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Melting Crystal, Quantum Torus and Toda Hierarchy

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Abstract Searching for the integrable structures of supersymmetric gauge theories and topological strings, we study melting crystal, which is known as random plane partition, from the viewpoint of integrable systems. We show that a series of partition functions of melting crystals gives rise to a tau function of the one-dimensional Toda hierarchy, where the models are defined by adding suitable potentials, endowed with a series of coupling constants, to the standard statistical weight. These potentials can be converted to a commutative sub-algebra of quantum torus Lie algebra. This perspective reveals a remarkable connection between random plane partition and quantum torus Lie algebra, and substantially enables to prove the statement. Based on the result, we briefly argue the integrable structures of five-dimensional $\mathscr{N}=1$ supersymmetric gauge theories and A-model topological strings. The aforementioned potentials correspond to gauge theory observables analogous to the Wilson loops, and thereby the partition functions are translated in the gauge theory to generating functions of their correlators. In topological strings, we particularly comment on a possibility of topology change caused by condensation of these observables, giving a simple example.

1 Introduction

An unanticipated but very exciting connection between the statistical mechanical problem of melting crystal, known as *random plane partition*, and A-model topological strings has been revealed (1), based on the topological vertex (2; 3). The topological vertex is a diagrammatical method which enables to compute all genus topological A-model string amplitudes for a certain class of local geometries.

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Fig. 1 The corner of the melting crystal and the corresponding plane partition (the three-dimensional Young diagram)

We can image the positive octant $\mathbb{Z}^3_{\geq 0} \subset \mathbb{R}^3$ occupied by unit cubes as the neighborhood of a corner of the crystal by putting unit cubes on the lattice points in the octant. The frozen crystal occupies the positive octant $\mathbb{Z}^3_{\geq 0}$. As the crystal melts, we remove atoms from the corner. We identify the configuration of crystal melting as the configuration of plane partition or the three-dimensional Young diagrams, as depicted in Fig. 1. Removing each atom contributes the factor $q = e^{-\frac{\mu}{T}}$ to the Boltzmann weight of the configuration, where T is the temperature and μ is the chemical potential. Heating up the crystal leads to melting of it.

Random plane partition also has a significant relation with five-dimensional $\mathcal{N} = 1$ supersymmetric gauge theories. Nekrasov's formula (4; 5) for five-dimensional $\mathcal{N} = 1$ supersymmetric $SU(N)$ Yang-Mills theory can be retrieved from the partition function of a random plane partition (6), where the model is interpreted as a q-deformed random partition. All genus topological A-model string amplitude for the local $SU(N)$ geometry is evaluated by the topological vertex and reproduces Nekrasov's formula for $\mathcal{N} = 2$ $SU(N)$ gauge theory (7; 8), as predicted in the geometric engineering.

It is shown in (5) that the Seiberg-Witten solutions (9) of four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories emerge through *random partition*, where Nekrasov's functions for four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories are understood as the partition functions of a random partition. The integrable structure of a random partition is elucidated in (10), and thereby the integrability of correlation functions among certain observables in four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories is explained.

Motivated by these results, we study in this article the integrable structure of a random plane partition in order to search for the integrable structures of supersymmetric gauge theories and topological strings.

A partition $\lambda = (\lambda_1, \lambda_2, \cdots)$ is a sequence of non-negative integers satisfying $\lambda_i \geq \lambda_{i+1}$ for all $i \geq 1$. Partitions are identified with the Young diagrams. The size is defined by $|\lambda| = \sum_{i\geq 1} \lambda_i$, which is the total number of boxes of the diagram. A plane partition π is an array of non-negative integers

$$\begin{matrix} \pi_{11} & \pi_{12} & \pi_{13} & \cdots \\ \pi_{21} & \pi_{22} & \pi_{23} & \cdots \\ \pi_{31} & \pi_{32} & \pi_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{matrix} \tag{1.1}$$

satisfying $\pi_{ij} \geq \pi_{i+1\,j}$ and $\pi_{ij} \geq \pi_{i\,j+1}$ for all $i, j \geq 1$. Plane partitions are identified with the three-dimensional Young diagrams. The three-dimensional diagram π is a set of unit cubes such that π_{ij} cubes are stacked vertically on each (i, j)-element of π. The size is defined by $|\pi| = \sum_{i,j\geq 1} \pi_{ij}$, which is the total number of cubes of the diagram. Diagonal slices of a plane partition π become partitions, as depicted in Fig. 2. Let $\pi(m)$ denote the partition along the m^{th} diagonal slice, where $m \in \mathbb{Z}$. In particular, $\pi(0) = (\pi_{11}, \pi_{22}, \cdots)$ is the main diagonal partition. This series of

Fig. 2 Plane partition (The three-dimensional Young diagram) (a) and the corresponding sequence of partitions (the two-dimensional Young diagrams) (b)

partitions satisfies the condition

$$\cdots \prec \pi(-2) \prec \pi(-1) \prec \pi(0) \succ \pi(1) \succ \pi(2) \succ \cdots, \tag{1.2}$$

where $\mu \succ \nu$ means the interlace relation between two partitions μ, ν,

$$\mu \succ \nu \iff \mu_1 \geq \nu_1 \geq \mu_2 \geq \nu_2 \geq \mu_3 \geq \cdots. \tag{1.3}$$

A statistical model of plane partitions is introduced by the following partition function:

$$Z \equiv \sum_{\pi} q^{|\pi|}, \tag{1.4}$$

where the sum is over all plane partitions. The parameter q is indeterminate satisfying $0 < q < 1$ and play the role of chemical potential (the energy of the removal of an atom from the crystal). Summing over plane partitions, we can obtain

$$\sum_{\pi} q^{|\pi|} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}. \tag{1.5}$$

The partition function of the model is the generating function of plane partitions, known as the McMahon function.

1.1 The models

We introduce a series of potentials for partitions and put the main diagonal partition of π in these potentials. For the later convenience, we introduce them as functions on charged partitions, which are partitions paired with integers. Let Φ_k $(k = 1, 2, \cdots)$ be the following functions on charged partitions:

$$\Phi_k(\lambda, p) = \sum_{i=1}^{\infty} q^{k(p+\lambda_i-i+1)} - \sum_{i=1}^{\infty} q^{k(-i+1)}, \tag{1.6}$$

where (λ, p) denotes a charged partition. Actually, the right hand side of this formula becomes a finite sum by cancellation of terms between the two sums. More precisely,

$$\Phi_k(\lambda, p) = \sum_{i=1}^{\infty} (q^{k(p+\lambda_i-i+1)} - q^{k(p-i+1)}) + q^k \frac{1-q^{pk}}{1-q^k}. \tag{1.7}$$

With each fixed value of p, these provide a series of potentials for partitions. These functions have been exploited in (10) from the four-dimensional gauge theory viewpoint, with q or q^k being replaced by a generating spectral parameter. Introducing the coupling constants $t = (t_1, t_2, \cdots)$, we write their combination as

$$\Phi_{(t;p)}(\lambda) = \sum_{k=1}^{\infty} t_k \Phi_k(\lambda, p). \tag{1.8}$$

The partition function of the random plane partition whose main diagonal partition is in the potential (1.8) is defined by

$$Z_p(t) \equiv \sum_{\pi} q^{|\pi|} e^{\Phi_{(t;p)}(\pi(0))}. \tag{1.9}$$

The model has an interpretation as a q-deformed random partition. To see this, note that, by virtue of the interlacing relations (1.2), the two series of partitions $\{\pi(m)\}_{m=0}^{\infty}$ and $\{\pi(-m)\}_{m=0}^{\infty}$ represent a pair T,T' of semi-standard Young tableaux of shape $\pi(0)$, in which the part of the m^{th} skew Young diagrams $\pi(\pm m)/\pi(\pm(m+1))$ is filled with $m+1$. The partition function can be thereby reorganized to a sum over the Young diagram $\lambda = \pi(0)$ and the pair T,T' of semi-standard Young tableaux of shape λ as

$$Z_p(t) = \sum_{\lambda} \sum_{T,T' : \text{shape}\,\lambda} q^T q^{T'} e^{\Phi_{(t;p)}(\lambda)}, \tag{1.10}$$

where $q^T = \prod_{m=0}^{\infty} q^{(m+\frac{1}{2})(|\pi(m)|-|\pi(m+1)|)}$ and $q^{T'} = \prod_{m=0}^{\infty} q^{(m+\frac{1}{2})(|\pi(-m)|-|\pi(-m-1)|)}$. The partial sum over the semi-standard tableaux gives the Schur function $s_\lambda(q^\rho) = s_\lambda(x_1,x_2,\cdots)$ specialized to $x_i = q^{i-\frac{1}{2}}$:

$$\sum_{T : \text{shape}\,\lambda} q^T = \sum_{T' : \text{shape}\,\lambda} q^{T'} = s_\lambda(q^\rho). \tag{1.11}$$

Therefore the partition function can be eventually expressed as

$$Z_p(t) = \sum_{\lambda} e^{\Phi_{(t;p)}(\lambda)} s_\lambda(q^\rho)^2. \tag{1.12}$$

The representation of $s_\lambda(q^\rho)$ in terms of the hook polynomial (11) allows us to write it further as

$$Z_p(t) = \sum_{\lambda} e^{\Phi_{(t;p)}(\lambda)} \left(\frac{q^{\frac{1}{2}|\lambda|+n_\lambda}}{\prod_{s\in\lambda}(1-q^{h(s)})} \right)^2, \tag{1.13}$$

where $h(s)$ denotes the hook length of the box $s \in \lambda$.

1.2 Main result

In this article, we study these models of random plane partition from the viewpoint of integrable systems. Our main result is that the following series of the partition functions is a tau function of an integrable hierarchy:

$$\tau(t;p) \equiv q^{\frac{1}{6}p(p+1)(2p+1)} Z_p(t), \qquad p \in \mathbb{Z}. \tag{1.14}$$

More precisely, we show in the text that $\tau(t;p)$ is a tau function of the one-dimensional Toda hierarchy, where the coupling constants $t = (t_1,t_2,\cdots)$ are interpreted as a single series of time variables of the one-dimensional Toda hierarchy.

In particular, $\tau(t;p)$ is shown to have a representation by two-dimensional free fermions or free bosons as

$$\tau(t;p) = e^{\sum_{k=1}^{\infty}\frac{t_k q^k}{1-q^k}}\,\langle p|\, e^{\frac{1}{2}\sum_{k=1}^{\infty}(-)^k t_k J_k}\,\mathbf{g}_\star\, e^{\frac{1}{2}\sum_{k=1}^{\infty}(-)^k t_k J_{-k}}\,|p\rangle. \qquad (1.15)$$

In the right hand side of this formula, $\mathbf{g}_\star$ is an element of $GL(\infty)$ and $J_{\pm k}$ denote the modes of the $U(1)$ current associated with the complex fermions. Thus $\tau(t;p)$ is a matrix element, taken between the Dirac sea of $U(1)$ charge p, of an element of $GL(\infty)$. The exponential operators on both sides of $\mathbf{g}_\star$ are generators of the commuting flows of the one-dimensional Toda hierarchy.

Once it is known that the partition function relates with the tau function, it becomes amenable to obtain an infinite set of non-linear differential equations that the partition function obeys. This is because tau functions of an integrable hierarchy satisfy an infinite set of non-linear differential equations. These non-linear differential equations are all encoded in the bilinear identity. It follows from the above that the partition functions satisfy the following bilinear identities in the one-dimensional Toda hierarchy:

$$\begin{aligned}&\oint \frac{dz}{2\pi i} z^{p-p'} e^{\frac{1}{2}\sum_{k\geq 1}(t_k-t'_k)z^k} Z_p(t-[z^{-1}])Z_{p'}(t'+[z^{-1}])\\ &= q^{(p+p'+1)(p-p'+1)}\oint \frac{dz}{2\pi i} z^{p-p'} e^{-\frac{1}{2}\sum_{k\geq 1}(t_k-t'_k)z^{-k}} Z_{p+1}(t+[z])Z_{p'-1}(t'-[z]),\end{aligned} \qquad (1.16)$$

where p,p' are arbitrary. The integral on the left hand side means taking the residue at $z=\infty$ and multiply it by -1; the integral on the right hand side is understood to be the residue at $z=0$. We also use notations like $t\pm[z]=(t_1\pm z, t_2\pm\frac{z^2}{2}, t_3\pm\frac{z^3}{3},\cdots)$. Towers of non-linear differential equations are obtained from (1.16) as the coefficients of the Taylor expansions along the diagonal $t=t'$, that is, the coefficients of the expansions of the bilinear identities in the variables $t_k-t'_k$. For instance, let $p=p'$. The first equation one gets in the tower is nothing but the Toda equation in Hirota's bilinear form,

$$D_{t_1}^2 Z_p\cdot Z_p = q^{2p+1}Z_{p+1}Z_{p-1}, \qquad (1.17)$$

where D denotes Hirota's derivative that is defined by $D_x f(x)\cdot g(x) = \lim_{y\to x}(\partial_x-\partial_y)f(x)g(y)$.

The partition functions also give rise to a solution of the modified KP hierarchy. The corresponding bilinear identities are read as

$$\oint \frac{dz}{2\pi i} z^{p-p'} e^{\sum_{k\geq 1}(t_k-t'_k)z^k} Z_p(t-[z^{-1}])Z_{p'}(t'+[z^{-1}]) = 0, \qquad (1.18)$$

where $p\geq p'$. These identities include in the towers the modified KP equations as well as the KP equation. For instance, let $p'=p-1$. The first equation one gets in this tower is

$$(D_{t_1}^2 - D_{t_2})Z_p(t)\cdot Z_{p-1}(t) = 0. \qquad (1.19)$$

This is the first equation of the modified KP hierarchy.

1.3 Transfer matrix approach

The main tool we use in the text is the transfer matrix formulation of the random plane partition. The hamiltonian picture is hinted from the interlace relations (1.2), which state that plane partitions are certain evolutions of partitions by the discretized time m. In particular, the transfer matrix formulation (12) makes it possible to express the partition function (1.9) in terms of two-dimensional conformal field theory ($2d$ free fermions).

Let $\psi(z)=\sum_{m\in\mathbb{Z}}\psi_m z^{-m-1}$ and $\psi^*(z)=\sum_{m\in\mathbb{Z}}\psi^*_m z^{-m}$ be complex fermions with the anti-commutation relations, $\{\psi_m,\psi^*_n\}=\delta_{m+n,0}$ and $\{\psi_m,\psi_n\}=\{\psi^*_m,\psi^*_n\}=0$. The Noether current of the $U(1)$ rotation is given by

$$J(z)=:\psi(z)\psi^*(z):=\sum_{m\in\mathbb{Z}} z^{-m-1}J_m, \tag{1.20}$$

where $:\ :$ denotes the normal ordering of fermions that is defined by

$$\psi(z)\psi^*(w)=:\psi(z)\psi^*(w):+\frac{1}{z-w}, \qquad |z|>|w|. \tag{1.21}$$

It is well known that partitions are realized as states of the fermion Fock space. In particular, charged partitions are realized by the states of the same $U(1)$ charges. For a charged partition (λ,p), the corresponding state is

$$|\lambda;p\rangle=\prod_{i=1}^{\infty}\psi_{i-\lambda_i-1-p}\psi^*_{-i+1+p}|p\rangle, \tag{1.22}$$

where $|p\rangle$ denotes the Dirac sea having the $U(1)$ charge p, and is defined by the conditions

$$\begin{aligned}\psi_m|p\rangle&=0 \ \text{ for } \ \forall m\geq -p,\\ \psi^*_m|p\rangle&=0 \ \text{ for } \ \forall m\geq p+1.\end{aligned} \tag{1.23}$$

Using the realization (1.22), it can be seen that the function (1.6) corresponds to the following fermion bilinear operator:

$$H_k \equiv \sum_{m\in\mathbb{Z}} q^{km} :\psi_{-m}\psi^*_m: . \tag{1.24}$$

Actually, the potential function (1.6) is reproduced as

$$H_k|\lambda;p\rangle=\Phi_k(\lambda,p)|\lambda;p\rangle. \tag{1.25}$$

These operators are commutative. The following combination reproduces the potential (1.8),

$$H(t)=\sum_{k=1}^{\infty}t_k H_k. \tag{1.26}$$

The transfer matrices (12) are vertex operators of the following forms:

$$\Gamma_+(m) = \exp\left(\sum_{k=1}^{+\infty} \frac{1}{k} q^{-k(m+\frac{1}{2})} J_k\right), \tag{1.27}$$

$$\Gamma_-(m) = \exp\left(\sum_{k=1}^{+\infty} \frac{1}{k} q^{k(m+\frac{1}{2})} J_{-k}\right), \tag{1.28}$$

where $J_{\pm k}$ are the modes of the $U(1)$ current. The matrix elements between partitions of different charges always vanishes, while those between partitions of the same charge are

$$\langle \lambda; p|\Gamma_+(m)|\mu; p\rangle = \begin{cases} q^{-(m+\frac{1}{2})(|\mu|-|\lambda|)} & \lambda \prec \mu \\ 0 & \text{otherwise,} \end{cases} \tag{1.29}$$

$$\langle \mu; p|\Gamma_-(m)|\lambda; p\rangle = \begin{cases} q^{(m+\frac{1}{2})(|\mu|-|\lambda|)} & \mu \succ \lambda \\ 0 & \text{otherwise.} \end{cases} \tag{1.30}$$

By comparing these formulas with the interlace relations (1.2), we see that $\Gamma_\pm(m)$ describes the evolutions of partitions. More precisely, the evolution at a negative time $m \leq -1$ is given by $\Gamma_+(m)$, while the evolution at a nonnegative time $m \geq 0$ is by $\Gamma_-(m)$.

Taking the hamiltonian picture of plane partitions, the partition function (1.9) can be reproduced in the transfer matrix formulation. Actually, following the same steps as we translated the partition function to the q-deformed random partition (1.10), but using the transfer matrices in place of the Schur functions, the partition function (1.9) is eventually expressed as

$$Z_p(t) = \langle p|\, G_+\, e^{H(t)}\, G_-\, |p\rangle, \tag{1.31}$$

where $G_\pm$ are the propagators which are responsible respectively to the negative time evolutions and the nonnegative time evolutions of partitions. These are the operators given by the following infinite products:

$$G_+ \equiv \prod_{m=-\infty}^{-1} \Gamma_+(m), \tag{1.32}$$

$$G_- \equiv \prod_{m=0}^{+\infty} \Gamma_-(m). \tag{1.33}$$

1.4 Quantum torus Lie algebra and random plane partition

Starting from the expression (1.31) of the partition function (1.9), we prove in the text that the series of the partition functions (1.14) is a tau function of the one-dimensional Toda hierarchy. Remarkable relations between random plane partition and quantum torus Lie algebra are revealed in the course of the proof. Throughout the text, we take the perspective that such a quantum Lie algebra is a hidden symmetry of the random plane partition.

We realize the quantum torus Lie algebra in terms of the complex fermions. Using this realization, we can regard the operators H_k as a commutative subalgebra of the quantum torus Lie algebra. The adjoint actions of the propagators $G_\pm$ on the Lie algebra generate automorphisms of the algebra. By taking advantage of such automorphisms, we provide a proof of the statement. Actually, among such automorphisms, we pay special attention to the shift symmetry that is the automorphism generated by the adjoint action of the product G_-G_+ or equivalently G_+G_-. By utilizing this symmetry, we can eventually express the partition function in the form (1.15). In particular, the element of $GL(\infty)$ in the formula (1.15) is given by

$$\mathbf{g}_\star \equiv q^{\frac{W}{2}}(G_-G_+)^2 q^{\frac{W}{2}}, \tag{1.34}$$

where W is a generator of the W_∞-algebra of the following form:

$$W \equiv \sum_{m\in\mathbb{Z}} m^2 : \psi_{-m}\psi_m^* : . \tag{1.35}$$

Using this formula, we finally confirm the statement by showing that $\mathbf{g}_\star$ actually realizes a solution of the one-dimensional Toda hierarchy.

Taking the formula (1.15), Virasoro/W-constraints on the partition function (1.9) and the tau function (1.14) can be obtained from the transformation of the W_∞-algebra by the adjoint action of $\mathbf{g}_\star$. In the same way, the transformation of the quantum torus Lie algebra by the adjoint action of $\mathbf{g}_\star$ gives rise to quantum torus analogues of the Virasoro/W-constraints on the partition function (1.9) and the tau function (1.14). As we argue subsequently, such quantum torus analogues of the Virasoro/W-constraints are also obtainable in five-dimensional $\mathcal{N}=1$ supersymmetric gauge theories and certain topological string amplitudes. These are reported in a separate publication (13).

1.5 Integrability of $5d$ $\mathcal{N}=1$ SUSY gauge theories from random plane partition

The random plane partition has a significant relation with five-dimensional $\mathcal{N}=1$ supersymmetric gauge theories (6; 14; 15; 16). Our study of the integrable structure of the random plane partition is motivated by a quest for the integrable structure of five-dimensional $\mathcal{N}=1$ supersymmetric gauge theories and topological strings.

1.5.1 Random plane partition and 5d $\mathcal{N}=1$ SUSY gauge theories.

The Nekrasov's functions for five-dimensional $SU(N)$ gauge theories (4; 5) are interpreted as partition functions of the random plane partition (6). Actually, the original partition function (1.4) reproduces Nekrasov's function for five-dimensional $U(1)$ gauge theory, by adding a simple chemical potential $Q^{|\pi(0)|}$ to the statistical weight. In the transfer matrix approach, that partition function becomes

$$Z^{5d\,U(1)} = \langle 0|\, G_+\, Q^{L_0}\, G_-\,|0\rangle, \tag{1.36}$$

where $L_0 \equiv \sum_{m\in\mathbb{Z}} m : \psi_{-m}\psi_m^* :$. The above matrix element is easily computed and leads

$$Z^{5d\,U(1)} = \prod_{n=1}^{+\infty} \frac{1}{(1-Qq^n)^n}. \tag{1.37}$$

Indeterminates q, Q in the right hand side of this formula must be interpreted in terms of the gauge theory parameters to reproduce Nekrasov's function for the five-dimensional gauge theory. The gauge theory lives on $\mathbb{R}^4 \times S^1_R$, where R denotes the radius of the circle, and has the dynamical scale Λ. The indeterminates are identified with these parameters by the relations

$$q = e^{-R\hbar}, \qquad Q = (R\Lambda)^2. \tag{1.38}$$

By shrinking the circle to a point, from Nekrasov's functions for five-dimensional gauge theories, one obtains the four-dimensional versions. Considering the relations (1.38), this indicates that the partition function (1.36) is a q-analogue of the four-dimensional version. To see this, we note that the four-dimensional limit of the right-hand side of (1.36) is obtained by employing the dressed propagators, $Q^{-\frac{1}{2}L_0}G_+Q^{\frac{1}{2}L_0}$ and $Q^{\frac{1}{2}L_0}G_-Q^{-\frac{1}{2}L_0}$. Actually, taking the relations (1.38), these dressed operators become nonsingular at the limit $R \to 0$, and eventually give the following operators:

$$\lim_{R\to 0} Q^{-\frac{1}{2}L_0}G_+Q^{\frac{1}{2}L_0} = \Lambda^{-L_0}\, e^{\frac{1}{\hbar}J_1}\, \Lambda^{L_0}, \tag{1.39}$$

$$\lim_{R\to 0} Q^{\frac{1}{2}L_0}G_-Q^{-\frac{1}{2}L_0} = \Lambda^{L_0}\, e^{\frac{1}{\hbar}J_{-1}}\, \Lambda^{-L_0}. \tag{1.40}$$

By using these formulas, one obtains

$$\begin{aligned}\lim_{R\to 0}\langle 0|\, G_+\, Q^{L_0}\, G_-\, |0\rangle &= \lim_{R\to 0}\langle 0|Q^{-\frac{1}{2}L_0}G_+Q^{\frac{1}{2}L_0}\,Q^{\frac{1}{2}L_0}G_-Q^{-\frac{1}{2}L_0}|0\rangle \\ &= \langle 0|\, e^{\frac{1}{\hbar}J_1}\,\Lambda^{2L_0}\, e^{\frac{1}{\hbar}J_{-1}}\,|0\rangle. \end{aligned} \tag{1.41}$$

The right-hand side of this formula is nothing but Nekrasov's function for four-dimensional $U(1)$ gauge theory.

1.5.2 Integrable structure of 5d $\mathcal{N}=1$ SUSY gauge theories.

The operator H_k has a counterpart in five-dimensional gauge theories. It corresponds to the Wilson loop operator encircling the circle k times (17),

$$O_k = \mathrm{Tr}\left\{Pe^{\oint dt A_4 + i\varphi}\right\}^k, \tag{1.42}$$

where A_4 and φ denote respectively the fifth component of the gauge field and the real scalar field in the vector multiplet. The generating function of the correlators among these observables becomes thereby the following analogue of (1.31):

$$Z_p^{5d\,U(1)}(t) = \langle p|\, G_+\, e^{H(t)}\, Q^{L_0} G_-\, |p\rangle. \tag{1.43}$$

In the same way as we stated on (1.14), the series of the generating functions (1.43) also gives rise to a tau function of the one-dimensional Toda hierarchy. In particular, the tau function has the following expression analogous to (1.15):

$$\tau^{5dU(1)}(t;p) = e^{\sum_{k=1}^{\infty}\frac{t_k q^k}{1-q^k}} \langle p| e^{\frac{1}{2}\sum_{k=1}^{\infty}(-)^k t_k J_k}\, \mathbf{g}_{\star}^{5dU(1)}\, e^{\frac{1}{2}\sum_{k=1}^{\infty}(-)^k t_k J_{-k}} |p\rangle, \quad (1.44)$$

where $\mathbf{g}_{\star}^{5dU(1)}$ is the element of $GL(\infty)$ given by

$$\mathbf{g}_{\star}^{5dU(1)} \equiv q^{\frac{W}{2}} (G_- G_+)\, Q^{L_0}\, (G_- G_+)\, q^{\frac{W}{2}}. \quad (1.45)$$

We note that this formula is easily generalized to the $SU(N)$ gauge theory, where the series of the corresponding generating functions of the correlators becomes a tau function of the one-dimensional Toda hierarchy.

In four-dimensions, such an integrable structure of $\mathcal{N}=2$ supersymmetric gauge theories has been found out (10; 18) among the generating functions of the higher Casimir operators $\mathrm{Tr}\phi^k$, where ϕ is the complex scalar in the vector multiplet, corresponding to $A_4 + i\varphi$ in five-dimensional theories. One might expect that the tau function (1.44) is a q-analogue of the tau function of the four-dimensional theory, just like the partition function (1.36) is the q-analogue. However, this is not the case. The relation between these two integrable structures is not straightforward. The subtlety can be found, for instance, in that the tau function (1.44) becomes trivial at the four-dimensional limit, owing to the degeneration of all the observables (1.42). Actually, when R is nearly zero, the generating function (1.43) behaves as

$$Z_p^{5dU(1)}(t) \sim R^{p(p+1)} e^{p\sum_{k=1}^{\infty} t_k} \langle 0| e^{\frac{1}{\hbar}J_1}\, \Lambda^{2L_0}\, e^{\frac{1}{\hbar}J_{-1}} |0\rangle. \quad (1.46)$$

1.6 Integrability of topological string amplitudes from random plane partition

In addition to the gauge theory interpretation, the partition function (1.36) has an interpretation as an all genus A-model topological string amplitude on $\mathcal{O} \oplus \mathcal{O}(-2) \to \mathbb{C}\mathrm{P}^1$. It is a non-compact toric Calabi-Yau threefold, often called a local geometry. The toric description is given by a fan Δ consisting of rational cones of dimensions ≤ 3 on $\mathbb{R}^3$. The topological vertex (2; 3) is a diagrammatical method to compute all genus A-model topological string amplitudes for such local geometries. The diagram can be drawn from a polyhedron which is obtained by taking duals to the cones in $\mathbb{R}^3$. For the local geometry $\mathcal{O} \oplus \mathcal{O}(-2) \to \mathbb{C}\mathrm{P}^1$, the relevant diagram is depicted in Fig. 3. The topological vertex computation based on the diagram gives all genus A-model topological string amplitude on $\mathcal{O} \oplus \mathcal{O}(-2) \to \mathbb{C}\mathrm{P}^1$ as

$$\mathcal{A}_{string}^{\mathcal{O}\oplus\mathcal{O}(-2)} = \prod_{n=1}^{+\infty} \frac{1}{(1-e^{-a}e^{-ng_{st}})^n}, \quad (1.47)$$

where a denotes the Kähler volume of the base $\mathbb{C}\mathrm{P}^1$, and g_{st} is the string coupling constant. Comparing two formulas (1.37) and (1.47), one sees that the partition

Fig. 3 The diagram for $\mathscr{O}\oplus\mathscr{O}(-2)\to\mathbb{C}\mathrm{P}^1$

function (1.36) becomes an all genus A-model topological string amplitude on $\mathscr{O}\oplus\mathscr{O}(-2)\to\mathbb{C}\mathrm{P}^1$, by the following identification of the parameters:

$$q=e^{-g_{st}},\qquad Q=e^{-a}. \tag{1.48}$$

As is the case of the five-dimensional gauge theory, we generalize the amplitude including the operators H_k in it, anticipating that a wound Euclidean brane along the M-theory circle[1] corresponds to the observable O_k of the gauge theories. We physically conjecture such a generalization by

$$\mathscr{A}_{string}^{\mathscr{O}\oplus\mathscr{O}(-2)}(t;p)\;=\;\langle p|\,G_+\,e^{H(t)}\,Q^{L_0}G_-\,|p\rangle, \tag{1.49}$$

where the right-hand side of this equation is same as the formula (1.43) but with the different interpretation (1.48). The series of the generating functions (1.49) gives the same tau function as is the case of the five-dimensional $U(1)$ gauge theory.

It seems rather rare that the one-dimensional Toda hierarchy shows up as an integrable structure of topological strings. One of such rare cases is the topological sigma model (in other words, the Gromov-Witten invariants) of $\mathbb{C}P^1$ (19; 20; 21; 22; 23; 24). On the other hand, the "relative" or "equivariant" versions of those Gromov-Witten invariants have a different integrable structure, namely, the two-dimensional Toda hierarchy (23; 25). It is remarkable that substantially the same quantum torus Lie algebra is used in the work of Okounkov and Pandharipande (25), but we have been unable to see whether there is a deep connection with our work. To obtain the generators $V_m^{(k)}$ of our quantum torus Lie algebra, one has to specialize the parameter z of Okounkov and Pandharipande's operators $\mathscr{E}_m(z)$ to $e^z=q^k$; such powers of q apparently play no role in the work of Okounkov and Pandharipande.

The two-dimensional Toda hierarchy is also known to arise in the generating function $\tau(x,\bar{x})=\sum_{\lambda,\mu}s_\lambda(x)s_\mu(\bar{x})c_{\lambda\mu\bullet}$ of the two-legged topological vertex $c_{\lambda\mu\bullet}$ (26). By changing variables from $x,\bar{x}$ to $T_k=\frac{1}{k}\sum_i x_i^k$, $\bar{T}_k=-\frac{1}{k}\sum_i\bar{x}_i^k$, $\tau(x,\bar{x})$ coincides with the special part $\tau^{2Toda}(T,\bar{T},0)$ of a tau function $\tau^{2Toda}(T,\bar{T},p)$ of the two-dimensional Toda hierarchy. As Zhou pointed out, the tau function $\tau^{2Toda}(T,\bar{T},p)$ has a fermionic representation in terms of an element g of $GL(\infty)$ of the form

$$g=q^{\frac{K}{2}}G_+G_-q^{\frac{K}{2}}, \tag{1.50}$$

where $K=\sum_m(m-\frac{1}{2})^2:\psi_{-m}\psi_m^*:$. Thus the building blocks of this tau function are similar to the foregoing partition functions of $U(1)$ gauge theory. The difference between K and W is almost negligible, and can be absorbed by rescaling of $T_k,\bar{T}_k$ and an exponential prefactor.

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[1] We thank Y. Hyakutake for suggesting such a possibility to us.

1.6.1 Emergence of geometry from condensation.

It is amazing to comment on a possibility of emergence of another geometry from the condensation in a local geometry.

Consider the generating function (1.49) for the case of $p = 0$, and write it simply as $\mathscr{A}^{\mathscr{O}\oplus\mathscr{O}(-2)}_{string}(t) \equiv \mathscr{A}^{\mathscr{O}\oplus\mathscr{O}(-2)}_{string}(t; p = 0)$. As explained in the beginning of Sect. 4, (1.49) has another representation of the following form:

$$\mathscr{A}^{\mathscr{O}\oplus\mathscr{O}(-2)}_{string}(t) = e^{\sum_{k=1}^{\infty} \frac{t_k q^k}{1-q^k}} \langle 0|\, e^{\sum_{k=1}^{\infty}(-)^k t_k J_k}\, \mathbf{g}_\star^{5d\,U(1)}\, |0\rangle. \tag{1.51}$$

This becomes a tau function of the KP hierarchy.

The generating function has a factorization in terms of the skew Schur functions. The matrix element in the right-hand side of formula (1.51) is factored by plugging the unity $1 = \sum_{p\in\mathbb{Z}} \sum_\lambda |\lambda; p\rangle\langle\lambda; p|$ so that it divides $\mathbf{g}_\star^{5d\,U(1)}$ into two parts. Owing to the charge conservation, the sum over p truncates to $p = 0$. Writing $|\lambda; 0\rangle$ simply as $|\lambda\rangle$, we can obtain

$$\langle 0|\, e^{\sum_{k=1}^{\infty}(-)^k t_k J_k}\, \mathbf{g}_\star^{5d\,U(1)}\, |0\rangle = \sum_\lambda \langle 0|\, e^{\sum_{k=1}^{\infty}(-)^k t_k J_k}\, q^{\frac{W}{2}} G_-\, |\lambda\rangle \times \langle\lambda|\, G_+ Q^{L_0} G_-\, |0\rangle. \tag{1.52}$$

Matrix elements in the right-hand side of this equation are expressed in terms of the skew Schur functions as follows:

$$\langle 0|\, e^{\sum_{k=1}^{\infty}(-)^k t_k J_k}\, q^{\frac{W}{2}} G_-\, |\lambda\rangle = \sum_\mu q^{\frac{\kappa(\mu)}{2} + \frac{|\mu|}{2}}\, s_\mu(x) s_{\mu/\lambda}(q^\rho), \tag{1.53}$$

$$\langle\lambda|\, G_+ Q^{L_0} G_-\, |0\rangle = \frac{Q^{|\lambda|} s_\lambda(q^\rho)}{\prod_{n=1}^{+\infty}(1 - Qq^n)^n}, \tag{1.54}$$

where variables are changed from t to x by $t_k = \frac{1}{k}\sum_i (-x_i)^k$. Combining these two formulas, the right hand side of formula (1.51) is expressed in terms of the skew Schur functions. Therefore, the factorization of the generating function, normalized by the original A-model topological string amplitude, can be written as

$$\frac{\mathscr{A}^{\mathscr{O}\oplus\mathscr{O}(-2)}_{string}(t)}{\mathscr{A}^{\mathscr{O}\oplus\mathscr{O}(-2)}_{string}(0)} = e^{\sum_{k=1}^{\infty} \frac{t_k q^k}{1-q^k}} \sum_\lambda Q^{|\lambda|} s_\lambda(q^\rho) \left\{ \sum_\mu q^{\frac{\kappa(\mu)}{2} + \frac{|\mu|}{2}}\, s_\mu(x) s_{\mu/\lambda}(q^\rho) \right\}. \tag{1.55}$$

We examine the condensation by choosing the coupling constants t at certain values. In particular, we take the following values.

$$t_k^\star = \frac{q^{\frac{k}{2}}}{k(1-q^k)}, \qquad k = 1, 2, \ldots. \tag{1.56}$$

Owing to the identification (1.48), $t^\star$ behave $\sim g_{st}^{-1}$ when g_{st} is nearly zero and therefore a possible condensation becomes nonperturbative in IIA superstrings.

Fig. 4 The diagram for two conifolds

However, one can compute the right hand side of formula (1.55). The computation yields eventually the amplitude as

$$\frac{\mathscr{A}_{string}^{\mathscr{O}\oplus\mathscr{O}(-2)}(t^\star)}{\mathscr{A}_{string}^{\mathscr{O}\oplus\mathscr{O}(-2)}(0)} = \prod_{n=1}^{+\infty}(1-Qq^{n+\frac{1}{2}})^n. \tag{1.57}$$

The right hand side of this formula coincides with all genus A-model topological string amplitude on the resolved conifold $\mathscr{O}(-1)\oplus\mathscr{O}(-1)\to\mathbb{C}\mathrm{P}^1$, where the Kähler volume of the base $\mathbb{C}\mathrm{P}^1$ is $a+\frac{1}{2}g_{st}$.

Emergence of the resolved conifold in (1.57) seems mysterious. However, this can be explained as follows. Let us consider the local geometry of coupled conifolds. Coupled conifolds can be obtained by patching torically together the foregoing local geometry and $\mathbb{C}^2$. The diagram of coupled conifolds is depicted in Fig. 4, where $Q_{1,2}$ are the Kähler parameters attached to the internal edge of the diagram. Each internal edges corresponds to $\mathbb{C}P^1$. The Kähler parameters are given by $Q_{1,2}=e^{-a_{1,2}}$, where $a_{1,2}$ denote the Kähler volumes of the corresponding $\mathbb{C}P^1$s. Based on this diagram, the topological vertex computation yields the all genus A-model topological string amplitude as

$$\mathscr{A}_{string}^{two\,conifolds} = \frac{\prod_{n=1}^{+\infty}(1-Q_1Q_2q^n)^n\cdot\prod_{n=1}^{+\infty}(1-Q_2q^n)^n}{\prod_{n=1}^{+\infty}(1-Q_1q^n)^n}. \tag{1.58}$$

It is remarkable that the topological string amplitude (1.58) appears by tuning the Kähler volumes $a_{1,2}$, as a building block of the generating function (1.51) at $t=t^\star$. Actually, the matrix element $\langle 0|\, e^{\sum_{k=1}^{\infty}(-)^k t_k J_k}\, \mathbf{g}_\star^{5dU(1)}\, |0\rangle$ is evaluated by using formulas (1.53),(1.54) and becomes eventually.

$$\langle 0|\, e^{\sum_{k=1}^{\infty}(-)^k t_k J_k}\, \mathbf{g}_\star^{5dU(1)}\, |0\rangle\Big|_{t=t^\star} = \mathscr{A}_{string}^{two\,conifolds}, \tag{1.59}$$

where the Kähler volumes $a_{1,2}$ in the right hand side of this formula are

$$a_1 = a, \qquad a_2 = \frac{1}{2}g_{st}. \tag{1.60}$$

This formula indicates that the condensation (1.56) changes the original geometry into two-conifolds with two-cycles having the Kähler volumes (1.60), and that the ratio (1.57) counts the worldsheet instantons wrapping the two-cycles simultaneously. The further issue will be reported in (13).

Organization of the article.

The purpose of this article is to show that the series of the partition functions (1.14) is a tau function of one-dimensional Toda hierarchy. We start Sect. 2 by giving the realization of the quantum torus Lie algebra using the complex fermions. In Sect. 3, we argue that automorphisms of the Lie algebra are generated by the adjoint actions of $G_\pm$. By using such automorphisms we confirm that the series of the partition functions (1.14) satisfy the Toda equation (1.17). In Sect. 4, we provide a proof of the statement.

2 Quantum Torus Lie Algebra

Let $V_m^{(k)}$, where $k = 0, 1, 2, \cdots$ and $m \in \mathbb{Z}$, be a set of operators which are defined by the following generating function for each k:

$$: \psi(q^{\frac{k}{2}} z) \psi^*(q^{-\frac{k}{2}} z) : = \sum_{m\in\mathbb{Z}} z^{-m-1} q^{-\frac{k}{2}} V_m^{(k)}. \tag{2.1}$$

This formula yields the following expression of $V_m^{(k)}$:

$$V_m^{(k)} = q^{\frac{k}{2}} \oint \frac{dz}{2\pi i} z^m : \psi(q^{\frac{k}{2}} z) \psi^*(q^{-\frac{k}{2}} z) :, \tag{2.2}$$

where the integral means taking the residue at $z = 0$. This integral can be evaluated by plugging the mode expansion of $\psi(z), \psi^*(z)$ into the generating function. Thereby, the right-hand side of Eq. (2.2) is read as

$$V_m^{(k)} = q^{-\frac{km}{2}} \sum_{n\in\mathbb{Z}} q^{kn} : \psi_{m-n} \psi_n^* : . \tag{2.3}$$

The operator H_k (1.24) and the $U(1)$ current J_m (1.20) are represented as

$$H_k = V_0^{(k)}, \qquad J_m = V_m^{(0)}. \tag{2.4}$$

Henceforth, taking the viewpoint of integrable systems, we call H_k ($k \geq 1$) hamiltonians.

We note that the normal ordering in the formula (2.1) is redundant when $k \neq 0$. Actually, the fermion bilinear form $-\psi^*(q^{-\frac{k}{2}} z)\psi(q^{\frac{k}{2}} z)$ is regularized solely by the point splitting[2] $z \to q^{\pm\frac{k}{2}} z$, without any normal ordering. Effect of the normal ordering is known from (1.21) and becomes the subtraction of a finite term as

$$-\psi^*(q^{-\frac{k}{2}} z)\psi(q^{\frac{k}{2}} z) =: \psi(q^{\frac{k}{2}} z)\psi^*(q^{-\frac{k}{2}} z) : -\frac{q^{\frac{k}{2}}}{z(1-q^k)}. \tag{2.5}$$

Thereby, the normal ordering only makes the finite gap between $V_0^{(k)}$ and that obtained without the normal ordering as follows:

$$V_0^{(k)} = -\sum_m q^{km} \psi_m^* \psi_{-m} + \frac{q^k}{1-q^k}. \tag{2.6}$$

The operators $V_m^{(k)}$ satisfy the quantum torus Lie algebra (the sine-algebra (27)). The following commutation relations can be found:

$$\left[V_m^{(k)}, V_n^{(l)}\right] = (q^{\frac{lm-kn}{2}} - q^{-\frac{lm-kn}{2}}) \left(V_{m+n}^{(k+l)} - \delta_{m+n,0} \frac{q^{k+l}}{1-q^{k+l}}\right), \tag{2.7}$$

where $k, l = 0, 1, \cdots$ and $m, n \in \mathbb{Z}$. The commutation relations (2.7) become the standard ones by shifting $V_0^{(k)} \to V_0^{(k)} - \frac{q^k}{1-q^k}$ for $k \neq 0$. The hamiltonians $H_k(k \geq$

[2] We take $0 < q < 1$.

0), where $H_0 \equiv V_0^{(0)}$ is included, generate a commutative sub-algebra of this Lie algebra. The non-negative modes $\{J_m\}_{m\geq 0}$ and the non-positive modes $\{J_{-m}\}_{m\geq 0}$ of the $U(1)$ current do as well. In addition to these sub-algebras, there are two more commutative sub-algebras that are generated respectively by $\{V_k^{(k)}\}_{k\geq 0}$ and $\{V_{-k}^{(k)}\}_{k\geq 0}$. All these sub-algebras are found to relate with one another.

The appearance of a quantum torus Lie algebra might be unexpected. However it can be explained from the viewpoint of the sine-algebra (27). The sine-algebra is obtained from $sl(N)$ by taking the large N limit of its trigonometrical basis. Let X, Y be two $N \times N$ unitary matrices given by

$$X = \sum_{i=1}^{N-1} E_{i,i+1} + E_{N,1}, \quad Y = \sum_{i=1}^{N} \omega^{i-1} E_{i,i}, \tag{2.8}$$

where ω is a N^{th} root of the unity. These two matrices satisfy the relation $XY = \omega YX$. Their non-commutative monomials $X^m Y^k$ for $0 \leq m < N$ and $0 \leq k < N$ give the trigonometrical basis of $sl(N)$. The sine-algebra is the Lie algebra of $X^m Y^k$ at $N = \infty$ and is identified with the quantum torus Lie algebra. It is the Lie algebra derived from a quantum two-torus (non-commutative two-torus), that is, an unital algebra with two generators U, V satisfying the relation $UV = qVU$. Here q is regarded as the non-commutative parameter. The trigonometrical basis $X^m Y^k$ is transmuted to the non-commutative monomials $U^m V^k$. Let us normalize them as follows:

$$v_m^{(k)} = q^{-\frac{km}{2}} U^m V^k. \tag{2.9}$$

These normalized ones satisfy the commutation relations (2.7), apart from the shift of the zero modes.

The sine-algebra is the algebra of trigonometric basis of $sl(\infty)$. Making use of the embedding $sl(\infty) \subset gl(\infty)$, as utilized in (2.8) for finite N, a trigonometric basis can be realized in terms of q-difference operators with respect to z. Among them, the fundamental operators are z and $q^{-zd/dz} = \exp(-\log q z \frac{d}{dz})$, which are the counterparts of X and Y. These two operators satisfy the relation $zq^{-zd/dz} = qq^{-zd/dz}z$. The normalized basis (2.9) of the sine-algebra correspond to

$$v_m^{(k)} = q^{-\frac{km}{2}} z^m q^{-kz\frac{d}{dz}}. \tag{2.10}$$

The operators $V_m^{(k)}$ are nothing but the second-quantizations of $v_m^{(k)}$ by means of the fermions.

$$V_m^{(k)} = \oint \frac{dz}{2\pi i} : \psi(z) v_m^{(k)} \psi^*(z) : . \tag{2.11}$$

Since we have restricted $k \geq 0$, the operators $V_m^{(k)}$ or the basis (2.10) generate only half of quantum torus Lie algebra. Needless to say, the operators for $k < 0$ are obtainable from the generating function (2.1) by choosing k as such, however those play no role in this article. The above half is, so to speak, a quantum cylinder Lie algebra to be obtained from a quantum cylinder. In the context of the random plane partition, this quantum cylinder becomes a classical cylinder $\mathbb{C}^*$ at the thermodynamic limit or the semi-classical limit; the Seiberg-Witten hyperelliptic curves of five-dimensional $\mathscr{N}=1$ supersymmetric gauge theories emerge as the double cover of the cylinder (14; 16). See (14; 16) for details.

3 One-Dimensional Toda Chain

In this section we show that the series of the partition functions (1.14) provide a solution of the one-dimensional Toda chain. We will identify the partition functions with dynamical variables ϕ_p on a $\mathbb{Z}$-lattice by

$$e^{\phi_p} \equiv \frac{\tau(t;p+1)}{\tau(t;p)} = q^{(p+1)^2}\frac{Z_{p+1}(t)}{Z_p(t)}, \qquad p \in \mathbb{Z}. \tag{3.1}$$

Then, writing $x = t_1$, the variables ϕ_p satisfy the following Toda equation:

$$\partial_x^2 \phi_p = e^{\phi_{p+1}-\phi_p} - e^{\phi_p - \phi_{p-1}}, \qquad p \in \mathbb{Z}. \tag{3.2}$$

3.1 Transformations by adjoint action

We argue transformations of the generators $V_m^{(k)}$ by the adjoint action of the propagators $G_\pm$. For this purpose, we conveniently start with the transformations of the fermions $\psi(z), \psi^*(z)$. These follow from their rotations by the transfer matrices $\Gamma_\pm(m)$, since $G_\pm$ are just the products of $\Gamma_\pm(m)$. The rotations are easily computed by recalling that each mode of the $U(1)$ current rotates the fermions as follows:

$$e^{\theta J_m}\psi(z)e^{-\theta J_m} = e^{\theta z^m}\psi(z), \qquad e^{\theta J_m}\psi^*(z)e^{-\theta J_m} = e^{-\theta z^m}\psi^*(z), \qquad \theta \in \mathbb{C}. \tag{3.3}$$

For instance, the rotation of $\psi(z)$ by $\Gamma_+(m)$ is computed as follows:

$$\begin{aligned} \Gamma_+(m)\,\psi(z)\,\Gamma_+(m)^{-1} &= \prod_{k=1}^{+\infty} e^{\frac{1}{k}q^{-k(m+\frac{1}{2})}J_k}\,\psi(z)\prod_{k=1}^{+\infty} e^{-\frac{1}{k}q^{-k(m+\frac{1}{2})}J_k} \\ &= e^{\sum_{k=1}^{+\infty}\frac{1}{k}z^k q^{-k(m+\frac{1}{2})}}\,\psi(z) \\ &= (1 - zq^{-(m+\frac{1}{2})})^{-1}\,\psi(z). \end{aligned} \tag{3.4}$$

By using this formula reiteratively according as (1.32), we can compute the adjoint action of G_+. In this way, we eventually obtain the following transformations of the fermions:

$$\begin{cases} G_+\psi(z)(G_+)^{-1} = \displaystyle\prod_{m=1}^{+\infty}(1 - zq^{m-\frac{1}{2}})^{-1}\,\psi(z), \\ G_+\psi^*(z)(G_+)^{-1} = \displaystyle\prod_{m=1}^{+\infty}(1 - zq^{m-\frac{1}{2}})\,\psi^*(z). \end{cases} \tag{3.5}$$

$$\begin{cases} (G_-)^{-1}\psi(z)G_- = \displaystyle\prod_{m=0}^{+\infty}(1 - z^{-1}q^{m+\frac{1}{2}})\,\psi(z), \\ (G_-)^{-1}\psi^*(z)G_- = \displaystyle\prod_{m=0}^{+\infty}(1 - z^{-1}q^{m+\frac{1}{2}})^{-1}\,\psi^*(z). \end{cases} \tag{3.6}$$

Nextly we examine the transformations of the generating function (2.1) of the quantum torus Lie algebra. These transformations are obtained by using formulas (3.5), (3.6), taking (2.5) into account. We illustrate the computation for the case of G_+:

$$\begin{aligned}
&G_+ :\psi(q^{\frac{k}{2}}z)\psi^*(q^{-\frac{k}{2}}z):(G_+)^{-1}\\
&= G_+\left\{-\psi^*(q^{-\frac{k}{2}}z)\psi(q^{\frac{k}{2}}z)+\frac{q^{\frac{k}{2}}}{z(1-q^k)}\right\}(G_+)^{-1}\\
&= -G_+\psi^*(q^{-\frac{k}{2}}z)(G_+)^{-1}G_+\psi(q^{\frac{k}{2}}z)(G_+)^{-1}+\frac{q^{\frac{k}{2}}}{z(1-q^k)}\\
&= -\prod_{m=1}^{k}(1-zq^{\frac{k+1}{2}-m})\ \psi^*(q^{-\frac{k}{2}}z)\psi(q^{\frac{k}{2}}z)+\frac{q^{\frac{k}{2}}}{z(1-q^k)}. \qquad (3.7)
\end{aligned}$$

The last line in (3.7) can be rewritten in terms of the generating function itself. We thus obtain

$$\begin{aligned}
&G_+\left\{:\psi(q^{\frac{k}{2}}z)\psi^*(q^{-\frac{k}{2}}z):-\frac{q^{\frac{k}{2}}}{z(1-q^k)}\right\}(G_+)^{-1}\\
&\quad=\prod_{m=1}^{k}(1-zq^{\frac{k+1}{2}-m})\left\{:\psi(q^{\frac{k}{2}}z)\psi^*(q^{-\frac{k}{2}}z):-\frac{q^{\frac{k}{2}}}{z(1-q^k)}\right\}. \qquad (3.8)
\end{aligned}$$

The similar computation goes as well for the case of G_- and gives

$$\begin{aligned}
&(G_-)^{-1}\left\{:\psi(q^{\frac{k}{2}}z)\psi^*(q^{-\frac{k}{2}}z):-\frac{q^{\frac{k}{2}}}{z(1-q^k)}\right\}G_-\\
&\quad=\prod_{m=1}^{k}(1-z^{-1}q^{-\frac{k+1}{2}+m})\left\{:\psi(q^{\frac{k}{2}}z)\psi^*(q^{-\frac{k}{2}}z):-\frac{q^{\frac{k}{2}}}{z(1-q^k)}\right\} \qquad (3.9)
\end{aligned}$$

The transformations of $V_m^{(k)}$ can be obtained from formulas (3.8), (3.9) by reading the coefficients of the Laurent expansions of the equations around $z=0$. It is evident but nevertheless surprising that $G_\pm$ generate automorphisms of the Lie algebra (2.7) by adjoint action. Let us concentrate on the transformations of the hamiltonians $H_k=V_0^{(k)}$. These are read from formulas (3.8), (3.9) as follows:

$$G_+V_0^{(k)}(G_+)^{-1}=q^{\frac{k}{2}}\oint\frac{dz}{2\pi i}\prod_{m=1}^{k}(1-zq^{\frac{k+1}{2}-m}):\psi(q^{\frac{k}{2}}z)\psi^*(q^{-\frac{k}{2}}z):, \qquad (3.10)$$

$$(G_-)^{-1}V_0^{(k)}G_-=q^{\frac{k}{2}}\oint\frac{dz}{2\pi i}\prod_{m=1}^{k}(1-z^{-1}q^{-\frac{k+1}{2}+m}):\psi(q^{\frac{k}{2}}z)\psi^*(q^{-\frac{k}{2}}z):, \qquad (3.11)$$

where the integrals denote taking the residue at $z=0$. These integrals can be evaluated by using the q-binomial theorem:

$$\prod_{m=1}^{k}(1+xq^m)=\sum_{i=0}^{k}q^{\frac{i(i+1)}{2}}\begin{bmatrix}k\\ i\end{bmatrix}_q x^i, \qquad (3.12)$$

where

$$\begin{bmatrix} k \\ i \end{bmatrix}_q = \frac{(q:q)_k}{(q:q)_i (q:q)_{k-i}}, \qquad (q:q)_n = (1-q)(1-q^2)\cdots(1-q^n). \quad (3.13)$$

By expanding the products in the right hand sides of Eqs. (3.10),(3.11), these are brought to the following form:

$$G_+ V_0^{(k)} (G_+)^{-1} = \sum_{i=0}^{k} (-)^i q^{-\frac{i(k-i)}{2}} \begin{bmatrix} k \\ i \end{bmatrix}_q V_i^{(k)}, \quad (3.14)$$

$$(G_-)^{-1} V_0^{(k)} G_- = \sum_{i=0}^{k} (-)^i q^{-\frac{i(k-i)}{2}} \begin{bmatrix} k \\ i \end{bmatrix}_q V_{-i}^{(k)}. \quad (3.15)$$

3.2 Toda equation

The transformations of $H_1 = V_0^{(1)}$ in formulas (3.14),(3.15) are read as

$$G_+ H_1 (G_+)^{-1} = V_0^{(1)} - V_1^{(1)}, \quad (3.16)$$

$$(G_-)^{-1} H_1 G_- = V_0^{(1)} - V_{-1}^{(1)}. \quad (3.17)$$

These transformations deal naturally in the evolution of the partition function (1.9) by the time $x = t_1$. The operator $G_+ e^{H(t)} G_-$ that appears in the expression (1.31) evolves according to

$$\partial_x (G_+ e^{H(t)} G_-) = G_+ H_1 (G_+)^{-1} G_+ e^{H(t)} G_- = (V_0^{(1)} - V_1^{(1)}) G_+ e^{H(t)} G_-. \quad (3.18)$$

It can be also written as

$$\partial_x (G_+ e^{H(t)} G_-) = G_+ e^{H(t)} G_- \, (G_-)^{-1} H_1 G_- = G_+ e^{H(t)} G_- \, (V_0^{(1)} - V_{-1}^{(1)}). \quad (3.19)$$

These two descriptions lead to

$$\partial_x Z_p(t) = \langle p | (V_0^{(1)} - V_1^{(1)}) G_+ e^{H(t)} G_- | p \rangle \quad (3.20)$$

$$= \langle p | G_+ e^{H(t)} G_- (V_0^{(1)} - V_{-1}^{(1)}) | p \rangle. \quad (3.21)$$

Let us derive the Toda Eq. (3.2). Owing to the identification (3.1), it suffices to prove the following identity:

$$Z_p \, \partial_x^2 Z_p - (\partial_x Z_p)^2 = q^{2p+1} Z_{p+1} Z_{p-1}, \qquad p \in \mathbb{Z}. \quad (3.22)$$

We first rewrite the left hand side of Eq. (3.22), using the expression (1.31) and applying formulas (3.18), (3.19) in it, as follows:

$$\begin{aligned} & Z_p \, \partial_x^2 Z_p - (\partial_x Z_p)^2 \\ & = Z_p \cdot \langle p | (V_0^{(1)} - V_1^{(1)}) G_+ e^{H(t)} G_- (V_0^{(1)} - V_{-1}^{(1)}) | p \rangle \\ & \quad - \langle p | (V_0^{(1)} - V_1^{(1)}) G_+ e^{H(t)} G_- | p \rangle \cdot \langle p | G_+ e^{H(t)} G_- (V_0^{(1)} - V_{-1}^{(1)}) | p \rangle. \end{aligned} \quad (3.23)$$

The matrix elements in the right-hand side of this equation can be translated to the fermion correlation functions, by replacing $V_0^{(1)} - V_{\pm 1}^{(1)}$ with the states generated by these operators. The corresponding states are

$$(V_0^{(1)} - V_{-1}^{(1)})\,|p\rangle = q\frac{1-q^p}{1-q}\,|p\rangle + q^{p+\frac{1}{2}}\,\psi_{-p-1}\psi_p^*\,|p\rangle \tag{3.24}$$

and its conjugate state. Thereby, we can eventually translate the right-hand side of Eq. (3.23) into the following combination of the correlation functions.

$$\begin{aligned} &Z_p\,\partial_x^2 Z_p - (\partial_x Z_p)^2 \\ &= q^{2p+1} Z_p \cdot \langle p|\,\psi_{-p}\psi_{p+1}^*\; G_+ e^{H(t)} G_-\;\psi_{-p-1}\psi_p^*\,|p\rangle \\ &\quad - q^{2p+1}\langle p|\,\psi_{-p}\psi_{p+1}^*\; G_+ e^{H(t)} G_-\,|p\rangle \cdot \langle p|\,G_+ e^{H(t)} G_-\;\psi_{-p-1}\psi_p^*\,|p\rangle. \end{aligned} \tag{3.25}$$

Wick's theorem shows that correlation functions of free fermions are factorized into products of their two point functions. The four point function in the right hand side of Eq. (3.25) is factorized into

$$\begin{aligned} &\frac{1}{Z_p}\langle p|\,\psi_{-p}\psi_{p+1}^*\; G_+ e^{H(t)} G_-\;\psi_{-p-1}\psi_p^*\,|p\rangle \\ &= \frac{1}{Z_p}\langle p|\,\psi_{p+1}^*\; G_+ e^{H(t)} G_-\;\psi_{-p-1}\,|p\rangle \cdot \frac{1}{Z_p}\langle p|\,\psi_{-p}\; G_+ e^{H(t)} G_-\;\psi_p^*\,|p\rangle \\ &\quad + \frac{1}{Z_p}\langle p|\,\psi_{-p}\psi_{p+1}^*\; G_+ e^{H(t)} G_-\,|p\rangle \cdot \frac{1}{Z_p}\langle p|\,G_+ e^{H(t)} G_-\;\psi_{-p-1}\psi_p^*\,|p\rangle. \end{aligned} \tag{3.26}$$

In the right hand side of this equation, the first term equals $Z_p^{-2} Z_{p+1} Z_{p-1}$, making use of the relations $|p+1\rangle = \psi_{-p-1}|p\rangle$ and $|p-1\rangle = \psi_p^*|p\rangle$. Thus, Wick's theorem leads to

$$\begin{aligned} &Z_p \cdot \langle p|\,\psi_{-p}\psi_{p+1}^*\; G_+ e^{H(t)} G_-\;\psi_{-p-1}\psi_p^*\,|p\rangle \\ &\quad = Z_{p+1}Z_{p-1} + \langle p|\,\psi_{-p}\psi_{p+1}^*\; G_+ e^{H(t)} G_-\,|p\rangle \cdot \langle p|\,G_+ e^{H(t)} G_-\;\psi_{-p-1}\psi_p^*\,|p\rangle. \end{aligned} \tag{3.27}$$

By plugging this formula into the right hand side of Eq. (3.25), we obtain $q^{2p+1} Z_{p+1} Z_{p-1}$. Thereby this completes the proof.

4 One-Dimensional Toda Hierarchy

In this section we prove that the series of the partition functions (1.14) is a tau function of the one-dimensional Toda hierarchy.

We first recall the theory of tau functions of the Toda hierarchy (28; 29; 30). The two-dimensional Toda hierarchy has two series of commuting flows and thereby two series of time variables, $T = (T_1, T_2, \cdots)$ and $\bar{T} = (\bar{T}_1, \bar{T}_2, \cdots)$, each of which describes each the commuting flows. Tau functions of the two-dimensional Toda hierarchy are admitted to have several expressions including the realization by means of free fermions or free bosons. The standard description in terms of free fermions is

$$\tau^{2\,Toda}(T,\bar{T};p) = e^{\sum_{k=1}^{\infty}(c_k T_k + \bar{c}_k \bar{T}_k)} \langle p|\, e^{\sum_{k=1}^{\infty} T_k J_k}\, g\, e^{-\sum_{k=1}^{\infty} \bar{T}_k J_{-k}}\, |p\rangle, \qquad (4.1)$$

where g is an element of $GL(\infty)$. c_k and $\bar{c}_k$ are numerical constants which originate in the ambiguity of the tau function. The two-dimensional Toda hierarchy reduces to the one-dimensional Toda hierarchy when the two-sided time evolutions degenerate. This reduction imposes the following constraint on the tau function:

$$\Big(\frac{\partial}{\partial T_k} + \frac{\partial}{\partial \bar{T}_k}\Big)\tau^{2\,Toda}(T,\bar{T};p) = 0, \qquad k = 1,2,\cdots. \qquad (4.2)$$

When the condition is fulfilled, the tau function translates to the tau function of the one-dimensional Toda hierarchy. Owing to the degeneration, the one-dimensional Toda hierarchy has one series of commuting flows. The corresponding time variables can be identified with $T = (T_1, T_2, \cdots)$. The tau function has the following expression in terms of free fermionss:

$$\tau^{1\,Toda}(T;p) = e^{\sum_{k=1}^{\infty} c_k T_k} \langle p|\, e^{\frac{1}{2}\sum_{k=1}^{\infty} T_k J_k}\, g\, e^{\frac{1}{2}\sum_{k=1}^{\infty} T_k J_{-k}}\, |p\rangle. \qquad (4.3)$$

The coupling constants $t = (t_1, t_2, \cdots)$ in the series of the partition functions (1.14) are eventually identified with the standard Toda time variables $T = (T_1, T_2, \cdots)$ by $T_k = (-)^k t_k$.

To implement the constraint (4.2), g is chosen to satisfy the constraint

$$J_k\, g = g\, J_{-k}, \qquad k = 1,2,\cdots. \qquad (4.4)$$

Under this condition, the foregoing expression of $\tau^{1\,Toda}(T;p)$ can be rewritten as

$$\tau^{1\,Toda}(T;p) = e^{\sum_{k=1}^{\infty} c_k T_k} \langle p|\, e^{\sum_{k=1}^{\infty} T_k J_k}\, g\, |p\rangle. \qquad (4.5)$$

This shows that $\tau^{1\,Toda}(T;p)$ is a tau function of the modified KP hierarchy as well.

4.1 Shift symmetry

Among automorphisms of the Lie algebra (2.7), we pay special attention to the following shift symmetry:

$$V_m^{(k)} - \delta_{m,0}\frac{q^k}{1-q^k} \;\longmapsto\; V_{m+k}^{(k)} - \delta_{m+k,0}\frac{q^k}{1-q^k}, \qquad k \geq 1, \qquad (4.6)$$

and $V_m^{(0)}$ are left unchanged. Using this symmetry, three commutative sub-algebras generated respectively by $\{V_0^{(k)}\}_{k\geq 0}$, $\{V_k^{(k)}\}_{k\geq 0}$ and $\{V_{-k}^{(k)}\}_{k\geq 0}$ become conjugate to one another.

The symmetry (4.6) becomes eventually one of the automorphisms generated by the adjoint action of $G_\pm$. Combining the transformations (3.8),(3.9), we find

$$\begin{aligned}(G_-G_+)&\left\{:\psi(q^{\frac{k}{2}}z)\psi^*(q^{-\frac{k}{2}}z): - \frac{q^{\frac{k}{2}}}{z(1-q^k)}\right\}(G_-G_+)^{-1}\\ &= (-)^k z^k\left\{:\psi(q^{\frac{k}{2}}z)\psi^*(q^{-\frac{k}{2}}z): - \frac{q^{\frac{k}{2}}}{z(1-q^k)}\right\}. \qquad (4.7)\end{aligned}$$

Taking account of the mode expansion (2.1), we can read the symmetry (4.6) from formula (4.7) in the following form:

$$(G_-G_+)\left(V_m^{(k)} - \delta_{m,0}\frac{q^k}{1-q^k}\right)(G_-G_+)^{-1} = (-)^k\left(V_{m+k}^{(k)} - \delta_{m+k,0}\frac{q^k}{1-q^k}\right). \qquad (4.8)$$

This formula shows that the conjugacy between the three sub-algebras is realized as their transformations by $(G_-G_+)^{\pm1}$. We shall describe such transformations in some detail. Let $k \geq 1$. Putting $m = 0$ in (4.8), we obtain

$$(G_-G_+)\left(V_0^{(k)} - \frac{q^k}{1-q^k}\right)(G_-G_+)^{-1} = (-)^k V_k^{(k)}. \qquad (4.9)$$

Similarly, putting $m = -k$ in (4.8), we obtain

$$(G_-G_+)^{-1}\left(V_0^{(k)} - \frac{q^k}{1-q^k}\right)(G_-G_+) = (-)^k V_{-k}^{(k)}. \qquad (4.10)$$

Let us consider two more commutative sub-algebras which are generated respectively by $\{V_k^{(0)}\}_{k\geq0}$ and $\{V_{-k}^{(0)}\}_{k\geq0}$. These sub-algebras become conjugate to the aforementioned three sub-algebras by the adjoint actions of $(G_-G_+)^{\pm1}$ and $q^{\pm\frac{W}{2}}$. To see this, note that W (1.35) is able to rotate $V_{\pm k}^{(k)}$ to $V_{\pm k}^{(0)}$ by the adjoint action. Actually, we have the following transformations.

$$q^{\frac{W}{2}} V_k^{(k)} q^{-\frac{W}{2}} = V_k^{(0)}, \qquad (4.11)$$

$$q^{-\frac{W}{2}} V_{-k}^{(k)} q^{\frac{W}{2}} = V_{-k}^{(0)}. \qquad (4.12)$$

To obtain these formulas, note that, by the adjoint action of $q^{\frac{W}{2}}$, the fermions transform as $q^{\frac{W}{2}}\psi_m q^{-\frac{W}{2}} = q^{\frac{m^2}{2}}\psi_m$ and $q^{\frac{W}{2}}\psi_m^* q^{-\frac{W}{2}} = q^{-\frac{m^2}{2}}\psi_m^*$. By using these transformations, the left hand side of (4.11) can be computed as

$$\begin{aligned}q^{\frac{W}{2}} V_k^{(k)} q^{-\frac{W}{2}} &= q^{-\frac{k^2}{2}}\sum_{n\in\mathbb{Z}} q^{kn} q^{\frac{W}{2}} :\psi_{k-n}\psi_n^*: q^{-\frac{W}{2}}\\ &= q^{-\frac{k^2}{2}}\sum_{n\in\mathbb{Z}} q^{kn+\frac{(k-n)^2}{2}-\frac{n^2}{2}} :\psi_{k-n}\psi_n^*:\\ &= \sum_{n\in\mathbb{Z}} :\psi_{k-n}\psi_n^*:, \qquad (4.13)\end{aligned}$$

which is nothing but $V_k^{(0)}$. Thus we obtain the formula (4.11). The similar computation leads to the formula (4.12).

The formulas (4.11), (4.12) in addition to (4.9), (4.10) show that all the five sub-algebras are conjugate to one another by the adjoint actions of $(G_-G_+)^{\pm 1}$ and $q^{\pm \frac{W}{2}}$. Therefore, the hamiltonian $H_k = V_0^{(k)}$ can transform to $J_{\pm k} = V_{\pm k}^{(0)}$ by the adjoint actions of $(G_-G_+)^{\pm 1}$ and $q^{\pm \frac{W}{2}}$. Actually, as seen from formulas (4.9), (4.11), $q^{\frac{W}{2}}(G_-G_+)$ rotates H_k to J_k, while $q^{-\frac{W}{2}}(G_-G_+)^{-1}$ rotates H_k to J_{-k}, as seen from formulas (4.10), (4.12). We write down these transformations in the following form convenient for the later use:

$$G_+\left(V_0^{(k)} - \frac{q^k}{1-q^k}\right)(G_+)^{-1} = (-)^k\big(q^{\frac{W}{2}}G_-\big)^{-1}J_k\,(q^{\frac{W}{2}}G_-), \tag{4.14}$$

$$(G_-)^{-1}\left(V_0^{(k)} - \frac{q^k}{1-q^k}\right)G_- = (-)^k\big(G_+q^{\frac{W}{2}}\big)J_{-k}\,(G_+q^{\frac{W}{2}})^{-1}. \tag{4.15}$$

4.2 The proof

4.2.1 New representation of the partition functions.

We derive the expression (1.15) of the partition function (1.9).

Let us first rewrite $G_+e^{H(t)}G_-$ as follows:

$$\begin{aligned} G_+e^{H(t)}G_- &= G_+e^{\frac{1}{2}H(t)}\,e^{\frac{1}{2}H(t)}G_- \\ &= G_+e^{\frac{1}{2}H(t)}(G_+)^{-1}G_+G_-\,(G_-)^{-1}e^{\frac{1}{2}H(t)}G_-. \end{aligned} \tag{4.16}$$

The transformations of $e^{\frac{1}{2}H(t)}$ by the adjoint actions of G_+,$(G_-)^{-1}$ in this expression can be evaluated by using formulas (4.14), (4.15). Eventually, these transformations are expressed as

$$G_+\,e^{\frac{1}{2}H(t)}\,(G_+)^{-1} = e^{\sum_{k=1}^\infty \frac{t_kq^k}{2(1-q^k)}}\,(q^{\frac{W}{2}}G_-)^{-1}\,e^{\frac{1}{2}\sum_{k=1}^\infty(-)^kt_kJ_k}\,(q^{\frac{W}{2}}G_-), \tag{4.17}$$

$$(G_-)^{-1}\,e^{\frac{1}{2}H(t)}\,G_- = e^{\sum_{k=1}^\infty \frac{t_kq^k}{2(1-q^k)}}\,(G_+q^{\frac{W}{2}})\,e^{\frac{1}{2}\sum_{k=1}^\infty(-)^kt_kJ_{-k}}\,(G_+q^{\frac{W}{2}})^{-1} \tag{4.18}$$

By plugging these expressions into the right hand side of (4.16), we obtain the following formula:

$$\begin{aligned} G_+e^{H(t)}G_- &= e^{\sum_{k=1}^\infty \frac{t_kq^k}{1-q^k}}\,(q^{\frac{W}{2}}G_-)^{-1}\,e^{\frac{1}{2}\sum_{k=1}^\infty(-)^kt_kJ_k} \\ &\quad\times \mathbf{g}_\star\,e^{\frac{1}{2}\sum_{k=1}^\infty(-)^kt_kJ_{-k}}\,(G_+q^{\frac{W}{2}})^{-1}, \end{aligned} \tag{4.19}$$

where $\mathbf{g}_\star$ is the element of $GL(\infty)$ given by (1.34).

Making use of this formula we arrange the expression (1.31) as

$$\begin{aligned}
Z_p(t) &= \langle p|G_+ e^{H(t)} G_-|p\rangle \\
&= e^{\sum_{k=1}^{\infty} \frac{t_k q^k}{1-q^k}} \langle p|\,(G_-)^{-1}\, q^{-\frac{W}{2}}\; e^{\frac{1}{2}\sum_{k=1}^{\infty}(-)^k t_k J_k}\; \mathbf{g}_\star\; e^{\frac{1}{2}\sum_{k=1}^{\infty}(-)^k t_k J_{-k}}\; q^{-\frac{W}{2}}\,(G_+)^{-1}\,|p\rangle \\
&= e^{\sum_{k=1}^{\infty} \frac{t_k q^k}{1-q^k}} \langle p|\,q^{-\frac{W}{2}}\; e^{\frac{1}{2}\sum_{k=1}^{\infty}(-)^k t_k J_k}\; \mathbf{g}_\star\; e^{\frac{1}{2}\sum_{k=1}^{\infty}(-)^k t_k J_{-k}}\; q^{-\frac{W}{2}}\,|p\rangle. \qquad (4.20)
\end{aligned}$$

Considering in the last line that the W-charge of the state $|p\rangle$ is $\frac{1}{6}p(p+1)(2p+1)$, we finally obtain the expression (1.15).

4.2.2 Reduction to one-dimensional Toda hierarchy.

Let us show that $\tau(t;p) = q^{\frac{1}{6}p(p+1)(2p+1)} Z_p(t)$ is a tau function of the one-dimensional Toda hierarchy. The key is the fact that $\mathbf{g}_\star$ satisfies (4.4). This can be seen by using formulas (4.9), (4.10), (4.11), (4.12) as follows:

$$\begin{aligned}
J_k\,\mathbf{g}_\star &= J_k q^{\frac{W}{2}}(G_-G_+)^2 q^{\frac{W}{2}} \\
&= q^{\frac{W}{2}} V_k^{(k)} (G_-G_+)^2 q^{\frac{W}{2}} \\
&= q^{\frac{W}{2}} (G_-G_+)\,(-)^k \left(V_0^{(k)} - \frac{q^k}{1-q^k}\right)(G_-G_+) q^{\frac{W}{2}} \\
&= q^{\frac{W}{2}} (G_-G_+)^2 V_{-k}^{(k)} q^{\frac{W}{2}} \\
&= q^{\frac{W}{2}} (G_-G_+)^2 q^{\frac{W}{2}} J_{-k} \\
&= \mathbf{g}_\star J_{-k}. \qquad (4.21)
\end{aligned}$$

The constraint (4.4) is equivalent to the constraint (4.2) on the tau function given by (4.1) taking $g = \mathbf{g}_\star$. This means that $\mathbf{g}_\star$ actually gives a solution of the one-dimensional Toda hierarchy. Therefore it follows from (4.3) that $\tau(t;p)$ is the corresponding tau function by the identifications $T_k = (-)^k t_k$. This completes the proof.

5 Conclusion and Discussion

We investigated a melting crystal, which is known as a random plane partition, from the viewpoint of integrable systems. We proved that a series of partition functions of melting crystals give rise to a tau function of the one-dimensional Toda hierarchy, where the models are defined by adding suitable potentials, endowed with a series of coupling constants, to the standard statistical weight. We showed that these potentials are converted to a commutative sub-algebra of a quantum torus Lie algebra. Further exploiting the underlying algebraic structure, a remarkable connection between the random plane partition and the quantum torus Lie algebra was revealed. This connection substantially enabled to prove the statement. Based on the result, we briefly studied the integrable structures of five-dimensional $\mathcal{N} = 1$ supersymmetric gauge theories and A-model topological strings. The aforementioned potentials correspond to gauge theory observables analogous to the Wilson loops, and thereby the partition functions are translated in the gauge theory to generating functions of their correlators. In topological strings, we particularly comment on the possibility of topology change caused by condensation of these observables, giving a simple example.

In four-dimensional $\mathscr{N} = 2$ supersymmetric gauge theories, the authors of (10) obtained the generating function of correlation functions of the higher Casimir operators in the fermionic form

$$Z_p^{AdU(1)}(x) = \langle p| e^{\frac{1}{\hbar}J_1} e^{\sum_{k=0}^{\infty} \frac{x_k}{(k+1)!}\mathscr{P}_{k+1}} e^{\frac{1}{\hbar}J_{-1}} |p\rangle, \tag{5.1}$$

where $\mathscr{P}_k$ are the fermion bilinear forms introduced by Okounkov and Pandharipande (23), and x_k are the coupling constants of the higher Casimirs of the gauge theory. The above partition function also appears in the Gromov-Witten theory as the generating function of the absolute Gromov-Witten invariants on $\mathbb{C}P^1$ (23; 18). Before this fermionic representation was presented, Getzler had conjectured, and later proven (21), that the generating function is a tau function of the one-dimensional Toda hierarchy. Getzler's proof is, however, fairly complicated and somewhat indirect, combining the Virasoro conjecture (24) with the partial result that the Toda equation holds on the subspace $x_2 = x_3 = \cdots = 0$. It will therefore be an interesting problem to give a more direct proof on the basis of the fermionic representation. A possible scenario will be, as we have done in the five-dimensional case, to find a suitable analogue of $\mathbf{g}_\star$ and to rewrite the foregoing fermionic representation into a standard form like (4.3) and (4.5). Unfortunately, as we remarked in the end of Sect. 1.5.2, the naive four-dimensional ($R \to 0$) limit of $\mathbf{g}_\star$ itself does not work. Since the role of $\mathbf{g}_\star$ reminds us of various "dressing operators" in the work of Okounkov and Pandharipande (23; 25), a correct four-dimensional analogue of $\mathbf{g}_\star$ might be hidden therein.

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