers, and hence stabilised topologically; the sine-Gordon solitons are prototypical *topological* solitons [45].

In fact the energy of the static 1-soliton solution may be found prior to knowledge of the exact solution via the Bogomolny bound [46]. Setting time derivatives to zero the Hamiltonian  $H$  for a single scalar field with some potential  $V$  is

$$H[\varphi] = \int_{-\infty}^{\infty} dx \left[ \frac{1}{2} \varphi'^2 + V \right] \quad (1.112)$$

which may be rearranged into the form

$$H[\varphi] = \frac{1}{2} \int_{-\infty}^{\infty} dx (\varphi' \mp \sqrt{2V})^2 \pm \int_{\varphi(-\infty)}^{\varphi(\infty)} \sqrt{2V} d\varphi. \quad (1.113)$$

Due to the square in the integrand of the first term we may say that

$$H[\varphi] \geq \pm \int_{\varphi(-\infty)}^{\varphi(\infty)} \sqrt{2V} d\varphi. \quad (1.114)$$

With the potential of sine-Gordon theory given by (1.111) and the values of  $\varphi(x)$  taking vacuum values  $\varphi(-\infty) = 2n\pi$ ,  $\varphi(\infty) = 2m\pi$ ,  $n, m \in \mathbb{Z}$ , this gives a bound on the energy of a static solution of

$$E \geq 2 \int_{2n\pi}^{2m\pi} \left| \sin\left(\frac{\varphi}{2}\right) \right| d\varphi = 8(m-n). \quad (1.115)$$

The bound is saturated when  $\varphi' = \pm\sqrt{2V(\varphi)}$ , as seen from (1.113). Consequently we have a first order differential equation for  $\varphi(x)$  which when solved may be boosted to produce a one parameter family of soliton solutions to the sine-Gordon equation (1.110). Specifically we have

$$\frac{d\varphi}{\left| \sin\left(\frac{\varphi}{2}\right) \right|} = \pm 2dx \quad (1.116)$$

$$\Rightarrow \text{sign}\left[\sin\left(\frac{\varphi}{2}\right)\right] \ln\left(\left|\tan\left(\frac{\varphi}{4}\right)\right|\right) = \pm(x - x_0). \quad (1.117)$$

If we pick  $n = 0$  and  $m = 1$ , i.e. a solution that interpolates between  $\varphi = 0$  and  $\varphi = 2\pi$ , then  $\text{sign}\left[\sin\left(\frac{\varphi}{2}\right)\right] = 1$  and the solution is

$$\varphi(x) = 4 \tan^{-1}(\exp(-(x - x_0))) \quad (1.118)$$

which has energy  $E = 8$  and is a kink centred at  $x = x_0$ .

A generic Bäcklund transformation is an equation (or set of equations) relating two fields  $f$  and  $g$ ,  $H(f, g) = 0$ , such that the fields  $f$  and  $g$  separately satisfy their own equations of motion,  $F(f) = 0$  and  $G(g) = 0$ . Hence if we know a solution  $f_0$  of  $F(f) = 0$  and the Bäcklund transformation  $H(f, g) = 0$  then we may in principle solve for a solution  $g_0$  of  $G(g) = 0$ .  $H(f, g)$  will involve a parameter named the spectral parameter which in relativistic theories will normally become the boost parameter of the new solution. When the equations  $F = 0$  and  $G = 0$  are identical then we have what is strictly referred to as an auto-Bäcklund transformation.

The (auto-)Bäcklund transformation for the sine-Gordon field in light-cone coordinates is given by the two equations

$$\frac{1}{2}(\psi + \varphi)_\zeta = \gamma^{-1} \sin\left(\frac{\psi - \varphi}{2}\right), \quad (1.119)$$

$$\frac{1}{2}(\psi - \varphi)_\eta = -\gamma \sin\left(\frac{\psi + \varphi}{2}\right), \quad (1.120)$$

where  $\psi_{\zeta\eta} = -\sin(\psi)$  and  $\varphi_{\zeta\eta} = -\sin(\varphi)$ . It is immediately clear from  $\gamma$ 's appearance with respect to the light-cone derivatives that it is a rapidity variable. A known solution to the sine-Gordon equation  $\varphi$  will now produce a new solution  $\psi = \psi(\gamma)$ .

It can be checked that the Bäcklund transformation equations are self consistent in that they imply the satisfaction of the sine-Gordon equation by  $\psi$  if it is satisfied by  $\varphi$ , and vice-versa. For example, taking  $\partial_\zeta$  of equation (1.120) and substituting from equation (1.119) we find

$$\frac{1}{2}(\psi - \varphi)_{\zeta\eta} = -\sin\left(\frac{\psi - \varphi}{2}\right) \cos\left(\frac{\psi + \varphi}{2}\right) \quad (1.121)$$

$$= -\frac{1}{2}(\sin(\psi) - \sin(\varphi)), \quad (1.122)$$

or

$$\psi_{\zeta\eta} + \sin(\psi) = \varphi_{\zeta\eta} + \sin(\varphi) \quad (1.123)$$

which obviously holds if both fields satisfy the sine-Gordon equation.

If we take a vacuum solution given by  $\varphi = 0$  then (1.119) and (1.120) give

$$\frac{d\psi}{\sin\left(\frac{\psi}{2}\right)} = \frac{2}{\gamma} d\zeta \quad \text{and} \quad \frac{d\psi}{\sin\left(\frac{\psi}{2}\right)} = -2\gamma d\eta \quad (1.124)$$

which together give

$$\psi(\zeta, \eta; \gamma) = 4 \tan^{-1} \left( \exp \left( \gamma^{-1} \zeta - \gamma \eta \right) \right) \quad (1.125)$$

which is the well known 1-soliton solution to sine-Gordon theory in light-cone coordinates; a boosted version of the static solution (1.118) obtained from the saturation of the Bogomolny bound.

The Bäcklund transformation equations also imply a conservation law. If we multiply equation (1.119) by the factor  $\sin\left(\frac{\psi+\varphi}{2}\right)$  and equation (1.120) by  $\sin\left(\frac{\psi-\varphi}{2}\right)$  then we obtain two equations involving identical products of sines but different powers of the parameter  $\gamma$ . We may consequently equate the two expressions

$$-\gamma \left\{ \frac{1}{2}(\psi + \varphi)_\zeta \sin\left(\frac{\psi + \varphi}{2}\right) \right\} = \gamma^{-1} \left\{ \frac{1}{2}(\psi - \varphi)_\eta \sin\left(\frac{\psi - \varphi}{2}\right) \right\} \quad (1.126)$$

which we further recognise as total derivatives,

$$-\gamma \left\{ \cos\left(\frac{\psi + \varphi}{2}\right) \right\}_\zeta = \gamma^{-1} \left\{ \cos\left(\frac{\psi - \varphi}{2}\right) \right\}_\eta. \quad (1.127)$$

and to be hence a light-cone conservation law for conserved currents

$$j_\zeta = \cos\left(\frac{\psi - \varphi}{2}\right), \quad j_\eta = \cos\left(\frac{\psi + \varphi}{2}\right). \quad (1.128)$$

From this law an infinite hierarchy of conserved charges may be derived. We begin by expanding the field  $\psi = \psi(\zeta, \eta; \gamma)$  in a series in  $\gamma$  about  $\gamma = 0$ ,

$$\psi(\zeta, \eta; \gamma) = \psi_0 + \psi'_0 \gamma + \frac{1}{2} \psi''_0 \gamma^2 + \dots \quad (1.129)$$

where the zero subscript means we have evaluated at  $\gamma = 0$  and the number of primes indicates the order of  $\partial_\gamma$ . Substituting this into the conservation law (1.127) we obtain a polynomial in  $\gamma$  which involves variously mixed derivatives with respect to both  $\zeta$  and  $\eta$  as well as with respect to  $\gamma$ . By separately setting these coefficients to zero at each order of  $\gamma$  we obtain an infinite sequence of conserved currents from which the charges may be obtained.

We have  $\psi_0 = \varphi$  immediately. To solve for the derivatives of  $\psi$  with respect to  $\gamma$ ,  $\{\psi'_0, \psi''_0, \dots\}$ , we can use the Bäcklund transformation equations again. From (1.119) we find

$$\psi'_0 = 2\varphi_\zeta, \quad \psi''_0 = 4\varphi_{\zeta\zeta}, \quad \psi'''_0 = 12\varphi_{\zeta\zeta\zeta}, \quad \dots \quad (1.130)$$

while from (1.120) we find

$$\psi'_{0\eta} = -\sin(\varphi), \quad \psi''_{0\eta} = -4\varphi_\zeta \cos(\varphi), \quad \dots. \quad (1.131)$$

At the first three orders of  $\gamma$  we have

$$\gamma^0 : \quad 0 = 0, \quad (1.132)$$

$$\gamma^1 : \quad \{\cos(\varphi)\}_\zeta + \left\{ -\frac{(\psi'_0)^2}{8} \right\}_\eta = 0, \quad (1.133)$$

$$\gamma^2 : \quad 0 = 0. \quad (1.134)$$

Every even order we expect a trivial identity and at every odd order an independent conservation law involving increasing numbers of derivatives. So, the first non-trivial relation, upon substituting for  $\psi'_0$ , re-expressing in terms of derivatives with respect to  $x$  and  $t$  and integrating with respect to  $x$  gives

$$\left[ \cos(\varphi) + \frac{1}{2}(\partial_t \varphi + \partial_x \varphi)^2 \right]_{-\infty}^{\infty} = \partial_t \int_{-\infty}^{\infty} dx \left\{ -\cos(\varphi) + \frac{1}{2}(\partial_t \varphi + \partial_x \varphi)^2 \right\}. \quad (1.135)$$

Assuming that the field  $\varphi \rightarrow 2n\pi$ ,  $n \in \mathbb{Z}$  as  $x \rightarrow \pm\infty$  and that the derivatives of  $\varphi$  vanish in the same limit then the left hand side is zero, leaving us with the expression for a conserved charge

$$Q_1 \equiv \int_{-\infty}^{\infty} dx \left\{ \frac{1}{2}(\partial_t \varphi)^2 + \frac{1}{2}(\partial_x \varphi)^2 - \cos(\varphi) \right\} + \int_{-\infty}^{\infty} dx (\partial_t \varphi \partial_x \varphi). \quad (1.136)$$

The combination appearing is the sum of the Hamiltonian  $H = \int dx \mathcal{H}$  (actually minus a constant contribution to the potential) and the momentum, which are separately the charges associated with the conserved currents expressed through

$$\partial_\mu T^{\mu\nu} = 0, \quad T^{00} = \mathcal{H}, \quad T^{01} = \partial_t \varphi \partial_x \varphi. \quad (1.137)$$

So the first conserved charges express translational symmetry in  $x$  and  $t$ . Higher conserved charges appearing at order  $\gamma^n$ ,  $n \geq 3$ , have a less straightforward physical interpretation.

A final comment is due as to the nature of the map between sine-Gordon and string solutions.

It is useful to see the Virasoro constraints (1.92) and definition of the sine-Gordon field (1.95) in the standard conformal gauge coordinates on the world sheet. The constraints become

$$\dot{X}^2 + X'^2 = 1, \quad \dot{X} \cdot X' = 0 \quad (1.138)$$

while the definition of the sine-Gordon field is

$$\dot{X}^2 = \cos^2 \left( \frac{\varphi}{2} \right), \quad X'^2 = \sin^2 \left( \frac{\varphi}{2} \right). \quad (1.139)$$

These equations tell us both that the parametrisation we have chosen gives  $\dot{X}^i$  as the transverse velocity of the string and that the absolute value of this velocity is controlled by the sine-Gordon field. At the same time the temporal and spatial parameterisation are linked such that when  $\dot{X}^2 = 1$  we must have a target space cusp point on the string,  $X'^i = 0$ .

In this way we obtain an intuition for the string motion described by the sine-Gordon field. Vacua of the sine-Gordon theory,  $\varphi = 2n\pi$ ,  $n \in \mathbb{Z}$  map to a point-like string moving with  $\dot{X}^2 = 1$ . We have already mentioned this solution to the sigma

model, equation (1.90) (with  $\omega = 1$ ). In the following section we shall learn more of its significance. We also see immediately that the kink solutions to sine-Gordon theory must in some way describe string solutions that interpolate between two point-like segments of string.

### 1.3.4 BMN limit, spin chains and the magnon spectrum

The string state corresponding to the operator

$$\text{Tr}(Z^n) \tag{1.140}$$

composed of  $n$  of one of the complex scalar fields  $Z$  (previously we used  $\phi^I \in \mathbb{R}$ ,  $I = 1, \dots, 6$ ) is the unique state with zero light-cone Hamiltonian; it has

$$\Delta - J = 0. \tag{1.141}$$

The SYM operator is chiral primary and (1.141) is the statement (1.81) made in section 1.2.3; the number of units of this single R-charge,  $n$ , equals the string's angular momentum,  $J$ .

When the number of scalar fields in the operator is taken to infinity its dimension and R-charge separately diverge (the energy and angular momentum of the string diverge) while (1.141) holds. The string picture is a point-like string moving around an equator of  $S^5$  at  $\dot{X}^2 = 1$ , which as we have seen in equation (1.90) above is a simple solution of the sigma model and the vacuum state of the sine-Gordon field.

Exciting the string about this state, about the light-cone vacuum or about the sine-Gordon vacuum, will then produce corrections to (1.141) that in general will be of the form

$$\Delta - J = f(J, \lambda) \tag{1.142}$$

and on the gauge theory side involves operators that are no longer chiral primaries. This was considered by Berenstein, Maldacena and Nastase [22]. The simplest such excitation is given by operators in which a single extra field (not a  $Z$ ) is placed among the  $Z$ 's of (1.140). Up to cyclic permutivity the order in which the new field is inserted is important and in general a phase dependent upon the position of the

insertion along the chain of  $Z$ 's must be included. For example, another complex scalar  $W$  is inserted amid a chain of  $J$   $Z$ 's as

$$\text{Tr}(Z^l W Z^{J-l}) e^{\frac{2\pi i m l}{J}}. \quad (1.143)$$

In the double limit that one component of the string's angular momentum  $J \rightarrow \infty$  and in the free string / planar limit  $N \rightarrow \infty$  (which with  $g_{\text{YM}}$  kept fixed so that the gauge 't Hooft coupling  $\lambda \rightarrow \infty$  also) Berenstein, Maldacena and Nastase then showed that those of these operators for which  $f(J, \lambda) = 1$  are in one to one correspondence with the modes of the string in flat space<sup>7</sup> when summed over all possible positions along the string. The chain of  $Z$ 's becomes the physical string while the insertions describe excitations on the world sheet.

Integrability raised its head when it was shown that the problem of diagonalising the planar Dilation operator [47] (and hence finding values of  $\Delta - J$ ) reduces to the problem of diagonalising the Hamiltonian of a certain type of known spin chain [23, 48]. Indeed the form of the operators considered above are immediately reminiscent of spin chains where the position of an operator is discretised and the ferromagnetic vacua appear as uniform choices of a single orientation of the fields at each site. From (1.143) the excitation has a momentum given by

$$p \equiv \frac{2\pi m}{J}. \quad (1.144)$$

In the language of spin chains the excitations created by the insertions of new operators ('flipped spins') are named *magnons* which may be seen as travelling on the world sheet of the physical string that emerges from the spin chain picture [49].

The remaining symmetry of a state created by the insertion of a single magnon into the spin chain vacuum turned out to be enough to calculate the S-matrix for the magnons along the spin chain (or along the string world sheet) up to an overall phase [50]. In addition the superconformal algebra determined the dispersion relation for

---

<sup>7</sup>In fact they achieved this result on a “pp wave” geometry which is the geometry felt by the string as it moves at the speed of light about the equator of  $S^5$ .

an individual magnon of spin chain momentum (or world sheet momentum)  $p$  to be

$$\Delta - J = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2\left(\frac{p}{2}\right)}. \quad (1.145)$$

This relation at first seemed strange because of the periodicity in  $p$  that while arising naturally from the discreteness of the spin chain seems alien to the world sheet of a string that should not be discrete. The resolution of this issue will be discussed in the following chapter.

# Chapter 2

## Giant states: semi-classical strings and branes in $AdS/CFT$

Semi-classical strings have provided a niche of the  $AdS/CFT$  duality in which often exact or analytic expressions may be obtained for observables [51–53]. We are interested in the anomalous conformal dimension of Yang-Mills operators and string energies where we would ideally like to obtain  $\Delta(\lambda) = E(\lambda)$  for all  $\lambda$ . Before the recent discovery of integrable sectors [32] of  $\mathcal{N} = 4$  SYM / strings on  $AdS_5 \times S^5$  almost the only sector where this may be achieved was  $\frac{1}{2}$  BPS (single trace, chiral primary) operators [4] dual to point-like supergravity string states where  $\Delta$  is protected from scaling with  $\lambda$ .

In this chapter we shall familiarise ourselves with string theory states that, through being integrable, have allowed a derivation of their properties that may be examined also on the field theory side of the duality at arbitrary  $\lambda$ . We will find focus on the giant magnons and giant gravitons with which the research reported in this thesis is concerned.

The giant magnons [9] are semi-classical string states with large angular momentum on  $S^5$  that could be thought of as generalisations of a class of string solutions with analogues in flat space that move rigidly with fixed angular momentum and this is how they will be introduced. In the standard 't Hooft limit strings are non-interacting ( $g_s \rightarrow 0$  as  $N \rightarrow \infty$  with  $\lambda$  fixed) and at large  $\lambda$  become classical. There is

therefore no string splitting and the configurations moving on the spherical subspace of  $AdS_5 \times S^5$  will be solutions of the classical  $O(6)$  sigma model whose equations were discussed in chapter 1. The states we are considering are semi-classical in the sense that they possess the same quantum numbers as the equivalent, properly quantum states but with certain quantum numbers taken to be very large [51].

The giant gravitons [10] are finite energy, stable D-branes on  $AdS_5 \times S^5$  which arise as the description of certain graviton states with large momentum in a compact direction, preserving the same supersymmetries as the point-graviton [54]. The world-volume itself is compact and will host a low energy effective theory that is of Born-Infeld type [19] plus a coupling to the Ramond-Ramond background [11]. We will describe how these branes, arising naturally in a type IIB string theory, then give rise to the presence of open strings which in chapter 4 we shall see may be interpreted as the ‘‘boundary giant magnons’’ ending on the giant graviton brane which are studied in chapter 3.

## 2.1 Spinning strings to giant magnons

String states are characterised by their conserved charges which in our case are the momenta associated with the isometry group of  $AdS_5 \times S^5$  which is  $SO(2, 4) \times SO(6)$ . If we use global coordinates on  $AdS_5$  then we have an energy  $E$  associated with the global time coordinate and two spins  $S_1, S_2$  while from the  $S^5$  part of the space we will have three independent momenta  $J_1, J_2, J_3$ . The Virasoro constraints typically then provide for a relation between the charges that allows for the energy to be expressed in terms of  $\lambda$  and the other charges [51].

In the following sections we will describe the giant magnons, motivating them by reference to spinning string configurations first in flat space and then on a spherical background. While the spinning string on  $\mathbb{R} \times S^5$  already describes a non-trivial operator of  $\mathcal{N} = 4$  SYM (that is, a non-chiral primary operator) creating excitations around the BMN vacuum whose anomalous conformal dimension scales as  $\sqrt{\lambda}$  [34] the giant magnon introduces a generalisation of this allowing for the description of

the fundamental degrees of freedom associated with these excitations.

### 2.1.1 Spinning strings

Beginning with spinning closed strings in a flat background we consider a solution where the string moves on  $\mathbb{R}^{1,2}$  with coordinates  $(t, x^\mu)$ ,  $\mu = 1, 2$ . The string action (1.82) in the conformal, static ( $\tau = \kappa t$ ) gauge is

$$S = -\frac{1}{4\pi\alpha'} \int_0^{2\pi} d\sigma \int dt (\dot{x}^2 - x'^2) \quad (2.1)$$

and the equations of motion are the wave equation

$$\ddot{x}_\mu - x''_\mu = 0 \quad (2.2)$$

together with Virasoro constraints  $\dot{x}^2 + x'^2 = 1$  and  $\dot{x} \cdot x' = 0$ .

The action (2.1) is invariant under shifts in  $t$  and  $\phi$ , hence we have the conserved charges

$$E = -\frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma p_t, \quad J = -\frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma p_\phi \quad (2.3)$$

where  $E$  is the target space energy and  $J$  the angular momentum.

Perhaps the simplest non-trivial closed string solution is the rigidly rotating folded string. In flat space this string famously reproduces the Regge trajectories of the hadrons [55] for which the lowest excited state for a given angular momentum  $J$  obeys  $E \propto \sqrt{J}$ . The flat space solution is

$$x_1 + ix_2 = r \sin(\sigma) e^{i\tau} \quad (2.4)$$

where the energy is related to the angular momentum as  $E = 2\pi T_0 \kappa = \sqrt{\frac{2}{\alpha'} J}$  and  $J = \frac{r^2}{2\alpha'}$  is the statement that the cusp (fold) points of the string must travel at the speed of light. At fixed values of the other string charges (here just fixed  $J$ ) the string energy satisfies

$$E^2 \propto \frac{1}{\alpha'} \quad (2.5)$$

so that this string is a semiclassical limit of states built from massive string excitations.

The generalisation of this configuration to strings moving on  $\mathbb{R} \times S^5$  [34] can be found, for example, as a limit of the spinning string on  $\mathbb{R} \times S^3$  [56, 57]. This string has two independent angular momenta on  $S^3$  and depending upon their relative values describes either the folded closed string we desire or a wrapped closed string.

Defining the angular frequencies in orthogonal directions on  $S^3$  to be  $\omega_1$  and  $\omega_2$  we construct the quantities

$$q \equiv \frac{\kappa^2 - \omega_1^2}{\omega_{12}^2}, \quad \omega_{21}^2 \equiv \omega_2^2 - \omega_1^2 \quad (2.6)$$

with  $\kappa$  as above. The folded string has  $q \leq 1$ . It is then possible to derive two equations involving the energy  $E$  and angular momenta  $J_1$  and  $J_2$  related via complete elliptic integrals<sup>1</sup>,

$$\frac{4q\lambda}{\pi^2} = \frac{E^2}{K(q)^2} - \frac{J_1^2}{E(q)^2}, \quad (2.7)$$

$$\frac{4\lambda}{\pi^2} = \frac{J_2^2}{(K(q) - E(q))^2} - \frac{J_1^2}{E(q)^2}. \quad (2.8)$$

If we take  $J_1 = 0$  (so  $\omega_1 = 0$ ) so that the remaining angular momentum is  $J_2 \equiv J$  then these reduce to

$$E = 2\frac{\sqrt{\lambda}}{\pi}\sqrt{q}K(q), \quad J = 2\frac{\sqrt{\lambda}}{\pi}(K(q) - E(q)). \quad (2.9)$$

with the parameter  $q = \kappa^2\omega^{-2}$ ,  $\omega_2 \equiv \omega$ .

We shall encounter all but identical expressions for open string analogues of the spinning string in chapter 3 where we shall also discuss the behaviour of  $E = E(J, \lambda)$ .

### 2.1.2 The Giant Magnon

The giant magnon [9] is the large  $\lambda$  string solution that is dual to the operator equivalent to a spin chain magnon [49, 53, 58] which is meant to be the fundamental excitation through which all other excited states of the planar theory may be expressed. A magnon exceeds the BPS bound  $\Delta - J = 0$  obeyed by chiral primary operators and carries a fixed value of momentum  $p$  along the spin chain, or alternatively along the world sheet of the string, which in the BMN limit [22] must be sent

---

<sup>1</sup>See Appendix A for the properties of elliptic integrals.

to zero as  $J \rightarrow \infty$ . In order to discover the giant magnon string itself Hofman and Maldacena [9] kept  $p$  fixed while  $J \rightarrow \infty$ :

$$J \rightarrow \infty, \quad p \text{ fixed}, \quad \lambda = g_s N = \text{fixed and large.} \quad (2.10)$$

In addition it is of course required that  $E - J$  be fixed. Keeping  $\lambda$  fixed, as opposed to allowing  $\lambda \rightarrow \infty$  as in the analysis of [22], means that we are able to sensibly discuss the interpolation to small  $\lambda$  where the field theory is perturbatively accessible.

With the knowledge that the BPS ground state satisfies  $E - J = 0$ , a point-like string on a light-like trajectory, an ansatz may be made to look for a configuration that has fixed world sheet momentum  $p$  and reproduces the relation (1.145) at large  $\lambda$ . It can then be shown that the configuration is the minimum energy one for a given  $p$  [9], which will be discussed below.

As we have discussed (at the end of section 1.3.3) by reference to our knowledge of sine-Gordon kink solutions that interpolate between vacua we expect string configurations that interpolate between point-like configurations with  $\dot{X}^2 = 1$  that we now know correspond to BMN vacua of the gauge theory. We therefore follow Hofman and Maldacena in making an ansatz for the string that satisfies these boundary conditions.

We require solutions to the closed string however, so that these two BMN-like points cannot be string end points. Nevertheless they may be treated as end points. As discussed in section 1.3.2, a conformal gauge string with  $J \rightarrow \infty$  and hence  $E = \Delta \rightarrow \infty$  must have a world sheet spatial coordinate with infinite range. Excitations on the world sheet must then feel themselves to be in the bulk, always an infinite distance from any possible end point, or after scattering with a finite number of other excitations to be eventually asymptotically well separated.

In this way we may look for solutions to a temporarily open string representing an isolated excitation on the world sheet. Two or more such open strings must then be linked end to end to create a genuine closed string. Such a string is then the asymptotic solution, long before or after the various individual excitations have interacted.

Taking the ansatz that the originally point-like string is excited in a single direction,  $\theta$ , transverse to its trajectory in the  $\phi$  direction and remains rigid ( $\theta$  is

independent of  $t$ ) then the motion takes place on  $\mathbb{R} \times S^2$  with metric

$$ds^2 = R^2 [-dt^2 + d\theta^2 + \sin^2(\theta)d\phi^2] \quad (2.11)$$

where  $t$  is once again the global time coordinate of  $AdS_5$ . Using the static gauge and parametrising the worldsheet spatial direction by  $\sigma = \tilde{\phi} = \phi - t$  the Nambu string action

$$S = -\frac{T}{2} \int d\tau d\sigma R^2 \sqrt{(\dot{X}^\mu X'_\mu)^2 - \dot{X}^2 X'^2} \quad (2.12)$$

must be expanded using

$$X^\mu = \begin{pmatrix} t \\ \sin(\theta) \\ \cos(\theta) \left\{ \cos(t) \cos(\tilde{\phi}) - \sin(t) \sin(\tilde{\phi}) \right\} \\ \cos(\theta) \left\{ \sin(t) \cos(\tilde{\phi}) + \cos(t) \sin(\tilde{\phi}) \right\} \end{pmatrix} \quad (2.13)$$

and after much cancellation becomes

$$S = -\frac{T}{2} \int dt d\sigma \sqrt{\cos^2(\theta) \theta'^2 + \sin^2(\theta)}. \quad (2.14)$$

Abbreviating  $s \equiv \sin(\theta)$  then the Euler-Lagrange equations are

$$s = s'' - \frac{s'(s's'' + ss')}{s'^2 + s^2} \quad (2.15)$$

$$\Rightarrow s^2 s'' - 2s s'^2 - s^3 = 0 \quad (2.16)$$

which upon substitution of  $u = \frac{1}{s}$  returns

$$u'' = -u, \quad \Rightarrow \quad u = A\cos(\sigma) + B\sin(\sigma). \quad (2.17)$$

Together with the boundary conditions that at the string end points  $\theta = \frac{\pi}{2}$  and at  $\sigma = 0$  we have some  $\theta|_{\sigma=0} = \theta_0$  then the solution is

$$\sin(\theta) = \frac{\sin(\theta_0)}{\cos(\sigma)}, \quad -\left(\frac{\pi}{2} - \theta_0\right) \leq \sigma \leq \frac{\pi}{2} - \theta_0. \quad (2.18)$$

This is Hofman and Maldacena's giant magnon dual to the fundamental magnon of the spin chain operators of planar  $\mathcal{N} = 4$  SYM. It is giant because it typically extends across target space to lengths  $\sim R$ .

In the comoving coordinates  $(\tilde{\phi}, \theta)$  the solution has the form  $\sin(\theta) \cos(\tilde{\phi}) = \sin(\theta_0) = \text{constant}$ , which is to say that when projected into the  $(X_5, X_6)$  plane (for

example, so that  $J_{56} \equiv J$ ) the giant magnon forms a straight line joining two points on the equator, or a chord. In fact the giant magnon is a semi-circle on  $S^2$  with its two ends on the equator and radius  $\cos(\theta_0)$ . In normal coordinates it rotates rigidly with  $\dot{\phi} = 1$ . See Figure 2.1.

This simple geometry of the solution could in fact be expected. If the equation of motion (2.16) is written in terms of the coordinates  $(x, y)$  in the plane through the equator of the sphere then each term is proportional to either  $\frac{dy}{dx}$ ,  $(\frac{dy}{dx})^2$  or  $\frac{d^2y}{dx^2}$  indicating that a simple solution will be  $y$  equal to a constant, and hence a semi-circular shape on the sphere. Any other chord within the disc will also be described by the same equation upon some constant rotation in the plane.

A more useful gauge in which to describe the giant magnon is the conformal (plus static) gauge. Here the Hofman-Maldacena limit can be thought of as a rescaling of the world sheet coordinates  $(\tau, \sigma) \rightarrow (\kappa\tau, \kappa\sigma) \equiv (x, t)$  such that these new coordinates are held fixed as

$$\kappa = \frac{\Delta}{\sqrt{\lambda}} \rightarrow \infty. \quad (2.19)$$

This defines the decompactifying limit whereby excitations on the world sheet may be considered to be asymptotically separated. The conformal gauge gives a constant energy density in  $x$  and so we know that away from the centre of the excitation as  $x \rightarrow \pm\infty$  the string carries a large amount of energy so that  $x \rightarrow \pm\infty$  are the BMN-like end points of the string at the equator. The two points in  $\sigma$  that would have been identified to form a closed string are now separated in  $x$  by an infinite distance and the two ‘ends’ either side of an excitation around a point  $x$  on the world sheet may be considered as genuine string end points; the decompactifying limit allows us to open up the closed string.

With  $S^2$  coordinates  $\{\tilde{\phi}, \theta\}$  now both functions of  $x$  and  $t$  the equations of motion can be simplified, one component giving

$$\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \tan(\tilde{\phi}) + 2v\gamma \tan(\tilde{\phi}) \frac{\partial}{\partial t} \tan(\tilde{\phi}) = 0 \quad (2.20)$$

which is solved by

$$\tan(\tilde{\phi}) = \frac{1}{v\gamma} \tanh(\zeta_-), \quad \zeta_- = \gamma(x - vt), \quad \gamma^2 = (1 - v^2)^{-1}. \quad (2.21)$$

With the boundary conditions  $\theta = \frac{\pi}{2}$  as  $x \rightarrow \pm\infty$  and  $\theta|_{x=0} = \theta_0$  then we can deduce that  $v\gamma = \tan(\theta_0)$ , or

$$\zeta_- = \frac{x - \sin(\theta_0)t}{\cos(\theta_0)}, \quad (2.22)$$

and the full co-moving string solution is

$$X^i = \begin{pmatrix} \cos(\theta_0) \operatorname{sech}(\zeta_-) \\ \sin(\theta_0) \\ \cos(\theta_0) \tanh(\zeta_-) \end{pmatrix}. \quad (2.23)$$

Again it is clear that one component in the  $(X_5, X_6)$  plane is a constant so that the giant magnon is a chord when projected down from the sphere. See Figure 2.1.

Using complex coordinates  $Z_1 = e^{it}(X_5 + iX_6) = re^{i(t-\phi)}$  and picking  $Z_2 = X_3$  so that the divergent angular momentum  $J \equiv J_{56}$  points in the  $Z_2$  direction we have an angular momentum density

$$j = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \mathcal{I} \left[ \frac{\partial \dot{Z}_1}{\partial \dot{\phi}} \frac{\partial \mathcal{L}}{\partial \dot{Z}_1} \right] = \mathcal{I} \left[ Z_1 \dot{\bar{Z}}_1 \right]. \quad (2.24)$$

Direct substitution of the solution confirms that the angular momentum  $J = T \int_{-\infty}^{\infty} j dx$  is divergent upon integration over the domain  $x \in (-\infty, \infty)$ . The energy is of course divergent also, being given by  $E = T \int_{-\infty}^{\infty} dx$ , but the difference is finite:

$$E - J = T \int_{-\infty}^{\infty} (1 - j) dx = T \cos(\theta_0) \int_{-\infty}^{\infty} \operatorname{sech}^2(\zeta_-) d\zeta_- \quad (2.25)$$

which with the crucial identification

$$\cos(\theta_0) = \sin\left(\frac{p}{2}\right) \quad (2.26)$$

and  $T = \frac{\sqrt{\lambda}}{2\pi}$  gives

$$E - J = \frac{\sqrt{\lambda}}{\pi} \sin\left(\frac{p}{2}\right). \quad (2.27)$$

This quantity is indeed the large  $\lambda$  limit of (1.145) as required. Note that  $p$  has the interpretation of the angle subtended by the giant magnon end points at the equator. This is then the “giant” explanation for the periodicity of the relation (1.145) in  $p$  given by Hofman and Maldacena.

Of course we must remember that the giant magnon is not on its own a valid closed string (it is not even closed!) so we must join a number of giant magnons

end to end such that the momenta, or angles  $p_i$ , obey  $\sum_i p_i = 2m\pi$ ,  $m \in \mathbb{Z}$ , hence closing the string. In this case, as would be expected for asymptotically separated excitations, the charges of the individual components simply add,

$$\Delta - J = \sum_i \frac{\sqrt{\lambda}}{\pi} \sin\left(\frac{p_i}{2}\right). \quad (2.28)$$

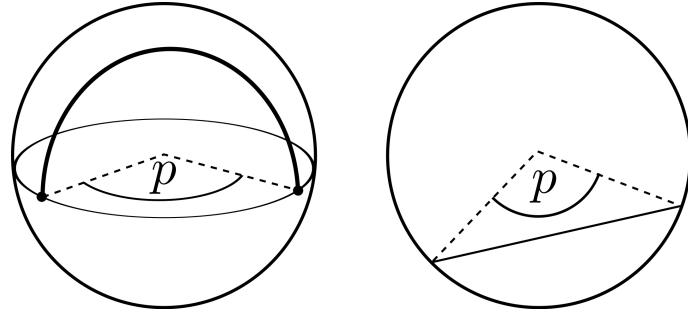


Figure 2.1: The geometry of the giant magnon. The string forms a semi-circular arc on  $S^2$  with its end points at the equator around which they move at the speed of light. The projection onto the plane is a chord subtending an angle of  $p$ , the magnon momentum.

The extra 1 under the square root in (1.145) should appear upon quantisation of the string and can be seen to do so in the plane wave limit [22] (at  $p \rightarrow 0$ ) where this has been possible. Nevertheless Hofman and Maldacena make an argument for fixed  $p$  and large  $\lambda$  that the exact relation is (1.145) by resort to the action of the superconformal algebra.

It is in the conformal gauge that we may also adopt the Pohlmeyer reduced description of the string to make contact with the sine-Gordon picture of a giant magnon. Using the definitions of the sine-Gordon field from equation (1.95) and taking derivatives of (2.23) we have

$$\cos(\varphi) = \tanh^2(\zeta_-) - \operatorname{sech}^2(\zeta_-) \quad (2.29)$$

or

$$\tan\left(\frac{\varphi}{4}\right) = e^{-\zeta_-}, \quad \zeta_- = \frac{x - \cos\left(\frac{p}{2}\right)t}{\sin\left(\frac{p}{2}\right)}, \quad (2.30)$$

which is indeed a 1-soliton kink solution of the sine-Gordon field. The soliton propagates with velocity  $v = \cos\left(\frac{p}{2}\right)$  on the world sheet. Recalling that the radius of the semicircular giant magnon is  $\cos(\theta_0) = \sin\left(\frac{p}{2}\right)$  we see that fast solitons (small  $p$ ) are small giant magnons that approach the BMN limit as  $p \rightarrow 0$  while slow solitons ( $p \rightarrow \pi$ ) are large giant magnons which for  $p = \pi$  become precisely one half of the spinning string discussed above.

We see that the decompactification of the world sheet afforded by taking the  $J \rightarrow \infty$  limit (with string tension fixed) now leads us to consider *bulk* sine-Gordon solitons. The sine-Gordon picture is at this point very useful for it allowed Hofman and Maldacena to calculate the overall phase factor  $\delta$  of the magnon scattering matrix

$$\mathcal{S} = e^{i\delta} \hat{\mathcal{S}} \quad (2.31)$$

in the large  $\lambda$  limit where the classical string description is appropriate.

The world sheet time variable (which in the static gauge is equal to the global *AdS* time) is also the time variable for the sine-Gordon field so that time differences translate from the sine-Gordon picture to the string picture. A semi-classical phase shift  $\delta$  for the world sheet excitation is then obtained from a time delay  $\Delta t$  due to scattering of the sine-Gordon solitons by integration with respect to the energy  $E$ . This is an instance of the semi-classical Levinson's theorem [9, 59],

$$\Delta t = \frac{\partial \delta}{\partial E}. \quad (2.32)$$

If we were calculating the phase for semi-classical sine-Gordon theory then  $E$  would be the energy of the scattering soliton which experiences the delay. We are however interested in the phase shift for excitations of the string for which the energy is not equal to that of the corresponding sine-Gordon soliton. In fact the energies are inverse to one another,

$$E_{\text{string}}^{\text{light-cone}} = E_{\text{string}} - J = \frac{1}{E_{\text{sine-Gordon}}} = \frac{\sqrt{\lambda}}{\pi} \sin\left(\frac{p}{2}\right). \quad (2.33)$$

The time delay  $\Delta t$  may be found by examination of the asymptotics of a 2-soliton solution. As  $t \rightarrow \pm\infty$  the solution tends to the linear superposition of two travelling single-solitons where the  $t \rightarrow +\infty$  configuration is delayed compared to the  $t \rightarrow -\infty$

configuration. Such a calculation will be presented explicitly for a 3-soliton solution in Chapter 5 so here we make a brief presentation of the (simplest) 2-soliton case. The 2-soliton kink / anti-kink solution from [60] for example, or (5.5), is<sup>2</sup>

$$\varphi = 4 \tan^{-1} \left( \frac{e^{-\zeta_1 + \frac{a_1}{2}} - e^{-\zeta_2 + \frac{a_2}{2}}}{1 + \tanh^2 \left( \frac{\theta_{12}}{2} \right) e^{-\zeta_1 - \zeta_2 + \frac{a_1 + a_2}{2}}} \right) \quad (2.34)$$

where  $\tanh \left( \frac{\theta_{12}}{2} \right) \equiv \tanh \left( \frac{\theta_1 - \theta_2}{2} \right)$  and

$$\zeta_1 = \cosh(\theta_1)x - \sinh(\theta_1)t, \quad \zeta_2 = \cosh(\theta_2)x - \sinh(\theta_2)t. \quad (2.35)$$

The initial positions  $a_1$  and  $a_2$  can be set to zero without harm, or alternatively the combination  $a_- \equiv \frac{a_1 - a_2}{2}$  may be set to zero while an overall translation in time may be used to remove the combination  $a_+ \equiv \frac{a_1 + a_2}{2}$ . If soliton 1 is the right-mover then the appropriate asymptotics are achieved by scalings

$$t \rightarrow -\infty, \quad x \rightarrow -\infty, \quad \zeta_1 \text{ fixed,} \quad \text{and} \quad (2.36)$$

$$t \rightarrow +\infty, \quad x \rightarrow +\infty, \quad \zeta_1 \text{ fixed.} \quad (2.37)$$

To perform the integral for  $\delta$  with respect to the string energy in terms of  $p_1$  we must re-express

$$\begin{aligned} \tanh \left( \frac{\theta_1 - \theta_2}{2} \right) &= \tanh \left( \frac{\tanh^{-1}(c_1) - \tanh^{-1}(c_2)}{2} \right) \\ &= \tanh \left( \frac{1}{4} \ln \left[ \frac{(1 + c_1)(1 - c_2)}{(1 - c_1)(1 + c_2)} \right] \right) \\ &= \frac{\left[ \frac{(1 + c_1)(1 - c_2)}{(1 - c_1)(1 + c_2)} \right]^{\frac{1}{2}} - 1}{\left[ \frac{(1 + c_1)(1 - c_2)}{(1 - c_1)(1 + c_2)} \right]^{\frac{1}{2}} + 1} \end{aligned} \quad (2.38)$$

$$= \sqrt{\frac{1 - \cos \left( \frac{p_1 - p_2}{2} \right)}{1 - \cos \left( \frac{p_1 + p_2}{2} \right)}}. \quad (2.39)$$

The result we find for the scattering time delay experienced by soliton 1 as it passes soliton 2 is then

$$\Delta t = \frac{\sin \left( \frac{p_1}{2} \right)}{\cos \left( \frac{p_1}{2} \right)} \ln \left\{ \frac{1 - \cos \left( \frac{p_1 - p_2}{2} \right)}{1 - \cos \left( \frac{p_1 + p_2}{2} \right)} \right\}. \quad (2.40)$$

---

<sup>2</sup>In [9] the calculation was performed by taking the simpler looking centre of mass solution and performing a boost.

Applying (2.32) we find

$$\delta(p_1, p_2) = -4g \left( \cos \frac{p_1}{2} - \cos \frac{p_2}{2} \right) \ln \left\{ \frac{\sin^2 \frac{p_1 - p_2}{4}}{\sin^2 \frac{p_1 + p_2}{4}} \right\} - 4gp_1 \sin \frac{p_2}{2}. \quad (2.41)$$

Up to an ambiguity to be discussed in chapter 5 this is the overall phase produced in [9] for the scattering of two magnons of arbitrary  $p$  in the bulk.

### 2.1.3 Finite $J$ giant magnons

The giant magnons above possess strictly infinite angular momentum  $J$ . We may consider finite angular momentum giant magnons [61–64] while remaining in the large  $\lambda$ ,  $N \rightarrow \infty$  limit however, retaining the classical dynamics and non-interaction of the string. What we lose by taking finite  $J$  is the infinite volume simplification: at finite  $J$  the dispersion relation (2.27) will be modified by finite size effects as we move from the theory on a plane to the theory on a cylinder, and the consideration of giant magnon excitations in isolation on the world sheet becomes entirely unphysical.

In the following chapter we will be concerned with finite  $J$  solutions for the *boundary* giant magnons that are very closely related to finite  $J$  giant magnons below. In chapter 4 we will see how the effective world volume theory of D3-branes at finite  $J$  reproduces their behaviour.

The analysis of giant magnons at finite  $J$  was performed by Arutyunov, Frolov and Zamaklar in [61]. The authors employed a uniform gauge [65]

$$x_+ = (1 - a)t + a\phi = \tau, \quad x_- = \phi - t, \quad p_+ = (1 - a)p_\phi - ap_t = \text{const.} \quad (2.42)$$

which is a generalisation of the light-cone gauge with gauge parameter  $a$  in which the Hamiltonian is always  $E - J$ . One consequence of describing a single giant magnon in a finite volume is that the solution and dispersion relation become gauge dependent, explicitly involving the parameter  $a$ . This is intuitively understood to be a consequence of opening the closed string up, therefore involving the selection of a special set of points on the formerly reparametrisation invariant world sheet.

For example, the dispersion relation for a single finite  $J$  giant magnon becomes

$$E - J = \frac{\sqrt{\lambda}}{\pi} \sin \left( \frac{p}{2} \right) \left\{ 1 - \frac{4}{e^2} \sin^2 \left( \frac{p}{2} \right) e^{-\mathcal{R}} + \dots \right\} \quad (2.43)$$

where the exponential correction is controlled by

$$\mathcal{R} = \frac{2\pi}{\sqrt{\lambda}} \frac{J}{\sin\left(\frac{p}{2}\right)} + ap \cot\left(\frac{p}{2}\right) \quad (2.44)$$

and the ellipsis stands for higher powers of the correction. For comparison with the conformal gauge we are to take  $a = 0$ . For a closed string made of multiple giant magnons we are again able to add the values for  $\Delta - J$  with the proviso that each of the  $p_i$  (and hence the size of the giant magnon) are now equal.

Of particular relevance for the work presented in the following chapter is the case of  $p = \pi$  which is equal to the half folded string, or ‘half-GKP’ solution (in reference to [34]), for which we have the  $a$ -independent quantity

$$E - J = \frac{\sqrt{\lambda}}{\pi} \left\{ 1 - \frac{4}{e^2} e^{-\mathcal{J}} + \dots \right\}, \quad \mathcal{J} = \frac{2\pi}{\sqrt{\lambda}} J. \quad (2.45)$$

To obtain the conformal (and static) gauge solution the authors of [61] make the ansatz

$$\tilde{\phi} = \tilde{\phi}(x - u\omega t), \quad \cos(\theta) = \cos(\theta)(x - u\omega t) \quad (2.46)$$

for some ‘light-cone’ coordinate  $\tilde{\phi} = \phi - \omega t$ . The conformal gauge velocity of the excitation is  $v = u\omega$  and we see that  $\omega$  is the angular frequency with which a point labelled by  $\tilde{\phi}$  on the string moves about the sphere. The solution for one of the coordinates is

$$\cos(\theta) = \frac{1}{\gamma\omega\sqrt{\eta}} \operatorname{dn}\left(\frac{1}{\sqrt{\eta}}\gamma(x - ut), \eta\right), \quad \eta = \frac{\gamma^2}{\omega^2}(1 - \omega^2 v^2). \quad (2.47)$$

Here,  $\operatorname{dn}$  is one of the Jacobi elliptic functions (see Appendix A) with  $\eta$  the elliptic modulus and  $\gamma^{-2} = (1 - u^2)$ . While  $\phi(x, t)$  is more complicated it is again true that the change in the angle  $\phi$  is equal to the world sheet momentum,  $\Delta\phi = p_{\text{ws}}$ .

That (2.45) is independent of the gauge parameter is a signal that perhaps it could be found without the use of the methods of [61]; it is after all almost just the value of  $E - J$  for the spinning folded string. Indeed, in chapter 3 we shall rederive this result (up to a factor of 2), including the elliptic solution (2.47), for the finite  $J$  boundary giant magnon, which is very similar to both of these configurations.

### 2.1.4 Further generalisations of the giant magnon

Another generalisation of the original giant magnons are those with multiple independent angular momenta on  $S^5$  [66–68]. We may have up to three independent angular momenta on  $S^5$ ; the rank of the isometry generating algebra  $\mathfrak{su}(4) \sim \mathfrak{so}(6)$  is three. Giant magnons with two such angular momenta, named dyonic giant magnons, were considered by Chen, Dorey and Okamura [66] and live on  $\mathbb{R} \times S^3$ . In lieu of soliton solutions for the  $O(4)$  string sigma model, equivalently the  $SU(2)$  principal chiral model, the authors employed knowledge of the Pohlmeyer reduced theory which in this case is the *complex* sine-Gordon model.

Complex sine-Gordon theory [70, 71] is a theory of a complex scalar field  $\psi$  which in light-cone coordinates  $x_{\pm} = (t \pm x)/2$  satisfies

$$\partial_{-+}^2 \psi + \bar{\psi} \frac{\partial_+ \psi \partial_- \psi}{1 - |\psi|^2} + \psi(1 - |\psi|^2) = 0. \quad (2.48)$$

Equivalently we may write this as a theory of two real scalars  $\varphi$  and  $\chi$  combined as  $\psi = \sin(\frac{\varphi}{2})e^{i\frac{\chi}{2}}$  to give the coupled equations of motion

$$\begin{aligned} \partial_{+-}^2 \varphi + \sin(\varphi) - \frac{\sin(\varphi)}{(1 + \cos(\varphi))^2} \partial_{-} \chi \partial_{+} \chi + \\ \frac{i}{(1 + \cos(\varphi))} \{ \sin(\varphi) \partial_{-+}^2 \chi + \partial_{-} \chi \partial_{+} \varphi + \partial_{+} \chi \partial_{-} \varphi \} = 0. \end{aligned} \quad (2.49)$$

Immediately we see that the limit  $\partial_{-} \chi = \partial_{+} \chi = 0$  returns the sine-Gordon equation in light-cone coordinates,  $\partial_{-+}^2 \varphi + \sin \varphi = 0$ .

The Pohlmeyer reduction for the sigma model on  $\mathbb{R} \times S^3$  [35] proceeds similarly to the  $O(3)$  sigma model / sine-Gordon case described in section 1.3.2 above by identifying scalar quantities invariant under the isometries of  $S^3$ . Along with the original sine-Gordon field defined through

$$\cos \varphi = \partial_{-} X^i \partial_{+} X^i \quad (2.50)$$

we have two more defined as

$$u \sin(\varphi) = \partial_{+}^2 X^i K^i, \quad v \sin(\varphi) = \partial_{-}^2 X^i K^i \quad (2.51)$$

where the Euclidean vector  $K^i$  is defined to be orthogonal to  $X^i$  itself and its derivatives by

$$K^i = \epsilon^{ijkl} X^j \partial_{-} X^k \partial_{+} X^l. \quad (2.52)$$

The quantities  $u$  and  $v$  are related to  $\varphi$  and  $\chi$  through

$$u = \partial_+ \chi \tan\left(\frac{\varphi}{2}\right), \quad v = -\partial_- \chi \tan\left(\frac{\varphi}{2}\right) \quad (2.53)$$

and the resulting equation of motion is (2.48).

Note that if we restrict the motion to  $\mathbb{R} \times S^2$  then the vector  $K^i$  defined in (2.52) is zero so that by equations (2.51) and (2.53) we will have constant  $\chi$  and hence the Pohlmeyer reduced theory is appropriately the sine-Gordon model by (2.49).

To derive the dyonic giant magnon solution the authors of [66] use the known 1-soliton solution to the complex sine-Gordon model and exploit the form of the string sigma model equations to create a linear system that the string target space coordinates must obey. The 1-soliton solution is

$$\psi = e^{i\mu} \frac{\cos(\alpha) e^{i \sin(\alpha) T}}{\cosh(\cos(\alpha)(X - X_0))} \quad (2.54)$$

with

$$X = \cosh(\theta)x - \sinh(\theta)t, \quad T = \cosh(\theta)t - \sinh(\theta)x \quad (2.55)$$

and constant phase  $\mu$ , charge parameter  $\alpha$  and shift  $X_0$ , while the sigma model equations (1.89) may be written

$$\partial_{-+}^2 X^i + \cos(\varphi) X^i = 0. \quad (2.56)$$

In terms of the boosted coordinates  $X$  and  $T$ , and complexified string coordinates  $Z_1$  and  $Z_2$ , we have

$$\partial_{-+}^2 Z^i + \left[ 1 - \frac{2 \cos^2(\alpha)}{\cosh^2(\cos(\alpha)X)} \right] Z^i = 0 \quad (2.57)$$

which is a known scattering problem for complex scalar particles in 1+1 dimensions<sup>3</sup>.

The solution satisfying the giant magnon boundary conditions

$$Z_1 \rightarrow e^{it \pm i \frac{p}{2}}, \quad Z_2 \rightarrow 0 \quad \text{as } x \rightarrow \pm\infty \quad (2.58)$$

turns out to be

$$Z_1 = \frac{1}{\sqrt{1+k^2}} (\tanh[\cos(\alpha)X] - ik) e^{it}, \quad (2.59)$$

$$Z_2 = \frac{1}{\sqrt{1+k^2}} \frac{1}{\cosh[\cos(\alpha)X]} e^{i \sin(\alpha) T} \quad (2.60)$$

---

<sup>3</sup>Assuming separation of variables equation (2.57) takes the form of a time-independent Schrödinger equation with a Rosen-Morse potential [66].

where the constant  $k$  is related to the charge parameter  $\alpha$  and rapidity  $\theta$  as  $k = \frac{\sinh(\theta)}{\cos(\alpha)}$ .

Calculating the energy and angular momenta the authors of [66] find the dyonic magnon dispersion relation

$$E - J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2} \sin^2\left(\frac{p}{2}\right)}. \quad (2.61)$$

Not only does this relation reproduce the correct (exact) relation (1.145) for the elementary magnon with  $J_2 = 1$  but it also agrees precisely with a preceding proposal [72] for the dispersion relation of operators with R-charges  $J_1$  and  $J_2$  in a  $U(1) \times U(1) \subset SU(2)$  subsector of the field theory. In fact the dyonic giant magnons are dual to operators that consist of BPS bound states of  $J_2$  elementary magnons [72].

That the bound states are BPS can be demonstrated easily for the simplest case of  $J_2 = 2$ . We should be able to bind two magnons together by complexifying their momenta  $p_k$ ,  $k = 1, 2$  and then to be BPS we should be able to simply add their energies  $\varepsilon(p_1)$  and  $\varepsilon(p_2)$  in order to get the exact bound state energy  $\varepsilon(p)$  equal to (2.61). This turns out to be correct,

$$\sqrt{1 + \frac{\lambda}{\pi^2} \sin^2\left(\frac{p}{4} + i\frac{v}{2}\right)} + \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2\left(\frac{p}{4} - i\frac{v}{2}\right)} = \sqrt{4 + \frac{\lambda}{\pi^2} \sin^2\left(\frac{p}{2}\right)}. \quad (2.62)$$

We already saw how the elementary excitations of the field theory in the planar limit [22] are realised as giant string configurations [9]. Now their bound states too have a simple string interpretation [72]. The bound states, which from the field theory perspective appear as poles of the scattering matrix of free magnons and must be added in to complete the spectrum, on the string side appear simply as strings executing motion on a larger subspace of  $AdS_5 \times S^5$ .

## 2.2 Born-Infeld effective theory and giant gravitons

In addition to the semiclassical states of fundamental strings that are considered in this thesis we also consider semiclassical states of certain D-branes. The low energy effective action on the world volume of a D-brane was presented in (1.4). Below we

will describe how soliton solutions of this world volume theory describe fundamental strings in target space and how by coupling the world volume theory in addition to the background 5-form flux present on  $AdS_5 \times S^5$  we may consider stable, finite energy D-branes from which we may also consider producing fundamental strings.

### 2.2.1 World volume solitons: BIon solutions

The electromagnetic theory living on the world volume of a  $Dp$ -brane may contain sources. In the linearised version of the theory, which contains Maxwell's theory, it is of course possible to find static, spherically symmetric (Coulomb) solutions for the field strength that correspond to the inclusion of sources on the world volume: electric and magnetic monopoles. These solutions carry over into the fully non-linear theory as soliton solutions where they have the name BIons [73, 74].

The BIons are in fact the world volume description of strings ending upon the brane that first revealed the  $Dp$ -branes as independent dynamical objects [11]. On  $D3$ -branes the electric sources are those carried by the end points of fundamental strings while magnetic sources represent the end points of  $D1$ -strings. The BIons are BPS solutions [75] of the world volume theory so it is no surprise that they also solve the linearised theory.

The prototypical BIon solution for a  $Dp$ -brane in 10 dimensional flat space can be found by taking a Coulomb-like ansatz for the linearised theory [73]. If we demand that the solution be BPS it is guaranteed to satisfy the non-linear equations of motion also. Taking only  $F_{0r} \neq 0$ , where  $r$  is the purely spatial radial direction on the world volume, with

$$A^0 = \frac{c_p}{r^{p-2}} \quad (2.63)$$

would suffice to describe an electric charge in pure Maxwell, satisfying Laplace's equation in  $p$  spatial dimensions away from the source itself,

$$\nabla^2 A^0 = \frac{1}{r^{p-1}} \frac{\partial}{\partial r} \left( r^{p-1} \frac{\partial A^0}{\partial r} \right) = 0. \quad (2.64)$$

However this configuration does not preserve any supersymmetries. If we additionally excite one of the scalars we may preserve half of the supersymmetries. Calling

this single real scalar  $x$  then its profile is given by

$$x = \frac{c_p}{r^{p-2}} \quad (2.65)$$

so that at the position of the source on the world volume there is a spike in the embedding of the brane into the background (which extends off to infinity in the  $x$  direction). This gives the shape of the fundamental string ending on the brane. Were we to allow this end point to move, that is to increase the energy above that of the BPS configuration, then higher derivative corrections or the full solution of the non-linear theory would be required.

To fix the value of  $c_p$  the energy of the configuration may be compared with that of a static fundamental string which should be simply proportional to its extension. Integrating over the world volume around the source and regulating the divergence by terminating the integral a distance  $\epsilon$  from  $r = 0$  produces the energy

$$E = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^{p-2}} T_p c_p^2 (p-2) \Omega_{p-1} \quad (2.66)$$

which is in fact proportional to  $x(\epsilon) = \frac{c_p}{\epsilon^{p-2}}$ , the extension of the string.  $c_p$  can then be chosen so that this energy is

$$E = \frac{1}{2\pi\alpha'} x(\epsilon). \quad (2.67)$$

Clearly the BIon describes a string of constant tension.

Other soliton solutions of world volume theories [73, 74, 76] enjoy interpretations as other interesting objects in target space. The magnetic monopole, as has already been mentioned, describes a D1-brane ending on the D3-brane. Other examples include a string-like soliton of the M5-brane theory which describes the intersection of an M2-brane with the M5-brane, or the non-BPS ‘‘catenoidal’’ solutions describing a  $D_p$ -brane / anti- $D_p$ -brane system where the possibility of obtaining a vanishing gauge field implies a net force on the world volume, and hence instability so that we in fact describe the decay of the brane / anti-brane system.

Another possibility of particular relevance to this thesis is to think of performing the reverse of pulling a string out of a D-brane: allowing a fundamental string to

blow up into a D-brane. A first example of this [77] was achieved by considering an F-string in the presence of a 3-form potential, to which a string does not couple at the perturbative level. By developing a D2-brane bulge however it may couple to the background via the standard electric coupling of the brane to its corresponding R-R field strength.

In the following subsection we shall examine an analogous occurrence for strings moving on the  $AdS_5 \times S^5$  background with a non-zero 4-form potential: the giant gravitons.

### 2.2.2 Giant gravitons from the effective action

The classical giant graviton solutions [10] were not first described by taking a fundamental string and allowing the creation of a D-brane bulge but instead by arguing that an initially point-like string, whose spherical harmonic modes on the  $S^5$  part of the background are graviton modes, should with large enough angular momentum around a direction in  $S^5$  blow up into a D3-brane wrapping an  $S^3 \subset S^5$ , supported against collapse by its interaction with the 4-form potential.

Without therefore wishing to describe a D-brane *plus* a string configuration we examine the DBI action with the inclusion of the coupling of the world volume to the R-R 4-form and look for solutions with spherical symmetry which are static up to a constant translation about the sphere. In chapter 4 we shall report on original research concerning the appearance of the DBI strings in this picture.

The Dirac-Born-Infeld action (1.4) on its own gives the coupling of a  $Dp$ -brane to the NS-NS string fields  $G_{\mu\nu}$ ,  $B_{\mu\nu}$  and  $\Phi$ . In backgrounds that contain R-R fields a  $Dp$  brane couples naturally via an ‘electric’ coupling of the  $(p+1)$ -form potential to the world volume of the  $Dp$ -brane [12],

$$\int_{\Omega_{Dp}} C_{p+1}. \quad (2.68)$$

The  $AdS_5 \times S^5$  background considered above possesses a non-zero 4-form potential  $C_4$  to which a D3-brane will couple, while the Kalb-Ramond field  $B_{\mu\nu} = 0$ . The low

energy effective action for a D3-brane in the  $AdS_5 \times S^5$  background is then

$$S = -T_3 \int_{\Omega_{D3}} \sqrt{-\det(\tilde{G}_{ab} + (2\pi l_s^2)F_{ab})} + T_3 \int_{\Omega_{D3}} C_4 \quad (2.69)$$

with  $\tilde{G}_{ab}$  defined in (1.5). If we only excite scalars that correspond to coordinates on  $\mathbb{R} \times S^5$  and adopt a static gauge then a metric on this space is

$$ds^2 = -R^2 dt^2 + \frac{R^2}{R^2 - r^2} dr^2 + (R^2 - r^2) d\phi^2 + r^2 d\Omega_3^2 \quad (2.70)$$

from which we may calculate  $\tilde{G}_{ab}$ . We will be more explicit in chapter 4.

The potential term is an integral over the world volume of the brane  $\Omega_{D3}$  of the background 4-form potential  $C_4$ . This can be rewritten in terms of the 5-form field strength by

$$S_{\text{Potential}} = \int_{\Omega_{D3}} C_4 = \int_{\Sigma} F_5 \quad (2.71)$$

where  $F_5 = dC_4$  and  $\Sigma$  is now the 5-manifold whose boundary is the 4 dimensional hypersurface swept out by the D3 brane.

As we are interested in solutions that in a co-rotating frame are static then the Lagrangian for the potential term may be simply written down as  $L_{\text{Potential}} = \frac{S_{\text{Potential}}}{\tau}$ ,  $\tau$  being a single rotation period,  $\tau = \frac{2\pi}{\dot{\phi}}$ ,  $\dot{\phi} = \text{constant}$ . As the constant 5-form field strength  $F_5$  could be written  $F_5 = B dVol_{S^5}$  where  $dVol_{S^5}$  is the volume form on  $S^5$ , then calculating the appropriate volume of  $\Sigma$  the Lagrangian for a D3-brane with angular momentum  $J$  in this background may be written

$$L = -T_{D3}\Omega_3 r^3 \sqrt{1 - (R^2 - r^2)\dot{\phi}^2} + \dot{\phi} N \frac{r^4}{R^4} \quad (2.72)$$

where the normalisation of  $B$  has been used,  $BR^5\Omega_5 = 2\pi N$ .

Given the complete spherical symmetry we desire of the target space picture of the D3-brane (we have only one scalar,  $r$ , on the world volume excited and it is independent of the world volume coordinates) then a solution will have the form  $r = r(J)$ . This is obtained now by minimising the energy  $E$  of the brane at fixed  $J$  (obtaining a genuinely BPS configuration). Eliminating  $\dot{\phi}$  with the expression for  $J$  obtained from (2.72) then the resulting relation between  $r$  and  $J$  is

$$r^2 = \frac{J}{N} R^2, \quad \text{with} \quad E_{\min} = \frac{J}{R}. \quad (2.73)$$

Clearly the giant graviton has a maximum radius of  $R$ , that of the background  $S^5$ , at which point the brane achieves the upper bound also upon its angular momentum,  $J = N$ . This is an instance of the “stringy exclusion principle” whereby the number of string states available in the UV is cut off by uniquely stringy, non-local effects [10].

At the other end of the spectrum of giant gravitons, when  $r \ll R$ , we will encounter high curvatures, or high frequency modes in the world volume theory. This is where we must not normally trust the effective action given by the Dirac-Born-Infeld approach. Nevertheless, it is often the case that BPS states are protected from the corrections that would be necessary under other circumstances. In this thesis however we need not be too concerned with the behaviour of the giant gravitons of very small radius.

We should note that giant gravitons that expand into the  $AdS$  part of the background are possible and have been named “dual giant gravitons” [54]. These states share many of the properties of the sphere giants, including the scaling of radius with angular momentum and the supersymmetries preserved, but because  $AdS$  space is non-compact there is no upper bound on the size or angular momentum of the  $AdS$  giants.

Finally, by taking the scalar  $r$  to depend non-trivially upon the world volume coordinates we may describe deformations of the giant graviton. Chapter 4 will describe the BIon-spike solutions of the giant graviton that when  $\dot{\phi} \neq 1$  reproduce the behaviour of the classical string solutions that are obtained in chapter 3, generalising previous work [84] on the  $J \rightarrow \infty$  limit.

### 2.2.3 Gauge theory dual to giant gravitons

The standard map from one particle states of strings to operators of the dual gauge theory is given by a basis of single trace operators [4],

$$\mathcal{O} \sim \text{Tr}(\mathcal{O}_{i_2}^{i_1} \mathcal{O}_{i_3}^{i_2} \dots \mathcal{O}_{i_{n-1}}^{i_{n-2}} \mathcal{O}_{i_n}^{i_{n-1}}) \quad (2.74)$$

where the number of single traces counts the particle number in the gravity Fock space. It was initially presumed that if the giant gravitons are identified as single gravitons then their corresponding operators will be found among the single traces. However this is not the case, and in fact at large R-charge ( $J$ ) the failure of the planar approximation invalidates this relationship [13].

As we have discussed, the giant gravitons possess an upper bound on their angular momentum  $J$  which under  $AdS/CFT$  corresponds to an upper bound on the R-charge of the SYM operator. While the single trace operators possess such a bound (the correct bound,  $J = N$ , given by the size  $N \times N$  of the adjoint representation matrices) there is another type of gauge invariant combination of the scalars that may be formed with exactly this bound: the sub-determinants. For the complex scalar  $Z_j^i$ ,  $i, j = 1, \dots, N$  for example the sub-determinant is defined as

$$\det_k(Z) \equiv \frac{1}{k!} \epsilon_{i_1 \dots i_k a_{k+1} \dots a_{N-k}} \epsilon^{j_1 \dots j_k a_{k+1} \dots a_{N-k}} Z_{j_1}^{i_1} \dots Z_{j_k}^{i_k}, \quad (2.75)$$

which is an operator with  $2 \times (N - k)$  free indices. This operator is also a chiral primary which matches what would be expected of the short representations of  $\frac{1}{2}$ -BPS states such as the ground state of the giant graviton is [13, 54].

The largest giant graviton, of angular momentum  $J = N$ , is given when  $k = N$  and the operator is simply the determinant. This is the *maximal* giant graviton. The  $S^3$  that it wraps is of equal radius to the background  $S^5$ . All giants given by sub-determinants  $k < N$  are referred to as *non-maximal* giant gravitons.

The language of the gauge theory dual to giant gravitons is generally used to talk about the important issue of their embedding into  $S^5$ . On its own a generic embedding of a giant graviton into  $S^5$  is physically equivalent to any other but in the presence of additional excitations it is of course necessary to specify their relative positions. In chapter 5 we will present results concerning the scattering of string solitons with large angular momentum off giant gravitons [78] and the relative orientation of the strings to the branes will then have physical consequences.

Given that the giant graviton wraps an  $S^3 \subset S^5$  and possesses angular momentum  $J$  about some axis then it suffices to specify the orientation of this axis. We

have already seen that each of the three complex scalars  $X$ ,  $Y$  and  $Z$  carry orthogonal units of  $SU(4)$  R-charge which are dual to units of  $SO(6)$  angular momentum,  $J_X$ ,  $J_Y$  and  $J_Z$ . It is then apparent that we can define the axis of rotation of the giant graviton (and, together with its radius, its complete embedding) by choice of the scalar field of which the (sub-)determinant is formed.

Considering  $S^3 \subset S^5 \subset \mathbb{R}^6$ , then we may say that whenever the radius of the giant wrapping  $S^3$  is less than the radius of  $S^5$  then the giant orbits the axis of its angular momentum in some plane  $\mathbb{R}^2 \subset \mathbb{R}^6$  in which it has no extension. Each of the three scalars defines one of the three possible orthogonal planes in  $\mathbb{R}^6$  so that specifying the scalar filling the sub-determinant picks out one of these orthogonal planes. The operator in (2.75) for example corresponds to what would be referred to in total as a “non-maximal,  $Z = 0$  giant graviton”.<sup>4</sup>

#### 2.2.4 Open strings on giant gravitons

The research presented in this thesis is concerned throughout with open strings that end upon giant graviton D-branes. While chapters 3 and 4 examine the Sine-Gordon / Nambu-Goto string and world volume description of the equivalent of a folded spinning string emerging from the D-brane, chapter 5 is concerned with the scattering of giant magnons from the string end points; the formerly closed string solitons now becoming solitons of a genuinely open string.

As well as the established integrability of closed strings on  $AdS_5 \times S^5$  [24, 32] the construction of integrable open strings has also been considered. [79–83] and our research on open strings and giant gravitons presented in chapter 5 ties into this scheme. We shall comment on the relevance to [79] in the discussion.

In the same way as a spinning string is extended by virtue of the centrifugal force balancing its tension, perhaps the simplest way to attach an open string to a giant graviton is to allow a string with angular momentum to pull itself out of the D-brane [84, 85]. This is the picture we shall have of the *boundary giant magnons* [14, 86] that are the subject of chapters 3 and 4 where we shall find two distinct types of

---

<sup>4</sup>This terminology is due to Hofman and Maldacena’s study of reflecting giant magnons [78].

such solutions.

The gauge theory dual to open strings attached to giant gravitons [78] must contain both the elements that we have previously described: something like the operators dual to giant magnons, the long spin-chains with their magnon impurities, (1.143), plus the operator dual to the giant graviton itself, (2.75). Such operators can be achieved for strictly non-maximal giant gravitons ( $k < N$ ) by removing the trace over the operator (1.143) and instead contracting with two of the indices of the sub-determinant. For  $N \rightarrow \infty$  this is effectively a string attached to a maximal giant.

The simplest cases of open strings supporting giant magnon excitations that are allowed to scatter with the giant graviton at one end point were considered in [78] where the giant gravitons were maximal. As well as a perturbative ( $\lambda \rightarrow 0$ ) analysis revealing the sub-leading S-matrix structure of the scatterings a strong coupling (large  $\lambda$ ) analysis was also performed, and despite not producing the string solutions themselves they were able to take advantage of the sine-Gordon connection illustrated in section 1.3.1 to derive expressions for the leading order scattering phase  $\delta$  when the magnon reflection matrix is written

$$\mathcal{R} = \mathcal{R}_0 \hat{\mathcal{R}} \quad (2.76)$$

and the overall factor applying the phase shift is  $\mathcal{R}_0 = e^{i\delta}$ .

The world sheet of an open string corresponds under Pohlmeyer reduction to the sine-Gordon model on the half-line, or ‘‘boundary sine-Gordon theory’’ [87, 88]. Using the terminology of the previous subsection to describe the orientation of the giant graviton with respect to a string whose angular momentum vector points out of the  $Z$ -plane, Hofman and Maldacena considered three physically distinct cases [78] involving a maximal  $Z = 0$  brane and two maximal  $Y = 0$  branes. These cases map to the boundary reflection of kinks and anti-kinks with the simplest boundary conditions: two pure Dirichlet cases and a pure Neumann case.

Strings attached to a  $Z = 0$  giant graviton require a boundary degree of freedom equivalent to the boundary field excited in the sine-Gordon theory when we have a Dirichlet boundary condition. This is seen in the string picture as a segment of string that must interpolate from the ‘BMN vacuum’ at the equator to the point where

the brane intersects the sphere upon which the string moves. Strings attached to a  $Y = 0$  giant graviton require no such extra degree of freedom<sup>5</sup>. This string degree of freedom is the boundary giant magnon of the following chapter.

These cases will be discussed in more detail in chapter 5 where the string solutions will actually be presented and the dictionary between the string and boundary sine-Gordon parameters established. We will then proceed to consider the non-trivial boundary conditions afforded by integrability which will allow us to describe the scattering of giant magnons with *non-maximal* giant gravitons [15].

---

<sup>5</sup>The scattering of magnons from these boundary conditions has been studied also within the context of Yangian symmetry [89].

# Chapter 3

## Boundary giant magnons

In this chapter we begin the presentation of original research. The aim is first to understand the boundary giant magnons from the perspective of sine-Gordon theory. The boundary giant magnons consist of open string segments attached to a D-brane intersecting the subspace  $S^2 \subset S^5$ . If the open string has giant magnon excitations upon it then this segment must interpolate between the brane and a BMN vacuum point at the equator of  $S^2$ . In the absence of giant magnons the boundary string segment may display a richer behaviour, studied here.

By the definitions in section 1.3.3 we can guess that we require for now only static sine-Gordon theory; the boundary giant magnons satisfy  $\ddot{X}^i = 0$  and so  $\dot{\varphi} = \ddot{\varphi} = 0$  by (1.139). We shall then map the solution set to solutions of the  $O(3)$  sigma model to find the boundary giant magnons themselves. We recover the known solutions plus a novel family of solutions with qualitatively different behaviour at finite  $J$ , corresponding neatly to the very similar full set of spinning string solutions.

### 3.1 Static Sine-Gordon Solutions on the Interval

In order to produce the boundary giant magnon string solutions at finite  $J$  we in this section find their Pohlmeyer reduced versions from the general solution to static sine-Gordon theory on the interval. Hence we take all time derivatives of  $\varphi$  to be zero, and to place the theory on the interval we will demand that  $0 \leq x \leq L$ . As we have set the time derivative of  $\varphi$  to zero we will have two Dirichlet boundary conditions

to consider at  $x = 0$  and  $L$ . We will focus on those for which  $\varphi(0) = \varphi(L) + 2m\pi$ ,  $m \in \mathbb{Z}$  as in the static case this corresponds to open strings that end on the same D-brane. The same solutions described here have been previously deployed [93, 94] in semiclassical analyses of sine-Gordon theory on the interval.

For static solutions on the interval we take  $\dot{\varphi} = \ddot{\varphi} = 0$  and will therefore need to find solutions to

$$\frac{d^2\varphi}{dx^2} = \sin(\varphi), \quad 0 \leq x \leq L. \quad (3.1)$$

The equation is separated and integrated once as

$$\begin{aligned} \frac{1}{2}d\varphi'^2 &= d\varphi \sin(\varphi) \\ \Rightarrow \varphi'^2 &= 2(c - \cos(\varphi)). \end{aligned} \quad (3.2)$$

At this point it is clear that we will have two types of solution depending upon the value of the constant of integration  $c$ ;

- if  $c > 1$  then at no point can  $\varphi' = 0$
- if  $|c| \leq 1$  we may obtain  $\varphi' = 0$

while for  $c < -1$  there is no real solution.

### 3.1.1 $c > 1$ Solution

We must solve

$$\int \frac{d\varphi}{\sqrt{c - \cos(\varphi)}} = \sqrt{2} dx, \quad c > 1. \quad (3.3)$$

The solution is found in terms of an elliptic integral of the first kind<sup>1</sup> to be

$$\begin{aligned} \frac{2}{\sqrt{c+1}} F(\delta, k) &= \sqrt{2}(x - x_0) \\ \delta &= \arcsin \left( \sqrt{\frac{(c+1)(1-\cos(\varphi))}{2(c-\cos(\varphi))}} \right), \quad k = \sqrt{\frac{2}{1+c}} \end{aligned} \quad (3.4)$$

or

$$F(\operatorname{am}(u, k), k) = u, \quad u = \frac{x - x_0}{k} \quad (3.5)$$

---

<sup>1</sup>See Appendix A for a list of the relevant properties of elliptic integrals and functions.

$\text{am}(u, k)$  being the Jacobi amplitude function, with  $k$  the elliptic modulus satisfying

$$0 \leq k < 1.$$

This solution is in implicit form however so to make it explicit in  $\varphi$  we write

$$\begin{aligned} \text{am}(u, k) &= \arcsin \left( \frac{1}{k} \sqrt{\frac{(1 - \cos(\varphi))}{(c - \cos(\varphi))}} \right) \\ \Rightarrow \sin(\text{am}(u, k)) &\equiv \text{sn}(u, k) = \frac{1}{k} \sqrt{\frac{(1 - \cos(\varphi))}{(c - \cos(\varphi))}}. \end{aligned} \quad (3.6)$$

With some rearrangement, including substituting in for  $c$  in terms of the modulus  $k$ , we find

$$\begin{aligned} \cos(\varphi) &= 2 \frac{\text{cn}^2(u, k)}{\text{dn}^2(u, k)} - 1 = 2\text{sn}^2(u + K, k) - 1 \\ \Rightarrow \cos^2\left(\frac{\varphi}{2}\right) &= \text{sn}^2(u + K, k). \end{aligned}$$

where  $K = K(k)$  is the complete elliptic integral of the first kind. We get finally

$$\sin^2\left(\frac{\varphi}{2} \pm \frac{\pi}{2}\right) = \sin^2(\text{am}(u + K, k))$$

or

$$\varphi = \pm\pi \pm 2 \text{ am}\left(\frac{x - x_0}{k} + K, k\right) \quad (3.7)$$

with signs uncorrelated. It can be checked that this satisfies the static sine-Gordon equation (3.1).

### 3.1.2 $|c| \leq 1$ Solution

Now we must solve

$$\int \frac{d\varphi}{\sqrt{c - \cos(\varphi)}} = \sqrt{2} dx, \quad |c| \leq 1, \quad c \neq -1. \quad (3.8)$$

This time the solution is of the form

$$\begin{aligned} -\sqrt{2}F(\gamma, q) &= \sqrt{2}(x - x_0), \\ \gamma &= \arcsin \left( \sqrt{\frac{(1 - \sin(\varphi - \frac{\pi}{2}))}{1 + c}} \right), \quad q = \sqrt{\frac{1+c}{2}} \end{aligned} \quad (3.9)$$

or

$$F \left( \operatorname{am} \left( u', \frac{1}{k} \right), \frac{1}{k} \right) = -(x - x_0), \quad u' = -(x - x_0). \quad (3.10)$$

The previously defined constant  $k$  now takes the values  $1 \leq k$  and we proceed to an explicit form of the solution.

$$\begin{aligned} \operatorname{am}(u', k^{-1}) &= \arcsin \left( \sqrt{\frac{(1 - \sin(\varphi - \frac{\pi}{2}))}{1 + c}} \right) \\ \Rightarrow \operatorname{sn}(u', k^{-1}) &= \sqrt{\frac{(1 - \sin(\varphi - \frac{\pi}{2}))}{1 + c}} \\ &= \frac{k}{\sqrt{2}} \sqrt{1 + \cos(\varphi)} \\ \Rightarrow \operatorname{sn} \left( u, \frac{1}{k} \right) &= \pm k \cos \left( \frac{\varphi}{2} \right), \quad u = (x - x_0) \end{aligned} \quad (3.11)$$

where in the last step we absorbed the negative sign in the argument  $u'$  into the free choice of signs on the right hand side. Finally then,

$$\varphi = 2 \arccos \left( \pm \frac{1}{k} \operatorname{sn} \left( x - x_0, \frac{1}{k} \right) \right). \quad (3.12)$$

Again, it can be checked that this satisfies the equation of motion (3.1).

### 3.1.3 Boundary Conditions

Our two qualitatively different general solutions to static sine-Gordon theory on the interval are

$$c > 1, \quad (0 \leq k < 1), \quad \varphi = \pm \pi \pm 2 \operatorname{am} \left( \frac{x - x_0}{k} + K, k \right) \quad (3.13)$$

$$|c| \leq 1, \quad (1 \leq k), \quad \varphi = 2 \arccos \left( \pm \frac{1}{k} \operatorname{sn} \left( x - x_0, \frac{1}{k} \right) \right) \quad (3.14)$$

with  $0 \leq x \leq L$  and  $c \neq -1$  in (3.14).

The boundary conditions with  $\varphi(x = 0) \equiv \varphi_B = \pi$  (and  $\varphi(x = L) = \pi + 2m\pi$ ,  $m \in \mathbb{Z}$ ) will be of special interest to us so we focus on these.

$$c > 1$$

Firstly, the amplitude is an increasing function in its argument

$$u + K = k^{-1}(x - x_0) + K \quad (3.15)$$

so that from the choice of signs in (3.13) it is clear that we have a solution increasing or decreasing in  $u$  from the values  $\varphi_B = \pm\pi$ .

Restricting to values of  $\varphi$  with  $\varphi(x = L) - \varphi(x = 0) \leq 2\pi$  and choosing all positive signs we have the boundary condition at  $x = 0$

$$\begin{aligned}\varphi_B &= \pi = \pi + 2 \operatorname{am} \left( \frac{-x_0}{k} + K, k \right) \\ \Rightarrow \operatorname{am} \left( \frac{-x_0}{k} + K, k \right) &= 0 \\ \Rightarrow x_0 &= kK(k)\end{aligned}\tag{3.16}$$

and at  $x = L$

$$\begin{aligned}3\pi &= \pi + 2 \operatorname{am} \left( \frac{L}{k}, k \right) \\ \Rightarrow \operatorname{am} \left( \frac{L}{k}, k \right) &= \pi\end{aligned}\tag{3.17}$$

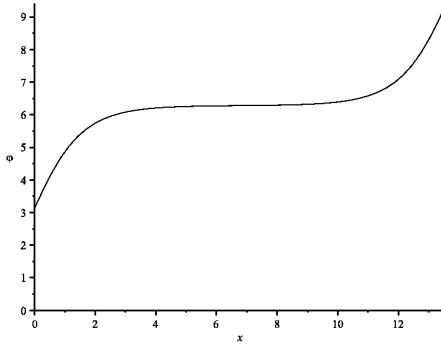
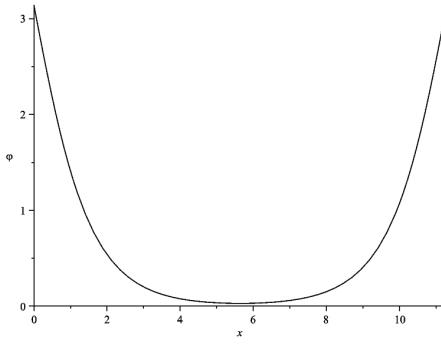
which using  $\operatorname{am}(u + 2K) = \operatorname{am}(u) + \pi$  gives us the length  $L$  of the interval in terms of the elliptic modulus  $k$  as

$$L = 2kK(k), \quad x_0 = \frac{L}{2}.\tag{3.18}$$

Generically, the solution increases, more rapidly at first, from  $\pi$  up to  $2\pi$  where the gradient  $\varphi'$  decreases but remains positive, and then steepens again up to  $3\pi$ . The solution is quasi-periodic with period  $L$ , increasing by a further  $2\pi$  for each  $L$  moved in  $x$ . When  $L \rightarrow 0$  ( $k \rightarrow 0$ ) the solution tends to an increasingly steep, straighter line from  $\pi$  to  $3\pi$  while for  $L \rightarrow \infty$  ( $k \rightarrow 1$ ) the solution increases to  $2\pi$  as  $x \rightarrow \infty$  becoming equal to half a kink solution of the bulk theory. The solution is plotted in Figure 3.1.

$$|c| \leq 1$$

In this case the solution (3.14) is literally periodic with  $0 \leq \varphi \leq 2\pi$  (or  $2m\pi \leq \varphi \leq 2(m+1)\pi$ ,  $m \in \mathbb{Z}$  if we'd chosen appropriately in (3.12)), the choice of sign

Figure 3.1: The  $c > 1$  solution plotted on  $x \in [0, L]$ .Figure 3.2: The  $|c| \leq 1$  solution plotted on  $x \in [0, L]$ .

reflecting the solution about  $\varphi = \pi$ . Taking the positive sign, at  $x = 0$  we have

$$\begin{aligned} \varphi_B &= \pi = 2 \arccos \left( \frac{1}{k} \operatorname{sn} \left( -x_0, \frac{1}{k} \right) \right) \\ \Rightarrow \frac{1}{k} \operatorname{sn} \left( -x_0, \frac{1}{k} \right) &= 0 \\ \Rightarrow x_0 &= 0 \end{aligned} \tag{3.19}$$

and if we take  $x = L$  to be the point at which the solution first returns to  $\varphi = \pi$  then

$$\begin{aligned} \pi &= 2 \arccos \left( \frac{1}{k} \operatorname{sn} \left( L, \frac{1}{k} \right) \right) \\ \Rightarrow \frac{1}{k} \operatorname{sn} \left( L, \frac{1}{k} \right) &= 0 \\ \Rightarrow L &= 2K \left( \frac{1}{k} \right). \end{aligned} \tag{3.20}$$

For generic  $L(k)$  the solution decreases from  $\varphi = \pi$  to some minimum at  $x = \frac{L}{2}$  and then increases again up to  $\varphi = \pi$ . For  $L \rightarrow \infty$  ( $k \rightarrow 1$ ) the solution approaches

$\varphi = 0$  as  $x \rightarrow \infty$  and again becomes half a kink solution of the bulk theory in the limit. In contrast to the  $c > 1$  case however when we take  $k \rightarrow \infty$  we have a minimum value of  $L$ :

$$L_{\min} = L(k = \infty) = 2K(0) = \pi. \quad (3.21)$$

The solution is plotted in Figure 3.2.

## 3.2 String Solutions and $\Delta - J$

We require solutions to the  $O(3)$  sigma-model equations<sup>2</sup> (1.89) which for this section we write in explicit (Euclidean) vector form,

$$\ddot{\vec{X}} - \vec{X}'' + (\dot{\vec{X}}^2 - \vec{X}'^2)\vec{X} = 0. \quad (3.22)$$

These arose by taking the conformal and static partial gauges on the world sheet of the Polyakov string and implementing the constraint  $\vec{X}^2 = 1$  by the method of Lagrange multipliers. The squares of target space vectors are understood to mean  $\vec{Y}^2 \equiv \vec{Y} \cdot \vec{Y} \equiv Y_i Y^i = Y^i Y^i$ , the index  $i$  taking spatial target space values in  $\mathbb{R}^3$ , into which  $S^2$  is embedded. We must also satisfy the Virasoro constraints arising from taking the conformal gauge, which together with the use of the static gauge, are

$$\dot{\vec{X}}^2 + \vec{X}'^2 = 1 \quad (3.23)$$

$$\dot{\vec{X}} \cdot \vec{X}' = 0. \quad (3.24)$$

This is achieved by mapping the solutions to sine-Gordon theory of the previous section into the string target space. We will be particularly interested in the energy  $\Delta$  of the string<sup>3</sup> and its angular momentum  $J$ . In section 4 we will re-express the angular momentum and energy in a general gauge in order to make a comparison with the brane picture.

---

<sup>2</sup>In this section we set the radius  $R$  of  $S^2$  (also of course the radius of  $S^5$  and  $AdS_5$ ) to one.

<sup>3</sup>Throughout this section and the next we use the symbol  $\Delta$  to denote string theory energies as a direct reference to the conformal dimension of operators in  $\mathcal{N} = 4$  SYM to which they correspond.

The map between sine-Gordon theory and the string sigma model is difficult to explicitly invert in general, however it is particularly easy to make the map in the case of solutions satisfying a condition due to Klose and McLoughlin [63] which allowed them to consider strings corresponding to various “2 phase” sine-Gordon solutions:

$$\partial_x \left( \frac{\partial_t \varphi}{\sin \left( \frac{\varphi}{2} \right)} \right) = 0. \quad (3.25)$$

Clearly any static solution will do. We therefore construct the strings corresponding to the two types of static sine-Gordon solutions following the method of Klose and McLoughlin, for which it is found that, remembering to take the radius of the sphere  $R = 1$  and with the coordinates

$$\vec{X} = (r \cos(\phi), r \sin(\phi), \sqrt{1 - r^2}) \quad (3.26)$$

the radius  $r$  is given by

$$r = -\frac{1}{\dot{\phi}} \cos \left( \frac{\varphi}{2} \right), \quad \phi = \dot{\phi} t \quad (3.27)$$

where

$$\dot{\phi} = \sqrt{\cos^2 \left( \frac{\varphi}{2} \right) + \left( \frac{\varphi'}{2} \right)^2} = \text{constant}. \quad (3.28)$$

From the definition

$$\dot{\vec{X}}^2 = \cos^2 \left( \frac{\varphi}{2} \right) \quad (3.29)$$

we see that the boundary condition  $\varphi_B = \pi$  corresponds to a stationary string endpoint, in fact stuck to the north pole of the sphere where there is taken to be a maximal “ $Z = 0$ ” giant graviton<sup>4</sup>. Similarly,  $\varphi = \pi + 2m\pi$ ,  $m \in \mathbb{Z}$  will map to a static point. We will focus on these “maximal” boundary conditions.

If we work out the angular velocity  $\dot{\phi}$  in each case we find that it is related to the parameter  $k$ , or  $c$ , as follows: for  $c > 1$ ,  $0 < k < 1$  and taking  $\dot{\phi}$  positive

$$\dot{\phi} = \sqrt{\operatorname{sn}^2 \left( \frac{x}{k}, k \right) + \frac{1}{k^2} \operatorname{dn}^2 \left( \frac{x}{k}, k \right)} = \frac{1}{k}; \quad \dot{\phi} > 1 \quad (3.30)$$

while for  $|c| \leq 1$  ( $c \neq -1$ ),  $k \geq 1$ , we have

$$\dot{\phi} = \sqrt{\frac{1}{k^2} \left[ \operatorname{sn}^2 \left( x, \frac{1}{k} \right) + \operatorname{cn}^2 \left( x, \frac{1}{k} \right) \right]} = \frac{1}{k}; \quad 0 < \dot{\phi} \leq 1. \quad (3.31)$$

---

<sup>4</sup>It is of course possible that there are other Dp-brane embeddings that result in identical string boundary conditions.

The full parameter range  $-1 < c$  therefore maps to the range of angular velocities  $\dot{\phi} > 0$ , or including  $c = -1$  ( $k$  strictly undefined) then  $\dot{\phi} \geq 0$ . Including now without harm the point  $c = 1$ ,  $k = 1$  in both solutions we have

$$c \geq 1 \Leftrightarrow \dot{\phi} \geq 1, \quad |c| \leq 1 \Leftrightarrow 0 \leq \dot{\phi} \leq 1$$

so that from here onwards we most usefully characterise our solutions by the value of  $\dot{\phi}$ .

### 3.2.1 $\dot{\phi} \geq 1$ solution

From equations (3.26) and (3.27) then we have

$$\vec{X} = \left( \frac{1}{\dot{\phi}} \cos(\dot{\phi}t) \operatorname{sn} \left( \dot{\phi}x, \frac{1}{\dot{\phi}} \right), \frac{1}{\dot{\phi}} \sin(\dot{\phi}t) \operatorname{sn} \left( \dot{\phi}x, \frac{1}{\dot{\phi}} \right), \operatorname{dn} \left( \dot{\phi}x, \frac{1}{\dot{\phi}} \right) \right). \quad (3.32)$$

It can be checked that this solution satisfies the string's sigma model equations of motion (3.22), both Virasoro constraints (3.23) and (3.24), and returns the correct form of  $\varphi(x)$  using the sine-Gordon map.

The solution is a folded ‘half a spinning string’ with both endpoints attached to the north pole. For  $\dot{\phi} \rightarrow 1$  the cusp point reaches to the equator, while for  $\dot{\phi} \rightarrow \infty$  the string shortens toward a point at the pole. The cusp point always maintains the condition

$$r\dot{\phi} = 1, \quad \Rightarrow \quad \cos \left( \frac{\varphi}{2} \right) = -1 \quad (3.33)$$

i.e. we always have a point at which  $\varphi = 2\pi$ , which is true of the sine-Gordon solution in this parameter range. The string is depicted in Figure 3.3.

Examining the third component of  $\vec{X}$  we see that  $\sqrt{1 - r^2} = \operatorname{dn} \left( \dot{\phi}x, k \right)$  which compares very favourably with this component of the finite  $J$  giant magon (2.47) presented in chapter 2.

We now compute the angular momentum of the string as

$$J = \frac{\sqrt{\lambda}}{2\pi} \int_0^L dx \vec{X} \wedge \dot{\vec{X}} = \frac{\sqrt{\lambda}}{2\pi} \frac{1}{\dot{\phi}} \int_{x=0}^{x=L} du \vec{X} \wedge \dot{\vec{X}}, \quad u = \dot{\phi}x. \quad (3.34)$$

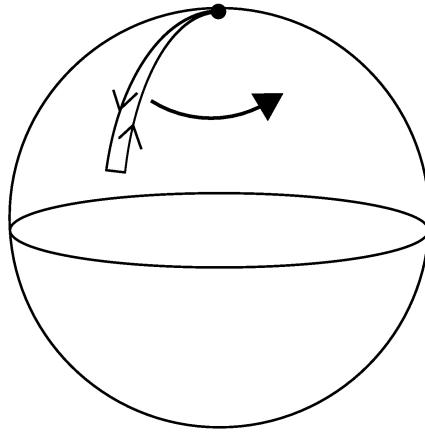


Figure 3.3: The  $\dot{\phi} > 1$  string depicted with halves of the string artificially separated for clarity. For  $\dot{\phi} = 1$  the cusp point reaches the equator while for  $\dot{\phi} \rightarrow \infty$  the string retreats to the north pole where it vanishes.

We have

$$\vec{X} \wedge \dot{\vec{X}} = \begin{pmatrix} \cos(\dot{\phi}t) \operatorname{sn}(u, k) \operatorname{dn}(u, k) \\ -\sin(\dot{\phi}t) \operatorname{sn}(u, k) \operatorname{dn}(u, k) \\ \frac{1}{\dot{\phi}} \operatorname{sn}^2(u, k) \end{pmatrix}. \quad (3.35)$$

Using the integrals

$$\int du \operatorname{sn}(u, k) \operatorname{dn}(u, k) = -\operatorname{cn}(u, k), \quad \int du \operatorname{sn}^2(u, k) = \frac{u - E(u, k)}{k^2}$$

where  $E(u, k)$  is the incomplete elliptic integral of the second kind and  $k = \frac{1}{\dot{\phi}}$  we find

$$\vec{J} = \frac{\sqrt{\lambda}}{2\pi} \left( \frac{2 \cos(\dot{\phi}t)}{\dot{\phi}}, \frac{2 \sin(\dot{\phi}t)}{\dot{\phi}}, 2(K(k) - E(k)) \right) \quad (3.36)$$

where  $E(k)$  is the *complete* elliptic integral of the second kind and  $K(k)$  that of the first. We will be interested in the  $z$ -component of the angular momentum, thus we take

$$J_1 \equiv J_z = \frac{\sqrt{\lambda}}{\pi} (K(k) - E(k)), \quad k = \frac{1}{\dot{\phi}}. \quad (3.37)$$

The energy  $\Delta$  of the string in the conformal gauge is just the length of the string in  $x$  multiplied by the string tension, or the length of the sine-Gordon interval times

the tension. Hence,

$$\begin{aligned}\Delta_1 &= \frac{\sqrt{\lambda}}{2\pi} L \\ &= \frac{\sqrt{\lambda}}{\pi} kK(k), \quad k = \frac{1}{\dot{\phi}}.\end{aligned}\tag{3.38}$$

### 3.2.2 $0 \leq \dot{\phi} \leq 1$ solution

This time equations (3.26) and (3.27) give us

$$\vec{X} = \left( \cos(\dot{\phi}t) \operatorname{sn}(x, \dot{\phi}), \sin(\dot{\phi}t) \operatorname{sn}(x, \dot{\phi}), -\operatorname{cn}(x, \dot{\phi}) \right). \tag{3.39}$$

Again, it can be checked that this solution satisfies the string's sigma model equations of motion (3.22), both Virasoro constraints (3.23) and (3.24), and returns the correct form of  $\varphi(x)$  using the sine-Gordon map.

The solution is qualitatively different from the first, being a rotating, *stretched* string between the north and south poles of the sphere. For  $\dot{\phi} \rightarrow 1$  it can in fact be seen that the two solutions become identical except that for  $\dot{\phi} > 1$  half the string is folded back onto the same hemisphere while for  $0 \leq \dot{\phi} \leq 1$  the string continues through the equator to the south pole. As  $\dot{\phi} \rightarrow 0$  we obtain just the stretched string on  $S^2$ . We have no cusp point; the corresponding sine-Gordon solution never passes through  $\varphi = 2m\pi$ ,  $m \in \mathbb{Z}$ . The solution is depicted in Figure 3.4.

Computing the angular momentum as above,

$$\vec{X} \wedge \dot{\vec{X}} = \begin{pmatrix} \dot{\phi} \cos(\dot{\phi}t) \operatorname{cn}(u, \dot{\phi}) \operatorname{sn}(u, \dot{\phi}) \\ -\dot{\phi} \sin(\dot{\phi}t) \operatorname{cn}(u, \dot{\phi}) \operatorname{sn}(u, \dot{\phi}) \\ \dot{\phi} \operatorname{sn}^2(u, \dot{\phi}) \end{pmatrix} \tag{3.40}$$

and we further need the integral

$$\int du \operatorname{cn}(u, m) \operatorname{sn}(u, m) = -\frac{\operatorname{dn}(u, m)}{m^2}, \quad u = x, \quad m = \dot{\phi}$$

giving

$$\vec{J} = \frac{\sqrt{\lambda}}{2\pi} \left( 0, 0, \frac{2(K(\dot{\phi}) - E(\dot{\phi}))}{\dot{\phi}} \right). \tag{3.41}$$

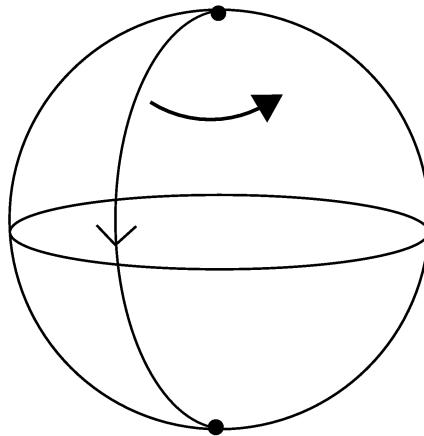


Figure 3.4: The  $0 \leq \dot{\phi} \leq 1$  string. The string is geometrically indistinguishable for different valid values of  $\dot{\phi}$ , apart from moving at different angular velocities.

The only non-zero component this time is the  $z$ -component:

$$J_2 \equiv J_z = \frac{\sqrt{\lambda}}{\pi} \frac{(K(\dot{\phi}) - E(\dot{\phi}))}{\dot{\phi}}. \quad (3.42)$$

Again, the energy is just proportional to the length of the string:

$$\begin{aligned} \Delta_2 &= \frac{\sqrt{\lambda}}{2\pi} L \\ &= \frac{\sqrt{\lambda}}{\pi} K(\dot{\phi}). \end{aligned} \quad (3.43)$$

### 3.2.3 Behaviour of anomalous dimension: $\Delta - J$

Both the energy and angular momentum of these solutions are divergent when  $\dot{\phi} \rightarrow 1$ , or  $k \rightarrow 1$ , as can be seen in Figures 3.5 and 3.6. Both solutions possess a limit in which the angular momentum vanishes, but only the  $\dot{\phi} > 1$  solution has a vanishing energy (as  $\dot{\phi} \rightarrow \infty$ , or  $k \rightarrow 0$ ) - the  $0 \leq \dot{\phi} \leq 1$  solution has a minimum energy which is that of a stretched string. In the previous section we remarked that the  $|c| \leq 1$  sine-Gordon solution possessed a minimum interval length given by

$$L_{\min} = 2K(0) = \pi \quad (3.44)$$

which in turn gives us a minimum energy

$$\Delta_{2,\min} = \frac{\sqrt{\lambda}}{2\pi} \pi = \frac{\sqrt{\lambda}}{2} \quad (3.45)$$

which is indeed the energy of a string of tension  $\sqrt{\lambda}/2\pi$  stretched to a target space length of  $\pi$ .

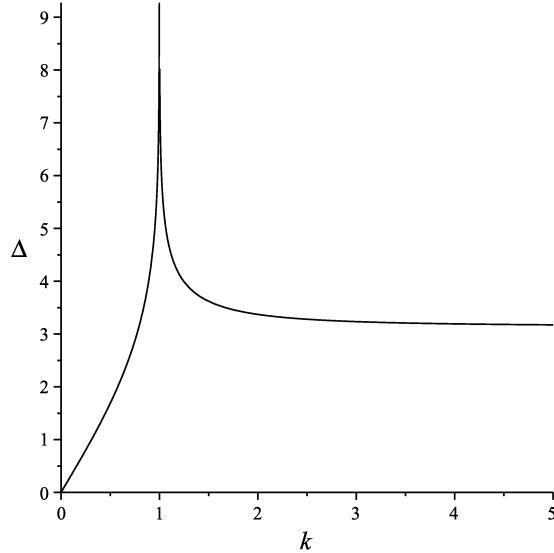


Figure 3.5: The energy in units of  $\sqrt{\lambda}/2\pi$  of both solutions plotted against  $k = \dot{\phi}^{-1}$ . The energy of the  $0 \leq \dot{\phi} \leq 1$  solution tends to that of the static, stretched string as  $\dot{\phi} \rightarrow 0$ , or  $k \rightarrow \infty$ .

Figure 3.7 plots  $\Delta - J$  for both solutions. The quantity is always finite, vanishing at  $\dot{\phi} \rightarrow \infty$  ( $k \rightarrow 0$ ) and tending to a constant value as  $\dot{\phi} \rightarrow 0$  ( $k \rightarrow \infty$ ). In the limit  $\dot{\phi} \rightarrow 1$  both solutions tend to the same configuration (bar the change of hemisphere of one half of the string) which is that with

$$\Delta - J = \frac{\sqrt{\lambda}}{\pi} \quad (3.46)$$

studied previously.

It is worth pausing to further comment upon the  $J \rightarrow 0$  behaviour of the  $\dot{\phi} \geq 1$  string which is so similar to the folded spinning string. We expect the small string at  $J \rightarrow 0$ ,  $\dot{\phi} \rightarrow \infty$  to reproduce the Regge-trajectories of the hadron spectrum,  $m^2 \propto \sqrt{J}$ , given that as the string shrinks to zero size it should not sense the curvature of the sphere, seeing flat space.

With  $k = \frac{1}{\dot{\phi}} \rightarrow 0$  we make use of the expressions for the energy (3.38) and angular momentum (3.37) of this solution along with the appropriate series representations

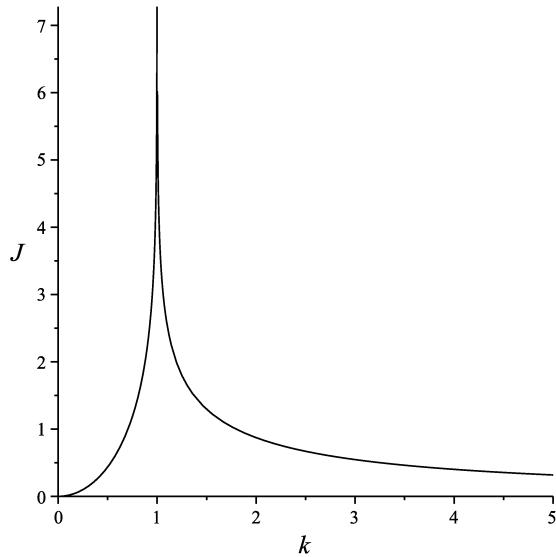


Figure 3.6: Angular momentum in units of  $\sqrt{\lambda}/2\pi$  plotted against  $k = \dot{\phi}^{-1}$ . The  $\dot{\phi} \geq 1$  solution lies to the left of the divergence and the  $0 \leq \dot{\phi} \leq 1$  solution lies to the right. For both  $\dot{\phi} \rightarrow 0$  ( $k \rightarrow \infty$ ) and  $\dot{\phi} \rightarrow \infty$  ( $k \rightarrow 0$ ) the angular momentum vanishes, in the former case because the stretched string is static, in the latter case because the string itself vanishes.

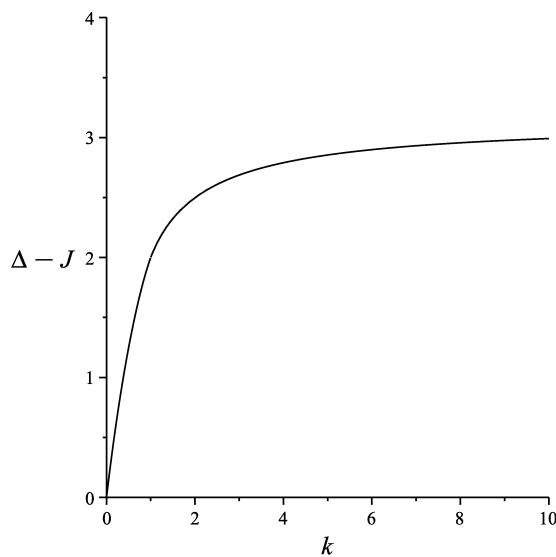


Figure 3.7: Plot of  $\Delta - J$  in units of  $\sqrt{\lambda}/2\pi$  for both solutions. Both are finite for all  $k$ , or  $\dot{\phi}$ , and in particular we move smoothly between the solutions as we go through  $\dot{\phi} = k = 1$ .

of the complete elliptic integrals (A.3.43) and (A.3.44) included in Appendix A. We find

$$\Delta = \frac{\sqrt{\lambda}}{\pi} \frac{\pi}{2} \left\{ 1 + \frac{k^2}{4} + \frac{9k^2}{64} \dots \right\} \quad (3.47)$$

$$= \frac{\sqrt{\lambda}}{2} k + \mathcal{O}(k^3) \quad (3.48)$$

while the angular momentum is

$$J = \frac{\sqrt{\lambda}}{\pi} \left( \frac{\pi}{2} \left\{ 1 + \frac{k^2}{4} + \frac{9k^4}{64} + \dots \right\} - \frac{\pi}{2} \left\{ 1 - \frac{k^2}{4} - \frac{3k^4}{64} - \dots \right\} \right) \quad (3.49)$$

$$= \frac{\sqrt{\lambda}}{4} k^2 + \mathcal{O}(k^4). \quad (3.50)$$

So for the  $J \rightarrow 0$  behaviour of the string energy we find

$$\Delta \approx \lambda^{\frac{1}{4}} \sqrt{J} \quad (3.51)$$

which does indeed reproduce the Regge trajectories. At the same time, with  $\lambda = R^4 \alpha'^{-2}$  we have that for fixed values of  $J$ ,  $\Delta^2 = m^2 \propto \frac{1}{\alpha'}$ , concurring with the expression for massive string states.

### 3.2.4 Finite $J$ corrections

We can examine the leading order finite  $J$  corrections to both of these solutions by expanding  $\Delta$  and  $J$  around  $\dot{\phi} = k = 1$ , with  $\epsilon \equiv \sqrt{1 - k^2}$ , finding  $\epsilon(J)$  and then resubstituting this back into our expressions for  $\Delta - J$ . Using the expansions found in (A.3.46) and (A.3.47) we have

$$K(\epsilon) \approx -\ln\left(\frac{\epsilon}{4}\right) + \frac{\epsilon^2}{4} \left( -\ln\left(\frac{\epsilon}{4}\right) - 1 \right) \quad (3.52)$$

$$E(\epsilon) \approx 1 + \frac{\epsilon^2}{2} \left( -\ln\left(\frac{\epsilon}{4}\right) - \frac{1}{2} \right) \quad (3.53)$$

so that we have from equation (3.37)

$$\begin{aligned} \frac{2\pi}{\sqrt{\lambda}} J_1 &= 2(K(k) - E(k)) \\ &\approx 2 \left( -\ln\left(\frac{\epsilon}{4}\right) - 1 \right) \\ \Rightarrow -\frac{2\pi}{\sqrt{\lambda}} J_1 &\approx \ln\left(\epsilon^2 \frac{e^2}{16}\right) \end{aligned} \quad (3.54)$$

giving  $\epsilon(J)$  as

$$\epsilon^2 = \frac{16}{e^2} e^{-\frac{2\pi}{\sqrt{\lambda}} J}. \quad (3.55)$$

From equations (3.37) and (3.38) we have

$$\begin{aligned} \frac{2\pi}{\sqrt{\lambda}}(\Delta_1 - J_1) &= 2[(k-1)K(k) + E(k)] \\ &\approx 2\left(1 - \frac{\epsilon^2}{2}\right)\left(-\ln\left(\frac{\epsilon}{4}\right) + \frac{\epsilon^2}{4}\left(-\ln\left(\frac{\epsilon}{4}\right) - 1\right)\right) \\ &\quad + 2 + \epsilon^2\left(-\ln\left(\frac{\epsilon}{4}\right) - \frac{1}{2}\right) \\ &\approx 2\left[1 - \frac{\epsilon^2}{4}\right] \end{aligned}$$

or

$$\Delta_1 - J_1 \approx \frac{\sqrt{\lambda}}{\pi} \left[1 - \frac{4}{e^2} e^{-\frac{2\pi}{\sqrt{\lambda}} J}\right] \quad (3.56)$$

matching the correction found by Bak [86] for the very same string configuration, which as remarked in the introduction is seen to be exactly two coincident halves of a normal giant magnon with  $p = \pi$ .

Note that if we perform the same analysis for the  $0 \leq \dot{\phi} \leq 1$  solution then, as seen from Figure 3.7, the correction should be equal in magnitude and opposite in sign,  $\Delta - J$  being continuous as we move between the two solutions.

This time then the angular momentum is

$$J_2 = \frac{\sqrt{\lambda}}{2\pi} \frac{2(K(\dot{\phi}) - E(\dot{\phi}))}{\dot{\phi}} \quad (3.57)$$

where  $\dot{\phi}$  is playing the role of the elliptic modulus. Hence we expand in  $\varepsilon = \sqrt{1 - \dot{\phi}^2}$  instead and at leading order find

$$\frac{2\pi}{\sqrt{\lambda}} J_2 \approx 2\left(-\ln\left(\frac{\varepsilon}{4}\right) - 1\right) \quad (3.58)$$

which similarly to (3.55) gives

$$\varepsilon^2 = \frac{16}{e^2} e^{-\frac{2\pi}{\sqrt{\lambda}} J}. \quad (3.59)$$

Then from equations (3.42) and (3.43) we have

$$\begin{aligned} \frac{2\pi}{\sqrt{\lambda}}(\Delta_2 - J_2) &= 2 \left( K(\dot{\phi}) - \frac{K(\dot{\phi}) - E(\dot{\phi})}{\dot{\phi}} \right) \\ &= 2 \left[ 1 - \frac{\varepsilon^2}{2} \ln \left( \frac{\varepsilon}{4} \right) - \frac{\varepsilon^2}{4} \right] + O(\varepsilon^4 \ln(\varepsilon)) \\ &\approx 2 \left[ 1 - \frac{\varepsilon^2}{2} \left( -\frac{2\pi}{\sqrt{\lambda}} J - 1 \right) - \frac{\varepsilon^2}{4} \right] + O(\varepsilon^4 \ln(\varepsilon)) \end{aligned} \quad (3.60)$$

becoming

$$\Delta_2 - J_2 \approx \frac{\sqrt{\lambda}}{\pi} \left[ 1 + \frac{4}{e^2} e^{-\frac{2\pi}{\sqrt{\lambda}} J} \right] + e^{-\frac{2\pi}{\sqrt{\lambda}} J} J. \quad (3.61)$$

Unlike in the  $\dot{\phi} > 1$  case we have a term not proportional to  $\sqrt{\lambda}$ . With the interpretation that the string solution is valid at large  $\lambda$  then the ‘extra’ term of order  $\mathcal{O}(\lambda^0)$  becomes negligible, such that

$$\Delta_2 - J_2 \approx \frac{\sqrt{\lambda}}{\pi} \left[ 1 + \frac{4}{e^2} e^{-\frac{2\pi}{\sqrt{\lambda}} J} \right] \quad (3.62)$$

which as predicted is the same as equation (3.56) but for a difference of sign in the correction term.

### 3.3 Conclusion

We began this chapter by finding the general solution to static sine-Gordon theory on the interval, producing two qualitatively different solutions depending upon the choice of a real-valued parameter. We chose boundary conditions that under the mapping defined by (1.139) we knew would produce fixed string end points. We then mapped these solutions to string solutions on  $\mathbb{R} \times S^2$  where they correspond to open string versions of the spinning string, or boundary giant magnons attached to  $Z = 0$  maximal giant gravitons.

The choice of the parameter appearing turns out to be the choice of the string’s angular frequency  $\dot{\phi}$  about  $S^2$  and the choice between  $0 \leq \dot{\phi} \leq 1$  and  $1 \leq \dot{\phi}$  separates two sets of qualitatively different open string solutions, one of which has been studied before [86]. The two types of solution converge at  $\dot{\phi} = 1$  which from the sine-Gordon perspective is when the length of the interval  $L$  on which the theory

is defined diverges, recovering two infinitely separated copies of the boundary kink solutions and corresponding boundary giant magnons.

We then calculated the values of  $\Delta - J$  which in the dual gauge theory gives the value of the anomalous dimension of the corresponding, non-chiral primary operators. Taking  $J$  large but finite with  $\dot{\phi} > 1$  we find exponential corrections to the (finite) difference between string energy and angular momentum which is the expected result given the similarity to a particular finite  $J$  giant magnon. For the new solutions at  $\dot{\phi} < 1$  we find corrections of equal magnitude but opposite sign. At  $\dot{\phi} = 0$  we recover the static stretched string on  $S^2$ , supported against collapse by maintaining Dirichlet boundary conditions at each end.

# Chapter 4

## World volume description

In this section we will examine the world volume theory of the giant graviton wrapping an  $S^3 \subset S^5$  in order to rediscover the range of behaviour exhibited by the finite  $J$  boundary giant magnons of the previous chapter. The strings appear in an analogous manner to that of the BIon spikes of D3-branes in flat space [73, 74] requiring both scalar and gauge fields to be excited. Related solutions for the giant graviton at  $\dot{\phi} = 1$  were constructed in [84, 85] while here we allow any  $\dot{\phi} \geq 0$ , and hence a finite  $J$  description.

We use the static gauge where the world volume time  $t = \tau$  is chosen to coincide with the target space time coordinate that appears as the global time variable (1.15) of the  $AdS$  part of  $AdS_5 \times S^5$ . With the abbreviations  $\rho^2 \equiv 1 - r^2$ ,  $s_4 \equiv \sin\sigma_4$  and  $s \equiv \sin\sigma$ , we may write the metric on  $\mathbb{R} \times S^5$  as

$$ds^2 = R^2 \left\{ -dt^2 + \frac{dr^2}{\rho^2} + r^2 d\phi^2 + \rho^2(d\sigma^2 + s^2 d\sigma_4^2 + s^2 s_4^2 d\sigma_5^2) \right\}. \quad (4.1)$$

where  $\{\sigma, \sigma_4, \sigma_5\}$  coordinate the space  $\Omega_3$  appearing in the metric (2.70). The D3-brane is taken to lie at the centre of  $AdS_5$  so that the world volume scalars associated with these directions are unexcited. We will mainly be concerned with the maximal giant graviton that wraps an  $S^3$  entirely transverse to the string coordinates (a  $Z = 0$  giant graviton).

## 4.1 D3-brane action and ansatz

The brane action, having a kinetic part given by (1.4) and a potential part given by the Chern-Simons coupling, is

$$S = -T_{D3} \left[ \int_{\Omega_{D3}} \sqrt{-\det(G_{ab} + 2\pi l_s^2 F_{ab})} + \int_{\Omega_{D3}} C_4 \right] \quad (4.2)$$

where  $G_{ab}$  is the metric induced<sup>1</sup> on the world volume of the D3 brane embedded in  $S^5$  so the indices  $a, b$  take one time and 3 spatial values;  $F_{ab}$  is the field strength tensor for the electromagnetic fields living on the world volume and there is no  $\tilde{B}_{ab}$  contribution.

We choose the embedding of the D3 brane to be given simply by the coordinates  $\{\sigma, \sigma_4, \sigma_5\}$  so that the coordinates  $\{r, \phi\}$  are transverse to the brane and equal to two of the scalar fields living thereon (there will be 4 more that will not concern us).

### 4.1.1 D3-brane equations of motion

We can simplify the explicit form of the action if we at this point make the ansatz that the scalar fields depend only upon the coordinate  $\sigma$ , with  $\dot{r} = 0$  and  $\dot{\phi} \neq 0$  but constant. Brane solutions will therefore be symmetric with respect to rotations by  $\sigma_4$  and  $\sigma_5$ . We will now denote with a prime the derivative with respect to  $\sigma$  and take a purely electric  $F_{ab}$  with only  $F_{\tau\sigma} = -F_{\sigma\tau} \neq 0$ .

As discussed in chapter 2 the potential term is an integral over the world volume of the brane  $\Omega_{D3}$  of the background 4-form potential  $C_4$ . We may re-write this in terms of the 5-form field strength as

$$S_{CS} = \int_{\Omega_{D3}} C_4 = \int_{\Sigma} F_5 \quad (4.3)$$

where  $F_5 = dC_4$  and  $\Sigma$  is now the 5-manifold whose boundary is the 4 dimensional hypersurface swept out by the D3 brane. The form of the potential used in [10] and given in (2.72) was obtained by an argument relating the action to a Lagrangian

---

<sup>1</sup>We omit the tilde previously used to distinguish the induced metric from the target space metric.

by dividing by a fixed time period. Below we check more explicitly that in working with a Lagrangian *density* we can trust the same expression.

Since the background flux has a constant density over the  $S^5$ ,  $F = B d\text{vol}_5$  so that

$$\begin{aligned} S_{\text{CS}} &= B \text{ vol}(\Sigma) \\ &= B \int_{\Sigma} R^5 r \rho^2 s^2 s_4 dr d\phi d\sigma d\sigma_4 d\sigma_5. \end{aligned}$$

We may immediately integrate over  $\sigma_4$  and  $\sigma_5$  then convert from an integral over  $r$ , at fixed  $\sigma$ , to  $\rho$  which goes from 0 up to the dimensionless radius of the brane  $\rho(\sigma)$ , and finally use  $d\phi = \dot{\phi} dt$  to get a Lagrangian density for the potential of

$$\mathcal{L}_{CS} = \pi B R^5 \sin^2 \sigma \rho^4 \dot{\phi}. \quad (4.4)$$

$B$  may be re-expressed using the flux quantisation condition for the 5-form on  $S^5$ ,  $BR^5 \Omega_5 = 2\pi N$ , while the D3-brane tension is

$$T_{p=3} = \frac{2\pi}{g_s (2\pi l_s)^{p+1}} \Big|_{p=3} \quad (4.5)$$

$$\begin{aligned} \Rightarrow T_3 &= \frac{1}{g_s (2\pi)^3 l_s^4} \\ &= \frac{N}{R^4 \Omega_3}, \quad \Omega_3 = 2\pi^2 \end{aligned} \quad (4.6)$$

where we used the relation between the forms of the fixed 't Hooft coupling  $\lambda$ ,

$$\lambda = \frac{R^4}{l_s^4} = 4\pi g_s N. \quad (4.7)$$

Hence we can write the potential term as

$$\mathcal{L}_{\text{CS}} = T_3 \Omega_2 R^4 \sin^2 \sigma (1 - r^2)^2 \dot{\phi}. \quad (4.8)$$

Putting these together the Lagrangian density in  $\sigma$  is

$$\mathcal{L} = -T \int d\sigma d\tau s^2 \left[ \rho^2 \sqrt{D} - \rho^4 \dot{\phi} \right] \quad (4.9)$$

with effective tension  $T = 4\pi R^4 T_3$  and

$$D = r'^2 \Lambda^2 + r^2 \phi'^2 - \tilde{F}^2 + \rho^4 \Lambda^2 \quad (4.10)$$

with rescaled electric field  $\tilde{F} = \left(\frac{2\pi l_s^2}{R^2}\right) F_{\tau\sigma}$  and we have defined

$$\Lambda \equiv \sqrt{\frac{1 - \dot{\phi}r^2}{1 - r^2}}. \quad (4.11)$$

Turning to the equations of motion we encounter only total derivatives with respect to  $\sigma$  due to the time independence of our ansatz, thus for example we have

$$\frac{\partial}{\partial\tau} \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = -Ts^2 \frac{\partial}{\partial\tau} \left( \frac{1}{2} \frac{1}{\sqrt{D}} \frac{\partial D}{\partial\dot{\phi}} - \rho^4 \right) = 0 \quad (4.12)$$

as everything is independent of time except  $\phi$  itself, which does not appear. Then the  $\phi$  equation of motion becomes

$$\phi : \quad \frac{d}{d\sigma} \left( \frac{s^2 \rho^2 r^2 \phi'}{\sqrt{D}} \right) = 0. \quad (4.13)$$

As we have already covered we would like to study a radial configuration so that  $\phi' = 0$  and the  $\phi$  equation is therefore satisfied automatically by these configurations. The equation of motion for the gauge component  $A_\sigma$  is similarly trivial while for the  $A_\tau$  component we have

$$A_\tau : \quad \frac{d}{d\sigma} \left( \frac{s^2 \rho^2 \tilde{F}}{\sqrt{D}} \right) = 0 \quad (4.14)$$

and for the  $r$  equation of motion we have

$$r : \quad \frac{d}{d\sigma} \left( \frac{s^2 \rho^2 r' \Lambda^2}{\sqrt{D}} \right) = s^2 \frac{\partial}{\partial r} \left( \rho^2 \sqrt{D} - \rho^4 \dot{\phi} \right). \quad (4.15)$$

The  $A_\tau$  equation can be integrated once to give a constant in  $\sigma$ :

$$\frac{s^2 \rho^2 \tilde{F}}{\sqrt{D}} = \kappa. \quad (4.16)$$

The constant is fixed by the electric flux quantisation condition<sup>2</sup>  $\Pi_A = k \in \mathbb{Z}$ ;

$$\begin{aligned} \Pi_A &= \frac{\partial\mathcal{L}}{\partial\dot{A}_\tau} = \frac{\partial\mathcal{L}}{\partial F_{\tau\sigma}} \frac{\partial F_{\tau\sigma}}{\partial A'_\tau} = -\frac{\partial\mathcal{L}}{\partial F_{\tau\sigma}} \\ &= -T \frac{s^2 \rho^2}{\sqrt{D}} \cdot \left( \frac{2\pi l_s^2}{R^2} \right)^2 F_{\tau\sigma} \end{aligned} \quad (4.17)$$

---

<sup>2</sup>The number of F-strings attaching to the brane is therefore given by the integer  $k$ .

and then using the  $A_\tau$  equation of motion, (4.14), we can write

$$T\kappa \left( \frac{2\pi l_s^2}{R^2} \right) = k, \quad \Rightarrow \quad \kappa = \frac{1}{T} \left( \frac{R^2}{2\pi l_s^2} \right) k = \frac{\sqrt{\lambda}}{4N} k \quad (4.18)$$

recovering the form of the constant introduced in [84].

The angular momentum density is found to be

$$\Pi_\phi = Ts^2 \left[ \frac{r^2 \dot{\phi}}{\sqrt{D}} (r'^2 + \rho^4) + \rho^4 \right] \quad (4.19)$$

and the Hamiltonian density is

$$\mathcal{H} = \Pi_\phi \dot{\phi} + \Pi_A F_{\tau\sigma} - \mathcal{L} \quad (4.20)$$

$$= \frac{Ts^2}{\sqrt{D}} (r'^2 + \rho^4). \quad (4.21)$$

By rearranging the  $A_\tau$  equation (4.14) we can find a general form of the field strength  $\tilde{F}$  in terms of the derivative  $r'$ :

$$\tilde{F} = \pm \kappa \Lambda \sqrt{\frac{r'^2 + \rho^4}{s^4 \rho^4 + \kappa^2}}. \quad (4.22)$$

Before proceeding to the case of general  $\dot{\phi}$  where we hope to recover the results of the previous chapter we shall first re-examine the classical string in the present coordinates, allowing us to refine our expectations. Then we shall first specialise to  $\dot{\phi} = 1$  to recover known results.<sup>3</sup>

### 4.1.2 Classical string

We are going to produce Born-Infeld configurations that have a target space interpretation of fundamental strings attaching to the giant graviton. In order that we are able to identify the appearance of the string correctly we first find expressions for the angular momentum and energy of the string in a more general gauge.

The solutions we are after live on an  $S^2$  of radius  $R$  which we choose to be coordinated by  $\{r, \phi\}$ , and are radially extended in  $r$ , so that denoting with a prime derivatives with respect to the world sheet spatial coordinate  $x$  we have  $r' \neq 0$  and

---

<sup>3</sup>Most of the results in section 4.2 are to be found in a slightly different form in [84].

$\phi' = 0$ . The strings move rigidly with  $\dot{r} = \ddot{\phi} = 0$ . With induced metric  $g_{ab}$  the Nambu-Goto Lagrangian density is [5]

$$\begin{aligned}\mathcal{L}_{\text{NG}} &= -\frac{1}{2\pi l_s^2} \sqrt{-\det g_{ab}} \\ &= -\frac{\sqrt{\lambda}}{2\pi} \Lambda r',\end{aligned}\quad (4.23)$$

The angular momentum density is

$$j_s = \frac{\sqrt{\lambda}}{2\pi} \frac{r^2 r'}{\rho^2} \frac{\dot{\phi}}{\Lambda}. \quad (4.24)$$

Given the dependence here only on  $r$ ,  $\dot{\phi}$  and a single power of  $r'$  we may integrate up immediately with respect to  $r$ . In light of the above results we should find two sets of solutions depending upon whether  $0 \leq \dot{\phi} \leq 1$  or  $\dot{\phi} \geq 1$ . If for  $\dot{\phi} \geq 1$  we take the upper and lower limits of  $r$  to be  $r = \dot{\phi}^{-1}$  and  $r = 0$  respectively, and for  $0 \leq \dot{\phi} \leq 1$  we take  $r = 1$  and  $r = 0$  (so that in both cases we shall obtain half of one of the strings above) then we find

$$J_s = -\frac{\sqrt{\lambda}}{2\pi} \int_{r_L}^{r_U} \frac{r^2}{\rho^2} \frac{\dot{\phi}}{\Lambda} dr \quad (4.25)$$

$$= \begin{cases} -\frac{\sqrt{\lambda}}{2\pi} \left( E\left(\frac{1}{\dot{\phi}}\right) - K\left(\frac{1}{\dot{\phi}}\right) \right), & \text{if } \dot{\phi} \geq 1 \\ -\frac{\sqrt{\lambda}}{2\pi} \frac{1}{\dot{\phi}} \left( E(\dot{\phi}) - K(\dot{\phi}) \right), & \text{if } 0 \leq \dot{\phi} \leq 1 \end{cases} \quad (4.26)$$

which is precisely as expected from the sine-Gordon derived results of chapter 3, namely results (3.37) and (3.42), without even requiring the solutions themselves. The overall minus signs reflect the fact that the world sheet spatial variable runs from the end of the string at or closest to the equator, or the *upper* limit of  $r$ , to the point with  $r = 0$ .

The Nambu string energy density  $h_s$  is given by

$$h_s = \frac{\sqrt{\lambda}}{2\pi} \frac{r'}{\rho^2} \frac{1}{\Lambda} \quad (4.27)$$

which when integrated for the string energy  $\Delta_s$  returns the results obtained above for any  $\dot{\phi} \geq 0$  and therefore returns equations (3.56) and (3.62), the same leading order correction to the quantity  $\Delta_s - J_s$ :

$$\Delta_s - J_s \approx \frac{\sqrt{\lambda}}{2\pi} \left\{ 1 \pm \frac{4}{e^2} e^{-\frac{2\pi}{\sqrt{\lambda}} J_s} \right\} \quad (4.28)$$

where the plus sign is for  $\dot{\phi} \geq 1$  and the minus sign is for  $0 \leq \dot{\phi} \leq 1$ . Note that this is in fact half of the  $\Delta - J$  calculated in the previous chapter simply because we have integrated over a single half of the full string.

Returning to the world volume we must now find solutions that reproduce the above results for the boundary giant magnons, namely the correct solution profiles and behaviours of  $\Delta - J$ . We begin with the  $J \rightarrow \infty$  limit,  $\dot{\phi} = 1$ .

## 4.2 $\dot{\phi} = 1$

One way to recover the  $\dot{\phi} = 1$  solution is to use the  $A_\tau$  equation (4.14) to write  $\sqrt{D} = \frac{s^2 \rho^2 \tilde{F}}{\kappa}$  and set  $\dot{\phi} = 1$  whereever else it appears so that the  $r$  equation becomes

$$\frac{d}{d\sigma} \left( \frac{r'}{\tilde{F}} \right) = -2r\rho^2 \left[ \left( \frac{s^2 \tilde{F}}{\kappa} \right)^2 - 1 \right]^2. \quad (4.29)$$

This can be satisfied if

$$\tilde{F} = r' \quad \text{and} \quad \frac{s^2 \tilde{F}}{\kappa} = \pm 1 \quad \Rightarrow \quad r' = \pm \frac{\kappa}{s^2}. \quad (4.30)$$

To check for consistency with the  $A_\tau$  equation we substitute  $\kappa = s^2 r'$  (and  $\dot{\phi} = 1$ ) into the general form of  $\tilde{F}$  given by equation (4.22) and this indeed returns us  $\tilde{F} = r'$ . Solving equation (4.30) gives the solution

$$r = c \mp \kappa \cot \sigma. \quad (4.31)$$

This is essentially the solution presented in [84] (and similarly in [85]) except that we have solved for  $r$  as opposed to one of the Cartesian coordinates  $\{x_1, x_2\}$  satisfying  $x_1^2 + x_2^2 = r^2$ , and we have set  $\phi' = 0$ . There  $x_1$  was taken to be constant, giving the familiar geometry of the giant magnon as a chord within the disc formed when projecting the  $S^2$  onto the  $\{x_1, x_2\}$  plane. Our analogous solution effectively has  $x_1 = 0$  so that the string/Bion protrudes radially from the giant graviton situated at  $r = 0$ . For the constant<sup>4</sup>  $c = 0$  the body of the D-brane sits at  $r = 0$  corresponding to a maximal giant graviton. The solution is plotted in Figure 4.1.

---

<sup>4</sup>The constant  $c$  appearing here is not the constant introduced in chapter 3.

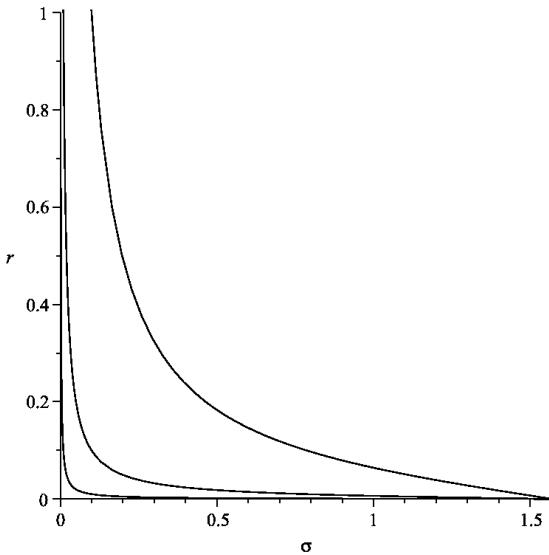


Figure 4.1: The  $\dot{\phi} = 1$  solution with  $c = 0$  and positive sign taken plotted for three values of  $\kappa$ :  $\kappa = 0.1$ ,  $\kappa = 0.01$ ,  $\kappa = 0.001$ . The solution has boundary conditions  $r\left(\frac{\pi}{2}\right) = 0$  and  $r(\sigma_0) = 1$ . As  $\kappa \rightarrow 0$  the solution tends to a constant for most of  $\sigma$  with a sudden spike as we approach  $\sigma_0 \rightarrow 0$ .

While the solution of [84] ran from negative to positive  $x_2(\sigma)$ ,  $r$  can only be positive or zero. However, by taking the appropriate sign we can maintain  $r \geq 0$ . In this way the solution is well defined on  $\sigma \in [\sigma_0, \pi - \sigma_0]$  and describes two spikes at either end of the range of  $\sigma$ . If the range of  $\sigma$  is  $\sigma_0 \leq \sigma \leq \pi - \sigma_0$  (and taking  $c = 0$ ) then we have that  $r(\sigma_0) = r(\pi - \sigma_0) = 1$ . We could choose the spikes to emerge in opposite directions, leading to precisely the previous solution with  $x_1 = 0$  and corresponding to the giant magnon string solution with world sheet momentum  $p = \pi$  (alternatively a sine-Gordon kink with velocity  $v = 1$ ). Or we could take both spikes to emerge in the same direction leading to two new solutions that are only physical for the strings that pass over the poles of the sphere, i.e.  $r = 0$ ; they are the ‘boundary giant magnons’, one of which is described in [86]. If  $x_3$  is the third Cartesian coordinate on  $S^2$  satisfying  $r^2 + x_3^2 = 1$  then we may choose either  $0 \leq \pm x_3 \leq 1$ , placing both spikes on just one of the hemispheres of  $S^2$ , or choose  $-1 \leq x_3 \leq 1$  so that the string passes from pole to pole through the equator.

From equation (4.18) we have that  $\kappa = \frac{\pi}{\sqrt{\lambda}} g_s$  so that with  $\lambda$  fixed  $\kappa$  controls

the string coupling. The Nambu string description in which the giant magnons live requires vanishing string coupling and hence  $\kappa \rightarrow 0$ . As  $\kappa \rightarrow 0$  the spikes on the brane become more pronounced, being concentrated into the points  $\sigma_0$  and  $\pi - \sigma_0$ , while in between these points the radius of the brane (that is, the radius  $R\rho$  of the  $S^2$  at each point in  $\sigma$ ) tends to a constant. In the limit then, the picture is of a spherical giant graviton with infinitesimally thin strings attached to its poles. In the same limit  $\sigma_0$ , for which  $r(\sigma_0) = r(\pi - \sigma_0) = 1$ , tends to zero.

Taking non-zero values of the constant  $c$  was not discussed in [84] but it would appear to describe boundary giant magnons attached to non-maximal  $Z = 0$  giant gravitons, at least for small values of  $c$  where we can trust the brane description of the graviton state. In sine-Gordon theory these solutions are described by taking boundary values of  $\varphi \neq \pi$ . The non-maximal but, as we shall see below, still BPS giant graviton now rotates in the  $\{x_1, x_2\}$  plane maintaining  $\dot{\phi} = 1$ .

This behaviour covers that which we have found for boundary giant magnons in the previous chapter, described by sine-Gordon theory on the interval at  $\dot{\phi} = 1$ , or  $L \rightarrow \infty$ . Below we shall find approximate solutions to the brane equations that reproduce the behaviour of the two types of boundary giant magnons at  $\dot{\phi} \neq 1$ . That is, we desire one family of brane solutions for which we obtain  $r = 1 \forall 0 \leq \dot{\phi} \leq 1$  and another family that has some maximum value of  $r$  when  $1 \leq \dot{\phi}$ . While the target space coordinate  $r$  for which we wish to solve is trivially identified in the string and brane pictures the independent variables on which it depends in each picture, a world sheet coordinate on the one hand and a world volume coordinate on the other, are not easily matched. We can however be confident that we have obtained matching solutions by the equality of their energy and angular momenta.

At  $\dot{\phi} = 1$  the factor  $\sqrt{D} = \rho^2$  and so the total angular momentum density given by (4.19) contains a term  $\propto r'$  and two that are not. If we take the  $r'$  term as the string's contribution to the angular momentum density, and take a single string ( $k = 1$ ), then integrating over one of the spikes on  $\sigma \in [\sigma_0, \frac{\pi}{2}]$  we get the divergent

quantity

$$\begin{aligned}
J|_{\dot{\phi}=1} &= T \int_{\sigma_0}^{\frac{\pi}{2}} \frac{s^2 r^2 r'^2}{\rho^2} d\sigma \\
&= T\kappa \int_{\sigma_0}^{\frac{\pi}{2}} \frac{r^2 r'}{\rho^2} d\sigma \\
&= -\frac{\sqrt{\lambda}}{2\pi} \int_0^1 \frac{r^2}{\rho^2} dr = J_s|_{\dot{\phi}=1}.
\end{aligned} \tag{4.32}$$

In the last step we used

$$r' = \frac{\kappa}{s^2}, \quad \kappa = \frac{\sqrt{\lambda}}{4N}, \quad T = \frac{2}{\pi}N, \quad \Rightarrow T\kappa = \frac{\sqrt{\lambda}}{2\pi} \tag{4.33}$$

which recovers the tension of the fundamental string from the brane picture. This angular momentum is equal to that of the Nambu string at  $\dot{\phi} = 1$ , as seen by substituting  $\dot{\phi} = 1$  into equations (4.24) and (4.25). Again, we have an overall minus sign difference to the sine-Gordon derived result simply because of the direction in which we integrated over the string, and because we integrated over only one spike we have only half of the sine-Gordon derived result. Below we will see that, given an approximation, this same term also appears to capture the string's contribution to the angular momentum density at  $\dot{\phi} \neq 1$ .

Computing  $\Delta - J$  we get

$$\begin{aligned}
(\Delta - J)|_{\dot{\phi}=1} &= T \int_{\sigma_0}^{\frac{\pi}{2}} \frac{s^2(1-r^2)r'^2}{\rho^2} d\sigma \\
&= T\kappa = \frac{\sqrt{\lambda}}{2\pi}
\end{aligned} \tag{4.34}$$

which is the finite result that we expect. As noted in [84], this contribution has arisen from the term  $\Pi_A F_{\tau\sigma}$  appearing in the Hamiltonian, equation (4.20); the contribution is entirely from the electric flux on the world volume, that is from the giant magnon string attached to the giant graviton, for which the contribution to  $\Delta - J$  is zero. Computing the brane angular momentum  $J_B$  as

$$\begin{aligned}
J_B|_{\dot{\phi}=1} &= T \int_{\sigma_0}^{\frac{\pi}{2}} s^2 \left[ \frac{r^2 \dot{\phi} \rho^4}{\sqrt{D}} + \rho^4 \right] \Big|_{\dot{\phi}=1} d\sigma \\
&= T \int_{\sigma_0}^{\frac{\pi}{2}} s^2 \rho^2 d\sigma, \quad T = \frac{2}{\pi}N,
\end{aligned} \tag{4.35}$$

and using the solution (4.31) gives

$$J_B \rightarrow N \text{ when } \sigma_0 \rightarrow 0 \text{ and } \kappa \rightarrow 0 \text{ keeping } \kappa \cot(\sigma) = 1 \text{ fixed,} \quad (4.36)$$

the final condition being required to comply with the boundary condition  $r(\sigma_0) = 1$ , as found in [84].  $J_B = N$  is what we should expect from the BPS giant graviton. In [10] McGreevy and Susskind treated the D3 brane with spherical symmetry and showed that the giants obey

$$\frac{J_B}{N} \leq \rho^2 \quad (4.37)$$

where  $\rho$  is the (constant) radius of the brane and BPS giants saturate the inequality. If the brane is maximal then  $\rho = 1$ . Looking back at the integral (4.35) and being conscious of the behaviour of the  $\dot{\phi} = 1$  solution which for  $\kappa \rightarrow 0$  tends to a solution with a constant  $\rho$  for  $\sigma \in [\sigma_0 \rightarrow 0, \frac{\pi}{2}]$  then we can see how we recover the BPS, spherical brane to which the string is attached, obtaining the same result as if we had taken  $\rho^2$  out of the integral as a constant;

$$\begin{aligned} J_B &= \frac{2}{\pi} N \int_{\sigma_0}^{\frac{\pi}{2}} s^2 \rho^2 d\sigma \\ &\simeq \frac{2}{\pi} N \rho^2 \int_0^{\frac{\pi}{2}} s^2 d\sigma = \frac{2}{\pi} N \rho^2 \cdot \frac{\pi}{2} = N \rho^2. \end{aligned} \quad (4.38)$$

With  $\rho = 1$  we have  $J_B = N$ .

With the energy of the brane given by an integral over the second term in equation (4.21) for the total Hamiltonian density,

$$\begin{aligned} \Delta_B|_{\dot{\phi}=1} &= T \int_{\sigma_0}^{\frac{\pi}{2}} \frac{s^2 \rho^4}{\sqrt{D}} \Big|_{\dot{\phi}=1} d\sigma \\ &= T \int_{\sigma_0}^{\frac{\pi}{2}} s^2 \rho^2 d\sigma \end{aligned} \quad (4.39)$$

which is equal to  $J_B|_{\dot{\phi}=1}$  as given by (4.35). That is, the brane has

$$\Delta_B - J_B = 0 \quad \text{at } \dot{\phi} = 1. \quad (4.40)$$

Below we shall see that, given  $\kappa \rightarrow 0$ , the same is true for any  $\dot{\phi} \geq 0$ .

Finally, we note that the field strength  $\tilde{F}$ , given by (4.31), as well as satisfying equation (4.14), satisfies Gauss's law on the world volume,

$$\text{div}_{S^3} F_{\tau a} = 0, \quad a \in \{\sigma, \sigma_4, \sigma_5\}. \quad (4.41)$$

We have only the  $\sigma$  component of the field strength and no dependence on the other world volume coordinates  $\{\sigma_4, \sigma_5\}$ , hence

$$\text{div}_{S^3} = \frac{1}{\sqrt{h}} \partial_a \sqrt{h}, \quad h \equiv \sin^4(\sigma) \sin^2(\sigma_4), \quad (4.42)$$

where  $h$  is the determinant of the metric on  $S^3$ , and the resulting factors of  $\sin(\sigma_4)$  cancel. From (4.31), the scalar  $r$  should then satisfy

$$\frac{1}{s^2} \frac{\partial}{\partial \sigma} \left( s^2 \frac{\partial}{\partial \sigma} r \right) = 0 \quad (4.43)$$

which indeed it does.

This is analogous to the behaviour of the BIon spikes of Callan and Maldacena [73] where the linearised theory gave the same configurations as the fully non-linear approach; Maxwell theory was sufficient to construct the (BPS) objects of the Born-Infeld theory. Similarly, the solutions presented in [85] were constructed from a linearised approach to the giant graviton. Being BPS the configuration is protected from receiving the corrections to the Born-Infeld approach that would normally be required when derivatives become large.

Defining the charge  $Q$  as that seen by the Maxwell fields on the world volume, then while the net charge is of course zero, by performing a surface integral over the  $S^2$  located at  $\sigma$  we can find the charge of one of the point charges located at the poles to get

$$\begin{aligned} Q &= \int_{S^2} \tilde{F} dA = s^2 \tilde{F} \Omega_2 \\ &= 4\pi\kappa. \end{aligned} \quad (4.44)$$

Next we consider  $\dot{\phi} \neq 1$ , or finite  $J$  configurations.

### 4.3 $\dot{\phi} \neq 1$

Away from  $\dot{\phi} = 1$  it appears that the brane equations are no longer exactly solvable. However upon making appropriate approximations a solution is available for any  $\dot{\phi} \geq 0$  that reproduces the Nambu string quantities derived above for both types of solution. The first approximation we will take focusses our attention close to the

poles of the deformed D-brane where the BIon spikes are to be found. The second is to take  $\kappa$  to be small, which as stated above means that we take the small string coupling, Nambu string limit.

First we examine the conserved quantities  $J$  and  $\Delta$ . Using the  $A_\tau$  equation (4.14) to eliminate  $\tilde{F}$  we write the factor  $\sqrt{D}$  as

$$\sqrt{D} = \pm s^2 \rho^2 \Lambda \sqrt{\frac{r'^2 + \rho^4}{s^4 \rho^4 + \kappa^2}}, \quad \Lambda \equiv \sqrt{\frac{1 - \dot{\phi}^2 r^2}{1 - r^2}}. \quad (4.45)$$

If we take the string's contribution to the angular momentum to again be the “ $r'$  term”, and take the positive sign in front of  $\sqrt{D}$ , then we have for a single spike

$$J = \int_{\sigma_0}^{\frac{\pi}{2}} \frac{Ts^2 r^2 \dot{\phi} r'^2}{\sqrt{D}} d\sigma = \int_{\sigma_0}^{\frac{\pi}{2}} \frac{Tr^2 \dot{\phi} r'^2}{\rho^2 \Lambda} \sqrt{\frac{s^4 \rho^4 + \kappa^2}{r'^2 + \rho^4}} d\sigma. \quad (4.46)$$

Focussing on small  $\sigma$  (or equivalently  $\sigma$  close to  $\pi$ ) we take  $s^4 \rho^4 \ll \kappa^2$ . Around the spike we expect  $r'$  to be large. In fact for  $\dot{\phi} = 1$  we had  $r' = \frac{\kappa}{s^2}$  so that when  $s^4 \rho^4 \ll \kappa^2$  (and remember  $0 \leq \rho \leq 1$ ) then  $|r'| \gg 1$ . If at  $\dot{\phi} \neq 1$  we expect there to remain a spiky solution that is continuously related to the  $\dot{\phi} = 1$  case then we take this to remain true for  $\dot{\phi} \neq 1$ .

An integral over  $\sigma$  with lower limit of  $\sigma_0$  will have to be cut off at some  $\hat{\sigma}$  satisfying  $\sin^4(\hat{\sigma}) \rho^4 \ll \kappa^2$ . For  $\sigma > \sigma_0$  there should be vanishing contribution to quantities coming from the string / BIon spike. Obviously a similar argument holds close to the other spike close to  $\sigma = \pi$ .

At leading order in this approximation then the factor  $\sqrt{D}$  satisfies

$$\frac{s^4 \rho^4}{\kappa^2} \ll 1, \quad \frac{\rho^4}{r'^2} \ll 1, \quad \Rightarrow \quad \sqrt{D} \approx s^2 \rho^2 \Lambda \frac{r'}{\kappa} \quad (4.47)$$

and the candidate string / BIon angular momentum becomes

$$\begin{aligned} J &= \frac{\sqrt{\lambda}}{2\pi} \int_{\sigma_0}^{\frac{\pi}{2}} \frac{r^2 \dot{\phi} r'^2}{\rho^2 \Lambda} \sqrt{\frac{s^4 \rho^4 + \kappa^2}{r'^2 + \rho^4}} d\sigma \\ &\approx \frac{\sqrt{\lambda}}{2\pi} \int_{\sigma_0}^{\hat{\sigma}} \frac{r^2 \dot{\phi} r'^2}{\rho^2 \Lambda} \frac{\kappa}{r'} d\sigma \end{aligned} \quad (4.48)$$

$$= -\frac{\sqrt{\lambda}}{2\pi} \int_{\hat{r}}^{r_U} \frac{r^2 \dot{\phi}}{\rho^2 \Lambda} dr \quad (4.49)$$

which is to be compared with equation (4.25) for the Nambu string's angular momentum. The lower limit here is  $\hat{r} \equiv r(\hat{\sigma})$ . Clearly if  $\hat{r} \rightarrow 0$  then  $J \rightarrow J_s$ , which

will be the case if after taking  $s^4\rho^4 \ll \kappa^2$  we allow  $\kappa \rightarrow 0$ ; we recover the Nambu string contribution in the zero string coupling (or large  $N$  't Hooft) limit which is by definition what we should expect.

So the leading order approximation to the angular momentum captures all of the detail of the Nambu string when  $\kappa \rightarrow 0$ , for any  $\dot{\phi}$ , and as shown above this integral can be performed and matched with the results obtained explicitly from the string solutions. Together with a similar simplification of the expression for the string / BIon energy  $\Delta$ ,

$$\Delta \approx \Delta_s, \quad \text{to leading } s^4\rho^4 \ll \kappa^2 \quad \text{and } \kappa \rightarrow 0$$

we will obtain precisely the finite  $J$  leading order corrections to  $\Delta - J$  presented previously, such as in equation (4.28), i.e.

$$\Delta - J \approx \frac{\sqrt{\lambda}}{2\pi} \left\{ 1 \pm \frac{4}{e^2} e^{-\frac{2\pi}{\sqrt{\lambda}} J} \right\} \quad (4.50)$$

and indeed the full spectrum of  $\Delta - J$  with  $\dot{\phi}$  discussed above.

Now we know the BIon spikes will return us the correct dispersion relations between the conserved quantities for any  $\dot{\phi}$ , our next step is to find solutions that display the same behaviour as encountered in section 3. The full equations appear intractable but we may be able to find approximate solutions given the above discussion.

To this end we once again eliminate  $\tilde{F}$  from the  $r$  equation of motion (4.15) using the  $A_\tau$  equation (4.14). Multiplying through by a factor of  $\sqrt{D}$  and using the approximation  $s^4\rho^4 \ll \kappa^2$  we expand to first order to get a left hand side (LHS) of

$$\text{LHS} \approx \rho^2 \Lambda s^2 r' \left( 1 + \frac{1}{2} \frac{\rho^4}{r'^2} - \frac{1}{2} \frac{s^4 \rho^4}{\kappa^2} \right) d_\sigma \left[ \Lambda \left( 1 - \frac{1}{2} \frac{\rho^4}{r'^2} + \frac{s^4 \rho^4}{\kappa^2} \right) \right]. \quad (4.51)$$

Remembering that we expect  $r' s^2$  to be of order one, we may use the same reasoning to take  $r'' \lesssim r'^2$ , then the remaining terms to first order in  $\frac{s^4 \rho^4}{\kappa^2}$  on the left hand side give

$$\begin{aligned} \text{LHS} \approx & \rho^2 \Lambda r' s^2 \left\{ \frac{rr'(1 - \dot{\phi}^2)}{\Lambda \rho^4} \left( 1 - \frac{1}{2} \frac{\rho^4}{r'^2} + \frac{1}{2} \frac{s^4 \rho^4}{\kappa^2} \right) - \left( -\frac{2rr'\rho^2}{r'^2} - \frac{\rho^4 r''}{r'^3} \right) \right. \\ & \left. + \frac{\Lambda}{2} \left( \frac{4s^3 \cos(\sigma) \rho^4}{\kappa^2} - \frac{4s^4 rr' \rho^2}{\kappa^2} \right) + \left( \frac{1}{2} \frac{\rho^4}{r'^2} - \frac{1}{2} \frac{s^4 \rho^4}{\kappa^2} \right) \frac{rr' \Omega^2}{\Lambda \rho^4} \right\} \end{aligned} \quad (4.52)$$

where  $\Omega^2 \equiv 1 - \dot{\phi}^2$ . From the right hand side (RHS) we get

$$\begin{aligned} \text{RHS} \approx & -2rs^2\Lambda^2 \frac{s^4\rho^4r'^2}{\kappa^2} \left( 1 + \frac{\rho^4}{r'^2} - \frac{s^4\rho^4}{\kappa^2} \right) \\ & + rs^2 \left( \frac{r'^2\Omega^2}{\rho^2} - \rho^4\Lambda^2 - \dot{\phi}^2\rho^4 \right) \pm 4rs^2\dot{\phi}\rho^4s^2\Lambda \frac{r'}{\kappa} \left( 1 + \frac{1}{2}\frac{\rho^4}{r'^2} - \frac{1}{2}\frac{s^4\rho^4}{\kappa^2} \right). \end{aligned} \quad (4.53)$$

The  $\pm$  in front of the final term on the right hand side (which has been ‘squared out’ elsewhere in the above expressions) comes from that in front of the  $\sqrt{D}$ , equation (4.45), and is important.

Next we take our second approximation:  $\kappa \rightarrow 0$ . Looking at the above,  $\kappa$  appears only in negative powers and the leading term (in  $\kappa$ ) is on the right hand side, being proportional to  $\kappa^{-4}$ . On its own we do not recover a realistic equation for  $r'$  so we include the next to leading term, which is also on the right hand side and is proportional to  $\kappa^{-3}$ . With the left hand side contributing only higher order terms in  $\kappa$  we hence are led to another first order differential equation for  $r$ ,

$$0 = \frac{2(s^4\rho^4)^2\Lambda^2rr'^2}{\kappa^4} \mp \frac{4\rho^4s^2\dot{\phi}\Lambda r'(\rho^4s^4)}{2\kappa^3} \quad (4.54)$$

or

$$r' = \pm \frac{\kappa}{s^2\Lambda} \frac{\dot{\phi}}{\rho^4} \quad (4.55)$$

which generalises equation (4.30) to  $\dot{\phi} \neq 1$ . At  $\dot{\phi} = 1$  the factor  $\Lambda = 1$  and we recover the correct equation for  $r'$ . The equation is again separable so that we get the *implicit* solutions

$$1 \leq \dot{\phi} : \quad E \left( r\dot{\phi}, \frac{1}{\dot{\phi}} \right) - W^2 F \left( r\dot{\phi}, \frac{1}{\dot{\phi}} \right) = c \pm \kappa \cot(\sigma) \quad (4.56)$$

$$0 \leq \dot{\phi} \leq 1 : \quad \frac{1}{\dot{\phi}} E(r, \dot{\phi}) = c \pm \kappa \cot(\sigma) \quad (4.57)$$

where in (4.56) we define  $W^2 \equiv 1 - \frac{1}{\dot{\phi}^2}$ . As in the  $\dot{\phi} = 1$  case a constant  $c$  appears on the right hand side and a choice of sign. The choice of sign is essential to be able to always maintain  $r \geq 0$ . As in the  $\dot{\phi} = 1$  case we still have the *two* spikes, one of which occurs close to  $\sigma = \pi$  where the  $\cot(\sigma)$  is large and negative, hence requiring the choice of the minus sign.

Again we encounter elliptic integrals, and the two solutions are the inverse mod-

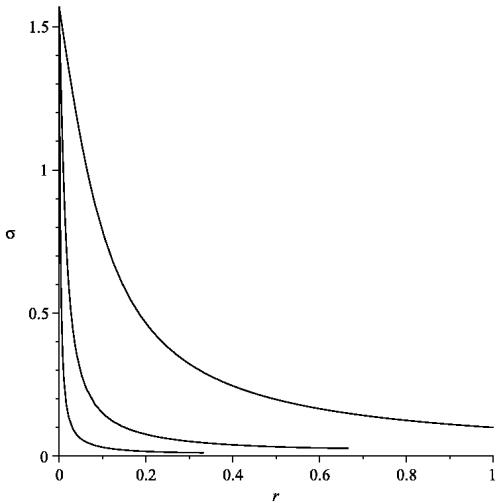


Figure 4.2: The  $\dot{\phi} \geq 1$  solution plotted at three different values of  $\dot{\phi}$  and three different values of  $\kappa$ . We should in fact always have  $\kappa$  small enough that the spike ( $r > 0$ ) is located at very small  $\sigma$ , but here we relax this condition so as to see the qualitative behaviour of the solution. The values are  $\dot{\phi} = 1$  and  $\kappa = 0.1$ ,  $\dot{\phi} = 1.5$  and  $\kappa = 0.01$ ,  $\dot{\phi} = 3$  and  $\kappa = 0.001$ .

ulus transform of one another<sup>5</sup>. While these are only implicit formulae for  $r(\sigma)$  we can rearrange for  $\sigma(r)$  to view their behaviour, even if inverted about  $r = \sigma$ . Figures 4.2 and 4.3 plot  $\sigma(r)$  for each solution. Remembering our approximations we should keep  $\kappa$  small, for which the solutions are very spiky, and we must only believe in  $\sin^2(\sigma) \ll \kappa$ , although as  $\kappa \rightarrow 0$  the whole of the spike fits into this region, with  $r' \rightarrow 0$  for all other values of  $\sigma$ .

For the  $\dot{\phi} \geq 1$  solution, with the lower limit of  $r$ ,  $\hat{r} \approx 0$ , we obtain a string that has  $0 \leq r \leq \frac{1}{\dot{\phi}}$  as expected; as we increase the angular velocity  $\dot{\phi}$  the end/cusp point retreats from the equator of the sphere at  $r = 1$ . Note that from equation (4.55) the spike has infinite gradient (in the world volume coordinate  $\sigma$ ) at its end/cusp point. This is for any  $\kappa$ , in contrast to the  $\dot{\phi} = 1$  solution for which this is true only at  $\kappa \rightarrow 0$ . It should be remembered though that the solution took small  $\kappa$  to begin with so that this change in behaviour between  $\dot{\phi} = 1$  and  $\dot{\phi} > 1$  is not visible in the

<sup>5</sup>See Appendix A for a review of the properties of elliptic integrals and functions including the inverse modulus transform.

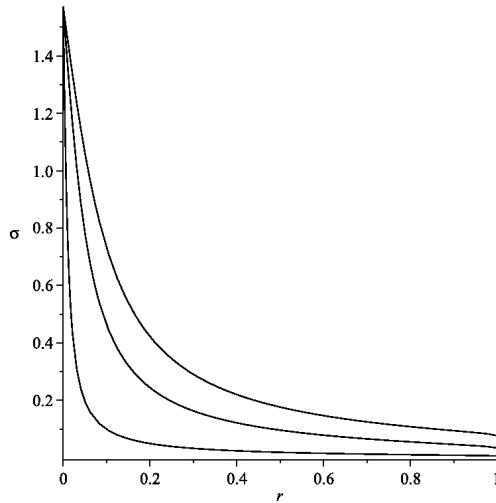


Figure 4.3: The  $0 \leq \dot{\phi} \leq 1$  solution plotted at three different values of  $\dot{\phi}$  at  $\kappa = 0.1$ . Again  $\kappa$  should be taken small enough that all of  $r > 0$  takes place at very small  $\sigma$ . For any  $\dot{\phi}$  in the range the solution always obtains  $r = 1$ , the equator of  $S^2$ , where the radius of the brane  $R\phi$  shrinks to zero size. Note that here  $\kappa$  is held *fixed*; as we decrease  $\dot{\phi}$  toward the stretched string at  $\dot{\phi} = 0$  the solution becomes spike-like *anyway*.

limit.

For the  $0 \leq \dot{\phi} \leq 1$  solution we obtain a string that has  $0 \leq r \leq 1$  as expected, always obtaining the point  $r = 1$  where the radius of the brane  $R\rho$  shrinks to zero size.

When we picture the  $S^2 \subset S^5$  on which we will find our strings appearing we remember that there is an extra  $S^3$  of varying size at each point. The giant graviton, which is deformed from the shape of an  $S^3$  but retains the same topology, intersects the this  $S^2$  along a line which when drawn on the  $S^2$  traces the profile of the string solutions of the previous chapter. At  $\kappa > 0$  ( $g_s > 0$ ) the string can therefore be thought of as having a thickness which expands into the remaining directions of  $S^5$ .

Note that in both cases the points at or closest to the equator at  $r = 1$  are at the poles of  $S^3$  on the world volume of the brane and would therefore seem to be the opposite end points of the stringy extensions of the brane. This is in apparent contrast to the string solutions derived above where these points are in the middle of the string, the end points as determined by the ends of the ranges of the spatial

world sheet coordinate being at the ‘north’ and ‘south’ poles of the  $S^2$ . However, the  $\dot{\phi} \geq 1$  string satisfies a free end point condition at its cusp point so that it can equally well be thought of as two coincident strings with one Dirichlet condition each at  $r = 0$  and one Neumann condition each at the points closest to the equator.<sup>6</sup>

In the  $0 \leq \dot{\phi} \leq 1$  case the string could equally well be thought of as two separate strings satisfying both Dirichlet conditions at  $r = 0$  and Dirichlet conditions at  $r = 1$  in the vertical direction. This would then appear to describe a string (pair) connecting a  $Z = 0$  giant graviton to a  $Y = 0$  giant graviton that meets the  $S^2$  upon which the string moves only at the equator,  $r = 1$ .

Given that our approximation allows us to consider almost all values of  $r$  with  $0 < r \leq 1$  we may think about the constant  $c$  once again. Clearly, if  $c = 0$  we have the finite  $J$  version of the boundary giant magnons attached to a maximal, BPS  $Z = 0$  giant graviton. However, for  $c \neq 0$  we would appear to be describing finite  $J$  boundary giant magnons attached to  $Z = 0$  giant gravitons that are non-maximal and *non-BPS* as the giant graviton itself must now orbit with  $\dot{\phi} \neq 1$ . From [10] we get the bound (4.37) which is saturated by BPS giants (for which  $\Delta - J = 0$ ) when  $\dot{\phi} = 1$  only.

A simple form of the electric field  $\tilde{F}$  can be given at this level of approximation<sup>7</sup>. Expanding (4.22) to terms of order 1 in  $s^4\kappa^{-2}$ , which is just the leading order  $\propto r'$ , and using (4.55) we get

$$\tilde{F} = \Lambda r' + O\left(\frac{s^2}{\kappa}\right) \quad (4.58)$$

$$\approx \frac{\kappa}{s^2} \dot{\phi} \quad (4.59)$$

This is the  $\dot{\phi} = 1$  result multiplied by  $\dot{\phi}$ . Applying the divergence operator (4.42)

---

<sup>6</sup>This is similar to the case of the  $J \rightarrow \infty$  giant magnon that must technically be thought of as part of a closed string but on its own passes for an isolated open string.

<sup>7</sup>i.e. the same level at which we return the classical string energy and angular momentum.

we again satisfy Gauss's law,

$$\frac{1}{s^2} \frac{d}{d\sigma} \left( s^2 \cdot \frac{\kappa \dot{\phi}}{s^2} \right) = 0. \quad (4.60)$$

We can work out the magnitude of each of the point charges located at the poles as in (4.44) once again to find

$$Q = 4\pi\kappa\dot{\phi}. \quad (4.61)$$

We have an extra factor of  $\dot{\phi}$  compared with the  $\dot{\phi} = 1$  case. So changing  $\dot{\phi}$  changes the magnitude of the electric charges that are seen by the Maxwell fields on the world volume.

Turning to the angular momentum of the brane once again, the  $\kappa \rightarrow 0$  limit squashes the spike into an infinitesimally small region close to  $\sigma = 0$  outside of which the brane tends to a constant radius of  $R\rho = R$  (i.e.  $\rho = 1$ ). If we call  $\sigma \in [\tilde{\sigma}, \frac{\pi}{2}]$  the region in which the brane has a constant radius to a good degree of approximation, and will hence satisfy  $\Delta_B - J_B \approx 0$ , then we may also want to worry how the brane contribution  $\Delta_B - J_B$  behaves in the region  $\sigma \in [\sigma_0, \hat{\sigma}]$  over which our solution is valid. In particular we would like to see that it is also zero so that we maintain the picture of a Nambu string attached to a BPS giant graviton in the  $\kappa \rightarrow 0$  limit. To this end we argue as follows.

Substituting for  $\tilde{F}$  in  $\sqrt{D}$  and examining the leading approximation,

$$\begin{aligned} \sqrt{D} &= s^2 \rho^2 \Lambda \sqrt{\frac{r'^2 + \rho^4}{s^4 \rho^4 + \kappa^2}} \\ &\approx s^2 \rho^2 \Lambda \cdot \frac{r'}{\kappa} \\ &\approx \dot{\phi} \rho^2. \end{aligned} \quad (4.62)$$

If we substitute this now into our expression for the brane angular momentum density in the region  $\sigma \in [\sigma_0, \hat{\sigma}]$  we find

$$\begin{aligned} J_B &= T \int_{\sigma_0}^{\hat{\sigma}} s^2 \left[ \frac{r^2 \dot{\phi} \rho^4}{\sqrt{D}} + \rho^4 \right] d\sigma \\ &\approx T \int_{\sigma_0}^{\hat{\sigma}} s^2 \rho^2 d\sigma. \end{aligned} \quad (4.63)$$

while the energy becomes

$$\Delta_B \approx \frac{T}{\dot{\phi}} \int_{\sigma_0}^{\hat{\sigma}} s^2 \rho^2 d\sigma \quad (4.64)$$

so that together with  $0 \leq \rho^2 \leq 1$  we find

$$\begin{aligned} |\Delta_B - J_B| &\leq T \left| \frac{1}{\dot{\phi}} - 1 \right| \int_{\sigma_0}^{\hat{\sigma}} s^2 d\sigma \\ &< T \left| \frac{1}{\dot{\phi}} - 1 \right| \frac{\hat{\sigma}^3}{3}. \end{aligned} \quad (4.65)$$

Now,  $\hat{\sigma}$  satisfied  $\sin^4(\hat{\sigma})\rho^4 \ll \kappa^2$  which will be true for any  $\rho$  if  $\sin^4(\hat{\sigma}) \ll \kappa^2$ , where  $\kappa = \frac{\sqrt{\lambda}}{4N}$ . With  $\sin(\hat{\sigma}) \ll 1$  we have that  $\hat{\sigma}^3 \ll \kappa^{\frac{3}{2}}$  so that with  $T = \frac{2}{\pi}N$  we have the bound

$$|\Delta_B - J_B| \ll \frac{2}{3\pi} \left| \frac{1}{\dot{\phi}} - 1 \right| \frac{\lambda^{\frac{3}{4}}}{4^{\frac{3}{2}}} \cdot \frac{1}{N^{\frac{1}{2}}}. \quad (4.66)$$

If we take  $N \rightarrow \infty$ , i.e.  $\kappa \rightarrow 0$ , with  $\lambda$  fixed (and  $\dot{\phi}$  strictly non-zero) then  $|\Delta_B - J_B| \rightarrow 0$  as would be expected of a BPS giant.

## 4.4 Conclusion

In this chapter we have examined the world volume theory of the giant graviton itself in order to re-discover our boundary strings of chapter 3 as BIon spike solutions of Born Infeld theory coupled to the background 4-form potential. Again, the known case of  $\dot{\phi} = 1$  was recovered and with an appropriate pair of approximations that focussed on the rapidly varying part of the brane in the limit of small string coupling we found solutions that reproduced the behaviour of both sets of string solutions and at leading order were sure to possess the same angular momenta and energies as the classical string. It also followed from the solutions that in the zero string coupling limit, or large  $N$  't Hooft limit, the total brane configuration appeared to become simply the sum of a boundary giant magnon and a BPS giant graviton.

We discussed the generalisation of the brane solutions that have non-zero constant  $c$  at both  $\dot{\phi} = 1$  and  $\dot{\phi} \neq 1$ . For small  $c$  at least, where the brane description of the giants is valid,  $\dot{\phi} = 1$  describes boundary giant magnons attached to a non-maximal, BPS,  $Z = 0$  giant graviton, while for  $\dot{\phi} \neq 1$  we have finite  $J$  boundary giant magnons attached to a non-maximal, *non*-BPS,  $Z = 0$  giant graviton.

# Chapter 5

## Giant magnons and non-maximal giant gravitons

The aim of this chapter is to describe the scattering of giant magnons with giant gravitons. We have seen in chapters 1 and 2 that a giant magnon consists of a highly excited ( $J \rightarrow \infty$ ) closed string corresponding to a (traced) spin chain operator of  $\mathcal{N} = 4$  SYM with a single impurity operator. We discussed the notion of an excitation representing this impurity propagating on the world sheet of the closed string. In chapters 2 and 4 we discussed giant gravitons which are stable, finite energy D3-branes on  $\mathbb{R} \times S^5$  to which we may attach strings with giant magnon characteristics.

We now open up the string with a magnon excitation on it, allowing it to encounter one of the string end points. The giant magnon was a solitonic solution of the string sigma model equations (1.89) and we will likewise require soliton solutions. In section 5.2.1 we take advantage of an existing N-soliton solution to these equations [98] (in fact for the  $O(4)$  sigma model or  $SU(2)$  principal chiral model) allowing us in section 5.2.2 to apply the method of images for the string to obtain new open string solutions. We establish a dictionary between the parameters appearing on the sine-Gordon side and on the string side that will allow us to calculate the magnon boundary scattering phase [78, 95].

Section 5.3 then details the integrable scattering of giant magnons with various giant gravitons.

In section 1.3.2 we reproduced the argument that there exists a one to one map between the classical solutions of a class of generalised sine-Gordon theories and those of the  $O(n)$  sigma model, describing a string on  $\mathbb{R} \times S^{n-1}$  in the conformal gauge. In order to discuss the one-to-one map between string solutions on  $\mathbb{R} \times S^2$  and those of boundary sine-Gordon theory we review its construction and solutions, paying heed to the asymptotic behaviour that will later be used to match the two sets of solutions.

Throughout this chapter it is sometimes necessary to abbreviate  $\sin \frac{p}{2}$  to  $s$  (and similarly for other trigonometric expressions). Where possible, particularly in results, we have left the full trigonometric expression in place for clarity but during workings we will often slip into the abbreviated notation.

## 5.1 Boundary sine-Gordon theory

As well as providing a brief review of the relevant aspects of boundary sine-Gordon theory this section develops an intuition for the scattering of solitons on the world sheet that will be used to describe the scattering of giant magnons with giant gravitons.

Sine-Gordon theory for a single scalar field  $\varphi(x, t) \in \mathbb{R}$  may be defined on the half-line [87] while preserving its integrability, i.e. conserving an infinite number of constants of motion and  $N$ -soliton solutions. The Lagrangian is modified by the inclusion of a boundary potential [87, 95], conventionally evaluated at  $x = 0$ , specifically

$$\mathcal{L} = \frac{1}{2} \{ \dot{\varphi}^2 - \varphi'^2 + g \cos(\beta \varphi) \} + M \cos \left( \frac{\beta(\varphi - \varphi_0)}{2} \right) \delta(x), \quad (5.1)$$

and then taking either  $x \in (-\infty, 0]$  or  $x \in [0, \infty)$ . Rescaling the field and coupling constant, and varying the action, then without loss of generality we may obtain the equation of motion

$$\ddot{\varphi} - \varphi'' = -\sin(\varphi) \quad (5.2)$$

plus the boundary condition

$$\varphi'|_{x=0} = M \sin \left( \frac{\varphi - \varphi_0}{2} \right) \Big|_{x=0}. \quad (5.3)$$

Solutions to the boundary theory may then be constructed by the method of images, whereby a three soliton solution to the bulk theory suffices to describe the in-going soliton, the out-going (reflected) soliton and a non-trivial boundary. We will follow the notation of [87].

The  $N$ -soliton solution to the bulk theory is expressed through the  $\tau$ -function as

$$\varphi(x, t) = 4 \arctan \left( \frac{\mathcal{I}(\tau)}{\mathcal{R}(\tau)} \right), \quad \tau \equiv \tau(x, t) \in \mathbb{C} \quad (5.4)$$

where the  $\tau$ -function itself is

$$\begin{aligned} \tau = \sum_{\mu_j=0,1} e^{-\frac{\pi}{2}(\sum_{j=1}^N \epsilon_j \mu_j)} \exp \left\{ - \sum_{j=1}^N \mu_j [\cosh(\theta_j)x + \sinh(\theta_j)t + 2a_j] \right. \\ \left. + 2 \sum_{1 \leq i < j \leq N} \mu_i \mu_j \ln \left( \tanh \left( \frac{\theta_i - \theta_j}{2} \right) \right) \right\}. \end{aligned} \quad (5.5)$$

This describes a mix of  $N$  kinks and / or anti-kinks, which are topologically stable solutions interpolating between the vacua of the theory at  $\varphi = 2m\pi$ ,  $m \in \mathbb{Z}$ .

The rapidity of the  $j$ th soliton at early and late times is  $\theta_j$ , the  $a_j$  are initial positions and the  $\epsilon_j = \pm 1$  determine whether the  $j$ th soliton is a kink or an anti-kink.

To perform the method of images we must take two solitons to have equal and opposite velocities and one to be stationary, therefore we take

$$\theta_1 \equiv \theta, \quad \theta_3 = -\theta, \quad \theta_2 = 0 \quad (5.6)$$

and with the definitions

$$\epsilon \equiv \epsilon_1 \epsilon_3, \quad \epsilon_0 \equiv \epsilon_2, \quad a_+ = a_1 + a_3, \quad a_- = a_1 - a_3, \quad b \equiv a_2 \quad (5.7)$$

$$v = \tanh(\theta), \quad \gamma = \cosh(\theta) \quad (5.8)$$

we construct the  $\tau$  function appropriate to the method of images as

$$\begin{aligned} \tau = 1 - \epsilon v^2 e^{-2\gamma x - a_+} - \epsilon_0 \kappa^2 e^{-(\gamma+1)x - b} F(t) \\ + i \left\{ e^{-\gamma x} F(t) + \epsilon_0 e^{-x - b} - \epsilon \epsilon_0 v^2 \kappa^4 e^{-(2\gamma+1)x - a_+ - b} \right\} \end{aligned} \quad (5.9)$$

where we have defined

$$\kappa \equiv \tanh\left(\frac{\theta}{2}\right). \quad (5.10)$$

The time dependence is confined to the factor  $F(t)$ ,

$$F(t) = \epsilon_1 e^{-v\gamma t - a_1} + \epsilon_3 e^{v\gamma t - a_3} \quad (5.11)$$

$$= \epsilon_1 e^{-\frac{1}{2}a_+} \left( e^{-v\gamma t - \frac{1}{2}a_-} + \epsilon e^{v\gamma t + \frac{1}{2}a_-} \right) \quad (5.12)$$

where the second form shows us that of the 3 parametric degrees of freedom entering via the initial positions  $a_j$  we may remove one by an overall translation in time of  $t \mapsto t - \frac{1}{2v\gamma}a_-$ .

The integrable boundary condition is determined by the two real parameters  $M$  and  $\varphi_0$ , which may be translated into the two parameters inherited as initial positions,  $a_+$  and  $b$ . The three discrete parameters  $\epsilon_1$ ,  $\epsilon$  and  $\epsilon_0$  are fixed by the choice of boundary conditions at both  $x = 0$  and  $x \rightarrow \pm\infty$ .

The asymptotic behaviour of the solution on  $x \in [0, \infty)$  consists of an in-going left-mover at  $t \rightarrow -\infty$  and an out-going right-mover at  $t \rightarrow +\infty$  while the elastic boundary returns to its initial state. To follow the left-mover we may take  $x \rightarrow +\infty$ ,  $t \rightarrow -\infty$  with  $\zeta_+ = \gamma(x + vt)$  fixed, leading to

$$\tan\left(\frac{\varphi}{4}\right) \rightarrow \epsilon_2 e^{-\zeta_+ - a_3}, \quad \text{or} \quad \sin\left(\frac{\varphi}{2}\right) \rightarrow \frac{\epsilon_2}{\cosh(\zeta_+ + a_3)} \quad (5.13)$$

and to follow the right-mover take  $x \rightarrow +\infty$ ,  $t \rightarrow +\infty$  with  $\zeta_- = \gamma(x - vt)$  fixed leading to

$$\tan\left(\frac{\varphi}{4}\right) \rightarrow \epsilon_1 e^{-\zeta_- - a_1}, \quad \text{or} \quad \sin\left(\frac{\varphi}{2}\right) \rightarrow \frac{\epsilon_1}{\cosh(\zeta_- + a_1)}. \quad (5.14)$$

We recognise these as single kink or anti-kink solutions proceeding left and right respectively, with initial positions  $a_3$  and  $a_1$ . Meanwhile the asymptotic behaviour of the central soliton is found at  $t \rightarrow \pm\infty$  by taking  $x$  fixed, giving

$$\tan\left(\frac{\varphi}{4}\right) \rightarrow -\epsilon_0 e^{x+b-2\ln(\kappa)}, \quad \text{or} \quad \sin\left(\frac{\varphi}{2}\right) \rightarrow \frac{-\epsilon_0}{\cosh(x+b-2\ln(\kappa))}. \quad (5.15)$$

Note that the position of the central soliton is given not just by the parameter  $b$  but depends upon the speed of the incoming soliton as  $x_0 = 2\ln(\kappa) - b$ . The parameters  $b$  and  $\varphi_0$  are related by

$$b = \ln \left[ -\epsilon_0 \kappa^2 \tan\left(\frac{\varphi_0}{4}\right) \right]. \quad (5.16)$$

We see that  $\epsilon_0 = \pm 1$  must be picked such that  $-\epsilon_0 \tan\left(\frac{\varphi_0}{4}\right)$  is positive and so is fixed by the boundary condition at  $x = 0$ .

By comparing the times at which we expect the soliton centre to be at  $x = 0$ , as extrapolated from its asymptotic in-going and out-going trajectories, we can deduce that the reflected soliton suffers a time shift (or phase shift) of  $\Delta(v\gamma t) = a_+$ . This is depicted in Figure 5.1. The form of  $a_+$  as a function of the soliton's velocity and the two boundary parameters can be calculated by substitution of the solution (5.9) into the boundary condition (5.3). In terms of the parameterisation introduced by the authors of [87],

$$M \cos\left(\frac{\varphi_0}{2}\right) = 2 \cosh(\zeta) \cos(\eta), \quad M \sin\left(\frac{\varphi_0}{2}\right) = 2 \sinh(\zeta) \sin(\eta) \quad (5.17)$$

where  $0 \leq \zeta < \infty$  and  $-\pi < \eta \leq \pi$  the phase shift can be written as

$$a_+ = \ln \left\{ -\epsilon \kappa^2 v^2 \left[ \frac{(\kappa^2 + \tan^2(\frac{\eta}{2})) (1 - \kappa^2 \tanh^2(\frac{\zeta}{2}))}{(1 + \kappa^2 \tan^2(\frac{\eta}{2})) (\kappa^2 - \tanh^2(\frac{\zeta}{2}))} \right]^{\pm 1} \right\}. \quad (5.18)$$

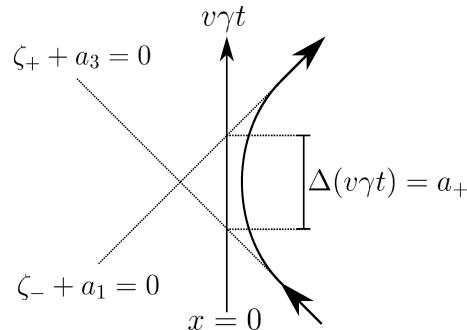


Figure 5.1: The time shift due to boundary scattering via the method of images.

Given  $a_+ \in \mathbb{R}$  then we see that the argument of the log is positive as long as the combination  $-\epsilon (\kappa^2 - \tanh^2(\frac{\zeta}{2}))$  is positive, according to which we must pick  $\epsilon = \pm 1$ . In fact, for a given selection of boundary parameters  $\{\zeta, \eta\}$  this leads to two distinct regimes of behaviour that depend upon the speed of the reflecting soliton. Remembering that  $\epsilon \equiv \epsilon_1 \epsilon_3$  then the two regimes comprise Dirichlet-like solutions where solitons reflect as solitons and Neumann-like solutions where solitons reflect as anti-solitons:

$$\begin{aligned} |\theta| < \zeta &\Rightarrow \epsilon = +1, \quad \text{Dirichlet regime} \\ |\theta| > \zeta &\Rightarrow \epsilon = -1, \quad \text{Neumann regime.} \end{aligned}$$

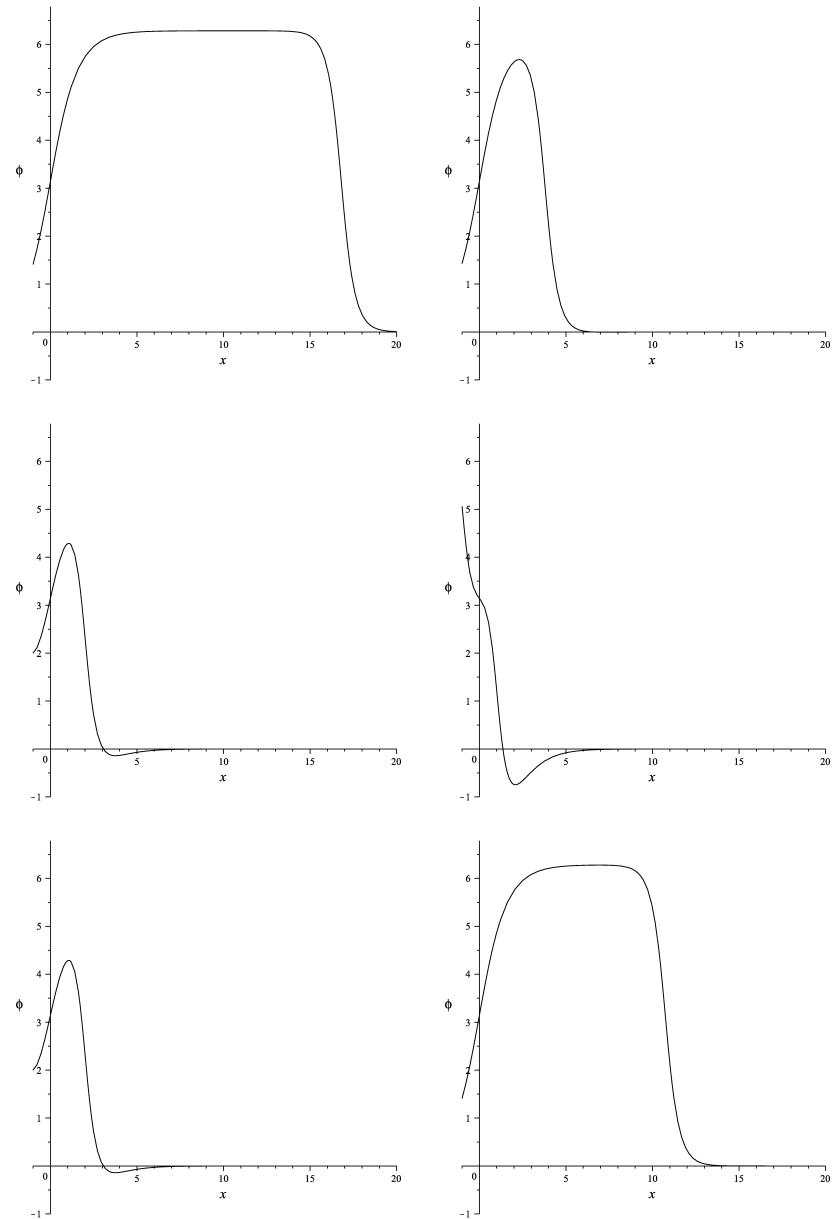


Figure 5.2: An example of sine-Gordon boundary / soliton scattering using the method of images. Time flows from upper left to lower right as you would read a page. In this instance a kink scatters with Dirichlet boundary conditions  $\varphi_0 = \pi$  at  $x = 0$  and  $\varphi \rightarrow 0$  as  $x \rightarrow \infty$ .

At the exact point between these two regimes, for  $|\theta| = \zeta$ , the phase delay diverges with the interpretation that the incoming soliton has been absorbed by the boundary.

When  $M$ , or equivalently  $\zeta$ , is large then a greater range of  $\theta$  falls into the Dirichlet regime so that the true Dirichlet condition is  $M \rightarrow \infty$ , which by (5.3) appropriately requires  $\varphi|_{x=0} = \varphi_0$  if  $\varphi'|_{x=0}$  is to be finite. For the Dirichlet condition  $a_+$  is given by

$$a_+^{\text{Dirichlet}} = \ln \left\{ (1-s)^2 \left[ \frac{1-s+(1+s)\tan^2\left(\frac{\varphi_0}{4}\right)}{1+s+(1-s)\tan^2\left(\frac{\varphi_0}{4}\right)} \right]^{\pm 1} \right\}, \quad s = \frac{1}{\gamma}. \quad (5.19)$$

$M = 0$ , or equivalently  $\zeta = 0$  and  $\eta = \frac{\pi}{2}$ , gives  $\varphi'|_{x=0} = 0$  and hence the true Neumann condition.

The choice in sign of the power in (5.18) reflects the choice in solutions we obtain for  $a_+$ , having solved a quadratic equation. The method of images in fact produces two valid integrable boundary solutions at the same time, one for  $x \in (-\infty, 0]$  and one for  $x \in [0, \infty)$ . We may generally swap one for the other by an appropriate combination of operations  $x \rightarrow -x$ ,  $\varphi \rightarrow -\varphi$  and  $\varphi \rightarrow \varphi \pm 2\pi$ .

Figure 5.2 plots an example of boundary / soliton scattering.

## 5.2 Open string on $\mathbb{R} \times S^2$

We will consider open strings obtained by carefully taking half of the  $N = 3$  magnon solution presented in [98] in such a way that the motions of the string end point thus produced correspond to the integrable boundary of sine-Gordon theory, with the giant magnon reflecting off this end point being the reflecting sine-Gordon soliton. We will need to include in the string solution the complex moduli discussed by the authors of [98] but not explicitly considered.

The string solutions we describe live on an  $S^2$  subspace of the  $S^5$  of  $AdS_5 \times S^5$  so that we will not be concerned with the spatial coordinates of the  $AdS_5$  part of the space; these are set to zero. The target space time coordinate is provided by the global time coordinate of  $AdS_5$  so that the strings move on  $\mathbb{R}_t \times S^2$ . We set the common radius  $R$  of  $AdS_5$  and  $S^5$  to be  $R = 1$ . As is often the case in the description of the giant magnon solutions we use the conformal and static partial

gauges so that the world sheet time variable is aligned with the global time variable and the range of the world sheet spatial variable  $x$  is infinite or semi-infinite due to the divergent energy of the string solutions.

Note that because the energy density of the string is constant in the conformal gauge and we take the solution supported on  $x \in [0, \infty)$  we keep the length of the string in this gauge fixed and hence the energy of the open string thus defined will be constant.

In section 5.2.1 we will include and discuss the properties of the moduli of the bulk ( $x \in (-\infty, \infty)$ ) solution restricted to  $\mathbb{R} \times S^2$ . In section 5.2.2 we apply the method of images and analyse the asymptotics of the solution in order to find the dictionary between the open string and boundary sine-Gordon solutions that is essential to this work.

### 5.2.1 N-magnon string solution on $\mathbb{R} \times S^2$ with moduli

The solution presented by Kalousios, Papathanasiou and Volovich [98] uses the dressing method [96, 97] to provide an  $N$ -magnon string solution that lives on  $\mathbb{R} \times S^3$ , determined by  $N$  complex parameters  $\lambda_i$ . If the coordinates on  $S^3$  are  $\{X_3, X_4, X_5, X_6\}$  then for complexified coordinates  $Z_1 = X_5 + iX_6$ ,  $Z_2 = X_3 + iX_4$  it is

$$Z_1 = \frac{e^{it}}{\prod_{l=1}^N |\lambda_l|} \frac{N_1}{D}, \quad (5.20)$$

$$Z_2 = \frac{-ie^{-it}}{\prod_{l=1}^N |\lambda_l|} \frac{N_2}{D} \quad (5.21)$$

with the indices  $i, j$  running over the ordered values  $1, \bar{1}, 2, \bar{2}, \dots, N, \bar{N}$ ,

$$D = \sum_{\mu=0,1} \left( \prod_{i < j}^{2N} \lambda_{ij} [\mu_i \mu_j + (\mu_i - 1)(\mu_j - 1)] \right) \left( \prod_{i=1}^{2N} \exp [\mu_i (2i\mathcal{Z}_i)] \right), \quad (5.22)$$

$$N_1 = \sum_{\mu=0,1} \left( \prod_{i < j}^{2N} \lambda_{ij} [\mu_i \mu_j + (\mu_i - 1)(\mu_j - 1)] \right) \left( \prod_{i=1}^{2N} [\mu_i \lambda_i] \exp [\mu_i (2i\mathcal{Z}_i)] \right), \quad (5.23)$$

$$N_2 = \sum_{\mu=0,1} \left( \prod_{i < j}^{2N} \lambda_{ij} [\mu_i \mu_j + (\mu_i - 1)(\mu_j - 1)] \right) \left( \prod_{i=1}^{2N} [-(\mu_i - 1) \lambda_i] \exp [\mu_i (2i\mathcal{Z}_i)] \right) \quad (5.24)$$

where  $\lambda_{ij} \equiv \lambda_i - \lambda_j$ . The world sheet coordinates enter through

$$\mathcal{Z}_i = \frac{z}{\lambda_i - 1} + \frac{\bar{z}}{\lambda_i + 1} \quad (5.25)$$

in which  $z = \frac{1}{2}(x - t)$  and  $\bar{z} = \frac{1}{2}(x + t)$  are the light-cone coordinates and the sums over  $\mu_i = 0, 1$  are subject to the condition that

$$\sum_{i=1}^{2N} \mu_i = \begin{cases} N, & \text{for } N_1, D \\ N + 1, & \text{for } N_2. \end{cases} \quad (5.26)$$

The parameters  $\lambda_k$  can be written  $\lambda_k = r_k e^{i \frac{p_k}{2}}$  where  $p_k$  is the magnon momentum. If we take  $|r_k| = 1$  then the solution reduces to the solution on  $\mathbb{R} \times S^2$  as  $Z_2$  becomes, up to a factor of  $ie^{4it}$  that cancels, pure real. Writing also

$$v_k = \cos\left(\frac{p_k}{2}\right), \quad \gamma_k = \frac{1}{\sin\left(\frac{p_k}{2}\right)} \quad (5.27)$$

the  $\mathcal{Z}_k$  defined in (5.25) then become

$$\begin{aligned} \mathcal{Z}_k &= \frac{-i\gamma_k}{2}x + \frac{i\gamma_k v_k}{2} + \frac{t}{2} \\ &= \frac{-i}{2}\zeta_k + \frac{t}{2} \end{aligned} \quad (5.28)$$

and then

$$e^{2i\mathcal{Z}_k} = e^{\zeta_k + it}, \quad \zeta_k \equiv \gamma_k(x - v_k t). \quad (5.29)$$

The  $v_k$  are then the velocities of the solitons moving on the world sheet, equivalently the velocities of the sine-Gordon kinks and anti-kinks to which they correspond.

The finite difference between the string energy  $E \equiv \Delta$  and angular momentum  $J$  that characterise the giant magnon solutions is here the sum of the values for  $N$  individual giant magnons,

$$\Delta - J = \frac{\sqrt{\lambda}}{\pi} \sum_{k=1}^N \sin \frac{p_k}{2} \quad (5.30)$$

and the solutions will satisfy both of the Virasoro constraints

$$\dot{X}_i \dot{X}_i + X'_i X'_i = 1, \quad i = 1, 2, 3 \quad (5.31)$$

$$\dot{X}_i X'_i = 0. \quad (5.32)$$

As discussed in [96, 98] the dressing method admits a generalisation of the solution whereby

$$e^{i\mathcal{Z}_k} \rightarrow w e^{i\mathcal{Z}_k}, \quad w \in \mathbb{C} \quad (5.33)$$

introducing an extra  $2N$  real parametric degrees of freedom, two of which may always be absorbed by shifts of  $x$  and  $t$ , thus

$$\mathcal{Z}_k \rightarrow \mathcal{Z}_k + i\ln(w_k). \quad (5.34)$$

This leaves  $2(N - 1)$  parameters to be moduli of the solution. Clearly the single magnon solution has no moduli. A 3-magnon solution would have four moduli, but by breaking spatial translational symmetry we shall be left a fifth. However, we will require that the solution be constrained from  $\mathbb{R} \times S^3$  to  $\mathbb{R} \times S^2$  so that three moduli become discretised leaving us two continuous moduli that will correspond to the two boundary parameters encountered in the sine-Gordon picture and three discrete soliton/anti-soliton parameters.

Including these parameters then as in (5.34) with  $w_k = \exp\left(\frac{\alpha_k}{2} + i\psi\right)$ , we have

$$e^{2i\mathcal{Z}_k} \rightarrow \epsilon_k e^{\zeta_k - \alpha_k + it}, \quad \alpha_k \in \mathbb{R} \quad (5.35)$$

where we used (5.29) and defined

$$\epsilon_k \equiv e^{-2i\psi}, \quad \psi \in \mathbb{R}. \quad (5.36)$$

The solution is then composed, via (5.20) to (5.24), of sums of products of factors (5.35) together with coefficients involving functions of the magnon momenta  $p_k$ . In this raw form, such as is included explicitly in an appendix in [98], the 3-magnon solution is difficult to analyse and so we first reduce it to products of trigonometric and hyperbolic functions. The general 3-magnon solution in such a form is included here in Appendix B. The form will further simplify with our specific application.

If we examine the solution in Appendix B it is clear that the factors  $\epsilon_k$  which appear variously in products of  $\epsilon_1\epsilon_2\epsilon_3$ ,  $\epsilon_i\epsilon_j$  and on their own, must be equal to  $\pm 1$  in

order that  $Z_2$  be real. Three of the six moduli we have introduced therefore become discretised in order that we constrain the solution to the 2-sphere:

$$\epsilon_k = \pm 1; \quad \psi = m \frac{\pi}{2}, \quad m \in \mathbb{Z}. \quad (5.37)$$

This recovers the soliton/anti-soliton parameters we met in the sine-Gordon picture and in fact reiterates the appearance of the topological solitons of sine-Gordon theory from the non-topological solitons of the complex sine-Gordon theory [70] to which the string theory on  $\mathbb{R} \times S^3$  corresponds.

Before we turn to the method of images we pause to comment on the generic behaviour of the  $N = 3$  solutions on  $\mathbb{R} \times S^2$ .

We recall that at early and late times giant magnons form semi-circular arcs with their end points ( $x \rightarrow \pm\infty$ ) on the equator which is defined by the axis about which the string has a divergent angular momentum. The radius of this arc is equal to  $\sin(\frac{p}{2})$  in terms of the world sheet momentum that creates the excitation, or  $\sqrt{1 - v^2}$  in terms of its speed on the world sheet or correspondingly of the sine-Gordon soliton. Fast solitons are therefore small, vanishing to a point on the equator at  $v = 1$ , while a stationary soliton crosses the pole of the sphere as half of a great circle.

For a single giant magnon the physics is invariant under  $Z_2 \rightarrow -Z_2$ , as is the physics of a single sine-Gordon kink or anti-kink. For multi-magnon states though we may choose which hemisphere each magnon lives upon, just as there is a difference between multi-soliton sine-Gordon solutions in the choice between composing kinks with kinks or anti-kinks. While the phase delays of scattering kinks and anti-kinks, and hence the scattering time delays of giant magnons with those on the same or opposite hemispheres, are identical, the solutions in each case produce qualitatively different behaviour.

The solution as presented in [98] effectively has  $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$  and  $\alpha_1 = \alpha_2 = \alpha_3 = 0$ , an example of which is plotted in Figure 5.3 at an early time. We may think that due to all  $\epsilon_k = +1$  then the corresponding sine-Gordon solution is a 3-kink. It is however a kink/anti-kink/kink solution, as ordered at  $t \rightarrow -\infty$ . In fact, when the momenta are ordered such that  $p_1 < p_2 < p_3$  then  $\epsilon_2$  in the string picture is

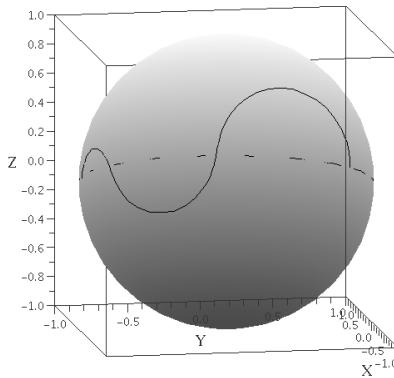


Figure 5.3: A 3-magnon solution at early time. The momenta are  $p_1 = \frac{\pi}{6}$ ,  $p_2 = \frac{\pi}{4}$  and  $p_3 = \frac{\pi}{3}$ . The moduli chosen are  $\epsilon_k = +1$  and  $\alpha_k = 0$ ,  $k = 1, 2, 3$ , and the corresponding sine-Gordon solution is kink/anti-kink/kink.

always the negative of the similar parameter coming from the sine-Gordon picture, and the ‘raw’ 3-magnon solution of [98], when restricted to the 2-sphere by taking all  $\lambda_k$  to have  $|\lambda_k| = 1$ , will always give alternating kink/anti-kink/kink or anti-kink/kink/anti-kink solutions. Had we therefore chosen  $\epsilon_2 = -1$  we would obtain a 3-magnon solution where each giant magnon sits on the same hemisphere and the corresponding sine-Gordon solution is kink/kink/kink. Such a choice is plotted in Figure 5.4.

Each of the inequivalent choices of  $\{\epsilon_k\}$  (those not related by each  $\epsilon_k \rightarrow -\epsilon_k$ , hence the coordinate  $Z_2 \rightarrow -Z_2$ ) display different finite time behaviour as the solitons scatter on the world sheet, but the qualitative difference is between those solutions with sine-Gordon topological charge  $|Q| = 1$  (such as kink/anti-kink/anti-kink) and those with  $|Q| = 3$  (such as kink/kink/kink).

The  $|Q| = 3$  solutions display cusps on the bulk of the string that persist at finite time with the result that the scattering appears as three giant magnons that smoothly grow or shrink such that their orders are reversed as we go from  $t \rightarrow -\infty$  to  $t \rightarrow +\infty$ . On the other hand those solutions with  $|Q| = 1$  display cusps that emerge from the equatorial end points of the string and race through to the opposite end point where they merge, sometimes annihilating with another cusp along the way before reappearing.

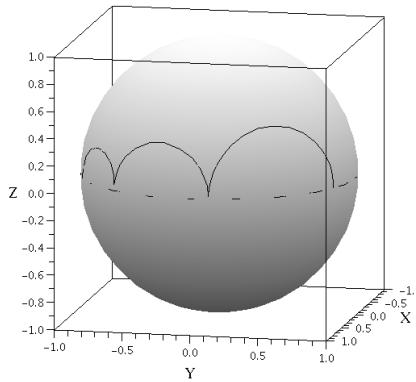


Figure 5.4: Another 3-magnon solution at early time. The momenta are again  $p_1 = \frac{\pi}{6}$ ,  $p_2 = \frac{\pi}{4}$  and  $p_3 = \frac{\pi}{3}$ , but this time the moduli chosen are  $\epsilon_1 = \epsilon_3 = +1$  and  $\epsilon_2 = -1$ .  $\alpha_k = 0$ ,  $k = 1, 2, 3$  again and the corresponding sine-Gordon solution is kink/kink/kink.

This behaviour of the cusps is given in the sine-Gordon theory by the behaviour of points where  $\varphi(x, t) = 2m\pi$ ,  $m \in \mathbb{Z}$ , as is seen from the relation

$$\sin^2 \frac{\varphi}{2} = X'_i X'_i. \quad (5.38)$$

Taking the kink / boundary scattering plotted in Figure 5.2 (Section 5.1) for instance, it may appear at first glance that  $\varphi \rightarrow 0$  from above as  $x \rightarrow \infty$ , especially at early and late times. However,  $\varphi \rightarrow 0$  from *below*  $\forall t$  as  $x \rightarrow \infty$  so that we must always have a point where  $\varphi = 0$ , and hence a cusp,  $X'_i = 0$ .

### 5.2.2 Method of images on $\mathbb{R} \times S^2$

In this subsection we will select the 3-magnon solution required by the method of images and find the string moduli in terms of the sine-Gordon boundary parameters so that we can say exactly which sine-Gordon integrable boundary configuration corresponds to which open string.

We proceed with the method of images then as in Section 5.1 by taking

$$v_1 \equiv v, \quad v_2 = 0, \quad v_3 = -v \quad (5.39)$$

$$\gamma_1 \equiv \gamma, \quad \gamma_2 = 1, \quad \gamma_3 = \gamma \quad (5.40)$$

or in terms of the magnon momenta, using (5.27), and the abbreviations  $s_k \equiv \sin\left(\frac{p_k}{2}\right)$ ,  $c_k \equiv \cos\left(\frac{p_k}{2}\right)$ ,

$$c_1 \equiv c, \quad c_2 = 0, \quad c_3 = -c \quad (5.41)$$

$$s_1 \equiv s, \quad s_2 = 1, \quad s_3 = s. \quad (5.42)$$

Note that  $v \rightarrow -v$  is equivalent to  $p \rightarrow 2\pi - p$ . Mirroring the notation adopted in the previous section we write

$$\alpha_+ \equiv \alpha_1 + \alpha_3, \quad \alpha_- = \alpha_1 - \alpha_3, \quad \alpha_2 \equiv \beta \quad (5.43)$$

and we note that in precisely the manner in which we were able to perform an overall time shift to absorb  $\alpha_-$  from the sine-Gordon solution, so we are able to absorb  $\alpha_-$  so that it does not appear.

Taking the general solution presented in Appendix B and inserting the above we obtain forms of  $D$ ,  $N_1$  and  $N_2$  from which factors of  $32c^2e^{3it}$  will cancel. A further factor of  $ie^{it}$  cancels from  $N_2$ . We produce below these simplified forms  $\tilde{D}$ ,  $\tilde{N}_1$  and  $\tilde{N}_2$  which are such that the ratios

$$\tilde{Z}_1 = \frac{\tilde{N}_1}{\tilde{D}} \quad \text{and} \quad \tilde{Z}_2 = \frac{\tilde{N}_2}{\tilde{D}} \quad (5.44)$$

give the string solution in a co-moving frame. In the solution below upper and lower signs correlate with the upper and lower cases and correspond to taking  $\epsilon = \pm 1$ . Similarly to the previous section we have used

$$\epsilon = \epsilon_1 \epsilon_3, \quad \epsilon_3 = \epsilon \epsilon_1 \quad (5.45)$$

which has been used to eliminate  $\epsilon_3$  in favour of  $\epsilon$ .

$$\begin{aligned} \tilde{D} = 2s^2 \epsilon_0 \cosh(x - \beta) + 8s\epsilon_1 \left\{ \begin{array}{l} -\cosh(v\gamma t) \cosh(\gamma x - \frac{\alpha_+}{2}) \\ +\sinh(v\gamma t) \sinh(\gamma x - \frac{\alpha_+}{2}) \end{array} \right\} \\ \pm \epsilon_0 [(1-s)^2 \cosh((2\gamma + 1)x - \alpha_+ - \beta) + (1+s)^2 \cosh((2\gamma - 1)x - \alpha_+ + \beta) \\ + 2\cosh(2v\gamma t)], \end{aligned} \quad (5.46)$$

$$\begin{aligned}
 \tilde{N}_1 = 8sc\epsilon_1 & \left\{ \begin{array}{l} +\sinh(v\gamma t)\sinh(\gamma x - \frac{\alpha_+}{2}) \\ -\cosh(v\gamma t)\cosh(\gamma x - \frac{\alpha_+}{2}) \end{array} \right\} \mp 4sc\epsilon_0\sinh(x - \beta)\sinh(2v\gamma t) \\
 & +i \left[ 2s^2\epsilon_0\sinh(x - \beta) + 8s^2\epsilon_1 \left\{ \begin{array}{l} -\cosh(v\gamma t)\sinh(\gamma x - \frac{\alpha_+}{2}) \\ +\sinh(v\gamma t)\cosh(\gamma x - \frac{\alpha_+}{2}) \end{array} \right\} \right. \\
 & \quad \left. \pm\epsilon_0 \left[ -(1-s)^2\sinh((2\gamma+1)x - \alpha_+ - \beta) \right. \right. \\
 & \quad \left. \left. +(1+s)^2\sinh((2\gamma-1)x - \alpha_+ + \beta) + 2(1-2c^2)\sinh(x - \beta)\cosh(2v\gamma t) \right] \right], \quad (5.47)
 \end{aligned}$$

$$\begin{aligned}
 \tilde{N}_2 = -6s^2 + 4s(1-s)\epsilon_0\epsilon_1 & \left\{ \begin{array}{l} +\cosh(v\gamma t)\cosh((1+\gamma)x - \beta - \frac{\alpha_+}{2}) \\ -\sinh(v\gamma t)\sinh((1+\gamma)x - \beta - \frac{\alpha_+}{2}) \end{array} \right\} \\
 & +4s(1+s)\epsilon_0\epsilon_1 \left\{ \begin{array}{l} +\cosh(v\gamma t)\cosh((1-\gamma)x - \beta + \frac{\alpha_+}{2}) \\ +\sinh(v\gamma t)\sinh((1-\gamma)x - \beta + \frac{\alpha_+}{2}) \end{array} \right\} \\
 & \mp 2 \left[ c^2\cosh(2\gamma x - \alpha_+) + \cosh(2v\gamma t) \right]. \quad (5.48)
 \end{aligned}$$

Our job is to make identifications between the displacement parameters on both sides which we achieve by examination of the asymptotics. We define

$$\zeta_1 \equiv \zeta_- \equiv \gamma(x - vt), \quad \zeta_3 \equiv \zeta_+ \equiv \gamma(x + vt) \quad (5.49)$$

and using equations (5.44) and (5.46) to (5.48) (with  $v\gamma t \rightarrow v\gamma t + \alpha_-$ ) we examine the asymptotic form of the right-mover with target space coordinates  $(X_-, Y_-, Z_-)$  to find

$$X_- \rightarrow \frac{-4sc \exp(-\zeta_- - \ln(1-s) + \alpha_1)}{\cosh(\zeta_- + \ln(1-s) - \alpha_1)} \quad (5.50)$$

$$Y_- \rightarrow \frac{-2\exp(\zeta_- + \ln(1-s) - \alpha_1) + 2(1-2c^2)\exp(\zeta_- + \ln(1-s) - \alpha_1)}{\cosh(\zeta_- + \ln(1-s) - \alpha_1)} \quad (5.51)$$

$$Z_- \rightarrow \frac{s\epsilon_1}{\cosh(\zeta_- + \ln(1-s) - \alpha_1)}. \quad (5.52)$$

As already proved by the authors of [98] this should amount to the simple solution for a single giant magnon rotated in the  $Z_1$  plane by an angle  $\frac{p}{2}$  from parallel with the  $X_5$ -axis. Performing the rotation thus

$$\begin{pmatrix} \tilde{X}_- \\ \tilde{Y}_- \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} X_- \\ Y_- \end{pmatrix} \quad (5.53)$$

we do indeed obtain the original giant magnon solution in co-moving coordinates,

$$\begin{aligned}\tilde{X}_- + i\tilde{Y}_- &= s \tanh(\zeta_- + \ln(1-s) - \alpha_1) - ic \\ Z_- &= \frac{\epsilon_1 s}{\cosh(\zeta_- + \ln(1-s) - \alpha_1)},\end{aligned}\quad (5.54)$$

together with a phase shift dependent upon  $p$  and, of course,  $\alpha_1$ ; compare (5.54) with (2.23). Dealing similarly with the left-mover with coordinates  $(X_+, Y_+, Z_+)$ , rotating in the opposite direction as

$$\begin{pmatrix} \tilde{X}_+ \\ \tilde{Y}_+ \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} X_+ \\ Y_+ \end{pmatrix} \quad (5.55)$$

we obtain

$$\begin{aligned}\tilde{X}_+ + i\tilde{Y}_+ &= -s \tanh(\zeta_+ + \ln(1-s) - \alpha_3) - ic \\ Z_+ &= \frac{\epsilon_3 s}{\cosh(\zeta_+ + \ln(1-s) - \alpha_3)},\end{aligned}\quad (5.56)$$

showing that the early and late time configurations are reflections of one another in the  $Y$ -axis of the co-moving frame, together with a potential mirroring in  $Z_2$  if  $\epsilon_3 = -\epsilon_1$ .

Taking  $t \rightarrow \pm\infty$  with  $x$  fixed we find the central soliton,

$$t \rightarrow -\infty, \quad (s \tanh(x - \beta), -c \tanh(x - \beta), -\epsilon_0 \operatorname{sech}(x - \beta)) \quad (5.57)$$

$$t \rightarrow +\infty, \quad (-s \tanh(x - \beta), -c \tanh(x - \beta), -\epsilon_0 \operatorname{sech}(x - \beta)). \quad (5.58)$$

The corresponding asymptotic sine-Gordon solutions are easily found once we ‘spin up’ these solutions by applying  $\tilde{Z}_1 \rightarrow e^{it}\tilde{Z}_1 = Z_1$ . Comparing with the asymptotics obtained in section 5.1,

$$\sin\left(\frac{\varphi}{2}\right) = \frac{\epsilon_3}{\cosh(\zeta_+ + a_3)} = \frac{\pm 1}{\cosh(\zeta_+ + \ln(1-s) - \alpha_3)} \quad (5.59)$$

and

$$\sin\left(\frac{\varphi}{2}\right) = \frac{\epsilon_1}{\cosh(\zeta_- + a_1)} = \frac{\pm 1}{\cosh(\zeta_- + \ln(1-s) - \alpha_1)} \quad (5.60)$$

giving

$$a_1 = \ln(1-s) - \alpha_1, \text{ and } a_3 = \ln(1-s) - \alpha_3, \quad (5.61)$$

or

$$a_+ = 2\ln(1 - s) - \alpha_+ \quad (5.62)$$

and for the central soliton

$$\sin\left(\frac{\varphi}{2}\right) = \frac{-\epsilon_0}{\cosh(x + b - 2\ln(\kappa))} = \frac{\pm 1}{\cosh(x - \beta)} \quad (5.63)$$

so

$$b = \ln\left(\frac{1 - s}{1 + s}\right) - \beta. \quad (5.64)$$

From (5.16) then we can relate  $\beta$  to  $\varphi_0$  as

$$\beta = -\ln\left(-\epsilon_0 \tan\left(\frac{\varphi_0}{4}\right)\right). \quad (5.65)$$

Relations (5.62) and (5.64), together with the identifications

$$\epsilon_1^{SG} = \epsilon_1^{\text{String}}, \quad \epsilon_3^{SG} = \epsilon_3^{\text{String}}, \quad -\epsilon_0 = \epsilon_2^{\text{String}}, \quad (5.66)$$

give us the dictionary by which we can match sine-Gordon boundary solutions to corresponding string solutions through the two real moduli that appear in each theory plus boundary conditions at infinity.

### 5.3 Scattering giant magnons off giant gravitons and semi-classical phase shifts

In this section we will consider explicit examples of giant magnons in scattering with giant gravitons, beginning with three maximal cases and then turning to the case of non-maximal ( $Y = 0$ ) giant gravitons. In each case we look for boundary parameters that produce the required string end point motions.

The coordinates of the embedding of  $S^5$  and  $S^2$  into  $\mathbb{R}^6$  shall be given by six real coordinates  $\{X_i\}$ ,  $i = 1, \dots, 6$  or three complex coordinates

$$W = X_1 + iX_2, \quad Y = X_3 + iX_4, \quad Z = X_5 + iX_6 \quad (5.67)$$

always satisfying

$$|W|^2 + |Y|^2 + |Z|^2 = 1. \quad (5.68)$$

The giant gravitons wrap an  $S^3 \subset S^5$  so that their orientations are specified by any two relations  $a_i X_i = b_i X_i = 0$  for arbitrary vectors  $a_i, b_i \in \mathbb{R}^6$ . We follow Hofman and Maldacena's convention [78] of taking the large angular momentum possessed by the strings and branes to be in the  $\{X_5, X_6\}$  direction and hence referring to the physically distinct orientations of the branes as either  $Y = 0$  or  $Z = 0$ .

We shall use the semi-classical Levinson's theorem [59] to calculate the leading contribution to the strong coupling scattering phase  $e^{i\delta}$ , given our knowledge of the time shift,

$$\Delta T_B = \frac{\partial \delta_B}{\partial E}, \quad (5.69)$$

due to integrable boundary scattering of sine-Gordon solitons of the Pohlmeyer reduced string. Collisions with the boundary are elastic, so the speeds of the incoming and outgoing solitons are equal, and so it makes sense to identify the time delay involved in scattering as the sine-Gordon phase delay  $a(p)$  divided by  $v\gamma$ ,

$$\Delta T_B = \frac{a}{v\gamma}, \quad \gamma = \frac{1}{\sqrt{1-v^2}} \quad (5.70)$$

$$= \frac{\sin \frac{p}{2}}{\cos \frac{p}{2}} a(p). \quad (5.71)$$

We then obtain the *magnon* phase shift  $\delta$  by integrating with respect to the *magnon* energy, remembering that, as discussed in [78], while the string and sine-Gordon pictures share their time coordinate exactly the energy of the string and sine-Gordon solitons differ. Hence

$$\delta^{\text{magnon}} = \int \frac{\sin \frac{p}{2}}{\cos \frac{p}{2}} a(p) dE(p)^{\text{magnon}}. \quad (5.72)$$

### 5.3.1 Scattering with maximal $Z = 0$ and $Y = 0$ giant gravitons

In this subsection we consider the moduli necessary to construct the solutions to three previously studied instances [78] of giant magnons scattering with maximal giant gravitons.

#### Maximal Y=0 cases

The scattering of giant magnons with a maximal  $Y = 0$  giant graviton was studied in [78] where it was shown that there are two distinct scenarios dependent upon the state of the magnon, or equivalently the relative orientation of the giant magnon with respect to the brane. As the string rotates in  $Z = X_5 + iX_6$  and lives on an  $S^2$  this relative orientation is decided by the choice of the string's third coordinate. The  $Y = 0$  brane coordinates satisfy

$$X_1^2 + X_2^2 + X_5^2 + X_6^2 = 1 \quad (5.73)$$

so that the intersection of this brane with a maximal radius  $S^2$  upon which the string moves is determined by whether we pick one of the  $Y$  coordinates ( $X_3$  or  $X_4$ ) or one of the  $W$  coordinates ( $X_1$  or  $X_2$ ) as the string's third coordinate. Choosing the string's  $S^2$  to be embedded in  $\{X_3, X_5, X_6\}$  would place it transverse to the world volume of the brane while taking  $\{X_1, X_5, X_6\}$  would place it within the world volume of the brane (i.e. by setting  $X_2 = 0$  on the brane). In both cases there is an obvious  $SO(2)$  symmetry in the choice of the orientation of the giant magnon, or  $U(1)$  symmetry of the magnon state. These two cases are now discussed separately.

The string that moves on  $\{X_3, Z\}$  was argued to be equivalent to a sine-Gordon soliton scattering from a boundary with a Neumann condition. That the string has to meet the brane at  $|Y| = X_3 = 0$  forces the end point to move on the equator of its  $S^2$ , which is the edge of the disc at  $X_3 = 0$  representing the brane. The string's coordinates at the end point on the brane will obey

$$X'_3|_{x=0} \neq 0, \quad Z'|_{x=0} = 0 \quad (5.74)$$

$$\dot{X}_3|_{x=0} = \ddot{X}_3|_{x=0} = 0, \quad \dot{Z}|_{x=0} \neq 0. \quad (5.75)$$

Differentiating the relation for the sine-Gordon field  $\varphi$  from (1.95) we get

$$\begin{aligned} \varphi' \sin \frac{\varphi}{2} \cos \frac{\varphi}{2} \Big|_{x=0} &= 2X'_i X''_i|_{x=0}, \quad i = 3, 5, 6 \\ &= -2X'_i \ddot{X}_i|_{x=0} \end{aligned} \quad (5.76)$$

$$= 0 \quad (5.77)$$

where in the middle step we used the sigma model equations of motion

$$\ddot{X}_i - X''_i + \cos(\varphi) X_i = 0 \quad (5.78)$$

along with the condition  $X_i X'_i = 0$  to avoid evaluating  $X''_i|_{x=0}$ . The conclusion is that  $\varphi'(x, t)|_{x=0} = 0$ ; a Neumann condition.

To construct the corresponding string solution we must find the values of  $\alpha_+$  and  $\beta$  that result from the Neumann sine-Gordon condition. The integrable boundary parameters take the values  $M = 0$  and  $\varphi_0 = 2m\pi$ ,  $m \in \mathbb{Z}$  in this case, or in terms of  $\zeta$  and  $\eta$  (equations (5.17)),  $\zeta = 0$  and  $\eta = \frac{\pi}{2}$ . As discussed in section 2, the Neumann condition requires that the discrete parameter  $\epsilon = -1$ , meaning that a soliton reflects as an anti-soliton. The value of the sine-Gordon phase delay  $a_+$  is then set by (5.18) to be

$$a_+^{\text{Neumann}} = 2\ln \left\{ \cos \frac{p}{2} \right\} \quad (5.79)$$

which means a time delay for boundary scattering  $\Delta T_B$  of

$$\Delta T_B = \frac{\sin \frac{p}{2}}{\cos \frac{p}{2}} a_+ = 2 \frac{\sin \frac{p}{2}}{\cos \frac{p}{2}} \ln \left\{ \cos \frac{p}{2} \right\}. \quad (5.80)$$

Via the dictionary (5.62) we have

$$\alpha_+^{\text{Neumann}} = \ln \left\{ (1 - \sin \frac{p}{2})^2 \cos^2 \frac{p}{2} \right\}. \quad (5.81)$$

The value of  $\beta$ , determined through (5.65) depends upon the boundary conditions at  $x \rightarrow \infty$  (if we are defining the half line to be  $0 \leq x$ ) but will always be divergent in the Neumann case because  $\beta = \beta(\varphi_0)$  controls the position of the central soliton, which for a Neumann boundary condition must be pushed off to infinity. So for example, with  $\varphi(x \rightarrow \infty) = 0$  and  $\epsilon_1 = -1$ , i.e. we send in an anti-kink, then

$$\beta^{\text{Neumann}} \rightarrow +\infty. \quad (5.82)$$

That we dispose of the central soliton this way means that we could treat this solution using only a two soliton solution of the string or sine-Gordon theories, and it was as such that this problem was considered in [78]. The leading contribution to the magnon boundary scattering phase was calculated and found to be

$$\delta_{B, \text{Neumann}}^{\text{Max } Y=0} = -8g \cos \frac{p}{2} \ln \left\{ \cos \frac{p}{2} \right\}. \quad (5.83)$$

The sine-Gordon phase delay on the other hand was given above in (5.79) which if used to calculate a  $\delta_B$  through the prescription (5.72) in fact produces an extra term on top of the Hofman and Maldacena result (5.83). Namely we get

$$\delta_{B, \text{Neumann}}^{\text{Max } Y=0} = -8g\cos\frac{p}{2}\ln\left\{\cos\frac{p}{2}\right\} + \delta' \quad (5.84)$$

where

$$\delta' = 8g\cos\frac{p}{2}. \quad (5.85)$$

The extra term is lacking in [78] as the time delay  $\Delta T_B$  the authors use is modified in order to present their results in the same language as those coming from the Bethe ansatz [26], making clear an ambiguity in the definition of the magnon S-matrix explained in [9]. In [26] the authors use a gauge in which the angular momentum density on the world sheet is a constant while in the conformal gauge here the energy density is a constant. The difference in gauge changes the measure of distance on the world sheet and results in an ambiguity in the time delay due to scattering.

The second maximal  $Y = 0$  case consists of giant magnons contained within the world volume of the brane, on coordinates  $\{X_1, Z\}$ . As such the string end point is free, and by (1.95) we have a Dirichlet sine-Gordon boundary condition with  $\varphi_0 = 2m\pi$ ,  $m \in \mathbb{Z}$ . Similarly to the first maximal  $Y = 0$  case we find that  $\beta \rightarrow \pm\infty$  (depending upon our exact choice of boundary conditions). If we choose to send in a soliton from  $x = \infty$  at  $t \rightarrow -\infty$  then  $\varphi_0 = 2\pi$  gives, by (5.19),

$$a_+^{\text{Dirichlet, } \varphi_0=2\pi} = 2\ln\left\{\cos\frac{p}{2}\right\}, \quad (5.86)$$

so that both maximal  $Y = 0$  cases return the same time delay, (5.80), and hence leading scattering phase, (5.84).

### Maximal $Z = 0$ case

This is the other orientation of a maximal giant graviton studied in [78] where it was argued that the corresponding sine-Gordon picture is that of kinks or anti-kinks in reflection with a boundary with Dirichlet condition  $\varphi_0 = \pi$ . Considering the embedding of the maximal brane we have

$$X_1^2 + X_2^2 + X_3^2 + X_4^2 = 1 \quad (5.87)$$

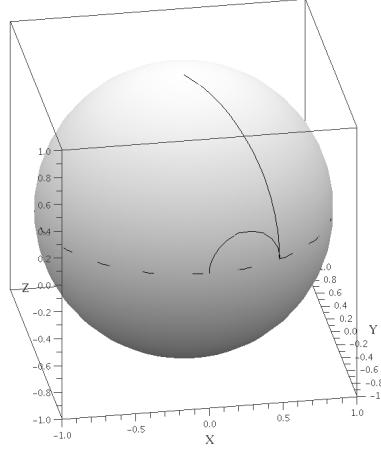


Figure 5.5: A giant magnon attached to a maximal  $Z = 0$  giant graviton by the ‘boundary giant magnon’ degree of freedom.

so that by (5.68) we have  $|Z| = 0$ . As always, the string moves on an  $S^2 \subset S^5$  of maximum radius 1, with two of its three directions being  $Z = X_5 + iX_6$ . We have a complete  $SO(4)$  symmetry under the choice of the third coordinate so taking it to be  $X_3$  the string coordinates must satisfy

$$X_3^2 + |Z|^2 = 1 \quad (5.88)$$

and so in order that the string meet the  $|Z| = 0$  brane we have  $X_3 = \pm 1$ . This fixes the end point of the string to a single point at which  $\dot{X}^2|_{x=0} = 0$ , so that the definition (1.95) tells us that the corresponding sine-Gordon boundary has the expected  $\varphi_0 = \pi$  Dirichlet condition. Figure 5.5 depicts the string at early time.

Next we want to know what values of the moduli / boundary parameters to take in order to meet these boundary conditions. We should expect  $\alpha_+ = \beta = 0$  as this returns the solution without moduli that was considered implicitly in [78]. We know that at  $t \rightarrow \pm\infty$  then  $Z_2|_{x=0} = X_3|_{x=0} = \pm 1$  which from equations (5.57) and (5.58) immediately tells us that  $\beta = 0$ . To convince ourselves that the end point remains fixed to the pole for all time we can demand, for instance, from (5.47) that  $Z_1|_{x=0} = 0$  by taking  $\tilde{N}_1|_{x=0} = 0 \forall t$ . Putting  $\beta = 0$  in (5.47) gives

$$\begin{aligned} \tilde{N}_1|_{x=0}^{\beta=0} &= -8sc\epsilon_1 \sinh(v\gamma t) \sinh\left(\frac{\alpha_+}{2}\right) \\ &+ i [\epsilon_0(1-s)^2 \sinh(\alpha_+) - \epsilon_0(1+s)^2 \sinh(\alpha_+)] . \end{aligned} \quad (5.89)$$

Equation (5.62) relates  $\alpha_+$  to the sine-Gordon phase delay as  $\alpha_+ = 2\ln(1-s) - a_+$  while equation (5.19) for  $a_+^{\text{Dirichlet}}$  with  $\varphi_0 = \pi$  then gives us

$$\alpha_+^{\text{SG-Dirichlet}}|_{\varphi_0=\pi} = 0. \quad (5.90)$$

Equation (5.89) then gives  $\tilde{N}_1 = 0 \forall t$  as required. So indeed  $\alpha_+ = \beta = 0$  for the maximal  $Z = 0$  case.

Figure 5.6 depicts the scattering of a giant magnon off the maximal  $Z = 0$  giant graviton. At early and late times we have a giant magnon attached at one end to a boundary giant magnon [14, 78, 86], an arc of the string that connects the equator to the brane that intersects the sphere at the pole. At finite time an excitation spins the string about the sphere keeping its end point attached to the pole and returning the configuration to the mirror image (in the co-moving frame) of the  $t \rightarrow -\infty$  configuration.

The leading contribution to the magnon boundary scattering phase as calculated in [78] is

$$\delta_B^{\text{Max } Z=0} = -4g\cos\frac{p}{2}\ln\left\{\cos\frac{p}{2}\left(\frac{1-\sin\frac{p}{2}}{1+\sin\frac{p}{2}}\right)\right\}. \quad (5.91)$$

The sine-Gordon phase delay on the other hand is given by (5.19) with  $\varphi_0 = \pi$ , giving

$$a(p)^{\text{Max } Z=0} = \ln\left\{(1-\sin\frac{p}{2})^2\right\} = \ln\left\{\cos^2\frac{p}{2}\left(\frac{1-\sin\frac{p}{2}}{1+\sin\frac{p}{2}}\right)\right\}. \quad (5.92)$$

If we obtain a  $\delta_B$  from this expression using (5.72) we again get extra terms on top of the Hofman and Maldacena result,

$$\delta_B^{\text{Max } Z=0} = \delta_B^{\text{HM}} + \delta' \quad (5.93)$$

where now

$$\delta' = 8g\cos\frac{p}{2} + 8g\left(\frac{p}{2} - \frac{\pi}{2}\right). \quad (5.94)$$

We have the same term  $-8g\cos\frac{p}{2}$  as appeared in the  $Y = 0$  cases along with a constant phase shift of  $-8g\frac{\pi}{2}$  and a term proportional to  $p$ , which is associated with the presence of the central soliton. The term may be understood as follows. Consideration in [9] of the bulk scattering of two magnons with momenta  $p_1$  and  $p_2$  gave a scattering phase

$$\delta(p_1, p_2) = -4g\left(\cos\frac{p_1}{2} - \cos\frac{p_2}{2}\right)\ln\left\{\frac{\sin^2\frac{p_1-p_2}{4}}{\sin^2\frac{p_1+p_2}{4}}\right\} - 4gp_1\sin\frac{p_2}{2} \quad (5.95)$$

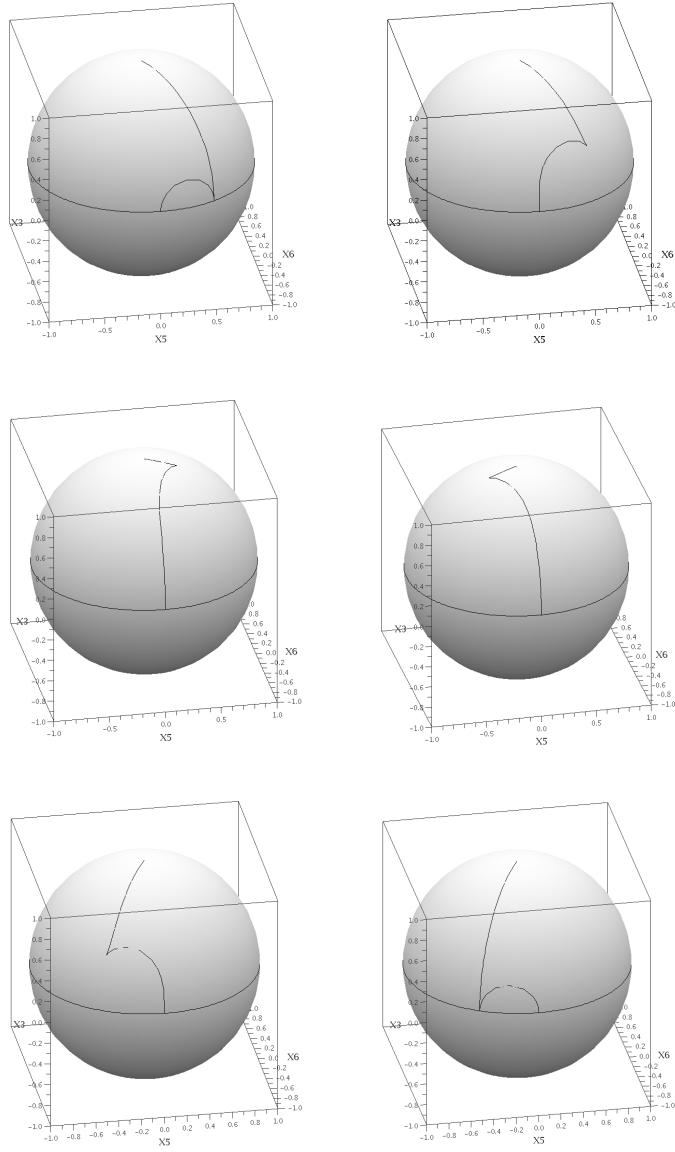


Figure 5.6: This sequence shows the scattering of a giant magnon of momentum  $p = \frac{\pi}{6}$  with a maximal  $Z = 0$  giant graviton in the co-moving frame. The time sequence starts at  $t = -20$  at the top left and proceeds as you would read a page to  $t = +20$  at the lower right. The cusp that sits on the equator at early times swings around the pole at which the brane is located. It returns to the equator at late times in a mirrored configuration.

obtained by integration of the time delay for the scattering of two bulk sine-Gordon solitons along the lines of (5.72). This contains a term proportional to  $p_1$  that does not appear in the large  $\lambda$  limit of the results of [26] and is associated with a difference in the measure of length given to the soliton on the world sheet. Our term above appears as the equivalent of this term with  $p_2 = \pi$ , which is the momentum of the central soliton.

### 5.3.2 Scattering with non-maximal $Y = 0$ giant gravitons

Among the pairs of integrable sine-Gordon boundary parameters  $\{\zeta, \eta\}$  we are able to find a family that produce string solutions we associate with open strings ending upon non-maximal  $Y = 0$  giant gravitons.

We describe the non-maximal  $Y = 0$  case where the giant magnon is transverse to the world volume of the brane. A  $Y = 0$  giant graviton of general radius  $\rho$  is described by the relations

$$|W|^2 + |Z|^2 = \rho^2, \quad 0 \leq \rho \leq 1. \quad (5.96)$$

for the world volume coordinates and

$$|Y|^2 + \rho^2 = 1 \quad (5.97)$$

for the embedding of the brane into  $S^5$ . As  $\rho$  is a constant and our string that lives on the 2-sphere is described by

$$X_3^2 + |Z|^2 = 1 \quad (5.98)$$

then the string must meet the brane at a constant  $X_3 = |Y| = \pm\sqrt{1 - \rho^2}$ .

In the maximal case  $\rho = 1$  the brane appears, as before, as a unit disc located at  $X_3 = 0$  and the string must meet it at the edge of the disc,  $|Z| = 1$ . For  $0 < \rho < 1$  this disc shrinks in radius but the string must still end at the edge of the disc. At early and late times then we will find the development of a boundary giant magnon connecting the scattering giant magnon ending at the equator  $|Z| = 1$  with the brane located at  $|Z| = \rho$ . See Figure 5.7. This non-maximal  $Y = 0$  case then begins to resemble the  $Z = 0$  case in which we encountered a boundary degree of

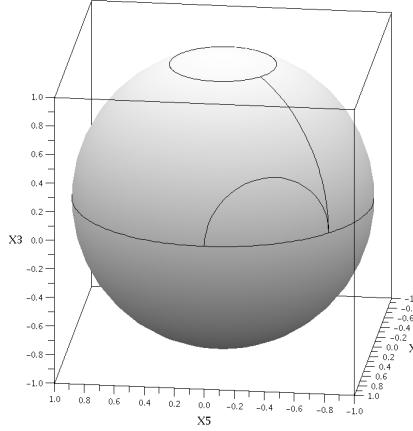


Figure 5.7: A giant magnon attached to a non-maximal  $Y = 0$  giant graviton whose world volume is transverse to the string, at early time. The giant magnon is attached via a boundary giant magnon analogously to the maximal  $Z = 0$  case.

freedom. Figure 5.8 depicts a scattering process with a giant magnon projected into the  $Z$ -plane.

We therefore seek a relation between the parameters of the string solution of the previous section such that  $X_3|_{x=0} = Z_2|_{x=0}$  is a constant. This value will be equal to that of  $Z_2|_{x=0}$  when we take  $t \rightarrow \pm\infty$ ,

$$Z_2(x = 0, t \rightarrow \pm\infty) = \frac{-\epsilon_2}{\cosh(\beta)}. \quad (5.99)$$

By equating the time dependent solution (with  $x = 0$ ) to this value and separately arranging for the coefficients of  $\cosh(v\gamma t)$  and  $\cosh(2v\gamma t)$  to vanish  $\forall t$  we obtain a trivial identity and the relations

$$\tanh\left(\frac{\alpha_+}{2}\right) = \frac{\tanh(\beta)}{\sin\frac{p}{2}}, \quad \text{for } \epsilon = +1 \quad (5.100)$$

$$\coth\left(\frac{\alpha_+}{2}\right) = \frac{\tanh(\beta)}{\sin\frac{p}{2}}, \quad \text{for } \epsilon = -1. \quad (5.101)$$

The string end point will live on a circle in the  $Z$ -plane with radius  $|Z| = \rho$  which from (5.99) must satisfy

$$\rho = \sqrt{1 - \frac{1}{\cosh^2(\beta)}} = |\tanh(\beta)| \quad (5.102)$$

Then we will have a value of  $\alpha_+$ , related to the scattering time delay, for a given magnon momentum  $p$  and giant graviton radius  $\rho$ .

Note that the value of  $\Delta - J$  for this solution is a constant. The end point at  $x = 0$  remains always in a radial orientation when projected into the  $Z$ -plane and so it is straightforward that we have zero flux of angular momentum  $J$  off the string end point. Indeed, if we had started by demanding

$$\frac{\partial J}{\partial t} = j|_{x=\infty} - j_{x=0} = 0, \quad (5.103)$$

where  $j$  is the angular momentum current density, along with satisfaction of the Virasoro constraint  $\dot{X}_i X'_i = 0$  then we would find ourselves asking for the same condition as produced (5.100) and (5.101). We have

$$\Delta - J = \frac{\sqrt{\lambda}}{\pi} \left\{ \sin \frac{p}{2} + \sqrt{1 - \rho^2} \right\} \quad (5.104)$$

which is equal to the sum of the values for the scattering giant magnon and the boundary giant magnon separately at early or late time.

The relations (5.100) and (5.101) indicate that  $\sin \frac{p_c}{2} = \tanh(\beta)$  is a critical momentum and by entering these values into the solution (5.44) - (5.48) we find that for a given size of non-maximal giant  $\rho$  the giant magnons with momentum  $p_c$  are such that the string finds itself perfectly balanced across the pole of  $S^2$  mid way through the scattering process. As such the time delay for the scattering diverges. From a sine-Gordon method of images perspective this divergence is due to the incoming soliton passing behind the boundary at  $x = 0$ , as depicted in Figure 5.1, and taking a large amount of time to scatter with the mirror soliton which we then find emerging from the boundary again as the string finally slips off the pole. When the time delay diverges the boundary value of the sine-Gordon field,  $\varphi(x, t)|_{x=0}$  must therefore fluctuate by a full  $2\pi$  from its asymptotic value of  $\varphi_0$  during the process.

We would like to see this divergence from the expression for the sine-Gordon phase delay  $a_+$  given in (5.18). We will also want to consider this quantity to calculate the semi-classical phase shift for the magnons so to these ends we first re-express (5.18).

Using

$$v^2 = 1 - s^2, \quad \kappa^2 = \frac{1 - s}{1 + s}, \quad s \equiv \sin \frac{p}{2}, \quad (5.105)$$

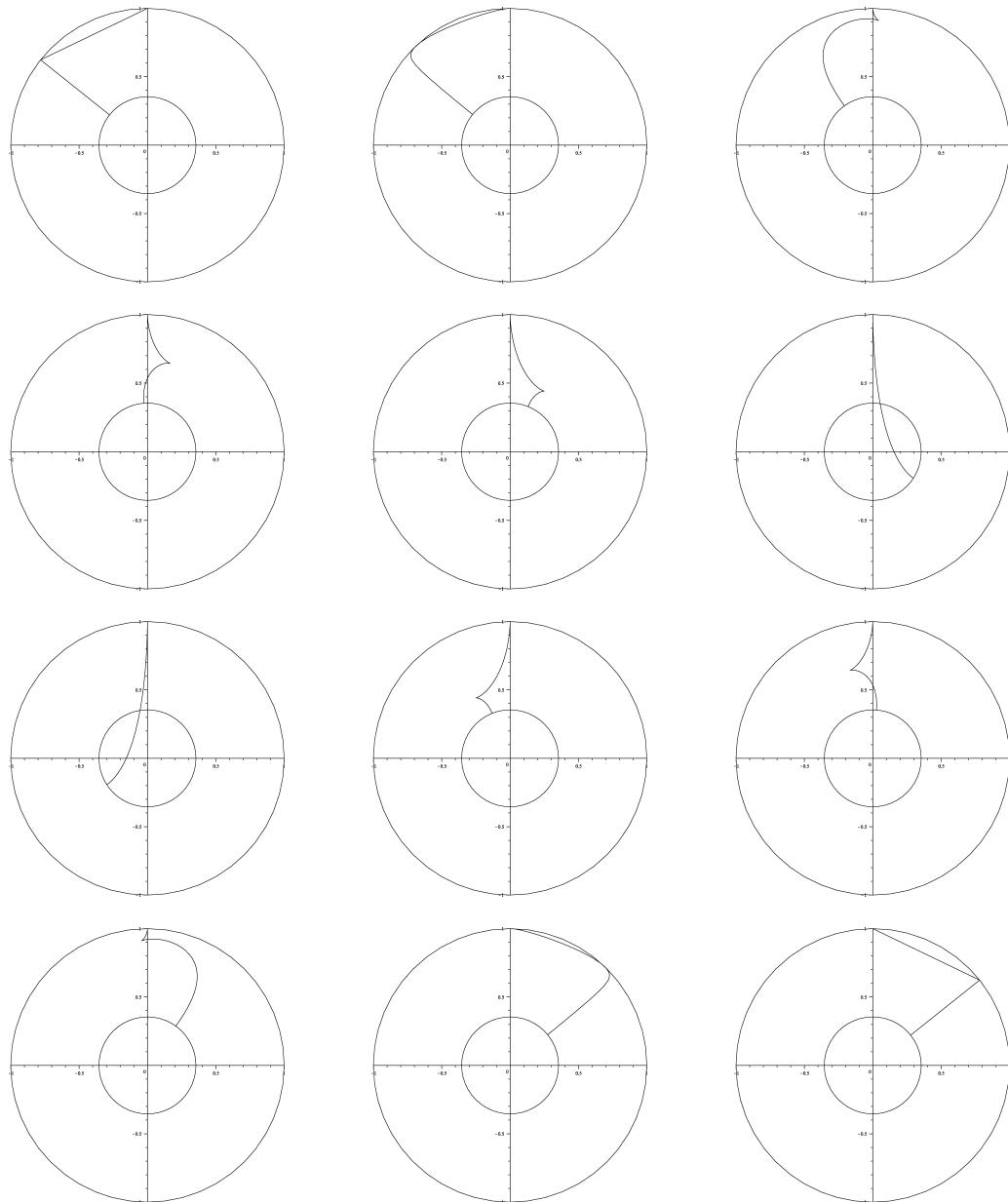


Figure 5.8: This sequence shows the scattering of a giant magnon with a non-maximal  $Y = 0$  giant graviton projected to the  $Z$  plane in which the graviton forms a disc of less than unit radius whose edge is depicted by a circle. At the top left we start with  $t = -20$  and, proceeding as you would read a page, at the bottom right we have  $t = +20$ .

we can write

$$a_+(p) = \ln \left\{ -\epsilon(1-s)^2 \left[ \left( \frac{1-s \cos(\eta)}{1+s \cos(\eta)} \right) \left( \frac{1+s \cosh(\zeta)}{1-s \cosh(\zeta)} \right) \right]^{\pm 1} \right\}. \quad (5.106)$$

Our relations (5.100) and (5.101) for circular end-point motion may then be used to eliminate one of the boundary parameters, although we must translate from  $\{\alpha_+, \beta\}$  to  $\{\zeta, \eta\}$  or  $\{M, \varphi_0\}$ , which in fact for our particular uses may be achieved very neatly as follows.

Selecting the positive power in (5.106) and using the relation between  $a_+$  and  $\alpha_+$  (5.62) we find

$$\tanh\left(\frac{\alpha_+}{2}\right) = -\frac{1-s^2 \cosh(\zeta) \cos(\eta)}{s(\cosh(\zeta) - \cos(\eta))}, \quad \text{for } \epsilon = +1 \quad (5.107)$$

$$\tanh\left(\frac{\alpha_+}{2}\right) = -\frac{s(\cosh(\zeta) - \cos(\eta))}{1-s^2 \cosh(\zeta) \cos(\eta)}, \quad \text{for } \epsilon = -1 \quad (5.108)$$

which are the inverse of one another. Then relating  $\beta$  to  $\varphi_0$  by (5.65) we find

$$\tanh(\beta) = \cos\left(\frac{\varphi_0}{2}\right) \quad (5.109)$$

so that for both  $\epsilon = \pm 1$  the conditions (5.100) and (5.101) for a circular end point motion gives us the single equation

$$-\frac{1-s^2 \cosh(\zeta) \cos(\eta)}{\cosh(\zeta) - \cos(\eta)} = \cos\left(\frac{\varphi_0}{2}\right). \quad (5.110)$$

This can now be used to write the phase delay in terms of the single boundary parameter  $\varphi_0$  and the momentum  $p$ .

Equation (5.106) may be expanded as

$$a_+(p) = \ln \left\{ -\epsilon(1-s)^2 \frac{1+s(\cosh(\zeta) - \cos(\eta)) - s^2 \cosh(\zeta) \cos(\eta)}{1-s(\cosh(\zeta) - \cos(\eta)) - s^2 \cosh(\zeta) \cos(\eta)} \right\}. \quad (5.111)$$

The definition

$$M \cos\left(\frac{\varphi_0}{2}\right) = 2 \cosh(\zeta) \cos(\eta) \quad (5.112)$$

then allows us to substitute immediately for the final terms of the numerator and denominator inside the log. The lone  $\cosh(\zeta)$  and  $\cos(\eta)$  would normally result in an untidy expression but in this instance we may use (5.110) and (5.112) to write

$$\cosh(\zeta) - \cos(\eta) = -\frac{1 - \frac{1}{2}s^2 M \cos\left(\frac{\varphi_0}{2}\right)}{\cos\left(\frac{\varphi_0}{2}\right)} \quad (5.113)$$

and, cancelling a common factor involving  $M$ , we find the sine-Gordon phase delay for the boundary condition corresponding to giant magnons reflecting from  $Y = 0$  giant gravitons can be written simply as

$$a_+(p)^{Y=0} = \ln \left\{ -\epsilon \left( 1 - \sin \frac{p}{2} \right)^2 \left( \frac{\cos \left( \frac{\varphi_0}{2} \right) - \sin \frac{p}{2}}{\cos \left( \frac{\varphi_0}{2} \right) + \sin \frac{p}{2}} \right) \right\}. \quad (5.114)$$

This expression must interpolate to the case applicable to the scattering of giant magnons off maximal  $Y = 0$  giant gravitons with  $\rho = 1$ , which corresponds to a Neumann boundary condition in sine-Gordon theory. From the discussion above we have

$$\rho = r = |\tanh(\beta)| = \left| \cos \left( \frac{\varphi_0}{2} \right) \right| \quad (5.115)$$

and so the phase delay in terms of the graviton radius and magnon momentum is equivalently

$$a_+(p)^{Y=0} = \ln \left\{ -\epsilon \left( 1 - \sin \frac{p}{2} \right)^2 \left( \frac{\rho + \sin \frac{p}{2}}{\rho - \sin \frac{p}{2}} \right) \right\}. \quad (5.116)$$

When  $\sin \frac{p}{2} < \rho$  we should take  $\epsilon = -1$ , a Neumann-like solution, and when  $\sin \frac{p}{2} > \rho$  take  $\epsilon = +1$ , a Dirichlet-like solution. When  $\rho = 1$  we get

$$a_+(p)^{\text{Maximal } Y=0} = 2 \ln \left\{ \cos \frac{p}{2} \right\} \quad (5.117)$$

which is indeed the Neumann phase delay produced above.

We can now calculate the semi-classical phase shift for the scattering of magnons with the non-maximal giant graviton by the integral

$$\delta_B^{Y=0} = 4g \int \frac{s}{\sqrt{1-s^2}} \ln \left\{ -\epsilon(1-s)^2 \left( \frac{\rho+s}{\rho-s} \right) \right\} ds. \quad (5.118)$$

Putting this into the form of the maximal, or Neumann, contribution and a term appearing when  $\rho < 1$  we could write

$$\delta_B^{Y=0} = 4g \int \left[ \frac{2s}{\sqrt{1-s^2}} \ln \left\{ \sqrt{1-s^2} \right\} + \frac{s}{\sqrt{1-s^2}} \ln \left\{ \left( \frac{1-s}{1+s} \right) \left( \frac{\rho+s}{\rho-s} \right) \right\} \right] ds \quad (5.119)$$

and the answer that results is

$$\delta_B^{Y=0} = -4g \cos \frac{p}{2} \ln \left\{ \cos^2 \frac{p}{2} \left( \frac{1-\sin \frac{p}{2}}{1+\sin \frac{p}{2}} \right) \left( \frac{\rho+\sin \frac{p}{2}}{\rho-\sin \frac{p}{2}} \right) \right\} + \delta'. \quad (5.120)$$

with

$$\delta' = 8g \cos \frac{p}{2} + 8g \left( \frac{p}{2} - \frac{\pi}{2} \right). \quad (5.121)$$

The extra factors that appear within the log for  $\rho < 1$  coincide with the appearance of the boundary degree of freedom we have described. In fact it is the same factor we saw in the maximal  $Z = 0$  case but dressed with an extra factor containing the parameter  $\rho = \left| \cos \left( \frac{\varphi_0}{2} \right) \right|$ . If we were to trust the  $\rho \rightarrow 0$  limit<sup>1</sup> when the  $Y = 0$  brane shrinks to zero size then the boundary giant magnon present at early and late times extends all the way to the pole of  $S^2$  giving a string solution identical to the maximal  $Z = 0$ , or  $\varphi_0 = \pi$  sine-Gordon Dirichlet, case with an identical scattering phase to match. For  $\rho = 1$  of course we recover the maximal  $Y = 0$  case.

The extra terms  $\delta'$  appear from the integrals associated with the pure Neumann piece and the undressed boundary piece, and are not to do with the added integrable boundary condition. Indeed, the extra terms are associated with the direct scattering of the method of images solitons while the new factor appearing in (5.120), or (5.116), represents simply the insertion of an added time delay to the scattering process such that the scattering of the in-going soliton and the mirror out-going soliton occurs some way behind the point  $x = 0$ , consistent with the origin of the modulus  $\alpha_+$  in the initial displacement parameters of the bulk solution. Seen as an added time delay  $\tilde{\Delta T}$  we can see immediately from (5.116) that this quantity is given by

$$\tilde{\Delta T} = \frac{\sin \frac{p}{2}}{\cos \frac{p}{2}} \ln \left\{ \frac{\rho + \sin \frac{p}{2}}{\rho - \sin \frac{p}{2}} \right\}. \quad (5.122)$$

### 5.3.3 Other open strings with integrable sine-Gordon boundary conditions

It is interesting to ask what string solutions correspond to other particular integrable sine-Gordon boundary conditions. For general  $\{\zeta, \eta\}$  the string end point will execute some motion that we do not associate with any obvious configuration of a D-brane embedded in  $S^5$ . We will also generically have  $j \neq 0$  so that the brane must accept some angular momentum from the string, albeit temporarily as in- and out-going solitons have the same energy and angular momentum.

---

<sup>1</sup>In the  $\rho \rightarrow 0$  limit the curvature of the giant graviton of course becomes large so that corrections to the supergravity description would generally become necessary [10].

We cannot rule out the possibility of embeddings of a giant graviton that are oblique to the  $\{X_5, X_6\}$  plane, for example such that its coordinates span the subspace

$$V = \{X_1, X_2, X_3, X_5\}, \quad (5.123)$$

as while the intersection of the brane's  $S^3$  and the string's  $S^2$  may be confined to a line in the  $X_5$  direction, the brane itself can of course still take a moving position in the  $X_6$  direction so that the string end point is 'free' to range over some region of the  $\{X_5, X_6\}$  plane which is the behaviour we generically find resulting from the method of images. Such a scattering of giant magnon and giant graviton would transfer angular momentum, as just described.

On the other hand such oblique embeddings may be ruled out by the non-existence of BPS brane solutions describing such embeddings; it may be that the operators dual to such branes are not of definite R-charge and that operators corresponding to these branes with strings of  $J_{56}$  angular momentum possess no supersymmetry.

### Strings with sine-Gordon Dirichlet boundary conditions

We might ask about the family of sine-Gordon Dirichlet boundary conditions, of which we have seen two, these representing all those strings whose end points have a constant speed (by (1.95)). It is possible to show that all such strings possess end points that, while maintaining a constant speed, execute circular motions in the co-moving frame with angular velocity in the  $Z$ -plane  $\dot{\theta} = 1$ . The angle of the axis of this circle with respect to the  $Z$ -plane is  $\frac{\varphi_0}{2}$ .

We prove this claim as follows. If we select the co-moving frame and rotate our solution by an angle of  $\frac{\pi}{2} - \frac{\varphi_0}{2}$  toward the vertical so that the circle upon which the end point moves is horizontal then we may proceed as in section 5.3.2 to obtain a relation between the moduli  $\alpha_+$  and  $\beta$  by demanding that the value of the new  $Z_2(x = 0)$  be constant for all time. We obtain

$$\tanh\left(\frac{\alpha_+}{2}\right) = \sin\frac{p}{2}\cos\frac{\varphi_0}{2} \quad (5.124)$$

which is exactly the form of  $\tanh\left(\frac{\alpha_+}{2}\right)$  we find by taking the form of  $\alpha_+$  due to the sine-Gordon Dirichlet phase delay (5.19) (with positive power) and dictionary (5.62).

We note that as  $\tanh(\beta) = \cos\frac{\varphi_0}{2}$  this relation is very similar to the condition (5.100) for motion on the non-maximal  $Y = 0$  brane when  $\epsilon = +1$ , as it must necessarily be in the Dirichlet case.

It would be tempting to think of this co-moving circle as a brane configuration intersecting  $S^2$  analogously to the non-maximal  $Y = 0$  giant of section 5.3.2 but rotated at an angle to the axis of large  $J$ . As such however the string end point would not meet the brane orthogonally: it is seen immediately at early and late times that while the intersection of the brane with the string's  $S^2$  lies at an angle to the equator ( $\forall t$ ) the string hangs radially from the point at which it meets the brane, and so lies at some angle different from  $\frac{\pi}{2}$  to the brane.

### Scattering with non-maximal $Z = 0$ giant gravitons?

For the case of a non-maximal  $Z = 0$  giant graviton the string end point must be some point away from the pole. For the radius of the graviton  $\rho \neq 1$  then a BPS brane must sit on a circle in the  $Z$ -plane, some  $0 < |Z| \leq 1$  with  $|Z| = \sqrt{1 - \rho^2}$ . This point now moves at a constant speed of  $\dot{\theta}\sqrt{1 - \rho^2}$  where we must know  $\dot{\theta}$ , the angular velocity about the circle in the  $Z$ -plane, or about the  $J$ -axis. We can show that  $\dot{\theta} = 1$  (when the radius of  $S^5$  is  $R = 1$ ) as follows.

In [10] McGreevy et al. used the Born-Infeld (plus Chern-Simons potential) effective action for giant gravitons wrapping  $S^3 \subset S^5$  with radius  $0 \leq \rho \leq 1$  to show that they possess angular momentum  $J$  given by<sup>2</sup>

$$J = N\rho^3 \frac{\dot{\theta}(1 - \rho^2)}{\sqrt{1 - \dot{\theta}^2(1 - \rho^2)}} + N\rho^4 \quad (5.125)$$

where  $N$  is here the number of units of 5-form flux on  $S^5$  (equivalently the number of colours in the gauge theory dual). Note  $J \propto N$ . Rearranging this expression for

---

<sup>2</sup>Equivalently, integrate (4.19) over the world volume with constant  $\rho$ .

$\dot{\theta}$  we have

$$\dot{\theta} = \frac{\left(\frac{J}{N} - \rho^4\right)}{\sqrt{\rho^6(1 - \rho^2)^2 + \left(\frac{J}{N} - \rho^4\right)^2(1 - \rho^2)}}. \quad (5.126)$$

If we then use the BPS condition for the radius of the brane at fixed  $J$ ,

$$\rho^2 = \frac{J}{N}, \quad (5.127)$$

then we find

$$\dot{\theta} = 1. \quad (5.128)$$

Now, supposing that in the large  $N$  limit the string has only angular momentum of order  $\sqrt{N}$  to give to the brane during scattering. Then we expect the brane to remain ‘heavy’, and to be unperturbed by the scattering of the giant magnon. The string end point must remain fixed to this point which proceeds about the circle  $|Z| = \sqrt{1 - \rho^2}$  at  $\dot{\theta} = 1$ .

In the co-moving frame with  $\dot{\theta} = 1$  then the string end point must remain completely fixed. Using the solution (5.20) - (5.24) it is then easy to see that we cannot describe such an end point motion using the method of images.

## 5.4 Conclusion

We have described the scattering of giant magnons with certain maximal and non-maximal giant gravitons by applying the method of images to obtain solutions to the boundary string sigma model from the bulk 3-soliton solution. We have also calculated the corresponding overall scattering phases at large  $\lambda$  through knowledge of the time delays for scattering in the sine-Gordon picture. As such we have addressed a question about the status of string solutions resulting from the less trivial integrable boundary conditions of the Pohlmeyer reduced string asked in [78], and contributed to the understanding of the map between sine-Gordon theory and the sigma model on the half-line [90–92].

In the most interesting case a giant magnon of arbitrary momentum  $p$  is scattered from a non-maximal  $Y = 0$  giant graviton whose world volume is entirely transverse to the string. It is seen that as we reduce the radius  $\rho$  of the giant graviton from its maximal value  $\rho = 1$  we find the development of a boundary degree of freedom, or

boundary giant magnon, akin to that found in the maximal  $Z = 0$  case. In fact the large  $\lambda$  boundary scattering phase is seen to resemble that obtained for scattering with a maximal  $Z = 0$  giant but for a factor that interpolates appropriately between the maximal  $Y = 0$  case without a boundary giant magnon and the maximal  $Z = 0$  case for which the boundary magnon stretches from the equator to the pole. (However, this does not correspond to the rotation of a maximal  $Y = 0$  giant into a maximal  $Z = 0$  giant; the string configurations thus produced are simply identical in both cases.)

We were not able to describe the scattering of giant magnons with non-maximal  $Z = 0$  giant gravitons using this method and it seems likely that this process is not integrable.

# Chapter 6

## Discussion

In this thesis we have been concerned with semi-classical strings and D-branes of type IIB string theory on subspaces of  $AdS_5 \times S^5$  that occur in solvable limits of the  $AdS/CFT$  duality relating these states to operators of  $\mathcal{N} = 4$  extended superconformal  $SU(N)$  Yang-Mills theory. Taking the 't Hooft large  $N$  limit in which gauge theory processes are dominated by planar contributions and in which the string theory becomes free we have focussed on sectors with large R-charge / angular momentum  $J$  on  $S^5$  where further simplifications are afforded. Spin-chain phenomena from the perspective of the gauge theory are given “giant” interpretations on the string side.

In particular we have studied the appearance of boundary giant magnons attached to (or emerging from) giant gravitons and the subsequent interaction of giant magnon excitations of the world sheet with giant gravitons at the boundary. Making extensive use of the Pohlmeyer reduced picture of the world sheet theory of strings on  $\mathbb{R} \times S^2$ , sine-Gordon theory, we have firstly found an extended family of boundary giant magnons to add to the literature by considering sine-Gordon theory on the interval. Secondly, from the theory in the bulk and on the half-line we have described novel scattering solutions of giant magnons with non-maximal giant gravitons. By resort to the world volume theory of the giant graviton the full behaviour of boundary giant magnons is corroborated through a set of BIon-like solutions.

The boundary giant magnons found in chapter 3 split into two families with

qualitatively different finite  $J$  behaviour, corresponding to two distinct types of static sine-Gordon solution on the interval. The limit  $J \rightarrow \infty$  is achieved when the angular frequency  $\dot{\phi} = 1$  and the boundary giant magnons become segments of true giant magnons with a point at the equator satisfying  $(X^i)' = 0$  and end points at a pole of  $S^2$  with  $\dot{X}^i = 0$ . The two solutions are distinguished then simply by the choice between folding the string back upon itself at the equator or continuing on to the opposite pole of the sphere. The  $J \rightarrow \infty$  sine-Gordon solutions are defined on an interval of divergent length; in the conformal gauge the energy of the string is proportional to the range of  $x$ .

By taking  $\dot{\phi} \neq 1$  in either direction we move away from  $J \rightarrow \infty$  to finite  $J$ . The  $\dot{\phi} > 1$  solution is the previously studied boundary giant magnon [86] which resembles the folded spinning string but cut in half with both end points attached at the same point (to what may be called a  $Z = 0$  giant graviton) at the pole. In this sense the  $\dot{\phi} > 1$  solution is related to the finite  $J$  giant magnon (with magnon momentum  $p = \pi$ ). For  $J$  large we have exponentially small corrections to the  $J \rightarrow \infty$  relation for the anomalous dimension,

$$\Delta_1 - J_1 \approx \frac{\sqrt{\lambda}}{\pi} \left[ 1 - \frac{4}{e^2} e^{-\frac{2\pi}{\sqrt{\lambda}} J} \right]. \quad (6.1)$$

This is equal to the correction for the finite  $J$  giant magnon [61] with  $p = \pi$ .

When  $0 < \dot{\phi} < 1$  the string cannot fold back upon itself; it is immediately obvious from the corresponding sine-Gordon solution that as there is no point at which  $\varphi$  achieves  $\varphi = 2n\pi$ ,  $n \in \mathbb{Z}$  then there is no cusp point at which  $(X^i)' = 0$ . Rather the string must continue on to the opposite pole where it meets the maximal giant graviton once again. The solution is similar to the second of the spinning closed string solutions on  $S^2$  discussed by Gubser, Klebanov and Polyakov [34] where it was observed that the equivalent closed, wrapped string would be unstable to ‘slipping off the side of the sphere’. Our solution is not unstable because it is stretched between two fixed points at the poles. At large  $J$  we find identical exponential corrections to the anomalous dimension but with the opposite sign,

$$\Delta_2 - J_2 \approx \frac{\sqrt{\lambda}}{\pi} \left[ 1 + \frac{4}{e^2} e^{-\frac{2\pi}{\sqrt{\lambda}} J} \right]. \quad (6.2)$$

At  $\dot{\phi} = 0$  the string is simply suspended between two poles of the sphere without

moving: it is the stretched string and we obtain the correct energy, equalling tension multiplied by target space length.

In the above analysis we chose a particular boundary condition of the sine-Gordon solutions that ensured both string end points would be stationary with  $\varphi = \pi$ . We could have chosen other boundary conditions for the static theory which would have resulted in strings whose end points moved at some constant speed about the sphere. This we could interpret as being due to the presence of a non-maximal ( $Z = 0$ ) giant graviton, which being non-maximal is forced to orbit the sphere at some distance from its orbital axis. Such configurations of the giant graviton are also stable, BPS states and these were discussed first in chapter 2 and then in chapter 4.

In chapter 4 we examined the world volume effective theory of the giant gravitons. By exciting one of the world volume scalar fields, allowing it to depend on one of the world volume coordinates in such a way that we preserve an  $SO(3)$  symmetry, we found BIon ‘spike’ solutions interpreted as boundary giant magnons. At  $\dot{\phi} = 1$  the solution again matches previously known results [84] and gives the correct anomalous dimension for a giant magnon. Moving to finite  $J$  at  $\dot{\phi} > 1$  and  $0 < \dot{\phi} < 1$  we found behaviour matching the string solutions of chapter 3.

The brane theory was analysed at finite  $N$  while the string results are presented with the  $N \rightarrow \infty$  limit (as  $g_s \rightarrow 0$  at fixed  $\lambda$ ) implicit. In fact the brane ‘spike’ solutions became singular in the  $N \rightarrow \infty$  limit: the width of the string in target space is determined by the constant

$$\kappa = \frac{\sqrt{\lambda}}{4N} = \frac{\pi}{\sqrt{\lambda}} g_s. \quad (6.3)$$

If we desire fixed ’t Hooft gauge coupling  $\lambda$  then  $\kappa$  is essentially the string coupling and the non-interacting string has zero width. At the same time the giant graviton tends to a perfectly spherical brane with two singular points at which the string protrudes / meets the brane.

To find a solvable limit of the highly non-linear brane equations at  $\dot{\phi} \neq 1$  we had to make a pair of approximations, one of which was  $\kappa \rightarrow 0$ . We had to be careful to first focus on regions of the brane close enough to the centre of the BIon. This

was achieved by looking at values of the world volume coordinate  $\sigma$ , which close to  $\sigma = 0$  acts as a radial coordinate, satisfying  $\sin^2(\sigma) \ll \kappa$ . Given the discussion just above this is clearly an intuitive approximation: we examine distances less than those that capture anything other than stringy parts of the brane.

We again found a simple generalisation would have been possible, equivalent to the choice of different boundary conditions for the sine-Gordon field. It is seen explicitly then that this can indeed be interpreted as describing strings attached to a non-maximal giant graviton; we have the corresponding non-maximal giant graviton solution!

Chapter 5 was concerned with the scattering of giant magnons from giant gravitons in the strict planar, and  $J \rightarrow \infty$  limits. By requiring ingoing giant magnons to remain intact upon scattering we restricted to integrable scattering processes so that each possible solution should correspond to a scattering of solitons in boundary sine-Gordon theory.

We applied the method of images in which a bulk 3 soliton solution is employed to produce the ingoing, outgoing and boundary degrees of freedom and where two real boundary parameters determine the range of behaviour exhibited. Mimicking this procedure with a 3 soliton string solution modified to include moduli we first restricted the solution on  $\mathbb{R} \times S^3$  to  $\mathbb{R} \times S^2$  which resulted in three discrete parameters that we were able to identify as the soliton / anti-soliton parameters necessary in a topological soliton theory. Then by examining the asymptotic behaviour of both theories we found a dictionary between the real parameters.

By requiring mixed string boundary conditions that allowed the string end point to move on an  $S^1$  at the intersection of the  $S^3$  wrapped by a  $Y = 0$  giant graviton and the  $S^2$  upon which the giant magnon moves we were able to produce the solution for a novel scattering process involving a *non-maximal* giant graviton. This was such that angular momentum was conserved.

In Mann and Vázquez's study of classical open string integrability [79] which attempted to construct an infinite hierarchy of conserved charges in a manner analogous to the closed string case [24] it was unclear if integrability could be maintained

for strings ending upon non-maximal giant gravitons. In fact a condition appeared on the exchange of angular momentum,

$$\partial_t J \sim -\frac{\sqrt{\lambda}}{\pi} \sin^2(\theta_0) \phi' \quad (6.4)$$

which is to be evaluated at the string end point. Our string /  $Y = 0$  non-maximal giant graviton solutions have  $\phi' = 0$  at the end point, hence  $\partial_t J = 0$ , as discussed and should be the only integrable solutions that do so. We have therefore found the solutions that obey the condition in [79] that allow the construction of the infinite hierarchy of conserved charges.

We were also able to calculate a magnon boundary scattering phase in manner similar to [78], taking advantage of the equality of time variables in the string picture and the Pohlmeyer reduced picture. We found a neat form of the phase in terms of the radius of the non-maximal giant graviton.

We discussed the possibility of our solution describing scattering with other configurations of branes. The  $Y = 0$  non-maximal case already covers every possible integrable case conserving the angular momentum of the string; all other integrable scattering processes will transfer angular momentum to the brane, albeit temporarily.

A problem with this idea is that in the strict planar limit it is hard to see how a string may have enough energy to move a brane: as we have seen in chapter 4 the energy of a giant graviton is by (4.21) and (4.6) proportional to

$$T = 4\pi R^4 T_3 = \frac{4\pi}{\Omega_3} N \quad (6.5)$$

while that of a string is by proportional to

$$T_\kappa = \frac{\sqrt{\lambda}}{2\pi} = T_s \quad (6.6)$$

which is independent of  $N$ . Now the standard 't Hooft limit we take to talk about classical non-interacting strings involves keeping  $\lambda = 4\pi g_s N$  fixed as  $N \rightarrow \infty$  and  $g_s \rightarrow 0$ . This implies that in the limit we are interested in the string should always be unable to budge a brane.

Nevertheless it is perhaps not impossible to consider a limit in which such things could occur while maintaining a classical, non-interacting picture of the string. The

BMN limit (section 1.3.4) involved taking  $\lambda \rightarrow \infty$  as  $N \rightarrow \infty$  with  $g_s$  therefore fixed. If  $g_s$  is kept small enough then the energy of the string could become comparable to that of the brane. We would of course then have to consider string splitting, but string splitting in a similar regime has been considered [99–101].

We also note that Berenstein, Correa and Vázquez have suggested the possibility of a D-brane instability [83] in which an appropriately excited string ending on a giant graviton is provoked by the acceleration of its non-geodesic motion into drawing an increasing amount of energy from the D-brane. The end result of this process was not clear as at some stage the string must violate planarity, absorbing of order  $N$  units of angular momentum so that we must begin to consider a D-brane description of the string. Is it nevertheless possible that an elastic version of such a process may be described by the solutions that we have obtained in this thesis?

# Bibliography

- [1] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [[arXiv:hep-th/9711200](https://arxiv.org/abs/hep-th/9711200)].
- [2] S. S. Gubser, I. R. Klebanov, A. M. Polyakov, “Gauge theory correlators from noncritical string theory,” *Phys. Lett.* **B428** (1998) 105-114. [[hep-th/9802109](https://arxiv.org/abs/hep-th/9802109)].
- [3] E. Witten, “Anti-de Sitter space and holography,” *Adv. Theor. Math. Phys.* **2** (1998) 253 [[arXiv:hep-th/9802150](https://arxiv.org/abs/hep-th/9802150)].
- [4] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, “Large N field theories, string theory and gravity,” *Phys. Rept.* **323** (2000) 183 [[arXiv:hep-th/9905111](https://arxiv.org/abs/hep-th/9905111)].
- [5] J. Polchinski, “String theory. Vol. 1: An introduction to the bosonic string,” <http://www.slac.stanford.edu/spires/find/hep/www?irn=4634799> (SPIRES entry) *Cambridge, UK: Univ. Pr. (1998) 402 p*
- [6] J. Polchinski, “String theory. Vol. 2: Superstring theory and beyond,” <http://www.slac.stanford.edu/spires/find/hep/www?irn=4634802> (SPIRES entry) *Cambridge, UK: Univ. Pr. (1998) 531 p*
- [7] G. ’t Hooft, “A Planar Diagram Theory for Strong Interactions,” *Nucl. Phys. B* **72** (1974) 461.
- [8] J. H. Schwarz, “Covariant Field Equations of Chiral N=2 D=10 Supergravity,” *Nucl. Phys. B* **226** (1983) 269.

- [9] D. M. Hofman and J. M. Maldacena, “Giant magnons,” *J. Phys. A* **39** (2006) 13095 [arXiv:hep-th/0604135].
- [10] J. McGreevy, L. Susskind and N. Toumbas, “Invasion of the giant gravitons from anti-de Sitter space,” *JHEP* **0006** (2000) 008 [arXiv:hep-th/0003075].
- [11] J. Polchinski, “Dirichlet Branes and Ramond-Ramond charges,” *Phys. Rev. Lett.* **75** (1995) 4724 [arXiv:hep-th/9510017].
- [12] C. V. Johnson, “D-branes”,  
<http://www.slac.stanford.edu/spires/find/hep/www?irn=5462010> (SPIRES entry) *Cambridge, USA: Univ. Pr. (2003) 548 p*
- [13] V. Balasubramanian, M. Berkooz, A. Naqvi *et al.*, “Giant gravitons in conformal field theory,” *JHEP* **0204** (2002) 034. [hep-th/0107119].
- [14] A. Ciavarella and P. Bowcock, “Boundary Giant Magnons and Giant Gravitons,” *JHEP* **1009** (2010) 072 [arXiv:1007.1674 [hep-th]].
- [15] A. Ciavarella, “Giant magnons and non-maximal giant gravitons,” *JHEP* **1101** (2011) 040 [arXiv:1011.1440 [hep-th]].
- [16] M. J. Duff, R. R. Khuri and J. X. Lu, “String solitons,” *Phys. Rept.* **259** (1995) 213 [arXiv:hep-th/9412184].
- [17] K. S. Stelle, “BPS branes in supergravity,” arXiv:hep-th/9803116.
- [18] J. L. Petersen, “Introduction to the Maldacena conjecture on AdS / CFT,” *Int. J. Mod. Phys. A* **14** (1999) 3597 [arXiv:hep-th/9902131].
- [19] R. G. Leigh, “Dirac-Born-Infeld Action from Dirichlet Sigma Model,” *Mod. Phys. Lett. A* **4** (1989) 2767.
- [20] S. Weinberg, “The quantum theory of fields. Vol. 3: Supersymmetry,”  
<http://www.slac.stanford.edu/spires/find/hep/www?irn=4384008> (SPIRES entry) *Cambridge, UK: Univ. Pr. (2000) 419 p*

- [21] G. T. Horowitz and A. Strominger, “Black strings and P-branes,” *Nucl. Phys. B* **360** (1991) 197.
- [22] D. E. Berenstein, J. M. Maldacena and H. S. Nastase, “Strings in flat space and pp waves from  $N = 4$  super Yang Mills,” *JHEP* **0204** (2002) 013 [arXiv:hep-th/0202021].
- [23] J. A. Minahan and K. Zarembo, “The Bethe-ansatz for  $N = 4$  super Yang-Mills,” *JHEP* **0303** (2003) 013 [arXiv:hep-th/0212208].
- [24] I. Bena, J. Polchinski and R. Roiban, “Hidden symmetries of the  $AdS_5 \times S^5$  superstring,” *Phys. Rev. D* **69** (2004) 046002 [arXiv:hep-th/0305116].
- [25] V. A. Kazakov, A. Marshakov, J. A. Minahan and K. Zarembo, “Classical/quantum integrability in AdS/CFT,” *JHEP* **0405** (2004) 024 [arXiv:hep-th/0402207].
- [26] G. Arutyunov, S. Frolov, M. Staudacher, “Bethe ansatz for quantum strings,” *JHEP* **0410** (2004) 016. [hep-th/0406256].
- [27] M. Staudacher, “The factorized S-matrix of CFT/AdS,” *JHEP* **0505** (2005) 054 [arXiv:hep-th/0412188].
- [28] S. W. Hawking and G. F. R. Ellis, “The Large scale structure of space-time,” <http://www.slac.stanford.edu/spires/find/hep/www?irn=6991262> (SPIRES entry) *Cambridge University Press, Cambridge, 1973*
- [29] P. Di Francesco, P. Mathieu and D. Senechal, “Conformal field theory,” <http://www.slac.stanford.edu/spires/find/hep/www?irn=3810356> (SPIRES entry) *New York, USA: Springer (1997) 890 p*
- [30] S. Weinberg, “The Quantum theory of fields. Vol. 1: Foundations,” <http://www.slac.stanford.edu/spires/find/hep/www?irn=3355144> (SPIRES entry) *Cambridge, UK: Univ. Pr. (1995) 609 p*
- [31] R. Haag, J. T. Lopuszanski and M. Sohnius, “All Possible Generators of Supersymmetries of the s Matrix,” *Nucl. Phys. B* **88** (1975) 257.

- [32] N. Beisert *et al.*, “Review of  $AdS/CFT$  Integrability: An Overview,” arXiv:1012.3982 [hep-th].
- [33] A. M. Polyakov, “Gauge fields and space-time,” *Int. J. Mod. Phys. A* **17S1** (2002) 119 [arXiv:hep-th/0110196].
- [34] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “A semi-classical limit of the gauge/string correspondence,” *Nucl. Phys. B* **636** (2002) 99 [arXiv:hep-th/0204051].
- [35] K. Pohlmeyer, “Integrable Hamiltonian Systems And Interactions Through Quadratic Constraints,” *Commun. Math. Phys.* **46** (1976) 207.
- [36] K. Pohlmeyer and K. H. Rehren, “Reduction Of The Two-Dimensional  $O(N)$  Nonlinear Sigma Model,” *J. Math. Phys.* **20** (1979) 2628.
- [37] R. D’Auria, T. Regge and S. Sciuto, “A general scheme for bidimensional models with associate linear set,” *Phys. Lett. B* **89** (1980) 363.
- [38] R. D’Auria, T. Regge and S. Sciuto, “Group theoretical construction of two-dimensional models with infinite set of conservation laws,” *Nucl. Phys. B* **171** (1980) 167.
- [39] I. Bakas, Q. H. Park and H. J. Shin, “Lagrangian Formulation of Symmetric Space sine-Gordon Models,” *Phys. Lett. B* **372** (1996) 45 [arXiv:hep-th/9512030].
- [40] M. Grigoriev and A. A. Tseytlin, “Pohlmeyer reduction of  $AdS_5 \times S^5$  superstring sigma model,” *Nucl. Phys. B* **800** (2008) 450 [arXiv:0711.0155 [hep-th]].
- [41] T. J. Hollowood and J. L. Miramontes, “Magnons, their Solitonic Avatars and the Pohlmeyer Reduction,” *JHEP* **0904** (2009) 060 [arXiv:0902.2405 [hep-th]].
- [42] B. Hoare, Y. Iwashita and A. A. Tseytlin, “Pohlmeyer-reduced form of string theory in  $AdS_5 \times S^5$ : semiclassical expansion,” *J. Phys. A* **42** (2009) 375204 [arXiv:0906.3800 [hep-th]].

- [43] T. J. Hollowood and J. L. Miramontes, “The Semi-Classical Spectrum of Solitons and Giant Magnons,” arXiv:1103.3148 [hep-th].
- [44] A. B. Zamolodchikov and A. B. Zamolodchikov, “Factorized s Matrices in Two-Dimensions as the Exact Solutions of Certain Relativistic Quantum Field Models,” *Annals Phys.* **120** (1979) 253.
- [45] N.S.Manton and P.Sutcliffe, “Topological solitons,” <http://www.slac.stanford.edu/spires/find/hep/www?irn=6000355> (SPIRES entry) *Cambridge, UK: Univ. Pr. (2004) 493 p*
- [46] E. B. Bogomol’nyi, *Sov. J. Nucl. Phys.* **24**, 449 (1976)
- [47] N. Beisert, “The Dilatation operator of  $\mathcal{N} = 4$  super Yang-Mills theory and integrability,” *Phys. Rept.* **405** (2005) 1 [arXiv:hep-th/0407277].
- [48] N. Beisert, “The complete one loop dilatation operator of  $\mathcal{N} = 4$  superYang-Mills theory,” *Nucl. Phys. B* **676** (2004) 3 [arXiv:hep-th/0307015].
- [49] M. Kruczenski, “Spin chains and string theory,” *Phys. Rev. Lett.* **93** (2004) 161602 [arXiv:hep-th/0311203].
- [50] N. Beisert, “The  $su(2|2)$  dynamic S-matrix,” *Adv. Theor. Math. Phys.* **12** (2008) 945 [arXiv:hep-th/0511082].
- [51] A. A. Tseytlin, “Review of *AdS/CFT* Integrability, Chapter II.1: Classical  $AdS_5 \times S^5$  string solutions,” arXiv:1012.3986 [hep-th].
- [52] A. A. Tseytlin, “Spinning strings and *AdS/CFT* duality,” arXiv:hep-th/0311139.
- [53] J. Plefka, “Spinning strings and integrable spin chains in the *AdS/CFT* correspondence,” *Living Rev. Rel.* **8** (2005) 9 [arXiv:hep-th/0507136].
- [54] M. T. Grisaru, R. C. Myers and O. Tafjord, “SUSY and Goliath,” *JHEP* **0008** (2000) 040 [arXiv:hep-th/0008015].

- [55] G. Veneziano, “An introduction to dual models of string interactions and their physical motivations,” *Phys. Rept.* **9** (1974) 199.
- [56] S. Frolov and A. A. Tseytlin, “Multispin string solutions in  $AdS_5 \times S^5$ ,” *Nucl. Phys. B* **668** (2003) 77 [arXiv:hep-th/0304255].
- [57] S. Frolov and A. A. Tseytlin, “Rotating string solutions:  $AdS$  / CFT duality in nonsupersymmetric sectors,” *Phys. Lett. B* **570** (2003) 96 [arXiv:hep-th/0306143].
- [58] J. A. Minahan, “Review of  $AdS/CFT$  Integrability, Chapter I.1: Spin Chains in  $\mathcal{N} = 4$  Super Yang-Mills,” arXiv:1012.3983 [hep-th].
- [59] R. Jackiw, G. Woo, “Semiclassical Scattering of Quantized Nonlinear Waves,” *Phys. Rev. D* **12** (1975) 1643.
- [60] M. J. Ablowitz and H. Segur, “Solitons and the inverse scattering transform,” <http://www.slac.stanford.edu/spires/find/hep/www?irn=1378953> (SPIRES entry) *Philadelphia, Usa: Siam ( 1981) 425p*
- [61] G. Arutyunov, S. Frolov and M. Zamaklar, “Finite-size effects from giant magnons,” *Nucl. Phys. B* **778** (2007) 1 [arXiv:hep-th/0606126].
- [62] D. Astolfi, V. Forini, G. Grignani and G. W. Semenoff, “Gauge invariant finite size spectrum of the giant magnon,” *Phys. Lett. B* **651** (2007) 329 [arXiv:hep-th/0702043].
- [63] T. Klose, T. McLoughlin, “Interacting finite-size magnons,” *J. Phys. A* **A41** (2008) 285401. [arXiv:0803.2324 [hep-th]].
- [64] Y. Hatsuda and R. Suzuki, “Finite-Size Effects for Dyonic Giant Magnons,” *Nucl. Phys. B* **800** (2008) 349 [arXiv:0801.0747 [hep-th]].
- [65] G. Arutyunov and S. Frolov, “Uniform light-cone gauge for strings in  $AdS_5 \times S^5$ : Solving  $SU(1|1)$  sector,” *JHEP* **0601** (2006) 055 [arXiv:hep-th/0510208].
- [66] H. Y. Chen, N. Dorey and K. Okamura, “Dyonic giant magnons,” *JHEP* **0609** (2006) 024 [arXiv:hep-th/0605155].

- [67] N. P. Bobev and R. C. Rashkov, “Multispin giant magnons,” *Phys. Rev. D* **74** (2006) 046011 [[arXiv:hep-th/0607018](https://arxiv.org/abs/hep-th/0607018)].
- [68] S. Ryang, “Three-spin giant magnons in  $AdS_5 \times S^5$ ,” *JHEP* **0612** (2006) 043.
- [69] K. Okamura and R. Suzuki, “A perspective on classical strings from complex sine-Gordon solitons,” *Phys. Rev. D* **75** (2007) 046001 [[arXiv:hep-th/0609026](https://arxiv.org/abs/hep-th/0609026)].
- [70] F. Lund and T. Regge, “Unified Approach To Strings And Vortices With Soliton Solutions,” *Phys. Rev. D* **14** (1976) 1524.
- [71] F. Lund, “Example Of A Relativistic, Completely Integrable, Hamiltonian System,” *Phys. Rev. Lett.* **38** (1977) 1175.
- [72] N. Dorey, “Magnon Bound States and the  $AdS/CFT$  Correspondence,” *J. Phys. A* **39** (2006) 13119 [[arXiv:hep-th/0604175](https://arxiv.org/abs/hep-th/0604175)].
- [73] C. G. Callan and J. M. Maldacena, “Brane dynamics from the Born-Infeld action,” *Nucl. Phys. B* **513** (1998) 198 [[arXiv:hep-th/9708147](https://arxiv.org/abs/hep-th/9708147)].
- [74] G. W. Gibbons, “Born-Infeld particles and Dirichlet p-branes,” *Nucl. Phys. B* **514** (1998) 603 [[arXiv:hep-th/9709027](https://arxiv.org/abs/hep-th/9709027)].
- [75] J. P. Gauntlett, J. Gomis and P. K. Townsend, “BPS bounds for world volume branes,” *JHEP* **9801** (1998) 003 [[arXiv:hep-th/9711205](https://arxiv.org/abs/hep-th/9711205)].
- [76] S. Lee, A. W. Peet and L. Thorlacius, “Brane waves and strings,” *Nucl. Phys. B* **514** (1998) 161 [[arXiv:hep-th/9710097](https://arxiv.org/abs/hep-th/9710097)].
- [77] R. Emparan, “Born-Infeld strings tunneling to D-branes,” *Phys. Lett. B* **423** (1998) 71 [[arXiv:hep-th/9711106](https://arxiv.org/abs/hep-th/9711106)].
- [78] D. M. Hofman and J. M. Maldacena, “Reflecting magnons,” *JHEP* **0711** (2007) 063 [[arXiv:0708.2272 \[hep-th\]](https://arxiv.org/abs/0708.2272)].
- [79] N. Mann, S. E. Vazquez, “Classical Open String Integrability,” *JHEP* **0704** (2007) 065. [[hep-th/0612038](https://arxiv.org/abs/hep-th/0612038)].

- [80] V. Balasubramanian, M. x. Huang, T. S. Levi and A. Naqvi, “Open strings from  $\mathcal{N} = 4$  super Yang-Mills,” *JHEP* **0208** (2002) 037 [arXiv:hep-th/0204196].
- [81] D. Berenstein and S. E. Vazquez, “Integrable open spin chains from giant gravitons,” *JHEP* **0506** (2005) 059 [arXiv:hep-th/0501078].
- [82] D. Berenstein, D. H. Correa and S. E. Vazquez, “Quantizing open spin chains with variable length: An example from giant gravitons,” *Phys. Rev. Lett.* **95** (2005) 191601 [arXiv:hep-th/0502172].
- [83] D. Berenstein, D. H. Correa and S. E. Vazquez, “A study of open strings ending on giant gravitons, spin chains and integrability,” *JHEP* **0609** (2006) 065 [arXiv:hep-th/0604123].
- [84] S. Hirano, “Fat magnon,” *JHEP* **0704** (2007) 010 [arXiv:hep-th/0610027].
- [85] D. Sadri and M. M. Sheikh-Jabbari, “Giant hedge-hogs: Spikes on giant gravitons,” *Nucl. Phys. B* **687** (2004) 161 [arXiv:hep-th/0312155].
- [86] D. Bak, “Zero Modes for the Boundary Giant Magnons,” *Phys. Lett. B* **672**, 284 (2009) [arXiv:0812.2645 [hep-th]].
- [87] H. Saleur, S. Skorik and N. P. Warner, “The Boundary sine-Gordon theory: Classical and semiclassical analysis,” *Nucl. Phys. B* **441** (1995) 421 [arXiv:hep-th/9408004].
- [88] P. Bowcock, G. Tzamtzis, “The Complex sine-Gordon model on a half line,” *JHEP* **0703** (2007) 047. [hep-th/0203139].
- [89] N. MacKay and V. Regelskis, “Yangian symmetry of the  $Y=0$  maximal giant graviton,” arXiv:1010.3761 [hep-th].
- [90] P. Bowcock, E. Corrigan, P. E. Dorey and R. H. Rietdijk, “Classically integrable boundary conditions for affine Toda field theories,” *Nucl. Phys. B* **445**, 469 (1995) [arXiv:hep-th/9501098].

- [91] E. Corrigan and Z. M. Sheng, “Classical integrability of the  $O(N)$  nonlinear sigma model on a half-line,” *Int. J. Mod. Phys. A* **12**, 2825 (1997) [arXiv:hep-th/9612150].
- [92] N. J. MacKay and C. A. S. Young, “Classically integrable boundary conditions for symmetric space sigma models,” *Phys. Lett. B* **588** (2004) 221 [arXiv:hep-th/0402182].
- [93] Z. Bajnok, L. Palla and G. Takacs, “(Semi)classical analysis of sine-Gordon theory on a strip,” *Nucl. Phys. B* **702** (2004) 448 [arXiv:hep-th/0406149].
- [94] G. Mussardo, V. Riva and G. Sotkov, “Semiclassical energy levels of sine-Gordon model on a strip with Dirichlet boundary conditions,” *Nucl. Phys. B* **705** (2005) 548 [arXiv:hep-th/0406246].
- [95] S. Ghoshal and A. B. Zamolodchikov, “Boundary S matrix and boundary state in two-dimensional integrable quantum field theory,” *Int. J. Mod. Phys. A* **9** (1994) 3841 [Erratum-ibid. A **9** (1994) 4353] [arXiv:hep-th/9306002].
- [96] M. Spradlin and A. Volovich, “Dressing the giant magnon,” *JHEP* **0610** (2006) 012 [arXiv:hep-th/0607009].
- [97] C. Kalousios, M. Spradlin and A. Volovich, “Dressing the giant magnon. II,” *JHEP* **0703** (2007) 020 [arXiv:hep-th/0611033].
- [98] C. Kalousios, G. Papathanasiou and A. Volovich, “Exact solutions for  $N$ -magnon scattering,” *JHEP* **0808** (2008) 095 [arXiv:0806.2466 [hep-th]].
- [99] R. Iengo and J. G. Russo, “Semiclassical decay of strings with maximum angular momentum,” *JHEP* **0303** (2003) 030 [arXiv:hep-th/0301109].
- [100] K. Peeters, J. Plefka and M. Zamaklar, “Splitting spinning strings in  $AdS/CFT$ ,” *JHEP* **0411** (2004) 054 [arXiv:hep-th/0410275].
- [101] K. Peeters, J. Plefka and M. Zamaklar, “Splitting strings and chains,” *Fortsch. Phys.* **53** (2005) 640 [arXiv:hep-th/0501165].

---

- [102] I. S. Gradshteyn, I. M. Ryzhik, “Table of Integrals, Series, and Products,” *Edited by A. Jeffrey and D. Zwillinger, Academic Press, New York, 7th edition, 2007*
- [103] P. F. Byrd, M. D. Friedman, “Handbook of elliptic integrals for engineers and physicists,” (*Die Grundlehren der mathematischen Wissenschaften, vol. 67.*) *Berlin, Springer, 1954*

# Appendix A

## Elliptic integrals and functions

The following elliptic functions and integrals are used in chapters 2, 3 and 4 and may be found described in great detail, for example, in [102] or [103].

### A.1 Elliptic integrals

The incomplete elliptic integral of the first kind is defined as

$$\int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^\varphi \frac{d\vartheta}{1-k^2 \sin^2(\vartheta)} \quad (\text{A.1.1})$$

$$= \int_0^u du \quad (\text{A.1.2})$$

$$= u \equiv \operatorname{sn}^{-1}(y, k) \equiv F(\varphi, k) \quad (\text{A.1.3})$$

where  $y = \sin(\varphi)$  and  $\varphi = \operatorname{am}(u)$  is named the amplitude function.

The incomplete elliptic integral of the second kind is defined as

$$\int_0^y \sqrt{\frac{1-k^2t^2}{1-t^2}} dt = \int_0^\varphi \sqrt{1-k^2 \sin^2(\vartheta)} d\vartheta \quad (\text{A.1.4})$$

$$= \int_0^u \operatorname{dn}^2(u) du \quad (\text{A.1.5})$$

$$= E(u) \equiv E(\operatorname{am}(u), k) \equiv E(\varphi, k). \quad (\text{A.1.6})$$

The constant  $k$  is named the elliptic modulus. In chapter 2 the “elliptic parameter”  $q \equiv k^2$  appears. A third elliptic integral may be defined but does not concern us in this thesis.

The *complete* elliptic integrals are special cases of the above when  $y = 1$ , or  $\varphi = \frac{\pi}{2}$ . The complete elliptic integral of the first kind is defined as

$$\int_0^{\frac{\pi}{2}} \frac{d\vartheta}{\sqrt{1 - k^2 \sin^2(\vartheta)}} = \int_0^K du \quad (A.1.7)$$

$$= F\left(\frac{\pi}{2}, k\right) \equiv K(k) \equiv K \quad (A.1.8)$$

while the complete elliptic integral of the second kind is defined as

$$\int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2(\vartheta)} d\vartheta = \int_0^K dn^2(u) du \quad (A.1.9)$$

$$= E\left(\frac{\pi}{2}, k\right) \equiv E(k) \equiv E. \quad (A.1.10)$$

Special values of these integral functions are

$$F(0, k) = 0, \quad E(0, k) = 0, \quad (A.1.11)$$

$$F(\varphi, 0) = \varphi, \quad E(\varphi, 0) = \varphi, \quad (A.1.12)$$

$$F(\varphi, 1) = \ln(\tan(\varphi) + \sec(\varphi)), \quad E(\varphi, 1) = \sin(\varphi). \quad (A.1.13)$$

$$F\left(\frac{\pi}{2}, 1\right) = K(1) = \infty, \quad (A.1.14)$$

$$E\left(\frac{\pi}{2}, 1\right) = E(1) = 1. \quad (A.1.15)$$

We also have

$$F(-\varphi, k) = -F(\varphi, k), \quad (A.1.16)$$

$$E(-\varphi, k) = -E(\varphi, k). \quad (A.1.17)$$

For values of the elliptic modulus  $k > 1$  we have the reciprocal modulus transform,

$$F(\varphi, k) = k_1 F(\varphi_1, k_1), \quad (A.1.18)$$

$$E(\varphi, k) = k_1 [k^2 E(\varphi_1, k_1) + k'^2 F(\varphi_1, k_1)] \quad (A.1.19)$$

where  $k'$  is the complementary modulus satisfying  $k'^2 = 1 - k^2$  and

$$k_1 = \frac{1}{k}, \quad \sin(\varphi_1) = k \sin(\varphi). \quad (A.1.20)$$

## A.2 Elliptic functions

The Jacobian elliptic functions are the inverses of the elliptic integrals. Using the amplitude function  $\varphi = \text{am}(u, k)$  the sines and cosines are defined as

$$\text{sn}(u, k) \equiv \sin(\varphi), \quad \text{cn}(u, k) \equiv \cos(\varphi). \quad (\text{A.2.21})$$

Further, a third function  $\text{dn}(u, k)$  is defined by

$$\text{dn}(u, k) \equiv \sqrt{1 - k^2 \sin^2(\varphi)}. \quad (\text{A.2.22})$$

Briefly, we write  $\text{sn}(u) \equiv \text{sn}(u, k)$  etc. The fundamental relations between these are

$$\text{sn}^2(u) + \text{cn}^2(u) = 1, \quad (\text{A.2.23})$$

$$k^2 \text{sn}^2(u) + \text{dn}^2(u) = 1, \quad (\text{A.2.24})$$

$$\text{dn}^2(u) - k^2 \text{cn}^2(u) = k'^2 \quad (\text{A.2.25})$$

$$k'^2 \text{sn}^2(u) + \text{cn}^2(u) = \text{dn}^2(u). \quad (\text{A.2.26})$$

We have

$$-1 \leq \text{sn}(u) \leq 1, \quad (\text{A.2.27})$$

$$-1 \leq \text{cn}(u) \leq 1, \quad (\text{A.2.28})$$

$$k' \leq \text{dn}(u) \leq 1. \quad (\text{A.2.29})$$

and special values

$$\text{am}(0) = 0, \quad (\text{A.2.30})$$

$$\text{sn}(0) = 0, \quad (\text{A.2.31})$$

$$\text{cn}(0) = 1, \quad (\text{A.2.32})$$

$$\text{dn}(0) = 1, \quad (\text{A.2.33})$$

$$\text{am}(K) = \frac{\pi}{2}, \quad (\text{A.2.34})$$

$$\text{sn}(K) = 1, \quad (\text{A.2.35})$$

$$\text{cn}(K) = 0, \quad (\text{A.2.36})$$

$$\text{dn}(K) = k'. \quad (\text{A.2.37})$$

The real periodicities and quasi-periodicities of the Jacobi elliptic functions are given by

$$\text{am}(u + 2mK) = m\pi + (-1)^m \text{am}(u), \quad (\text{A.2.38})$$

$$\text{sn}(u + 2mK) = (-1)^m \text{sn}(u), \quad (\text{A.2.39})$$

$$\text{cn}(u + 2mK) = (-1)^m \text{cn}(u), \quad (\text{A.2.40})$$

$$\text{dn}(u + 2mK) = \text{dn}(u). \quad (\text{A.2.41})$$

### A.3 Series representations

The complete elliptic integrals may be expanded about different limits of the modulus  $k$ . In this thesis we make use of the series representations of  $K(k)$  and  $E(k)$  about  $k = 0$  and  $k = 1$ . Around  $k = 1$  the expansion parameter is the complementary elliptic modulus

$$k' \equiv \sqrt{1 - k^2}. \quad (\text{A.3.42})$$

Around  $k = 0$  we have

$$K(k) = \frac{\pi}{2} \left\{ 1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \dots + \left[ \frac{(2n-1)!!}{2^n n!} \right] k^{2n} + \dots \right\}, \quad (\text{A.3.43})$$

$$E(k) = \frac{\pi}{2} \left\{ 1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - \dots - \left[ \frac{(2n-1)!!}{2^n n!} \right] \frac{k^{2n}}{(2n-1)} - \dots \right\}. \quad (\text{A.3.44})$$

The notation  $!!$  is defined by

$$(2n-1)!! \equiv 1 \cdot 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1), \quad (-1)!! = (0)!! = 1. \quad (\text{A.3.45})$$

Around  $k = 1$  ( $k' = 0$ ) we have

$$K(k') = \ln \left( \frac{4}{k'} \right) + \frac{1}{4} \left\{ \ln \left( \frac{4}{k'} \right) - 1 \right\} k'^2 + \dots \quad (\text{A.3.46})$$

$$E(k') = 1 + \frac{1}{2} \left\{ \ln \left( \frac{4}{k'} \right) - \frac{1}{2} \right\} k'^2 + \dots. \quad (\text{A.3.47})$$

## Appendix B

### N=3 solution on $\mathbb{R} \times S^2$

The notation here is that introduced at the start of sub-section 5.2.2 together with  $\tilde{\zeta}_k \equiv \zeta_k - \alpha_k$ . The components  $D$ ,  $N_1$  and  $N_2$  are to be composed as in (5.20) and (5.21).

$$\begin{aligned}
 D = 16e^{3it} \{ & -4s_2s_3(c_1 - c_2)(c_1 - c_3)\epsilon_1 \cosh(\tilde{\zeta}_1) \\
 & + 4s_1s_3(c_1 - c_2)(c_2 - c_3)\epsilon_2 \cosh(\tilde{\zeta}_2) - 4s_1s_2(c_1 - c_3)(c_2 - c_3)\epsilon_3 \cosh(\tilde{\zeta}_3) \\
 & + \epsilon_1\epsilon_2\epsilon_3 \left[ (1 - c_{1-2})(1 - c_{1-3})(1 - c_{2-3}) \cosh(\tilde{\zeta}_1 + \tilde{\zeta}_2 + \tilde{\zeta}_3) \right. \\
 & \quad + (1 - c_{1-2})(1 - c_{1+3})(1 - c_{2+3}) \cosh(\tilde{\zeta}_1 + \tilde{\zeta}_2 - \tilde{\zeta}_3) \\
 & \quad + (1 - c_{1+2})(1 - c_{1-3})(1 - c_{2+3}) \cosh(\tilde{\zeta}_1 - \tilde{\zeta}_2 + \tilde{\zeta}_3) \\
 & \quad \left. + (1 - c_{1+2})(1 - c_{1+3})(1 - c_{2-3}) \cosh(-\tilde{\zeta}_1 + \tilde{\zeta}_2 + \tilde{\zeta}_3) \right] \} \tag{B.0.1}
 \end{aligned}$$

$$\begin{aligned}
 N_1 = & 16e^{3it} \left\{ -4c_1s_2s_3(c_1 - c_2)(c_1 - c_3)\epsilon_1 \cosh(\tilde{\zeta}_1) \right. \\
 & + c_2s_1s_3(c_1 - c_2)(c_2 - c_3)\epsilon_2 \cosh(\tilde{\zeta}_2) - 4c_3s_1s_2(c_1 - c_3)(c_2 - c_3)\epsilon_3 \cosh(\tilde{\zeta}_3) \\
 & + \epsilon_1\epsilon_2\epsilon_3 \left[ c_{1+2+3}(1 - c_{1-2})(1 - c_{1-3})(1 - c_{2-3}) \cosh(\tilde{\zeta}_1 + \tilde{\zeta}_2 + \tilde{\zeta}_3) \right. \\
 & + c_{1+2-3}(1 - c_{1-2})(1 - c_{1+3})(1 - c_{2+3}) \cosh(\tilde{\zeta}_1 + \tilde{\zeta}_2 - \tilde{\zeta}_3) \\
 & + c_{1-2+3}(1 - c_{1+2})(1 - c_{1-3})(1 - c_{2+3}) \cosh(\tilde{\zeta}_1 - \tilde{\zeta}_2 + \tilde{\zeta}_3) \\
 & \left. \left. + c_{-1+2+3}(1 - c_{1+2})(1 - c_{1+3})(1 - c_{2-3}) \cosh(-\tilde{\zeta}_1 + \tilde{\zeta}_2 + \tilde{\zeta}_3) \right] \right\} \\
 & + i \left[ -4s_1s_2s_3 \left( (c_1 - c_2)(c_1 - c_3)\epsilon_1 \sinh(\tilde{\zeta}_1) \right. \right. \\
 & \left. \left. - (c_1 - c_2)(c_2 - c_3)\epsilon_2 \sinh(\tilde{\zeta}_2) + (c_1 - c_3)(c_2 - c_3)\epsilon_3 \sinh(\tilde{\zeta}_3) \right) \right. \\
 & + \epsilon_1\epsilon_2\epsilon_3 \left[ s_{1+2+3}(1 - c_{1-2})(1 - c_{1-3})(1 - c_{2-3}) \sinh(\tilde{\zeta}_1 + \tilde{\zeta}_2 + \tilde{\zeta}_3) \right. \\
 & + s_{1+2-3}(1 - c_{1-2})(1 - c_{1+3})(1 - c_{2+3}) \sinh(\tilde{\zeta}_1 + \tilde{\zeta}_2 - \tilde{\zeta}_3) \\
 & + s_{1-2+3}(1 - c_{1+2})(1 - c_{1-3})(1 - c_{2+3}) \sinh(\tilde{\zeta}_1 - \tilde{\zeta}_2 + \tilde{\zeta}_3) \\
 & \left. \left. + s_{-1+2+3}(1 - c_{1+2})(1 - c_{1+3})(1 - c_{2-3}) \sinh(-\tilde{\zeta}_1 + \tilde{\zeta}_2 + \tilde{\zeta}_3) \right] \right] \} \quad (B.0.2)
 \end{aligned}$$

$$\begin{aligned}
 N_2 = & 32ie^{4it} \left\{ -s_1s_2s_3 [(1 - c_{1+2})(1 - c_{1-2}) \right. \\
 & \left. + (1 - c_{1+3})(1 - c_{1-3}) + (1 - c_{2+3})(1 - c_{2-3})] \right. \\
 & + s_1(c_1 - c_2)(c_1 - c_3)\epsilon_2\epsilon_3 \left[ (1 - c_{2-3}) \cosh(\tilde{\zeta}_2 + \tilde{\zeta}_3) + (1 - c_{2+3}) \cosh(\tilde{\zeta}_2 - \tilde{\zeta}_3) \right] \\
 & - s_2(c_1 - c_2)(c_2 - c_3)\epsilon_1\epsilon_3 \left[ (1 - c_{1-3}) \cosh(\tilde{\zeta}_1 + \tilde{\zeta}_3) + (1 - c_{1+3}) \cosh(\tilde{\zeta}_1 - \tilde{\zeta}_3) \right] \\
 & \left. + s_3(c_1 - c_3)(c_2 - c_3)\epsilon_1\epsilon_2 \left[ (1 - c_{1-2}) \cosh(\tilde{\zeta}_1 + \tilde{\zeta}_2) + (1 - c_{1+2}) \cosh(\tilde{\zeta}_1 - \tilde{\zeta}_2) \right] \right\} \quad (B.0.3)
 \end{aligned}$$