PROCEEDINGS OF SCIENCE

Modave lectures on Noncommutative Geometry and its applications to physics

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I had the pleasure to give a series of lectures dedicated to Noncommutative Geometry and its applications to physics during the XVIII Modave Summer School in Mathematical Physics. These notes contain the material I presented during the School.

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1. Introduction

These lecture notes contain material I presented during the XVIII Modave Summer School in Mathematical Physics. My intention was to give a brief introduction to the subject of Noncommutative Geometry, accessible to Ph.D. students in physics, without assuming comprehensive mathematical background. Due to the time constraints, certain choices of material covered during the lectures had to be done. I am aware of the fact that these choices are biased by my personal research interests. These notes are not intended to substitute the extensive literature on the subject, and I recommend the reader familiarize themselves with both original scientific and review papers as well as existing textbooks. Some of them one can find in the list of references, e.g. [1–5]. This list is by no means comprehensive and I apologize to all the authors whose articles or books are not listed therein. Despite the fact that these lectures were intended to provide a gentle introduction to the subject, I decided to include also several results which are very recent, especially the ones that I was working on during my Ph.D. studies at Jagiellonian University under the supervision of prof. Andrzej Sitarz. The choice of the presented material is then biased by the content of my Ph.D. thesis [6], especially due to the fact that I was presenting these lectures during the week before my Ph.D. defense. Some more technical discussions and a more complete list of references for further reading can be found also therein. In these notes, I am trying to avoid using very sophisticated techniques, concentrating on physical applications. However, mathematical formalism is necessary to fully understand the importance of the discussed notions. Instead of starting with axioms, I try to motivate them. In my opinion, this approach has the advantage that it presents the subject as an active area of research and motivates the reader to call into question the made assumptions. Instead of assuming a series of God-given axioms, I prefer to challenge revealed truths. It turns out, as is usually the case in physics, that such a procedure allows for the existence of models with intriguing properties. In my research on Noncommutative Geometry, I have seen this many times.

The organization of these notes is as follows. The content of each section corresponds roughly to one lecture from a series given during the workshop. I start by posing a question about the meaning of geometry. In section 2 I motivate the use of Noncommutative Geometry, ending up with the notion of a spectral triple obtained by the consideration of pre-Fredholm modules and their role in differential calculus. To establish these objects, we formulate topological notions in C^*-algebraic language, represent them in terms of certain operators, describe the correspondence between vector bundles and modules, and discuss the notion of noncommutative differential calculus. For the latter, we briefly discuss the role of Hopf algebra structures and bicovariance for noncommutative differential forms. One step further and we will be in the land of Fredholm modules allowing for the algebraic formulation of differential calculus. This section is concluded with the definition of a spectral triple as a specific example of a Fredholm module. In section 3 I discuss several types of spectral triples, starting with finite-dimensional examples and discussing their classification. Then I briefly present different additional structures that can be added on top of a bare spectral triple. This includes grading, the real structure as well as a series of further conditions between these objects. Finally, the example of the canonical spectral triple as a link with classical geometry is described, together with the famous Connes' reconstruction theorem and its consequences. The aim of section 4 is to answer the question of how one can derive physical action out of the geometric data. Starting from the bosonic spectral action and its asymptotic expansion, the main idea behind the spectral action principle is presented. I then briefly present computational techniques based on symbolic calculus for pseudodifferential operators and the role of Wodzicki residue. Next, the construction of almost-commutative geometries is presented. Their role in considerations of gauge theories within the spectral geometry framework is then discussed and illustrated by the standard derivation of effective action for Yang-Mills theories. In section 5 these methods are used to describe the Standard Model of particle physics. I briefly discuss the appearance of the Higgs field in this formulation and comment on some potential issues that might arise from this construction. As a next step, I summarize in section 6 how to include pseudo-Riemannian structures for finite spectral triples and the consequences of its existence in the case of the Standard Model and also for its particular extension, the Pati-Salam model. Finally, section 7 contains a short list of other possible applications of Noncommutative Geometry in physical models. Due to the time constraint, I wasn't able to present the details during the given lectures, and these applications are then only briefly summarized. This list is of course not complete and only some potential applications are chosen to illustrate the plenty of possibilities. The choice is very subjective and biased by my own research interest. It is by no means dictated by the importance of a particular application, etc.

2. What is geometry?

Before starting any discussion about possible applications of generalized or noncommutative geometries to physical models, one has to first answer the question stated in the title of this section. The answer looks seemingly obvious, however, it really depends on the perspective. Having only experience with school geometry, one can naively think that the only things, it tells us about, are shapes of figures or distances between their specific points. Up some level of complexity, it is, in essence, true, but it is not the only thing we can associate with the notion of geometry. For our purpose, the geometry will be thought of as a sum of two ingredients. The first of them is a topology, which in particular tells us about how the points of our spaces are separated by open sets. It also takes into account continuous deformations of objects under consideration and introduces the concept of homeomorphism and homeomorphic equivalence. Several topological invariants can tell us how particular spaces differ from each other. The second ingredient of geometry can be called a *metric geometry*. It contains information about the distances between points which can be measured using a metric, which is a basic object in Riemannian geometry. Here we also have the concept of differential manifolds, and vector bundles over them as well as their sections defining vector fields, which are fundamental objects in many physical models. Continuing along these lines one introduces the notion of connections on bundles (e.g. related to gauge fields in physics) and their curvature.

In this section, I will briefly discuss some of the aforementioned concepts and describe their equivalent formulation that allows for generalizations. The first notion that has to be explored is topology. For a reason that will be clear later, I will need only locally compact Hausdorff spaces. The space $\mathcal{M}$ is locally compact if each point has a compact neighborhood. We can think of $\mathcal{M}$ as a collection of points $\{x\}_{x\in\mathcal{M}}$ with a set of conditions imposed on them. But can we forget about the points? Is it possible to fully characterize (class of) topological spaces without referring directly to the notion of a point? The answer turns out to be positive and leads to the concept of *pointless topology* (or, after discussing metricity, *pointless geometry* [7]). But is this topology indeed pointless or it has some nontrivial implications and interesting applications? The fact we can algebraically describe the space without directly using its points has an advantage in that this construction can be naturally generalized and allows for highly nontrivial extensions of the usual down-to-earth meaning of geometry.

In order to show how this works in practice, we start with the observation that instead of looking at the space we can equally well consider maps defined on it. It should be not surprising since it is exactly what we are doing in physics. For most practical purposes, we are working with fields defined on a given space. Moreover, the physical observables also are functions on physical space or spaces of parameters, e.g. on the phase space in classical mechanics. Also, the distances, etc. are at the end functions that associate numbers with pairs of points. Therefore, this perspective should not be anything strange for a physicist, but realizing this fact was the first step towards a generalizable algebraic formulation of the notion of geometry.

Let us then start by taking $\mathcal{M}$ to be a compact space. I will later show how to generalize the main results for only locally compact spaces, but at this moment we can restrict to this class of spaces to simplify the discussion. We can construct the space $C(\mathcal{M})$ of all continuous maps from $\mathcal{M}$ to $\mathbb{C}$. This set has many interesting features, in particular, of an algebraic nature. First of all, notice that for any two functions f_1, f_2 from $C(\mathcal{M})$ also the sum $f_1 + f_2$ belongs to $C(\mathcal{M})$. Moreover, if $\lambda \in \mathbb{C}$ and $f \in C(\mathcal{M})$, then $\lambda f \in C(\mathcal{M})$. Furthermore, these two operations are associative. In other words, $C(\mathcal{M})$ is a vector space. But this is not the only structure that $C(\mathcal{M})$ is naturally equipped with. One can also introduce point-wise multiplication, i.e. for $f_1, f_2 \in C(\mathcal{M})$ we define $(f_1 \cdot f_2)(x) := f_1(x) f_2(x)$ with $x \in \mathcal{M}$. This leads to an algebra structure on the space of complex-valued continuous functions of $\mathcal{M}$. What is remarkable, the multiplication operation is commutative, i.e. $f_1 \cdot f_2 = f_2 \cdot f_1$. As we will see shortly, this feature will be of crucial importance. However, this is not the final word one can say about this space. Since we are working with complex-valued functions, there is yet another natural operation one can consider. Mainly, for any $f \in C(\mathcal{M})$ we define $f^* \in C(\mathcal{M})$ by $f^*(x) := \overline{f(x)}$, $x \in \mathcal{M}$, where $\overline{z}$ stands for the usual complex conjugation in $\mathbb{C}$. Therefore, up to now, we know that $C(\mathcal{M})$ is a commutative $*$-algebra. So far so good, but there is even more one can do. The function $\|\cdot\| : C(\mathcal{M}) \to \mathbb{C}$ defined by $\|f\| = \sup_{x \in \mathcal{M}} |f(x)|$ is a well-defined norm on $C(\mathcal{M})$, so that $(C(\mathcal{M}), \|\cdot\|)$ is a Banach algebra. This means that it is complete, i.e. every Cauchy sequence of points in $C(\mathcal{M})$ has a limit in $C(\mathcal{M})$, and for all $f, g \in C(\mathcal{M})$ we have an inequality $\|fg\| \leq \|f\| \|g\|$. At this point, one can ask about further relations between the Banach algebra structure and the $*$-structure. Indeed, it turns out that

$$\forall f \in C(\mathcal{M}) \ \|f\|^2 = \|f^* f\|, \tag{1}$$

which is a defining relation of a C^*-algebra. To sum up, we know now that $C(\mathcal{M})$ is a commutative C^*-algebra. In other words, there is a natural way of attaching a C^*-algebra to a given compact space. One can then generalize this construction to locally compact spaces by considering $C_0(\mathcal{M})$, the algebra of continuous functions vanishing at infinity.

Let us now explore a little bit the world of C^*-algebras. To be precise, a C^*-algebra $\mathcal{A}$ is a Banach algebra which has a $*$-structure s.t. for all $a \in \mathcal{A}$ we have $\|a^* a\| = \|a\|^2$. As a first exercise, I suggest proving that the norm $\|\cdot\|$ on a $\mathcal{A}$ is unique. Please also make a precise sense of this statement. One can easily show that the following are natural examples of C^*-algebras: $\mathbb{C}$, $M_n(\mathbb{C})$,

$\bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C})$ and $B(\mathcal{H})$ for a Hilbert space $\mathcal{H}$. Notice that not all of them are commutative. This is the feature that will allow us to discuss noncommutative topologies. But before that, we need to introduce and discussed further notions and their basic properties. The first of them is related to the observation that if $\mathcal{M}$ was compact, then $C(\mathcal{M})$ naturally contains a unit element with respect to the multiplication, i.e. it is an unital commutative C^*-algebra, while for only locally compact space $\mathcal{M}$, the commutative C^*-algebra $C_0(\mathcal{M})$ do not have any such an element. However, there is a natural construction, called unitization, that allows for adding a unit to a given C^*-algebra $\mathcal{A}$. It is done by considering $\mathcal{A}^+ = \mathcal{A} \times \mathbb{C}$ with a multiplication $(a, \lambda)(b, \rho) := (ab + \lambda b + \rho a, \lambda\rho)$. Then one can easily check that $\mathcal{A}^+$ is an unital C^*-algebra with $1_+ = (0, 1)$ as a unit. Why this construction is important? To answer this question, we need to remind a one-point compactification known from classical topology. To a locally compact Hausdorff space, $\mathcal{M}$ we associate its Alexandroff compactification by *adding a point at infinity*, $\mathcal{M}^+ = \mathcal{M} \sqcup \{\infty\}$ and defining the resulting topology by adding to the collection of open sets on $\mathcal{M}$ the family $\{(\mathcal{M} \setminus C) \cup \{\infty\} : C - \text{compact in } \mathcal{M}\}$. One can show [1] that the Alexandroff compactification of the underlying space is related to the unitization of the corresponding C^*-algebras by $C_0(\mathcal{M})^+ = C(\mathcal{M}^+)$. As a result, for many purposes, we can assume from the very beginning that the space $\mathcal{M}$ is compact.

Above we have demonstrated that one can naturally assign a commutative C^*-algebra to a locally compact Hausdorff space. In other words, we have established that there exists *an arrow*

$$\textbf{Topology} \rightsquigarrow \mathbf{C^*}\textbf{-algebras}. \tag{2}$$

Moreover, this assignment extends also to the level of morphisms between the spaces, leading to the conclusion that one can think of it in the language of category theory and functorial properties. I will not discuss here this aspect but the reader familiar with category theory is highly invited to formulate the above statement rigorously by providing the precise description of involved categories and the functorial mapping. The natural question that arises at this moment is the existence of an arrow that goes in the opposite direction. Moreover, are these two arrows inverses of each other? That is, is it possible to associate a topological space to a given C^*-algebra in a way that applying the resulting construction to a C^*-algebra obtained as $C(\mathcal{M})$ reproduces exactly the space $\mathcal{M}$? The answer is affirmative within the class of commutative C^*-algebras. Before showing this, we need to introduce some mathematical notions. Firstly, we need a notion of a character on a Banach algebra $\mathcal{A}$. It is defined as a non-zero homomorphism $\mu : \mathcal{A} \to \mathbb{C}$. The space of all characters on $\mathcal{A}$ is denoted by $M(\mathcal{A})$ and, if $\mathcal{A}$ is commutative, it is called the Gelfand spectrum of $\mathcal{A}$. On this space, there is a natural topology, the so-called Gelfand topology, which is the relative topology determined by the inclusion $M(\mathcal{A}) \hookrightarrow B_{\mathcal{A}^*}(0, 1)$, where the unit ball on the right is compact in the weak* topology by Banach-Alaoglu theorem. One then can show that $M(\mathcal{A})$ equipped with this topology is a locally compact Hausdorff space [1]. Therefore, there is a way to produce topological spaces out of commutative C^*-algebras, and moreover, we end up within exactly the same class of topological spaces we started with. It remains to argue that the two constructions we have are indeed inverses of each other. To formulate the precise theorem we need a notion of a Gelfand transformation, which is defined as a map

$$\mathcal{A} \ni a \longmapsto \hat{a} \in C_0(M(\mathcal{A})), \tag{3}$$

where $\hat{a} : M(\mathcal{A}) \ni \mu \longmapsto \mu(a) \in \mathbb{C}$, the so-called Gelfand transform of an element a. The fundamental Gelfand-Naĭmark theorem then says that the above Gelfand transformation is an isometric $*$-isomorphism $\mathcal{A} \cong C_0(M(\mathcal{A}))$ [1]. In other words, there is a one-to-one correspondence between locally compact Hausdorff spaces and commutative C^*-algebras,

$$\textbf{Topology} \leftrightsquigarrow \textbf{Commutative C}^*\textbf{-algebras}. \tag{4}$$

Furthermore, under the above identification, the compactness of the topological space corresponds to the fact that the algebra is unital. This discussion motivates the natural generalization one can introduce. Mainly, suppose we forget about the adjective on the right-hand side of the above correspondence. What is then on the left-hand size? In other words, we would like to have the following picture:

$$\textbf{Noncommutative Topology} \leftrightsquigarrow \textbf{Noncommutative C}^*\textbf{-algebras}. \tag{5}$$

One can then think of noncommutative C^*-algebras as algebras of functions on something that can be referred to as *noncommutative space*. This is the main idea behind Noncommutative Geometry. Again, there is also a rigorous categorical version of the above statement and the reader is invited to explore the formulation of the above correspondence using this language. Knowing that for the classical case, we can forget about the points and work purely on the algebraic level, one can then extend this to the category of noncommutative algebras and think of them as generalized geometries.

Let us now look more closely at the algebraic side. First of all, arbitrary C^*-algebras seem to be quite general and one may wonder how artificial is this notion. Physicists used to work with a more concrete picture, or more precisely, more concrete realizations or representations of mathematical objects. Fortunately, such a formulation exists also for these algebras and it is given by the famous GNS representation [8]. Any C^*-algebra (commutative or not) can be represented as an algebra of bounded operators on a certain separable Hilbert space $\mathcal{H}$. In other words, its elements can be understood as operators acting on $\mathcal{H}$. Therefore, by the aforementioned duality, we can now encode topology in terms of some (properties of) bounded operators.

Of course, this is not the end of the story. Our goal is to describe not only the topology but we need the information about distances and differential structures in our space to be able to really talk about its geometry. It seems like the following two aspects have to be discussed:

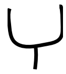
(a) We need smoothness to be able to compute certain derivatives. In many places in the classical Riemannian geometry, one has to differentiate functions to compute certain quantities like curvatures, etc. To encode these ideas in the algebraic language we expect that having an algebraic version of smoothness will be necessary.

(b) The fundamental objects in Riemannian geometry are sections of vector bundles. This includes vector fields, differential forms, and any tensor fields. On the other hand, these are also the notions that naturally appear in physical models. Therefore, we can expect that having established their algebraic version could shed a light on possible applications of Noncommutative Geometry in physics.

Regarding the first aspect, we can make use of the fact that smooth functions form a dense $*$-subalgebra $C^\infty(\mathcal{M})$ of $C(\mathcal{M})$. This is a consequence of the classical Stone-Weierstrass theorem.

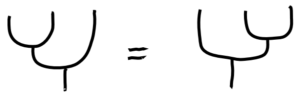
This suggests that one could use $*$-algebras instead of C^*-algebras, but having in mind that there is a larger C^*-algebra in which the $*$-algebra is densely embedded. This is how this issue is (usually) solved in Noncommutative Geometry. The second aspect is more subtle. First of all, the standard way of translating bundle-like notions into a noncommutative world is by making use of the Serre-Swan theorem [1]. Roughly speaking, it establishes a correspondence between vector bundles and modules over the algebra of functions on the manifold. The object of a particular interest is the $C^\infty(\mathcal{M})$-bimodule $\Omega^1(\mathcal{M})$ of differential 1-forms. It is naturally equipped with a differential operator $d : C^\infty(\mathcal{M}) \to \Omega^1(\mathcal{M})$ that satisfies the Leibniz rule. The last property allows in particular for the extension of the $\Omega^1(\mathcal{M})$ into higher-order differential forms, resulting in the construction of the differential complex $\Omega^\bullet(\mathcal{M})$ that can be also understood in terms of differential graded algebra (DGA). The algebraic analog of the space of 1-forms is first-order differential calculus (FODC). It is a pair $(\Omega^1(\mathcal{A}), d)$ of a bimodule over $\mathcal{A}$, $\Omega^1(\mathcal{A})$, and a linear map d from $\mathcal{A}$ into $\Omega^1(\mathcal{A})$ s.t. the Leibniz rule is satisfied, i.e. for all $a, b \in \mathcal{A}$ we have $d(ab) = (da)b + a(db)$. We stress that since, in general, the algebra $\mathcal{A}$ is not necessarily commutative, we have to be careful about the order in the multiplication operation. For functions over the manifold $\mathcal{M}$ this was not the case, and indeed it simplified a lot of the subtleties existing in the noncommutative world. We remark that since it is possible that more than one noncommutative generalization can give the same commutative limit, it is usually not obvious what is the right choice for the noncommutative analog of classical objects. The noncommutative world is therefore much richer than the one we used to work with in classical geometry. It might also happen that a straightforward generalization of commutative notions that are known to be uniquely defined by a given set of axioms, in the noncommutative framework may lead to a whole family of allowed objects. Therefore, the question of how many first-order differential calculi are allowed on a given noncommutative algebra is then fully justified. One can also ask about their classification. The answer can be compactly formulated for the class of bicovariant calculi and is known as Woronowicz theorem [9]. In its formulation, the role played but yet another algebraic structure, Hopf algebras, is crucial. We will now do a small detour and give a brief presentation of these objects. It turns out that Hopf algebras are significant only for the differential calculi, but appear in many different branches of Noncommutative Geometry and its application to physics. Later on, we will discuss some other aspects related to the so-called Quantum Riemannian Geometry [10] and also very briefly the role of certain Hopf algebra structures in the renormalization problems in quantum field theories [11].

In order to define the notion of Hopf algebra we need to first introduce tools that allow for rephrasing algebraic definitions in a much more accessible form. We then use diagrammatic techniques. To illustrate how they work, we begin with the notion of an unital algebra, which is a vector space $\mathcal{A}$ equipped with two maps: multiplication $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and a unit $\eta : \mathbb{C} \ni \lambda \mapsto \lambda 1_{\mathcal{A}} \in \mathcal{A}$. These maps are supposed to satisfy a few compatibility conditions which are equivalent to the statement that the multiplication has to be associative and the multiplication by unit acts like an identity operation. These conditions can be written purely algebraically but I prefer another way of representing them using a graphical method (see e.g. [12] for a more detailed discussion and further examples). The multiplication map can be represented by the following diagram

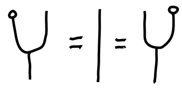


which has a simple intuitive interpretation when one reads it from the top to the bottom: take two elements, say $a, b \in \mathcal{A}$, and multiply them obtaining, as a result, a new element $m(a \otimes b) \in \mathcal{A}$. For the identity map, the corresponding diagram is

which again can be easily interpreted as an insertion of the unit in the algebra $\mathcal{A}$. The associativity of the multiplication is then represented as

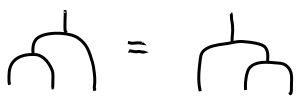


while the fact that η is the unit can be schematically drawn as follows:

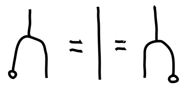


Equipped with the graphical tools we are ready to discuss further algebraic structures which will finally lead us to the notion of Hopf algebra. As a next step, we need an object which is dual to algebra. The reader familiar with some basics of category theory should easily guess the name of this object: a coalgebra. $\mathcal{A}$ is called a coalgebra if it possesses a map $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, called a comultiplication or a coproduct, and a map $\varepsilon : \mathcal{A} \to \mathbb{C}$, called a counit, that satisfy conditions *dual* to the ones for multiplication and unit. To draw them one can simply place a mirror along the horizontal line and read them from the ones for the algebra. More precisely, the maps Δ and ε are represented, respectively, by the following diagrams:

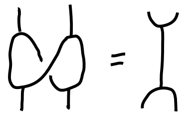
The coproduct satisfies the coassociativity condition,



and together with the counit, they satisfy



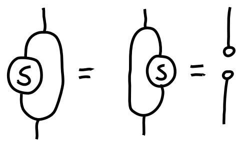
Imagine now a situation on a vector space $\mathcal{A}$ we have an algebra structure given by (m, η) and a coalgebra structure defined by the maps Δ and ε. It would be great if these two are not completely independent but rather they are compatible with each other. The notion which arises in such a situation is the one of bialgebra. We say that $\mathcal{A}$ is a bialgebra if it is both algebra and coalgebra, and the conditions represented in the following diagrams are fulfilled:



and

In the first diagram, the new structure appears. Notice that we have to interchange two elements of $\mathcal{A}$ and to represent this operation diagrammatically we have introduced a new symbol. Working within the category of vector spaces it is not necessary to be very careful about the intersection of the lines since in this case the interchange is given simply by the flip operation $a \otimes b \mapsto b \otimes a$, and it is involutive. However, it might happen that in other categories this is not the case, and a more general notion is needed. This leads to the consideration of the so-called braided monoidal categories. In such a situation, the braiding Ψ that generalized the flip map does not need to satisfy $\Psi^2 = \mathrm{id}$. Such structures play an important role in the theory of anyons, where the braiding has a physical meaning of interchanging particles on a two-dimensional surface and it is related to the statistics of such collective excitations. In these lecture notes, I will not discuss this undeniably exciting topic, and the interested reader is highly invited to consult e.g. [13] for further details of the mathematical background and applications in physical systems.

So far we have defined the notion of a bialgebra. Suppose $(\mathcal{A}, m, \eta, \Delta, \varepsilon)$ is such an object and consider the space $\mathrm{End}(\mathcal{A})$ of endomorphisms on $\mathcal{A}$. For any two elements f, g of the latter, the convolution $f * g$ is defined through $f * g = m(f \otimes g)\Delta$. One easily checks that this operation defines a multiplication on $\mathrm{End}(\mathcal{A})$. Since id is also an endomorphism on $\mathcal{A}$, one can ask if this map has an inverse with respect to the convolution. If this is true, the bialgebra $\mathcal{A}$ is called a Hopf algebra. In other words, a Hopf algebra $\mathcal{A}$ is a bialgebra together with a map $S : \mathcal{A} \to \mathcal{A}$ s.t.



The map S is called an antipode. As an exercise, I suggest demonstrating that on a given bialgebra there exists at most one Hopf algebra structure, i.e. the antipode is unique.

As a first example consider a finite group G and a ring R, and let $\mathcal{A}$ be the group ring $R[G]$. For example, $\mathbb{C}[G] = \left\{ \sum_i \lambda_i g_i \,:\, \lambda_i \in \mathbb{C},\, g_i \in G \right\}$. The multiplication and a unit are naturally inherited from the group structure on G. For the coproduct, we can take $\Delta(g) = g \otimes g$, while $\varepsilon(g) = 1_R$ and $S(g) = g^{-1}$. These maps are then extended by linearity. One can easily check that they indeed define a Hopf algebra. We remark that for a generic Hopf algebra $\mathcal{A}$ an element $a \in \mathcal{A}$ is called group-like if $\Delta(a) = a \otimes a$.

For a second example, we again start with a finite group G but then we defined $\mathcal{A}$ to be the space $C(G)$. This has a natural algebra structure, as we discussed before. The coproduct can be introduced using the group structure on G. Mainly, we define $\Delta(f)(x, y) = f(xy)$. Its coassociativity follows from the associativity of the multiplication in G. Next, $\varepsilon(f) = f(1_G)$ and

$S(f)(x) = f(x^{-1})$. Again, it is a simple exercise to show the remaining conditions for being a Hopf algebra.

From the above examples, one can expect that there is a deeper relationship between groups and Hopf algebras. Indeed, Hopf algebras are sometimes called *noncommutative analogs* of groups, and the notion of *Hopf algebra symmetry* often appears in physical literature (see e.g. [14]) as a generalization of the usual group symmetry. We also remark that some authors use the name *quantum groups* for Hopf algebras. It is, however, not a fully precise statement. In this context, it is better to call them Drinfeld–Jimbo-type quantum groups [15]. There is another notion of quantum groups that involves also the C^*-algebraic structure originating from Woronowicz compact matrix quantum groups [9]. I will come back to the last version of quantum groups and quantum spaces at the end of these notes and I will very briefly discuss their potential applications in the modern theory of quantum information.

Having defined Hopf algebras, one can follow this route and, by a complete analogy from the theory of algebras, consider the notion of modules over Hopf algebras, etc., but now there are more structures that have to be compatible with each other. However, since we have in hand also *co*-objects, we can also think of comodules, etc., that would involve the coproduct and counit, so that we will deal with coactions.

Suppose we are interested in FODC over a Hopf algebra $\mathcal{A}$. It would be great if these calculi will not be too wild, i.e. they should satisfy certain compatibility conditions with the algebraic structures we have. This regularity condition is known as bicovariance and was studied by Woronowicz [9]. He was then able to classify such calculi. The crucial object that appears in this classification is the so-called universal FODC. It is defined by taking $\Omega^1_u(A)$ to be the kernel of the multiplication map m on $\mathcal{A}$, and the differential operator is defined by $d_u(a) = a \otimes 1 - 1 \otimes a$. Obviously, it is an example of a FODC since the Leibniz rule is satisfied. But why it was called *universal*? The reason is that for any $\mathcal{A}$-bimodule $\Omega^1(\mathcal{A})$ defining a FODC $(\Omega^1(\mathcal{A}), d)$ over $\mathcal{A}$, there exists a unique map $\iota_d : \Omega^1_u(\mathcal{A}) \to \Omega^1(\mathcal{A})$ s.t. $d = \iota_d \circ d_u$. In other words, every FODC can be obtained from the universal one.

Having a FODC calculus, one can then try to define higher forms in a way compatible with the Leibniz rule. We will not discuss here the details which can be found e.g. in [1]. The take-home message from the above discussion is that one can introduce in this way analogs of differential forms in the noncommutative world. The noncommutative space in this language is just a (dense subset of a) C^*-algebra we started with. We know that we can think of its elements as bounded operators on a certain Hilbert space $\mathcal{H}$. In other words, there exists a representation $\pi : \mathcal{A} \to B(\mathcal{H})$. It would be great if we can also encode the differential forms in the same language. Therefore, the natural question arises: when it is possible to think of differential forms as bounded operators on the same Hilbert space $\mathcal{H}$? It turns out that the answer is affirmative if we can find an unbounded self-adjoint operator F on $\mathcal{H}$ s.t. it is involutive and its commutators with all $\pi(a)$, $a \in \mathcal{A}$ are bounded as operators on $\mathcal{H}$. Then $\pi(a_0)[F, \pi(a_1)] \dots [F, \pi(a_n)]$ defines the representation π_F of $\Omega^\bullet(\mathcal{A})$ on the same Hilbert space $\mathcal{H}$. This observation motivated the notion of a Fredholm module which allows for an algebraic formulation of differential calculus [1]. A Fredholm module is essentially a pair (π, F) of a representation of $\pi : \mathcal{A} \to B(\mathcal{H})$ and an operator F that satisfies the aforementioned conditions (for the commutators, instead of being bounded one usually assumes here their compactness). One can also allow for less restrictive conditions. Mainly, we do not demand that the above requirements are strictly satisfied but we allow for small deviations. In this world, the word *small* means *up to compact operators*. More precisely, instead of e.g. assuming that F is selfadjoint, we demand only that $\pi(a)(F-F^*)$ is a compact operator for any $a \in \mathcal{A}$. Similarly for the other conditions. This leads to the notion of the pre-Fredholm module. Equivalence classes of such objects are central in the so-called K-homology theory, which together with the K-theory forms a framework that unifies all the notions we need - the KK-theory. I will not discuss in these lecture notes any more sophisticated aspects of this branch of mathematics. The interested reader can consult e.g. [1, 16] and references therein. Let us stress here that by satisfying the properties that allow for the representation of differential forms, pre-Fredholm modules are usually understood as a generalized version of geometries. For our purposes, we will need only their special cases for which the operator F is obtained from a more concrete construction. We will see that this type of example is very natural and appears in my place in theoretical physics.

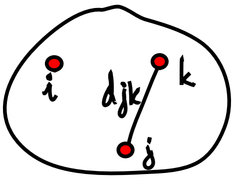
Having announced and motivated the object of interest, let us finally define it. Suppose that on the Hilbert space $\mathcal{H}$, on which an unital $*$-algebra $\mathcal{A}$ is (faithfully) represented, we have given an (essentially, i.e. on a dense domain,) self-adjoint operator $\mathcal{D}$ which has compact resolvent (i.e. a map $\lambda \mapsto (\mathcal{D}-\lambda)^{-1}$) and every commutator $[\mathcal{D}, \pi(a)]$ is bounded (in the *essentially self-adjoint* case: can be extended to a bounded operator), $a \in \mathcal{A}$. Under these assumptions, the operator $F = \mathcal{D}(1+\mathcal{D}^2)^{-1/2}$ defines a pre-Fredholm module. In other words, the triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ can be understood as a generalized version of geometry, since it contains information about the differential structure on a noncommutative manifold $\mathcal{A}$. Such a triple is called a **spectral triple**. In these lecture notes, we will identify *geometry* with a *spectral triple*. In the next sections, I will discuss this notion in more detail, study the canonical examples and present some applications to physics.

3. Spectral Triples and all that

The goal of this section is to briefly discuss the concept of spectral triples and further motivate their use. Firstly, following mostly the approach presented in [2, 17], I will describe how to algebraically encode distances on finite spaces. This will lead us to a version of Connes' distance formula for finite geometries. As one can guess, for finite spaces one can fully classify spectral triples describing possible geometries. I will briefly discuss how this can be achieved using decorated graphs. The reader interested in a more detailed discussion is invited to consult [18]. Yet another approach, based on purely algebraic considerations of matrix algebras can be found in [19]. The second approach will be especially useful in the next sections when I will show some existing proposals on how one can include the Lorentzian structure into the language of spectral geometry.

The rest of this section is dedicated to the discussion of general spectral triples and their certain subclasses. I will present an overview of several additional structures that one can include on top of a spectral triple. They are motivated either by geometrical properties known from classical Riemannian theory or are imposed based on physical considerations. The chosen way of presenting this part of the material is highly biased by the way of thinking one can find in my Ph.D. thesis [6]. The approach of *not assuming too much* will manifest itself many times in these lecture notes. We will see how the classical manifolds fit into the general framework and how to algebraically compute distances on them.

We start our discussion of spectral triples with the simplest possible examples - finite spaces. Let $X = \{1, \ldots, N\}$ be a space consisting of N points. To simplify the notation we write here k instead of x_k. According to the ideas of Noncommutative Geometry, we should replace X by the algebra $\mathcal{A}$ of functions on X. Since the space is finite, we deal here with a discrete topology and do not need to care about subtle topological aspects that will play a role in the case of e.g. Riemannian manifolds. Since X has N points, any function a defined on it will be completely specified by N numbers: $a(1), \ldots, a(N)$. We would like to represent these functions as operators acting on a certain Hilbert space. Since $|X| = N$, the natural choice for the Hilbert space is $\mathcal{H} = \mathbb{C}^N$. How does the function a act on $\mathcal{H}$? This representation is chosen to be given by an action of a diagonal matrix on a vector from $\mathcal{H}$. In other words, we identify $a \in \mathcal{A}$ with the matrix $\mathrm{diag}(a(1), \ldots, a(N)) \in M_N(\mathbb{C}) = \mathrm{End}(\mathcal{H})$. In the spirit of Gelfand duality, we identify A with X. The set X can, in principle, correspond to several configurations of N points that differ e.g. by the relative distances between them. Let then $d_{jk} = \mathrm{dist}(j, k)$ be the distance between the points j and k:



Denote by d the collection of all such numbers for points in X, i.e. $\mathsf{d} = \{d_{jk} \,:\, j, k \in X\}$. Of course, not every family of real numbers will lead to a set of distances on finite space. A finite subset of $\mathbb{R}$ forms a family d if and only if

- $d_{jk} = d_{kj}$,
- $d_{jk} \geq 0$,
- $d_{jk} = 0 \Leftrightarrow j = k$,
- $d_{jl} \leq d_{jk} + d_{kl}$.

This is nothing else than the usual conditions in the definition of a metric. We would like to find an operator $\mathcal{D}$ acting on $\mathcal{H}$ from which the family d can be deduced. In other words, the goal is to encode information about the metric structure in terms of operators acting on the same space on which the algebra is represented. Since we are working now with finite spaces, the operators we are dealing with are just matrices and all the computations reduce to manipulations of matrices. As a first example, we start with the simplest and simultaneously non-trivial example - the two-point space, $X = \{1, 2\}$. In this situation we have $\mathcal{H} = \mathbb{C}^2$ and the representation of $\mathcal{A}$ on $\mathcal{H}$ is given by $\pi(a) = \mathrm{diag}(a(1), a(2))$. Let us define the operator $\mathcal{D}$ to be of the form

$$\mathcal{D} = \begin{pmatrix} 0 & \frac{1}{d_{12}} \\ \frac{1}{d_{12}} & 0 \end{pmatrix}. \tag{6}$$

A simple computation shows that

$$[\mathcal{D}, \pi(a)] = \frac{1}{d_{12}} \begin{pmatrix} 0 & a(2) - a(1) \\ a(1) - a(2) & 0 \end{pmatrix}. \tag{7}$$

How to extract the distance d_{12} from it? Notice first that the norm of this matrix is $\frac{1}{d_{12}}|a(1)-a(2)|$. Therefore, computing $[\mathcal{D},\pi(a)]$ and knowing the algebra $\mathcal{A}$, we can in principle get the information about the distance between the two points. This result generalizes to any finite space. Indeed, it turns out that for any $N \geq 2$ there exists a Hilbert space $\mathcal{H}_N$, a corresponding representation π_N of $\mathcal{A}$ on $\mathcal{H}_N$, and a Hermitian matrix $\mathcal{D}_N \in \mathrm{End}(\mathcal{H}_N)$ s.t.

$$\|[\mathcal{D}_N,\pi_N(a)]\| = \max_{j\neq k}\left\{\frac{1}{d_{jk}}|a(j)-a(k)\right\}. \tag{8}$$

This result can be proven by induction and be found in [2]. Obviously,

$$\sup_{a\in\mathcal{A}}\{|a(j)-a(k)| \,:\, \|[\mathcal{D}_N,\pi_N(a)]\|\leq 1\} \leq d_{jk}, \tag{9}$$

but also the opposite inequality holds as one can easily show by constructing a concrete function a that saturates the above upper bound on the supremum (for details see again [2]). This leads to an explicit formula to compute the distances d_{jk} out of the algebraic data $(\mathcal{A}=\mathbb{C}^N,\mathcal{H}_N,\mathcal{D}_N)$. One can easily check that this system defines a spectral triple. The commutators $d(a) := [\mathcal{D},\pi(a)]$ play a special role. It is an easy exercise to show that d defined in this way satisfies the Leibniz rule and therefore leads to the construction of the space of differential 1-forms, $\Omega^1_{\mathcal{D}}(\mathcal{A}) := \{\sum_k a_k[D,b_k] : a_k, b_k \in \mathcal{A}\}$, where we have omitted the symbol π of representation identifying the elements of the algebra with the corresponding operators on the Hilbert space. This notation will be used from now on as long as this will not lead to any confusion. The set $\Omega^1_{\mathcal{D}}(\mathcal{A})$ is known in the literature as the space of *Connes' 1-forms*. This construction can be performed for more general spectral triples.

In the above discussion, we have seen that finite spaces lead to examples of finite spectral triples, i.e. the ones with finite-dimensional algebra $\mathcal{A}$. The natural question that arises is if we can describe all possible spectral triples with finite-dimensional algebras. The answer to this question turns out to be affirmative as was shown by Krajewski [18] and Paschke and Sitarz [19]. Using the fact that any unital $*$-algebra which can be faithfully represented on a finite-dimensional Hilbert space is a direct sum of matrix algebras, $\mathcal{A}=\bigoplus_{i=1}^N M_{n_i}(\mathbb{C})$, one can then use a corresponding decomposition of the Hilbert space $\mathcal{H}=\bigoplus_{i=1}^N \mathbb{C}^{n_i}\otimes V_i$ with $\dim(V_i)$ being the multiplicity of the representation n_i, to study possible operators satisfying required set of conditions. One can do this purely algebraically or use diagrammatic techniques known as Krajewski diagrams. For the latter, we will illustrate this in a simple example with $\mathcal{A}=M_{n_1}(\mathbb{C})\oplus M_{n_2}(\mathbb{C})$ and $\mathcal{H}=(\mathbb{C}^{n_1}\otimes V_1)\oplus(\mathbb{C}^{n_2}\otimes V_2)$ with $\dim(V_1)=1$ and $\dim(V_2)=2$. The pair $(\mathcal{A},\mathcal{H})$ encodes the vertices of a graph. In our example, this gives two vertices - one per each n_j, $j=1,2$. The second of them, however, has to be doubled because the dimension of the corresponding space V_2 is equal to 2. The operator D can be written as $D=\sum_{i,j} D_{i,j}$ with $D_{ij}:\mathbb{C}^{n_i}\otimes V_i \to \mathbb{C}^{n_j}\otimes V_j$ s.t. $D_{ij}=D^*_{ji}$. Every such non-zero D_{ij} gives rise to an edge of the graph with vertices $\{n_j\}_{j=1}^N$. In other words, edges of this graph *are decorated* by the D_{ij} maps. It turns out that there is a one-to-one correspondence between decorated graphs (i.e. ordered pairs (G,Λ) with G being a finite graph and $\Lambda\subseteq\mathbb{N}_+$) and finite spectral triples *up to unitary equivalence*, where we say that two spectral triples $(\pi_1:\mathcal{A}\to\mathcal{H}_1,\mathcal{D}_1)$ and $(\pi_2:\mathcal{A}\to\mathcal{H}_2,\mathcal{D}_2)$ with the same algebra $\mathcal{A}$ are unitarily equivalent if there exists a unitary matrix U s.t. $U\pi_a(a)U^*=\pi_2(a)$ and

$U\mathcal{D}_1U^* = \mathcal{D}_2$. For a precise statement and further examples e.g. [2] and references therein. The main message of this discussion is that one can reduce the problem of classification of finite spectral triples to the classification of decorated graphs. Of course, demanding that our spectral triple satisfies further conditions will reflect in the structure of the corresponding Krajewski diagram. These graphical methods are especially useful when our triple is equipped with other structures that can be formulated in terms of the existence of further operators that satisfy certain commutativity conditions with the already existing ones, in particular with the $\mathcal{D}$ operator.

As advertised above, our next goal is to add additional structures on $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. Here we do not longer assume that we are working with finite triples. I will often refer to these extra structures as *decorations*. This is because, in principle, we do not need to have them always but there are situations in which the presence of these structures mimics certain properties known from classical geometry. The first such decoration is the grading. It is implemented by a self-adjoint operator $\gamma = \gamma^* \in B(\mathcal{H})$ which squares to an identity operator, commutes with the representation of the algebra $\mathcal{A}$, and anticommutes with the operator $\mathcal{D}$. A spectral triple equipped with such a grading operator is called even. If no such γ exists the name of an odd triple is used. The presence of such grading essentially means that we can divide the content of the Hilbert space into two parts and adjust the action of the operator $\mathcal{D}$ accordingly. We will shortly see that the presence of this grading in the case of the triple describing the Standard Model of particle physics will be related to the fact of having two chiralities of particles. This will allow distinguishing on the algebraic level left-handed particles from right-handed ones.

The second structure that is usually present on spectral triples is related to the fact that an algebra $\mathcal{A}$ possesses a $*$-structure. This leads to the conclusion that it would be great to have some operator on the corresponding Hilbert space that implements an involution. One can do this by using an antilinear isometry J on $\mathcal{H}$. Again, we have to take into account compatibility conditions with the remaining structures in the spectral triple. In particular, we have to demand that $\mathcal{D}J = \epsilon J\mathcal{D}$, $J^2 = \epsilon'\mathrm{id}$ and $J\gamma = \epsilon''\gamma J$ with $\epsilon, \epsilon', \epsilon'' = \pm 1$. Since each of the signs can take two possible values, we have eight allowed combinations. The choice of these signs corresponds to the choice of the so-called KO-dimension, which is a number modulo eight (having a much deeper meaning in the KR-theory, but this discussion is much beyond the scope of this introductory lecture notes). The operator J defines a *real structure* and the resulting triple is called real. The reader familiar with the Tomita-Takesaki modular theory [20] should immediately formulate a question if the presence of the J operator allows for implementing a bimodule structure on $\mathcal{H}$. Having a real structure allows for, in addition to the left action,

$$\mathcal{A} \times \mathcal{H} \ni (a, \phi) \longmapsto a\psi \in \mathcal{H}, \tag{10}$$

defining also the right one:

$$\mathcal{H} \times \mathcal{A} \ni (\psi, a) \longmapsto Ja^*J^{-1}\psi \in \mathcal{H}. \tag{11}$$

The existence of the bimodule structure means that the above two actions commute with each other, i.e. the so-called zeroth order condition is satisfied:

$$\forall a, b \in \mathcal{A}\ [a, Jb^*J^{-1}] = 0. \tag{12}$$

This condition does not involve the operator $\mathcal{D}$ but only relates the real structure with the representations of the algebra $\mathcal{A}$. Now, it is time for adding the operator $\mathcal{D}$ into the game. The so-called first-order condition,

$$\forall a, b \in \mathcal{A} \ \left[[\mathcal{D}, a], Jb^* J^{-1}\right] = 0, \tag{13}$$

is the one that is very often imposed on a spectral triple. It essentially relates 1-forms obtained from the left representation of the algebra $\mathcal{A}$ with the right representation. If it is understood in this way, one can easily generalize the first-order condition to the situation when the real structure is not present but we have given only two independent representations: π_L and π_R (the left and right one, respectively). The role of the 1-forms for particular representations turns out to be crucial in the formulation of other classes of spectral triples. Having defined the space $\Omega^1_{\mathcal{D}}(\mathcal{A})$ of Connes' 1-forms, one can construct the algebra generated by $\mathcal{A}$ and $\Omega^1_{\mathcal{D}}(\mathcal{A})$ understood as a complex C^*-subalgebra of $B(\mathcal{H})$. The resulting algebra is denoted by $Cl_{\mathcal{D}}(\mathcal{A})$ and called a Clifford algebra for the given spectral triple. It can be used to formulate further conditions one can impose on a given spectral triple [21, 22]. The so-called second-order condition can be formulated by demanding that $JCl_{\mathcal{D}}(\mathcal{A})J^{-1}$ is contained in the commutant of $Cl_{\mathcal{D}}(\mathcal{A})$ in $B(\mathcal{H})$. One can also ask when the above inclusion is actually equality. If it is the case, we say that we have a Hodge spectral triple or that the spectral triple satisfies the Hodge property. This notion originates from the consideration of the spectral triple defined as follows. Let $(\mathcal{M}, g)$ be an oriented closed Riemannian manifold equipped with a Hermitian vector bundle $E \to \mathcal{M}$. Let $c : \Gamma^\infty(\mathcal{M}, T^*_{\mathbb{C}}\mathcal{M} \otimes E) \to \Gamma^\infty(\mathcal{M}, E)$ be the unitary Clifford action and suppose that ∇^E is a connection compatible with g. Then one can define $\mathcal{A} = C^\infty(\mathcal{M})$, $\mathcal{H} = L^2(\mathcal{M}, E)$ and $\mathcal{D} = c \circ \nabla^E$. Taking $E = \bigwedge^\bullet T^*_{\mathbb{C}}\mathcal{M}$, the operator $\mathcal{D}$ corresponds to the Hodge-de Rham operator known from the theory of differential forms.

We are now ready to present the example that will motivate the use of spectral triples as generalized geometries. This, the so-called *canonical spectral triple*, is constructed as follows. Let $\mathcal{M}$ be a closed four-dimensional Riemannian manifold, with metric g, equipped with a spin structure. As an algebra we take $\mathcal{A} = C^\infty(\mathcal{M})$, the algebra of smooth functions on the given manifold $\mathcal{M}$. Let $\mathcal{S} \to \mathcal{M}$ be a spinor bundle over $\mathcal{M}$ that gives the spin structure. The Hilbert space is taken to consist of square-integrable sections of this bundle, $\mathcal{H} = L^2(\mathcal{M}, \mathcal{S})$. Since $\mathcal{M}$ is a Riemannian manifold it possesses the unique Levi-Civita connection, which can be then lifted to the spin bundle leading to the spin connection ω. Using the associated Clifford algebra with generators γ^μ satisfying $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I_4$, one can then construct an operator, whose action on sections of the spinor bundle is locally given by

$$\mathcal{D}_{\mathcal{M}} = i\gamma^\mu(\partial_\mu + \omega_\mu). \tag{14}$$

It is called the Dirac operator for $\mathcal{M}$. Notice that the Clifford algebra $Cl_4(\mathcal{M})$ naturally possesses a grading γ which in this case is given by the γ_5 operator. Moreover, we have therein a charge conjugate operator C which can be used to define a real structure J. To summarize, the system $(C^\infty(\mathcal{M}), L^2(\mathcal{M}, \mathcal{S}), \mathcal{D}_{\mathcal{M}}, \gamma, J)$ forms an even real spectral triple of KO-dimension 4. Moreover, it satisfies the first-order condition. In this context, the first-order condition means that the Dirac operator is a first-order differential operator. We remark that the canonical spectral triple can be obtained from the previous construction with the bundle E taking to be the spinor bundle $\mathcal{S}$.

The canonical spectral triple described above is of particular interest due to Connes' reconstruction theorem [23]. Its essence is that starting with a spectral triple that has commutative algebra as one of the defining data, one can reconstruct a spin manifold s.t. the canonical triple constructed out of the geometric data on this manifold agrees with the triple one started with. Of course, one has to impose several additional requirements on the initial triple in order to be able to rigorously formulate this equivalence. For the purpose of these lectures, I will not discuss the details here, and the interested reader is invited to consult existing reviews on this topic e.g. [2, 24] or original papers, e.g. [25]. The take-home message from the reconstruction theorem is that there is a one-to-one correspondence between *suitably regular* manifolds and *suitably regular* commutative spectral triples. In other words, such commutative spectral triples are algebraic analogs of these geometries. This can be taken as a down-to-earth motivation for studying these algebraic objects, without referring to the deeper understanding in terms of pre-Fredholm modules and the language of KK-theory. Out of the Dirac operator, one can of course construct a corresponding pre-Fredholm module following the method presented before. However, from now on we think of the canonical triple as a toy model for the nature of what we would like to describe using more sophisticated constructions. The picture of Noncommutative Geometry is then a natural consequence of this example. We can think of general spectral triples as being an algebraic description of *noncommutative manifolds* and the operator $\mathcal{D}$ for a general spectral triple is called a *Dirac operator*.

Let us discuss yet another aspect of the canonical triple. Since it corresponds to the classical Riemannian manifold, one can expect that there should exist a way to compute the distances between points on these manifolds, and, if the reconstruction is indeed correct, the obtained metric should be related to the usual geodesic distance. It is indeed the case. We have already seen that for finite spectral triples, there exists Connes' distance formula that involves the Dirac operator and the algebra itself, and it reconstructs the metric on the finite space. This prescription turns out to be true also in the context of the reconstruction theorem, where instead of algebras of functions on finite spaces, we consider algebras $\mathcal{A}$ of smooth functions on a given manifold. The Connes' distance formula takes the form

$$d(x, y) = \sup_{f \in \mathcal{A}} \{ |f(x) - f(y)| \, : \, \| [\mathcal{D}, f] \| \leq 1 \} \, , \tag{15}$$

and indeed the distances computed in this way coincide with the geodesic distances on $\mathcal{M}$.

Finally, we stress that even if there exists a set of axioms under which a given commutative spectral triple corresponds to a certain manifold, it does not imply that we have to assume these requirements also for the case of more general noncommutative spectral triples. Indeed, there are known many examples of objects that one would like to call *noncommutative* spaces but they are beyond this class. For a short review of this problem, I refer to [6] and references therein. For this reason, when considering spectral triples that potentially can have some interesting physical applications we are not forced to assume all the conditions that were crucial to establishing Connes' reconstruction theorem. In principle, there is no reason to restrict our attention only to objects that *are very similar* to classical geometries. The physics does not need to be like that! One can equally well think of more *exotic* spaces and their potential applications for physical models. At the philosophical level, one can think of this point of view as replacing the semi-classical approach to quantum mechanics with its purely quantum version. Of course, this is only a very naive resemblance but I think that it appropriately illustrates the main point. The Noncommutative Geometry was *derived* by making analogies with the classical one, but having established the dictionary we can try to weaken the link with the commutative case. This is yet another way of saying that are trying to *do not assume too much*. I will show later how different modifications of the usual approach to geometry can lead to interesting physical phenomena.

4. Where is physics?

In the previous sections, I presented a very brief introduction to the ideas behind the concept of Noncommutative Geometry. The use of certain algebraic structures was motivated by looking closely at objects present in classical geometry and trying to reformulate their properties in purely algebraic terms. As advertised in the title of these lectures, these constructions should have some interesting applications to physical models. So, where the physics is hidden? In order to answer this question, or at least propose an answer that could possibly satisfy mathematically oriented physicists, we first have to stress once again the main concept we discussed in the previous sections. Mainly, the choice of geometry is nothing else than the choice of a spectral triple. This crucial idea in the presented approach to Noncommutative Geometry, if treated seriously, leads to nontrivial consequences. Suppose, we agree with this statement and let us then consider Einstein's point of view on gravity. According to it, gravity is just geometry. In other words, Einstein's general theory of relativity tells us that the theory of gravity can be described using purely geometric terms. We, therefore, have an example of a physical theory whose content is equivalent to the knowledge of the geometry of a certain space. Following this way of thinking one can propose to use other geometries, maybe even noncommutative ones, to describe and understand other physical theories. In order to do so, we have to first establish how Einstein's theory can be encoded in algebraic objects associated with a given manifold. One way of doing this is by using the observation that physical theories are mostly formulated in the language of Lagrangians or actions, very often valid only on some characteristic effective energy scale. This is also the case for general relativity. The Einstein-Hilbert action leads to Einstein's equations describing the dynamical content of this theory. The aforementioned procedure is roughly speaking the main idea behind the so-called spectral action principle. Since the geometry is given by a spectral triple, we can try to find a way of constructing certain functional out of these data and interpret it as an action for some physical theory. More precisely, let $\Lambda > 0$ be a constant that we would like to interpret as an effective energy scale on which the action or Lagrangian should be a valid description of our theory. Out of the Dirac operator $\mathcal{D}$ we construct its renormalized version, $\mathcal{D}_\Lambda = \frac{|\mathcal{D}|}{\Lambda}$. Using this new operator we can e.g. easily answer the question of the number of eigenvalues of $\mathcal{D}$ that are smaller than Λ. This can be achieved by computing $\mathrm{Tr}\chi_{[0,1]}(\mathcal{D}_\Lambda)$, where $\chi_{[0,1]}$ is the characteristic function of the unit interval. This is the simplest example of *spectral functionals*. It turns out that its mild modification gives us what we want to extract the physical Langragian from a given spectral triple! This is achieved by defining the so-called bosonic spectral action,

$$S_b(\mathcal{D}) = \mathrm{Tr} f(\mathcal{D}_\Lambda), \tag{16}$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth function s.t. the operator $f(\mathcal{D}_\Lambda)$ is of a trace class. In practice, one uses a smooth approximation of $\chi_{[0,1]}$. The computation of the spectral action from the very definition is usually a complicated task. But it turns out that one does not need to do it in order to extract interesting information. Indeed, under certain regularity conditions on the spectral triple (see e.g. [2]) one can compute the asymptotic expansion as $\Lambda \to \infty$ of the bosonic spectral action. The coefficients in this expansion are expressed in terms of the coefficients in the heat kernel expansion, $\mathrm{Tr} e^{-t\mathcal{D}^2} = \sum_\alpha c_\alpha t^\alpha$. Let us come back to the canonical spectral triple associated with a given manifold and compute the leading terms of the asymptotic expansion. It turns out that the result is just the Einstein-Hilbert action for the general theory of relativity (with a cosmological constant term) formulated on the manifold we started with. This observation is the essence of the spectral action principle. We can start with any spectral triple for which the asymptotic expansion of the bosonic spectral action exists, and then we interpret the leading terms in this expansion as effective physical action. In other words, spectral action is a way to produce physical models out of noncommutative geometries. This leads to a generalized version of the statement that *physics is a geometry*.

The above way of extracting the leading terms is, however, not the only one. There exists another possibility which is based on manipulating symbols of differential operators acting on certain bundles. To be more precise, let $E \to \mathcal{M}$ be a finite-dimensional vector bundle over a given closed manifold. In (the sections of) this bundle one can consider the action of pseudodifferential operators. To understand this class of objects one can start by defining them of $\mathbb{R}^n$, and then use a standard way of generalizing definitions locally from $\mathbb{R}^n$ into manifolds and bundles over them. The action of such an operator on a test function u is of the form

$$P(x,D)u(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} P(x,\xi)\hat{u}(\xi)d\xi, \tag{17}$$

i.e. the integral representation in terms of the Fourier transform is used. In order to have the above expression well-defined one has to assume something about the function $P(x,\xi)$, i.e. it has to belong to a certain class. One of the typical conditions is that this function belongs to a so-called Hörmander class, i.e.

$$|\partial_\xi^\alpha \partial_x^\beta P(x,\xi)| \leq C_{\alpha\beta}(1+|\xi|)^{m-|\alpha|}. \tag{18}$$

For the purpose of these lectures, I assume that some conditions are imposed to make all the operations mathematically legal and I will not discuss the technical details here. For the practical applications, we will use concrete differential operators - usually, a square of the Dirac operator - and all the subtle points raised here will be irrelevant. But for the purpose of presenting some crucial properties, we will stay for a moment in this more general class of operators. One can show that the set of pseudodifferential operators actually is an algebra. To illustrate what it means in practice, let us take first a simple differential operator, say $\partial_1^2 + 2\partial_1$. In the Fourier representation, we essentially replace every differential operator by its *symbol* via $\partial_j \to i\xi_j$, leading to $-\xi_2^2 + i\xi_1$. The leading term is called the principal symbol, and if it is invertible on $\{(x,\xi) : \xi \neq 0\}$, then the operator is called elliptic. The multiplication in the algebra of operators mimics the rule for computing the derivative of a composition of functions. Let $\sigma_P(x,\xi) = \sum_\alpha \sigma_{P,\alpha}(x)\xi^\alpha$ and $\sigma_Q(x,\xi) = \sum_\alpha \sigma_{Q,\beta}(x)\xi^\beta$ be two symbols corresponding to operators P and Q, respectively. Here α and β are multi-indices. Then

$$\sigma_{PQ}(x,\xi) = \sum_\gamma \frac{(-1)^{|\gamma|}}{\gamma!}\partial_\gamma^\xi \sigma_P(x,\xi)\partial_\gamma^x \sigma_Q(x,\xi). \tag{19}$$

Let $\mathcal{D}$ be the Dirac operator on a certain manifold and define $P = \mathcal{D}^2$. This operator is elliptic and

$$\sigma_P(x,\xi) = \mathfrak{a}_2 + \mathfrak{a}_1 + \mathfrak{a}_0, \tag{20}$$

where $\mathfrak{a}_j$ is homogeneous in ξ's in order j. For this concrete example, $\mathfrak{a}_2$ is obtained from P by collecting all the terms that contain second derivatives are replacing them with the corresponding symbols. Similarly for other $\mathfrak{a}_j$'s. As an exercise, I suggest showing that for the symbol of the operator P^{-1} we have

$$\begin{aligned}
&\mathfrak{b}_0 = \mathfrak{a}_2^{-1},\\
&\mathfrak{b}_1 = -\left(\mathfrak{b}_0\mathfrak{a}_1 - i\partial_a^\xi(\mathfrak{b}_0)\partial_a^x(\mathfrak{a}_2)\right)\mathfrak{b}_0,\\
&\mathfrak{b}_2 = -\left(\mathfrak{b}_1\mathfrak{a}_1 + \mathfrak{b}_0\mathfrak{a}_0 - i\partial_a^\xi(\mathfrak{b}_0)\partial_a^x(\mathfrak{a}_1) - i\partial_a^\xi(\mathfrak{b}_1)\partial_a^x(\mathfrak{a}_2) - \frac{1}{2}\partial_a^\xi\partial_b^\xi(\mathfrak{b}_0)\partial_a^x\partial_b^x(\mathfrak{a}_2)\right)\mathfrak{b}_0,
\end{aligned} \tag{21}$$

where now $\mathfrak{b}_j$ is the homogeneous part of order $-2-j$.

The algebra of pseudodifferential operators has a nice feature, showed for the first time by M. Wodzicki [26], that there is exactly one normalized trace on it, which can be explicitly computed in terms of integrals over a cosphere bundle with a certain symbol of a given operator. This trace is called the Wodzicki residue and is denoted by $\mathrm{Wres}(\cdot)$. It turns out [27] that the heat kernel coefficients are related to Wodzicki residua of certain powers of the Dirac operator. In particular, the leading terms of the bosonic spectral action are then

$$S_b(\mathcal{D}) \sim \Lambda^4 \mathrm{Wres}(\mathcal{D}^{-4}) + c\Lambda \mathrm{Wres}(\mathcal{D}^{-2}), \tag{22}$$

with some constant c. For the canonical spectral triple, these terms correspond to the Einstein-Hilbert action together with the cosmological constant term. One can now imagine a situation in which starting with another spectral triple could lead to some modified gravity models. Their effective actions can be simply read from these leading terms and the whole problem reduces to the computation of Wodzicki residua for $\mathcal{D}^{-4}$ and $\mathcal{D}^{-2}$. This idea was used in [28, 29] to produce models that go beyond Einstein's general relativity and share some features with bimetric theories. The crucial step in the computations was based on the aforementioned formulas for $\mathfrak{b}_j$'s.

An important remark is in place here. Notice that all of these models are Euclidean. One needs some kind of Wick rotation scheme to get physically (cosmologically) interesting results. However, the purely Lorentzian formulation of the spectral action principle seems to be still beyond the applicability of known techniques and existing machinery. There exists, however, a zoo of proposals for pseudo-Riemannian generalizations of spectral triples - some of them will be discussed later in these lectures. Nevertheless, the main problem with the spectral action is the lack of ellipticity of the operator $\mathcal{D}^2$. Instead, this operator is hyperbolic in the Lorentzian setup and the usual heat kernel methods are ill-defined since the corresponding symbol is non-invertible.

4.1 Towards particle physics: AC-manifolds

In the previous section, we briefly motivated the use of the machinery of spectral geometry to derive potential action functionals for physical models. In order to do this, we concentrated on Einstein's vision of gravity as geometrical theory. Since one can derive its corresponding (Euclidean version of) Lagrangian from the canonical spectral triple associated with (suitably regular) Riemannian space, the natural expectation was that following this idea one can derive also modified gravity models by changing the initial data - a spectral triple. On the other hand, it is not clear how to derive other than gravity models that are of practical interest. In this part of the lecture,

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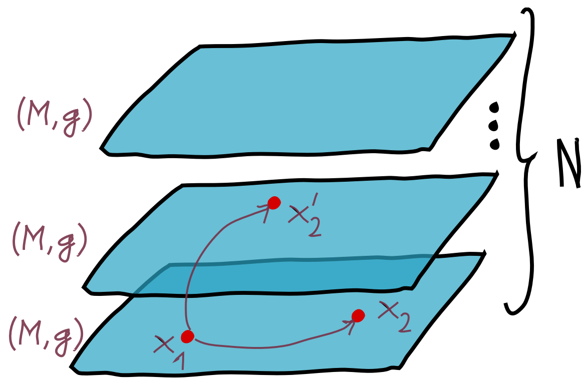
we will briefly discuss how gauge theories fit in the framework of Noncommutative Geometry. We start with an observation that gravity itself can be understood as a gauge model and the symmetry group consists of diffeomorphisms of the space. We would like to describe field theories so that they should live on a given manifold. In other words, there is always a space that plays a role of a background. This quite oversimplified picture is beyond the main idea of almost-commutative geometries - finite extensions of classical spaces. We would like to think of them as Cartesian product $\mathcal{M}\times F$ of the usual manifold with some finite dimensional space. One can easily recognize this construction as an analogy to the Kaluza-Klein models. In the latter, one uses a circle S^1 instead of a finite space. Notice that $\dim(\mathcal{M}\times F) = \dim(\mathcal{M}) + \dim(F) = \dim(\mathcal{M})$ since F is zero-dimensional, but for the Kaluza-Klein model we have $\dim(\mathcal{M}\times S^1) = \dim(\mathcal{M}) + 1$. This seemingly small difference will have important consequences. The simplest example of the space F is $F = \mathbb{Z}_N \subseteq S^1$ with $N > 0$. Then $\mathcal{A}_F = C^\infty(F) = \mathbb{C}^N$ and, as a result, $\mathcal{A} = C^\infty(\mathcal{M}\times F) \cong C^\infty(M, \mathbb{C}^N) \cong C^\infty(\mathcal{M})\otimes\mathbb{C}^N$. Since we would like to formulate the model from the very beginning on the algebraic level, our starting point should be the choice of an algebra $\mathcal{A}$. In the case of almost-commutative models, we then take it to be of the form $\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathcal{A}_F$, with F being a finite-dimensional algebra, i.e. $\mathcal{A}_F = \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C})$. From this construction, one can guess that the full spectral triple for such types of geometries should be (properly understood) a product of the canonical triple associated with the manifold and some finite one. This is indeed the case as we will see shortly. Before discussing these details, let us first look again more closely at the case with $F = \mathbb{Z}_N$. For $N = 2$ we have $\mathcal{M}\times\mathbb{Z}_2 = \mathcal{M}\oplus\mathcal{M}$ and consequently $\mathcal{A} = C^\infty(\mathcal{M}) \oplus C^\infty(\mathcal{M})$, i.e. we have two disjoint copies of the same Riemannian space $(\mathcal{M}, g)$. This picture generalizes to any $N > 0$:



We can now imagine a situation in which we would like to move within this product space starting e.g. from point x_1 in the above figure. There are essentially two possible directions to move: reaching points on the same layer we started with, e.g. x_2, or moving to the other layer, say to point x_2'. This very intuitive picture can be formalized in terms of vector bundles and connections over such a product space. This will be of crucial importance for the discussion of gauge fields in the framework of spectral geometry, using certain principal bundles. We will discuss this aspect more carefully in a moment, but first, let us remark on some naive expectations one can have from this picture. First of all, we should be able to derive actions for gauge theories using spectral techniques. Secondly, the possibility of a movement in the finite direction should produce some effective fields on the level of the asymptotic expansion of a spectral action. We will see that this is indeed the case, and, in particular, the Higgs field will appear exactly in this way.

Let us now finally define the almost-commutative geometry. We have already seen the choice for the algebra, $\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathcal{A}_F$. Since it is just a tensor product, the Hilbert space on which we would like to represent it has an analogous structure, i.e. we take $\mathcal{H} = L^2(\mathcal{M}, \mathcal{S}) \otimes \mathcal{H}_F$. That was the easiest part. Now, we introduce the Dirac operator to this product geometry. It has to be done in a self-consistent way. The choice is motivated by the expected form of a Laplace-type operator on a tensor product. Notice that since the square of the canonical Dirac operator on a Riemannian manifold is a Laplace-type operator, we should require that for the Dirac operator $\mathcal{D}$ on the product geometry we should have $\mathcal{D}^2 = \mathcal{D}_\mathcal{M}^2 + \mathcal{D}_F^2$. One can easily check that the choice $\mathcal{D}_\mathcal{M} \otimes \mathcal{D}_F$ does not work and it has to be modified. Let us remark that the canonical triple was naturally equipped with a grading given by the γ_5 operator in the associated Clifford algebra. It is therefore naturally demanded that tensoring with the even finite triple should also lead to an even product. Therefore the existing grading will restrict possible choices for the product Dirac operator, and this choice has to be done in a way that the product Dirac operator anticommutes with the product grading. I will not discuss all of the nuances here, referring the reader to [30] for a detailed discussion. For the purpose of these lectures, we assume that $\dim(\mathcal{M}) = 4$ and present the finite result:

$$\mathcal{D} = \mathcal{D}_\mathcal{M} \otimes \mathrm{id} + \gamma_5 \otimes \mathcal{D}_F, \qquad \gamma = \gamma_5 \otimes \gamma_F. \tag{23}$$

If, moreover, the real structures are taken into account, the one on the product geometry is simply $J = J_\mathcal{M} \otimes J_F$.

4.2 Gauge theories from spectral action

Having defined almost-commutative geometries, we will now show how they can be used to encode gauge theories. The first observation, already mentioned previously, is that the symmetries of a Riemannian manifold are described by the group of their diffeomorphisms. More precisely, $\mathrm{Diff}(\mathcal{M}) \cong \mathrm{Aut}(C^\infty(\mathcal{M}))$. Therefore, one can naively define $\mathrm{Diff}(\mathcal{M} \times F)$ as the automorphism group of $C^\infty(\mathcal{M}) \otimes \mathcal{A}_F$. But this is not the full symmetry group. There are also transformations induced by the unitaries of the algebra that implements the *inner symmetries*. We can think of them as the gauge group associated with the almost-commutative geometry. Under these inner symmetries, the Dirac operator transforms as $D \mapsto D_U = UDU^*$, where $U = uJuJ^{-1}$ with a unitary element u in the product algebra. The full symmetry group is then

$$\{U = uJuJ^{-1} : u \in \mathcal{U}(\mathcal{A})\} \rtimes \mathrm{Diff}(\mathcal{M}). \tag{24}$$

One can expect that having two spectral triples which are related by a certain symmetry transformation, the resulting physical action should not depend on this choice. There should be then a way of identifying two spectral triples and exploring this new type of *gauge freedom*. The appropriate way of doing this turns out to be based on the notion of Morita equivalence. It goes back to the notion of Morita equivalence for algebras, which is expressed in terms of bimodules over them (for details see [2]), and then this can be generalized to the whole data in a spectral triple. Of particular importance is the so-called Morita self-equivalence condition which, in particular, restricts the number of possible Hermitian connections $\nabla : \mathcal{A} \to \Omega^1(\mathcal{A})$ associated with $d(\cdot) = [\mathcal{D}, \cdot\,]$ to the ones of the form $d + \omega$ with $\omega = \omega^* \in \Omega^1_{\mathcal{D}}(\mathcal{A})$. This corresponds to replacing the bare Dirac operator $\mathcal{D}$ by the whole family of them:

$$\mathcal{D}_\omega = \mathcal{D} + \omega + \epsilon J \omega J^{-1}. \tag{25}$$

The resulting operator is called a fluctuation of the Dirac operator. It essentially takes into account the gauge freedom we have when we construct physical theories in the language of spectral geometry. We then expect that e.g. Yang-Mills type terms in the physical action should appear if we use the fluctuated Dirac operator in the spectral action and perform the asymptotic expansion for it. Let us remark also that, effectively, the fluctuated Dirac operator is constructed by simply adding an arbitrary 1-form to the bare one. This again illustrates the importance of the set of 1-forms in Noncommutative Geometry.

In the remaining part of this section, I will summarize the steps one has to perform to finally derive an effective bosonic action for Yang-Mills type theories. This will closely follow the presentation in [2] and the reader is invited to consult this book for detailed proofs of all the statements below. However, most of them are, essentially, based on straightforward algebraic manipulations and the reader is more than welcome to try to demonstrate these results first by themselves and then check these derivations in the existing literature.

We start with the almost-commutative geometry with $\dim(\mathcal{M}) = 4$ and a finite algebra chosen so that its unitaries correspond *more or less* to the gauge group of the Yang-Mills-type theory we would like to describe. This statement can be made more precise but for our purposes, we don't need to go into these technical details. The statement *more or less* should be understood here that we do not really care if we take $U(N)$ or $SU(N)$, or their quotients by some discrete group. This can be fixed later on. We then have an almost-commutative manifold, and for it, we can compute the set of Connes' 1-forms $\Omega^1_{\mathcal{D}}(\mathcal{A})$. Let $\omega = a[\mathcal{D}, b]$ be some 1-form. By linearity, it is enough to take just one summand here. By computing the commutators with the continuous part and the finite one, we can parametrize this 1-form in the following way:

$$\omega = \gamma^\mu \otimes A_\mu + \sigma \otimes \phi, \tag{26}$$

where $\phi = a[\mathcal{D}_F, b]$. Since we need to compute $\mathcal{D}_\omega$, the fluctuated Dirac operator, we first need

$$\gamma^\mu \otimes A_\mu + \epsilon J \gamma^\mu \otimes A_\mu J^{-1} \tag{27}$$

which we denote by $\gamma^\mu \otimes B_\mu$. Defining

$$\Phi = D_F + \phi + J_F \phi J_F, \qquad \nabla^E = \nabla^S \otimes \mathrm{id} + i \mathrm{id} \otimes B, \tag{28}$$

one can finally write

$$\mathcal{D}_\omega = -i\gamma^\mu \nabla^E_\mu + \gamma_5 \otimes \Phi. \tag{29}$$

Its square, i.e. the associated Laplace-type operator, is of the form $D^2_\omega = \Delta^E + F$ with

$$\Delta^E = -g^{\mu\nu}(\nabla^E_\mu \nabla^E_\nu - \Gamma^\rho_{\mu\nu}\nabla^E_\rho), \tag{30}$$

$$F = -\frac{1}{4} R \otimes 1 - 1 \otimes \Phi^2 + \frac{i}{2}\gamma^\mu\gamma^\nu \otimes F_{\mu\nu} - i\gamma_5\gamma^\mu \otimes D_\mu \Phi, \tag{31}$$

where $D_\mu \Phi = \partial_\mu \Phi + i[B_\mu, \Phi]$. Then one can proceed with the computation of the asymptotic expansion for the spectral action. Again, I will present only the final answer and the method of derivation can be found in e.g. [2]. Finally, we get

$$\mathrm{Tr} f\left(\mathcal{D}_{\omega,\Lambda}\right) \sim \int_{\mathcal{M}} \sqrt{g} d^4x \left(\dim(\mathcal{H}_F)\mathcal{L}_{\mathcal{M}} + \mathcal{L}_B + \mathcal{L}_\Phi\right) \tag{32}$$

with

$$\mathcal{L}_M = \frac{f_4\Lambda^4}{2\pi^2} - \frac{f_2\Lambda^2}{24\pi^2}R + \frac{f(0)}{16\pi^2}\left(\frac{\Delta R}{30} - \frac{7}{360}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{1}{45}R_{\mu\nu}R^{\mu\nu} + \frac{1}{180}R^2\right), \tag{33}$$

$$\mathcal{L}_B = \frac{f(0)}{24\pi^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}), \tag{34}$$

$$\mathcal{L}_\Phi = -\frac{2f_2\Lambda^2}{4\pi^2}\mathrm{Tr}(\Phi^2) + \frac{f(0)}{8\pi^2}\mathrm{Tr}(\Phi^4) + \frac{f(0)}{24\pi^2}\Delta\mathrm{Tr}(\Phi^2) + \frac{f(0)}{48\pi^2}R\mathrm{Tr}(\Phi^2) + \frac{f(0)}{8\pi 62}\mathrm{Tr}\left[(D_\mu\Phi)(D^\mu\Phi)\right], \tag{35}$$

where $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + i[B_\mu, B_\nu]$ and f_j denotes the $(j-1)$th moment of the function f. In addition to the geometric part $\mathcal{L}_\mathcal{M}$, we easily recognize the Yang-Mills action. The role of the field Φ will become clear in the next section when we will discuss an application of these general formulas to the Standard Model of particle physics.

5. The Standard Model

As advertised in the previous section we discuss here an application of the spectral action method to the Standard Model of particle physics. The way of presenting this material is rather standard and can be found in many classical textbooks - see e.g. the ones listed in the list of references. Here I will concentrate on the main ideas behind the Connes-Chamseddine construction, rather than presenting the detailed computations which can be easily found in the aforementioned references.

We start with the choice of algebra. As it was mentioned in the previous section, this should correspond to the choice of the gauge group for the model. Since the Standard Model has (up to a finite subgroup) the gauge group of the form $U(1) \times SU(2) \times SU(3)$, the simplest natural choices for the finite algebra is $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, where $\mathbb{H}$ denotes the algebra of quaternions. The second question we have to answer is the choice of a finite Hilbert space. Which one is the proper one? To find the answer we look at the fermionic spectral action, which essentially has a form $S_f(\mathcal{D}) \sim \langle\Psi|\mathcal{D}|\Psi\rangle$ with $\Psi \in \mathcal{H}$. By comparing this expression with the one known from field theory, one can guess that a reasonable choice would be by taking $\mathcal{H}_F$ to have a dimension equal to the number of independent fermionic species in the model. So, what are they? In the standard construction by A. Connes and A. Chamseddine, this is taken to be 96. Why? We think of each fermion with a fixed chirality and charge as an element of a basis of $\mathcal{H}_F$. In this picture, for every antiparticle, we have associated a basis vector that is independent of the one for the corresponding particle. Also, both the number of colors for quarks and the number of generations are taken into account. As a grading operator on the finite space we take the standard chirality γ represented as an operator on $\mathcal{H}_F$. Also the real structure J_F is the natural one that essentially (up to complex conjugation to make it antilinear) interchanges particles with their antiparticles. Having an algebra and a Hilbert space, we should now, according to the general prescription, define a representation of this algebra on the Hilbert space. In the Connes-Chamseddine construction, it is done as follows. First, assume for simplicity that there is only one generation of particles. One will, later on, extend the representation diagonally in order to take into account also the number of generations. Following the standard notation, we denote an element of $\mathcal{A}_F$ by (λ, q, m), where λ is a complex number, q stands for a quaternion, and m belongs to $M_3(\mathbb{C})$. Then such an element is represented on both the leptonic sector and for each quark color as an operator $\begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} \oplus q$. Its action on antileptons is taken to be just a multiplication by $\bar{\lambda}$, whereas, on the antiquark sector, the corresponding operator is $\mathrm{id}_4 \otimes m$. Notice that in the latter case, the color structure is already included.

It remains to find a Dirac operator for the finite part of the product triple. This is a nontrivial question and the standard Connes-Chamseddine choice is as follows. Decomposing the Hilbert space $\mathcal{H}_F$ according to

$$\mathcal{H}_F = \left(\left(\mathbb{C}^2_R \oplus \mathbb{C}^2_L \right) \otimes \mathbb{C}^{4\circ} \right) \oplus \left(\mathbb{C}^4 \otimes \left(\mathbb{C}^{2\circ}_R \oplus \mathbb{C}^{2\circ}_L \right) \right), \tag{36}$$

the Dirac operator is taken to be

$$\mathcal{D} = \begin{pmatrix} S & T^* \\ T & S \end{pmatrix} \tag{37}$$

with the S operator on the leptonic sector being given by

$$S_l = \begin{pmatrix} 0 & 0 & \Upsilon_\nu^* & 0 \\ 0 & 0 & 0 & \Upsilon_e^* \\ \Upsilon_\nu & 0 & 0 & 0 \\ 0 & \Upsilon_e & 0 & 0 \end{pmatrix}, \tag{38}$$

whereas for quarks one chooses

$$S_q \otimes \mathrm{id}_3 = \begin{pmatrix} 0 & 0 & \Upsilon_u^* & 0 \\ 0 & 0 & 0 & \Upsilon_d^* \\ \Upsilon_u & 0 & 0 & 0 \\ 0 & \Upsilon_d & 0 & 0 \end{pmatrix} \otimes \mathrm{id}_3. \tag{39}$$

The T operator is non-zero only for ν_R and given by $T\nu_R = \Upsilon_R \bar{\nu}_R$. In the case of three generations of particles, Υ's are promoted to 3×3 matrices.

The immediate consequence of the chosen representation can be observed when restrict to the subspace spanned by $\{\nu_L, \nu_R, \bar{\nu}_L, \bar{\nu}_R\}$. In the case of one generation, we identify Υ_ν with m_ν and similarly for Υ_R, $\Upsilon_R = m_R$. The spectrum of the operator $\mathcal{D}_F$ restricted to this subspace can be easily computed and in the limit $m_\nu \ll m_R$ one gets the following eigenvalues: $\pm m_R$ and $\pm \frac{m_\nu^2}{m_R}$. This leads to the celebrated seesaw mechanism: there exists a heavy neutrino with mass m_R and a light neutrino with mass $\frac{m_\nu^2}{m_R}$.

In general, the matrices $\Upsilon_e, \Upsilon_\nu, \Upsilon_u, \Upsilon_d$ have the physical meaning of matrices of Yukawa parameters, while Y_R corresponds to the Majorana mass matrix.

The natural question that arises is the uniqueness of the choice of the finite part of the Dirac operator. It turns out that the answer is negative. There is plenty of choices that are self-consistent. A lot of arguments were invented to reasonably reduce the number of possible choices but not all of them are fully convincing. In the next section, I will show how introducing the Lorenztian structure on the finite part of the spectral triple can be used to reduce the number of such choices. For a moment we simply accept the above choice and present its consequences.

First of all, as one can expect, the spectral action technique produces the bosonic part of the classical action for the Standard Model. It is, in general, given on a curved background, but for our purpose, we will assume a flat torus as a background geometry to simplify the discussion. In this case, the resulting Lagrange density is of the form $\mathcal{L} = 96\mathcal{L}_M + \mathcal{L}_A + \mathcal{L}_H$ with

$$\mathcal{L}_A = \frac{f(0)}{\pi^2}\left(\frac{10}{3}\Lambda_{\mu\nu}\Lambda^{\mu\nu} + \mathrm{Tr}(Q_{\mu\nu}Q^{\mu\nu}) + \mathrm{Tr}(V_{\mu\nu}V^{\mu\nu})\right), \tag{40}$$

$$\mathcal{L}_H = \frac{bf(0)}{2\pi^2}|H|^4 + \frac{ef(0) - 2af_2\Lambda^2}{\pi^2}|H|^2 + \frac{af(0)}{2\pi^2}|D_\mu H|^2, \tag{41}$$

where I have omitted the terms of $\mathcal{L}_H$ that introduce a constant shift (and can contribute to a cosmological constant). Here

$$a = \mathrm{Tr}\left(\Upsilon_\nu^*\Upsilon_\nu + \Upsilon_e^*\Upsilon_e + 3\Upsilon_u^*\Upsilon_u + 3\Upsilon_d^*\Upsilon_d\right), \tag{42}$$

$$b = \mathrm{Tr}\left((\Upsilon_\nu^*\Upsilon_\nu)^2 + (\Upsilon_e^*\Upsilon_e)^2 + 3(\Upsilon_u^*\Upsilon_u)^2 + 3(\Upsilon_d^*\Upsilon_d)^2\right) \tag{43}$$

and

$$e = \mathrm{Tr}\left(\Upsilon_R^*\Upsilon_R\Upsilon_\nu^*\Upsilon_\nu\right), \tag{44}$$

Λ, Q and V corresponds to gauge fields for the groups $U(1)$, $SU(2)$ and $SU(3)$, respectively, and H is an effective parametrization of the field Φ - for details see e.g. [2–4, 17].

Next, one writes Q_μ on the basis of Pauli matrices, V_μ on the basis of Gell-Mann matrices, and parametrizes the corresponding coefficients of these expansions by $Q_\mu^a = \frac{1}{2}g_2W_\mu^a$ and $V_\mu^i = \frac{1}{2}g_3G_\mu^i$, where coupling parameters are explicitly introduced. We also add the one for the Λ field: $\Lambda_\mu = \frac{1}{2}g_1Y_\mu$. Finally, expressing the Lagrangian in terms of these new fields and demanding that the kinetic energy terms have the canonical form we obtain a GUT-like relation $g_3^2 = g_2^2 = \frac{5}{3}g_1^2$ and the potential term for the H field is

$$\mathcal{L}_{\mathrm{pot}} = \frac{b\pi^2}{2a^2f(0)}|H|^4 - \frac{2af_2\Lambda^2 - ef(0)}{af(0)}|H|^2, \tag{45}$$

which is nothing else than the Higgs potential. Therefore, the spectral action produces not only the gauge fields but also the Higgs field. If we come back to the picture of almost-commutative geometry in which we can *move within layer* or *move between layers*, we recognize that this is a way in which the spectral geometry unifies the Higgs field and the gauge fields. Gauge fields can be interpreted in this very pictorial formulation as a result of a movement within one layer, while the Higgs field is a gauge field for the interlayer movements.

For $2af_2\Lambda^2 > ef(0)$ the Higgs minimum is at $\frac{2a^2f_2\Lambda^2 - aef(0)}{b\pi^2} \equiv v^2$. Using the standard parametrization $H = u(x)\begin{pmatrix} v + h(x) \\ 0 \end{pmatrix}$ we can then rewrite the Lagrangian in terms of the h field and find a relation between the Higgs mass and the parameter v. Moreover, introducing $M_W = \frac{1}{2}vg_2$ and $M_Z = \frac{1}{2}v\sqrt{g_1^2+g_2^2}$, after some algebraic manipulations one can relate the Higgs mass m_h with the W-boson mass M_W and the quartic interaction parameter λ for the Higgs field: $m_h^2 = \frac{4\lambda M_W^2}{3g_2^2}$. Now, using the renormalization group techniques, starting from the Λ_{GUT} energy scale, one can show [2, 4] that 167GeV $\leq m_h \leq$ 176GeV. This value is different than the experimental one, but it is of the right order of magnitude. There were several attempts to avoid this discrepancy. It is beyond the scope of these notes to discuss this problem. Let me mention only that it results in plenty of possible extensions of the Standard Model which have several intriguing features and non-trivial physical consequences.

Before finishing this section, let me briefly comment on other potential issues that are present in the formulation of the Standard Model within this framework. First of all, notice that in the Hilbert space $\mathcal{H} = L^2(\mathcal{M},\mathcal{S})\otimes\mathcal{H}_F$ we are counting multiple times certain degrees of freedom. Indeed both parts of this tensor product contain information about the chiralities and particle-antiparticle pairs. As a result, there are effectively four times more degrees of freedom than needed. This is a famous *fermion doubling problem* and one needs to eliminate this redundancy to obtain the proper physical Lagrangian. There are several approaches to solving this issue. Let me mention here some classical ones and I will come back to this problem in one of the forthcoming sections. To eliminate the chirality doubling one can introduce certain projections to reduce the full Hilbert space. The other factor of two can be cured by modifying the fermionic spectral action by inserting a real structure operator. Moreover, there are also approaches to this problem that relates it to the presence of the Lorentzian structure [31].

6. Towards pseudo-Riemannian spectral triples

In this section, we briefly discuss a method that allows for the reduction of possible Dirac operators describing the content of the Standard Model and its extension. This discussion is mostly based on the results obtained in [32–34]. We work here only with finite spectral triples. We start our discussion with the definition of a pseudo-Riemannian spectral triple proposed in [32, 33]. A pseudo-Riemannian spectral triple of signature (p,q) is a tuple $(\mathcal{A},\mathcal{H},\mathcal{D},J,\gamma,\beta)$, where

- $\mathcal{A}$ and $\mathcal{H}$ are as in the Euclidean case.
- If $p+q$ is divisible by 2 then there exists a grading γ.
- Both zeroth and first-order conditions hold.
- β is an additional grading that defines the Krein structure on the Hilbert space.
- $D^\dagger = (-1)^p\beta\mathcal{D}\beta$, i.e. the Dirac operator is self-adjoint with respect to the Krein structure.
- $\mathcal{D}\gamma = -\gamma\mathcal{D}$.
- $\mathcal{D}J = \epsilon J\mathcal{D}$, $J^2 = \epsilon'\mathrm{id}$ and $J\gamma = \epsilon''\gamma J$. The choice of $(\epsilon,\epsilon',\epsilon'')$ defines the KO-dimension being $p-q \pmod 8$.
- $\beta\gamma = (-1)^p\gamma\beta$

- $\beta J = (-1)^{\frac{p(p-1)}{2}} \epsilon^p J\beta$.

Moreover, if

$$\sum_{i=1}^{k} \left(Ja^i J^{-1}\right) a_0^i [\mathcal{D}, a_1^i] \ldots [\mathcal{D}, a_n^i] = \begin{cases} \gamma, & p+q \text{ even}, \\ \mathrm{id}, & p+1 \text{ odd}, \end{cases} \tag{46}$$

then this geometry is called orientable. If a similar representation exists for β then the geometry is called time-orientable. The above definition is motivated by the presence of analogous structures in the usual Clifford algebra of signature (p, q). For example, in the case of $Cl_{1,3}$, the role of β is played by the γ^0 matrix, as one can easily guess.

Suppose one gives us a pseudo-Riemannian finite spectral triple as above. It turns out that it is possible to construct out of it a pair of two Riemannian ones. One can accomplish this by simply modifying the Dirac operator. These two choices correspond to $\mathcal{D}_+ = \frac{1}{2}(\mathcal{D} + \mathcal{D}^\dagger)$ and $\mathcal{D}_- = \frac{i}{2}(\mathcal{D} - \mathcal{D}^\dagger)$. Redefining in an appropriate way also the real structure, we finally take $\mathcal{D}_E = \mathcal{D}_+ + \mathcal{D}_-$ and we end up with a Riemannian spectral triple. Some of the properties of the resulting triple can be then interpreted as a shadow of the existence of the pseudo-Riemannian one from which it was constructed.

Applying this idea to the spectral triple describing the Standard Model and considering possible Dirac operators $\mathcal{D}$ and gradings β that implement time orientation, we end up [33] with the conclusion that the only pseudo-Riemmanian structure β that is physically allowed is the one that implements the grading distinguishing between leptons and quarks. The only Dirac operator that is consistent with this structure is exactly the one that does not allow for leptoquarks and allows only for an extension by a sterile neutrino. Let us now briefly describe how this result was obtained. In order do to so, it is convenient to use another representation of the Hilbert space. Mainly, we organize its elements according to $\mathcal{H}_F = F \oplus F^*$ with $\mathcal{H}_F \ni \begin{pmatrix} v \\ w \end{pmatrix}$, where

$$v = \begin{pmatrix} \nu_R & u_R^1 & u_R^2 & u_R^3 \\ e_R & d_R^1 & d_R^2 & d_R^3 \\ \nu_L & u_L^1 & u_L^2 & u_L^3 \\ e_L & d_L^1 & d_L^2 & d_L^3 \end{pmatrix}, \qquad w = \begin{pmatrix} \overline{\nu}_R & \overline{e}_R & \overline{\nu}_L & \overline{e}_L \\ \overline{u}_R^1 & \overline{d}_R^1 & \overline{u}_L^1 & \overline{d}_L^1 \\ \overline{u}_R^2 & \overline{d}_R^2 & \overline{u}_L^2 & \overline{d}_L^2 \\ \overline{u}_R^3 & \overline{d}_R^3 & \overline{u}_L^3 & \overline{d}_L^3 \end{pmatrix}. \tag{47}$$

Here the upper index refers to the color of a quark. The real structure is simply given by $\begin{pmatrix} v \\ w \end{pmatrix} \mapsto \begin{pmatrix} w^* \\ v^* \end{pmatrix}$. In order to efficiently work with the operators on $\mathcal{H}_F$ one notices that $\mathrm{End}(\mathcal{H}_F) \cong M_4(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes M_4(\mathbb{C})$. This allows for writing both the grading and a representation of the algebra in a compact form:

$$\gamma = \begin{pmatrix} 1_2 & \\ & -1_2 \end{pmatrix} \otimes e_{11} \otimes 1_4 + 1_4 \otimes e_{22} \otimes \begin{pmatrix} -1_2 & \\ & 1_2 \end{pmatrix}, \tag{48}$$

$$\pi(\lambda, q, m) = \begin{pmatrix} \lambda & & \\ & \bar{\lambda} & \\ & & q \end{pmatrix} \otimes e_{11} \otimes 1_4 + \begin{pmatrix} \lambda & \\ & m \end{pmatrix} \otimes e_{22} \otimes 1_4, \tag{49}$$

where e_{ij} is the matrix with 1 in entry (i, j) and zeros otherwise.

This representation is convenient to work with Dirac operators and to try to classify possible choices. First of all, the requirement of the first order condition reduces the number of possibilities to the once of the form $\mathcal{D} = \mathcal{D}_0 + J_F \mathcal{D}_0 J_F^{-1}$ with

$$\mathcal{D}_0 = \begin{pmatrix} 0 & M \\ M^\dagger & 0 \end{pmatrix} \otimes e_{11} \otimes e_{11} + \begin{pmatrix} 0 & N \\ N^\dagger & 0 \end{pmatrix} \otimes e_{11} \otimes (1 - e_{11}) + \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \otimes e_{12} \otimes e_{11} + \begin{pmatrix} A^\dagger & 0 \\ B^\dagger & 0 \end{pmatrix} \otimes e_{21} \otimes e_{11}, \tag{50}$$

where $M, N, A, B \in M_2(\mathbb{C})$. The standard Connes-Chamseddine choice corresponds to $M = S_l$, $N = S_q$, $A = T$, and $B = 0$. It turns out that the only physically acceptable choice of the β grading is

$$\beta = \pi(1, 1_2, -1_3) J_F \pi(1, 1_2, -1_3) J_F^{-1}, \tag{51}$$

and Dirac operators compatible with it are of the above form with $B = 0$ and A satisfying $A = A \cdot \mathrm{diag}(1, -1)$ and there are no restrictions on M and N. In particular, the first two constraints mean that only a sterile neutrino is allowed as an extension of the standard picture. Moreover, no leptoquarks are allowed since there is no such mixing on the level of the Dirac operator. In this picture, this is a consequence of the existence of the pseudo-Riemannian structure on the finite spectral triple. Moreover, it turns out that the resulting triple satisfies the Hodge property.

The above analysis was extended in [34] to the family of Pati-Salam models. They are potential extensions of the Standard Model with the gauge group $SU(2) \times SU(2) \times SU(4)$. The finite algebra is $\mathcal{A}_F = \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_4(\mathbb{C})$ which again can be represented on $M_4(\mathbb{C}) \oplus M_4(\mathbb{C})$. There are, however, two natural choices for grading. One of them is the previously discussed γ, while the second one is

$$\gamma_* = \begin{pmatrix} 1_2 & \\ & -1_2 \end{pmatrix} \otimes e_{11} \otimes \begin{pmatrix} 1 & \\ & -1_3 \end{pmatrix} + \begin{pmatrix} -1 & \\ & 1_3 \end{pmatrix} \otimes e_{22} \otimes \begin{pmatrix} 1_2 & \\ & -1_2 \end{pmatrix}, \tag{52}$$

which corresponds to the situation in which left-handed leptons have the same parity as right-handed quarks. One can again classify all possible pseudo-Riemannian structures and Dirac operators compatible with them. This leads to the conclusion that the full Pati-Salam algebra has to be reduced to $\mathbb{H}_L \oplus \mathbb{H}_R \oplus \mathbb{C} \oplus M_3(\mathbb{C})$ and no leptoquarks are allowed. This corresponds to the family of Left-Right Symmetric models with the gauge group $SU(2)_R \times SU(2)_L \times SU(3) \times U(1)_{B-L}$.

7. Further applications - selected topics

In this last section, I will briefly present further selected applications of noncommutative geometries in physics. This list is highly incomplete and all the technical details are omitted. The purpose of this section is to illustrate that tools of Noncommutative Geometry can be successfully applied in a variety of situations.

7.1 Quantum Riemannian Geometry

Here I will present another approach to Noncommutative Geometry based on the Hopf-algebraic structures associated with the algebra we are working with as well as with the module of differentials 1-forms. The details of what I will only briefly mention here can be found in a comprehensive book by E. Beggs and S. Majid [10] which I highly recommend.

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For a given (Hopf) algebra $\mathcal{A}$ we are considering a FODC $(\Omega^1(\mathcal{A}), d)$. The choice of this calculus is essentially a starting point in most of the approaches known as *Quantum Riemannian Geometry* (QRG). We would like to define geometric-like objects for purely quantum (which in this framework is usually a synonym of being noncommutative) geometry. This should be done on the level of FODC. In the remaining part of this subsection, I will show how some of the classical geometric objects can be formulated in this language. We start with the observation that $\Omega^1(\mathcal{A})$ is a bimodule over $\mathcal{A}$. Then one can define a linear connection on it as a pair of maps (∇, σ) s.t.

- $\nabla : \Omega^1(\mathcal{A}) \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ is a linear map.
- $\sigma : \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ is a bimodule map, called a generalized braiding.

These two maps are subject to the following conditions:

- $\nabla(a\omega) = da \otimes_{\mathcal{A}} \omega + a\nabla\omega$,
- $\nabla(\omega a) = (\nabla\omega)a + \sigma(\omega \otimes_{\mathcal{A}} da)$,

for all $a \in \mathcal{A}$ and $\omega \in \Omega^1(\mathcal{A})$. As for classical geometry, one would like to have a notion of a Levi-Civita connection. In order to achieve this goal we need first the definition of a torsion. It is defined as a linear map $\Omega^1(\mathcal{A}) \to \Omega^2(\mathcal{A})$ given by $T_\nabla = \wedge \circ \nabla - d$. Next, the metric has to be defined. Within the QRG it is taken as a pair of an element $g = g^{(1)} \otimes g^{(2)} \in \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ and a bimodule map $(\cdot\,,\,\cdot) : \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \to \mathcal{A}$ s.t. it is *an inverse* of g in the following sense:

$$\forall \omega \in \Omega^1(\mathcal{A}) \quad (\omega, g^{(1)})g^{(2)} = \omega = g^{(1)}(g^{(2)}, \omega). \tag{53}$$

We say that a given connection is compatible with the metric g if

$$(\nabla \otimes \mathrm{id})g + (\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \nabla)g = 0. \tag{54}$$

One would like to define a Levi-Civita as a torsion-free ($T_\nabla = 0$) and metric-compatible one. However, in contrast to the classical situation, this does not lead to a unique solution. There are plenty of possible connections that satisfy these two conditions. More requirements are necessary. I will not discuss them here and refer to e.g. [10] and also to [35] for concrete examples. This illustrates that the noncommutative world is much richer.

The next objects one can construct are related to the curvature of space. For example, the Riemannian curvature is defined as a linear map from $\Omega^1(\mathcal{A})$ into $\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ given by $R_\nabla = (d \otimes \mathrm{id} - \mathrm{id} \wedge \nabla)\nabla$. The definition of the Ricci curvature is more subtle and requires a bimodule map ι that lifts 2-forms into a tensor product on 1-forms. It is highly non-unique and for this reason, the Ricci curvature depends on this choice for ι. But having chosen this map once, one can then proceed and define also Ricci scalar as well as the Einstein tensor.

Of course, the natural question of comparison between these geometric quantities and the ones obtained from Connes' spectral approach arises. Under certain assumptions they can agree - this topic is however beyond the scope of these lectures and can be found e.g. in [10].

7.2 Quantum spaces for quantum information

Here I will just present an idea of how quantum (noncommutative) spaces can be used in quantum information theory. Let us first imagine a classical scenario. We have two players trying to convince an external referee that a certain object has some well-defined properties. It can be, for example, some graphs can be colored by using a given number of colors, or there exists a homomorphism or isomorphism between two objects. In information theory, these problems are referred to as classical two-players games. Let us call these players, as usual Alice and Bob, and suppose that the referee has two sets of questions: X from which the question is chosen for Alice, and Y - the set of questions for Bob. Suppose the referee chooses $x \in X$ and $y \in Y$. Now, Alice can give her answer from a set A, say $a \in A$, while Bob chooses $b \in B$. The number $p(a, b|x, y)$ is a conditional probability describing such a single round. It turns out that certain problems can be reformulated in a way such that the answers for them will be affirmative if and only if there exists a winning strategy for some game. For example, the chromatic number for a graph can be defined as the minimal one for which there exists a winning strategy for some two-player game. Suppose now that the players are no longer classical, but quantum. By it, we mean that they share some quantum, most probably entangled state, and instead of providing answers from finite sets, they perform quantum measurements. This can be reformulated in terms of some C^*-algebras, and therefore there is a natural generalization to the case where the sets of questions and/or answers are also quantum, i.e. they are described by some noncommutative C^*-algebras. I will not discuss the details here since they are much beyond the scope of these introductory notes. A more comprehensive discussion can be found e.g. in [36].

7.3 Hopf algebras and renormalization in QFT

Most of the renormalizability problems that arise in quantum field theories are quite well-understood on the physical level. However, rigorous mathematical treatment of them requires sophisticated methods that are far beyond the traditional academic courses. In this very short subsection, I will present an idea known as the Connes-Kreimer approach to renormalization [11] that uses certain algebraic structures of Hopf algebras to encode combinatorial procedures known from quantum field theories.

It is based on the observation that starting from a commutative graded bialgebra that is connected, one can define on it antipodal map. In other words, such a bialgebra is always a Hopf algebra and, moreover, there exists a recursive formula for this antipodal map S. Let me briefly recall here the definition of a connected commutative graded bialgebra. As a vector space, it is given by $H = \bigoplus_{n\geq 0} H_n$, where $H_0 = \mathbb{C}$. Moreover, the counit is a zero map on $\bigoplus_{n\geq 1} H_n$. Furthermore,

$$m(H_m \otimes H_n) \subseteq H_{m+n}, \qquad \Delta(H_n) \subseteq \bigoplus_{p+q=n} H_p \otimes H_q. \tag{55}$$

If H is such a bialgebra, then one can define S recursively. I will not write its precise form here but illustrate this with a concrete example.

Let us take a field theory, for concreteness: let it be a bosonic ϕ^n one, but this construction can be also generalized to e.g. QED, etc. The standard way to compute physical quantities is by using Feynmann diagrams. Let us then take H to be a set of one-particle irreducible diagrams Γ.

We can construct formal linear combinations of them and produce a vector space on H. Moreover, the disjoint sum of such diagrams defines a natural multiplication structure, while

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\substack{\gamma \subset \Gamma \\ \text{1PI, proper}}} \gamma \otimes \Gamma/\gamma \tag{56}$$

defined a coproduct. Here Γ/γ denotes the diagram obtained from Γ by shrinking its proper subdiagram γ to a point. There is also a natural counit structure defined to be zero on all non-trivial diagrams. One can then check that all the axioms of being connected commutative graded bialgebra are satisfied and therefore there exists an antipode map that makes it into a Hopf algebra. In this case, it is given by

$$S(\Gamma) = -\Gamma - \sum_{\gamma \subset \Gamma} S(\gamma)\Gamma/\gamma. \tag{57}$$

One can easily recognize the combinatorial structure of the BPHZ renormalization scheme here! The problem of renormalization can be then studied using this language. I refer the interested reader to [37] for further details of this approach.

7.4 Non-product geometries

Here I will briefly comment on applications of non-product spectral geometries to both the Standard Model and cosmology. It was actually the main topic of my Ph.D. thesis. Let me start with a simple observation that for a Lorentzian manifold, the fermionic spectral action can be written equivalently in two possible ways. We can write either $\int_{\mathcal{M}} \bar{\psi} \mathcal{D} \psi$ or $\int_{\mathcal{M}} \psi^\dagger \widetilde{\mathcal{D}} \psi$, where $\mathcal{D} = \gamma^0 \mathcal{D}$ is the Krein shift of the Lorentzian Dirac operator $\mathcal{D}$. One can impose certain conditions on the Krein shift instead of on the original Lorentzian operator. This idea can be applied also to the triple that is aimed to describe the physical Standard Model.

First, we have to choose the Hilbert space. By reverse engineering the physical model, we take it to consist of

$$\begin{pmatrix} \nu_R & u_R^1 & u_R^2 & u_R^3 \\ e_R & d_R^1 & d_R^2 & d_R^3 \\ \nu_L & u_L^1 & u_L^2 & u_L^3 \\ e_L & d_L^1 & d_L^2 & d_L^3 \end{pmatrix} \tag{58}$$

but now every entry is a Weyl spinor field defined on $\mathcal{M}$. The algebra $C^\infty(\mathbb{C}) \otimes \mathcal{A}_F$ is the standard one, but we are now choosing two different representations of it on the Hilbert space. We do not impose real structure and define left and right representations independently, $\pi_L(\lambda, q, m) = \begin{pmatrix} \lambda & & \\ & \bar{\lambda} & \\ & & q \end{pmatrix}$ and $\pi_R(\lambda, q, m) = \begin{pmatrix} \lambda & \\ & m^T \end{pmatrix}$. The Dirac operator is taken as the sum of the Lorentzian one on the manifold $\mathcal{M}$ and a finite endomorphism $\mathcal{D}_F$ from $M_4(\mathbb{C}) \otimes \mathrm{id} \otimes M_4(\mathbb{C})$. Now each part of this Dirac operator has opposite commutativity relation with the natural grading. From the very construction, there is no fermion doubling here.

The main idea now is to impose certain conditions on the Krein shift of this full operator and deduce from them the properties of the finite part. Starting from the first-order condition, modified accordingly to work with two independent representations - $[[\mathcal{D},\pi_L],\pi_R] = 0$, we deduce that there is no color symmetry breaking [38]. Moreover, the $\mathcal{D}_F$ operator has to be of the form

$$\mathcal{D}_F = \begin{pmatrix} & M_l \\ M_l^\dagger & \end{pmatrix} \otimes \mathrm{id} \otimes e_{11} + \begin{pmatrix} & M_q \\ M_q^\dagger & \end{pmatrix} \otimes \mathrm{id} \otimes (1_4 - e_{11}). \tag{59}$$

If there is no massless neutrino for three generations this spectral triple has the Hodge property. Furthermore, the lack of real structure can be interpreted as the source of CP violation in the physical Standard Model.

One can also try to compute the spectral action [39] for such a model. But since it is purely Lorentzian, at the moment this can be done only for some simplified situations. For example, we can start with the spatial and static case and compute the spectral action for such Krein shift. The obtained result agrees with the physical action (under these simplifying assumptions). One can also start with the Wick-rotated version of the full Lorentzian Dirac operator and in this case, the new topological θ-term appears in the electroweak sector.

This non-product structure requires further investigations but already the aforementioned results suggest its interesting potential applications in particle physics. It would be interesting to relate this structure to other existing proposals, e.g. [40, 41].

Another potential application of non-product geometries is to describe modified gravity models that might have intriguing cosmological consequences. In order to present the main idea let me come back to the intuitive picture of multi-layer systems for almost-commutative geometry. For the two-point model, i.e. with $\mathcal{A}_F = \mathbb{C}^2$, the resulting geometry was described as $(\mathcal{M}, g) \oplus (\mathcal{M}, g)$. This picture can be easily modified by replacing metric g in the second copy with some other one, say f. In the corresponding Dirac operator, we have to then replace the diagonal part $\mathrm{diag}(\mathcal{D},\mathcal{D})$ by $\mathrm{diag}(\mathcal{D}_g,\mathcal{D}_f)$. One can expect that from the gravity sector of the spectral action, we will have an effective action containing, in addition to the usual Einstein-Hilbert terms for the two metrics, also an interaction term between them. This suggests that spectral methods can be potentially used to describe Hassan-Rosen bimetric gravity models [42]. This idea was discussed in detail in [43] and it turns out that the effective potential has a different form but it shares a lot of features with the bimetric models. One can now try different modifications of the classical triples and try to derive other modified gravity models. This shows that Noncommutative Geometry, and especially spectral methods, can have more intriguing applications in cosmology.

8. Conclusions

In these short lecture notes, I have briefly discussed some subjectively chosen aspects of Noncommutative Geometry together with their certain applications in modern physics. The presented material of course did not cover the whole range of this very broad subject. I tried to show only the main ideas and some applications in a way that is accessible to Ph.D. students in physics, and for this reason, some of the material was oversimplified with respect to classical textbooks. Once again, I apologize to the authors whose ideas and results were not mentioned in these introductory notes. I also owe an apology to those whose results have not been properly cited - where possible I have tried to use references to textbooks I value for the style of presentation rather than to original source articles. I hope that the students attending my lectures as well as readers reading these notes

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had fun playing with noncommutative worlds and used the aforementioned textbooks to consult the material that I have only very briefly mentioned here.

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