Gravitational wave luminosity and net momentum flux in head-on mergers of black holes: Radiative patterns and mode-mixing

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We show that gravitational wave radiative patterns from a point test particle falling radially into a Schwarzschild black hole, as derived by Davis, Ruffini, Press and Price[1], are present in the nonlinear regime of head-on mergers of black holes. We use the Bondi-Sachs characteristic formulation and express the gravitational wave luminosity and the radiated energy in terms of the *news* functions. Our treatment is made in the realm of Robinson-Trautman dynamics, with characteristic initial data corresponding to the head-on merger of two black holes. We consider mass ratios in the range $0.01 \leq \alpha \leq 1$. We obtain the exponential decay with ℓ of the total energy contributed by each multipole ℓ. The total rescaled radiated energy $E_W^{\rm total}/m_0\alpha^2$ decreases linearly with decreasing α, yielding for the point particle limit $\alpha \to 0$ the value $\simeq 0.0484$, about five times larger than the result of Davis et al.[1] We also analyze the mode decomposition of the net momentum flux and the associated impulse of the gravitational waves emitted, resulting in an adjacent-even-odd mode-mixing pattern with the dominant contribution coming from the mixed mode $(2, 3)$. We obtain the exponential decay with ℓ of the total gravitational wave impulse contributed by each $(\ell, \ell + 1)$ mixed mode.

Keywords: Black hole head-on mergers; Gravitational wave luminosity; Net momentum flux; Radiative patterns.

1. Introduction

The collision and merger of two black holes are among the astrophysical sources which produce gravitational waves in the strong field regime and are therefore of crucial interest for the present direct observations made by the LIGO/VIRGO consortium[2].

In the realm of general relativity the production and extraction of gravitational waves in processes involving black holes have been investigated basically within three complementary approaches, most of them connected to binary black hole inspirals: Post-Newtonian (PN) approximations[3], numerical relativity[4] and the close-limit-approximation (CLA) supplemented with PN calculations[5], as well as combinations of these approaches.

Our treatment[6] is based on the Bondi-Sachs (BS) energy-momentum conservation laws in the characteristic formulation[7], that regulate the gravitational wave radiative transfer processes of the system, in the realm of Robinson-Trautman (RT) spacetimes[8]. The characteristic initial data constructed for the RT dynamics already present a global apparent horizon so that the dynamics covers the post-merger phase of the system, which represents one of the most dynamic parts of the evolution, up to the final configuration of the remnant black hole[9].

The cornerstone of our approach is the dependence of the Bondi-Sachs net four-momentum wave flux[10] on the *news* functions – which are the basic quantities characterizing the gravitational wave degrees of freedom of the system and are, by definition, quantities of spin-weight[11] $s = -2$. This decomposition is exact in the nonlinear regime, leading to an accurate evaluation of the even parity signals and of their relative contribution to the physical quantities involved in the radiative processes of the system. All these features are discussed in the following sections.

Throughout the paper we use geometrical units $G = c = 1$.

2. Robinson-Trautman Spacetimes

Robinson-Trautman (RT) spacetimes[8] are asymptotically flat solutions of Einstein's vacuum equations that describe the exterior gravitational field of a bounded system radiating gravitational waves. The RT metric can expressed as

$$ds^2 = \left(\lambda(u,\theta,\phi) - \frac{2m_0}{r} - 2r\frac{P_{,u}}{P}\right)du^2 + 2dudr - \frac{r^2}{P^2(u,\theta,\phi)}d\Omega^2, \tag{1}$$

where r is an affine parameter defined along the shearfree null geodesics determined by the vector field $\partial/\partial r$. Here $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, where $\lambda(u,\theta,\phi)$ is the Gaussian curvature of the surfaces (u =const, r =const) defined by

$$\lambda(u,\theta,\phi) = P^2 + \frac{P^2}{\sin\theta}\left(\sin\theta\,\frac{P_{,\theta}}{P}\right)_{,\theta} + \frac{P^2}{\sin^2\theta}\left(\frac{P_{,\phi}}{P}\right)_{,\phi}. \tag{2}$$

m_0 is the only dimensional parameter of the spacetime and fixes the energy and length scales of the system. For the stationary case m_0 corresponds to the rest mass of the black hole with respect to an asymptotic Lorentz frame at the future null infinity. Einstein's equations yield

$$12m_0\ P_{,u} + P^3\left(\frac{(\lambda_{,\theta}\sin\theta)_{,\theta}}{\sin\theta} + \frac{\lambda_{,\phi\phi}}{\sin^2\theta}\right) = 0. \tag{3}$$

In the above, the subscripts u, θ and ϕ preceded by a comma denote derivatives with respect to u, θ, ϕ, respectively. Eq. (3), denoted RT equation, governs the dynamics of the gravitational field (which is totally contained in the metric function $P(u,\theta,\phi)$) and propagates the initial data $P(u_0,\theta,\phi)$ from a given initial characteristic surface $u = u_0$.

An important feature of RT spacetimes that establishes its radiative character arises from the expression of its curvature tensor that, in a suitable semi-null tetrad basis, assumes the form

$$R_{ABCD} = \frac{N_{ABCD}}{r} + \frac{III_{ABCD}}{r^2} + \frac{II_{ABCD}}{r^3}, \tag{4}$$

where the scalar quantities N_{ABCD}, III_{ABCD} and II_{ABCD} are of the algebraic type N, III and II, respectively, in the Petrov classification of the curvature tensor[12].

Considering axial symmetry, the curvature tensor components in the above basis that contribute to N_{ABCD} are $R_{0303} = -R_{0202} = -D(u,\theta)/r + \mathcal{O}(1/r^2)$ where

$$D(u,\theta,\phi) = -P^2\partial_u \left(\frac{c_{,u}}{P}\right), \quad \text{with} \quad c_{,u}(u,\theta) = \frac{1}{2}\left(\partial^2_{\theta\theta} - \cot\theta\partial_\theta\right)P(u,\theta). \tag{5}$$

The *news* function $c_{,u}$ is one of the fundamental objects in our analysis. Generally speaking, it informs the presence of gravitational waves being emitted from a given bounded source and observed at the wave zone (Petrov type N).

3. The Bondi-Sachs Conservation Laws: The Gravitational Wave Luminosity and the Net Gravitational Wave Impulse

From the supplementary vacuum Einstein equations in the BS integration scheme together with the outgoing radiation condition, the BS four-momentum conservation laws for axisymmetric RT spacetimes are given by[10]

$$\frac{dP^\mu(u)}{du} = -\frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^\pi \frac{1}{P}\, l^\mu (c_{,u})^2 \sin\theta\, d\theta, \tag{6}$$

where $P^\mu(u)$ is the BS four-momentum. In the above the four vector $l^\mu = (1,0,0,\cos\theta)$ defines the generators of the translations of the BMS group in the temporal and Cartesian z axes of an asymptotic Lorentz frame at future null infinity[13]. The luminosity of the gravitational waves emitted is given by the right-hand side of (6).

The integration in u of Eq. (6) results for $\mu = 0$ and $\mu = z$, respectively, in the total energy $E_W(u)$ and the net impulse $I^z_W(u)$ of the gravitational wave emission, given as

$$E_W(u) = \frac{1}{2}\int_{u_0}^{u} du \int_0^\pi \frac{(c_{,u})^2}{P}\sin\theta\, d\theta, \quad I^z_W(u) = \frac{1}{2}\int_{u_0}^{u} du \int_0^\pi \frac{(c_{,u})^2}{P}\cos\theta\sin\theta\, d\theta, \tag{7}$$

where u_0 is the initial time. Due to the axisymmetry, the components $I^x_W(u) = 0 = I^y_W(u)$ for all u.

4. Initial data and numerical evolution

The initial data to be used was derived in Aranha et al.[9] and that can be interpreted as representing two instantaneously Schwarzschild black holes in head-on merger along the z axis, at $u = u_0$,

$$P(u_0,\theta) = \left(\frac{\alpha_1}{\sqrt{\cosh\gamma + \cos\theta\sinh\gamma}} + \frac{\alpha_2}{\sqrt{\cosh\gamma - \cos\theta\sinh\gamma}}\right)^{-2}. \tag{8}$$

In the derivation of (8) it turns out that $\alpha = \alpha_2/\alpha_1$ is the mass ratio of the Schwarzschild masses of the initial data, as seen by an asymptotic observer. In the remaining of the paper we will take $\alpha_1 = 1$ and denote $\alpha_2 = \alpha$, the mass ratio. This data already has a single apparent horizon so that the evolution covers the post-merger regime up to the final configuration, when the gravitational wave emission ceases. For α sufficiently small the data may be considered as a perturbation of a Schwarzschild black hole in the RT dynamics.

The initial data (8) is evolved numerically via the RT equation (3), which is integrated using a Galerkin method with a Legendre polynomial projection basis space adapted to the axisymmetric RT dynamics. The implementation of the Galerkin method, as well as its accuracy and stability for long time runs, is described in details in Section V of Aranha et al.[14] Exhaustive numerical experiments show that after a sufficiently long time $u \sim u_f$ all the modal coefficients of the Galerkin expansion become constant up to 12 significant digits, corresponding to the final time of computation u_f. At u_f the gravitational wave emission is considered to effectively cease and we obtain $P(u_f, \theta)$ that, in all cases, can be approximated as

$$P(u_f, \theta) = P_f(\cosh \gamma_f + \cos\theta \sinh \gamma_f). \tag{9}$$

This final configuration corresponds to a Schwarzschild black hole boosted along the z axis, with a final boost parameter γ_f and a final Bondi rest mass m_0/P_f^3. In all cases $\gamma_f < \gamma$ and $P_f < 1$.

5. The mode decomposition of the radiative content of the gravitational wave emission: Energy patterns

We are now led to examine the total energy $E_W^{\rm total}$ carried out of the system by the gravitational waves emitted, expressed as

$$E_W^{\rm total} = \sum_{\ell \geq 2} E_{W\ell} = \sum_{\ell \geq 2} \frac{1}{4\pi} \int_{u_0}^{u_f} N_{\ell 0}^2(u) du. \tag{10}$$

where

$$N_{\ell 0}(u) = 2\pi \int_0^{\pi} \Big(\frac{c_{,u}(\theta, u)}{\sqrt{P}} \Big) \; {}_{-2}\mathcal{Y}_{\ell 0}(\theta) \sin\theta \; d\theta. \tag{11}$$

Here, $\mathcal{Y}_{\ell 0}(\theta)$ is the spin-weight spherical harmonics[11] with $s = -2$ for the axial case.

Differentiating Eq. (10) with respect to u gives the mode decomposition of the luminosity, corresponding to the exact RT equivalent of the Moncrief-Zerilli formula[15] for the radiated luminosity of a particle falling radially into a Schwarzschild black hole. For the mass ratios examined in our numerical simulations, we obtain that the radiated energy per multipole ℓ decays exponentially with ℓ, as illustrated in Fig. 1 (left) for three values of α where we display the log-linear plot the points $E_{W\ell}/m_0$ versus ℓ for $\ell = 2...6$. The best fit curve to these points corresponds to the simple exponential law

$$E_{W\ell}/m_0 = \mathcal{A} \; e^{-\mathcal{B}\ell}. \tag{12}$$

The normalized rms error between the best fit straight lines of the log-linear plots and the points is of the order of, or smaller than 0.25%. This behavior is maintained up to $\alpha = 0.7$; a complete survey is given in Aranha et al.[6]

Its is remarkable that the exponential pattern of the plots – first observed in the computation by Davis et al.[1] of the gravitational radiation from a point test particle falling radially into a Schwarzschild black hole – extends to the nonlinear regime of head-on mergers for mass ratios at least up to $\alpha = 0.7$. In this sense, for the mass ratios considered in Fig. 1 (left), the initial data (8) may be considered to actually correspond to a perturbed Schwarzschild black hole. A complete survey is given in Aranha et al.[6]

The total radiated energy $E_W^{\rm total}/m_0$, for several mass ratios up to $\alpha = 0.3$, exhibits a simple linear relation with α,

$$E_W^{\rm total}/m_0\alpha^2 = 0.143275\ \alpha + 0.048462. \qquad (13)$$

The values for $\alpha = 0.01$, not shown in Fig. 1 (left), correspond to $E_W^{\rm total}/(m_0\alpha^2) =$ 0.0500813. The straight line is the best fit of the points with a normalized rms deviation $\simeq 0.35\%$. In the point particle limit ($\alpha \to 0$) we obtain $E_W^{\rm total}/m_0\alpha^2 =$ 0.048462, about five times larger than the value ~ 0.0104 of Davis et al.[1]

6. The total impulse imparted to the system via emission of gravitational waves: The adjacent-even-odd mode mixing

In the same vein we now examine the net impulse applied to the system due to the emission of gravitational waves. Following the last section, it can be expressed by

$$I_W^z(u_f) \equiv \sum_{\ell\geq 2} I^z_{W(\ell,\ell+1)} = \sum_{\ell\geq 2} \frac{1}{2\pi}\sqrt{\frac{(\ell+3)\ (\ell-1)}{(2\ell+1)(2\ell+3)}} \int_{u_0}^{u_f} N_{\ell 0}(u) N_{\ell+1,0}(u)du \qquad (14)$$

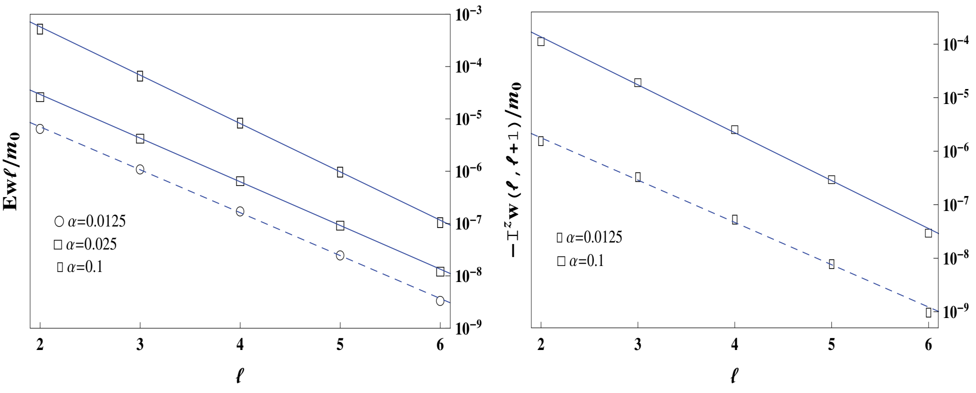

Fig. 1. *Left*: Plot of the points $E_{W\ell}/m_0$ versus ℓ, for the several mass ratios α considered. This pattern – first observed by Davis et al.[1] in the gravitational radiation of a point test particle falling radially into a Schwarzschild black hole – is seen to be still maintained in the nonlinear regime of head-on merger of two black holes. *Right*: Log-linear plots of the mixed-mode impulses $-I^z_{W(\ell,\ell+1)}/m_0$ versus ℓ, for the mass ratios $\alpha = 0.0125,\ 0.1$. The points are accurately fitted by straight lines, showing the exponential decay of the impulses with ℓ.

that corresponds to the mode decomposition of the total net impulse into adjacent-even-odd mixed modes $I^z_{W(\ell,\ell+1)}$, which can be evaluated from the $N_{\ell 0}(u)$ given as in (11). We remark that the right-hand-side of (14) can be evaluated independently, so that we may obtain the impulse per mode $I^z_{W(\ell,\ell+1)}$ and the percentage of the contribution of each $(\ell, \ell+1)$ mixed-mode to the total net impulse imparted to the system. This can be seen in Fig. 1 (right).

This mode-mixing effect for the total momentum fluxes and the associated recoil velocities was first reported by Moncrief[15] for small odd-parity axisymmetric perturbations in the Oppenheimer-Snyder collapse models, and by Lousto and Price[16] for even-parity axisymmetric perturbations on a Schwarzschild black hole by a particle falling radially.

Final Comments

The extension of this behavior for the whole mass ratio range $0 < \alpha \leq 1$ received a detailed examination in Ref. 6, both for the radiated energy per mode as well as for the mixed mode decomposition of the net momentum flux and impulse. In this reference we also examined the luminosity and the energy carried out by gravitational waves for increasing boost parameters $\gamma = [0.5, 0.6, 0.7, 0.8, 1.3]$. We obtain that the head-on mergers become more energetic as γ increases, while the time duration of the GW bursts $\Delta u/m_0 \simeq 2$ remains approximately constant with increasing γ.

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References

1. M. Davis, R. Ruffini. W. H. Press and R. H. Price, Phys. Rev. Lett. **27**, 1466 (1971).
2. LIGO detection papers at https://www.ligo.caltech.edu/page/detection-companion-papers.
3. L. Blanchet, Living Rev. Relativity **9**, 4 (2006).
4. T. W. Baumgarte and S. L. Shapiro, *Numerical Relativity: solving Einstein's equations in the computer*, (Cambridge University Press, Cambridge, 2010).
5. C.F. Sopuerta, N.Yunes, P. Laguna, Phys. Rev. D **74**, 124010 (2006) [Erratum: Phys. Rev. D **75**, 069903(E) (2007)].
6. R. F. Aranha, I. Damião Soares and E. V. Tonini, Phys. Rev. D **94**, 064017 (2016).
7. H. Bondi, M.G.J. van der Berg and A.W.K. Metzner, Proc. R. Soc. Lond. A **269**, 21 (1962).
8. I. Robinson, A. Trautman, Proc. R. Soc. Lond. A **265**, 463 (1962).

9. R. F. Aranha, I. Damião Soares and E. V. Tonini, Phys. Rev. D **81**, 104005 (2010).
10. R.F. Aranha, I. Damião Soares, E.V. Tonini, Class. Quantum Gravity **30**, 025014 (2013).
11. J. N. Goldberg, A. J. MacFarlane, E. T. Newman, F. Rohrlich and E. C. G. Sudarshan, J. Math. Phys. **8**, 2155 (1967).
12. A. Z. Petrov, *Sci. Nat. Kazan State University* **114**, 55 (1954).
13. R. K. Sachs , Phys. Rev. **128**, 2851 (1962).
14. R.F. Aranha, I. Damião Soares, E.V. Tonini. Phys. Rev. D **85**, 024003 (2012).
15. V. Moncrief, Ann. Phys. NY **88**, 323 (1974).
16. C. O. Lousto and R. H. Price, Phys. Rev. D **56**, 6439 (1997).