FIELD THEORY OF RELATIVISTIC STRINGS

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§1. Introduction

Recent development of the dual resonance model is reviewed[1].

First, the quantum mechanics of a relativistic free string is presented by putting the emphasis on the gauge invariance and the Poincaré invariance. The invariances are guaranteed only when the space-time dimension D is 26 (D=10 in another model) and the Regge intercept α_0 equals 1.

Second, the second quantized formalism of the string model is demonstrated. By this we mean the theory of many pieces of strings interacting each other, which is analogous to the Dyson's formalism in the local field theory. The basic interactions between (closed and/or open) strings are determined from the continuity condition of the world sheets swept out by the string motion. The five basic interactions are shown to be necessary.

So far, the second quantized formalism is possible only in a special gauge, i.e., in the light cone gauge.

Third, we show the equivalence of the string theory to the dual resonance model. The scattering amplitude corresponding to each Feynman diagram can be mapped onto the amplitude in the dual resonance model. The five basic interactions introduced from the geometrical reason are shown to be necessary and sufficient to reproduce the covariant dual resonance model.

§2. Quantum Mechanics of a String

Classical Theory: The motion of a classical string is governed by the action

$$I = \int_\Delta \mathcal{L}\, d\sigma\, d\tau = (1/\alpha')\int_\Delta \sqrt{(\dot{X}X')^2 - \dot{X}^2 X'^2}\; d\sigma\, d\tau \qquad (2.1)$$

where $X^\mu(\tau, \sigma)$ represents a point on a world sheet swept out by a string motion in a D-dimensional space-time ($\mu=0, \cdots, D-1$). The set of parameters (σ, τ) is an arbitrary coordinate on the world sheet Δ, and $\dot{X}^\mu = \partial X^\mu/\partial\tau$, $X^{\mu\prime} = \partial X^\mu/\partial\sigma$ and $X^2 = \sum_{\mu=0}^{D-1} X^\mu X_\mu$ [Fig.1]. The integrand $\mathcal{L}\, d\sigma\, d\tau$ is the infinitesimal surface element of Δ. The action principle implies that the classical motion of string is determined in such a way that the area of world sheet is made minimum, a generalized Fermat's principle.

The action (2.1) has an invariance under the reparametrization

$$\{\sigma, \tau\} \longrightarrow \{\sigma'(\sigma,\tau), \tau'(\sigma,\tau)\} \tag{2.2}$$

where σ' and τ' are arbitrary continuous functions. In such a case, the velocities $\dot{X}^\mu(\sigma, \tau)$ cannot be solved with respect to the canonical momenta

$$P^\mu(\sigma,\tau) \equiv \frac{\partial \mathcal{L}}{\partial \dot{x}_\mu(\sigma,\tau)} . \tag{2.3}$$

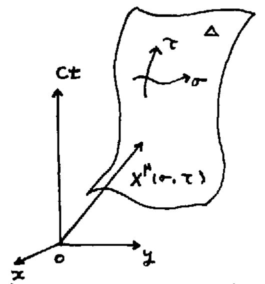


Fig.1. The world sheet.

The momenta, however, satisfy a set of "gauge conditions",

$$P^2(\sigma,\tau) + X'^2(\sigma,\tau) = 0 , \quad P(\sigma,\tau)X'(\sigma,\tau) = 0 . \tag{2.4}$$

Owing to (2.4), the mechanical system cannot be solved unless we specify a special gauge. The gauge we choose is the light cone gauge:

$$P^+ \equiv (P^0 + P^{D-1})/\sqrt{2} = 1 , \quad X^+ \equiv (X^0 + X^{D-1})/\sqrt{2} = \tau \tag{2.5}$$

which, when combined with (2.4), determine other components;

$$P^- \equiv (P^0 - P^{D-1})/\sqrt{2} = (\vec{P}^2 + \vec{X}'^2)/2 , \quad X^{-\prime} \equiv (X^{0\prime} - X^{D-1\prime})/\sqrt{2} = \vec{P}\cdot\vec{X}' . \tag{2.6}$$

Consequently, independent variables for string are the transverse components of coodinates $\vec{X}(\sigma, \tau) = (X^i(\sigma, \tau) | i=1,2,\cdots D-2)$, the zero frequency modes of light cone components $x^\pm$, and their canonical conjugates. The equation of motions, therefore, should be solved for $\vec{X}(\sigma,\tau)$ under the boundary condition

$$\vec{X}'(\sigma,\tau)\Big|_{\partial\Delta} = 0 \tag{2.7}$$

for an open string, or the periodicity condition

$$\vec{X}(\sigma,\tau) = \vec{X}(\sigma+\sigma_0, \tau) \tag{2.8}$$

for a closed string.

Quantum Theory: In going to the quantum mechanics, we first assume that $X^\pm$ except for the zero-modes, are functions of other independent quantities due to (2.5) and (2.6). The state vector which describes the string motion must be a functional of $\vec{X}$, and $x^\pm$, the zero-modes of $X^\pm(\sigma, \tau)$: $\Phi[\vec{X}, x^+, x^-]$. For later convenience we will introduce the following Fourier components of Φ with respect to x^-:

$$\Phi[\vec{X}, x^+, x^-] = \int_0^\infty dp^+ [\Phi_{p^+}[\vec{X}, x^+] e^{ip^+x^-} + c.c.] . \tag{2.9}$$

We have already imposed all the conditions (2.4)-(2.6) but the zero-mode condition of (2.6), which should be imposed as an equation of motion on Φ. In the Schödinger representation, the condition turns out to be

$$[i\,\partial/\partial x^+ - (1/2)\int_0^{\pi\alpha} d\sigma :\{-\delta^2/\delta\vec{X}^2(\sigma) + \vec{X}'^2(\sigma)\}: - \frac{\alpha_0}{\alpha}]\Phi_{p^+} = 0 , \tag{2.10}$$

where we have used

$$P^- = i\,\partial/\partial x^+ , \quad \vec{P}(\sigma) = -i\,\delta/\delta\vec{X}(\sigma) , \text{ and } \alpha \equiv 2p^+ .$$

A great advantage of the approach presented above consists in the ghost elimination. The time component coordinate $X^0(\sigma, \tau)$ has a negative norm if it is quantized covariantly. The trouble has been overcome by taking advantage of gauge freedom, because X^0 has been eliminated by (2.5), and (2.6). This method is analogous to the Coulomb gauge quantization in the usual quantum electrodynamics.

Contrary to the ghost elimination, the Lorentz covariance, or the Poincaré covariance is not manifest in the above treatment. GGRT[2] was the first who confirmed the covariance by the explicit construction of the Poincaré generators. From the geometrical meaning of $X^{\mu}(\sigma, \tau)$ and $P^{\mu}(\sigma, \tau)$ one can guess that

$$P^{\mu} = \sqrt{1/2\pi} \int_0^{\pi\alpha} P^{\mu}(\sigma)\, d\sigma$$

$$M^{\mu\nu} = 1/4 \int_0^{\pi\alpha} [\{X^{\mu}(\sigma), P^{\nu}(\sigma)\} - \{X^{\nu}(\sigma), P^{\mu}(\sigma)\}] \quad . \tag{2.11}$$

In constructing P^{μ}, and $M^{\mu\nu}$, the explicit solutions (2.5) and (2.6) has to be used with careful attention to the normal ordering for operators. Against expectation, GGRT found that the Poincaré invariance is possible only when D=26, and $\alpha_0=1$, where α_0 is the intercept of the Regge trajectory. The mass spectrum is given by

$$M^2 \equiv P_{\mu}^2 = \sum_{\ell=1}^{\infty} \ell\, \vec{a}_{\ell}^{\dagger} \vec{a}_{\ell} - 1 \quad . \tag{2.12}$$

The reason why D=26 can be trace back to the normal ordering of operators in (2.10) and (2.11) when the system is quantized. The operators $\vec{a}_{\ell}^{\dagger}$ and $\vec{a}_{\ell}$ are quantized coefficients in normal mode expansions of $\vec{X}(\sigma, \tau)$ and $\vec{P}(\sigma, \tau)$.

Finally, we note that a similar discussion is possible for a closed string with the cyclic boundary condition (2.8).

§3. The Second Quantization

In the previous section we discussed the quantum mechanics of a single string, and showed that the motion can be described in the Hilbert space spanned by x^{μ}, $\vec{X}(\sigma, \tau)$ and $\vec{P}(\sigma, \tau)$.

In order to construct a theory of interacting strings, one has to consider an infinite direct products of these Hilbert spaces:

$$\mathcal{H} = \prod_{i=1}^{\infty} \mathcal{H}_i \tag{3.1}$$

to each of which a string belongs. The purpose in this section is to formulate the perturbation theory for the many strings interacting each other.

Free Strings: Let us begin with the second quantization of interaction free strings. The wave function Φ_{p+} introduced in (2.9) now should be quantized as a field operator. The Lagrangian which provides (2.16) is given by

$$L_0 = \int_0^{\infty} dP_+ \int_0^{\pi\alpha} d\sigma \int \mathcal{D}\vec{X}\; \Phi_{p+}[X] \left\{ i\frac{\partial}{\partial x_+} - :\frac{1}{2}\left(-\delta^2/\delta\vec{X}^2(\sigma) + \vec{X}'^2(\sigma)\right): - \frac{\alpha_0}{\alpha} \right\} \Phi_{p+}[X] \quad . \tag{3.2}$$

The field Φ_{p^+}, then, has the following expansion:

$$\Phi_{p^+}[X] = (2\pi)^{-(D-2/2)} \int d\vec{p} \, \Sigma_{\{n_\ell\}} A_{\vec{p}, p^+, \{n_\ell\}} f_{\vec{p}, p^+, \{n_\ell\}}[X] \tag{3.3}$$

where the annihilation operator $A_{\vec{p}, p^+, \{n_\ell\}}$ (creation operator $A^+_{\vec{p}, p^+, \{n_\ell\}}$) destructs (creates) a whole string with excitation $\{n_\ell\}$.

The propagator for the string is, then, defined as

$$G_{p^+}[X, Y] = \langle\langle 0 | \Phi_{p^+}[X] \Phi^+_{q^+}[Y] | 0 \rangle\rangle . \tag{3.4}$$

The Fourier transformation of (3.4) turns out to be

$$G_{p^+}[P, Q] = \int \prod_{\sigma, \tau} \mathcal{D}\vec{X}(\sigma, \tau) \exp\left\{ - \int_{\Delta_p} \mathcal{L} \, d\sigma d\tau + i \int_{\tau=\tau_1} \vec{P}(\sigma)\vec{X}(\sigma, \tau_1) d\sigma + i \int_{\tau=\tau_2} \vec{Q}(\sigma) \vec{X}(\sigma, \tau_2) d\sigma \right\} \tag{3.5}$$

where

$$\mathcal{L} = (1/2)[\dot{\vec{X}}^2(\sigma, \tau) - \vec{X}'^2(\sigma, \tau)] \quad \text{and} \quad x^+ = -i\tau , \tag{3.6}$$

and the integration region Δ_p is given in Fig.2. Note that the propagator (3.4) is now expressed as a path integral on the two dimentional domain Δ_p, which can be interpreted as an image of the world sheet on a complex plane.

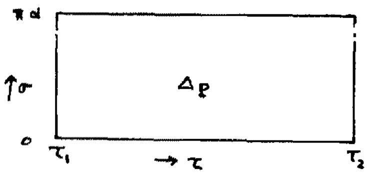


Fig.2. The propagator.

Interactions

The string picture has another advantage in determining the interactions between strings. Following geometrical observations provide us with an intuitive method to obtain the five basic interactions, which turn out to be sufficient to reproduce DRM.

(i) Suppose a piece of string $\vec{X}_1(\sigma_1, \tau)$ flies in the space time, and splits into two pieces $\vec{X}_2(\sigma_2, \tau)$ and $\vec{X}_3(\sigma_3, \tau)$ at $ix^+=\tau$. The world sheet swept out by this process will be the one shown in Fig.3. In the classical picture, the continuity of the world sheet requires that, at $ix_+=\tau$,

$$\begin{aligned} \vec{X}_1(\sigma_1) - \vec{X}_2(\sigma_2) &= 0 , \quad \text{if} \quad 0 \le \sigma_1 \le \pi\alpha_2 \\ \vec{X}_1(\sigma_1) - \vec{X}_3(\sigma_3) &= 0 , \quad \text{if} \quad \pi\alpha_2 \le \sigma_1 \le \pi(\alpha_2+\alpha_3) \end{aligned} \tag{3.7}$$

Similary, the momentum density distributed on the string must flow smoothly along the world sheet;

$$\begin{aligned} \vec{P}_1(\sigma_1) + \vec{P}_2(\sigma_2) &= 0 , \quad \text{if} \quad 0 \le \sigma_1 \le \pi\alpha_2 \\ \vec{P}_1(\sigma_1) + \vec{P}_2(\sigma_2) &= 0 , \quad \text{if} \quad \pi\alpha_2 \le \sigma_1 \le \pi(\alpha_2+\alpha_3) , \end{aligned} \tag{3.8}$$

where the direction of momentum is taken to be positive for inward.

In quantum mechanics, the above condition should be satisfied when the left hand sides of (3.7) and (3.8) are operated on the vertex function, provided $\vec{P}_i = -i\delta/\delta\vec{x}_i$. The relations are then considered to be a set of homogeneous differential equations to the vertex. The solution is unique except for a normalization constant. The formal solution turns out to be

$$L_1 = g\int \prod_{i=1}^{3} \mathcal{D}\vec{X}_i\, \Phi^{\dagger}_{p_1^+}[X_1]\,\Phi_{p_2^+}[X_2]\,\Phi_{p_3^+}[X_3] \prod_i \frac{dP_{+i}}{\sqrt{2P_{+i}}}\,\delta(P_{+1}-P_{+2}-P_{+3})$$
$$\times \prod_{\sigma} \delta[\vec{X}_1(\sigma_1) - \theta(\sigma_1-\pi\alpha_2)\vec{X}_3(\sigma_3) - \theta(\pi\alpha_2-\sigma_1)\vec{X}_2(\sigma_2)] + h.c., \quad (3.9)$$

$$\prod_{\sigma} \delta[\vec{X}_1 - \theta\vec{X}_3 - \theta\vec{X}_2] = g\lim_{\tau_i\to\tau}\int \mathcal{D}\vec{X}_1\cdots\mathcal{D}\vec{X}_3 \exp\{-\int_{\Delta_V}\mathcal{L}\,d\sigma d\tau\}\prod_{i=1}^{3}\prod_{\sigma}\delta[\vec{X}(\sigma,\tau_i)-\vec{X}_i(\sigma)] \quad . \quad (3.10)$$

The last expression is the path integral formula and $\mathcal{L}$ in the exponent is given by (3.6). The domain Δ_V is shown in Fig.3.

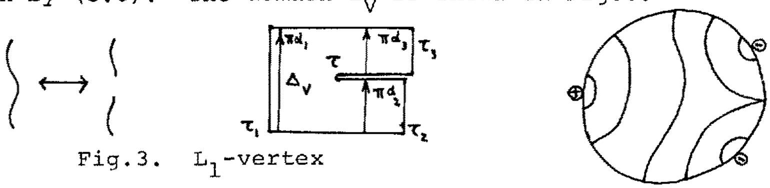


Fig.3. L_1-vertex

(ii) Another possible interaction will be the case when two pieces of strings collide at their intermediate points and undergo a rearrangement [Fig.4]. The continuity condition of the world sheet again determines the interaction Lagrangian, whose explicit form will be found in Ref.3.

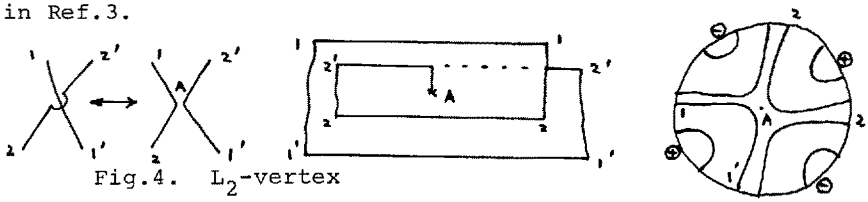


Fig.4. L_2-vertex

Assuming that the local structure of the string interactions is either the type of L_1 or L_2, we look for all other possibilities. Then, we have following three interactions.

(iii) L_3: The open-to-closed transition. Fig.5.

(iv) L_4: The open-to-an-open-and-a-closed transition. Fig.6.

(v) L_5: The closed-to-two-closed transition. Fig.7.

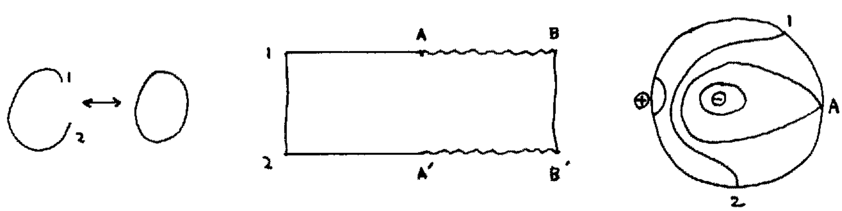


Fig.5. L_3-vertex. On the light cone sheet, AB is identified with A'B'. The equi-potential line 1-2 in the right fig. is mapped onto the vertical line 1-2 in the middle.

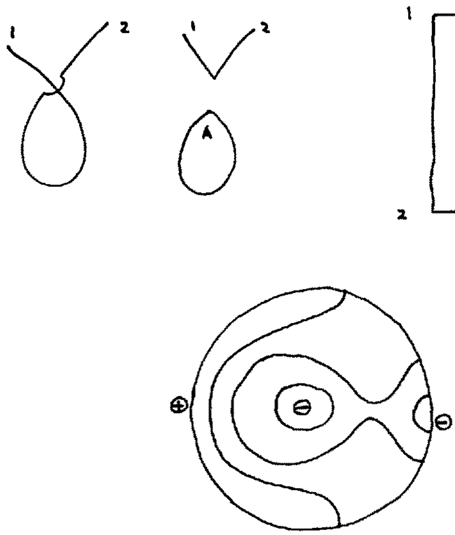


Fig.6. L_4-vertex. The two lines AB(A'B') are identified each other.

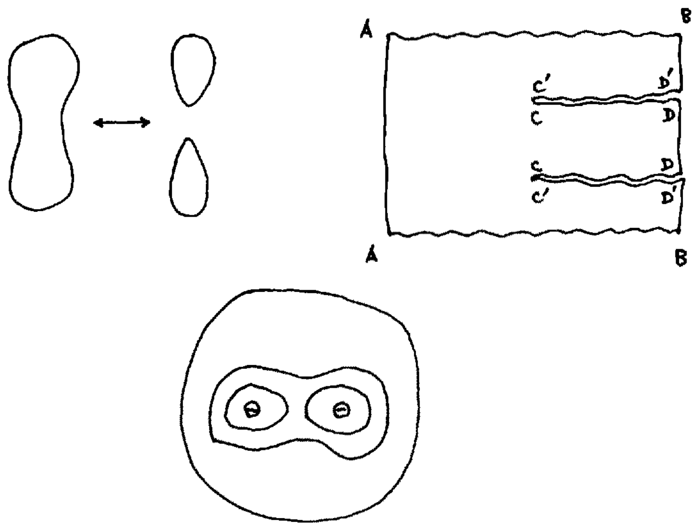


Fig.7. L_5-vertex. The two lines AB(CD and C'D') are identified so that any vertical line forms a loop in the right top fig.

Corresponding interaction Lagrangians can be written down from the continuity condition. The explicit forms are found in Ref.3.

Scattering matrix

The Lagrangian for interacting strings is now given by

$$L = L_o + L_c + \sum_{i=1}^{5} L_i \quad , \tag{3.11}$$

where L_o and L_c are the interaction free parts of the open and the closed strings, respectively.

We introduce the interaction representation operators as

$$V_i(\tau) = -e^{-H_o\tau} L_i \, e^{H_o\tau} \tag{3.12}$$

where H_o is the unperturbed Hamiltonian determined from L_o+L_c in (3.11). It will be worthwhile to point out that the interaction Lagrangians L_i in (3.11) are all local with respect to the time x^+, and with no derivative. This is the great advantage of the light cone gauge. In covariant gauges the interaction term becomes generally non-local, which destroys the possibility of using the canonical quantization method.

The transition matrix is now given by

$$U(\tau_f, \tau_i) = \sum_{n=0}^{\infty} \int_{\tau_i}^{\tau_f} d\tau_1 \int^{\tau_1} d\tau_2 \cdots \int^{\tau_{n-1}} d\tau_n \prod_{i=1}^{n} V(\tau_i) \tag{3.13}$$

where

$$V(\tau) = \sum_{i=1}^{5} V_i(\tau) \quad .$$

As is well known in the usual local field theory the Dyson-Wick contraction automatically provides us with any matrix element. The results will be summerlized as follows.

1) Each way of contractions of operators gives a light cone Feynman diagram.

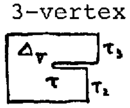
2) Associated with each internal line propagator, we obtain

$$G_{p^+}[P_1, P_2] = \boxed{\quad \Delta_P \quad} \tag{3.14}$$

where, by the rectangle, we mean the path integral over the domain Δ_P defined in (3.5).

3) Associated with each 3-vertex L_1, we obtain

$$\Gamma[P_1, P_2, P_3] = \lim_{\tau_i \to \tau} \tag{3.15}$$



where the polygon again means the path integral over the domain Δ_V. Similarly, associated with other interactions L_2-L_5, we obtain corresponding path integrals.

4) Associated with each incoming external line, we obtain (for a lowest excited state)

$$G_{p^+}[P_1, P_2]\, e^{\tau_i E_{\vec{P}, P_+, \{0\}}} = \boxed{\quad \Delta_P \quad}\, e^{\tau_i E_{\vec{P}, P_+, \{0\}}} \tag{3.16}$$

As for the outgoing external line, the Hermitian conjugate of (3.16) appears.

5) When each factor in 1)-4) are combined according to the Feynman diagram, we obtain a path integral over the region, say,

$$= e^{\sum_{j=1}^{4} \tau_j P_j^-} \int \exp\left\{-\int_\Delta \mathcal{L}\, d\sigma d\tau - i\int \sum_j \vec{X}_j(\sigma_j, \tau_j)\vec{P}_j(\sigma_j)\, d\sigma_j\right\} \mathfrak{D}\vec{X}(\sigma,\tau) \tag{3.17}$$

where we used a new notation; $P_j^-=E_j$ for incomming strings and $P_j^-=-E_j$ for outgoing strings. The second factor in (3.17) can be explicitly performed if the Newmann function for the given domain Δ is known. As for the first factor we use the following formula:

$$\begin{aligned} \tau_j &= 2\sum_i \int N_\Delta(\sigma_j, \tau_j \,;\, \sigma_i, \tau_i) P_i^+ d\sigma_i \\ &= 2 \oint_{\partial\Delta} N_\Delta(\rho, \rho') P^+(\rho') d\rho' \end{aligned} \tag{3.18}$$

where $N_\Delta(\rho, \rho')$ with $\rho=\tau+i\sigma$ is the Neumann function on the domain Δ.

6) Substituting these Neumann function expressions into (3.17), we will finally obtain a <u>formally</u> covariant expression for the scattering matrix element

$$\begin{aligned} T = \sum_{\substack{\text{over}\\ \text{contractions}}} \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 \cdots \int_{-\infty}^{\tau_{n-1}} d\tau_n \; J_\Delta(\tau_1 \cdots \tau_n) \cdot \\ \cdot \exp\left\{ \oint_{\partial\Delta} \oint P_\mu(\rho) N_\Delta(\rho, \rho') P^\mu(\rho') d\rho\, d\rho' \right\} \end{aligned} \tag{3.19}$$

where N_Δ, of course, depends on τ_i. Although we have not discussed the path integration measure factor, from which $J(\tau_1 \cdots \tau_n)$ is determined, the explicit forms are known for simple scattering diagrams. $J_\Delta(\tau_1 \cdots \tau_n)$ does depend on the space-time dimension D and the intercept α_0 also, and the theory is consistent only when D=26, and α_0=1.

§4. General Mapping

One will notice that the formula (3.19) is quite similar to the analogue model representation of DRM. In fact, a DRM amplitude is given by (3.19), provided, however, that the integration region Δ is not the light cone world sheet but that Δ is a certain simply or multiply connected compact domain in a complex z-plane. All the conformally equivalent domains are supposed to be equivalent.

What we should prove are, first, the existence of mapping of DRM amplitudes to (3.19), and second, that all the DRM amplitudes are reproduced by the combination of the five basic interactions L_1, L_2, ⋯, and L_5 introduced in §3.

We shall outline the proof below.

The mapping of the Neumann function defined above to the one in (3.19) is rather well known in fluid dynamics. Let $N_R(z, z')$ be a Neumann function for a given (simply or multiply) connected domain R. For a given source of the light cone momentum $P^+ = \frac{1}{2}(P^0 + P^{D-1})$:

$$P^+(z) = \sum_i p_{+i}\, \delta(z - z_i) \tag{4.3}$$

we can make an harmonic function

$$U_R(z) = (1/2\pi)\oint_{\partial R} N_R(z,z')P^+(z')dz' \qquad (4.4)$$

Associated with $U_R(z)$, let us define a complex harmonic function on R by

$$\rho_R = Z_R(z) = U_R(z) + iV_R(z) \qquad (4.5)$$

where V_R is a function determined by the Canchy-Riemann relations

$$\partial U_R(z)/\partial x = \partial V_R(z)/\partial y$$
$$\partial U_R(z)/\partial y = -\partial V_R(z)/\partial x\ . \qquad (4.6)$$

In the fuild dynamics U_R and V_R are known as the potential and the velocity potential, respectively.

The mapping we are looking for is given by (4.5). In fact an equi-potential line in R is mapped by (4.5) onto a vertical line in the light cone diagram. A stream line in R is mapped onto a horizontal line in the light cone diagram.

Finally, we show that DRM can be reproduced with the five basic interactions $L_1, \cdots L_5$. The necessity can be easily understood by looking the diagrams Fig.3-Fig.7. Since the mapping (4.5) transforms an equipotential line to a vertical string line on the light cone Feynman diagram, one will see the necessity of the four string interaction in Fig.5.

Other interactions are explained in similar ways. The sufficiency is given as follows. Suppose, for example, a six string interaction existed. Owing to the conformal invariance, three out of six z_i's (source position in (4.3)) can be fixed at any point. Therefore, the other threes z_4, z_5, and z_6 should be integration parameters in DRM amplitude. By the mapping the contact point P is mapped on the interaction point τ in the light cone diagram, which now is the new integration parameter. Since this point τ has only two degrees of freedom, the Jacobian for $(z_4, z_5, z_6)\to\tau$, must be zero on a non-zero measure. Since the mapping is conformal, (4.5) must be a constant. This is a contradiction.

For other string interactions, a similar argument works. The five interactions $L_1 \cdots L_5$, therefore, are sufficient to reproduce DRM.

References

1) Recent reviews for dual resonance models are, for example, J. Scherk, NYU preprint (to be published in Rev. Mod. Phys.) and S. Mandelstam "Dual-Resonance Models" Physics Report, 13C no.6 (1974).

2) J. Goldstone, P. Goddard, C. Rebbi and C. Thorn, Nucl. Phys. B56, 109 (1973).

3) M. Kaku and K. Kikkawa, Phys. Rev. 10D, 1110 (1974) and ibid. 10D, 1823 (1974).

Discussions

Matsuda: You have discussed here the mesonic strings with your five fundamental string interactions. What about the baryonic strings? In particular, how many topological interactions do you have for the latter? And also, can you bring the classical topology of the baryonic strings directory into the quantized version consistently?

Kikkawa: For the first two questions, see X. Artru, Nucl. Phys. B85, 442 (1975). As for the last, the answer is that the field theoretical formulation is not simple, because we have an infinite number of strings which are topologically different.

Takabayashi: Is your interaction theory Lorentz invariant?

Kikkawa: Yes. See S. Mandelstam, Nucl. Phys. B83, 413 (1974).