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D3-brane supergravity solutions from Ricci-flat metrics on canonical bundles of Kähler–Einstein surfaces

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Abstract

D3 brane solutions of type IIB supergravity can be obtained by means of a classical Ansatz involving a harmonic warp factor, $H(\mathbf{y}, \bar{\mathbf{y}})$ multiplying at power $-1/2$ the first summand, i.e., the Minkowski metric of the D3 brane world-sheet, and at power $1/2$ the second summand, i.e., the Ricci-flat metric on a six-dimensional transverse space $\mathcal{M}_6$, whose complex coordinates y are the arguments of the warp factor. Of particular interest is the case where $\mathcal{M}_6 = \text{tot}[K\,[(\mathcal{M}_B)]$ is the total space of the canonical bundle over a complex Kähler surface $\mathcal{M}_B$. This situation emerges in many cases while considering the resolution à la Kronheimer of singular manifolds of type $\mathcal{M}_6 = \mathbb{C}^3/\Gamma$, where $\Gamma \subset \text{SU}(3)$ is a discrete subgroup. When $\Gamma = \mathbb{Z}_4$, the surface $\mathcal{M}_B$ is the second Hirzebruch surface endowed with a Kähler metric having $\text{SU}(2) \times \text{U}(1)$ isometry. There is an entire class $\text{Met}(\mathcal{FV})$ of such cohomogeneity one Kähler metrics parameterized by a single function $\mathcal{FK}(\mathfrak{v})$ that are best described in the Abreu–Martelli–Sparks–Yau (AMSY) symplectic formalism. We study in detail a two-parameter subclass $\text{Met}(\mathcal{FV})_{\text{KE}} \subset \text{Met}(\mathcal{FV})$ of Kähler–Einstein metrics of the aforementioned class, defined on manifolds that are homeomorphic to $S^2 \times S^2$, but are singular as complex manifolds. Actually, $\text{Met}(\mathcal{FV})_{\text{KE}} \subset \text{Met}(\mathcal{FV})_{\text{ext}} \subset \text{Met}(\mathcal{FV})$ is a subset of a four parameter subclass $\text{Met}(\mathcal{FV})_{\text{ext}}$ of cohomogeneity one extremal Kähler metrics originally introduced by Calabi in 1983 and translated by Abreu into the AMSY action-angle formalism. $\text{Met}(\mathcal{FV})_{\text{ext}}$ contains also a two-parameter subclass $\text{Met}(\mathcal{FV})_{\text{ext}\mathbb{F}_2}$ disjoint from $\text{Met}(\mathcal{FV})_{\text{KE}}$ of extremal smooth metrics on the second Hirzebruch surface that we rederive using constraints on period integrals of the Ricci 2-form. The Kähler–Einstein nature of the metrics in $\text{Met}(\mathcal{FV})_{\text{KE}}$ allows the construction of the Ricci-flat metric on their canonical bundle via the Calabi Ansatz, which we recast in the AMSY formalism deriving some new elegant formulae. The metrics

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in $\mathrm{Met}(\mathcal{FV})_{\mathrm{KE}}$ are defined on the base manifolds of $U(1)$ fibrations supporting the family of Sasaki–Einstein metrics SEmet_5 introduced by Gauntlett et al. (Adv Theor Math Phys 8:711–734, 2004), and already appeared in Gibbons and Pope (Commun Math Phys 66:267–290, 1979). However, as we show in detail using Weyl tensor polynomial invariants, the six-dimensional Ricci-flat metric on the *metric cone* of $\mathcal{M}_5 \in \mathrm{Met(SE)}_5$ is different from the Ricci-flat metric on $\mathrm{tot}[K\,[(\mathcal{M}_{\mathrm{KE}})]$ constructed via Calabi Ansatz. This opens new research perspectives. We also show the full integrability of the differential system of geodesics equations on $\mathcal{M}_B$ thanks to a certain conserved quantity which is similar to the Carter constant in the case of the Kerr metric.

Keywords D3-brane supergravity solutions · Ricci-flat metrics · Kähler-Einstein metrics · quotient singularity resolutions

Mathematics Subject Classification 32Q20 · 32Q25 · 81T30 · 83E50

Contents

1 Introduction

The paper [6] reported, within the context of quiver gauge theories, on some advances about a special aspect of the gauge/gravity correspondence, i.e., *the relevance of the generalized Kronheimer construction*[1] [8, 9] to the resolution of $\mathbb{C}^3/\Gamma$ singularities. In particular, on the basis of the structure of exact D3-brane supergravity solutions, the issue of the construction of a Ricci-flat metric on a smooth resolution Y^Γ of $\mathbb{C}^3/\Gamma$ was considered. The general framework there and in this paper is the problem of establishing holographic dual pairs, made of

1. a gauge theory living on a D3-brane world volume;
2. a classical D3-brane solution of type IIB supergravity in $D = 10$.

Quiver gauge theories have been extensively studied in the literature in this connection, see, e.g., [7, 23–25, 44]. Indeed, the quiver diagram is a powerful tool which encodes the data of a Kähler quotient describing the geometry of the six directions transverse to the brane. The possibility of testing the holographic principle [26–28, 38, 42] and resorting to the supergravity side of the correspondence in order to perform, *classically and in the bulk*, quantum calculations that pertain to the boundary gauge theory, is tightly connected with the quiver approach whenever the classical brane solution has a conformal point corresponding to a limiting geometry of the type

$$M_D = \mathrm{AdS}_{p+2} \times \mathrm{SE}^{D-p-2}.$$

[1] By "generalized Kronheimer construction," we mean taking the Kähler quotient together with the construction of the metric that it naturally carries, thus generalizing Kronheimer's construction of the resolutions of the ADE singularities together with a hyperkähler metric on them [40]. In the case at hand of course the metric is not hyperkähler.

Here AdS_{p+2} denotes the anti-de Sitter space of dimension $p + 2$, while SE^{D-p-2} is a Sasaki–Einstein manifold of dimension $D - p - 2$ [22].

A special class of quivers is that of McKay quivers, which are associated with the resolution of $\mathbb{C}^n/\Gamma$ quotient singularities by means of a Kronheimer-like or GIT construction [3, 40, 41], where Γ is a discrete subgroup in $\mathrm{SU}(n)$. The case $n = 2$ corresponds to ALE manifolds, the discrete group Γ being given by the ADE classification.[2]

The case $n = 3$ was studied from the mid 90 s [17–19, 35, 36, 39, 45–47]. The three-dimensional McKay correspondence provides a group theoretical prediction of the cohomology groups $\mathrm{H}^\bullet(Y^\Gamma, \mathbb{Q})$ of a crepant resolution Y^Γ of the quotient singularity $\mathbb{C}^3/\Gamma$. Such complex 3-folds carry a Ricci-flat Kähler metric whose asymptotics at infinity depends on whether the locus in $\mathbb{C}^3$ having nontrivial isotropy is compact (i.e., the origin) or not. In the first case one has an ALE asymptotics, while in the second case the asymptotics depends on the direction: this is the Quasi-ALE case (see, e.g., [37], Thm. 3.3). As it was stressed in [6], *the Ricci-flat metric is not necessarily the metric determined by the Kähler quotient.*

In [6] the relevant conceptual landscape was summarized as follows. The finite group $\Gamma \subset SU(3)$ singles out a McKay quiver diagram which determines

1. the gauge group $\mathcal{F}_\Gamma$;
2. the matter field content Φ^I of the gauge theory;
3. the representation of the gauge group factor on the matter fields Φ^I;
4. the possible (mass)-deformations of the superpotential $\mathcal{W}(\phi)$.

The Ricci-flat metric on Y in principle can be inferred, by means of a Monge–Ampère equation, from the Kähler metric on the exceptional compact divisor in the resolution of $\mathbb{C}^3/\Gamma$, which in turn is determined by the McKay quiver through the Kronheimer construction. In particular, as discussed in [6] for the case $\mathbb{C}^3/\mathbb{Z}_4$, although the Kronheimer metric on Y^Γ is not Ricci-flat, yet its restriction to the exceptional divisor provides the appropriate starting point for an iterative solution of a Monge–Ampère equation which determines the Ricci-flat metric.

1.1 The reversed view point of this paper

The explicit solution of the Monge–Ampère equation for the case of the smooth Y^Γ resolution of the orbifold $\mathbb{C}^3/\mathbb{Z}_4$, namely the summation of the series implied by the iterative approach, proved to be quite unmanageable. Only the case of a partial resolution of the singularity, which is the canonical bundle of the (singular) weighted projective space $\mathbb{WP}[1, 1, 2]$, is quite accessible, and was discussed in [6] utilizing the powerful AMSY (Abreu–Martelli–Sparks–Yau) symplectic formalism of [1, 10, 43], whose simple results, however, cannot always be explicitly transferred back to the standard complex formalism of Kähler geometry, as such a transcription involves the inversion of higher transcendental functions.

On the other hand, Calabi's paper [11] provides, in the standard complex formalism, a recipe for constructing a Kähler metric g_E on the total space of a holomorphic vector

[2] For a recent review of these matters see chapter 8 of [30].

bundle $E \to \mathcal{M}$, where $\mathcal{M}$ is a compact Kähler manifold, satisfying the following conditions:

C1: the restriction of g_E to the space tangent to the zero section of E coincides with a given Kähler–Einstein (KE) metric $g_{\mathcal{M}}$ on $\mathcal{M}$;
C2: the horizontal spaces given by the Chern connection of the metric g_E are the orthogonal complements of the tangent spaces to the fibers of E with respect to g_E;
C3: g_E restricts on every fiber of E to an Hermitian metric.

In the case of the singular variety $\mathbb{C}^3/\mathbb{Z}_4$, the resolved variety is the total space of the canonical bundle of the second Hirzebruch surface $\mathbb{F}_2$, and since $\mathbb{F}_2$ admits no KE metric, the Calabi Ansatz cannot be applied. On the other hand, as we recall in Sect. 1.3.1, the resolution of singularities admits no infinitesimal deformation of the complex structure, namely no (2, 1)-forms, and this is an obstacle for introducing fluxes in the supergravity D3-brane solution. In this paper, making use of the AMSY formalism, we explore the possibility of singling out Kähler metrics, serving as the starting point for the Calabi ansatz, within a family, here called $\mathrm{Met}(\mathcal{FV})$, of 4D Kähler metrics. This family was already introduced in paper [6], on the basis of previous results of Gauntlet, Martelli, Sparks and Waldram [32]. The distinctive features of the family $\mathrm{Met}(\mathcal{FV})$ are the following:

(a) the metrics in the family $\mathrm{Met}(\mathcal{FV})$ are *parameterized by a single function of one variable* $\mathcal{FK}(\mathfrak{v})$;
(b) for any choice of $\mathcal{FK}(\mathfrak{v})$, the corresponding metric $g_{\mathcal{FK}}$ is *Kählerian*;
(c) for any choice of $\mathcal{FK}(\mathfrak{v})$, the *isometry group* of $g_{\mathcal{FK}}$ is $\mathrm{SU}(2) \times \mathrm{U}(1)$ and the underlying 4-manifold $\mathcal{M}_B$ has cohomogeneity one;
(d) $\mathrm{Met}(\mathcal{FV})$ includes both *the Kronheimer metric* for the smooth surface $\mathbb{F}_2$ and for the weighted projective plane $\mathbb{WP}[1, 1, 2]$.

Relying on the AMSY formalism it is straightforward to impose the condition that $g_{\mathcal{FK}}$ is an Einstein metric. This yields a linear differential equation for the function $\mathcal{FK}(\mathfrak{v})$ whose general integral contains two parameters (integration constants)

$$0 \le \lambda_1 < \lambda_2 < \infty, \tag{1.1}$$

so that one arrives at a two-parameter subclass of Kähler Einstein metrics

$$\mathrm{Met}(\mathcal{FV})_{\mathrm{KE}} \subset \mathrm{Met}(\mathcal{FV})$$

each singled out by a choice of the parameters in Eq. (1.1) and defined on a 4-manifold that we label with the same parameters $\mathcal{M}_B^{[\lambda_1,\lambda_2]}$ (actually all these manifolds are homeomorphic to $S^2 \times S^2$, but the metrics and the related complex structures are singular; more precisely, the metrics display two conical singularities, as we show in Sect. 6.3). These metrics already appeared in [34] and also [32], see eq. (4.4) there (a change of coordinates is necessary to compare the two families of metrics).[3]

[3] We thank Dario Martelli who called our attention to the relation of these four-dimensional KE metrics with the family of five-dimensional Sasaki–Einstein metrics introduced in eq.(4.1) of the same paper [32].

Actually, the family of metrics $\mathrm{Met}(\mathcal{FV})$ is a subclass of a 4-parameter family $\mathrm{Met}(\mathcal{FV})_{\mathrm{ext}}$ of extremal Kähler metrics in four real dimensions that was derived by Calabi in 1982 [12].[4]

The family $\mathrm{Met}(\mathcal{FV})_{\mathrm{ext}}$ was discussed by Abreu in [2] using the AMSY action-angle coordinate formalism. In Sect. 4, we explain the precise form of the family $\mathrm{Met}(\mathcal{FV})_{\mathrm{ext}}$, and using the global constraints on the periods of the Ricci 2-form characterizing the $\mathbb{F}_2$ surface, we show that the 4-parameter family contains two disjoint subfamilies:

$$\mathrm{Met}(\mathcal{FV})_{\mathrm{ext}} \supset \mathrm{Met}(\mathcal{FV})_{\mathrm{KE}} \bigcup \mathrm{Met}(\mathcal{FV})_{ext\mathbb{F}_2}$$

where $\mathrm{Met}(\mathcal{FV})_{\mathrm{KE}}$ is the aforementioned family of KE metrics on the manifolds $\mathcal{M}_B^{[\lambda_1,\lambda_2]}$ with two conical singularities, while $\mathrm{Met}(\mathcal{FV})_{ext\mathbb{F}_2}$ is a 2-parameter family of extremal Kähler metrics on the smooth $\mathbb{F}_2$ Hirzebruch surface. A relevant question is whether the Kronheimer Kähler metric $g_{Kro}^{\mathbb{F}_2}$ on $\mathbb{F}_2$ is extremal or not. The answer is that it belongs to the family $\mathrm{Met}(\mathcal{FV})$, as we already know, yet it does not belong to the subclass $\mathrm{Met}(\mathcal{FV})_{\mathrm{ext}}$ as we explicitly show in the sequel.

The good news is that the Calabi Ansatz, originally formulated in the standard complex formalism, has a particularly simple and elegant transcription into the AMSY formalism, leading to compact general formulas for the Ricci-flat metrics on the 2-parameter class of KE manifolds $\mathcal{M}_B^{[\lambda_1,\lambda_2]}$. This allows us to write an explicit exact D3-brane solution of Type IIB supergravity for each given manifold.

It is in conjunction with the construction of this Ricci-flat metric that another important question arises. One might think that, in view of the relation between the KE metrics $g_{4-\mathrm{KE}}^{[\lambda_1,\lambda_2]}$ and the Sasaki–Einstein metrics $g_{5-\mathrm{Sasaki}}^{[\lambda_1,\lambda_2]}$ introduced in [32], the Ricci-flat metric obtained from the Calabi Ansatz (in action-angle coordinates) with $g_{4-\mathrm{KE}}^{[\lambda_1,\lambda_2]}$ as an input, and the Ricci-flat metric obtained from the metric cone on the Sasaki–Einstein $g_{5-\mathrm{Sasaki}}^{[\lambda_1,\lambda_2]}$ should be the same metric, modulo change of coordinates. Actually, this natural guess is false and we demonstrate it explicitly in Sect. 12 with direct and indirect arguments that utilize invariants constructed with the Weyl tensor. So one cannot freely assume that the Sasaki–Einstein metrics $g_{5-\mathrm{Sasaki}}^{[\lambda_1,\lambda_2]}$ of [32] can be utilized to determine the gauge theory dual to D3-brane solution of supergravity constructed via Calabi Ansatz.

Notwithstanding the relation between $g_{4-\mathrm{KE}}^{[\lambda_1,\lambda_2]}$ and $g_{5-\mathrm{Sasaki}}^{[\lambda_1,\lambda_2]}$ the reversed problem of the dual pair of theories is still there. We have, by construction, the exact classical solution of supergravity based on the Ricci-flat metric. In order to find the other member of the pair, namely the corresponding four-dimensional gauge theory, we should be able to derive the spectrum of the theory from a suitable quiver and a holomorphic superpotential. This part of the problem is open.

[4] We thank Miguel Abreu for informing us of this fact. Extremal metrics are by definition those that extremize Calabi's functional $\int \mathcal{R}_s[g]^2 \,\mathrm{vol}(g)$, where $\mathcal{R}_s$ is the scalar invariant curvature; the notion was introduced by Calabi in [12]. For a modern introduction, see, e.g., [48].

1.2 The geometry of the fourfolds

From a geometric point of view, it should be stressed that all 4-manifolds in the $\mathrm{Met}(\mathcal{FV})$ family we find in our analysis that are homeomorphic (not necessarily diffeomorphic) to the product $S^2 \times S^2$. Concerning the complex structure, for some choices of the $\mathcal{FK}(\mathfrak{v})$ function we obtain the second Hirzebruch surface $\mathbb{F}_2$; for other choices we have the aforementioned a 2-parameter family of KE manifolds, which of course must be singular. Our analysis in Sect. 6.3 shows indeed that they have a conical singularity as we have already stressed.

1.3 Further issues

1.3.1 About (2, 1)-forms

According to the result proved by Ito and Reid [17, 18, 36] and based on the concept of age the conjugacy classes in the group Γ,[5] the homology cycles of Y^Γ are all algebraic, so that all notrivial cohomology groups are of type (q, q). There is a correspondence between the cohomology classes of type (q, q) and the discrete group conjugacy classes of age q:

$$\dim H^{1,1}(Y^\Gamma) = \#\ \text{of junior conjugacy classes in } \Gamma;$$
$$\dim H^{2,2}(Y^\Gamma) = \#\ \text{of senior conjugacy classes in } \Gamma.$$

Note that $H^{3,3} = 0$ as Y^Γ is noncompact.

In [6] it was emphasized that the absence of harmonic forms of type (2, 1), i.e., the absence of infinitesimal deformations of the complex structure, is a serious obstacle to the construction of supergravity D3-brane solutions based on Y^Γ that have transverse 3-form fluxes.

It was pointed out in [6] that by means of the Gaussian integration of certain scalar fields predicted by the McKay Γ quiver, and clearly distinguished from the other scalar fields on a group theoretical basis, one gets a new quiver diagram that is not directly associated with a discrete group, yet follows from the McKay Γ quiver in a unique way. In this way the group theoretical approach allows one to identify deformations of the superpotential and hence allows for "deformations" of the crepant resolution. By construction, these deformed varieties will have nontrivial harmonic (2,1) forms.

In this way, one goes beyond the McKay correspondence. Both physically and mathematically this is quite interesting and provides a new viewpoint on several results, some of them well known in the literature. Most of the latter are about cyclic groups Γ and rely on the powerful tools of toric geometry. Yet the generalized Kronheimer construction applies also to non-Abelian groups $\Gamma \subset \mathrm{SU}(3)$ and so do Ito-Reid's results. Hence, available mass deformations are encoded also in the McKay quivers of non-Abelian groups Γ and one might explore the geometry of the transverse manifolds emerging in these cases.

[5] For a recent review of these matters within a general framework of applications to brane gauge theories see [8, 9].

In Sect. A, we give a negative solution to this problem for what concerns the varieties we consider in this paper, for which the group $H^{2,1}$ vanishes. As one may expect, we find that the manifolds do not carry nontrivial self-dual (2, 1)-forms. From the point of view of supergravity, this means that we can only build classical D3-brane solutions with a 5-form flux but no 3-form fluxes.

1.3.2 Sasaki–Einstein manifolds

Given the exact explicit solution for the Ricci-flat metric on the total spaces of the canonical bundles $\mathrm{tot}(\mathcal{K}_{\mathcal{M}_B^{[\lambda_1,\lambda_2]}})$, we can investigate their behavior at large distances from the corresponding base manifold $\mathcal{M}_B^{[\lambda_1,\lambda_2]}$. In principle this procedure should single out Sasaki–Einstein 5-manifold $\widetilde{\mathcal{M}_{SE}}$ which, at this point, we expect to be different from those considered [32]. The $\widetilde{\mathcal{M}_{SE}}$ might have some singularities, typically corresponding to the modding out of some discrete group $\widetilde{\Gamma}$ related to the original Γ in the ancestor orbifold $\mathbb{C}^3/\Gamma$.

2 D3-solutions of type IIB supergravity

For the reader's convenience, in this section we concisely collect the main formulas related to the D3-brane solution of Type IIB supergravity. For more explanations, the reader is referred to section 2 of [6].

We separate the ten coordinates of space-time into the following subsets[6]:

$$x^M = \begin{cases} x^\mu,\ \mu = 0,\ldots,3 & \text{real coordinates of the 3-brane world volume} \\ y^\tau,\ \alpha = 1,2,3 & \text{complex coordinates of the } Y \text{ variety} \end{cases}$$

and we make the following Ansatz for the metric:

$$\begin{aligned} ds^2_{[10]} &= H(\mathbf{y},\bar{\mathbf{y}})^{-\frac{1}{2}}\left(-\eta_{\mu\nu}\,dx^\mu\otimes dx^\nu\right) + H(\mathbf{y},\bar{\mathbf{y}})^{\frac{1}{2}}\left(\mathbf{g}^{\mathrm{RFK}}_{\alpha\bar{\beta}}\,dy^\alpha\otimes d\bar{y}^{\bar{\beta}}\right) \\ ds^2_Y &= \mathbf{g}^{\mathrm{RFK}}_{\alpha\bar{\beta}}\,dy^\alpha\otimes d\bar{y}^{\bar{\beta}} \\ \eta_{\mu\nu} &= \mathrm{diag}(+,-,-,-) \end{aligned} \tag{2.1}$$

where $\mathbf{g}^{\mathrm{RFK}}$ is the Kähler metric of the manifold Y:

$$\mathbf{g}^{\mathrm{RFK}}_{\alpha\bar{\beta}} = \partial_\alpha\,\partial_{\bar{\beta}}\,\mathcal{K}^{\mathrm{RFK}}(\mathbf{y},\bar{\mathbf{y}}), \tag{2.2}$$

[6] Latin indices are always frame indices referring to the vielbein formalism. Furthermore, we distinguish the 4 directions of the brane volume by using Latin letters from the beginning of the alphabet while the 3 complex transversal directions are denoted by Latin letters from the middle and the end of the alphabet. For the coordinate indices we utilize Greek letters and we do exactly the reverse: early Greek letters $\alpha,\beta,\gamma,\delta,\ldots$ refer to the 3 complex transverse directions while Greek letters from the second half of the alphabet $\mu,\nu,\rho,\sigma,\ldots$ refer to the D3 brane world volume directions as it is customary in $D=4$ field theories.

the real function $\mathcal{K}^{\mathrm{RFK}}(\boldsymbol{y},\bar{\boldsymbol{y}})$ being a suitable Kähler potential. It follows that

$$\det(g_{[10]}) = H(\mathbf{y},\bar{\mathbf{y}})\det(\mathbf{g}^{\mathrm{RFK}}).$$

Actually, the formalism which is best suited for our aims is the AMSY symplectic one, rather than using holomorphic coordinates. In terms of the vielbein, the Ansatz (2.1) corresponds to

$$V^A = \begin{cases} V^a = H(\mathbf{y},\bar{\mathbf{y}})^{-1/4}\,\mathrm{d}x^a & a = 0,1,2,3 \\ V^\ell = H(\mathbf{y},\bar{\mathbf{y}})^{1/4}\,\mathbf{e}^\ell & \ell = 4,5,6,7,8,9 \end{cases}$$

where $\mathbf{e}^\ell$ are the vielbein 1-forms of the manifold Y. The structure equations are (the hats denote quantities computed without the warp factor, i.e., with $H = 1$)

$$\begin{aligned} 0 &= d\,\mathbf{e}^i - \widehat{\omega}^{ij} \wedge \mathbf{e}^k\,\eta_{jk} \\ \widehat{R}^{ij} &= d\widehat{\omega}^{ij} - \widehat{\omega}^{ik} \wedge \widehat{\omega}^{\ell j}\,\eta_{k\ell} = \widehat{R}^{ij}_{\ell m}\,\mathbf{e}^\ell \wedge \mathbf{e}^m. \end{aligned}$$

The relevant property of the Y metric that we use in solving Einstein equations is that it is Ricci-flat:

$$\widehat{R}^{im}_{\ell m} = 0.$$

To derive our solution and discuss its supersymmetry properties, we need the explicit form of the spin connection for the full ten-dimensional metric (2.1) and the corresponding Ricci tensor. From the torsion equation, one can uniquely determine the solution:

$$\begin{aligned} \omega^{ab} &= 0 \\ \omega^{a\ell} &= \tfrac{1}{4}\,H^{-3/2}\,\mathrm{d}x^a \eta^{\ell k}\,\partial_k H \\ \omega^{\ell m} &= \widehat{\omega}^{\ell m} + \Delta\omega^{\ell m} \quad ; \quad \Delta\omega^{\ell m} = -\tfrac{1}{2}\,H^{-1}\,\mathbf{e}^\ell\,\eta^{mk}\,\partial_k H \end{aligned}$$

Inserting this result into the definition of the curvature 2-form we obtain

$$\begin{aligned} R^a_b &= -\tfrac{1}{8}\left[H^{-3/2}\Box_{\mathbf{g}} H - H^{-5/2}\,\partial_k H \partial^k H\right]\delta^a_b \\ R^a_\ell &= 0 \\ R^m_\ell &= \tfrac{1}{8}H^{-3/2}\Box_{\mathbf{g}} H \delta^m_\ell - \tfrac{1}{8}H^{-5/2}\partial_s H \partial^s H \delta^m_\ell + \tfrac{1}{4}H^{-5/2}\partial_\ell H \partial^m H \end{aligned}$$

where for any function $f(\boldsymbol{y},\bar{\boldsymbol{y}})$ on Y the equation

$$\Box_{\mathbf{g}}\, f(\mathbf{y},\bar{\mathbf{y}}) = \frac{1}{\sqrt{\det\mathbf{g}}}\left(\partial_\alpha\left(\sqrt{\det\mathbf{g}}\;\mathbf{g}^{\alpha\bar{\beta}}\,\partial_{\bar{\beta}}\, f\right)\right)$$

defines the Laplace–Beltrami operator with respect to the Ricci-flat metric (2.2); we have omitted the superscript RFK just for simplicity—on the supergravity side of the correspondence we shall only use the Ricci-flat metric and there will be no ambiguity.

The equations of motion for the scalar fields φ and $C_{[0]}$ and for the 3-form field strengths $F^{NS}_{[3]}$ and $F^{RR}_{[3]}$ can be better analyzed using the complex notation. Defining, as it is explained in [6] above:

$$\begin{aligned} \mathcal{H}_{\pm} &= \pm 2\, e^{-\varphi/2} F^{NS}_{[3]} + \mathrm{i}2\, e^{\varphi/2}\, F^{RR}_{[3]} \quad \Rightarrow \quad \overline{\mathcal{H}_{+}} = -\mathcal{H}_{-} \\ P &= \tfrac{1}{2}\, \mathrm{d}\varphi - \mathrm{i}\tfrac{1}{2}\, e^{\varphi}\, F^{RR}_{[1]} \end{aligned}$$

but also setting in our Ansatz

$$\varphi = 0 \quad ; \quad C_{[0]} = 0$$

we reduce the equations for the complex 3-forms to

$$\begin{aligned} \mathcal{H}_{+} \wedge \star\mathcal{H}_{+} &= 0 \\ d\star\mathcal{H}_{+} &= \mathrm{i}\, F^{RR}_{[5]} \wedge \mathcal{H}_{+} \end{aligned}$$

while the equation for the 5-form becomes

$$\mathrm{d}\star F^{RR}_{[5]} = \mathrm{i}\, \frac{1}{8}\, \mathcal{H}_{+} \wedge \mathcal{H}_{-}$$

The Ansatz for the complex 3-forms of type IIB supergravity is given below and is inspired by what was done in [4, 5] in the case where $Y = \mathbb{C} \times \mathrm{ALE}_{\Gamma}$:

$$\mathcal{H}_{+} = \Omega^{(2,1)}$$

where $\Omega^{(2,1)}$ lives on Y and satisfies

$$\star_{\mathbf{g}}\, \mathrm{Q}^{(2,1)} = -i\, \mathrm{Q}^{(2,1)}$$

As shown in [6] this guarantees that

$$\mathcal{H}_{+} \wedge \star_{10}\, \mathcal{H}_{+} = 0.$$

The Ansatz for $F^{RR}_{[5]}$ is

$$\begin{aligned} F^{RR}_{[5]} &= \alpha\, (U + \star_{10}\, U) \\ U &= d\left(H^{-1}\, \mathrm{Vol}_{\mathbb{R}^{(1,3)}}\right) \end{aligned}$$

where α is a constant to be determined later. By construction, $F^{RR}_{[5]}$ is self-dual and its equation of motion is trivially satisfied. What is not guaranteed is that also the Bianchi identity is fulfilled. Imposing it results into a differential equation for the function $H\,(\boldsymbol{y}, \bar{\boldsymbol{y}})$. Indeed, we obtain

$$d\, F^{RR}_{[5]} = \alpha\, \Box_{\mathbf{g}}\, H(\boldsymbol{y}, \bar{\boldsymbol{y}}) \; \times \; \mathrm{Vol}_{Y}$$

where

$$\mathrm{Vol}_{Y} = \sqrt{\det \mathbf{g}}\, \frac{1}{(3!)^2}\, \epsilon_{\alpha\beta\gamma}\, \mathrm{d}y^{\alpha} \wedge \mathrm{d}y^{\beta} \wedge \mathrm{d}y^{\gamma} \; \wedge \epsilon_{\bar{\alpha}\bar{\beta}\bar{\gamma}}\, \mathrm{d}\bar{y}^{\bar{\alpha}} \wedge \mathrm{d}\bar{y}^{\bar{\beta}} \wedge \mathrm{d}\bar{y}^{\bar{\gamma}}$$

is the volume form of the transverse six-dimensional manifold, i.e., the total space of the canonical bundle $K\,[\mathcal{M}_B]$. With our Ansatz we obtain

$$\begin{aligned} \tfrac{1}{8}\,\mathcal{H}_+ \wedge \mathcal{H}_- &= \mathbb{J}\,(\mathbf{y},\bar{\mathbf{y}}) \times \mathrm{Vol}_Y \\ \mathbb{J}\,(\mathbf{y},\bar{\mathbf{y}}) &= -\,\frac{1}{72\sqrt{\det\mathbf{g}}}\;\Omega_{\alpha\beta\bar{\eta}}\;\bar{\Omega}_{\bar{\delta}\bar{\theta}\gamma}\;\epsilon^{\alpha\beta\gamma}\;\epsilon^{\bar{\eta}\bar{\delta}\bar{\theta}} \end{aligned}$$

and we conclude that

$$\Box_{\mathbf{g}}\,H = -\frac{1}{\alpha}\,\mathbb{J}\,(\mathbf{y},\bar{\mathbf{y}}) \tag{2.3}$$

This is the main differential equation to which the entire construction of the D3-brane solution can be reduced. In [6] it was shown that the parameter α is determined by Einstein's equations and is fixed to $\alpha = 1$. With this value the field equations for the complex three forms simplify and reduce to the condition that $\Omega^{2,1}$ should be closed, and then, being anti-self-dual also co-closed, namely harmonic:

$$\tilde{\Omega}^{(2,1)} = \star_{\mathbf{g}}\Omega^{(2,1)} = -\,\mathrm{i}\,\Omega^{(2,1)} \quad ; \quad d\star_{\mathbf{g}}\Omega^{(2,1)} = 0 \quad ; \quad d\Omega^{(2,1)} = 0$$

In other words the solution of type IIB supergravity with 3-form fluxes exists *if and only if* the transverse space admits *closed and imaginary anti-self-dual forms* $\Omega^{(2,1)}$, as we already stated.

Summarizing, in order to construct a D3-brane solution of type IIB supergravity we need:

(a) to find a Ricci-flat Kähler metric $\mathbf{g}_{RFK}$ on the transverse 6D space Y;
(b) to verify if in the background of the metric $\mathbf{g}_{RFK}$ there exists a nonvanishing linear space of anti-self-dual (2,1)-forms $\Omega^{(2,1)}$. In the case of a positive answer, the 3-form $\mathcal{H}_+$ will be a linear combination of such forms; otherwise it will be zero.
(c) to solve the Laplacian equation for the harmonic function H which is homogeneous if there are no 3-form fluxes, otherwise it is inhomogeneous as in Eq. (2.3).

In the next section, we describe the AMSY symplectic formalism that will be propedeutic to the derivation of the KE family of 4D manifolds and later on to the derivation of the Ricci-flat metrics on their canonical bundles.

3 The AMSY symplectic formulation

Following the discussions and elaborations of [6] based on [1, 10, 43], given the Kähler potential of a toric complex n-dimensional Kähler manifold $\mathcal{K}(|z_1|, .., |z_n|)$, where $z_i = e^{x_i + i\Theta_i}$ are the complex coordinates, introducing the moment variables

$$\mu^i = \partial_{x_i}\mathcal{K}$$

we can obtain the so named symplectic potential by means of the Legendre transform:

$$G\,(\mu_i) = \sum_i^n x_i\,\mu^i \;-\; \mathcal{K}(|z_1|, .., |z_n|) \tag{3.1}$$

where one assumes that $\mathcal{K}$ only depends on the modules of the z coordinates to achieve $\mathrm{U}(1)^n$ invariance. The main issue involved in the use of Eq. (3.1) is the inversion transformation that expresses the coordinates x_i in terms of the moments μ^i. Once this is done one can calculate the metric in moment variables utilizing the Hessian:

$$G_{ij} = \frac{\partial^2}{\partial\mu^i\partial\mu^j}G\,(\mu) \tag{3.2}$$

and its matrix inverse. Call the n angles by Θ_i. Complex coordinates better adapted to the complex structure tensor can be defined a

$$u_i = e^{z_i} = \exp[x_i \;+\; \mathrm{i}\Theta_i]$$

The Kähler 2-form has the following universal structure:

$$\mathbb{K} \;=\; \sum_{i=1}^n \mathrm{d}\mu^i \wedge \mathrm{d}\Theta_i$$

and the metric is expressed as

$$\mathrm{d}s^2_{\mathrm{symp}} = \mathbf{G}_{ij}\mathrm{d}\mu^i\,\mathrm{d}\mu^j \;+\; \mathbf{G}^{-1}_{ij}\mathrm{d}\Theta^i\,\mathrm{d}\Theta^j \tag{3.3}$$

4 Kähler metrics with SU(2) x U(1) isometry

In this paper, we are interested, to begin with, in Kähler metrics in two complex dimensions, $n = 2$, where the complex coordinates u, v enter the Kähler potential $\mathcal{K}_0(\varpi)$ only through the real combination

$$\varpi = \left(1+\mid u\mid^2\right)^2\;\mid v\mid^2, \tag{4.1}$$

which guarantees invariance under $\mathrm{SU}(2)\times\mathrm{U}(1)$ transformations realized as

$$\begin{aligned} &\text{if } \mathbf{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SU}(2) \quad \text{then} \quad \mathbf{g}\,(u, v) = \left(\tfrac{a\,u+b}{c\,u+d},\quad v\;(c\,u+d)^2\right);\\ &\text{if } \mathbf{g} = \exp(i\,\theta_1) \;\in\; \mathrm{U}(1) \quad \text{then} \quad \mathbf{g}\,(u, v) = (u,\quad \exp(i\,\theta_1)\,v)\,. \end{aligned} \tag{4.2}$$

The above realization of the isometry captures the idea that, at least locally, the manifold is an S^2 fibration over S^2 (u being a coordinate on the base and v a fiber coordinate), although their global topology might be different and have some kind of singularities.

Two cases are of particular interest within such a framework, namely

(a) the singular weighted projective plane $\mathbb{WP}[1, 1, 2]$;
(b) the second Hirzebruch surface $\mathbb{F}_2$.

In the case of the singular variety $\mathbb{WP}[1, 1, 2]$ we have a nonKähler–Einstein metric that emerges from a partial resolution of the $\mathbb{C}^3/\mathbb{Z}_4$ singularity within the generalized Kronheimer construction (see [9],[6]) whose explicit Kähler potential is the following one:

$$\mathcal{K}_0^{Kr}(\varpi) = \frac{9}{4}\left(\frac{3\varpi + \sqrt{\varpi(\varpi+8)}}{\varpi + \sqrt{\varpi(\varpi+8)}} + \log\left(\varpi + \sqrt{\varpi(\varpi+8)} + 4\right) - 2 - 4\log(2)\right) \tag{4.3}$$

On the other hand the Kähler potential (4.3) is the particular case $\alpha = 0$ of a one-parameter family of Kähler potentials obtained from the Kronheimer construction:

$$\begin{aligned} K^{Kr}[\varpi, \alpha] = -\frac{9}{16}\Bigg(&-4(\alpha+1)\log\left[\sqrt{\alpha^2+6\alpha\varpi+\varpi^2+8\varpi}+3\alpha+\varpi+4\right] \\ &-\frac{4\left(\alpha\left(\sqrt{\alpha^2+6\alpha\varpi+\varpi(\varpi+8)}+2\varpi+1\right)+\sqrt{\alpha^2+6\alpha\varpi+\varpi(\varpi+8)}+\alpha^2+3\varpi\right)}{\sqrt{\alpha^2+6\alpha\varpi+\varpi(\varpi+8)}+\alpha+\varpi} \\ &+4\alpha\log\sqrt{\frac{\sqrt{\alpha^2+6\alpha\varpi+\varpi(\varpi+8)}+\alpha+\varpi}{\sqrt{\varpi}}}+8+16\log 2\Bigg) \end{aligned} \tag{4.4}$$

that for $\alpha > 0$ generate *bona fide* Kähler metrics on the second Hirzebruch surface $\mathbb{F}_2$.

4.1 A family of 4D Kähler metrics

Having mentioned the two explicit examples of Eqs. (4.3),(4.4), now, using the AMSY approach, we discuss in more general terms a class of real 4D Kähler manifolds, that we call $\mathcal{M}_B$. These are endowed with a metric invariant under the $\mathrm{SU}(2) \times \mathrm{U}(1)$ isometry group acting as in Eq. (4.2). The class of these manifolds is singled out by the above assumption that, in the complex formalism, their Kähler potential $\mathcal{K}_0(\varpi)$ is a function only of the invariant ϖ defined in Eq. (4.1). Then the explicit form of the Kähler potential $\mathcal{K}_0(\varpi)$ cannot be worked out analytically in all cases since the inverse Legendre transform involves the roots of higher order algebraic equations; yet, using the ϖ-dependence assumption, the Kähler metric can be explicitly worked out in the symplectic coordinates and has a simple and very elegant form—actually the metric depends on a single function of one variable $\mathcal{FK}(\mathfrak{v})$ which encodes all the geometric properties and substitutes $\mathcal{K}_0(\varpi)$. Posing all the questions in this symplectic language allows one to calculate all the geometric properties of the spaces in the class under consideration and leads also to new results and to a more systematic overview of the already known cases. We choose to treat the matter in general, by utilizing a *local approach* where we discuss the differential equations in a given open dense coordinate patch u, v, and we address the question of its global topological and algebraic structure only a posteriori, once the metric as been found in the considered chart, just as one typically does in General Relativity.

The symplectic structure of the metric on $\mathcal{M}_B$ is exhibited in the following way:

$$ds^2_{\mathcal{M}_B} = \mathbf{g}_{\mathcal{M}_B|\mu\nu}\, dq^\mu\, dq^\nu \quad ; \quad q^\mu = \{\mathfrak{u}, \mathfrak{v}, \phi, \tau\} \quad ; \quad \mathbf{g}_{\mathcal{M}_B} = \left(\begin{array}{c|c} \mathbf{G}_{\mathcal{M}_B} & \mathbf{0}_{2\times 2} \\ \hline \mathbf{0}_{2\times 2} & \mathbf{G}^{-1}_{\mathcal{M}_B} \end{array}\right) \tag{4.5}$$

where the Hessian $\mathbf{G}_{\mathcal{M}_B}$ is defined by:

$$\mathbf{G}_{\mathcal{M}_B} = \partial_{\mu^i}\, \partial_{\mu^j}\, G_{\mathcal{M}_B} \quad ; \quad \mu^i = \{\mathfrak{u}, \mathfrak{v}\} \tag{4.6}$$

and

$$G_{\mathcal{M}_B} = G_0(\mathfrak{u}, \mathfrak{v}) \; + \; \mathcal{D}(\mathfrak{v}) \tag{4.7}$$

$$G_0\,(\mathfrak{u}, \mathfrak{v}) = \left(\mathfrak{v} - \frac{\mathfrak{u}}{2}\right) \log(2\mathfrak{v} - \mathfrak{u}) + \frac{1}{2}\mathfrak{u}\log(\mathfrak{u}) - \frac{1}{2}\mathfrak{v}\log(\mathfrak{v}) \tag{4.8}$$

The specific structure (4.7), (4.8) is the counterpart within the symplectic formalism, via Legendre transform, of the assumption that the Kähler potential $\mathcal{K}_0(\varpi)$ depends only on the ϖ variable.

After noting this important point, we go back to the discussion of $\mathcal{M}_B$ geometry and stress that with the given isometries its Riemannian structure is completely encoded in the boundary function $\mathcal{D}(\mathfrak{v})$. All the other items in the construction are as follows. For the Kähler form we have

$$\mathbb{K}^{\mathcal{M}_B} = 2\;(d\mathfrak{u} \wedge d\phi + d\mathfrak{v} \wedge d\tau) \;=\; \mathbf{K}^{\mathcal{M}_B}_{\mu\nu}\, dq^\mu \wedge dq^\nu; \quad \mathbf{K}^{\mathcal{M}_B} = \left(\begin{array}{c|c} \mathbf{0}_{2\times 2} & \mathbf{1}_{2\times 2} \\ \hline -\mathbf{1}_{2\times 2} & \mathbf{0}_{2\times 2} \end{array}\right)$$

and for the complex structure we obtain

$$\mathfrak{J}^{\mathcal{M}_B} = \mathbf{K}^{\mathcal{M}_B}\, \mathbf{g}^{-1}_{\mathcal{M}_B} = \left(\begin{array}{c|c} \mathbf{0}_{2\times 2} & \mathbf{G}_{\mathcal{M}_B} \\ \hline -\mathbf{G}^{-1}_{\mathcal{M}_B} & \mathbf{0}_{2\times 2} \end{array}\right)$$

Explicitly, the 2×2 Hessian is the following:

$$\mathbf{G}_{\mathcal{M}_B} = \begin{pmatrix} -\frac{\mathfrak{v}}{\mathfrak{u}^2 - 2\mathfrak{u}\mathfrak{v}} & \frac{1}{\mathfrak{u} - 2\mathfrak{v}} \\ \frac{1}{\mathfrak{u} - 2\mathfrak{v}} & \frac{-2\mathfrak{v}(\mathfrak{u}-2\mathfrak{v})\mathcal{D}''(\mathfrak{v}) + \mathfrak{u} + 2\mathfrak{v}}{2\mathfrak{v}(2\mathfrak{v} - \mathfrak{u})} \end{pmatrix}$$

$$\mathbf{G}^{-1}_{\mathcal{M}_B} = \begin{pmatrix} \frac{\mathfrak{u}\left(-2\mathfrak{v}(\mathfrak{u}-2\mathfrak{v})\mathcal{D}''(\mathfrak{v}) + \mathfrak{u} + 2\mathfrak{v}\right)}{\mathfrak{v}\left(2\mathfrak{v}\mathcal{D}''(\mathfrak{v}) + 1\right)} & \frac{2\mathfrak{u}}{2\mathfrak{v}\mathcal{D}''(\mathfrak{v}) + 1} \\ \frac{2\mathfrak{u}}{2\mathfrak{v}\mathcal{D}''(\mathfrak{v}) + 1} & \frac{2\mathfrak{v}}{2\mathfrak{v}\mathcal{D}''(\mathfrak{v}) + 1} \end{pmatrix}$$

The family of metrics (4.5) is parameterized by the choice of a unique one-variable function:

$$f(\mathfrak{v}) \equiv \mathcal{D}''(\mathfrak{v}) \tag{4.9}$$

and is worth being considered in its own right.

4.2 The inverse Legendre transform

Before proceeding further with the analysis of this class of metrics, it is convenient to consider the inverse Legendre transform and see how one reconstructs the Kähler potential on $\mathcal{M}_B$. The inverse Legendre transform provides the Kähler potential through the formula:

$$\mathcal{K}_0 = x_u\, \mathfrak{u} + x_v\, \mathfrak{v} - G_{\mathcal{M}_B}(\mathfrak{u}, \mathfrak{v}) \tag{4.10}$$

where $G_{\mathcal{M}_B}(\mathfrak{u}, \mathfrak{v})$ is the base-manifold symplectic potential defined in Eq. (4.7), and

$$x_u = \partial_\mathfrak{u}\, G_{\mathcal{M}_B}(\mathfrak{u}, \mathfrak{v})\,; \quad x_v = \partial_\mathfrak{v}\, G_{\mathcal{M}_B}(\mathfrak{u}, \mathfrak{v})$$

which explicitly yields:

$$x_u = \frac{1}{2}\left(\log(\mathfrak{u}) - \log(2\mathfrak{v} - \mathfrak{u})\right); \quad x_v = \mathcal{D}'(\mathfrak{v}) + \log(2\mathfrak{v} - \mathfrak{u}) - \frac{1}{2}\log(\mathfrak{v}) + \frac{1}{2} \tag{4.11}$$

Using Eq. (4.11) in Eq. (4.10) we immediately obtain the explicit form of the base-manifold Kähler potential as a function of the moment $\mathfrak{v}$:

$$\mathcal{K}_0 = \mathfrak{K}_0(\mathfrak{v}) = \mathfrak{v}\left(\mathcal{D}'(\mathfrak{v}) + \frac{1}{2}\right) - \mathcal{D}(\mathfrak{v}) \tag{4.12}$$

The problem is that we need the Kähler potential $\mathcal{K}_0$ as a function of the invariant ϖ. Utilizing Eq. (4.11) it is fairly easy to obtain the expression of ϖ in terms of the moment $\mathfrak{v}$ for a generic function $\mathcal{D}(\mathfrak{v})$ that codifies the geometry of the base-manifold, obtaining

$$\varpi = (1 + \exp[2\, x_u])^2 \exp[2\, x_u] = \Omega(\mathfrak{v}) = 4\mathfrak{v}\exp[\,2\,\partial_\mathfrak{v}\mathcal{D}(\mathfrak{v}) + 1]$$

If one is able to invert the function $\Omega(\mathfrak{v})$, the original Kähler potential of the base-manifold can be written as:

$$\mathcal{K}_0(\varpi) = \mathfrak{K}_0 \circ \Omega^{-1}(\varpi)$$

The inverse function $\Omega^{-1}(\varpi)$ can be written explictly in some simple cases, but not always, and this inversion is the main reason why certain Kähler metrics can be much more easily found in the AMSY symplectic formalism which deals only with real variables than in the complex formalism. Since nothing good comes without paying a price, the metrics found in the symplectic approach require that the ranges of the variables $\mathfrak{u}$ and $\mathfrak{v}$ should be determined, since it is just in those ranges that the topology and algebraic structure of the underlying manifold is hidden; indeed, the ranges of $\mathfrak{u}$ and $\mathfrak{v}$ define a convex closed *polytope* in the $\mathbb{R}^2$ plane that encodes very precious information about the structure of the underlying manifold.

5 The Ricci tensor and the Ricci form

Calculating the Ricci tensor for the family of metrics (4.5) we obtain the following structure:

$$\mathrm{Ric}^{\mathcal{M}_B}_{\mu\nu} = \left(\begin{array}{c|c} \mathbf{P}_\mathrm{U} & \mathbf{0}_{2\times 2} \\ \hline \mathbf{0}_{2\times 2} & \mathbf{P}_\mathrm{D} \end{array}\right)$$

The expressions for $\mathbf{P}_\mathrm{U}$ and $\mathbf{P}_\mathrm{D}$ are quite lengthy and we omit them. We rather consider the Ricci 2-form defined by:

$$\mathbb{R}\mathrm{ic}_{\mathcal{M}_B} = \mathbf{Ric}^{\mathcal{M}_B}_{\mu\nu}\, dq^\mu \wedge dq^\nu$$

where:

$$\mathbf{Ric}^{\mathcal{M}_B} = \mathrm{Ric}^{\mathcal{M}_B}\,\mathfrak{J}^{\mathcal{M}_B} = \left(\begin{array}{c|c} \mathbf{0}_{2\times 2} & \mathbf{R} \\ \hline -\mathbf{R}^T & \mathbf{0}_{2\times 2} \end{array}\right)$$

and

$$\begin{aligned}
\mathbf{R} &= \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \\
r_{11} &= \frac{2\mathfrak{v}\left(\mathfrak{v}f'(\mathfrak{v})+2\mathfrak{v}f(\mathfrak{v})^2+f(\mathfrak{v})\right)-1}{2\mathfrak{v}(2\mathfrak{v}f(\mathfrak{v})+1)^2} \\
r_{12} &= 0 \\
r_{21} &= \frac{2\mathfrak{u}\mathfrak{v}\left(\mathfrak{v}^2(2\mathfrak{v}f(\mathfrak{v})+1)f''(\mathfrak{v})-4\mathfrak{v}^3f'(\mathfrak{v})^2+4\mathfrak{v}f'(\mathfrak{v})-4\mathfrak{v}^2f(\mathfrak{v})^3-2\mathfrak{v}f(\mathfrak{v})^2+3f(\mathfrak{v})\right)+\mathfrak{u}}{2\mathfrak{v}^2(2\mathfrak{v}f(\mathfrak{v})+1)^3} \\
r_{22} &= \frac{f(\mathfrak{v})\left(2\mathfrak{v}^2\left(\mathfrak{v}f''(\mathfrak{v})+f'(\mathfrak{v})\right)+3\right)+\mathfrak{v}\left(\mathfrak{v}f''(\mathfrak{v})-4\mathfrak{v}^2f'(\mathfrak{v})^2+5f'(\mathfrak{v})\right)+2\mathfrak{v}f(\mathfrak{v})^2}{(2\mathfrak{v}f(\mathfrak{v})+1)^3} \\
\mathbf{R}^T &= \mathbf{P}_\mathrm{D}\,\mathbf{G}^{-1}_{\mathcal{M}_B}
\end{aligned} \tag{5.1}$$

The last equation is not a definition but rather a consistency constraint (the Ricci tensor must be skew-symmetric).

5.1 A two-parameter family of KE metrics for $\mathcal{M}_B$

An interesting and legitimate question is whether this family of cohomogeneity one metrics that we named Met$(\mathcal{FV})$ in the Introduction contains KE ones. The answer is positive, and they make up a two-parameter subfamily. As we already claimed in the Introduction and as it will be shown in the next section, such a KE family Met$(\mathcal{FV})_{\mathrm{ext}}$ is a subfamily of a 4-parameter family of extremal metrics Met$(\mathcal{FV})_{\mathrm{ext}}$. Let us first directly retrieve the KE family from the appropriate differential constraint. A metric is KE if the Ricci 2-form is proportional to the Kähler 2-form:

$$\mathbb{R}\mathrm{ic}^{\mathcal{M}_B} = \frac{k}{4}\,\mathbb{K}^{\mathcal{M}_B} \tag{5.2}$$

where k is a constant. This amounts to requiring that the 2×2 matrix $\mathbf{R}$ displayed in Eq. (5.1) be proportional via $\frac{k}{4}$ to the identity matrix $\mathbf{1}_{2\times 2}$. This condition implies differential constraints on the function $f(\mathfrak{v})$ that are uniquely solved by the following function:

$$f(\mathfrak{v}) = \frac{-3\beta + k\mathfrak{v}^3 + 3\mathfrak{v}^2}{-2k\mathfrak{v}^4 + 6\mathfrak{v}^3 + 6\beta\mathfrak{v}} \tag{5.3}$$

the parameter β being the additional integration constant, while k is defined by Eq. (5.2). To retrieve the original symplectic potential $\mathcal{D}(\mathfrak{v})$ one has just to perform a double integration in the variable $\mathfrak{v}$. The explicit calculation of the integral requires a summation over the three roots $\lambda_{1,2,3}$ of the following cubic polynomial:

$$P(x) = x^3 - \frac{3x^2}{k} - \frac{3\beta}{k} \tag{5.4}$$

whose main feature is the absence of the linear term. Hence, a beautiful way of parameterizing the family of KE metrics is achieved by using as parameters two of the three roots of the polynomial (5.4). Let us call the independent roots λ_1 and λ_2. The polynomial (5.4) is reproduced by setting:

$$k = \frac{3(\lambda_1 + \lambda_2)}{\lambda_1^2 + \lambda_2\lambda_1 + \lambda_2^2}, \quad \beta = -\frac{\lambda_1^2\lambda_2^2}{\lambda_1^2 + \lambda_2\lambda_1 + \lambda_2^2}, \quad \lambda_3 = -\frac{\lambda_1\lambda_2}{\lambda_1 + \lambda_2} \tag{5.5}$$

Substituting (5.5) in Eq. (5.3) we obtain

$$f(\mathfrak{v}) = -\frac{\lambda_1\mathfrak{v}^2(\lambda_2+\mathfrak{v})+\lambda_2\mathfrak{v}^2(\lambda_2+\mathfrak{v})+\lambda_1^2(\lambda_2^2+\mathfrak{v}^2)}{2\mathfrak{v}(\mathfrak{v}-\lambda_1)(\mathfrak{v}-\lambda_2)(\lambda_2\mathfrak{v}+\lambda_1(\lambda_2+\mathfrak{v}))} \tag{5.6}$$

which is completely symmetrical in the exchange of the two independent roots λ_1, λ_2. Utilizing the expression (5.6) the double integration is easily performed, and we obtain the explicit result, where we omitted irrelevant linear terms:

$$\begin{aligned}\mathcal{D}^{\mathrm{KE}}(\mathfrak{v}) = &-\frac{\left(\lambda_1^2 + \lambda_2\lambda_1 + \lambda_2^2\right)(\mathfrak{v} - \lambda_1)\log(\mathfrak{v} - \lambda_1)}{\lambda_1^2 + \lambda_2\lambda_1 - 2\lambda_2^2} \\ &-\frac{\left(\lambda_1^2 + \lambda_2\lambda_1 + \lambda_2^2\right)(\mathfrak{v} - \lambda_2)\log(\mathfrak{v} - \lambda_2)}{-2\lambda_1^2 + \lambda_2\lambda_1 + \lambda_2^2} \\ &+\frac{\left(\lambda_1^2 + \lambda_2\lambda_1 + \lambda_2^2\right)(\lambda_2\mathfrak{v} + \lambda_1(\lambda_2 + \mathfrak{v}))\log(\lambda_2\mathfrak{v} + \lambda_1(\lambda_2 + \mathfrak{v}))}{(\lambda_1 + \lambda_2)\left(2\lambda_1^2 + 5\lambda_2\lambda_1 + 2\lambda_2^2\right)} - \frac{1}{2}\mathfrak{v}\log(\mathfrak{v})\end{aligned} \tag{5.7}$$

Comparing with the original papers on the AMSY formalism [1, 43], we note that the full symplectic potential for the 4-manifold $\mathcal{M}_B$ has precisely the structure of what is there called *natural symplectic potential*

$$G_{\mathrm{natural}} = \sum_{\ell=1}^{r} c_\ell\, p_\ell(\mathfrak{u}, \mathfrak{v}) \times \log\left[p_\ell(\mathfrak{u}, \mathfrak{v})\right]; \quad p_\ell(\mathfrak{u}, \mathfrak{v}) = \mathrm{linearfunctionsofthemoments}$$

(with $r = 4$ in our case). The only difference is that in [1] the coefficients c_ℓ are all equal while here they differ one from the other in a precise way that depends on the parameters λ_1, λ_2 defining the metric and the argument of the logarithms. As we are going to discuss later on, the same thing happens also for the non-KE metric on the second Hirzebruch surface $\mathbb{F}_2$ derived from the Kronheimer construction.

Next, we turn to a general discussion of the properties of the metrics in Met$(\mathcal{FV})$ and to the organization of the latter into special subfamilies. This will also clarify the precise location of the KE metrics in the general landscape.

6 Properties of the family of metrics

Let us then perform a complete study of the considered class of four-dimensional metrics. We do not start from a given manifold but rather from the family of metrics Met$(\mathcal{FV})$ parameterized by the choice of the function $\mathcal{FK}(\mathfrak{v})$ of one variable $\mathfrak{v}$, given explicitly in coordinate form. The first tasks we are confronted with are the definition of the maximal extensions of our coordinates, and the search of possible singularities in the metric and/or in the Riemannian curvature, which happens to be the cleanest probing tool. Secondly we can calculate integrals of the Ricci and Kähler 2-forms. All this information is easily computed since everything reduces to the evaluation of a few integral-differential functionals of the function $\mathcal{FK}(\mathfrak{v})$. Thirdly, we can construct geodesics relatively to the given metric and try to explore their behavior. This is probably the finest and most accurate tool to visualize the geometry of a manifold. We can also integrate the complex structure and find explicitly the complex coordinates.

We begin by observing that all the metrics deriving from the symplectic potential defined by Eqs. (4.7), (4.8)[7] admit a general form which we display below:

$$\mathrm{d}s^2_{\mathcal{M}_B} = \frac{\mathrm{d}\mathfrak{v}^2}{\mathcal{FK}(\mathfrak{v})} + \mathcal{FK}(\mathfrak{v})\,[\mathrm{d}\phi(1-\cos\theta)+\mathrm{d}\tau]^2 + \mathfrak{v}\underbrace{\left(\mathrm{d}\phi^2\sin^2\theta+\mathrm{d}\theta^2\right)}_{S^2\text{ metric}} \tag{6.1}$$

where we have defined

$$\mathcal{FK}(\mathfrak{v}) = \frac{2\mathfrak{v}}{2\mathfrak{v}\mathcal{D}''(\mathfrak{v})+1} \tag{6.2}$$

This expression for the metric is obtained performing a convenient change of variable:

$$\mathfrak{u} \to (1-\cos\theta)\ \mathfrak{v} \quad ; \quad \theta \in [0,\pi] \tag{6.3}$$

which automatically takes into account that $\mathfrak{u} \leq 2\,\mathfrak{v}$. Furthermore, this change of variables clearly reveals that all the three-dimensional sections of $\mathcal{M}_B$ obtained by fixing $\mathfrak{v} = \text{const}$ are S^1 fibrations on S^2 which is consistent with the isometry $\mathrm{SU}(2)\times\mathrm{U}(1)$. Indeed, all the spaces $\mathcal{M}_B$ have cohomogeneity equal to one and the moment variable $\mathfrak{v}$ is the only one whose dependence is not fixed by isometries.

The next important point is that the metric (6.1) is positive definite only in the interval of the positive $\mathfrak{v}$-axis where $\mathcal{FK}(\mathfrak{v}) \geq 0$. Let us name the lower and upper endpoints of such interval $\mathfrak{v}_{\min}$ and $\mathfrak{v}_{\max}$, respectively. If the interval $[\mathfrak{v}_{\min}, \mathfrak{v}_{\max}]$ is

[7] In this section which deals only with the base manifold $\mathcal{M}_B$ and where there is no risk of confusion we drop the suffix 0 in the moment variables, in order to make formulas simpler.

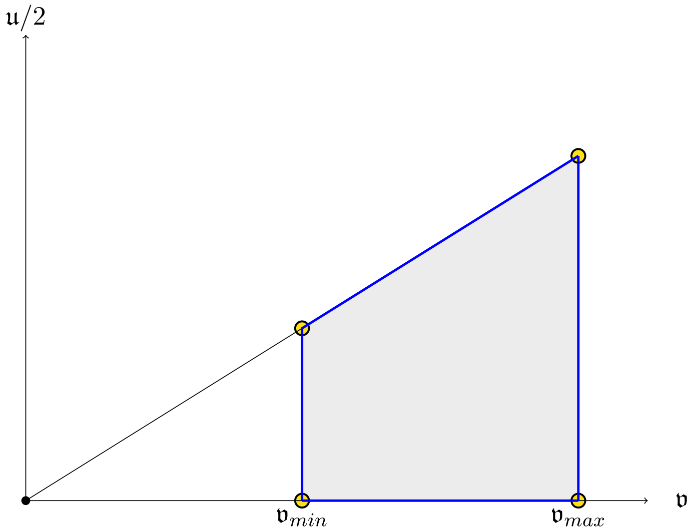

Fig. 1 The universal polytope in the $\mathfrak{v}$, $\frac{\mathfrak{u}}{2}$ plane for all the metrics of the $\mathcal{M}_B$ manifolds considered in this paper and defined in equation (6.1)

finite, then the space $\mathcal{M}_B$ is compact and the domain of the coordinates $\mathfrak{u}$, $\mathfrak{v}$ is provided by the trapezoidal polytope displayed in Fig. 1.

Our two main examples, which both correspond to the same universal polytope of Fig. 1, are provided by the case of the one-parameter family of *Kronheimer metrics* on the $\mathbb{F}_2$-surface, studied in [6, 9], whose Kähler potential was recalled in Eq. (4.4)) and by the family $\text{Met}(\mathcal{FV})_{\text{ext}}$ of *extremal Kähler metrics* due to Calabi and mentioned in the Introduction. In addition, within the first class, we have the degenerate case where the parameter α goes to zero and the trapezoid degenerates into a triangle. That case corresponds to the singular space $\mathcal{M}_B = \mathbb{WP}[1, 1, 2]$ (a weighted projective plane).

Extremal metrics are defined in the present cohomogeneity one case by the differential equation (see [2]):

$$\frac{\partial^2}{\partial \mathfrak{v}^2} \mathcal{R}_s(\mathfrak{v}) = 0 \tag{6.4}$$

where $\mathcal{R}_s(\mathfrak{v})$ is the scalar curvature.

Inserting in Eq. (6.4) the expression of $\mathcal{R}_s$ calculated later in Eq. (6.17) we obtain the following linear differential equation of order four:

$$\frac{\mathfrak{v}^2 \left(-\mathcal{FK}^{(3)}(\mathfrak{v})\right) + 2\mathfrak{v}\mathcal{FK}''(\mathfrak{v}) - 2\mathcal{FK}'(\mathfrak{v}) + 2}{\mathfrak{v}^3} - \frac{1}{2}\mathcal{FK}^{(4)}(\mathfrak{v}) = 0$$

whose general integral contains four integration constants (we name them $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$) and can be written as follows:

$$\mathcal{FK}_{\text{ext}}(\mathfrak{v}) = \frac{-\mathcal{A} + 8\,\mathcal{C}\mathfrak{v}^3 - 16\,\mathcal{D}\mathfrak{v}^4 + 4\mathfrak{v}^2 - 2\mathfrak{v}\,\mathcal{B}}{4\mathfrak{v}} \tag{6.5}$$

The explicit expression (6.5) is very much inspiring and useful. The function $\mathcal{FK}_{\text{ext}}(\mathfrak{v})$ is rational and it is the quotient of a quartic polynomial with a fixed coefficient of the quadratic term divided by the linear polynomial $4\,\mathfrak{v}$. A convenient way of parameterizing the entire family of metrics is therefore in terms of the four roots $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, as we did in Sect. 5.1 for the cubic polynomial of Eq. (5.4) (see Eq. (5.5)). Combining Eq. (5.3) with Eqs. (4.9) and (6.2) we obtain the expression of the function $\mathcal{FK}^{\text{KE}}$ corresponding to the KE metrics:

$$\mathcal{FK}^{\text{KE}}(\mathfrak{v}) = \frac{3\beta - k\mathfrak{v}^3 + 3\mathfrak{v}^2}{3\mathfrak{v}} \tag{6.6}$$

Comparing eq. (6.6) with eq. (6.5) we see that the KE metrics belong to the family of extremal metrics and are singled out by the constraint

$$\mathcal{A} = -4\beta \quad ; \quad \mathcal{B} = 0 \quad ; \quad \mathcal{C} = \frac{k}{6} \quad ; \quad \mathcal{D} = 0 \tag{6.7}$$

The most relevant aspect of eq. (6.7) is the suppression of the quartic and linear terms ($\mathcal{D} = \mathcal{B} = 0$), which fixes the number of free roots to two, as we know, the third being fixed in terms of λ_1, λ_2.

We have summarized the relevant choices of the function $\mathcal{FK}(\mathfrak{v})$ in Table 1. Inspecting this table we see that the functions $\mathcal{FK}^{\mathbb{F}_2}_{Kro}(\mathfrak{v})$ and $\mathcal{FK}^{\text{KE}}(\mathfrak{v})$ show strict similarities but also a difference which is expected to account for different topologies. In both cases the function $\mathcal{FK}(\mathfrak{v})$ is the ratio of a cubic polynomial having three real roots, two positive and one negative, and of a denominator that has no zeros in the $[\mathfrak{v}_{\min}, \mathfrak{v}_{\max}]$ interval. In the KE case there is a simple pole at $\mathfrak{v} = 0$ while for $\mathbb{F}_2$ (which is not KE) the denominator has two zeros and therefore $\mathcal{FK}(\mathfrak{v})$ has two simple poles at

$$\mathfrak{v}_{poles} = \frac{9}{32}\left[(3\alpha + 4) \pm 2\sqrt{2}\sqrt{\alpha^2 + 3\alpha + 2}\right]$$

These poles are out of the interval $[\mathfrak{v}_{\min}, \mathfrak{v}_{\max}]$ for any positive $\alpha > 0$, namely these poles do not correspond to points of the manifold $\mathcal{M}_B$, just as it is the case for the single pole $\mathfrak{v} = 0$ in the KE case. One also notes that the function $\mathcal{FK}^{\mathbb{F}_2}_{\text{ext}}(\mathfrak{v})$ cannot be reduced to the form (6.5) by any choice of the parameters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$, so that the smooth Kähler metric induced on the second Hirzebruch surface by the Kronheimer construction ([6, 9]) is not an extremal metric. A consistency check comes from the evaluation on $\mathcal{FK}^{\mathbb{F}_2}_{\text{ext}}(\mathfrak{v})$ of the scalar curvature provided by Eq. (6.17). In this case the scalar curvature is in no way linear in $\mathfrak{v}$, being a rational function of degree 6 in the numerator and of degree 7 in the denominator. The same is true of the limiting case $\alpha \to 0$. The last case ($\mathcal{FK}(\mathfrak{v}) = \mathfrak{v}$) corresponds to a metric cone on the 3-sphere, i.e., $\mathbb{C}^2/\mathbb{Z}_2$ with a flat metric. The case $\mathcal{FK}^{\text{KE}}_0$ will be discussed in Sect. 6.3.

Finally in Table 1 we observe the choice of the function

$$\mathcal{FK}^{\mathbb{F}_2}_{\text{ext}}(\mathfrak{v}) = \frac{(a-\mathfrak{v})(b-\mathfrak{v})\left(a^2(3b-\mathfrak{v}) + a\left(b^2 + 4b\mathfrak{v} + 3\mathfrak{v}^2\right) + b\mathfrak{v}(b+\mathfrak{v})\right)}{\mathfrak{v}\left(a^3 + 3a^2b - 3ab^2 - b^3\right)} \tag{6.8}$$

Table 1 $\mathcal{FK}$

$\mathcal{FK}^{\mathbb{F}_2}_{Kro}(\mathfrak{v}) = \frac{\left(1024\mathfrak{v}^2-81\alpha^2\right)(32\mathfrak{v}-9(3\alpha+4))}{16\left(81\alpha^2+1024\mathfrak{v}^2-576(3\alpha+4)\mathfrak{v}\right)}$	$\mathfrak{v}_{\min} = \frac{9\alpha}{32}$	$\mathfrak{v}_{\max} = \frac{9}{32}(3\alpha+4)$	$\alpha > 0$
[4mm] $\mathcal{FK}^{\mathbb{WP}[1,1,2]}_{Kro}(\mathfrak{v}) = \frac{\mathfrak{v}(8\mathfrak{v}-9)}{4\mathfrak{v}-9}$	$\mathfrak{v}_{\min} = 0$	$\mathfrak{v}_{\max} = \frac{9}{8}$	$\alpha = 0$
$\mathcal{FK}_{\text{ext}}(\mathfrak{v}) = \frac{-\mathcal{A}+8\mathcal{C}\mathfrak{v}^3-16\mathcal{D}\mathfrak{v}^4+4\mathfrak{v}^2-2\mathfrak{v}\mathcal{B}}{4\mathfrak{v}}$	$\mathfrak{v}_{\min} = \lambda_1^r$	$\mathfrak{v}_{\max} = \lambda_2^r$	$0 < \lambda_1^r < \lambda_2^r$
[4mm] $\underbrace{\mathcal{FK}^{\text{KE}}(\mathfrak{v})}_{\mathcal{B}=\mathcal{D}=0} = -\frac{(\mathfrak{v}-\lambda_1)(\mathfrak{v}-\lambda_2)(\lambda_2\mathfrak{v}+\lambda_1(\lambda_2+\mathfrak{v}))}{\left(\lambda_1^2+\lambda_2\lambda_1+\lambda_2^2\right)\mathfrak{v}}$	$\mathfrak{v}_{\min} = \lambda_1$	$\mathfrak{v}_{\max} = \lambda_2$	$0 < \lambda_1 < \lambda_2$
$\mathcal{FK}^{\text{KE}}_0(\mathfrak{v}) = \frac{\mathfrak{v}(\lambda_2-\mathfrak{v})}{\lambda_2}$	$\mathfrak{v}_{\min} = 0$	$\mathfrak{v}_{\max} = \lambda_2$	$\lambda_2 > 0$
[4mm] $\mathcal{FK}^{cone}(\mathfrak{v}) = \mathfrak{v}$	$\mathfrak{v}_{\min} = 0$	$\mathfrak{v}_{\max} = \infty$	
[4mm] $\mathcal{FK}^{\mathbb{F}_2}_{\text{ext}}(\mathfrak{v}) = \frac{(a-\mathfrak{v})(b-\mathfrak{v})\left(a^2(3b-\mathfrak{v})+a\left(b^2+4b\mathfrak{v}+3\mathfrak{v}^2\right)+b\mathfrak{v}(b+\mathfrak{v})\right)}{\mathfrak{v}\left(a^3+3a^2b-3ab^2-b^3\right)}$	$\mathfrak{v}_{\min} = a$	$\mathfrak{v}_{\max} = b$	$b > a > 0$

That above in Eq. (6.8) is a particular case of the general case $\mathcal{FK}_{\text{ext}}(\mathfrak{v})$, corresponding to the following choice of the parameters:

$$\begin{aligned}\mathcal{A} &= -\frac{4a^2b^2(3a+b)}{a^3+3a^2b-3ab^2-b^3}; \; \mathcal{B} = \frac{8a^3b}{a^3+3a^2b-3ab^2-b^3}\\ \mathcal{C} &= -\frac{2a^2}{a^3+3a^2b-3ab^2-b^3}; \; \mathcal{D} = \frac{3a+b}{4\left(-a^3-3a^2b+3ab^2+b^3\right)}\end{aligned} \tag{6.9}$$

where the parameters a, b are real, positive and naturally ordered $b > a > 0$. Wherefrom does the special form (6.9) originate? We claim that the metric defined by the function (6.8) is a smooth metric on the smooth $\mathbb{F}_2$ surface. The algebraic constraints that reduce the four parameters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ to the form (6.9) are derived from the conditions, already preliminarily discussed in [9] and specifically worked out in section 8.2.2 of [6], on the periods of the Ricci two-form localized on the standard toric homology cycles C_1 and C_2 of$\mathbb{F}_2$. Utilizing Eqs. (6.19) and (6.20) derived in the next Sect. 6.1 we have (see eq.(8.14) of [6]):

$$\mathbb{R}ic\;|_{C_1} = \mathfrak{A}(\mathfrak{v})\;\sin[\theta]\,\mathrm{d}\theta \wedge \mathrm{d}\phi; \quad \mathbb{R}ic\;|_{C_2} = \mathfrak{C}(\mathfrak{v})\,\mathrm{d}\theta \wedge \mathrm{d}\phi$$

the relevant functions being given in (6.20). The conditions on the periods are as follows:

$$\frac{1}{2\pi}\int_{C_1}\mathbb{R} = 0; \quad \frac{1}{2\pi}\mathbb{R} = 2 \tag{6.10}$$

that, as shown in section 8.2.2 of are automatically verified in [6]) are automatically verified by the Kronheimer metric.

The two conditions (6.10) become two statements on the functions $\mathfrak{A}(\mathfrak{v}), \mathcal{D}(\mathfrak{v})$ defined in eq. (6.20) and completely determined in terms of the function $\mathcal{FK}(\mathfrak{v})$ and its derivatives:

$$\mathfrak{A}[\mathfrak{v}_{\min}] = 0; \quad \int_{\mathfrak{v}_{\min}}^{\mathfrak{v}_{\max}} \mathfrak{D}[\mathfrak{v}]\mathrm{d}\mathfrak{v} = 2 \tag{6.11}$$

The result provided in Eq. (6.8) corresponding to the parameter choice (6.9) is deduced in the following way. First we re-parameterize the function (6.5) in terms of the four roots of the quartic polynomial appearing in the numerator that we name $\mu_1, \mu_2, \mu_3, \mu_4$ obtaining:

$$\begin{aligned}\mathcal{FK}_{\text{ext}}(\mathfrak{v}) = &\frac{\mu_1\left(\mathfrak{v}\left(\mu_3+\mathfrak{v}\right)\left(\mu_4+\mathfrak{v}\right)+\mu_2\left(\mu_3\left(\mathfrak{v}-\mu_4\right)+\mathfrak{v}\left(\mu_4+\mathfrak{v}\right)\right)\right)}{2\mathfrak{v}\left(\mathfrak{v}-\mu_1\right)\left(\mathfrak{v}-\mu_2\right)\left(\mathfrak{v}-\mu_3\right)\left(\mathfrak{v}-\mu_4\right)}\\ &+\frac{\mu_2\left(\mu_3+\mathfrak{v}\right)\left(\mu_4+\mathfrak{v}\right)+\mathfrak{v}\left(\mathfrak{v}\left(\mu_4-\mathfrak{v}\right)+\mu_3\left(\mu_4+\mathfrak{v}\right)\right)}{2\left(\mathfrak{v}-\mu_1\right)\left(\mathfrak{v}-\mu_2\right)\left(\mathfrak{v}-\mu_3\right)\left(\mathfrak{v}-\mu_4\right)}\end{aligned} \tag{6.12}$$

Secondly we rename $\mu_2 = a$,$\mu_3 = b$ deciding that $0 < a < b < \infty$ and we calculate the two conditions (6.11) using the function $\mathcal{FK}_{\text{ext}}(\mathfrak{v})$ in Eq. (6.12) as an input. We get a system of quadratic algebraic equations for the remaining roots μ_1, μ_4 that has the following solutions

$$\begin{aligned}\mu_2 &= a; \quad \mu_3 = b \\ \mu_1 &= \frac{a^2 - \left(b^2 \pm \sqrt{a^4 - 44a^3 b - 10a^2 b^2 + 4ab^3 + b^4}\right) - 4ab}{6a + 2b} \\ \mu_4 &= \frac{a^2 - \left(b^2 \mp \sqrt{a^4 - 44a^3 b - 10a^2 b^2 + 4ab^3 + b^4}\right) - 4ab}{6a + 2b}\end{aligned} \tag{6.13}$$

Substitution of Eq. (6.13) into Eq. (6.12) produces the function $\mathcal{FK}^{\mathbb{F}_2}_{\text{ext}}(\mathfrak{v})$ presented in (6.12) and recalled in Table 1. Furthermore, long as the roots μ_1, μ_4 as given above roots are complex conjugate of each other or, being real, do not fall in the interval $[a, b]$, the Kähler metric generated by the function $\mathcal{FK}^{F_2}_{\text{ext}}(\mathfrak{v})$ is smooth and well defined on the second Hirzebruch surface $\mathbb{F}_2$. The domain where this happens in the plane a, b can be easily studied looking at the discriminant under the square root in (6.13).

6.1 Vielbein formalism and the curvature 2-form of $\mathcal{M}_B$

The metric (6.1) is in diagonal form so it is easy to write a set of vierbein 1-forms. Indeed, if we set

$$\mathbf{e}^i = \left\{\frac{\mathrm{d}\mathfrak{v}}{\sqrt{\mathcal{FK}(\mathfrak{v})}},\ \sqrt{\mathcal{FK}(\mathfrak{v})}\ [\mathrm{d}\phi(1-\cos\theta) + \mathrm{d}\tau]\,,\ \sqrt{\mathfrak{v}}\,\mathrm{d}\theta,\ \sqrt{\mathfrak{v}}\,\mathrm{d}\phi\ \sin\theta\right\} \tag{6.14}$$

the line element (6.1) reads

$$\mathrm{d}s^2_B = \sum_{i=1}^{4} \mathbf{e}^i \otimes \mathbf{e}^i$$

Furthermore, we can calculate the matrix vielbein and its inverse quite easily, obtaining:

$$\begin{aligned}\mathbf{e}^i &= E^i_\mu\, \mathrm{d}y^\mu \quad ; \quad y^\mu = \{\mathfrak{v}, \theta, \phi, \tau\} \\ E^i_\mu &= \begin{pmatrix} \frac{1}{\sqrt{\mathcal{FK}(\mathfrak{v})}} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\mathcal{FK}(\mathfrak{v})}(1-\cos\theta) & \sqrt{\mathcal{FK}(\mathfrak{v})} \\ 0 & \sqrt{\mathfrak{v}} & 0 & 0 \\ 0 & 0 & \sqrt{\mathfrak{v}}\sin\theta & 0 \end{pmatrix} \\ E^\nu_j &= \begin{pmatrix} \sqrt{\mathcal{FK}(\mathfrak{v})} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{\mathfrak{v}}} & 0 \\ 0 & 0 & 0 & \frac{\csc(\theta)}{\sqrt{\mathfrak{v}}} \\ 0 & \frac{1}{\sqrt{\mathcal{FK}(\mathfrak{v})}} & 0 & \frac{(\cos\theta - 1)\csc\theta}{\sqrt{\mathfrak{v}}} \end{pmatrix}\end{aligned}$$

By means of the MATHEMATICA package VIELBGRAV23[8] we can easily calculate the Levi-Civita spin connection and the curvature 2-form from the definitions

$$0 = \mathfrak{T}^i = d\mathbf{e}^i + \omega^{ij} \wedge \mathbf{e}^j \quad ; \quad \mathfrak{R}^{ij} = d\omega^{ij} + \omega^{ik} \wedge \omega^{kj} = \mathcal{R}^{ij}_{k\ell}\, \mathbf{e}^k \wedge \mathbf{e}^\ell$$

obtaining

$$\begin{aligned}
\mathfrak{R}^{12} &= -\frac{\mathcal{FK}''(\mathfrak{v})}{2}\mathbf{e}^1 \wedge \mathbf{e}^2 - \frac{(\mathfrak{v}\mathcal{FK}'(\mathfrak{v}) - \mathcal{FK}(\mathfrak{v}))}{2\mathfrak{v}^2}\mathbf{e}^3 \wedge \mathbf{e}^4 \\
\mathfrak{R}^{13} &= -\frac{(\mathfrak{v}\mathcal{FK}'(\mathfrak{v}) - \mathcal{FK}(\mathfrak{v}))}{4\mathfrak{v}^2}\mathbf{e}^1 \wedge \mathbf{e}^3 - \frac{(\mathfrak{v}\mathcal{FK}'(\mathfrak{v}) - \mathcal{FK}(\mathfrak{v}))}{4\mathfrak{v}^2}\mathbf{e}^2 \wedge \mathbf{e}^4 \\
\mathfrak{R}^{14} &= \frac{(\mathfrak{v}\mathcal{FK}'(\mathfrak{v}) - \mathcal{FK}(\mathfrak{v}))}{4\mathfrak{v}^2}\mathbf{e}^2 \wedge \mathbf{e}^3 - \frac{(\mathfrak{v}\mathcal{FK}'(\mathfrak{v}) - \mathcal{FK}(\mathfrak{v}))}{4\mathfrak{v}^2}\mathbf{e}^1 \wedge \mathbf{e}^4 \\
\mathfrak{R}^{23} &= \frac{(\mathfrak{v}\mathcal{FK}'(\mathfrak{v}) - \mathcal{FK}(\mathfrak{v}))}{4\mathfrak{v}^2}\mathbf{e}^1 \wedge \mathbf{e}^4 - \frac{(\mathfrak{v}\mathcal{FK}'(\mathfrak{v}) - \mathcal{FK}(\mathfrak{v}))}{4\mathfrak{v}^2}\mathbf{e}^2 \wedge \mathbf{e}^3 \\
\mathfrak{R}^{24} &= -\frac{(\mathfrak{v}\mathcal{FK}'(\mathfrak{v}) - \mathcal{FK}(\mathfrak{v}))}{4\mathfrak{v}^2}\mathbf{e}^1 \wedge \mathbf{e}^3 - \frac{(\mathfrak{v}\mathcal{FK}'(\mathfrak{v}) - \mathcal{FK}(\mathfrak{v}))}{4\mathfrak{v}^2}\mathbf{e}^2 \wedge \mathbf{e}^4 \\
\mathfrak{R}^{34} &= \frac{(\mathfrak{v} - \mathcal{FK}(\mathfrak{v}))}{\mathfrak{v}^2}\mathbf{e}^3 \wedge \mathbf{e}^4 - \frac{(\mathfrak{v}\mathcal{FK}'(\mathfrak{v}) - \mathcal{FK}(\mathfrak{v}))}{2\mathfrak{v}^2}\mathbf{e}^1 \wedge \mathbf{e}^2
\end{aligned} \tag{6.15}$$

Equation (6.15) shows that the Riemann tensor $\mathcal{R}^{ij}_{k\ell}$ is constructed in terms of only three functions:

$$\mathcal{CF}_1(\mathfrak{v}) = \mathcal{FK}''(\mathfrak{v}) \quad ; \quad \mathcal{CF}_2(\mathfrak{v}) = \frac{(\mathfrak{v}\mathcal{FK}'(\mathfrak{v}) - \mathcal{FK}(\mathfrak{v}))}{\mathfrak{v}^2} \quad ; \quad \mathcal{CF}_3(\mathfrak{v}) = \frac{(\mathfrak{v} - \mathcal{FK}(\mathfrak{v}))}{\mathfrak{v}^2} \tag{6.16}$$

If these functions are regular in the interval $[\mathfrak{v}_{\min}, \mathfrak{v}_{\max}]$ the Riemann tensor is well defined and finite in the entire polytope of Fig. 1 and $\mathcal{M}_B$ should be a smooth compact manifold. From the expression (6.15) the MATHEMATICA CODE immediately derives the Riemann and Ricci tensors and the curvature scalar. This latter reads as follows:

$$\mathcal{R}_s = -\frac{\mathfrak{v}\mathcal{FK}''(\mathfrak{v}) + 2\mathcal{FK}'(\mathfrak{v}) - 2}{2\mathfrak{v}} \tag{6.17}$$

and its form was used above to define the extremal metrics. Similarly in the anholonomic vielbein basis, the Ricci tensor takes the following form:

$$\mathcal{R}_{ij} = \begin{pmatrix} \frac{\mathcal{FK}(\mathfrak{v}) - \mathfrak{v}(\mathfrak{v}\mathcal{FK}''(\mathfrak{v}) + \mathcal{FK}'(\mathfrak{v}))}{4\mathfrak{v}^2} & 0 & 0 & 0 \\ 0 & \frac{\mathcal{FK}(\mathfrak{v}) - \mathfrak{v}(\mathfrak{v}\mathcal{FK}''(\mathfrak{v}) + \mathcal{FK}'(\mathfrak{v}))}{4\mathfrak{v}^2} & 0 & 0 \\ 0 & 0 & -\frac{\mathfrak{v}(\mathcal{FK}'(\mathfrak{v}) - 2) + \mathcal{FK}(\mathfrak{v})}{4\mathfrak{v}^2} & 0 \\ 0 & 0 & 0 & -\frac{\mathfrak{v}(\mathcal{FK}'(\mathfrak{v}) - 2) + \mathcal{FK}(\mathfrak{v})}{4\mathfrak{v}^2} \end{pmatrix}$$

All the metrics in the considered family are of cohomogeneity one and have the same isometry; furthermore, they are all Kähler and share the same Kähler 2-form that can

[8] VIELBGRAV23 is a MATHEMATICA package for the calculation of the spin connection the curvature 2-form and the intrinsic components of the Riemann tensor in vielbein formalism. Constantly updated, it was originally written by one us (P.F.), almost thirty years ago. It can be furnished upon request and it will be at disposal on the De Gruyter site for the readers of the forthcoming book [31].

be written as follows:

$$\mathbb{K} = \mathrm{d}\mathfrak{u} \wedge \mathrm{d}\phi + \mathrm{d}\mathfrak{v} \wedge \mathrm{d}\tau = \mathbf{e}^1 \wedge \mathbf{e}^2 + \mathbf{e}^3 \wedge \mathbf{e}^4 = \frac{1}{2} \mathfrak{J}_{ij} \mathbf{e}^i \wedge \mathbf{e}^j \tag{6.18}$$

where:

$$\mathfrak{J}^i_j = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \delta^{ik} \mathfrak{J}_{kj}$$

is the complex structure in flat indices. Utilizing $\mathfrak{J}^i_j$ the Ricci 2-form is defined by:

$$\mathbb{R}\mathrm{ic} = \mathbb{R}_{ij} \mathbf{e}^i \wedge \mathbf{e}^j \quad ; \quad \mathbb{R}_{ij} = \mathcal{R}_{i\ell} \mathfrak{J}^\ell_j$$

and explicitly one obtains:

$$\mathbb{R}\mathrm{ic} = \mathfrak{A}(\mathfrak{v}) \sin[\theta] \mathrm{d}\theta \wedge \mathrm{d}\phi + \mathfrak{B}(\mathfrak{v}) (1 - \cos[\theta]) \mathrm{d}\mathfrak{v} \wedge \mathrm{d}\phi + \mathfrak{C}(\mathfrak{v}) \mathrm{d}\mathfrak{v} \wedge \mathrm{d}\tau \tag{6.19}$$

In Eq. (6.19) the functions of $\mathfrak{v}$ are the following ones:

$$\begin{aligned} \mathfrak{A}(\mathfrak{v}) &= -\frac{\mathfrak{v}\left(\mathcal{FK}'(\mathfrak{v}) - 2\right) + \mathcal{FK}(\mathfrak{v})}{2\mathfrak{v}} \\ \mathfrak{B}(\mathfrak{v}) &= -\frac{\mathcal{FK}(\mathfrak{v}) - \mathfrak{v}\left(\mathfrak{v}\mathcal{FK}''(\mathfrak{v}) + \mathcal{FK}'(\mathfrak{v})\right)}{2\mathfrak{v}^2} \\ \mathfrak{C}(\mathfrak{v}) &= \frac{\mathfrak{v}^2\left(-\mathcal{FK}''(\mathfrak{v})\right) - \mathfrak{v}\mathcal{FK}'(\mathfrak{v}) + \mathcal{FK}(\mathfrak{v})}{2\mathfrak{v}^2} \end{aligned} \tag{6.20}$$

6.1.1 The $\mathbb{F}_2$ Kronheimer case

In the $\mathbb{F}_2$ case with the "Kronheimer" metric we have:

$$\begin{aligned} \mathcal{CF}_1^{\mathbb{F}_2}(\mathfrak{v}) &= \frac{331776(\alpha+1)(\alpha+2)\left(729\alpha^2(3\alpha+4)+16384\mathfrak{v}^3-3888\alpha^2\mathfrak{v}\right)}{\left(81\alpha^2+1024\mathfrak{v}^2-576(3\alpha+4)\mathfrak{v}\right)^3} \\ \mathcal{CF}_2^{\mathbb{F}_2}(\mathfrak{v}) &= -\frac{9}{32\mathfrak{v}^2\left(81\alpha^2+1024\mathfrak{v}^2-576(3\alpha+4)\mathfrak{v}\right)^2} \times \left(6561\alpha^4(3\alpha+4) + 1048576(3\alpha+4)\mathfrak{v}^4\right. \\ &\quad \left. -1179648\alpha^2\mathfrak{v}^3 + 497664\alpha^2(3\alpha+4)\mathfrak{v}^2 - 93312\alpha^2(3\alpha+4)^2\mathfrak{v}\right) \\ \mathcal{CF}_3^{\mathbb{F}_2}(\mathfrak{v}) &= \frac{\mathfrak{v} - \dfrac{\left(1024\mathfrak{v}^2 - 81\alpha^2\right)(32\mathfrak{v} - 9(3\alpha+4))}{16\left(81\alpha^2 + 1024\mathfrak{v}^2 - 576(3\alpha+4)\mathfrak{v}\right)}}{\mathfrak{v}^2} \end{aligned}$$

The three functions $\mathcal{CF}_{1,2,3}^{\mathbb{F}_2}(\mathfrak{v})$ are smooth in the interval $\left(\frac{9\alpha}{32}, \frac{9}{32}(3\alpha+4)\right)$ and they are defined at the endpoints: see for instance Fig. 2A.

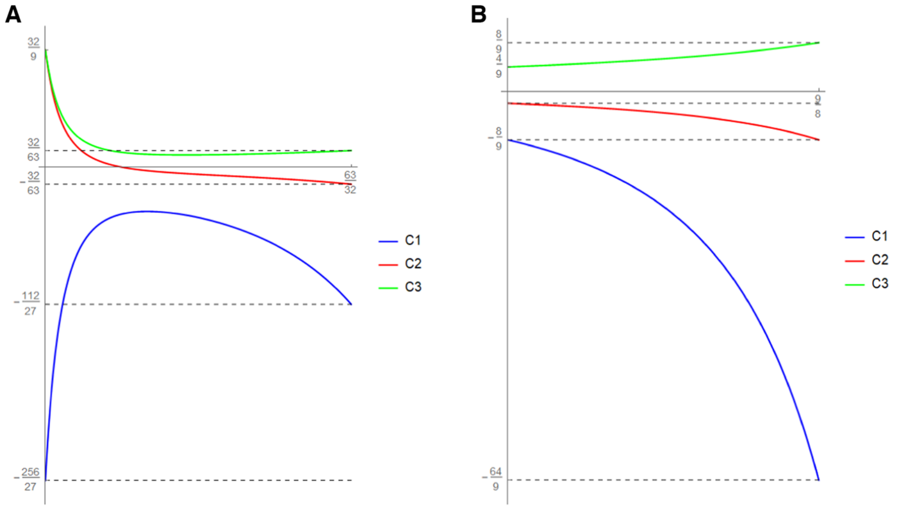

Fig. 2 **A** (left): Plot of the three functions $\mathcal{CF}^{\mathbb{F}_2}_{1,2,3}(\mathfrak{v})$ entering the intrinsic Riemann curvature tensor for the "Kronheimer" metric on $\mathbb{F}_2$ with the choice of the parameter $\alpha = 1$. **B** (right): Plot of the three functions $\mathcal{CF}^{\mathbb{WW}112}_{1,2,3}(\mathfrak{v})$ entering the intrinsic Riemann curvature tensor for the Kronheimer metric on $\mathbb{WP}[1,1,2]$ with the choice of the parameter $\alpha = 0$. Comparing this picture with the one on the left, we see the discontinuity. In all smooth cases, the functions $\mathcal{CF}^{\mathbb{F}_2}_{2,3}$ attain the same value in the lower endpoint of the interval while for the singular case of the weighted projective space, the initial values of $\mathcal{CF}^{\mathbb{WP}[1,1,2]}_{2,3}(\mathfrak{v})$ are different

Indeed, the values of the three functions at the endpoints are

$$
\begin{aligned}
\mathcal{CF}^{\mathbb{F}_2}_{1,2,3}(\mathfrak{v}_{\min}) &= \left\{-\frac{128(\alpha+1)}{9\alpha(\alpha+2)}, \frac{32}{9\alpha}, \frac{32}{9\alpha}\right\} \\
\mathcal{CF}^{\mathbb{F}_2}_{1,2,3}(\mathfrak{v}_{\max}) &= \left\{-\frac{32(3\alpha+4)}{9(\alpha^2+3\alpha+2)}, -\frac{32}{27\alpha+36}, \frac{32}{9(3\alpha+4)}\right\}
\end{aligned}
\tag{6.21}
$$

The singularity which might be developed by the space corresponding to the value $\alpha = 0$ is evident from Eq. (6.21). The intrinsic components of the Riemann curvature seem to have a singularity in the lower endpoint of the interval, for $\alpha = 0$.

6.1.2 The case of the singular manifold $\mathbb{WP}[1,1,2]$

In the previous section, we utilized the wording *seem to have a singularity* for the components of the Riemann curvature in the case of the space $\mathbb{WP}[1,1,2]$ since such a singularity in the curvature actually does not exist. The space $\mathbb{WP}[1,1,2]$ has indeed a singularity at $\mathfrak{v} = 0$ but it is very mild since the intrinsic components of the Riemann curvature are well behaved in $\mathfrak{v} = 0$ and have a finite limit. It depends on the way one does the limit $\alpha \to 0$. If we first compute the value of the curvature 2-form at the endpoints for generic α and then we do the limit $\alpha \to 0$ we see the singularity that is evident from Eq. (6.21). On the other hand, if we first reduce the function $\mathcal{FK}(\mathfrak{v})$ to its $\alpha = 0$ form we obtain:

$$\mathcal{FK}^{\mathbb{WP}[1,1,2]}(\mathfrak{v}) = \frac{\mathfrak{v}(8\mathfrak{v}-9)}{4\mathfrak{v}-9}$$

and the corresponding functions appearing in the curvature are:

$$\mathcal{CF}^{\mathbb{WP}[1,1,2]}_{1,2,3}(\mathfrak{v}) = \left\{\frac{648}{(4\mathfrak{v}-9)^3}, -\frac{18}{(9-4\mathfrak{v})^2}, \frac{4}{9-4\mathfrak{v}}\right\}$$

which are perfectly regular in the interval [0, 9/8] and have finite value at the endpoints (see Fig. 2B).

6.1.3 The case of the KE manifolds

In the case of the KE metrics the function $\mathcal{FK}(\mathfrak{v})$ is

$$\mathcal{FK}^{\mathrm{KE}}(\mathfrak{v}) = -\frac{(\mathfrak{v}-\lambda_1)(\mathfrak{v}-\lambda_2)(\lambda_1\lambda_2+(\lambda_1+\lambda_2)\mathfrak{v})}{\left(\lambda_1^2+\lambda_2\lambda_1+\lambda_2^2\right)\mathfrak{v}}$$

and the corresponding functions entering the intrinsic components of the Riemann curvature are

$$\mathcal{CF}^{\mathrm{KE}}_{1,2,3}(\mathfrak{v}) = \left\{-\frac{2\left(\lambda_1^2\lambda_2^2+(\lambda_1+\lambda_2)\mathfrak{v}^3\right)}{\left(\lambda_1^2+\lambda_2\lambda_1+\lambda_2^2\right)\mathfrak{v}^3}, \frac{2\lambda_1^2\lambda_2^2-(\lambda_1+\lambda_2)\mathfrak{v}^3}{2\left(\lambda_1^2+\lambda_2\lambda_1+\lambda_2^2\right)\mathfrak{v}^3}, \frac{\lambda_1^2\lambda_2^2+(\lambda_1+\lambda_2)\mathfrak{v}^3}{\left(\lambda_1^2+\lambda_2\lambda_1+\lambda_2^2\right)\mathfrak{v}^3}\right\}$$

and the interval of variability of the moment coordinate $\mathfrak{v}$ is the following $\mathfrak{v} \in [\lambda_1, \lambda_2]$. Correspondingly the boundary values are

$$\mathcal{CF}^{\mathrm{KE}}_{1,2,3}(\mathfrak{v}_{\min}) = \left\{-\tfrac{2}{\lambda_1}, \tfrac{1}{\lambda_1} - \tfrac{3(\lambda_1+\lambda_2)}{2(\lambda_1^2+\lambda_2\lambda_1+\lambda_2^2)}, \tfrac{1}{\lambda_1}\right\}$$
$$\mathcal{CF}^{\mathrm{KE}}_{1,2,3}(\mathfrak{v}_{\max}) = \left\{-\tfrac{2}{\lambda_2}, \tfrac{1}{\lambda_2} - \tfrac{3(\lambda_1+\lambda_2)}{2(\lambda_1^2+\lambda_2\lambda_1+\lambda_2^2)}, \tfrac{1}{\lambda_2}\right\}$$

We can use the case $\lambda_1 = 1, \lambda = 2$ as a standard example. In this case the behavior of the three functions is displayed in Fig. 3

6.1.4 The case of the extremal Kähler metric on the second Hizebruch surface $\mathbb{F}_2$

Finally we consider the case of the extremal metric on F_2 discussed in the previous pages and defined by the function (6.12). In this case the three functions (6.16) parameterizing the curvature 2-form and hence the intrinsic components of the Riemann tensor are the following ones:

$$\mathcal{CF}^{F2\mathrm{ext}}_1 = \frac{2a^2b^2(3a+b)-8a^2\mathfrak{v}^3+6\mathfrak{v}^4(3a+b)}{\mathfrak{v}^3(a-b)\left(a^2+4ab+b^2\right)}$$

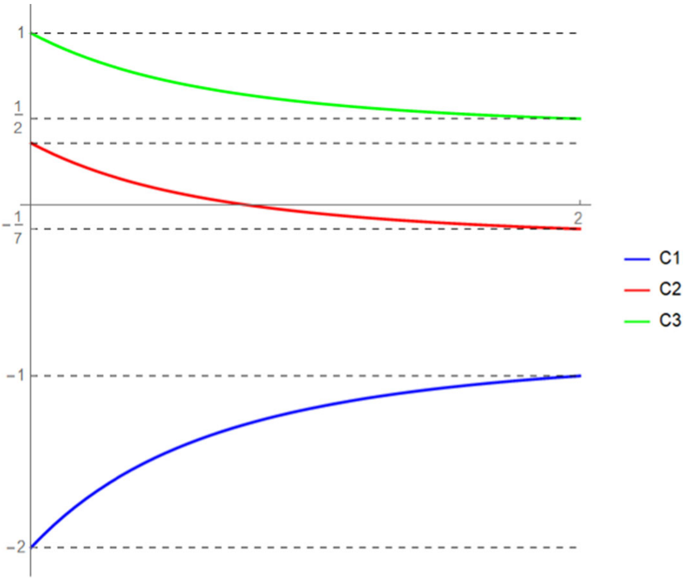

Fig. 3 Plot of the three functions $\mathcal{CF}^{\text{KE}}_{1,2,3}(\mathfrak{v})$ entering the intrinsic Riemann curvature tensor for the KE metric with the choice of the parameter $\lambda_1 = 1, \lambda_2 = 2$

$$\mathcal{CF}_2^{F2\text{ext}} = \frac{a^3 b(2\mathfrak{v} - 3b) - a^2\left(b^3 + 2\mathfrak{v}^3\right) + 3a\mathfrak{v}^4 + b\mathfrak{v}^4}{\mathfrak{v}^3(a-b)\left(a^2 + 4ab + b^2\right)}$$
$$\mathcal{CF}_3^{F2\text{ext}} = \frac{4a^3 b\mathfrak{v} - a^2 b^2(3a+b) + 4a^2\mathfrak{v}^3 + \mathfrak{v}^4(-(3a+b))}{\mathfrak{v}^3(a-b)\left(a^2 + 4ab + b^2\right)}$$

A plot of the three functions for the extremal $\mathbb{F}_2$ metric, to be compared with the analogous plot relative to the Kronheimer metric (Fig. 2A) on the same manifold is shown in Fig. 4.

6.2 The complex structure and its integration

We can easily convert the complex structure into curved indices using the vierbein and its inverse:

$$\mathfrak{J}^\mu_\nu = E^\mu_i\, \mathfrak{J}^i_j\, E^j_\nu = \begin{pmatrix} 0 & 0 & -\mathcal{FK}(\mathfrak{v})\,(\cos\theta + 1) & \mathcal{FK}(\mathfrak{v}) \\ 0 & 0 & \sin\theta & 0 \\ 0 & -\csc\theta & 0 & 0 \\ -\frac{1}{\mathcal{FK}(\mathfrak{v})} & \tan\frac{\theta}{2} & 0 & 0 \end{pmatrix}$$

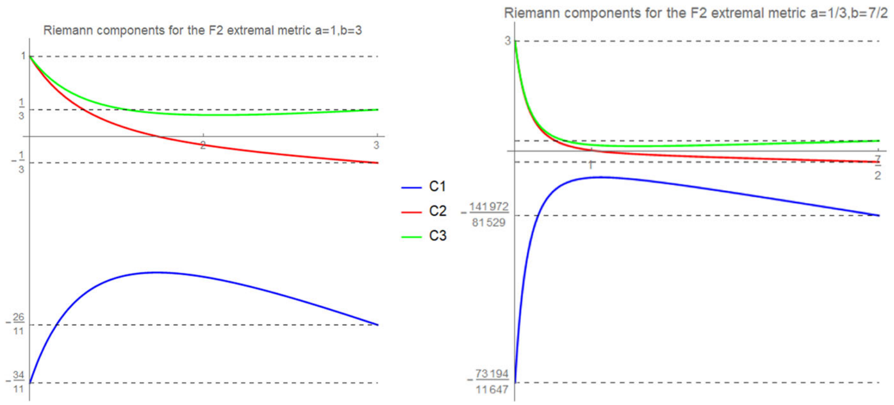

Fig. 4 Plot of the three functions $\mathcal{CF}^{F2ext}_{1,2,3}(\mathfrak{v})$ entering the intrinsic Riemann curvature tensor for the extremal Kähler metrics on F_2 with two different choices of the parameter $a = 1$, $b = 2$ and $a = 1/3$, $b = 7/2$

Since $\mathfrak{J}^2 = -\mathbf{1}_{4\times 4}$ the eigenvalues of $\mathfrak{J}$ are $\pm i$ and the eigenvectors are the rows of the following matrix:

$$\mathfrak{a}^i_\mu = \begin{pmatrix} \frac{i}{\mathcal{FK}(\mathfrak{v})} & -i\tan\frac{\theta}{2} & 0 & 1 \\ 0 & i\csc\theta & 1 & 0 \\ -\frac{i}{\mathcal{FK}(\mathfrak{v})} & i\tan\frac{\theta}{2} & 0 & 1 \\ 0 & -i\csc\theta & 1 & 0 \end{pmatrix}$$

We obtain the eigendifferentials by defining:

$$\mathrm{da}^i = i\,\mathfrak{a}^i_\mu\,\mathrm{d}x^\mu \quad ; \quad \mathrm{d}x^\mu = \{\mathrm{d}\mathfrak{v},\ \mathrm{d}\theta,\ \mathrm{d}\phi,\ \mathrm{d}\tau\}$$

The essential thing is that the eigendifferentials are all closed and that the first two are the complex conjugate of the second two:

$$\mathrm{dda}^i = 0 \quad (i = 1,\ldots,4) \quad ; \quad \mathrm{da}^1 = \overline{\mathrm{da}^3} \quad ; \quad \mathrm{da}^2 = \overline{\mathrm{da}^4}$$

This allows us to define the two complex variables u and v, by setting:

$$\begin{aligned} \mathrm{da}^3 &= \mathrm{d}\log[v] \\ \mathrm{da}^4 &= \mathrm{d}\log[u] \end{aligned}$$

In this way one obtains the universal result:

$$u = e^{i\phi}\tan\frac{\theta}{2} \quad ; \quad v = \frac{1}{2}e^{i\tau}\,(\cos\theta + 1)\;H(\mathfrak{v})$$

where:

$$H(\mathfrak{v}) = \exp\left[\int \frac{1}{\mathcal{FK}(\mathfrak{v})}\,\mathrm{d}\mathfrak{v}\right]$$

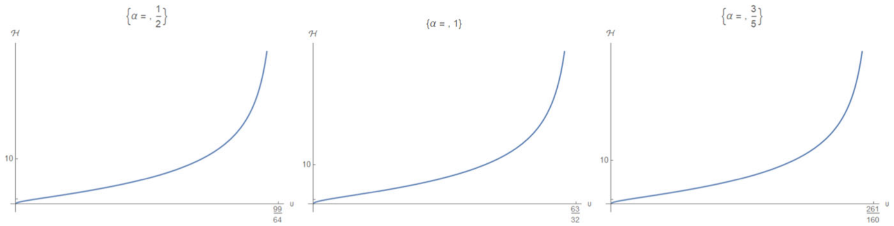

Fig. 5 Plot of three examples of the $H^{\mathbb{F}_2}_{Kro}(\mathfrak{v})$ function for three different choices of the parameter α

Hence, the whole difference between the various spaces is encoded in the properties of the function $H(\mathfrak{v})$ which is obviously defined up to a multiplicative constant due to the additive integration constant in the exponential.

6.2.1 The function $H(\mathfrak{v})$ for the Kronheimer metric on the smooth $\mathbb{F}_2$ surface

In the case of the Kronheimer metric on $\mathbb{F}_2$ we obtain

$$H^{\mathbb{F}_2}_{Kro}(\mathfrak{v}) = i\sqrt{\frac{1024\mathfrak{v}^2 - 81\alpha^2}{-32\mathfrak{v} + 9(3\alpha + 4)}}$$

The factor i can always be reabsorbed into a shift of $\pi/2$ of the phase τ and the function $\mathcal{H}^{\mathbb{F}_2}(\mathfrak{v})$ is positive definite in the finite interval $\left[\frac{9\alpha}{32}, \frac{9}{32}(3\alpha + 4)\right]$ and goes from 0 to $+\infty$ for all positive values of $\alpha > 0$. See Fig. 5 for some examples.

What is important is the monotonic behavior of the function $H^{\mathbb{F}_2}(\mathfrak{v})$, which guarantees that the two coordinates u, v describe a copy of $\mathbb{C}^2$ and hence define a dense open chart in the compact manifold $\mathbb{F}_2$.

6.2.2 The function $H(\mathfrak{v})$ for the extremal Calabi metric on the smooth $\mathbb{F}_2$ surface

Just for comparison we can consider the $H(\mathfrak{v})$ function also for the extremal metrics on F_2 defined by the function in (6.8). We obtain:

$$H^{\mathbb{F}_2}_{\text{ext}}(\mathfrak{v}) = \frac{\exp\left(-\frac{(a-b)(3a+b)\tan^{-1}\left(\frac{-a^2+4ab+6a\mathfrak{v}+b^2+2b\mathfrak{v}}{\sqrt{-a^4+44a^3b+10a^2b^2-4ab^3-b^4}}\right)}{\sqrt{-a^4+44a^3b+10a^2b^2-4ab^3-b^4}}\right)}{\sqrt{\frac{b-\mathfrak{v}}{a-\mathfrak{v}}}}$$

The behavior is absolutely similar as one can see in Fig. 6.

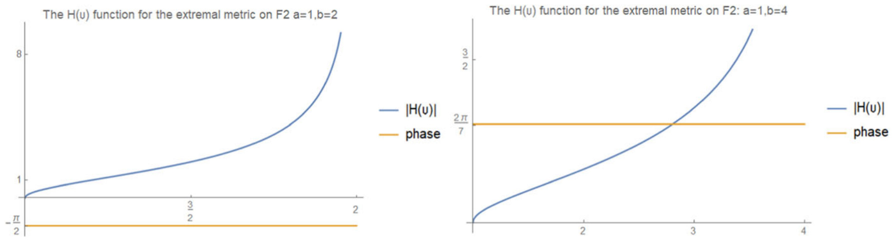

Fig. 6 Plot of two examples of the $H^{\mathbb{F}_2}_{\text{ext}}(\mathfrak{v})$ function for two different choices of the parameters a, b

6.2.3 The function $H(\mathfrak{v})$ for the KE metrics

In the case of the KE metrics in an equally easy way we obtain the following result:

$$H^{\text{KE}}(\mathfrak{v}) = \exp\left[-\left(\lambda_1^2 + \lambda_2\lambda_1 + \lambda_2^2\right)\left(\frac{\log(\mathfrak{v}-\lambda_1)}{\lambda_1^2+\lambda_2\lambda_1-2\lambda_2^2} + \frac{\log(\mathfrak{v}-\lambda_2)}{-2\lambda_1^2+\lambda_2\lambda_1+\lambda_2^2} - \frac{\log(\lambda_2\mathfrak{v}+\lambda_1(\lambda_2+\mathfrak{v}))}{2\lambda_1^2+5\lambda_2\lambda_1+2\lambda_2^2}\right)\right]$$

The structure of the function is similar to that of the $\mathbb{F}_2$ case, since there is a zero of the function in the lower limit $\mathfrak{v} \to \lambda_1$ and a pole in the upper limit $\mathfrak{v} \to \lambda_2$, yet this time the exponents of the pole and of the zero are rational numbers depending on the choice of the roots $\lambda_{1,2}$; similarly it happens for the third factor associated with the third root which is located out of the basic polytope (see Fig. 1). Our canonical example $\lambda_1 = 1, \lambda_2 = 2$ helps to illustrate the general case; with this choice we obtain

$$H^{\text{KE}}(\mathfrak{v})\,|_{\lambda_1=1,\lambda_2=2} = \frac{(\mathfrak{v}-1)^{7/5}(3\mathfrak{v}+2)^{7/20}}{(\mathfrak{v}-2)^{7/4}} = e^{i\frac{\pi}{4}} \times \underbrace{\frac{(\mathfrak{v}-1)^{7/5}(3\mathfrak{v}+2)^{7/20}}{(2-\mathfrak{v})^{7/4}}}_{\mathcal{H}^{\text{KE}}(\mathfrak{v})}$$

where, once again, the constant phase factor can be reabsorbed by a constant shift of the angular variable τ and what remains of $\mathcal{H}^{\text{KE}}(\mathfrak{v})$ is a positive definite function of $\mathfrak{v}$ in the interval $[1, 2]$ that has the same feature of its analog in the $\mathbb{F}_2$ cases, namely it maps, smoothly and monotonically, the finite interval $(1, 2)$ into the infinite interval $(0, +\infty)$. The behavior of this function is displayed in Fig. 7.

6.3 The structure of $\mathcal{M}_3$ and the conical singularity

Let us anticipate the main argument which we will develop further on. The two real manifolds defined by the restriction to the dense chart $\mathfrak{u}, \mathfrak{v}, \phi, \tau$, of the surface $\mathbb{F}_2$ and of the manifold $\mathcal{M}_B^{\text{KE}}$ are fully analogous. Cutting the compact four manifold into $\mathfrak{v} = $ const slices we always obtain the same result, namely a three manifold $\mathcal{M}_3$ with

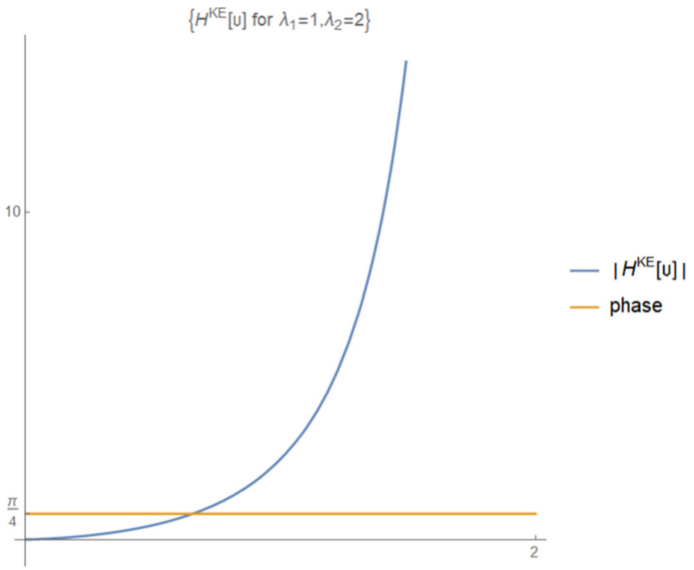

Fig. 7 Plot of the $H(\mathfrak{v})$ function in the KE case with the choice $\lambda_1 = 1, \lambda_2 = 2$

the structure of a circle fibration on S^2:

$$\mathcal{M}_B \supset \mathcal{M}_3 \xrightarrow{\pi} S^2 \quad ; \quad \forall p \in S^2 \quad \pi^{-1}(p) \sim S^1$$

The metric on $\mathcal{M}_3$ is the standard one for fibrations:

$$ds^2_{\mathcal{M}_3} = \mathfrak{v}\left(d\phi^2 \sin^2\theta + d\theta^2\right) + \mathcal{FK}(\mathfrak{v})\left[d\phi(1-\cos\theta) + d\tau\right]^2 \tag{6.22}$$

The easiest way to understand $\mathcal{M}_3$ is to study its intrinsic curvature by using the dreibein formalism. Referring to equation (6.22) we introduce the following dreibein 1-forms:

$$\boldsymbol{\epsilon}^1 = \sqrt{\mathfrak{v}}\, d\theta \quad ; \quad \boldsymbol{\epsilon}^2 = \sqrt{\mathfrak{v}}\, \sin\theta d\phi \quad ; \quad \boldsymbol{\epsilon}^3 = \sqrt{\mathcal{FK}(\mathfrak{v})}\left[d\phi(1-\cos\theta) + d\tau\right]$$

The fixed parameter $\mathfrak{v}$ plays the role of the squared radius of the sphere S^2 while $\sqrt{\mathcal{FK}(\mathfrak{v})}$ weights the contribution of the circle fiber defined over each point $p \in S^2$. At the endpoints of the intervals $\mathcal{FK}(\mathfrak{v}_{min}) = \mathcal{FK}(\mathfrak{v}_{max}) = 0$ the fiber shrinks to zero.

Using the standard formulas of differential geometry and once again the MATHEMATICA packgage VIELBGRAV23 we calculate the spin connection and the curvature 2-form. We obtain:

$$\mathfrak{R} = \left(\begin{array}{cc|c} 0 & \frac{(4\mathfrak{v}-3\mathcal{FK}(\mathfrak{v}))}{4\mathfrak{v}^2}\,\boldsymbol{\epsilon}^1\wedge\boldsymbol{\epsilon}^2 & \frac{\mathcal{FK}(\mathfrak{v})}{4\mathfrak{v}^2}\,\boldsymbol{\epsilon}^1\wedge\boldsymbol{\epsilon}^3 \\ -\frac{(4\mathfrak{v}-3\mathcal{FK}(\mathfrak{v}))}{4\mathfrak{v}^2}\,\boldsymbol{\epsilon}^1\wedge\boldsymbol{\epsilon}^2 & 0 & \frac{\mathcal{FK}(\mathfrak{v})}{4\mathfrak{v}^2}\,\boldsymbol{\epsilon}^2\wedge\boldsymbol{\epsilon}^3 \\ \hline -\frac{\mathcal{FK}(\mathfrak{v})}{4\mathfrak{v}^2}\,\boldsymbol{\epsilon}^1\wedge\boldsymbol{\epsilon}^3 & -\frac{\mathcal{FK}(\mathfrak{v})}{4\mathfrak{v}^2}\,\boldsymbol{\epsilon}^2\wedge\boldsymbol{\epsilon}^3 & 0 \end{array}\right) \tag{6.23}$$

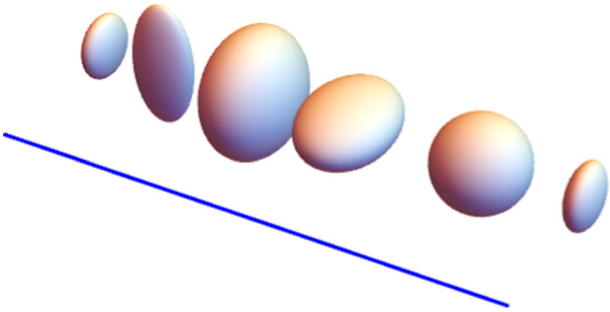

Fig. 8 A conceptual picture of the $\mathcal{M}_B$ spaces that include also the second Hirzebruch surface. The finite blue segment represent the $\mathfrak{v}$-variable varying from its minimum to its maximum value. Over each point of the line we have a three-dimensional space $\mathcal{M}_3$ which is homeomorphic to a 3-sphere but is variously deformed at each different value $\mathfrak{v}$. At the initial and final points of the blue segment, the three-dimensional space degenerates into an S^2 sphere. Graphically, we represent the deformed 3-sphere as an ellipsoid and the 2-sphere as a flat filled circle

The Riemann curvature 2-form in flat indices has constant components and if the coefficient $\frac{(4\mathfrak{v}-3\mathcal{FK}(\mathfrak{v}))}{4\mathfrak{v}^2}$ were equal to the coefficient $\frac{\mathcal{FK}(\mathfrak{v})}{4\mathfrak{v}^2}$ the 2-form in Eq. (6.23) would be the standard Riemann 2-curvature of the homogeneous space $\mathrm{SO}(4)/\mathrm{SO}(3)$, namely the 3-sphere S^3. What we learn from this easy calculation is that every section $\mathfrak{v} = \text{constant}$ of $\mathcal{M}_B$ is homeomorphic to a 3-sphere endowed with a metric that is not the maximal symmetric one with isometry $\mathrm{SU}(2) \times \mathrm{SU}(2)$ but a slightly deformed one with isometry $\mathrm{SU}(2) \times \mathrm{U}(1)$: in other words we deal with a 3-sphere deformed into the three-dimensional analog of an ellipsoid. At the endpoints of the $\mathfrak{v}$-interval, the ellipsoid degenerates into a sphere since the third dreibein $\boldsymbol{\epsilon}$ vanishes. A conceptual picture of the full space $\mathcal{M}_B$ is provided in picture Fig. 8.

6.3.1 Global properties of $\mathcal{M}_3$

Expanding on the global properties of $\mathcal{M}_3$, we describe it as a magnetic monopole bundle over S^2, and prove that the corresponding monopole strength is $n = 2$. We start from the definition of the action of the SU(2) isometry (4.2) and describe the 2-sphere S^2 spanned by θ and ϕ as $\mathbb{CP}^1$ with projective coordinates $U^0,\ U^1$:

$$U^0 = r \sin\left(\frac{\theta}{2}\right) e^{i\,\frac{\gamma+\phi}{2}}\,, \quad U^1 = r\cos\left(\frac{\theta}{2}\right) e^{i\,\frac{\gamma-\phi}{2}} \tag{6.24}$$

where $0 \le \theta \le \pi, 0 \le \phi < 2\pi, 0 \le \gamma < 4\pi$. In the North patch $\mathcal{U}_N$, $U^1 \neq 0$ and the sphere is spanned by the stereographic coordinate $u_N = U^0/U^1$, while in the south patch $\mathcal{U}_S$, $U^0 \neq 0$ and the stereographic coordinate $u_S = U^1/U^0$. The transformation properties (4.2) define a line bundle whose local trivializations about the two poles are:

$$\phi_N^{-1}(\mathcal{U}_N) = (u_N,\ v_N) = \left(\frac{U^0}{U^1},\ \xi\,(U^1)^2\right)\,,$$

$$\phi_S^{-1}(\mathcal{U}_S) = (u_S,\, v_S) = \left(\frac{U^1}{U^0},\, \xi\,(U^0)^2\right),$$

where ξ is a complex number in the fiber not depending on the patch. As $(U^0,\, U^1)$ transform linearly under the an SU(2)-transformation:

$$\begin{pmatrix} U^0 \\ U^1 \end{pmatrix} \to \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} U^0 \\ U^1 \end{pmatrix},$$

the fiber coordinate v transforms so that $(1+|u_N|^2)^2|v_N|^2$ and $(1+|u_S|^2)^2|v_S|^2$, in $\mathcal{U}_N$ and $\mathcal{U}_S$, respectively, are invariant. The transition function on the fiber reads, at the equator $\theta = \pi/2$:

$$t_{NS} = \left(\frac{U^1}{U^0}\right)^2 = e^{-2i\phi} = e^{-in\phi},$$

implying that the U(1)-bundle associated with the phase of v (i.e., the submanifold of the Kähler–Einstein space at constant $|v|$), is a monopole bundle with monopole strength $n = 2$. This has to be contrasted with the Hopf-fibering description of S^3, for which the local trivializations have fiber components $U^1/|U^1|$ and $U^0/|U^0|$ in the two patches, respectively, and t_{NS} at the equator is $U^1/U^0 = e^{-i\phi}$. In this case, the monopole strength is $n = 1$. We can verify that the manifold at constant $|v|$ is a Lens space $S^3/\mathbb{Z}_2$ also by direct inspection of the metric. This is done in "Appendix B".

6.3.2 Conical singularities and regularity of $\mathbb{F}_2$

Let us analyze the exact form of the singularities (when they are present). The restriction of the metric to a fiber spanned by $\mathfrak{v}$ and $\tau \in (0, 2\pi)$ is

$$\mathrm{d}s^2 = \frac{\mathrm{d}\mathfrak{v}^2}{\mathcal{FK}(\mathfrak{v})} + \mathcal{FK}(\mathfrak{v})\,\mathrm{d}\tau^2\,. \tag{6.25}$$

Let λ denote one of the two roots $\lambda_1,\ \lambda_2$ of $\mathcal{FK}(\mathfrak{v})$. Close to λ, to first order in $\mathfrak{v}$, in the KE case, the metric (6.25) is flat and features a deficit angle signaling a conifold singularity. This singularity is absent in the $\mathbb{F}_2$ cases, as expected. To show this let us Taylor expand $\mathcal{FK}(\mathfrak{v})$ about λ:

$$\mathcal{FK}(\mathfrak{v}) = \mathcal{FK}'(\lambda)(\mathfrak{v}-\lambda) + O((\mathfrak{v}-\lambda)^2).$$

We can verify that:

$$\begin{array}{lll}
\text{KE} & : \mathcal{FK}'_{\text{KE}}(\lambda_1) = \frac{(\lambda_2-\lambda_1)(\lambda_1+2\lambda_2)}{\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2} & ;\ \mathcal{FK}'_{\text{KE}}(\lambda_2) = \frac{(\lambda_1-\lambda_2)(2\lambda_1+\lambda_2)}{\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2}\,, \\
\mathbb{F}_2\text{Kronheimer} & : \mathcal{FK}'_{\mathbb{F}_2|Kro}\left(\frac{9\alpha}{32}\right) = 2 & ;\ \mathcal{FK}'_{\mathbb{F}_2|Kro}\left(\frac{9(3\alpha+4)}{32}\right) = -2 \\
\mathbb{F}_2\text{Extremal} & : \mathcal{FK}'_{\mathbb{F}_2|ext}(a) = 2 & ;\ \mathcal{FK}'_{\mathbb{F}_2|ext}(b) = -2
\end{array}$$

Next we replace the first-order expansion of this function in the fiber metric:

$$\mathrm{d}s^2 = \frac{\mathrm{d}\mathfrak{v}^2}{\mathcal{FK}'(\lambda)(\mathfrak{v}-\lambda)} + \mathcal{FK}'(\lambda)(\mathfrak{v}-\lambda)\,\mathrm{d}\tau^2\,,$$

and write it as a flat metric in polar coordinates:

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \beta^2\, r^2\, \mathrm{d}\tau^2\,.$$

One can easily verify that:

$$r = 2\sqrt{\frac{\mathfrak{v}-\lambda}{\mathcal{FK}'(\lambda)}}\,, \quad \beta = \frac{|\mathcal{FK}'(\lambda)|}{2}\,.$$

Defining $\tilde{\varphi} = \beta\,\tau$, we can write the fiber metric as follows:

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\tilde{\varphi}^2.$$

Now the polar angle varies in the range: $\tilde{\varphi} \in [0,\ 2\pi\,\beta]$. If $\beta < 1$ we have a deficit angle:

$$\Delta\phi = 2\pi(1-\beta).$$

Let us see what this implies in the various possible cases of Table 1.

1. In the case of the $\mathbb{F}_2$ manifold one has $|\mathcal{FK}'(\lambda)| = 2$ and $\beta = 1$, both for the Kronheimer metric and for the extremal one of the Calabi family, so there is no conical singularity, as expected.
2. In the case of $\mathbb{WP}[1,1,2]$ we have

$$\mathcal{FK}(\lambda) = \frac{32\lambda^2 - 144\lambda + 81}{(4\lambda-9)^2}.$$

   For the limiting value $\lambda = 0$ we obtain $\beta = \frac{1}{2}$, i.e., a $\mathbb{C}^2/\mathbb{Z}_2$ singularity, while for $\lambda = \frac{9}{8}$ we have $\beta = 1$, i.e., no singularity, as we expected as $\mathbb{WP}[1,1,2]$ is an orbifold $\mathbb{P}^2/\mathbb{Z}_2$ with one singular point.
3. In the KE manifold case considering $\lambda = \lambda_1$, we have:

$$\mathcal{FK}'(\lambda_1) = \frac{(\lambda_2-\lambda_1)(\lambda_1+2\lambda_2)}{\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2} = -1+\frac{3\lambda_2^2}{\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2} < -1+\frac{3\lambda_2^2}{\lambda_2^2} = 2 \;\Rightarrow\; \beta < 1\,,$$

   and $|\mathcal{FK}'(\lambda_2)| < |\mathcal{FK}'(\lambda_1)|$, so that $\beta < 1$ also at λ_2. The manifold has two conical singularities, both in the same fiber of the projection to one of the S^2's. One of the singularities will be an orbifold singularity of type $\mathbb{C}^2/\mathbb{Z}_n$ if the corresponding value of β is

$$\beta = 1 - \frac{1}{n}.$$

It is interesting to note that when this happens, the form of the function $\mathcal{FK}$ does not allow the other singularity to be of this type as well, as the corresponding integer m should satisty

$$m = \frac{4n}{2 + 5n \pm \sqrt{9n^2 + 12n - 12}}$$

which is not satisfied by any pair (m, n) where both m, n are integers greater than 1; so the singular fiber can never be a football or a spindle.

4. We discuss the case $\lambda_1 = 0$. In this case

$$\mathcal{FK}^0(\mathfrak{v}) = \frac{\mathfrak{v}(\lambda_2 - \mathfrak{v})}{\lambda_2} .$$

If we focus on the fiber metric:

$$ds^2 = \frac{d\mathfrak{v}^2}{\mathcal{FK}_0(\mathfrak{v})} + \mathcal{FK}_0(\mathfrak{v})\, d\tau^2 . \tag{6.26}$$

We can easily verify that in the coordinates $\tilde{\theta} \in [0, \pi]$ and $\tilde{\varphi} \in [0, \pi)$ defined by

$$\mathfrak{v}(\tilde{\theta}) = R^2 \sin^2\left(\frac{\tilde{\theta}}{2}\right) \le R^2 = \lambda_2 , \quad \tilde{\varphi} = \frac{\tau}{2},$$

where $R = \sqrt{\lambda_2}$, the fiber metric (6.26) becomes

$$ds^2 = R^2 \left(d\tilde{\theta}^2 + \sin^2(\tilde{\theta})\, d\tilde{\varphi}^2\right) .$$

Since $\tilde{\varphi} = \tau/2 \in [0, \pi)$, the fiber is the splindle $S^2/\mathbb{Z}_2$. Topologically, the entire 4-manifold is still $S^2 \times S^2$.

5. For $\mathcal{FK}(\mathfrak{v}) = \mathfrak{v}$ we get $\beta = \frac{1}{2}$, in accordance with the fact that the variety in this case is $\mathbb{C}^2/\mathbb{Z}_2$.

In the cases 3 and 4 the singular locus is of the form $S^2 \times p_\pm$, where $p_\pm$ are the "poles" of the fibers of the projection $S^2 \times S^2 \to S$. In complex geometric terms, it is a pair of divisors, both isomorphic to $\mathbb{P}^1$.

7 Complex structures

In this section, we study the complex structures corresponding to the KE case, i.e., the cases corresponding to the function $\mathcal{FK}^{\text{KE}}$. These are singular KE manifolds of complex dimension 2, homeomorphic to $S^2 \times S^2$. To make the analysis completely quantitative, let us choose a value of the parameter α and two values of λ_1, λ_2 so that the basic polytope becomes exactly identical in the two cases.

We choose the value $\alpha = \frac{4}{9}$ so that the endpoints of the interval in the pure $\mathbb{F}_2$ Kronheimer case are:

$$\mathfrak{v}_{\min} = \frac{1}{8} \quad ; \quad \mathfrak{v}_{\max} = \frac{3}{2}$$

and using the previously discussed procedure we obtain the complex v coordinate for the $\mathbb{F}_2$ pure case:

$$v_{\mathbb{F}_2} = \exp\left[i\left(\tau + \frac{\pi}{4}\right)\right] \times \frac{1+\cos\theta}{2} \times \sqrt{\frac{64\mathfrak{v}^2 - 1}{3 - 2\mathfrak{v}}}$$

In the same way we obtain the complex v variable for the Kähler Einstein case:

$$v_{\mathrm{KE}} = \exp\left[i\left(\tau + \frac{157}{154}\pi\right)\right] \times \frac{1+\cos\theta}{2} \times \frac{\left(\mathfrak{v} - \frac{1}{8}\right)^{157/275}\left(\frac{3\mathfrak{v}}{2} + \frac{1}{8}\left(\mathfrak{v} + \frac{3}{2}\right)\right)^{157/350}}{\left(\frac{3}{2} - \mathfrak{v}\right)^{157/154}}$$

Obviously there is no holomorphic way of writing v_{KE} in terms of $v_{\mathbb{F}_2}$ or viceversa. This can be immediately seen in the following way. Taking the ratio of the two coordinates v we obtain:

$$(i)^{\frac{237}{154}} \times \frac{v_{\mathbb{F}_2}}{v_{\mathrm{KE}}} = 4 \times \frac{2^{1877/3850}(3 - 2\mathfrak{v})^{40/77}\sqrt{64\mathfrak{v}^2 - 1}}{(8\mathfrak{v} - 1)^{157/275}(26\mathfrak{v} + 3)^{157/350}} \tag{7.1}$$

Moreover with some manipulations we can write

$$\mathfrak{v} = \frac{1}{64}\left(-\varpi_{\mathbb{F}_2} \pm \sqrt{\varpi_{\mathbb{F}_2}^2 + 192\,\varpi_{\mathbb{F}_2} + 64}\right) \tag{7.2}$$

Inserting Eq. (7.2) into Eq. (7.1) we see that the complex coordinate v_{KE} is not a holomorphic function of the complex coordinates $u_{\mathbb{F}_2}$, $v_{\mathbb{F}_2}$.

This argument can be used for all values of α. We can always choose the independent roots $\lambda_{1,2}$ so that the interval of the moment variable $\mathfrak{v}_{\min}$,$\mathfrak{v}_{\max}$ coincides in the Kähler Einstein case and in the $\mathbb{F}_2$ Kronheimer or extremal cases. In the dense open chart that we are using $\mathbb{F}_2$ and the manifold that admits a KE metric have different complex structures. The plot of the two functions $\mathcal{H}(\mathfrak{v})$ is displayed in Fig. 9.

The remaining problem is therefore the following. When we utilize the Kronheimer metric for $\mathbb{F}_2$ written in real variables in the open chart provided by the coordinates $\mathfrak{u}, \mathfrak{v}, \phi, \tau$ we know that the closure of such a dense chart is the second Hirzebruch surface. The same can be said if we utilize the extremal metric. In the same open real chart we have also a KE metric: the question is, *What is the closure of such an open real chart compatible with the KE metric?*. The important point to keep in mind while trying to answer such a question is that, topologically, the Hirzebruch surface is just $S^2 \times S^2$. What makes this real manifold Hirzebruch is the complex structure induced by the holomorphic embedding in $\mathbb{P}^1 \times \mathbb{P}^2$ as an algebraic variety. Yet the complex structure of $\mathbb{F}_2$ is different and incompatible with the complex structure compatible with the KE metric defined in the same open chart. This must be the guiding principle.

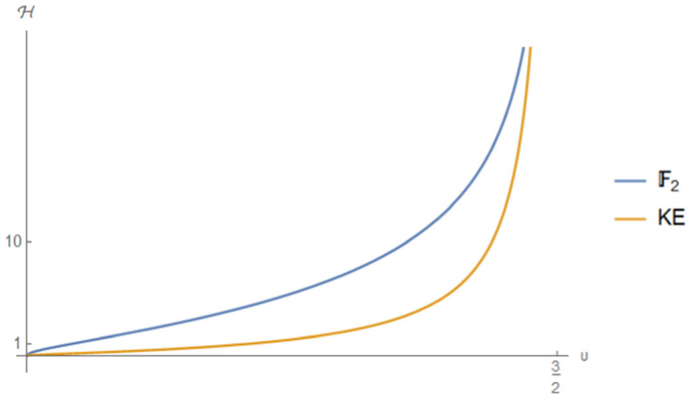

Fig. 9 Comparison of the plots of $\mathcal{H}^{\mathbb{F}_2}(\mathfrak{v})$ for the pure $\mathbb{F}_2$ case with $\mathcal{H}^{\text{KE}}(\mathfrak{v})$ for the KE case, when they are calibrated to insist on the same interval $\left[\frac{1}{8}, \frac{3}{2}\right]$

7.1 The homeomorphism with $S^2 \times S^2$

We want to analyze in detail the homeomorphism with the space $S^2 \times S^2$ in the KE case. Prior to that let us stress, once again, that all the manifolds we are considering are KE by construction, as:

(a) There is a candidate Kähler form written as

$$\mathbb{K} = K_{ij}\mathbf{e}^i \wedge \mathbf{e}^j$$

where the one forms $\mathbf{e}^i$ are a tetrad representation of the considered metric:

$$ds^2_{\mathcal{M}_B} = g_{\mu\nu}dx^\mu \, dx^\nu = \delta_{ij} \, \mathbf{e}^i \, \mathbf{e}^j$$

(b) The candidate form is closed:

$$d\mathbb{K} = 0$$

(c) The component tensor of the Kähler form in flat indices K_{ij} satisfies the condition:

$$K_{ij} \, K_{jk} = -\delta_{ik}$$

This guarantees that for each metric we can construct the corresponding complex structure tensor:

$$\mathfrak{J}^\mu_\nu = E^i_\mu \, K_{ij} \, \delta^{jk} \, E^\nu_k \quad ; \quad \mathfrak{J}^2 = -\text{Id}$$

and by construction the metric is Hermitian with respect to that complex structure.

The Kähler form is the same for all the family of metrics (6.1) and each metric chooses the complex structure with respect to which it is Hermitian. In principle, these complex structures are all different, yet some of them might be compatible, as it is the case for the one-parameter family of metrics on the Hirzebruch surface, that share the same complex structure and can be described in terms of the same complex coordinates. However, in the previous section we have already shown that the complex structure selected by one of the KE metrics is certainly incompatible with that of the Hirzebruch surface and this removes any possible conceptual clash. On the other hand, there is no obstacle to the fact that the underlying real manifold of the Hirzebruch case and of the (singular) KE case might be homeomorphic and this is what we want to show.

In the next lines, we argue how to construct explicitly such a homeomorphism. First of all, by looking at the metric in Eq. (6.1) we see that the first 2-sphere is already singled out in the standard coordinates θ and ϕ. As for the second sphere the azimuthal angle is already identified in the coordinate τ. It remains to be seen that the coordinate $\mathfrak{v}$ in the finite closed range $[\mathfrak{v}_{\min}\,,\ \mathfrak{v}_{\max}]$ is in one-to-one continuous correspondence with a new right ascension angle χ.

7.2 Behavior of the function $\sqrt{\mathcal{FK}(\mathfrak{v})}$

To this effect the main point is that the function $\sqrt{\mathcal{FK}(\mathfrak{v})}$ should be upper limited by the value 1 in the interval $[\mathfrak{v}_{\min}\,,\ \mathfrak{v}_{\max}]$, it should grow monotonically from 0 to a maximum value $a_0 \leq 1$, attained at $\mathfrak{v} = \mathfrak{v}_0$ and then it should decrease monotonically from a_0 to 0 in the second part of the interval $[\mathfrak{v}_0\,,\ \mathfrak{v}_{\max}]$. Under such conditions the inverse function arcsin can be applied unambiguously to $\sqrt{\mathcal{FK}(\mathfrak{v})}$ and we can obtain a one-to-one continuous map between the coordinate $\mathfrak{v}$ and a new right ascension angle χ. The homeomorphism is encoded in the following relation where the function $\mathfrak{h}(\mathfrak{v})$ is continuous and monotonous only under the above carefully specified conditions.

$$\chi = \mathfrak{h}(\mathfrak{v}) = \frac{\pi \arcsin\left(\sqrt{\mathcal{FK}(\mathfrak{v})}\right)}{2\arcsin\left(a_0\right)} + \Theta\left(\mathfrak{v} - \mathfrak{v}_0\right)\,\pi\,\left(1 - \frac{\arcsin\left(\sqrt{\mathcal{FK}(\mathfrak{v})}\right)}{\arcsin\left(a_0\right)}\right) \tag{7.3}$$

In the above formula, the symbol $\Theta(x)$ denotes the well-known Heaviside step function that vanishes for $x < 0$ and evaluates to 1 for $x > 0$.

The relevant fact is that both for the case of the Hirzebruch surface metric and for KE ones the above specified conditions are verified and the homeomorphism (7.3) can be written. We examine in detail one instance of the first and one instance of the second case, having verified that in each class the chosen examples represent the behavior of all members of the same class. For the Hirzebruch case we set the parameter $\alpha = 1$ and we obtain:

$$\mathcal{FK}^{\mathbb{F}_2}\left(\mathfrak{v}\right) = \frac{\left(32\mathfrak{v} - 63\right)\left(1024\mathfrak{v}^2 - 81\right)}{16\left(1024\mathfrak{v}^2 - 4032\mathfrak{v} + 81\right)} \tag{7.4}$$

For the KE case we choose, as we already did in previous sections, $\lambda_1 = 1$, $\lambda_2 = 2$ and we obtain:

$$\mathcal{FK}^{\mathrm{KE}}\left(\mathfrak{v}\right) = -\frac{(\mathfrak{v} - 2)(\mathfrak{v} - 1)(3\mathfrak{v} + 2)}{7\mathfrak{v}} \tag{7.5}$$

The behavior of the function $\sqrt{\mathcal{FK}(\mathfrak{v})}$ in the two cases and of the associated homeomorphism on the right ascension angle is shown in Figs. 10and 11

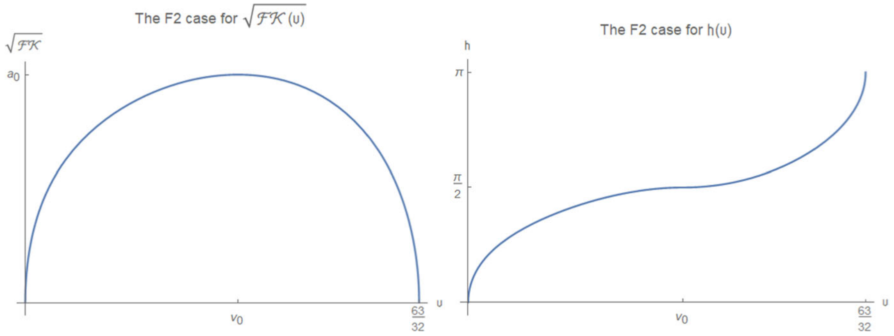

Fig. 10 On the left the plot of the function $\sqrt{\mathcal{FK}^{\mathbb{F}_2}(\mathfrak{v})}$ corresponding to $\alpha = 1$ and explicitly displayed in Eq. (7.4). On the right the corresponding function $\mathfrak{h}(\mathfrak{v})$ providing the homeomorphism to the right ascension angle

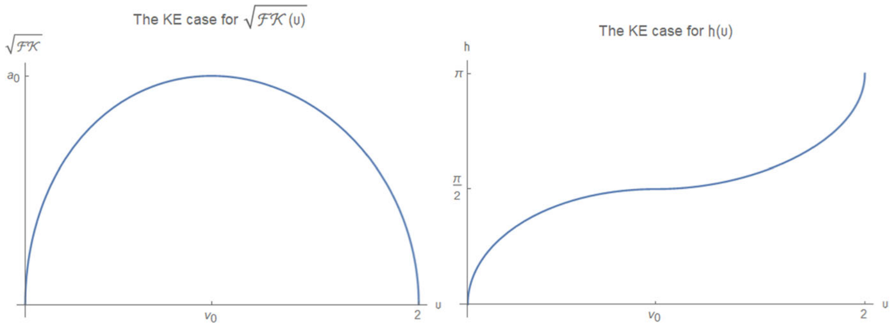

Fig. 11 On the left the plot of the function $\sqrt{\mathcal{FK}^{\text{KE}}(\mathfrak{v})}$ corresponding to $\lambda_1 = 1, \lambda_2 = 2$ and explicitly displayed in Eq. (7.5). On the right the corresponding function $\mathfrak{h}(\mathfrak{v})$ providing the homeomorphism to the right ascension angle

On the basis of the above lore in order to explore the behavior of a function, a vector field or whatever different geometric object in the neighborhood of the North pole of the second sphere one has at one's disposal a well-defined transition function from the coordinates $\mathfrak{v}_N$, τ_N to the coordinates $\mathfrak{v}_S$, τ_S:

$$\begin{aligned} \mathfrak{v}_S &= \mathfrak{h}^{-1}\left(\mathfrak{h}(\mathfrak{v}_N) - \pi\right) \\ \tau_S &= -\tau_N + \pi \end{aligned}$$

The inversion of the function $\mathfrak{h}$ is highly nontrivial since it is a combination of transcendental and algebraic functions, yet it can always be done numerically, if needed. As for the metric itself there is no need since the curvature is nonsingular in the point $\mathfrak{v}_{\text{max}}$ and the geodesics are also well behaved in all limits.

The conclusion is that the underlying manifold of the Kähler Einstein metrics is $S^2 \times S^2$, as in the case of $\mathbb{F}_2$.

8 Liouville vector field on $\mathcal{M}_B$ and the contact structure on $\mathcal{M}_3$

Next we go back to consider general properties of the metric (6.1) that,utilizing Eq. (6.3) we also rewrite in terms of the coordinates $\mathfrak{u}, \mathfrak{v}, \phi, \tau$:

$$ds^2_{\mathcal{M}_B} = \frac{d\mathfrak{v}^2}{\mathcal{FK}(\mathfrak{v})} + \frac{\mathfrak{u}(2\mathfrak{v}-\mathfrak{u})}{\mathfrak{v}}d\phi^2 + \frac{(\mathfrak{v}d\mathfrak{u}-\mathfrak{u}d\mathfrak{v})^2}{\mathfrak{u}\mathfrak{v}(2\mathfrak{v}-\mathfrak{u})} + \frac{\mathcal{FK}(\mathfrak{v})}{\mathfrak{v}^2}\left(\mathfrak{u}\,d\phi + \mathfrak{v}\,d\tau\right)^2 \tag{8.1}$$

The corresponding Kähler 2-form is provided by equation (6.18) that for reader's convenience we copy here below:

$$\mathbb{K} = d\mathfrak{u} \wedge d\phi + d\mathfrak{v} \wedge d\tau$$

The pair $(\mathcal{M}_B, \mathbb{K})$ constitutes a symplectic manifold, independently from the Riemannian structure provided by the metric (8.1). For symplectic manifolds, there exists the notion of Liouville vector fields (see for instance [29, 33]) defined as follows. The vector field $\mathbf{L} \in \Gamma[T\mathcal{M}_B, \mathcal{M}_B]$ is a Liouville vector field if

$$\mathcal{L}_{\mathbf{L}}\,\mathbb{K} = \mathbb{K} \tag{8.2}$$

where $\mathcal{L}_{\mathbf{V}}$ denotes the Lie derivative along the specified vector field $\mathbf{V}$. Utilizing Cartan's formula for the Lie derivative, we get

$$\mathcal{L}_{\mathbf{L}}\mathbb{K} = i_{\mathbf{L}}\,d\mathbb{K} + d\,(i_{\mathbf{L}}\,\mathbb{K}) = d\,(i_{\mathbf{L}}\,\mathbb{K}) = \mathbb{K}$$

A very simple Liouville field for the symplectic manifold $(\mathcal{M}_B, \mathbb{K})$ is the following one:

$$\mathbf{L} = \mathfrak{u}\,\frac{\partial}{\partial\mathfrak{u}} + \mathfrak{v}\,\frac{\partial}{\partial\mathfrak{v}}$$

as one can immediately verify.

Another result in symplectic geometry, (see [13–15, 20, 21, 29, 33]) states that a $(2n+1)$-submanifold $\mathcal{Z} \subset \mathcal{M}_B$ of a $(2n+2)$-dimensional symplectic manifold $(\mathcal{M}_B, \mathbb{K})$ that is transverse to a Liouville field $\mathbf{L}$ is a contact manifold with contact structure

$$\Omega = i_{\mathbf{L}}\,\mathbb{K}$$

In view of this theorem, the interpretation of the $\mathcal{M}_3^{\mathfrak{v}}$ manifolds, extensively discussed in previous sections, that have the topology of S^3 and are all transverse to the Liouville field since they correspond to fixed values of the coordinate $\mathfrak{v}$, becomes clear. They constitute the leaves of a foliation of the symplectic manifold $(\mathcal{M}_B, \mathbb{K})$ in diffeomorphic contact manifolds whose contact form is:

$$\Omega = \mathfrak{u}\,d\phi + \mathfrak{v}\,d\tau$$

On each leave $\mathfrak{v} = \text{const}$ we have:

$$d\Omega = d\mathfrak{u} \wedge d\phi \quad ; \quad d\Omega \wedge \Omega = \mathfrak{v}\,d\mathfrak{u} \wedge d\phi \wedge d\tau = \text{const} \times \text{Vol}_3^{\mathfrak{v}}$$

8.1 The Reeb field and Beltrami equation

It is now interesting to calculate the normalized Reeb field associated with the contact form Ω. This is possible since the symplectic manifold $(\mathcal{M}_B, \mathbb{K})$ is endowed with the Riemannian structure provided by the metric (6.22). Expanding the one-form Ω along the coordinate differentials:

$$\Omega = \Omega_\mu \, \mathrm{d}y^\mu \quad ; \quad y^\mu = \{\theta, \phi, \tau\}$$

we find:

$$\Omega_\mu = \{0, \mathfrak{v}(1 - \cos(\theta)), \mathfrak{v}\}$$

Utilizing the inverse of the metric tensor defined by the line element (6.22) namely:

$$g^{\mu\nu} = \begin{pmatrix} \frac{1}{\mathfrak{v}} & 0 & 0 \\ 0 & \frac{\csc^2(\theta)}{\mathfrak{v}} & \frac{(\cos(\theta)-1)\csc^2(\theta)}{\mathfrak{v}} \\ 0 & \frac{(\cos(\theta)-1)\csc^2(\theta)}{\mathfrak{v}} & \frac{1}{\mathcal{FK}(\mathfrak{v})} + \frac{\tan^2\left(\frac{\theta}{2}\right)}{\mathfrak{v}} \end{pmatrix}$$

we can raise the index of Ω_μ and we obtain the components of a normalized Reeb vector field:

$$U^\mu = g^{\mu\nu}\Omega_\nu = \left\{0, 0, \frac{\mathfrak{v}}{\mathcal{FK}(\mathfrak{v})}\right\} \quad \Rightarrow \quad \mathbf{U} = \frac{\mathfrak{v}}{\mathcal{FK}(\mathfrak{v})}\,\partial_\tau$$

such that:

$$\Omega\,(\mathbf{U}) = 1 \quad ; \quad i_{\mathbf{U}}\,\mathrm{d}\Omega = 0$$

It is a notable fact that the above Reeb vector field automatically satisfies Beltrami equation. Indeed, it is known that every contact structure in 3 dimensions admits a contact form and an associated Reeb field that satisfies Beltrami equation (see for instance [20, 29]) yet it is remarkable that the choice of the Liouville vector field (8.2) immediately selects a Beltrami Reeb field. The verification of our statement is almost immediate if we utilize the formulation of Beltrami equation introduced in [16] (see also [29]), namely:

$$\mathrm{d}\Omega^{\mathbf{U}} = \lambda\, i_{\mathbf{U}}\,\mathrm{Vol}_3 \tag{8.3}$$

where Vol_3 denotes the volume 3-form of the considered 3-manifold, $\Omega^{\mathbf{U}}$ is the contact form that admits $\mathbf{U}$ as normalized Reeb field and $\lambda \in \mathbb{R}$ is the Beltrami eigenvalue. In our case the volume form is:

$$\mathrm{Vol}_3 = \boldsymbol{\epsilon}^1 \wedge \boldsymbol{\epsilon}^2 \wedge \boldsymbol{\epsilon}^3 = \mathfrak{v}\sqrt{\mathcal{FK}(\mathfrak{v})}\,\sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi \wedge \mathrm{d}\tau$$

and Eq. (8.3) is satisfied with eigenvalue:

$$\lambda = -\frac{1}{\mathfrak{v}}.$$

9 Geodesics for the family of manifolds $\mathcal{M}_B$

In this section, we study the general problem of calculating the geodesics for the class of metrics (6.1). Sometimes the differential system determining the geodesics is completely integrable and this allows one to reduce it to first order and to quadratures, obtaining in this way the complete set of all geodesics—-leaving apart the practical problem of inverting transcendental functions, which possibly can be accomplished with numerical methods. An example is the Kerr metric where there is a hidden first integral (the Carter constant) which can be revealed by the use of the Hamilton–Jacobi formulation, and allows for complete integration.

We show in this section that for any choice of the function $\mathcal{FK}(\mathfrak{v})$ the geodesic dynamical system associated with the metrics (6.1) is fully integrable and admits a hidden Carter constant, another first integral in addition to the Hamiltonian, which allows to write the full system of geodesic lines for all the metrics in the class, in particular for the $\mathbb{F}_2$ surface and for the KE manifolds brought to attention in this paper.

9.1 The geodesic equation

We take for Lagrangian functional the square of the arc length

$$\mathcal{L} = \frac{1}{2}\,\mathcal{FK}(\mathfrak{v})\left(\dot{\phi}\,(1-\cos\theta)+\dot{\tau}\right)^2 + \frac{\dot{\mathfrak{v}}^2}{\mathcal{FK}(\mathfrak{v})} + \mathfrak{v}\left(\dot{\phi}^2\sin^2\theta+\dot{\theta}^2\right).$$

As usual, the Euler–Lagrange equations take the standard form if the Lagrangian satisfies the constraint

$$\mathcal{L}\mid_{\text{ongeodesics}} = \frac{k}{4}$$

for some $k > 0$; in this way parameter along the curves is the arc length s. In a mechanical analogy, k is the energy.

9.1.1 Cyclic variables and conserved momenta

The angles ϕ and τ are cyclic variables (due to toric symmetry) which leads to two first integral of the motion, which we call $\ell,\ m$, and can be represented in a synthetic way:

$$\begin{pmatrix}\ell\\ m\end{pmatrix} = \begin{pmatrix}p_\phi\\ p_\tau\end{pmatrix} = \mathfrak{M}\begin{pmatrix}\dot{\phi}\\ \dot{\tau}\end{pmatrix},\quad \mathfrak{M} = \frac{1}{2}\begin{pmatrix}8\mathcal{FK}(\mathfrak{v})\sin^4\frac{\theta}{2}+2\mathfrak{v}\sin^2\theta & -2\mathcal{FK}(\mathfrak{v})(\cos\theta-1)\\ -2\mathcal{FK}(\mathfrak{v})(\cos\theta-1) & 2\mathcal{FK}(\mathfrak{v})\end{pmatrix}$$

9.1.2 The Hamiltonian

We perform the Legendre transform in order to obtain the Hamiltonian H:

$$H = \dot{\phi}\,p_\phi + \dot{\tau}\,p_\tau + \dot{\theta}\,p_\theta + \dot{\mathfrak{v}}\,p_{\mathfrak{v}} - \mathcal{L}$$

getting

$$H(q,p)=\frac{1}{2}\left(\frac{p_\tau^2}{\mathcal{FK}(\mathfrak{v})}+\mathcal{FK}(\mathfrak{v})p_{\mathfrak{v}}^2+\frac{\csc^2\theta\left[(\cos\theta-1)p_\tau+p_\phi\right]^2+p_\theta^2}{\mathfrak{v}}\right)$$

where

$$p=\left\{p_\phi,\,p_\tau,\,p_\theta,\,p_{\mathfrak{v}}\right\}\quad;\quad q\;=\;\{\phi,\;\tau,\;\theta,\;\mathfrak{v}\}$$

are the momenta and coordinates.

As it is always the case in the geodesic problem, the Hamiltonian has the structure

$$H=g^{ij}(q)\,p_i\,p_j$$

having denoted by $g^{ij}(q)$ the inverse metric tensor.

9.1.3 The reduced Lagrangian and the reduced Hamiltonian

Having singled out two first integrals of the motion ℓ, $\mathfrak{m}$, it is convenient to introduce a reduced Lagrangian for the two residual degrees of freedom $\mathfrak{v},\theta$ that, geometrically, correspond to the two angles of *right ascension* of the 2-spheres composing the underlying differentiable manifold (see Sect. 7.1). The reduction in the Lagrangian is obtained by replacing the velocities of the cyclic coordinates $q^{\mathfrak{c}}$ with the corresponding momenta $p_{\mathfrak{c}}$ that are constant of the motion, namely ℓ, m:

$$\mathcal{L}_{\text{red}}=\frac{\mathcal{FK}(\mathfrak{v})\left(\dot{\theta}^2\,\mathfrak{v}^2\;+\;\csc^2\theta\,(m\cos\theta-m+\ell)^2\right)\;+\;m^2\mathfrak{v}\;+\;\mathfrak{v}\,\dot{\mathfrak{v}}^2}{2\mathfrak{v}\mathcal{FK}(\mathfrak{v})}$$

Performing the Legendre transform we obtain the reduced Hamiltonian

$$\begin{aligned}H_{\text{red}}&=p_{\mathfrak{v}}\,\dot{\mathfrak{v}}\;+\;p_\theta\,\dot{\theta}-\mathcal{L}_{red}\\&=\frac{1}{2}\left(-\frac{m^2}{\mathcal{FK}(\mathfrak{v})}+\mathcal{FK}(\mathfrak{v})p_{\mathfrak{v}}^2-\frac{\csc^2\theta(m\cos\theta-m+\ell)^2-p_\theta^2}{\mathfrak{v}}\right)\end{aligned}\tag{9.1}$$

where

$$p_{\mathfrak{v}}=\frac{\dot{\mathfrak{v}}}{\mathcal{FK}(\mathfrak{v})},\qquad p_\theta=\mathfrak{v}\,\dot{\theta}$$

9.1.4 The Carter constant and the reduction to quadratures

Considering now the reduced system with four Hamiltonian variables we have a nice surprise: there is an additional function of the q and p that is in involution with the Hamiltonian and therefore constitutes an additional conserved quantity, yielding in this way the complete integrability of the system. Since it is the analog of the Carter constant for the Kerr metric we call it the *Carter Hamiltonian* and we denote it with the letter $\mathcal{C}$:

$$\mathcal{C}=\csc^2\theta\;(m\cos\theta-m+\ell)^2\;-\;p_\theta^2\tag{9.2}$$

An immediate calculation shows that the Carter function has vanishing Poisson bracket with the reduced Hamiltonian:

$$\{\mathcal{C},\ H_{red}\} = \sum_{i=1}^{2}\left(\frac{\partial\mathcal{C}}{\partial q^i}\frac{\partial H_{red}}{\partial p_i} - \frac{\partial\mathcal{C}}{\partial p_i}\frac{\partial H_{red}}{\partial q^i}\right) = 0$$

Hence, on any solution of the equations motion (that is along geodesics) both the principal Hamiltonian H_{red} and $\mathcal{C}$ must assume constant values that we call, respectively, $\mathcal{E}$ (the energy) and K (the Carter constant):

$$H_{red} = \mathcal{E} \quad ; \quad \mathcal{C} = K \tag{9.3}$$

Using Eqs. (9.2) and (9.1) we can solve algebraically Eq. (9.3) for the two momenta $p_{\mathfrak{v}}$ and p_θ and we get the following two first-order differential equations:

$$\begin{aligned}\frac{\mathrm{d}\theta}{\mathrm{d}s} &= \frac{\sqrt{\csc^2\theta\,\left(\cos 2\theta\,\left(K+m^2\right) - K + 3m^2 + 2\ell^2 + 4m\cos\theta(\ell-m) - 4m\ell\right)}}{\sqrt{2}\mathfrak{v}}\\ \frac{\mathrm{d}\mathfrak{v}}{\mathrm{d}s} &= \frac{\sqrt{\mathcal{FK}(\mathfrak{v})(K+2\mathfrak{v}\mathcal{E}) + m^2\mathfrak{v}}}{\sqrt{\mathfrak{v}}}\end{aligned} \tag{9.4}$$

Eliminating the derivatives with respect to s we finally obtain the differential equation of the "orbit"

$$\frac{\mathrm{d}\theta}{\mathrm{d}\mathfrak{v}} = \frac{\sqrt{\csc^2(\theta)\left(\cos(2\theta)\left(K+m^2\right) - K + 3m^2 + 2\ell^2 + 4m\cos(\theta)(\ell-m) - 4m\ell\right)}}{\sqrt{2}\sqrt{\mathfrak{v}}\sqrt{\mathcal{FK}(\mathfrak{v})(K+2\mathfrak{v}\mathcal{E}) + m^2\mathfrak{v}}} \tag{9.5}$$

which can be reduced to quadratures:

$$\begin{aligned}\Lambda\,(\theta) &= \int \frac{\mathrm{d}\theta}{\sqrt{\csc^2(\theta)\left(\cos(2\theta)\left(K+m^2\right) - K + 3m^2 + 2\ell^2 + 4m\cos(\theta)(\ell-m) - 4m\ell\right)}}\\ \Sigma\,(\mathfrak{v}) &= \int \frac{\mathrm{d}\mathfrak{v}}{\sqrt{2}\sqrt{\mathfrak{v}}\sqrt{\mathcal{FK}(\mathfrak{v})(K+2\mathfrak{v}\mathcal{E}) + m^2\mathfrak{v}}}\end{aligned}$$

The solution of the geodesic problem is provided by giving the dependence of the variable $\mathfrak{v}$ on the *right ascension angle* θ of the first sphere:

$$\mathfrak{v} = \Sigma^{-1} \circ \Lambda\,(\theta)\,, \quad \theta = \Lambda^{-1} \circ \Sigma\,(\mathfrak{v})$$

Both functions Λ and Σ are transcendental and the inversion problem can be solved only numerically, except for some special cases as we are going to illustrate in the next section.

We conclude this section by noting that the existence of the Carter conserved Hamiltonian is probably an implicit consequence of the larger non-Abelian isometry of the original metric. The two first integrals ℓ, $\mathfrak{m}$ follow from the toric symmetry

U(1) × U(1). The Carter constant is indirectly linked to the extension to SU(2) of one of the two U(1)'s. If we had SU(2) × SU(2) isometry, then the metric would be the direct product of two Fubini Study metrics. With SU(2) × U(1) isometry, we have the hybrid case where one sphere is the fiber and the other is the base manifold.

9.2 Irrotational geodesics

The function $\Lambda(\theta)$ can be calculated explicitly in the general case ($\ell \leq 0$, $(m) \leq 0$) and we obtain:

$$
\begin{aligned}
\Lambda(\theta) &= \tfrac{N(\theta)}{D(\theta)}\\
N(\theta) &= \sqrt{\cos(2\theta)\left(K+m^2\right) - K + 3m^2 + 2\ell^2 + 4m\cos(\theta)(\ell - m) - 4m\ell}\\
&\quad\times\left(-\arctan\left[\frac{\sec^2\left(\frac{\theta}{2}\right)\left(\cos(\theta)\left(K+m^2\right)+m(\ell-m)\right)}{\sqrt{K+m^2}\sqrt{\sec^4\left(\frac{\theta}{2}\right)(m\cos(\theta)-m+\ell)^2-4K\tan^2\left(\frac{\theta}{2}\right)}}\right]\right)\\
D(\theta) &= (\cos(\theta)+1)\sqrt{K+m^2}\sqrt{\sec^4\left(\frac{\theta}{2}\right)(m\cos(\theta)-m+\ell)^2-4K\tan^2\left(\frac{\theta}{2}\right)}
\end{aligned}
$$

while the integral defining the function $\Sigma(\mathfrak{v})$ in the general case does not evaluate to a combination of known special functions—neither for the $\mathbb{F}_2$ metric nor for KE metrics.

Although it can be done, it is rather cumbersome to write explicit computer codes for the numerical computation of the function Σ in the general case and for the needed inverse of the function Λ. Hence, at this stage it is difficult to present explicit geodesics with trivial angular momenta. The alternative is the numerical integration of the pair of first-order equations (9.4) but also here we meet some difficulties since the differential system is *stiff*[9] and without special care and an in-depth study of the phase space, the standard integration routines run into divergences and fail to provide solutions both for the KE and the Hirzebruch case. This is not surprising given the analogy with the Kerr metric. Indeed, the study of Kerr geodesics is a wide field, there is a large variety of types of geodesics and each requires nontrivial computational efforts to be worked out.

Yet in our case things enormously simplify if we consider *irrotational geodesics* defined as those where $\ell = m = 0$ and only the Carter constant $\mathcal{C}$ and the energy $\mathcal{E}$ label the curve. Geometrically this corresponds to the fact that the azimuthal angles ϕ, τ span a two-dimensional torus T^2. Pursuing the analogy with General Relativity, irrotational geodesics are the analogs of the *radial geodesics* utilized in cosmology and in the study of the causal structure of spacetimes where one preserves only time t and radial distance r. The analogs of t and r are in our case the variables $\mathfrak{v}$ and θ, namely, the two ascension angles of $S^2 \times S^2$.

[9] The term "stiff" comes from numerical analysis and denotes a differential equation or differential system whose numerical solution is unstable unless the step size is taken to be very small.

By suppressing angular momenta things simplify drastically. The orbit Eq. (9.5) reduces to

$$\frac{d\theta}{d\mathfrak{v}} = \frac{\sqrt{-K}}{\sqrt{\mathfrak{v}}\sqrt{\mathcal{FK}(\mathfrak{v})(K+2\mathfrak{v}\mathcal{E})}},$$

which implies

$$\theta = \mathcal{FM}(\mathfrak{v}, K, \mathcal{E}) = \int \frac{\sqrt{-K}}{\sqrt{\mathfrak{v}}\sqrt{\mathcal{FK}(\mathfrak{v})(K+2\mathfrak{v}\mathcal{E})}} d\mathfrak{v}. \tag{9.6}$$

The good news is that in the KE case (with the choice $\lambda_1 = 1$, $\lambda_2 = 2$), the integral of Eq. (9.6) can be explicitly evaluated obtaining

$$\begin{aligned}
\mathcal{FM}^{\mathrm{KE}}(\mathfrak{v}, K, \mathcal{E}) &= \tfrac{N(\mathfrak{v},K,\mathcal{E})}{D(\mathfrak{v},K,\mathcal{E})}\\
N(\mathfrak{v}, K, \mathcal{E}) &= \sqrt{7}\sqrt{-K}(\mathfrak{v}-2)(\mathfrak{v}-1)\sqrt{\tfrac{(3\mathfrak{v}+2)(K+4\mathcal{E})}{K+2\mathfrak{v}\mathcal{E}}}\\
&\quad \times F\left(\arcsin\left(\frac{\sqrt{-\frac{(3K-4\mathcal{E})(\mathfrak{v}-2)}{K+2\mathcal{E}\mathfrak{v}}}}{2\sqrt{2}}\right) | \tfrac{8(K+2\mathcal{E})}{3K-4\mathcal{E}}\right)\\
D(\mathfrak{v}, K, \mathcal{E}) &= \sqrt{\mathfrak{v}}\sqrt{-\tfrac{(\mathfrak{v}-2)(3K-4\mathcal{E})}{K+2\mathfrak{v}\mathcal{E}}}\sqrt{\tfrac{(\mathfrak{v}-1)(K+4\mathcal{E})}{K+2\mathfrak{v}\mathcal{E}}}\\
&\quad \times\sqrt{-\tfrac{(3\mathfrak{v}^3-7\mathfrak{v}^2+4)(K+2\mathfrak{v}\mathcal{E})}{\mathfrak{v}}}
\end{aligned}$$

where by $F(z|h)$ we have denoted the F elliptic function.

In the case of the Hirzebruch surface metric, the integral in Eq. (9.6) does not evaluate to known special functions, yet it can be easily computed numerically, allowing one to draw the geodesic curves in the $\mathfrak{u}$, $\mathfrak{v}$-plane: the parametric form is

$$\{(1 - \cos[\mathcal{FM}(\mathfrak{v}, K, \mathcal{E})])\ \mathfrak{v}\ ,\ \mathfrak{v}\}$$

as follows from Eq. (6.3). Choosing various different values of the energy and of the Carter constant we obtain the curves shown in Fig. 12.

In both cases the irrotational geodesics are smooth and approach value $\mathfrak{v}_{\max}$, that is, the North pole of the second sphere. The only difference is that in the Hirzebruch case they reach $\mathfrak{v}_{\max}$ at various values of the right ascension of the first 2-sphere while in the KE case they tend to reach the North Pole of the second sphere arriving simultaneously at the North Pole of the first sphere.

10 The Calabi Ansatz and the AMSY symplectic formalism

Having studied in some detail the KE base manifolds $\mathcal{M}_B^{\mathrm{KE}}$, we turn now to main issue of this paper, namely, the construction of a Ricci-flat metric on their canonical bundle. We turn to the method introduced by Calabi, which, however, only works for KE base manifolds.

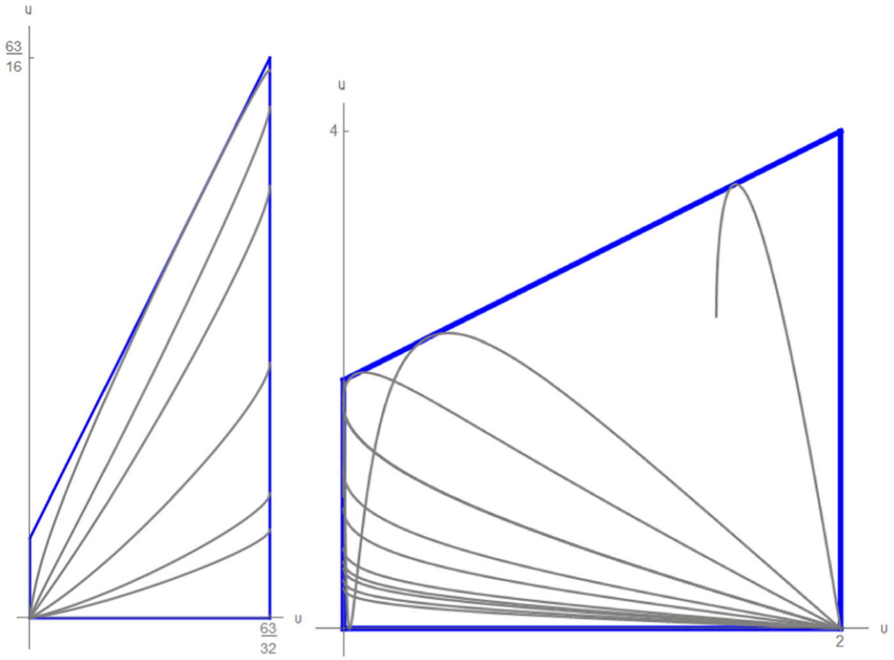

Fig. 12 On the left a plot of some irrotational geodesics for the case of the Hirzebruch surface. On the right plot of the same type of geodesics for the KE metric

As we anticipated in the Introduction, the Calabi ansatz method will produce a Ricci-flat metric on the canonical bundle tot $[K\,[\mathcal{M}_{\mathrm{KE}}]]$ that one might be tempted to consider diffeomorphic to the Ricci-flat metric on the metric cone over the Sasaki–Einstein manifolds of [32]; as we anticipated this is not true, notwithstanding the very close relation of the Kähler Einstein manifolds discussed above with the five-dimensional Sasaki–Einstein manifolds of [32]. We have so far postponed the comparison of the KE metrics discussed in the above sections with the base manifolds of the five-dimensional SE fibrations of [32] since such a comparison will be done more appropriately just in one stroke together with the comparison of the six-dimensional Ricci-flat metrics.

10.1 Calabi's Ansatz

Calabi's paper [11] introduces the following Ansatz for the local Kähler potential $\mathcal{K}(z, \bar{z}, w, \bar{w})$ of a Kähler metric g_E on the total space of a holomorphic vector bundle $E \to \mathcal{M}$, where $\mathcal{M}$ is a compact Kähler manifold satisfying the conditions already stated in the Introduction:

$$\mathcal{K}(z, \bar{z}, w, \bar{w}) = \mathcal{K}_0(z, \bar{z}) + U(\lambda) \tag{10.1}$$

where $\mathcal{K}_0(z,\bar{z})$ is a Kähler potential for $g_{\mathcal{M}}$, ($z^i, i = 1, \dots \dim_{\mathbb{C}}\mathcal{M}$) being the complex coordinates of the base manifold) and U is a function of a real variable λ, which we shall identify with the function

$$\lambda = \mathcal{H}_{\mu\bar{\nu}}(z,\bar{z})\, w^{\mu}\, w^{\bar{\nu}} = \parallel w \parallel^2 \tag{10.2}$$

(the square norm of a section of the bundle with respect to a fiber metric $\mathcal{H}_{\mu\bar{\nu}}(z,\bar{z})$). If θ is the Chern connection on E, canonically determined by the Hermitian structure $\mathcal{H}$ and the holomorphic structure of E, its local connections forms can be written as

$$\theta^{\lambda}_{\nu} = \sum_i dz^i\, L^{\lambda}_{i|\nu}$$

where

$$L^{\lambda}_{i|\nu} = \sum_{\bar{\mu}} \mathcal{H}^{\lambda\bar{\mu}} \frac{\partial}{\partial z^i} \mathcal{H}_{\nu\bar{\mu}}; \quad [\mathcal{H}^{\lambda\bar{\mu}}] = ([\mathcal{H}_{\lambda\bar{\mu}}]^{-1})^T$$

The curvature 2-form Θ of the connection θ is given by:

$$\Theta^{\lambda}_{\nu} = \sum_{i,\bar{j}} dz^i \wedge \mathrm{d}\bar{z}^{\bar{j}}\, S^{\lambda}_{i\bar{j}|\nu}; \quad S^{\lambda}_{i\bar{j}|\nu} = \frac{\partial}{\partial \bar{z}^{\bar{j}}} L^{\lambda}_{i|\nu}$$

The Kähler metric g_E corresponding to the Kähler potential $\mathcal{K}$ can be written as follows:

$$\partial\bar{\partial}\mathcal{K} = \sum_{i,\bar{j}} \Big[g_{i\bar{j}} + \lambda\, U'(\lambda) \sum_{\lambda,\nu,\bar{\mu}} \mathcal{H}_{\sigma\bar{\mu}} S^{\sigma}_{i\bar{j}|\rho} w^{\rho} \bar{w}^{\bar{\mu}} \Big] dz^i \mathrm{d}\bar{z}^{J} + \sum_{\sigma,\bar{\mu}} \Big[U'(\lambda) + \lambda\, U''(\lambda) \Big] \mathcal{H}_{\sigma\bar{\mu}}\, \nabla w^{\sigma}\, \nabla \bar{w}^{\bar{\mu}}.$$

If E is a line bundle, then the above equation reduces to

$$\partial\bar{\partial}\mathcal{K} = \sum_{i,\bar{j}} [g_{i\bar{j}} + \lambda U'(\lambda)\, S_{i\bar{j}}] dz^i \mathrm{d}\bar{z}^{\bar{j}} + [U'(\lambda) + \lambda U''(\lambda)] \mathcal{H}(z,\bar{z}) \nabla w \nabla \bar{w}$$

where $\lambda = \mathcal{H}(z,\bar{z})\, w\bar{w}$ is the nonnegative real quantity defined in Eq. (10.2) and ∇w denotes the covariant derivative of the fiber coordinate with respect to Chern connection θ:

$$\nabla w = dw + \theta\, w$$

10.2 Ricci-flat metrics on canonical bundles

Now we assume that E is the canonical bundle $K_{\mathcal{M}}$ of a Kähler surface $\mathcal{M}$ ($\dim_{\mathbb{C}} \mathcal{M} = 2$). The total space of $K_{\mathcal{M}}$ has vanishing first Chern class, i.e., it is a noncompact Calabi–Yau manifold, and we may try to construct explicitly a Ricci-flat metric on it. Actually, following Calabi, we can reduce the condition that g_E is Ricci-flat to a differential equation for the function $U(\lambda)$ introduced in Eq. (10.1). Note that under the present assumptions S is a scalar-valued 2-form on $\mathcal{M}$.

Since our main target is the construction of a Ricci-flat metric on the space $\mathrm{tot}(\mathcal{K}_{\mathcal{M}_B^{\mathrm{KE}}})$, where $\mathcal{M}_B^{\mathrm{KE}}$ denotes any of the KE manifolds discussed at length in previous sections, we begin precisely with an analysis of that case which will allow to derive a general form of $U(\lambda)$ as a function of the moment $\mathfrak{w}$ associated with the U(1) group acting by phase transformations of the fiber coordinate w, and of certain coefficients A, B, F that are determined in terms of the Kähler potential $\mathcal{K}_0$ of the base manifold $\mathcal{M}$. Consistency of the Calabi Ansatz requires that these coefficients should be constant, which happens in the case of base manifolds equipped with Kähler Einstein metrics. KE metrics do not exist on Hirzebruch surfaces and the Calabi Ansatz is not applicable in this case. As we discuss in the sequel, there exists a Ricci-flat metric on the canonical bundle of a singular blow-down of $\mathbb{F}_2$, namely the weighted projective plane $\mathbb{WP}[1, 1, 2]$, which is known in the AMSY symplectic toric formalism of [1] and [43]. If we were able to do the inverse Legendre transform we might reconstruct the so far missing Kähler potential and get inspiration on possible generalizations of the Calabi Ansatz. Hence, we are going to pay a lot of attention to both formulations, the Kähler one and the symplectic one.

10.3 Calabi Ansatz for 4D Kähler metrics with SU(2) × U(1) isometry

The Calabi Ansatz can be applied with success or not according to the structure of the Kähler potential $\mathcal{K}_0$ for the base manifold $\mathcal{M}$ and the algebraic form of the invariant combination Ω of the complex coordinates u, v which is the only real variable from which the Kähler potential $\mathcal{K}_0 = \mathcal{K}_0(\Omega)$ is assumed to depend. On the other hand Ω encodes the group of isometries which is imposed on the Kähler metric of $\mathcal{M}$.

In the case of the metrics discussed in Sect. 4, having $\mathrm{SU}(2) \times \mathrm{U}(1)$ isometry, the invariant is chosen to be

$$\Omega = \varpi$$

where ϖ was defined in Eq. (4.1). This choice guarantees the isometry of the Kähler metric $g_{\mathcal{M}}$ against the group $\mathrm{SU}(2) \times \mathrm{U}(1)$ with the action described in Eq. (4.2). Hence, we focus on such manifolds and we consider a Kähler potential for $\mathcal{M}$ that for the time being is a generic function $\mathcal{K}_0(\varpi)$ of the invariant variable. In this case the determinant of the Kähler metric $g_{\mathcal{M}}$ has an explicit expression in terms of $\mathcal{K}_0(\varpi)$

$$\det\left(g_{\mathcal{M}}\right) = 2\varpi\mathcal{K}_0'(\varpi)\left(\varpi\mathcal{K}_0''(\varpi)+\mathcal{K}_0'(\varpi)\right)$$

while the determinant of the Ricci tensor takes the form

$$\det\left(\mathrm{Ric}_{\mathcal{M}}\right) = \frac{N_{Ric}}{D_{Ric}}$$

$$\begin{aligned} N_{Ric} = {} & 2\left(\varpi^2\mathcal{K}_0''(\varpi)^2+\mathcal{K}_0'(\varpi)^2+\varpi\mathcal{K}_0'(\varpi)\left(\varpi\mathcal{K}_0^{(3)}(\varpi)+4\mathcal{K}_0''(\varpi)\right)\right)\times \\ & \times\left(-\varpi^3\mathcal{K}_0''(\varpi)^4+\varpi^2\mathcal{K}_0'(\varpi)\left(\varpi\mathcal{K}_0^{(3)}(\varpi)-\mathcal{K}_0''(\varpi)\right)\mathcal{K}_0''(\varpi)^2\right. \\ & +\varpi\mathcal{K}_0'(\varpi)^2\left(-\varpi^2\mathcal{K}_0^{(3)}(\varpi)^2-\mathcal{K}_0''(\varpi)^2+\varpi\left(\varpi\mathcal{K}_0^{(4)}(\varpi)+2\mathcal{K}_0^{(3)}(\varpi)\right)\mathcal{K}_0''(\varpi)\right) \\ & \left.+\mathcal{K}_0'(\varpi)^3\left(3\mathcal{K}_0''(\varpi)+\varpi\left(\varpi\mathcal{K}_0^{(4)}(\varpi)+5\mathcal{K}_0^{(3)}(\varpi)\right)\right)\right) \\ D_{Ric} = {} & \mathcal{K}_0'(\varpi)^3\left(\varpi\mathcal{K}_0''(\varpi)+\mathcal{K}_0'(\varpi)\right)^3 \end{aligned}$$

and the scalar curvature

$$R_s = \mathrm{Tr}\left(\mathrm{Ric}_{\mathcal{M}}\, g_{\mathcal{M}}^{-1}\right)$$

is

$$\begin{aligned} R_s &= \tfrac{N_s}{D_s} \\ N_s &= \mathcal{K}_0{}'(\varpi)^3 + \varpi^3 \mathcal{K}_0{}''(\varpi)^2\left(2\varpi\mathcal{K}_0{}^{(3)}(\varpi) + 5\mathcal{K}_0{}''(\varpi)\right) \\ &\quad + \varpi^2\mathcal{K}_0{}'(\varpi)\left(-\varpi^2\mathcal{K}_0{}^{(3)}(\varpi)^2 + 9\mathcal{K}_0{}''(\varpi)^2 + \varpi\left(\varpi\mathcal{K}_0{}^{(4)}(\varpi) + 4\mathcal{K}_0{}^{(3)}(\varpi)\right)\mathcal{K}_0{}''(\varpi)\right) \\ &\quad + \varpi\mathcal{K}_0{}'(\varpi)^2\left(9\mathcal{K}_0{}''(\varpi) + \varpi\left(\varpi\mathcal{K}_0{}^{(4)}(\varpi) + 6\mathcal{K}_0{}^{(3)}(\varpi)\right)\right) \\ D_s &= \varpi\mathcal{K}_0{}'(\varpi)\left(\varpi\mathcal{K}_0{}''(\varpi) + \mathcal{K}_0{}'(\varpi)\right)^3 \end{aligned}$$

Given these base-manifold data, we introduce a Kähler potential for a metric on the canonical bundle $\mathrm{tot}\,[K\,(\mathcal{M})]$ in accordance with the Calabi Ansatz, namely

$$\mathcal{K}\,(\varpi,\lambda) = \mathcal{K}_0(\varpi) \;+\; U(\lambda); \quad \lambda = \underbrace{\exp\left[\mathcal{P}(\varpi)\right]}_{\text{fibermetric}\mathcal{H}(\varpi)}\; \mid w\mid^2 = \parallel w \parallel^2$$

where λ is the square norm of a section of the canonical bundle and $\exp\left[\mathcal{P}(\varpi)\right]$ is some fiber metric. The determinant of the corresponding Kähler metric g_E on the total space of the canonical bundle is

$$\begin{aligned} \det g_E &= 2\varpi\,\Sigma(\lambda)e^{\mathcal{P}(\varpi)}\Sigma'(\lambda)\left(\varpi\mathcal{K}_0''(\varpi)\mathcal{P}'(\varpi) + \mathcal{K}_0'(\varpi)\left(\varpi\mathcal{P}''(\varpi) + 2\mathcal{P}'(\varpi)\right)\right) \\ &\quad + 2\varpi\, e^{\mathcal{P}(\varpi)}\Sigma'(\lambda)\mathcal{K}_0'(\varpi)\left(\mathcal{K}_0'(\varpi) + \varpi\mathcal{K}_0''(\varpi)\right) \\ &\quad + 2\varpi\,\Sigma(\lambda)^2 e^{\mathcal{P}(\varpi)}\Sigma'(\lambda)\mathcal{P}'(\varpi)\left(\varpi\mathcal{P}''(\varpi) + \mathcal{P}'(\varpi)\right) \end{aligned}$$

where we set

$$\Sigma(\lambda) = \lambda\, U'(\lambda)$$

If we impose the Ricci-flatness condition, namely, that the determinant of the metric g_E is a constant which we can always assume to be one since any other number can be reabsorbed into the normalization of the fiber coordinate w, by integration we get

$$\lambda = \frac{1}{48}\left(A\,\mathfrak{w}^3 + 2B\,\mathfrak{w}^2 + 4\,F\,\mathfrak{w}\right) \tag{10.3}$$

where we have set

$$\begin{aligned} \Sigma(\lambda) &= 2\,\mathfrak{w} \\ A &= 4\varpi\, e^{\mathcal{P}(\varpi)}\mathcal{P}'(\varpi)\left[\varpi\mathcal{P}''(\varpi) + \mathcal{P}'(\varpi)\right] \\ B &= 6\varpi\, e^{\mathcal{P}(\varpi)}\left[\varpi\mathcal{K}_0''(\varpi)\mathcal{P}'(\varpi) + \mathcal{K}_0'(\varpi)\left(\varpi\mathcal{P}''(\varpi) + 2\mathcal{P}'(\varpi)\right)\right] \\ F &= 12\varpi\, e^{\mathcal{P}(\varpi)}\mathcal{K}_0'(\varpi)\left[\mathcal{K}_0'(\varpi) + \varpi\mathcal{K}_0''(\varpi)\right] \end{aligned} \tag{10.4}$$

In our complex three-dimensional case, setting

$$x_u = \log \mid u \mid, \quad x_v = \log \mid v \mid, \qquad x_w = \log \mid w \mid,$$

the corresponding three moments can be named with the corresponding gothic letters, and we have

$$\mathfrak{u} = \partial_{x_u} \mathcal{K}(\varpi, \lambda) , \quad \mathfrak{v} = \partial_{x_v} \mathcal{K}(\varpi, \lambda) , \quad \mathfrak{w} = \partial_{x_w} \mathcal{K}(\varpi, \lambda) .$$

As the fiber coordinate w appears only in the function $U(\lambda)$ via the squared norm λ, we have

$$\mathfrak{w} = 2\,\lambda\, U'(\lambda) = \Sigma(\lambda)$$

and this justifies the position (10.4). At this point the function $U(\lambda)$ can be easily determined by first observing that, in view of Eq. (10.3) we can also set

$$U(\lambda) = \mathrm{U}\,(\mathfrak{w})$$

and we can use the chain rule

$$\partial_{\mathfrak{w}}\, \mathrm{U}\,(\mathfrak{w}) = \frac{\mathfrak{w}\lambda'(\mathfrak{w})}{2\lambda(\mathfrak{w})} = \frac{3A\mathfrak{w}^2 + 4B\mathfrak{w} + 4F}{2A\mathfrak{w}^2 + 4B\mathfrak{w} + 8F}$$

which by integration yields the universal function

$$\mathrm{U}(\mathfrak{w}) = -\frac{2\sqrt{4\,A\,F - B^2}\,\arctan\left(\frac{A\,\mathfrak{w}+B}{\sqrt{4\,A\,F-B^2}}\right) + B\,\log\left(A\,\mathfrak{w}^2 + 2\,B\mathfrak{w} + 4\,F\right) - 3\,A\,\mathfrak{w}}{2\,A} \tag{10.5}$$

The function $U(\lambda)$ appearing in the Kähler potential can be obtained by substituting for the argument $\mathfrak{w}$ in (10.5) the unique real root of the cubic equation (10.3), namely:

$$\begin{aligned}\mathfrak{w} = & \frac{\sqrt[3]{8\sqrt{\left(162A^2\lambda + 9ABF - 2B^3\right)^2 - 4\left(B^2 - 3AF\right)^3} + 1296A^2\lambda + 72ABF - 16B^3}}{3\sqrt[3]{2}A} \\ & + \frac{4\sqrt[3]{2}\left(B^2 - 3AF\right)}{3A\sqrt[3]{8\sqrt{\left(162A^2\lambda + 9ABF - 2B^3\right)^2 - 4\left(B^2 - 3AF\right)^3} + 1296A^2\lambda + 72ABF - 16B^3}} - \frac{2B}{3A}\end{aligned}$$

10.3.1 Consistency conditions for the Calabi Ansatz

In order for the Calabi Ansatz to yield a solution of the Ricci-flatness condition it is necessary that the universal function $\mathrm{U}(\mathfrak{w})$ in Eq. (10.5) should depend only on $\mathfrak{w}$, which happens if and only if the coefficients A, B, F are constant. In the case under consideration, where the invariant combination of the complex coordinate u, v is the one provided by ϖ as defined in Eq. (4.1), imposing such a consistency condition would require the solution of three ordinary differential equations for two functions

$\mathcal{P}(\varpi)$ and $\mathcal{K}_0(\varpi)$, namely:

$$\begin{aligned} k_1 &= 4\varpi e^{\mathcal{P}(\varpi)}\mathcal{P}'(\varpi)\left[\varpi\mathcal{P}''(\varpi)+\mathcal{P}'(\varpi)\right] \\ k_2 &= 6\varpi e^{\mathcal{P}(\varpi)}\left[\varpi\mathcal{K}_0''(\varpi)\mathcal{P}'(\varpi)+\mathcal{K}_0'(\varpi)\left(\varpi\mathcal{P}''(\varpi)+2\mathcal{P}'(\varpi)\right)\right] \\ k_3 &= 12\varpi e^{\mathcal{P}(\varpi)}\mathcal{K}_0'(\varpi)\left[\mathcal{K}_0'(\varpi)+\varpi\mathcal{K}_0''(\varpi)\right] \end{aligned} \tag{10.6}$$

where $k_{1,2,3}$ are three constants. It is clear from their structure that the crucial differential equation is the first one. If we could find a solution for it then it would suffice to identify the original Kähler potential $\mathcal{K}_0(\varpi)$ with a linear function of $\mathcal{P}(\varpi)$ and we could solve the three of them. So far we were not able to find any analytical solution of these equations but if we could find one, we still should verify that the Kähler metric following from such $\mathcal{K}_0$ is a good metric on the Hirzebruch surface $\mathbb{F}_2$.

On the contrary for the well known Kähler potentials obtained from the Kronheimer construction that define a one-parameter family of *bona fide* Kähler metrics on $\mathbb{F}_2$ and were discussed in [8, 9], namely those presented in Eqs. (4.4) and (10.6) cannot be satisfied and no Ricci-flat metric on the canonical bundle can be obtained by means of the Calabi Ansatz.

10.3.2 The general case with the natural fiber metric $\mathcal{H} = \frac{1}{\det(g_{\mathcal{M}})}$

If we consider the general case of a toric two-dimensional compact manifold $\mathcal{M}$ with a Kähler metric $g_{\mathcal{M}}$ derived from a Kähler potential of the form:

$$\mathcal{K}_0 = \mathcal{K}_0\left(|u|^2, |v|^2\right)$$

choosing as fiber metric the natural one for the canonical bundle, namely setting:

$$\lambda = \mathcal{H}\,|w|^2 = \frac{1}{\det\,(g_{\mathcal{M}})}\,|w|^2$$

and going through the same steps as in Sect. 10.3 we arrive at an identical result for the function $\mathrm{U}(\mathfrak{w})$ as in Eq. (10.5) but with the following coefficients:

$$A = 2\,\frac{\det\,(\mathrm{Ric}_{\mathcal{M}})}{\det\,(g_{\mathcal{M}})}, \quad B = 3\,\mathrm{Tr}\left(\mathrm{Ric}_{\mathcal{M}}\,g_{\mathcal{M}}^{-1}\right), \quad F = 6$$

It clearly appears why the Calabi Ansatz works perfectly if the starting metric on the base manifold is KE. In that case the Ricci tensor is proportional to the metric tensor:

$$\mathrm{Ric}_{i\bar{j}} = \kappa\, g_{i\bar{j}} \tag{10.7}$$

and we get:

$$\det\,(\mathrm{Ric}_{\mathcal{M}}) = \kappa^2 \det\,(g_{\mathcal{M}})\,, \quad \mathrm{Tr}\left(\mathrm{Ric}_{\mathcal{M}}\,g_{\mathcal{M}}^{-1}\right) = 2\,\kappa$$

which implies:

$$A = 2\,\kappa^2, \quad B = 6\,\kappa, \quad F = 6. \tag{10.8}$$

10.4 The AMSY symplectic formulation for the Ricci-flat metric on tot $[K(\mathcal{M}_B)]$

According to the discussion of the AMSY symplectic formalism presented in Sect. 3, given the Kähler potential of a toric complex three manifold $\mathcal{K}(|u|, |v|, |w|)$, we can define the moments

$$\mathfrak{u} = \partial_{x_u}\mathcal{K}, \quad \mathfrak{v} = \partial_{x_v}\mathcal{K}, \quad \mathfrak{w} = \partial_{x_w}\mathcal{K}$$

and we can obtain the symplectic potential by means of the Legendre transform:

$$G(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = x_u\,\mathfrak{u} + x_v\,\mathfrak{v} + x_w\,\mathfrak{w} - \mathcal{K}(|u|, |v|, |w|) \tag{10.9}$$

The main issue in the use of Eq. (10.9) is the inversion transformation that expresses the coordinates $x_i = \{x_u, x_v, x_w\}$ in terms of the three moments $\mu^i = \{\mathfrak{u}, \mathfrak{v}, \mathfrak{w}\}$. Once this is done one can calculate the metric in moment variables utilizing the Hessian as explained in Sect. 3. Relying once again on the results of that section we know that the Kähler 2-form has the following universal structure:

$$\mathbb{K} = \mathrm{d}\mathfrak{u} \wedge \mathrm{d}\phi + \mathrm{d}\mathfrak{v} \wedge \mathrm{d}\tau + \mathrm{d}\mathfrak{w} \wedge \mathrm{d}\chi$$

and the metric is expressed as displayed in Eq. (3.3))

10.4.1 The symplectic potential in the case with SU(2) × U(1) × U(1) isometries

In the case where the Kähler potential has the special structure which guarantees an SU(2) × U(1) × U(1) isometry, namely it depends only on the two variables ϖ (see Eq. (4.1))) and $|w|^2$, also the symplectic potential takes a more restricted form. Indeed, we find

$$G(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = \underbrace{\left(\mathfrak{v} - \frac{\mathfrak{u}}{2}\right)\log(2\mathfrak{v} - \mathfrak{u}) + \frac{1}{2}\mathfrak{u}\log(\mathfrak{u}) - \frac{1}{2}\mathfrak{v}\log(\mathfrak{v})}_{\text{universalpart}\,G_0(\mathfrak{u},\mathfrak{v})} + \underbrace{\mathcal{G}(\mathfrak{v}, \mathfrak{w})}_{\text{variablepart}} \tag{10.10}$$

where $\mathcal{G}(\mathfrak{v}, \mathfrak{w})$ is a function of two variables that encodes the specific structure of the metric. Note that when we freeze the fiber moment coordinate $\mathfrak{w}$ to some fixed constant value, for instance 0, the function $\mathcal{G}(\mathfrak{v}, 0) = \mathcal{D}(\mathfrak{v})$ can be identified with the boundary function that appears in Eqs. (4.7),(4.8), namely in the symplectic potential for the Kähler metric of the base manifold.

With the specific structure (10.10) of the symplectic potential we obtain the following form for the Hessian (3.2):

$$\mathbf{G} = \begin{pmatrix} -\frac{\mathfrak{v}}{\mathfrak{u}^2-2\mathfrak{u}\mathfrak{v}} & \frac{1}{\mathfrak{u}-2\mathfrak{v}} & 0 \\ \frac{1}{\mathfrak{u}-2\mathfrak{v}} & \frac{-2\mathfrak{v}(\mathfrak{u}-2\mathfrak{v})\mathcal{G}^{(2,0)}(\mathfrak{v},\mathfrak{w})+\mathfrak{u}+2\mathfrak{v}}{2\mathfrak{v}(2\mathfrak{v}-\mathfrak{u})} & \mathcal{G}^{(1,1)}(\mathfrak{v}, \mathfrak{w}) \\ 0 & \mathcal{G}^{(1,1)}(\mathfrak{v}, \mathfrak{w}) & \mathcal{G}^{(0,2)}(\mathfrak{v}, \mathfrak{w}) \end{pmatrix}$$

11 The general form of the symplectic potential for the Ricci flat metric on tot $\left[K\left(\mathcal{M}_B^{\mathrm{KE}}\right)\right]$

Having seen that KE metrics do indeed exist in the form described in Eqs. (4.5), (4.6), it is natural to inquire how we can utilize the Calabi Ansatz to write immediately the symplectic potential for a Ricci-flat metric on the canonical bundle of $\mathcal{M}_B^{\mathrm{KE}}$ without going through the process of inverting the Legendre transform. Namely, we would like to make the back and forth trip via inverse and direct Legendre transform only once and in full generality rather than case by case. Our goal is not only a simplification of the computatinal steps but it also involves a conceptual issue. Indeed, when we introduce intermediate steps that rely on the variable ϖ whose range is $[0, +\infty)$ we necessarily have to choose a branch of a cubic equation whose coefficients are determined by the root parameters $\lambda_{1,2}$. On the contrary, if we are able to determine directly the symplectic potential in terms of the symplectic coordinates, then we can explore the behavior of the metric and of its curvature on the full available range of variability of these latter and we learn more on the algebraic and topological structure of the underlying manifold.

So let us anticipate the final result of our general procedure. As we did in the previous section we assume that the Ricci form of $\mathcal{M}_B$ is proportional to the Kähler form via a coefficient

$$\kappa = \frac{k}{4} \tag{11.1}$$

as in eqs.(5.2),(10.7). The complete symplectic potential for the Ricci-flat metric on $\mathcal{M}_T = \mathrm{tot}\,[K\,(\mathcal{M}_B)]$ has the following structure:

$$\begin{aligned}
G_{\mathcal{M}_T^{\mathrm{KE}}}(\mathfrak{u},\mathfrak{v},\mathfrak{w}) &= G_0(\mathfrak{u},\mathfrak{v}) + \mathcal{G}^{\mathrm{KE}}(\mathfrak{v},\mathfrak{w})\\
G_0(\mathfrak{u},\mathfrak{v}) &= \left(\mathfrak{v}-\tfrac{\mathfrak{u}}{2}\right)\log[2\mathfrak{v}-\mathfrak{u}] + \tfrac{1}{2}\mathfrak{u}\log[\mathfrak{u}] - \tfrac{1}{2}\mathfrak{v}\log[\mathfrak{v}]\\
\mathcal{G}^{\mathrm{KE}}(\mathfrak{v},\mathfrak{w}) &= \left(\tfrac{\kappa\mathfrak{w}}{2}+1\right)\mathcal{D}^{\mathrm{KE}}\left(\tfrac{2\mathfrak{v}}{\kappa\mathfrak{w}+2}\right) - \tfrac{1}{2}\mathfrak{v}\log\left(\tfrac{\kappa\mathfrak{w}}{2}+1\right) + \tfrac{1}{2}\mathfrak{w}\log(\mathfrak{w})\\
&\quad + \frac{(\kappa\mathfrak{w}+3)\log(\kappa\mathfrak{w}(\kappa\mathfrak{w}+6)+12)}{2\kappa} + \frac{\sqrt{3}\arctan\left(\frac{\kappa\mathfrak{w}+3}{\sqrt{3}}\right)}{\kappa}
\end{aligned} \tag{11.2}$$

where the second equation is a repetition for the reader's convenience of Eq. (4.8) and $\mathcal{D}^{\mathrm{KE}}(\mathfrak{v}_0)$ is the boundary function defined in equation (5.7); the relation between the two independent roots $\lambda_{1,2}$ and the parameter κ ia provided by Eqs. (5.5) and (11.1). The reason while we have used the argument

$$\mathfrak{v}_0 = \frac{2\mathfrak{v}}{\kappa\mathfrak{w}+2} \tag{11.3}$$

is that the symplectic variable $\mathfrak{v}_0$ associated with the base-manifold metric and the symplectic variable $\mathfrak{v}$ associated with the metric on the canonical bundle $\mathcal{M}_T^{\mathrm{KE}}$ are not the same; their relation is precisely that in Eq. (11.3) which is a direct consequence of the Calabi Ansatz as we explain below.

11.1 Derivation of the formula for $\mathcal{G}^{\mathsf{KE}}(\mathfrak{v},\mathfrak{w})$

The general formula (11.2) is a direct yield of the direct Legendre transform after the Calabi Ansatz:

$$G_{\mathcal{M}_T^{\mathrm{KE}}}(\mathfrak{u},\mathfrak{v},\mathfrak{w}) = x_u\,\mathfrak{u} + x_v\,\mathfrak{v} + x_w\,\mathfrak{w} - \mathcal{K}_0(\mathfrak{v}_0) - U(\lambda) \tag{11.4}$$

where

$$\begin{aligned}
\lambda &= \tfrac{w\,\bar{w}}{\det g_{\mathcal{M}_B}} = \mathrm{const}\,\times\, w\,\bar{w}\,\;\exp\left[\kappa\,\mathcal{K}_0(\mathfrak{v}_0)\right] \\
&= \Lambda(\mathfrak{w}) = \tfrac{1}{24}\mathfrak{w}\left(\kappa^2\mathfrak{w}^2 + 6\kappa\mathfrak{w} + 12\right) \\
\frac{\mathfrak{w}}{2} &= \lambda\,U'(\lambda) \\
U(\lambda) = \mathbb{U}(\mathfrak{w}) &= \frac{-3\log\left(2\left(\kappa^2\mathfrak{w}^2 + 6\kappa\mathfrak{w} + 12\right)\right) + 3\kappa\mathfrak{w} - 2\sqrt{3}\arctan\left(\frac{\kappa\mathfrak{w}+3}{\sqrt{3}}\right)}{2\kappa} \\
\mathcal{K}_0(\mathfrak{v}_0) &= \mathfrak{v}_0\mathcal{D}'(\mathfrak{v}_0) - \mathcal{D}(\mathfrak{v}_0) + \tfrac{\mathfrak{v}_0}{2}
\end{aligned} \tag{11.5}$$

The last two lines in Eq. (11.5) were derived earlier, respectively, in Eqs. (10.5), (4.12). The explicit form of $\mathbb{U}(\mathfrak{w})$ follows from Eq. (10.5) using the KE condition, namely Eq. (10.8). From the above relations, one easily obtains the relations

$$\begin{aligned}
\mathfrak{u}_0 &= \frac{2\mathfrak{u}}{k\mathfrak{w}+2}, \quad \mathfrak{v}_0 = \frac{2\mathfrak{v}}{k\mathfrak{w}+2} \\
x_w &= \tfrac{1}{2}\left\{\log\left[\Lambda(\mathfrak{w})\right] - \kappa\,\mathcal{K}_0(\mathfrak{v}_0)\right\}
\end{aligned} \tag{11.6}$$

The first two relations can be understood as follows. The momenta $\mathfrak{u}_0, \mathfrak{v}_0$ are, by definition

$$\mathfrak{u}_0 = \partial_{x_u}\mathcal{K}_0 \quad ; \quad \mathfrak{v}_0 = \partial_{x_v u}\mathcal{K}_0$$

while we have

$$\mathfrak{u} = \partial_{x_u}\mathcal{K} \quad ; \quad \mathfrak{v} = \partial_{x_v}\mathcal{K}$$

By the Calabi Ansatz we get:

$$\mathfrak{u} = \mathfrak{u}_0 + \partial_{x_u}U(\lambda) = \mathfrak{u}_0 + \partial_{xu}\lambda\,\partial_\lambda U(\lambda) = \mathfrak{u}_0 + \kappa\partial_{xu}\mathcal{K}_0\,\lambda\partial_\lambda U(\lambda) = \mathfrak{u}_0\left(1 + \frac{\kappa}{2}\,\mathfrak{w}\right)$$

A completely analogous calculation can be done for the case of $\mathfrak{v}$. Finally let us note that the coordinate x_u, x_v were already resolved in terms of u_0, v_0 in Eqs. (4.11):

$$x_u = \frac{1}{2}\left(\log(\mathfrak{u}_0) - \log(2\mathfrak{v}_0 - \mathfrak{u}_0)\right); \, x_v = \mathcal{D}'(\mathfrak{v}_0) + \log(2\mathfrak{v}_0 - \mathfrak{u}_0) - \frac{1}{2}\log(\mathfrak{v}_0) + \frac{1}{2} \tag{11.7}$$

The information provided in Eqs. (11.5)–(11.7) is sufficient to complete the Legendre transform (11.4) and retrieve the very simple and elegant result encoded in Eq. (11.2).

To check the correctness of the general formula (11.2) we have explicitly calculated, by means of the MATHEMATICA package METRICGRAV[10], the Ricci tensor for a few cases of $\mathcal{M}_T^{[\lambda_1,\lambda_2]}$, always finding zero.

11.2 The example of the metric [2, 1]

Here we present the explicit form in symplectic coordinates of the Ricci-flat metric on the canonical bundle of the KE manifold $\mathcal{M}_B^{[1,2]}$, namely that determined by the choice: $\lambda_1 = 1, \lambda_2 = 2$. We get:

$$
\begin{aligned}
ds^2_{M_T^{[1,2]}} = {} & d\phi^2\left(-\frac{\mathfrak{u}^2(9\mathfrak{w}+14)^2}{343\mathfrak{v}^3}-\frac{16464\mathfrak{u}^2}{(9\mathfrak{w}+14)^4}+2\mathfrak{u}\right)\\
& +\frac{d\mathfrak{v}^2\left(\mathfrak{u}\left(2058\mathfrak{v}^3+(9\mathfrak{w}+14)^3\right)-686\mathfrak{v}^3(9\mathfrak{w}+14)\right)}{\mathfrak{v}(2\mathfrak{v}-\mathfrak{u})(7\mathfrak{v}-9\mathfrak{w}-14)(14\mathfrak{v}-9\mathfrak{w}-14)(21\mathfrak{v}+9\mathfrak{w}+14)}\\
& +2d\tau d\phi\left(-\frac{\mathfrak{u}(9\mathfrak{w}+14)^2}{343\mathfrak{v}^2}-\frac{16464\mathfrak{u}\mathfrak{v}}{(9\mathfrak{w}+14)^4}+\mathfrak{u}\right)+\frac{d\mathfrak{u}d\mathfrak{v}}{\mathfrak{u}-2\mathfrak{v}}+\frac{d\mathfrak{u}(\mathfrak{u}d\mathfrak{v}-\mathfrak{v}d\mathfrak{u})}{\mathfrak{u}(\mathfrak{u}-2\mathfrak{v})}\\
& +\frac{36\mathfrak{u}\mathfrak{w}\left(27\mathfrak{w}^2+126\mathfrak{w}+196\right)d\chi d\phi}{(9\mathfrak{w}+14)^3}+d\tau^2\left(-\frac{16464\mathfrak{v}^2}{(9\mathfrak{w}+14)^4}-\frac{(9\mathfrak{w}+14)^2}{343\mathfrak{v}}+\mathfrak{v}\right)\\
& +\frac{6174\mathfrak{v}^2 d\mathfrak{v}d\mathfrak{w}}{(7\mathfrak{v}-9\mathfrak{w}-14)(14\mathfrak{v}-9\mathfrak{w}-14)(21\mathfrak{v}+9\mathfrak{w}+14)}\\
& +\frac{d\mathfrak{w}^2\left(5647152\mathfrak{v}^3-343\mathfrak{v}^2(9\mathfrak{w}+14)^4+(9\mathfrak{w}+14)^6\right)}{2\mathfrak{w}(9\mathfrak{w}+14)\left(27\mathfrak{w}^2+126\mathfrak{w}+196\right)(7\mathfrak{v}-9\mathfrak{w}-14)(14\mathfrak{v}-9\mathfrak{w}-14)(21\mathfrak{v}+9\mathfrak{w}+14)}\\
& +\frac{36\mathfrak{v}\mathfrak{w}\left(27\mathfrak{w}^2+126\mathfrak{w}+196\right)d\tau d\chi}{(9\mathfrak{w}+14)^3}+\frac{2\mathfrak{w}\left(27\mathfrak{w}^2+126\mathfrak{w}+196\right)d\chi^2}{(9\mathfrak{w}+14)^2}
\end{aligned}
$$

12 The Ricci-flat metric on tot $\left[K\left(\mathcal{M}_B^{\mathrm{KE}}\right)\right]$ versus the metric cone on the Sasakian fibrations on $\mathcal{M}_B^{\mathrm{KE}}$

This last section is probably the most relevant one since it clarifies an unexpected distinction that opens new directions of investigations. To develop our argument it is appropriate to begin by reformulating the geometry of the Ricci-flat metric derived from the Calabi ansatz in terms of vielbeins. In that language the comparison with the Sasaki–Einstein metrics of [32] will become much more transparent.

With some straightforward yet cumbersome algebraic analysis we have verified that, after the change of variables (6.3), the Ricci-flat metric in action-angle coordinates coming from the symplectic potential $G_{\mathcal{M}_T^{\mathrm{KE}}}$, as displayed in (11.2), can be rewritten

[10] METRICGRAV just as VIELBGRAV23 is a Mathematica package written by one of us (P.F.) almost thirty years ago and constantly updated. It will be available from the site of the publisher De Gruyter to the readers of the forthcoming book [31].

as a sum of squares in terms of a set of six vielbein one-forms V^i:

$$\mathrm{d}s^2_{\mathcal{M}_T^{\mathrm{KE}}} = \sum_{i=1}^{6} \left(V^i\right)^2 = \underbrace{\delta_{ij}\, V^i_\mu\, V^j_\nu}_{\text{metric}\mathbf{g}_{\mu\nu}} \mathrm{d}y^\mu \mathrm{d}y^\nu \quad ; \quad \underbrace{y^\mu = \{\theta, \mathfrak{v}, \mathfrak{w}, \phi, \tau, \chi\}}_{\text{coordinates}} \quad (12.1)$$

The explicit general structure of the sechsbein V^i, whose matrix of components V^i_μ must be invertible and reproduces the metric (12.1) is the following one:

$$\begin{aligned}
V^1 &= \sqrt{\mathfrak{v}}\,\mathrm{d}\theta \\
V^2 &= \sqrt{\mathfrak{v}}\,\mathrm{d}\phi\,\sin(\theta) \\
V^3 &= \frac{\mathrm{d}\mathfrak{v}}{\mathfrak{A}(\mathfrak{v},\mathfrak{w})} \\
V^4 &= \mathfrak{B}(\mathfrak{v},\mathfrak{w})\,[\mathrm{d}\mathfrak{w} + \mathfrak{C}(\mathfrak{v},\mathfrak{w})\,\mathrm{d}\mathfrak{v}] \\
V^5 &= \mathfrak{D}(\mathfrak{v},\mathfrak{w})\,[\mathrm{d}\tau + (1-\cos(\theta))\,\mathrm{d}\phi] \\
V^6 &= \mathfrak{E}(\mathfrak{v},\mathfrak{w})\,[\mathrm{d}\chi + \mho(\mathfrak{v},\mathfrak{w})\,[\,\mathrm{d}\tau + (1-\cos(\theta))\,\mathrm{d}\phi\,]]
\end{aligned} \quad (12.2)$$

For the metric given by the symplectic potential in (11.2) the six functions $\mathfrak{A}(\mathfrak{v},\mathfrak{w})$, $\mathfrak{B}(\mathfrak{v},\mathfrak{w})$, $\mathfrak{C}(\mathfrak{v},\mathfrak{w})$, $\mathfrak{D}(\mathfrak{v},\mathfrak{w})$, $\mathfrak{E}(\mathfrak{v},\mathfrak{w})$, $\mho(\mathfrak{v},\mathfrak{w})$ are explicitly given below, where, in order to make formulae more easily readable, we have renamed $\lambda_1 = \alpha$, $\lambda_2 = \beta$:

$$\begin{aligned}
\mathfrak{A}(\mathfrak{v},\mathfrak{w}) &= -\frac{i\sqrt{64\mathfrak{v}^3(\alpha+\beta)\left(\alpha^2+\alpha\beta+\beta^2\right)^6 - 4\mathfrak{v}^2\left(\alpha^2+\alpha\beta+\beta^2\right)^3\left(2\left(\alpha^2+\alpha\beta+\beta^2\right)+3\mathfrak{w}(\alpha+\beta)\right)^4 + \alpha^2\beta^2\left(2\left(\alpha^2+\alpha\beta+\beta^2\right)+3\mathfrak{w}(\alpha+\beta)\right)^6}}{2\sqrt{\mathfrak{v}}\left(\alpha^2+\alpha\beta+\beta^2\right)^{3/2}\left(2\left(\alpha^2+\alpha\beta+\beta^2\right)+3\mathfrak{w}(\alpha+\beta)\right)^2} \\
\mathfrak{B}(\mathfrak{v},\mathfrak{w}) &= \frac{\sqrt{N_{\mathfrak{B}}(\mathfrak{v},\mathfrak{w})}}{\sqrt{D_{\mathfrak{B}}(\mathfrak{v},\mathfrak{w})}} \\
N_{\mathfrak{B}}(\mathfrak{v},\mathfrak{w}) &= 64\mathfrak{v}^3(\alpha+\beta)\left(\alpha^2+\alpha\beta+\beta^2\right)^6 - 4\mathfrak{v}^2\left(\alpha^2+\alpha\beta+\beta^2\right)^3\left(2\left(\alpha^2+\alpha\beta+\beta^2\right)+3\mathfrak{w}(\alpha+\beta)\right)^4 + \alpha^2\beta^2\left(2\left(\alpha^2+\alpha\beta+\beta^2\right)+3\mathfrak{w}(\alpha+\beta)\right)^6 \\
D_{\mathfrak{B}}(\mathfrak{v},\mathfrak{w}) &= 2\mathfrak{w}\left(8\left(\alpha^2+\alpha\beta+\beta^2\right)^3 + 9\mathfrak{w}^3(\alpha+\beta)^3 + 24\mathfrak{w}^2\left(\alpha^2+\alpha\beta+\beta^2\right)(\alpha+\beta)^2 + 24\mathfrak{w}\left(\alpha^2+\alpha\beta+\beta^2\right)^2(\alpha+\beta)\right) \\
&\quad \times\left(8\mathfrak{v}^3(\alpha+\beta)\left(\alpha^2+\alpha\beta+\beta^2\right)^3 - 4\mathfrak{v}^2\left(\alpha^2+\alpha\beta+\beta^2\right)^3\left(2\left(\alpha^2+\alpha\beta+\beta^2\right)+3\mathfrak{w}(\alpha+\beta)\right)\right. \\
&\quad \left.+\alpha^2\beta^2\left(2\left(\alpha^2+\alpha\beta+\beta^2\right)+3\mathfrak{w}(\alpha+\beta)\right)^3\right) \\
\mathfrak{C}(\mathfrak{v},\mathfrak{w}) &= \frac{24\mathfrak{v}^2\mathfrak{w}(\alpha+\beta)\left(\alpha^2+\alpha\beta+\beta^2\right)^3\left(8\left(\alpha^2+\alpha\beta+\beta^2\right)^3 + 9\mathfrak{w}^3(\alpha+\beta)^3 + 24\mathfrak{w}^2\left(\alpha^2+\alpha\beta+\beta^2\right)(\alpha+\beta)^2 + 24\mathfrak{w}\left(\alpha^2+\alpha\beta+\beta^2\right)^2(\alpha+\beta)\right)}{64\mathfrak{v}^3(\alpha+\beta)\left(\alpha^2+\alpha\beta+\beta^2\right)^6 - 4\mathfrak{v}^2\left(\alpha^2+\alpha\beta+\beta^2\right)^3\left(2\left(\alpha^2+\alpha\beta+\beta^2\right)+3\mathfrak{w}(\alpha+\beta)\right)^4 + \alpha^2\beta^2\left(2\left(\alpha^2+\alpha\beta+\beta^2\right)+3\mathfrak{w}(\alpha+\beta)\right)^6} \\
\mathfrak{D}(\mathfrak{v},\mathfrak{w}) &= \frac{\sqrt{-8\mathfrak{v}^3(\alpha+\beta)\left(\alpha^2+\alpha\beta+\beta^2\right)^3 + 4\mathfrak{v}^2\left(\alpha^2+\alpha\beta+\beta^2\right)^3\left(2\left(\alpha^2+\alpha\beta+\beta^2\right)+3\mathfrak{w}(\alpha+\beta)\right) - \alpha^2\beta^2\left(2\left(\alpha^2+\alpha\beta+\beta^2\right)+3\mathfrak{w}(\alpha+\beta)\right)^3}}{2\sqrt{\mathfrak{v}\left(\alpha^2+\alpha\beta+\beta^2\right)^3\left(2\left(\alpha^2+\alpha\beta+\beta^2\right)+3\mathfrak{w}(\alpha+\beta)\right)}} \\
\mathfrak{E}(\mathfrak{v},\mathfrak{w}) &= \sqrt{2}\sqrt{\frac{\mathfrak{w}\left(4\left(\alpha^2+\alpha\beta+\beta^2\right)^2 + 3\mathfrak{w}^2(\alpha+\beta)^2 + 6\mathfrak{w}\left(\alpha^3+2\alpha^2\beta+2\alpha\beta^2+\beta^3\right)\right)}{\left(2\left(\alpha^2+\alpha\beta+\beta^2\right)+3\mathfrak{w}(\alpha+\beta)\right)^2}} \\
\mho(\mathfrak{v},\mathfrak{w}) &= \Omega(\mathfrak{v},\mathfrak{w}) \equiv \frac{3\mathfrak{v}(\alpha+\beta)}{2\left(\alpha^2+\alpha\beta+\beta^2\right)+3\mathfrak{w}(\alpha+\beta)}
\end{aligned} \quad (12.3)$$

12.1 Properties of the sechsbein and comparison with Sasakian 5-manifolds

The general structure of the *sechsbein* in (12.2) and (12.3) is very interesting since it highlights the double fibration structure of the underlying manifold $\mathcal{M}_T$ which is a line-bundle (the canonical one) on the base manifold $\mathcal{M}_B$ which, in its turn, is a (singular) $\mathbb{P}^1$ fibration over a base $\mathbb{P}^1$:

$$\mathcal{M}_T \xrightarrow{\pi_1} \mathcal{M}_B \xrightarrow{\pi_2} \mathbb{P}^1$$

The projection onto the base manifold is produced by considering the limit $\mathfrak{w} \to 0$. Very much informative is the development in series of the sechsbein for small values of the coordinate $\mathfrak{w}$. The limit is regular for $\mathfrak{w} = 0$; two of the sechsbein (V^4, V^6) vanish and the remaining four 1-forms V^1, V^2, V^3, V^5 attain the values $\mathbf{e}^3, \mathbf{e}^4, \mathbf{e}^1, \mathbf{e}^2$ corresponding to the vierbein of the Kähler–Einstein four-dimensional metrics as given in Eq. (6.14) with the function $\mathcal{FK}^{\text{KE}}(\mathfrak{v})$ as given in Table 1. At order $\sqrt{\mathfrak{w}}$ there is no deformation of the base-manifold vierbein, but the two vielbein corresponding to the fiber directions do appear. At order $\mathfrak{w}$ we see the beginning of the deformation of the base-manifold vierbein. Precisely we find:

$$\begin{aligned}
V^1 &= \mathbf{e}^3 \\
V^2 &= \mathbf{e}^4 \\
V^3 &= \mathbf{e}^1 && + \mathfrak{w}\, \Delta \mathbf{e}^1 + \mathcal{O}(\mathfrak{w}^2) \\
V^4 &= 0 \; + \sqrt{\mathfrak{w}}\, \Delta \mathbf{\Phi}_{\mathfrak{w}} + \mathcal{O}(\mathfrak{w}^{3/2}) \\
V^5 &= \mathbf{e}^2 && + \mathfrak{w}\, \Delta \mathbf{e}^2 + \mathcal{O}(\mathfrak{w}^2) \\
V^6 &= 0 \; + \sqrt{\mathfrak{w}}\, \Delta \mathbf{\Phi}_{\chi} + \mathcal{O}(\mathfrak{w}^{3/2})
\end{aligned}$$

where the deformations of the base-manifold vielbein are as follows:

$$\begin{aligned}
\Delta \mathbf{e}^1 &= -\frac{3(\alpha+\beta)\sqrt{\frac{\mathfrak{v}}{\alpha^2+\alpha\beta+\beta^2}}\left(\alpha^2\beta^2 - 2\mathfrak{v}^3(\alpha+\beta)\right)}{2((\mathfrak{v}-\alpha)(\beta-\mathfrak{v})(\alpha\beta+\mathfrak{v}(\alpha+\beta)))^{3/2}}\, \mathrm{d}\mathfrak{v} \\
\Delta \mathbf{e}^2 &= \frac{3(\alpha+\beta)\left(\mathfrak{v}^3(\alpha+\beta) - 2\alpha^2\beta^2\right)}{4\left(\alpha^2+\alpha\beta+\beta^2\right)^{3/2}\sqrt{\mathfrak{v}(\mathfrak{v}-\alpha)(\beta-\mathfrak{v})(\alpha\beta+\mathfrak{v}(\alpha+\beta))}} \\
&\quad [\mathrm{d}\tau + (1-\cos\theta)\ \mathrm{d}\phi]
\end{aligned}$$

and the initial fiber-vielbein are instead the following ones:

$$\begin{aligned}
\Delta \mathbf{\Phi}_{\mathfrak{w}} &= \frac{\mathrm{d}\mathfrak{w}}{\sqrt{2\mathfrak{w}}} + \frac{3\mathfrak{v}^2\,(\alpha+\beta)}{(\mathfrak{v}-\alpha)(\mathfrak{v}-\beta)(\alpha\beta+\mathfrak{v}(\alpha+\beta))}\, \mathrm{d}\mathfrak{v} \\
\Delta \mathbf{\Phi}_{\chi} &= \sqrt{2}\left(\mathrm{d}\chi + \frac{3\mathfrak{v}(\alpha+\beta)}{2\left(\alpha^2+\alpha\beta+\beta^2\right)}[\mathrm{d}\tau + (1-\cos\theta)\ \mathrm{d}\phi]\right)
\end{aligned} \tag{12.4}$$

12.2 Comparison with the Sasaki–Einstein metrics

It is now the appropriate moment to consider the Sasaki–Einstein metrics introduced in [32]. In the coordinates utilized by those authors we have:

$$\mathrm{d}s^2_{SE_5} = \frac{\mathrm{d}y^2(1-cy)}{2\left(a+2cy^3-3y^2\right)} + \frac{\left(a+2cy^3-3y^2\right)(c\mathrm{d}\phi\cos(\theta)+\mathrm{d}\psi)^2}{18(1-cy)}$$

$$+\frac{1}{6}(1-cy)\left(\mathrm{d}\theta^2+\mathrm{d}\phi^2\sin^2\theta\right)+\frac{\mathbf{\Phi}_{\mathrm{SE}}^2}{9}$$
$$\mathbf{\Phi}_{\mathrm{SE}}=[\mathrm{d}\xi\ +\ y\,(\mathrm{d}\psi+c\mathrm{d}\phi\cos\theta+\mathrm{d}\psi)-\mathrm{d}\phi\cos\theta] \tag{12.5}$$

If in Eq. (12.5) we apply the following coordinate transformation and renaming of the parameters:

$$a\ \to\ \frac{1}{4}\left(3\beta k^2+4\right),\quad c\ \to\ 1,\quad y\to 1-\frac{k\mathfrak{v}}{2},\quad \psi\ \to\ -\tau-\phi \tag{12.6}$$

we find the following interesting result:

$$\mathrm{d}s^2_{SE_5}=\frac{k}{12}\,\widehat{ds}^2_5$$
$$\widehat{ds}^2_5=ds^2_{KE_4}+\frac{4}{3k}\,\mathbf{\Phi}^2_{\mathrm{SE}} \tag{12.7}$$
$$ds^2_{KE_4}=\frac{\mathrm{d}\mathfrak{v}^2}{\mathcal{FK}_{\mathrm{KE}}(\mathfrak{v})}\ +\ \mathcal{FK}_{\mathrm{KE}}(\mathfrak{v})\,[\mathrm{d}\tau+(1-\cos\theta)\ \mathrm{d}\phi]^2\ +\ \mathfrak{v}\left(\mathrm{d}\phi^2\sin^2\theta+\mathrm{d}\theta^2\right) \tag{12.8}$$
$$\mathcal{K}_{\mathrm{KE}}(\mathfrak{v})=\frac{3\beta-k\mathfrak{v}^3+3\mathfrak{v}^2}{3\mathfrak{v}}$$

As one sees the line element $ds^2_{KE_4}$ in (12.8) exactly coincides with the line element of the Kähler Einstein metrics discussed in previous sections and presented in eq. (6.1). It remains to be seen what is the appearance of the 1-form $\mathbf{\Phi}_{\mathrm{SE}}$ after the transformation (12.6). If we add also the coordinate transformation:

$$\xi\ \to\ p\,\chi\ +\ \tau\ +\ \phi;\quad p\in\mathbb{R} \tag{12.9}$$

we find:

$$\mathbf{\Phi}_{\mathrm{SE}}\ =\ p\left[\mathrm{d}\chi\ +\ \frac{k}{2p}\,\mathfrak{v}\ (\mathrm{d}\tau+(1-\cos\theta)\ \mathrm{d}\phi)\right] \tag{12.10}$$

Comparing Eq. (12.10) with (12.4) we see that we obtain:

$$\mathbf{\Phi}_{\mathrm{SE}}\ \propto\ \Delta\mathbf{\Phi}_{\chi}$$

if we choose the constant p as we do below:

$$p\ =\ k\,\frac{\alpha^2+\alpha\,\beta+\beta^2}{3\,(\alpha\ +\ \beta)}\ =\ 1 \tag{12.11}$$

Hence, using Eqs. (12.6), (12.9), (12.11), (5.5) we can conclude that the Sasaki–Einstein metric of [32] with the choice of the parameters a, c provided in (12.6) is proportional through the constant $\frac{k}{12}$ to the following five-dimensional metric:

$$\widehat{ds}_5^2 = \underbrace{ds^2_{KE_4}}_{\text{KEmetricon}\mathcal{M}_B} + \frac{4}{3\,k}\underbrace{\left(\mathrm{d}\chi + \Omega(\mathfrak{v},0)(\mathfrak{v},\mathfrak{w})\left[\mathrm{d}\tau + (1-\cos(\theta))\,\mathrm{d}\phi\right]\right)^2}_{\lim_{\mathfrak{w}\to 0}\frac{V^6}{\sqrt{2\mathfrak{w}}}} \tag{12.12}$$

As we have explicitly checked, the metric is an Einstein metric, since its Ricci tensor in intrinsic components takes the following form:

$$\mathcal{R}ic[\widehat{g}_5] = \frac{k}{6}\,\delta_{ij}$$

and on an Einstein space the standard metric of the metric cone is certainly Ricci-flat. Hence, writing a new sechsbein:

$$\mathbb{E}_{cone} = \{\mathbf{E}^1 \ldots, \mathbf{E}^6\}$$

where:

$$\begin{aligned} \mathbf{E}^1 &= R\,\mathbf{e}_3; & \mathbf{E}^2 &= R\,\mathbf{e}_4 \\ \mathbf{E}^3 &= R\,\mathbf{e}_1; & \mathbf{E}^4 &= R\,\mathbf{e}_2 \\ \mathbf{E}^5 &= R\sqrt{\frac{4}{3\,k}}\left(\mathrm{d}\chi + \frac{k}{2}\mathfrak{v}\left[\mathrm{d}\tau + (1-\cos(\theta))\,\mathrm{d}\phi\right]\right); & \mathbf{E}^6 &= 2\sqrt{\frac{3}{k}}dR \end{aligned} \tag{12.13}$$

With the Mathematica Code VIELBGRAV23, we have calculated the curvature 2-form $\mathfrak{R}^{ab}_{con}$ and the intrinsic components of the Riemann tensor $\mathcal{R}ie_{cone}$ for the metric provided by the sechsbein (12.13). As the metric is Ricci-flat, the Riemann tensor coincides with the Weyl tensor $\mathcal{W}_{cone}(R,\mathfrak{v})$. Similarly, we have done for the Ricci-flat metric constructed with the Calabi ansatz, using the sechsbein defined in (12.2) with the functions displayed in (12.3). In this way, we have obtained the curvature 2-form $\mathfrak{R}^{ab}_{CA}$ and the intrinsic components of the Weyl tensor associated with the Calabi Ansatz metric $\mathcal{W}_{CA}(\mathfrak{w},\mathfrak{v})$. In order to make the comparison more precise, the sechsbein (12.2) has been reordered in a similar way to the ordering utilized in the metric cone case:

$$\mathbb{E}_{CA} = \{V^1, V^2, V^3, V^5, V^6, V^4\}$$

The two metrics exactly coincide on the base-manifold $\mathcal{M}_B$ that is one of the KE manifolds with two conical singularities discussed at length in the previous sections of the present paper and are both Ricci-flat in six dimensions, yet from all what we said in the present section they seem to be intrinsically different, since the Sasakian Einstein five-dimensional metric as described in (12.12) cannot be obtained by fixing the fiber coordinate $\mathfrak{w}$ to some appropriate constant value. Yet one might think that there exists some clever change of coordinates that can map one metric into the other. To show that this is not the case, we resorted to the calculation of Weyl tensor invariants for the two metrics to compare them. Furthermore, we calculated polynomial 2-form curvature invariants, in particular the following 6-forms:

$$\begin{aligned} \mathrm{Ch} &= \mathfrak{R}^{ab} \wedge \mathfrak{R}^{bc} \wedge \mathfrak{R}^{ca} \; = \; Tr\,(\mathfrak{R} \wedge \mathfrak{R} \wedge \mathfrak{R}) \\ \mathrm{E} &= \mathfrak{R}^{ab} \wedge \mathfrak{R}^{cd} \wedge \mathfrak{R}^{fg}\, \epsilon_{abcdfg} \end{aligned} \tag{12.14}$$

The considered Weyl invariants are instead the following ones:

$$\boxed{\begin{aligned} Quad_1 &= \mathcal{W}[abij]\mathcal{W}[ijab] \\ \mathrm{Cub}_1 &= \mathcal{W}[abij]\mathcal{W}[ijpq]\mathcal{W}[pqab] \\ \mathrm{Cub}_2 &= \mathcal{W}[ijpq]\mathcal{W}[rpsq]\mathcal{W}[rsij] \\ \mathrm{Cub}_3 &= \mathcal{W}[ijpq]\mathcal{W}[rpsq]\mathcal{W}[risj] \\ Quart_1 &= \mathcal{W}[abij]\mathcal{W}[ijpq]\mathcal{W}[pqmn]\mathcal{W}[mnab] \\ Quart_2 &= \mathcal{W}[i,j,p,q]\mathcal{W}[p,r,q,s]\mathcal{W}[r,m,s,n]\mathcal{W}[m,n,i,j] \end{aligned}} \tag{12.15}$$

The result of the calculations with one special choice of α, β is displayed in Table 2.

Inspecting this table, one realizes that, as anticipated in the Introduction, the Ricci-flat metric constructed by means of the Calabi Ansatz translated into action-angle variables is different from that associated with the Sasaki–Einstein metric recalled in (12.5). The strongest evidence is given by vanishing of the invariant E in one case and the nonzero and coordinate dependent structure of the same invariant in the second case. However, also the other invariants corroborate the same evidence. One can calculate the value of the coordinate R in terms of $\mathfrak{v}$, $\mathfrak{w}$ by equating the quadratic invariant $Quad_1$ of the two metrics. Substituting the result in the other invariants the two expressions do not agree for the other invariants. This being clarified one can try to understand the reason of the disagreement. In the Sasaki Einstein approach, (12.7) corresponds to the standard construction of the metric on a U(1) bundle. To the metric of the base manifold, in the present case $\mathcal{M}_B^{\mathrm{KE}}$, one adds the square of a 1-form Φ of the type $\Phi = d\tau + \mathrm{U}(1) - \mathrm{connectionon}\mathcal{M}_B$. The remaining coordinate is the radial one R. In the Calabi ansatz approach, instead, one directly constructs the line bundle on the base manifold. A Sasaki–Einstein manifold could be retrieved a posteriori through the construction of a sphere bundle $R = |w|$, where the complex coordinate w is the canonical bundle fiber coordinate. The two procedures do not commute. The investigation of the Sasaki–Einstein manifold underlying the constructed Ricci-flat metric via Calabi ansatz is a new research project that we leave for the future. Similarly the identification of a gauge four-dimensional gauge theory dual to the here presented D3–brane solution of Type IIB supergravity is one of the goals we plan to pursue in the near future.

12.3 Completeness

The metrics in the family $\mathrm{Mat}(\mathcal{FV})_{\mathrm{KE}}$ are complete [2], so that, applying Theorem 4.3 in Calabi's paper [11], we would get that the Ricci-flat six-dimensional metric on the

Table 2 Polynomial invariants defined in (12.14) and (12.15) evaluated in the case of the metric cone over the Sasaki–Einstein manifold and in the Calabi Ansatz case of Ricci-flat metrics. Since the calculations are very long when the parameters are left symbolic, we evaluated the invariants in the reference case $\alpha = 1, \beta = 2$

Inv	metric cone over the Sasaki manifold	Calabi Ansatz for the Ricci-flat metric
Ch	0	0
E	0	$\frac{20736\left(63780651422208\mathfrak{v}^6+(9\mathfrak{w}+14)^{12}\right)}{343\mathfrak{v}^6(9\mathfrak{w}+14)^{12}} \times \mathrm{Vol}$
$Quad_1$	$\frac{96}{49R^4\mathfrak{v}^6}$	$\frac{6(9\mathfrak{w}+14)^4}{117649\mathfrak{v}^6}+\frac{8131898880}{(9\mathfrak{w}+14)^8}$
Cub_1	$\frac{384}{343R^6\mathfrak{v}^9}$	$\frac{6(9\mathfrak{w}+14)^6}{40353607\mathfrak{v}^9}+\frac{432}{343\mathfrak{v}^6}+\frac{348097316216832}{(9\mathfrak{w}+14)^{12}}$
Cub_2	$\frac{192}{343R^6\mathfrak{v}^9}$	$\frac{3(9\mathfrak{w}+14)^6}{40353607\mathfrak{v}^9}+\frac{216}{343\mathfrak{v}^6}+\frac{174048658108416}{(9\mathfrak{w}+14)^{12}}$
Cub_3	$\frac{96}{343R^6\mathfrak{v}^9}$	$\frac{3(9\mathfrak{w}+14)^6}{80707214\mathfrak{v}^9}+\frac{108}{343\mathfrak{v}^6}+\frac{87024329054208}{(9\mathfrak{w}+14)^{12}}$
$Quart_1$	$\frac{4608}{2401R^8\mathfrak{v}^{12}}$	$\frac{18(9\mathfrak{w}+14)^8}{13841287201\mathfrak{v}^{12}}+\frac{576(9\mathfrak{w}+14)^2}{117649\mathfrak{v}^9}+\frac{20736}{\mathfrak{v}^6(9\mathfrak{w}+14)^4}+\frac{31631121143724146688}{(9\mathfrak{w}+14)^{16}}$
$Quar_2$	$\frac{1152}{2401R^8\mathfrak{v}^{12}}$	$\frac{9(9\mathfrak{w}+14)^8}{27682574402\mathfrak{v}^{12}}+\frac{144(9\mathfrak{w}+14)^2}{117649\mathfrak{v}^9}+\frac{5184}{\mathfrak{v}^6(9\mathfrak{w}+14)^4}+\frac{7907780285931036672}{(9\mathfrak{w}+14)^{16}}$

canonical bundle is complete. However, one should check that Calabi's theorem also applies in the singular case.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

A The issue of (2,1)-forms

In this appendix, we consider the issue of the (2,1)-forms, giving a proof that no (anti)-self-dual (2, 1) forms exist on the KE manifolds previously studied. A (2,1)-form can be written as

$$\Omega_{ij\bar{k}}\,(z,\bar{z})\,dz^i \wedge dz^j \wedge \mathrm{d}\bar{z}^{\bar{k}}$$

The dual (2,1)-form is

$$\star_g \Omega \;=\; \tilde{\Omega}_{\ell p\bar{q}}\,(z,\bar{z})\,dz^\ell \wedge dz^p \wedge \mathrm{d}\bar{z}^{\bar{q}}$$

where:

$$\tilde{\Omega}_{\ell p\bar{q}}\,(z,\bar{z}) \;=\; \frac{1}{\sqrt{\det g}}\, g_{\ell\bar{m}}\, g_{p\bar{n}}\, g_{\bar{q}s}\, \epsilon^{\bar{m}\bar{n}\bar{k}} \epsilon^{ijs}\, \Omega_{ij\bar{k}}\,(z,\bar{z})$$

Hence, the (anti)-self-duality condition is expressed by the equation:

$$\pm\, i\, \tilde{\Omega}_{\ell p\bar{q}}\,(z,\bar{z}) \;=\; \Omega_{\ell p\bar{q}}\,(z,\bar{z})$$

Given the complex structure tensor $\mathfrak{J}$ and its eigenvectors, one writes the complex differentials:

$$dz^i \;=\; \omega^i_\ell(\mu)\mathrm{d}\mu^\ell + i\,\mathrm{d}\Theta^i \quad ; \quad \mathrm{d}\bar{z}^i \;=\; \omega^i_\ell(\mu)\mathrm{d}\mu^\ell - i\,\mathrm{d}\Theta^i$$

where the real 1-forms $\omega^i \;\equiv\; \omega^i_\ell(\mu)\mathrm{d}\mu^\ell$ depending only on moment variables are defined by the complex structure tensor and hence by the explicit form of the metric in terms of the Hessian. Using this formalism, a (2,1)-form is written as

$$\begin{aligned}\Omega^{(2,1)} &= \Omega_{ij|k}\,(\mu)\;(\omega^i + i\,\mathrm{d}\Theta^i)\wedge(\omega^j + i\,\mathrm{d}\Theta^j)\wedge(\omega^k - i\,\mathrm{d}\Theta^k)\\ &= Q_{IJK}\,(\mu)\;\mathrm{d}y^I\wedge\mathrm{d}y^J\wedge\mathrm{d}y^K\end{aligned} \tag{A.1}$$

where $y^I = \{\mu^i, \Theta^j\}$ is the complete set of the $2n$ real coordinates (moments and angles). The complex functions $Q_{IJK}\,(\mu)$ depend on the real variables μ. The (anti)self-duality condition is most easily written in the symplectic formalism as the determinant of the metric tensor in symplectic coordinates is just 1. One gets

$$Q_{IJK} = \pm i\,\epsilon_{IJKPQR}\,Q_{PQR} \tag{A.2}$$

The original components $\Omega_{ij|k}\,(\mu)$ are supposed to be complex-valued functions of their real arguments which means that we have a total of 9 complex-valued functions, namely a total of 18 real functions:

$$r(\mu) = \{f_1(\mu),\ldots,f_9(\mu),g_1(\mu),\ldots,g_9(\mu)\} \tag{A.3}$$

Explicitly we obtain

$$\begin{aligned}\Omega = \; 2\Big(&(f_3 - f_5 - f_7 + ig_3 - ig_5 - ig_7)\,\mathrm{d}\tau\wedge\mathrm{d}\chi\wedge\omega^1 - 2\,(f_8 + ig_8)\,\mathrm{d}\tau\wedge\mathrm{d}\chi\wedge\omega^2\\ &-2\,(f_9 + ig_9)\,\mathrm{d}\tau\wedge\mathrm{d}\chi\wedge\omega^3 + 2\,(f_1 + ig_1)\,\mathrm{d}\tau\wedge\mathrm{d}\phi\wedge\omega^1 + 2\,(f_2 + ig_2)\,\mathrm{d}\tau\wedge\mathrm{d}\phi\wedge\omega^2\\ &+(f_3 + f_5 - f_7 + ig_3 + ig_5 - ig_7)\,\mathrm{d}\tau\wedge\mathrm{d}\phi\wedge\omega^3 + 2\,(g_2 - if_2)\,\mathrm{d}\tau\wedge\omega^1\wedge\omega^2\\ &+(-if_3 - if_5 - if_7 + g_3 + g_5 + g_7)\,\mathrm{d}\tau\wedge\omega^1\wedge\omega^3 + 2\,(g_8 - if_8)\,\mathrm{d}\tau\wedge\omega^2\wedge\omega^3\\ &+(-if_3 - if_5 + if_7 + g_3 + g_5 - g_7)\,\mathrm{d}\chi\wedge\omega^1\wedge\omega^2 + 2\,(g_6 - if_6)\,\mathrm{d}\chi\wedge\omega^1\wedge\omega^3\\ &+2\,(g_9 - if_9)\,\mathrm{d}\chi\wedge\omega^2\wedge\omega^3 - 2\,(f_4 + ig_4)\,\mathrm{d}\phi\wedge\mathrm{d}\chi\wedge\omega^1\\ &-(f_3 + f_5 + f_7 + ig_3 + ig_5 + ig_7)\,\mathrm{d}\phi\wedge\mathrm{d}\chi\wedge\omega^2 - 2\,(f_6 + ig_6)\,\mathrm{d}\phi\wedge\mathrm{d}\chi\wedge\omega^3\\ &+2\,(g_1 - if_1)\,\mathrm{d}\phi\wedge\omega^1\wedge\omega^2 + 2\,(g_4 - if_4)\,\mathrm{d}\phi\wedge\omega^1\wedge\omega^3\\ &+(if_3 - if_5 - if_7 - g_3 + g_5 + g_7)\,\mathrm{d}\phi\wedge\omega^2\wedge\omega^3\\ &+(-if_3 + if_5 - if_7 + g_3 - g_5 + g_7)\,\mathrm{d}\tau\wedge\mathrm{d}\phi\wedge\mathrm{d}\chi\\ &+\omega^1\wedge\omega^2\wedge\omega^3\,(f_3 - f_5 + f_7 + ig_3 - ig_5 + ig_7)\Big)\end{aligned} \tag{A.4}$$

which is the most general expression for a (2,1)-form expressed in the real symplectic coordinate basis. Expanding each of the closed one-forms in the differentials $\mathrm{d}\mu^i$ of the moments one obtains the explicit form of the 20 components $Q_{IJK}(\mu)$ mentioned in (A.1). For instance in our standard example $\lambda_1 = 1, \lambda_2 = 2$ we have:

$$\begin{aligned}
\omega^1 &= \tfrac{\mathrm{d}\mathfrak{v}-\mathfrak{v}\mathrm{d}\mathfrak{u}}{\mathfrak{u}(\mathfrak{u}-2\mathfrak{v})}\\
\omega^2 &= \tfrac{N_2}{D_2}\\
N_2 &= 3087\mathfrak{v}^3\mathrm{d}\mathfrak{w}(\mathfrak{u}-2\mathfrak{v}) - \mathfrak{u}\mathrm{d}\mathfrak{v}\left(2058\mathfrak{v}^3 + (9\mathfrak{w}+14)^3\right)\\
&\quad +\mathfrak{v}\mathrm{d}\mathfrak{u}\left(2058\mathfrak{v}^3 - 343\mathfrak{v}^2(9\mathfrak{w}+14)\right.\\
&\quad \left.+(9\mathfrak{w}+14)^3\right) + 686\mathfrak{v}^3(9\mathfrak{w}+14)\mathrm{d}\mathfrak{v}\\
D_2 &= \mathfrak{v}(\mathfrak{u}-2\mathfrak{v})\left(2058\mathfrak{v}^3 - 343\mathfrak{v}^2(9\mathfrak{w}+14) + (9\mathfrak{w}+14)^3\right)\\
\omega^3 &= \tfrac{N_3}{D_3}\\
N_3 &= 6174\mathfrak{v}^2\mathrm{d}\mathfrak{v} + \tfrac{\mathrm{d}\mathfrak{w}\left(5647152\mathfrak{v}^3 - 343\mathfrak{v}^2(9\mathfrak{w}+14)^4 + (9\mathfrak{w}+14)^6\right)}{\mathfrak{w}\left(243\mathfrak{w}^3+1512\mathfrak{w}^2+3528\mathfrak{w}+2744\right)}\\
D_3 &= 2\left(2058\mathfrak{v}^3 - 343\mathfrak{v}^2(9\mathfrak{w}+14) + (9\mathfrak{w}+14)^3\right)
\end{aligned} \tag{A.5}$$

and by substitution one straightforwardly obtains the Q_{IJK} components whose expression is too lengthy to be displayed. In general, a complex-valued 3-form has the following structure:

$$\begin{aligned}
\Omega^{[3]} = &\,(X_{20}+iY_{20})\,\mathrm{d}\tau\wedge\mathrm{d}\phi\wedge\mathrm{d}\chi + (X_{10}+iY_{10})\,\mathrm{d}\mathfrak{u}\wedge\mathrm{d}\tau\wedge\mathrm{d}\chi + (X_8+iY_8)\,\mathrm{d}\mathfrak{u}\wedge\mathrm{d}\tau\wedge\mathrm{d}\phi\\
&+(X_3+iY_3)\,\mathrm{d}\mathfrak{u}\wedge\mathrm{d}\mathfrak{v}\wedge\mathrm{d}\tau + (X_1+iY_1)\,\mathrm{d}\mathfrak{u}\wedge\mathrm{d}\mathfrak{v}\wedge\mathrm{d}\mathfrak{w} + (X_4+iY_4)\,\mathrm{d}\mathfrak{u}\wedge\mathrm{d}\mathfrak{v}\wedge\mathrm{d}\chi\\
&+(X_2+iY_2)\,\mathrm{d}\mathfrak{u}\wedge\mathrm{d}\mathfrak{v}\wedge\mathrm{d}\phi + (X_6+iY_6)\,\mathrm{d}\mathfrak{u}\wedge\mathrm{d}\mathfrak{w}\wedge\mathrm{d}\tau + (X_7+iY_7)\,\mathrm{d}\mathfrak{u}\wedge\mathrm{d}\mathfrak{w}\wedge\mathrm{d}\chi\\
&+(X_5+iY_5)\,\mathrm{d}\mathfrak{u}\wedge\mathrm{d}\mathfrak{w}\wedge\mathrm{d}\phi + (X_9+iY_9)\,\mathrm{d}\mathfrak{u}\wedge\mathrm{d}\phi\wedge\mathrm{d}\chi + (X_{16}+iY_{16})\,\mathrm{d}\mathfrak{v}\wedge\mathrm{d}\tau\wedge\mathrm{d}\chi\\
&+(X_{14}+iY_{14})\,\mathrm{d}\mathfrak{v}\wedge\mathrm{d}\tau\wedge\mathrm{d}\phi + (X_{12}+iY_{12})\,\mathrm{d}\mathfrak{v}\wedge\mathrm{d}\mathfrak{w}\wedge\mathrm{d}\tau + (X_{13}+iY_{13})\,\mathrm{d}\mathfrak{v}\wedge\mathrm{d}\mathfrak{w}\wedge\mathrm{d}\chi\\
&+(X_{11}+iY_{11})\,\mathrm{d}\mathfrak{v}\wedge\mathrm{d}\mathfrak{w}\wedge\mathrm{d}\phi + (X_{15}+iY_{15})\,\mathrm{d}\mathfrak{v}\wedge\mathrm{d}\phi\wedge\mathrm{d}\chi + (X_{19}+iY_{19})\,\mathrm{d}\mathfrak{w}\wedge\mathrm{d}\tau\wedge\mathrm{d}\chi\\
&+(X_{17}+iY_{17})\,\mathrm{d}\mathfrak{w}\wedge\mathrm{d}\tau\wedge\mathrm{d}\phi + (X_{18}+iY_{18})\,\mathrm{d}\mathfrak{w}\wedge\mathrm{d}\phi\wedge\mathrm{d}\chi
\end{aligned} \tag{A.6}$$

where the X_i and Y_i are real functions of the momenta μ. The self-duality condition (A.2) reduces to an algebraic relation that expresses all the Y_i in terms of the X_i, precisely:

$$\begin{array}{llll}
Y_1 = \pm\ \ X_{20} & & Y_{11} = \pm\ \ X_{10}\\
Y_2 = \pm\ \ X_{19} & & Y_{12} = \pm -X_9\\
Y_3 = \pm -X_{18} & & Y_{13} = \pm -X_8\\
Y_4 = \pm -X_{17} & & Y_{14} = \pm -X_7\\
Y_5 = \pm -X_{16} & & Y_{15} = \pm -X_6\\
Y_6 = \pm\ \ X_{15} & & Y_{16} = \pm\ \ X_5\\
Y_7 = \pm\ \ X_{14} & & Y_{17} = \pm\ \ X_4\\
Y_8 = \pm\ \ X_{13} & & Y_{18} = \pm\ \ X_3\\
Y_9 = \pm\ \ X_{12} & & Y_{19} = \pm -X_2\\
Y_{10} = \pm -X_{11} & & Y_{20} = \pm -X_1
\end{array}$$

The choice of the $\pm$ sign corresponding to self/anti-self duality, respectively. Comparing Eq. (A.4) with Eq. (A.6) and using Eq. (A.5), one obtains the 20 X_i and the 20 Y_i of a generic (2,1)-form as linear combination of the 18 free parameter functions (A.3) with coefficients that are rational functions of the moment μ. The self-duality constraint is a set of 20 linear equations on the 18 parameters. Obviously, unless the rank of the 20×18 matrix is less than 18, there are no nontrivial solutions. We have indeed verified that the 20 equations do not have nontrivial solutions for the standard case $\lambda_1 = 1, \lambda_2 = 2$ and for some other choices of the parameters. Hence, no harmonic self-dual (2,1) forms exist on this KE background and we have exact D3-brane solutions without 3-form fluxes. [2]

B Metric on the Lens space $L(k;\ 1) \sim S^3/\mathbb{Z}_k$

Let us consider the embedding of S^3 in $\mathbb{C}^2$ described by the two complex coordinates in (6.24), which satisfy:

$$|U^0|^2 + |U^1|^2 = r^2.$$

The metric can be written in the following form:

$$\mathrm{d}s^2 = |dU^0|^2 + |dU^1|^2 = \frac{r^2}{4}\,(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2) + \frac{r^2}{4}\,(\mathrm{d}\gamma + \cos(\theta)\,\mathrm{d}\phi)^2\,.$$

Recall that $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$, $0 \le \gamma < 4\pi$. One can indeed verify that, being

$$\sqrt{|g|} = \frac{r^3}{8}\,\sin(\theta),$$

the above bounds on the angles yield the correct value of the volume of S^3:

$$\mathrm{Vol}(S^3) = \int \sqrt{|g|}\,\mathrm{d}\theta\mathrm{d}\phi\mathrm{d}\gamma = 2\pi^2\,r^3.$$

Consider now the Lens space $L(k;\ 1) \sim S^3/\mathbb{Z}_k$ obtained by performing quotient of S^3 by the group $\mathbb{Z}_k$ acting as

$$\begin{pmatrix} U^0 \\ U^1 \end{pmatrix} \to \begin{pmatrix} e^{\frac{2\pi i}{k}} & 0 \\ 0 & e^{\frac{2\pi i}{k}} \end{pmatrix} \begin{pmatrix} U^0 \\ U^1 \end{pmatrix}\,.$$

This amounts to identifying

$$\gamma \sim \gamma + \frac{4\pi}{k},$$

and has the effect of dividing by k the interval of values of γ, so that, after the identification, γ varies in the range

$$\gamma \in \left(0,\,\frac{4\pi}{k}\right).$$

Therefore, the effect is to make the replacement $\gamma = \psi/k$ in the metric, where $0 \leq \psi < 4\pi$:

$$\begin{aligned} ds^2 &= \frac{r^2}{4}\,(d\theta^2 + \sin^2(\theta)d\phi^2) + \frac{r^2}{4}\,(d\psi/k + \cos(\theta)\,d\phi)^2 \\ &= \frac{r^2}{4}\,(d\theta^2 + \sin^2(\theta)d\phi^2) + \frac{r^2}{4k^2}\,(d\psi + k\,\cos(\theta)\,d\phi)^2 \,. \end{aligned}$$

One can verify that the curvature is just the same (locally it amounts to a reparameterization), though globally the space becomes $S^3/\mathbb{Z}_k$. This identifies $L(k;\ 1) \sim S^3/\mathbb{Z}_k$ with the monopole of charge k. Comparing the above metric with the one in (6.1) at constant $|v|$, we see that the matching of the fiber metric requires $k = 2$ and $\tau = \psi/2$.

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