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Sharing quantum nonlocality under Hawking effect of a Schwarzschild black hole

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Abstract Sharing nonlocality refers to whether the post-measurement state in a Bell test can be reused to demonstrate nonlocality among multiple observers conducting sequential measurements. It has become one of fascinating and challenging topics in the past decade. In this study, we shift the nonlocality sharing scenario to near the Schwarzschild black hole to explore how Hawking radiation affects sequential nonlocality sharing. In the ideal case without Hawking radiation, Alice and Bob share a maximally entangled pure state. Bob measures his half and then passes it to a second Bob, who repeats the measurement process, and so on. Previous studies have shown that Bob can perform an infinite number of measurements to achieve nonlocality sharing with Alice. If only Alice or Bob is affected by Hawking radiation, the number of shares becomes finite and depends on the Hawking temperature. Unfortunately, when both are exposed to Hawking radiation, sharing nonlocality becomes nearly impossible. However, we can overcome this limitation by introducing auxiliary entangled sources. By adding several observers sharing an entangled state with Alice, we form a star network. Our results indicate that with these auxiliary sources, sequential network nonlocality sharing is achievable within this star network.

1 Introduction

Developing quantum technologies for space and space-related applications is of particular importance as it can support space exploration [1–4]. This includes the research and development of technologies in the areas of communication and sensing [5]. However, conducting quantum experiments in space requires considering the effects of the space environment such as extreme temperatures, magnetic fields, and cosmic radiation [1]. Black holes can emit radiation through quantum effects, now known as Hawking radiation [6]. Hawking radiation theory suggests that near the event horizon of a black hole, quantum fluctuations can cause virtual particle pairs to separate, with some particles escaping as radiation while introducing thermal noise. Previous research has predominantly concentrated on how the Hawking effect influences diverse quantum sources, including quantum steering [7], coherence [8], entanglement [9–11], Bell nonlocality [12–14], dynamical measurement's uncertainty [15], and the entropy uncertainty relation [16–19].

In the ideal scenario, to recycle Bell nonlocality, Silva et al. first introduced the nonlocality sharing scenario [20], which investigates whether the postmeasurement state in a Bell experiment can preserve its nonlocality after sequential measurements by an observer. The process is schematically illustrated in Fig. 3. They showed that at most two sequential observers (Bobs) could share nonlocality with a single Alice through weak measurements. Since then, sharing Bell nonlocality has been extensively studied [20–26]. Brown et al. provided the conditions for infinite sharing of nonlocality when the initial state is a maximally entangled pure state [23]. Fei et al. extended this to high-dimensional shared initial states, offering conditions for infinite sharing [24]. Yang et al. investigated the impact of noise on sharing [25]. Subsequently, it has also been shown that standard projective measurements can be employed for nonlocality sharing [27–29]. Researchers have also experimentally demonstrated that nonlocality sharing is achievable using weak measurements [26]. However, none of the aforementioned investigates have taken Hawking radiation into account.

In this study, we shift our attention to the influence of Hawking radiation on a sequential nonlocality sharing scenario, as illustrated in Fig. 3. Initially, we consider the case

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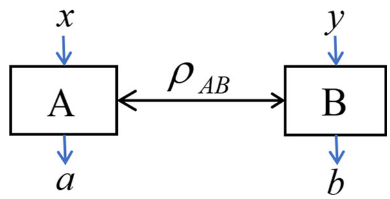


Fig. 1 Bell scenario. Alice (A) and Bob (B) share a bipartite state ρ_{AB}, where x and y represent the inputs, and a and b represent the corresponding outcomes

where only one party (either Alice or Bob) freely falls toward the event horizon. The study finds that Hawking radiation limits the number of shares. Specifically, when the Hawking temperature is high, the initial state shared by Alice and Bob may become local. Subsequently, we examine the scenario where both Alice and Bob are affected by the black hole's event horizon. When the Hawking temperature is within a certain range, nonlocality sharing between Alice and Bob can occur at most once. To overcome this limitation, we introduce additional sources to form a star network. Upon analyzing the calculation results, we find that the more branches the star network has, the more times Bob can perform measurements while still preserving nonlocality.

2 Preliminaries

2.1 Detection of nonlocality

In this subsection, we review the results on the Bell scenario shown in Fig. 1. The quantum nonlocality of a bipartite state ρ_{AB} can be proved by violating Clauser–Horne–Shimony–Holt (CHSH) Bell inequality [30]

$$S_{\mathrm{CHSH}} := \frac{1}{2}(\langle A_0B_0\rangle + \langle A_0B_1\rangle + \langle A_1B_0\rangle - \langle A_1B_1\rangle) \leq 1, \tag{1}$$

where $\langle A_xB_y\rangle = \sum_{a,b}(-1)^{a+b}p(a,b|x,y)$ represents the expectation value of the observations of Alice and Bob when their inputs are x and y ($x,y \in \{0,1\}$), and outcomes are a and b ($a,b \in \{0,1\}$), respectively. Here, A_i and B_i are observables satisfying $r(A_i) \leq 1$ and $r(B_i) \leq 1$ for $i = 0, 1$, and $r(A)$ denotes the spectral radius of matrix A. The probability distributions are given by

$$p(a,b|x,y) = \mathrm{tr}[(A_{a|x} \otimes B_{b|y})\rho_{AB}], \tag{2}$$

where $\sum_a(-1)^aA_{a|x} = A_x$ and $\sum_b(-1)^bB_{b|y} = B_y$ for the Alice's measurement set $\{A_{a|x}\}_{a,x}$ and the Bob's one $\{B_{b|y}\}_{b,y}$.

It is worth noting that $S_{AB}^{\max} = \sqrt{\tau_1^{AB} + \tau_2^{AB}}$ represents the maximal CHSH value S_{CHSH} for the state ρ_{AB},

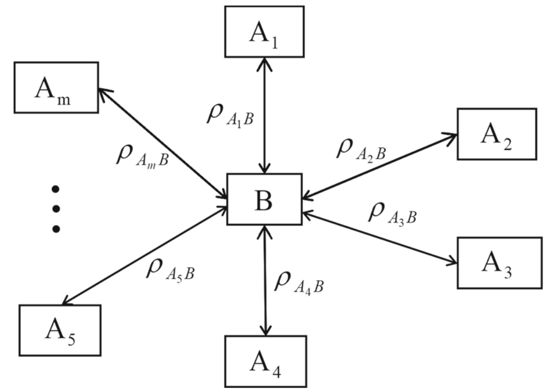


Fig. 2 Star network scenario with m sources ($\rho_{A_1B}, \rho_{A_2B}, \ldots, \rho_{A_mB}$) and $m+1$ parties ($A_1, A_2, \ldots, A_m$, B)

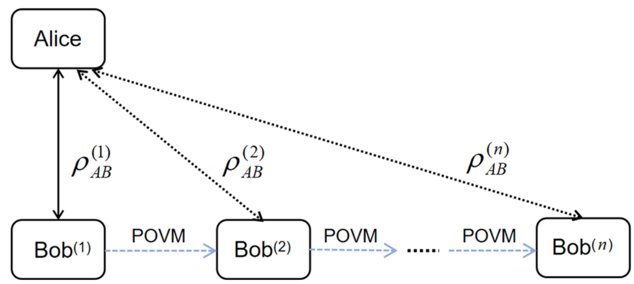


Fig. 3 A bipartite entangled state $\rho_{AB}^{(1)}$ is initially shared between Alice and Bob$^{(1)}$. Bob$^{(1)}$ performs a positive operator-valued measurement (POVM) on his part. Denote by $\rho_{AB}^{(2)}$ the postmeasurement state. Bob$^{(2)}$ also performs a POVM, and this process continues until the Bob$^{(n)}$. One want to ask how many Bobs can share the nonlocality with Alice at most

as determined by the Horodecki criterion [31]. Here, τ_1^{AB} and τ_2^{AB} denote the two larger non-negative eigenvalues of $(T(\rho_{AB}))^{\dagger}T(\rho_{AB})$, where $T(\rho_{AB})$ is the correlation matrix of ρ_{AB}.

References [32,33] have also investigated the nonlocal correlations generated by the star network shown in Fig. 2. The network state is given by $\rho_{\mathrm{net}} = \rho_{A_1B} \otimes \rho_{A_2B} \otimes \cdots \otimes \rho_{A_mB}$. Reference [33] has demonstrated that when inequality

$$S_{\mathrm{starnet}}^{\max} = \sqrt{\left(\prod_{i=1}^{m}\tau_1^{A_iB}\right)^{\frac{1}{m}} + \left(\prod_{i=1}^{m}\tau_2^{A_iB}\right)^{\frac{1}{m}}} > 1 \tag{3}$$

is satisfied, the correlations generated by the network state in the star network are nonlocal. Here, $\tau_1^{A_iB}$ and $\tau_2^{A_iB}$ denote the two larger non-negative eigenvalues of $(T(\rho_{A_iB}))^{\dagger}T(\rho_{A_iB})$.

2.2 Nonlocality sharing scenario

The sequential nonlocality sharing scenario is depicted in Fig. 3. Initially, Alice and Bob$^{(1)}$ share a 2-qubit entangled state $\rho_{AB}^{(1)}$. When Bob$^{(1)}$ performs the measurements determined by a binary input $Y^{(1)} = y$, resulting in a binary outcome $B^{(1)} = b$, the postmeasurement state between Alice and Bob$^{(2)}$ is given by the Lüders rule [34] i.e.,

$$\rho_{AB}^{(2)} = \frac{1}{2}\sum_{b,y}(I_2 \otimes \sqrt{B_{b|y}^{(1)}})\rho_{AB}^{(1)}(I_2 \otimes \sqrt{B_{b|y}^{(1)}}), \tag{4}$$

where $B_{b|y}^{(1)}$ is the positive operator-valued measurement (POVM) corresponding to outcome b for Bob$^{(1)}$'s measurement with input y. Repeating this process, the state $\rho_{AB^{(k)}}$ shared by Alice and Bob$^{(k)}$ can be computed as

$$\rho_{AB}^{(k)} = \frac{1}{2}\sum_{b,y}(I_2 \otimes \sqrt{B_{b|y}^{(k-1)}})\rho_{AB}^{(k-1)}(I_2 \otimes \sqrt{B_{b|y}^{(k-1)}}), \tag{5}$$

where $B_{b|y}^{(k-1)}$ is the POVM for the outcomes b of Bob$^{(k-1)}$'s measurements with input y.

In the sharing scenario, the primary issue is to determine the maximum number of measurements Bob can perform while still ensuring the persistence of nonlocality. When $\rho_{AB}^{(1)} = |\phi^+\rangle\langle\phi^+|$ ($|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$), the POVM strategies introduced in Ref. [34] allow arbitrarily many independent Bobs to share the nonlocality with the single Alice. These POVM strategies take the following forms

$$B_{0|0}^{(k)} := \frac{1}{2}(I_2 + \sigma_3), \tag{6}$$

$$B_{0|1}^{(k)} := \frac{1}{2}(I_2 + \gamma_k\sigma_1), \tag{7}$$

where σ_j $(j = 1, 2, 3)$ are Pauli matrices, $\theta \in (0, \pi/4]$ and $\eta_k \in (0, 1)$ for any $k = 1, 2, \ldots, n$.

2.3 Dirac fields in the Schwarzschild space-time

In this analysis, we select a fermionic initial state to align with recent quantum correlation studies in relativistic contexts that often focus on Dirac particles [7,8,35]. For simplicity, we set the gravitational constant G, the Planck constant $\hbar$, the speed of light c, and the Boltzmann constant $k_B = 1$ in our subsequent discussion.

To define the vacuum state of the curved space-time for fermions, one can begin with the following Dirac equation:

$$(i\gamma^a e_a^\mu D_\mu - m)\Phi = 0, \tag{8}$$

where m is the fermion mass, γ^a are the Dirac matrices, e_a^μ is *vierbein*, $D_\mu = \partial_\mu - \frac{i}{4}w_\mu^{ab}\sigma_{ab}$, $\sigma_{ab} = \frac{i}{2}\{\gamma_a, \gamma_b\}$, w_μ^{ab} is the spin connection, and Φ represents a spinor field.

Consider the Schwarzschild black hole metric:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{9}$$

Here, r is the black hole's radius, and M is its mass. The Dirac equation solutions in regions I (the Universe, physically accessible) and II (inside the black hole, physically inaccessible) are [36]:

$$\Phi_{\rm k}^{I+} = \xi e^{-iwu}, \quad \Phi_{\rm k}^{II+} = \xi e^{iwu}, \tag{10}$$

where k is the wave vector used to label the modes, ξ is a four-component Dirac spinor formed from spinorial spherical harmonics, w is the monochromatic frequency of the Dirac field, $u = t - r_*$ with the tortoise coordinate $r_* = r + 2M \ln\frac{r-2M}{2M}$.

To construct a complete basis for analytic modes with positive energy, we employ Kruskal coordinates to perform analytical continuation following the Damour–Ruffini method [37]. The resulting Dirac fields are expanded in the appropriate Kruskal basis as follows:

$$\Phi = \int d{\rm k}\frac{1}{\sqrt{2\cosh(4\pi Mw)}} \times [c_{\rm k}^I\Phi_{\rm k}^{I+} + c_{\rm k}^{II}\Phi_{\rm k}^{II+} + d_{\rm k}^{I\dagger}\Phi_{\rm k}^{I-} + d_{\rm k}^{II\dagger}\Phi_{\rm k}^{II-}], \tag{11}$$

where $c_{\rm k}^{\bf I}$ and $d_{\rm k}^{{\bf I}\dagger}$ with ${\bf I} = (I, II)$ are the fermion annihilation operators and antifermion creation operators acting on the Kruskal vacuum. The $\{+, -\}$ superscripts on the kets indicate the particle and antiparticle vacua, respectively.

The Bogoliubov transformation can be used to connect the operators in black hole and Kruskal space-time [38]. Specifically, the vacuum and excited states in black hole coordinates correspond to the Kruskal two-mode squeezed states:

$$\begin{aligned} |0\rangle_{\rm k}^+ &\to \lambda_-|0_{\rm k}\rangle_I^+|0_{-\rm k}\rangle_{II}^- + \lambda_+|1_{\rm k}\rangle_I^+|1_{-\rm k}\rangle_{II}^-, \\ |1\rangle_{\rm k}^+ &\to |1_{\rm k}\rangle_I^+|0_{-\rm k}\rangle_{II}^-, \end{aligned} \tag{12}$$

with the Bogoliubov coefficients $\lambda_\pm = (e^{\pm\frac{w}{T}} + 1)^{-\frac{1}{2}}$, where $T = \frac{1}{8\pi M}$ is the Hawking temperature [39]. For simplicity, we take $\omega = 1$, $|n_{\rm k}\rangle_I^+ = |n\rangle_I$ and $|n_{-\rm k}\rangle_{II}^- = |n\rangle_{II}$.

3 Nonlocality sharing near a Schwarzschild black hole

This section investigates a specific nonlocality sharing scenario where the initial state is $\rho_{AB}^{(1)} = |\phi^+\rangle\langle\phi^+|$ ($|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$). The analysis focuses on three cases: (i) the case where Bob traverses near the event horizon of the black hole as shown in Fig. 4a, (ii) the case where Alice traverses near the event horizon of the black hole as shown in Fig. 4b, and (iii) the case where both Alice and Bob traverse the event horizon of the black hole as shown in Fig. 4c.

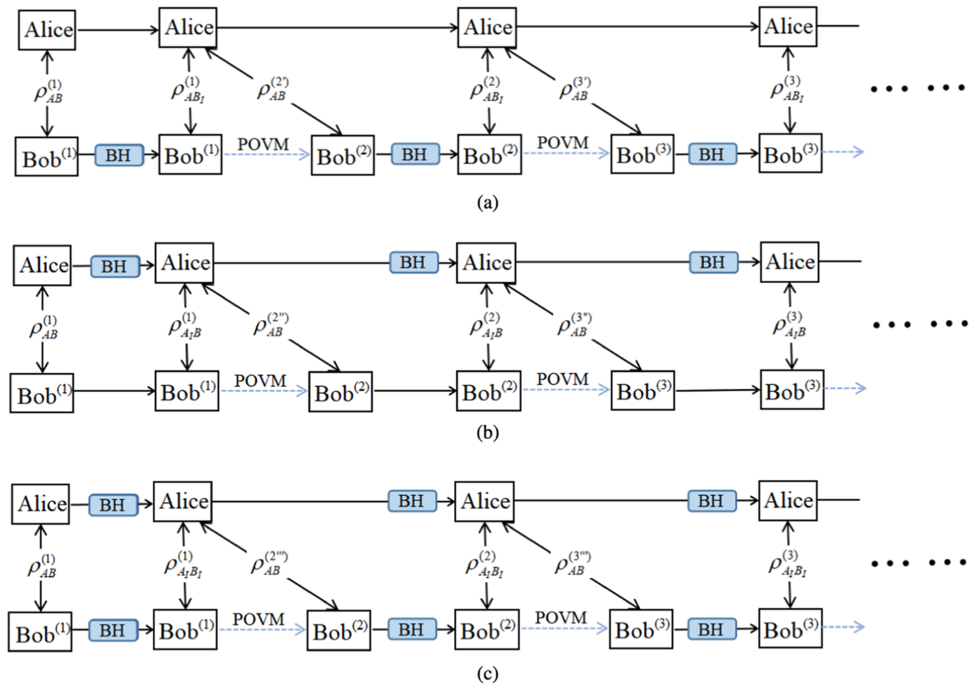


Fig. 4 Schematic of the nonlocality sharing scenario under the influence of a Schwarzschild black hole. **a** The particle of Alice is located outside the black hole (briefly BH), while the particle of Bob has traversed the event horizon of the BH. **b** The particle of Bob is located outside the BH, while the particle of Alice has traversed the event horizon of the BH. **c** Both particles have traversed the event horizon of the BH

3.1 The black hole mode for Bob

For the sharing scenario depicted in Fig. 4a, we use the Kruskal basis shown in Eq. (12) for Bob, while keeping Alice stationary. This allows us to reformulate the complete tripartite quantum state $\rho^{(1)}_{AB_IB_{II}}$, which involves subsystems A and B_I (observed by Alice and Bob, respectively) and subsystem B_{II} (observed by anti-Bob inside the black hole). Specifically, we can obtain

$$\rho^{(1)}_{AB_IB_{II}} = |\psi_{AB_IB_{II}}\rangle\langle\psi_{AB_IB_{II}}|, \tag{13}$$

where

$$|\psi_{AB_IB_{II}}\rangle = \frac{1}{\sqrt{2}}(\lambda_-|000\rangle + \lambda_+|011\rangle + |110\rangle) \tag{14}$$

with $\lambda_\pm = (e^{\pm\frac{w}{T}} + 1)^{-\frac{1}{2}}$ and $T = \frac{1}{8\pi M}$.

Since the black hole's interior is causally isolated from its exterior, Alice and Bob cannot access the modes within it. By tracing out the inaccessible mode B_{II}, we derive the reduced density matrix

$$\begin{aligned}\rho^{(1)}_{AB_I} =& \frac{1}{2}(\lambda_-^2|00\rangle\langle 00| + \lambda_-|00\rangle\langle 11| \\ &+ \lambda_+^2|01\rangle\langle 01| + \lambda_-|11\rangle\langle 00| + |11\rangle\langle 11|) \\ =& \frac{1}{4}(I_2\otimes I_2 + \frac{\lambda_-^2-\lambda_+^2-1}{2}I_2\otimes\sigma_z + \lambda_-\sigma_x\otimes\sigma_x \\ &- \lambda_-\sigma_y\otimes\sigma_y + \frac{\lambda_-^2-\lambda_+^2+1}{2}\sigma_z\otimes\sigma_z)\end{aligned} \tag{15}$$

Considering the scenario where Bob$^{(1)}$ performs the measurements shown in Eqs. (6) and (7), specifically $B^{(1)}_{0|0} := \frac{1}{2}(I_2+\sigma_3)$ and $B^{(1)}_{0|1} := \frac{1}{2}(I_2+\gamma_1\sigma_1)$. It follows from Eq. (4) that the form of the post-measurement state is

$$\begin{aligned}\rho^{(2')}_{AB} =& \frac{2+\sqrt{1-\gamma_1^2}}{4}\rho^{(1)}_{AB_I} + \frac{1}{4}(I_2\otimes\sigma_z)\rho^{(1)}_{AB_I}(I_2\otimes\sigma_z) \\ &+ \frac{1-\sqrt{1-\gamma_1^2}}{4}(I_2\otimes\sigma_x)\rho^{(1)}_{AB_I}(I_2\otimes\sigma_x) \\ =& \frac{1}{4}(I_2\otimes I_2 + b_3^{(2)}I_2\otimes\sigma_z + t_1^{(2)}\sigma_x\otimes\sigma_x \\ &+ t_2^{(2)}\sigma_y\otimes\sigma_y + t_3^{(2)}\sigma_z\otimes\sigma_z)\end{aligned} \tag{16}$$

with $b_3^{(2)} = \frac{1+\sqrt{1-\gamma_1^2}}{2}\frac{\lambda_-^2-\lambda_+^2-1}{2}$, $t_1^{(2)} = \frac{\lambda_-}{2}$, $t_2^{(2)} = -\lambda_-\frac{\sqrt{1-\gamma_1^2}}{2}$, and $t_3^{(2)} = \lambda_-^2\frac{1+\sqrt{1-\gamma_1^2}}{2}$.

Since Bob is still near the event horizon at this time, Eq. (12) can still be used to obtain $\rho^{(2)}_{AB_IB_{II}}$. By tracing out the inaccessible mode B_{II}, we derive the reduced density matrix

$$\begin{aligned}\rho^{(2)}_{AB_I} =& \frac{1}{4}(I_2\otimes I_2 + I_2\otimes(b_3^{(2)}\lambda_-^2 - \lambda_+^2)\sigma_z \\ &+ t_1^{(2)}\lambda_-\sigma_x\otimes\sigma_x + t_2^{(2)}\lambda_-\sigma_y\otimes\sigma_y \\ &+ t_3^{(2)}\lambda_-^2\sigma_z\otimes\sigma_z).\end{aligned} \tag{17}$$

Continuing with the above steps, Bob performs measurements in the form of Eqs. (6) and (7) successively. And by taking the partial trace over the modes inside the black hole, we can obtain the correlation matrix of the quantum state after Bob has performed $n-1$ measurements as

$$T(\rho_{AB_I}^{(n)}) = \begin{pmatrix} \left(\frac{\lambda_-}{2}\right)^{n-1} & 0 & 0 \\ 0 & -F_n\left(\frac{\lambda_-}{2}\right)^{n-1} & 0 \\ 0 & 0 & G_n\left(\frac{\lambda_-^2}{2}\right)^{n-1} \end{pmatrix}, \quad (18)$$

where $F_n = \prod_{i=1}^{n-1}\sqrt{1-\gamma_i^2}$ and $G_n = \prod_{i=1}^{n-1}(1+\sqrt{1-\gamma_i^2})$. Here $n \geq 2$.

By using the form of the correlation matrix of $\rho_{AB_I}^{(n)}$ shown in Eq. (18), we can obtain its maximum CHSH value as

$$S_{AB_I^n}^{\max} = \sqrt{\left(\frac{\lambda_-}{2}\right)^{2n-2} + \max\left\{F_n^2\left(\frac{\lambda_-}{2}\right)^{2n-2}, G_n^2\left(\frac{\lambda_-^2}{2}\right)^{2n-2}\right\}}. \quad (19)$$

For the sake of analysis, it is assumed that the measurements performed by each Bob are identical, i.e., $\gamma_1 = \gamma_2 = \cdots = \gamma_{n-1} = \gamma$. In this case, the quantum state that is formed after $n-1$ measurements can be specifically characterized as

$$S_{AB_I^n}^{\max} = \sqrt{\left(\frac{\lambda_-}{2}\right)^{2n-2} + Q(n,\gamma)}, \quad (20)$$

where

$$Q(n,\gamma) = \max\left\{\tilde{F}_n\left(\frac{\lambda_-}{2}\right)^{2n-2}, \tilde{G}_n\left(\frac{\lambda_-^2}{2}\right)^{2n-2}\right\} \quad (21)$$

with $\lambda_- = (e^{-\frac{1}{T}}+1)^{-\frac{1}{2}}$, $\tilde{F}_n = (1-\gamma^2)^{n-1}$ and $\tilde{G}_n = (1+\sqrt{1-\gamma^2})^{2n-2}$.

To obtain the maximum number of shares, we calculate the maximum CHSH value $S_{AB_I^n}^{\max}$ for the quantum state $\rho_{AB_I}^{(n)}$ with $n = 2, 3, \ldots, 6$. These values are shown in Fig. 5. As the Hawking temperature T increases, quantum nonlocality diminishes. From Fig. 6, we observe that a smaller γ in Bob's measurement allows for more nonlocality shares. When γ is large, nonlocality disappears after Bob's second measurement.

3.2 The black hole mode for Alice

In the sharing scenario shown in Fig. 4b, only Bob performs measurements successively. This section considers the case where Alice, who does not perform measurements successively, is near the event horizon of the black hole and provides the conditions for sharing nonlocal correlations n times. Using the Eq. (12), we get

$$\rho_{A_IA_{II}B}^{(1)} = |\psi_{A_IA_{II}B}\rangle\langle\psi_{A_IA_{II}B}|, \quad (22)$$

where

$$|\psi_{AB_IB_{II}}\rangle = \frac{1}{\sqrt{2}}(\lambda_-|000\rangle + \lambda_+|110\rangle + |101\rangle) \quad (23)$$

with $\lambda_\pm = (e^{\pm\frac{w}{T}}+1)^{-\frac{1}{2}}$ and $T = \frac{1}{8\pi M}$. At this point, the subsystems A_I and B, observed by Alice and Bob, are considered, along with the subsystem A_{II}, which is observed by anti-Alice inside the black hole. Given that the interior of the black hole is causally disconnected from its exterior, thus the physically accessible quantum state is

$$\begin{aligned}\rho_{A_IB}^{(1)} =& \mathrm{tr}_{A_{II}}(\rho_{A_IA_{II}B}^{(1)}) \\ =& \frac{1}{2}(\lambda_-^2|00\rangle\langle 00| + \lambda_-|00\rangle\langle 11| \\ &+ \lambda_+^2|10\rangle\langle 10| + \lambda_-|11\rangle\langle 00| + |11\rangle\langle 11|) \\ =& \frac{1}{4}(I_2\otimes I_2 + \frac{\lambda_-^2-\lambda_+^2-1}{2}\sigma_z\otimes I_2 + \lambda_-\sigma_x\otimes\sigma_x \\ &- \lambda_-\sigma_y\otimes\sigma_y + \frac{\lambda_-^2-\lambda_+^2+1}{2}\sigma_z\otimes\sigma_z)\end{aligned} \quad (24)$$

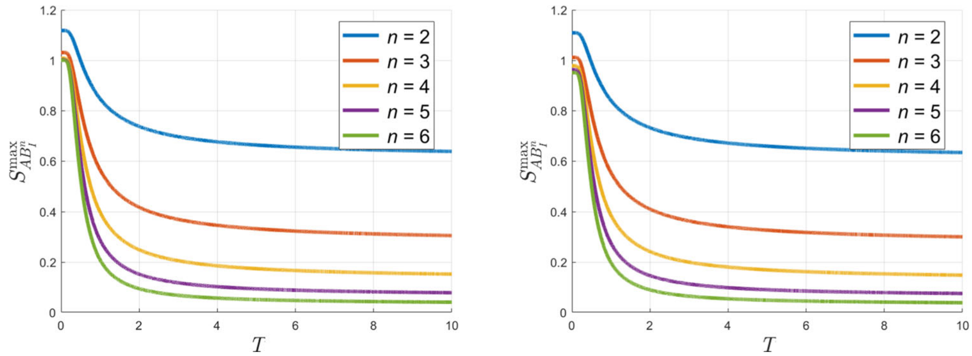
After Bob performs the measurement $B_{0|0}^{(1)} := \frac{1}{2}(I_2+\sigma_3)$ and $B_{0|1}^{(1)} := \frac{1}{2}(I_2+\gamma_1\sigma_1)$, the post-measurement state takes the form

$$\begin{aligned}\rho_{AB}^{(2'')} =& \frac{2+\sqrt{1-\gamma_1^2}}{4}\rho_{A_IB}^{(1)} + \frac{1}{4}(I_2\otimes\sigma_z)\rho_{A_IB}^{(1)}(I_2\otimes\sigma_z) \\ &+ \frac{1-\sqrt{1-\gamma_1^2}}{4}(I_2\otimes\sigma_x)\rho_{A_IB}^{(1)}(I_2\otimes\sigma_x) \\ =& \frac{1}{4}(I_2\otimes I_2 + a_3^{(2)}\sigma_z\otimes I_2 + t_1^{(2)}\sigma_x\otimes\sigma_x \\ &+ t_2^{(2)}\sigma_y\otimes\sigma_y + t_3^{(2)}\sigma_z\otimes\sigma_z)\end{aligned} \quad (25)$$

with $a_3^{(2)} = \frac{\lambda_-^2-\lambda_+^2+1}{2}$, $t_1^{(2)} = \frac{\lambda_-}{2}$, $t_2^{(2)} = -\lambda_-\frac{\sqrt{1-\gamma_1^2}}{2}$, and $t_3^{(2)} = \lambda_-^2\frac{1+\sqrt{1-\gamma_1^2}}{2}$. Since Alice is near the event horizon of the black hole, by using Eq. (12), we can obtain the form of $\rho_{A_IA_{II}B}^{(2)}$ and

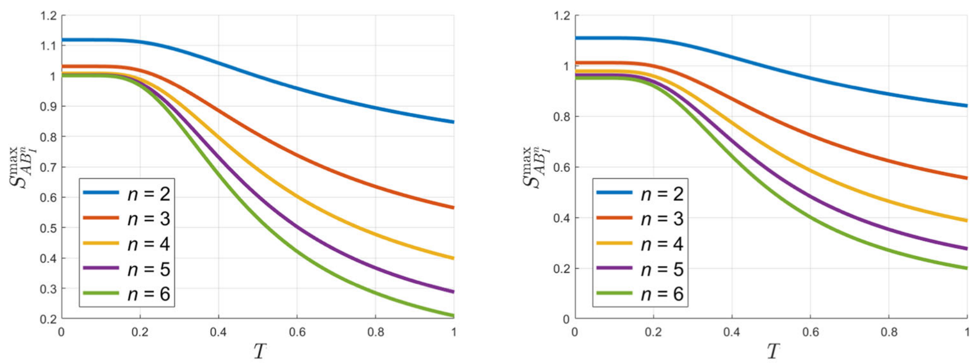
$$\begin{aligned}\rho_{A_IB}^{(2)} =& \mathrm{tr}_{A_{II}}(\rho_{A_IA_{II}B}^{(2)}) \\ =& \frac{1}{4}(I_2\otimes I_2 + (a_3^{(2)}\lambda_-^2 - \lambda_+^2)\sigma_z\otimes I_2\end{aligned}$$

Fig. 5 The x-axis represents the Hawking temperature T, and the y-axis represents the value of $S_{AB_I^n}^{\max}$

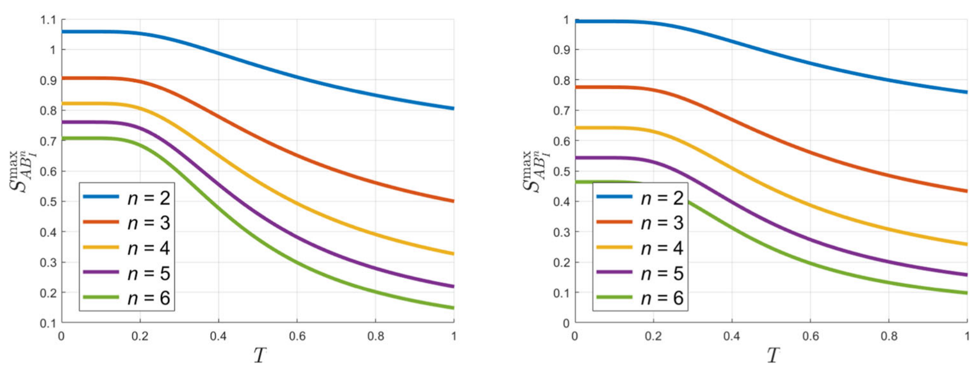


(a) $\gamma = 0.002$ (b) $\gamma = 0.2$

Fig. 6 The x-axis represents the Hawking temperature T, and the y-axis represents the value of $S_{AB_I^n}^{\max}$



(a) $\gamma = 0.002$ (b) $\gamma = 0.2$



(c) $\gamma = 0.5$ (d) $\gamma = 0.7$

$$+ t_1^{(2)}\lambda_-\sigma_x \otimes \sigma_x + t_2^{(2)}\lambda_-\sigma_y \otimes \sigma_y + t_3^{(2)}\lambda_-^2\sigma_z \otimes \sigma_z). \tag{26}$$

Following this procedure, we can always obtain the state $\rho_{AB}^{(n'')}$ after Bob has performed $n-1$ measurements, as well as the physically accessible $\rho_{A_I B}^{(n)}$. The correlation matrix of $\rho_{A_I B}^{(n)}$ can be written as

$$T(\rho_{A_I B}^{(n)}) = \begin{pmatrix} \left(\frac{\lambda_-}{2}\right)^{n-1} & 0 & 0 \\ 0 & -F_n\left(\frac{\lambda_-}{2}\right)^{n-1} & 0 \\ 0 & 0 & G_n\left(\frac{\lambda_-^2}{2}\right)^{n-1} \end{pmatrix}, \tag{27}$$

where $F_n = \prod_{i=1}^{n-1}\sqrt{1-\gamma_i^2}$ and $G_n = \prod_{i=1}^{n-1}(1+\sqrt{1-\gamma_i^2})$. Clearly, we can observe that $T(\rho_{AB_I}^{(n)}) = T(\rho_{A_I B}^{(n)})$.

By comparing the forms of Eqs. (18) and (27), it is evident that the correlation matrix of the postmeasurement state $\rho_{AB_I}^{(n)}$

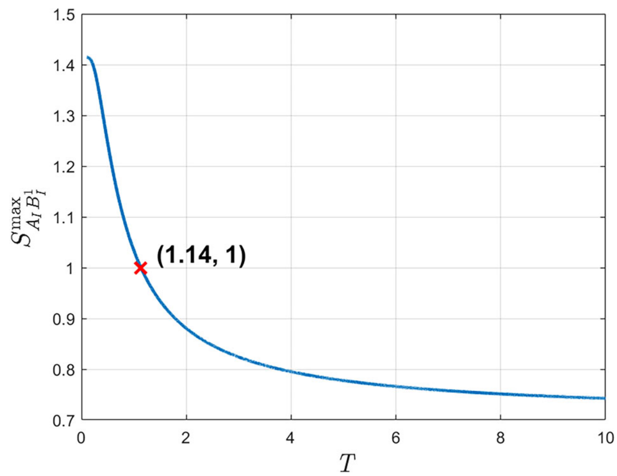


Fig. 7 The x-axis represents the Hawking temperature T, and the y-axis represents the value of $S^{\max}_{A_I B_I^1}$

and $\rho^{(n)}_{A_I B}$ remains unchanged, regardless of whether Alice or Bob is near the event horizon, as long as only Bob performs sequential measurements in the sharing scenario. Consequently, the analysis of Figs. 5 and 6 can still be employed to illustrate nonlocality sharing. We can draw the conclusion that the number of nonlocality sharing increases when the Hawking temperature is low and the measurement parameter γ decreases.

3.3 The black hole mode for both Alice and Bob

Assume that both Alice and Bob traverse the event horizon of the black hole in sharing scenario. The schematic diagram is shown in Fig. 4c. Applying the transformation of Eq. (12) to both Alice and Bob, we derive a four-partite state $\rho^{(1)}_{A_I A_{II} B_I B_{II}}$. This state involves modes A_I and B_I observed by Alice and Bob, and modes A_{II} and B_{II} observed by anti-Alice and anti-Bob in the interior of a black hole. Tracing over the inaccessible modes A_{II} and B_{II} gives the reduced density matrix

$$\begin{aligned}\rho^{(1)}_{A_I B_I} &= \mathrm{tr}_{A_{II} B_{II}}(\rho^{(1)}_{A_I A_{II} B_I B_{II}})\\ &= \frac{1}{4}(I_2\otimes I_2 + b_3^{(1)} I_2\otimes\sigma_z + t_1^{(1)}\sigma_x\otimes\sigma_x\\ &\quad + t_2^{(1)}\sigma_y\otimes\sigma_y + t_3^{(1)}\sigma_z\otimes\sigma_z),\end{aligned} \tag{28}$$

where $b_3^{(1)} = \lambda_-^2 - 1$, $t_1^{(1)} = \lambda_-^2$, $t_2^{(1)} = -\lambda_-^2$, $t_3^{(1)} = 2\lambda_-^4 - 2\lambda_-^2 + 1$. Let the maximal CHSH value of $\rho^{(1)}_{A_I B_I}$ be $S^{\max}_{A_I B_I^1}$. Figure 7 shows the value of $S^{\max}_{A_I B_I^1}$ at different Hawking temperatures T when $\gamma = 0.005$. This means that a Hawking temperature $T \in [0, 1.14]$ is required to ensure the shared initial state is nonlocal.

If Bob$^{(1)}$ performs the measurement $B^{(1)}_{0|0} := \frac{1}{2}(I_2 + \sigma_3)$ and $B^{(1)}_{0|1} := \frac{1}{2}(I_2 + \gamma_1\sigma_1)$, then we get

$$\begin{aligned}\rho^{(2''')}_{AB} &= \frac{2+\sqrt{1-\gamma_1^2}}{4}\rho^{(1)}_{A_I B_I} + \frac{1}{4}(I_2\otimes\sigma_z)\rho^{(1)}_{A_I B_I}(I_2\otimes\sigma_z)\\ &\quad + \frac{1-\sqrt{1-\gamma_1^2}}{4}(I_2\otimes\sigma_x)\rho^{(1)}_{A_I B_I}(I_2\otimes\sigma_x)\\ &= \frac{1}{4}(I_2\otimes I_2 + b_3^{(2)} I_2\otimes\sigma_z + t_1^{(2)}\sigma_x\otimes\sigma_x\\ &\quad + t_2^{(2)}\sigma_y\otimes\sigma_y + t_3^{(2)}\sigma_z\otimes\sigma_z),\end{aligned} \tag{29}$$

where $b_3^{(2)} = b_3^{(1)}\frac{1+\sqrt{1-\gamma_1^2}}{2}$, $t_1^{(2)} = \frac{t_1^{(1)}}{2}$, $t_2^{(2)} = \frac{t_2^{(1)}}{2}\sqrt{1-\gamma_1^2}$, and $t_3^{(3)} = \frac{t_3^{(1)}}{2}(1+\sqrt{1-\gamma_1^2})$. Since both Alice and Bob have traversed the event horizon of the black hole, we can obtain

$$\begin{aligned}\rho^{(2)}_{A_I B_I} &= \frac{1}{4}(I_2\otimes I_2 + (b_3^{(2)}\lambda_-^2 - \lambda_+^2) I_2\otimes\sigma_z\\ &\quad + t_1^{(2)}\lambda_-^2\sigma_x\otimes\sigma_x + t_2^{(2)}\lambda_-^2\sigma_y\otimes\sigma_y\\ &\quad + [t_3^{(2)}\lambda_-^4 - (b_3^{(2)}\lambda_-^2 - \lambda_+^2)\lambda_+^2]\sigma_z\otimes\sigma_z).\end{aligned} \tag{30}$$

In the sharing scenario, we consider that Bob sequentially performs the measurements indicated by Eqs. (6) and (7), given that both Alice and Bob are in the vicinity of the black hole's event horizon. Consequently, after Bob has performed $n-1$ measurements, we can always obtain $\rho^{(n)}_{A_I B_I}$. The correlation matrix of $\rho^{(n)}_{A_I B_I}$ can be calculated to be

$$T(\rho^{(n)}_{A_I B_I}) = \begin{pmatrix} \lambda_-^2\left(\frac{\lambda_-^2}{2}\right)^{n-1} & 0 & 0\\ 0 & -F_n\lambda_-^2\left(\frac{\lambda_-^2}{2}\right)^{n-1} & 0\\ 0 & 0 & H_n\end{pmatrix}, \tag{31}$$

where

$$\begin{aligned}H_n &= t_3^{(n)}\lambda_-^4 - (b_3^{(n)}\lambda_-^2 - \lambda_+^2)\lambda_+^2,\\ F_n &= \prod_{i=1}^{n-1}\sqrt{1-\gamma_i^2},\\ t_3^{(n)} &= \frac{1+\sqrt{1-\gamma_{n-1}^2}}{2}[t_3^{(n-1)}\lambda_-^4 - (b_3^{(n-1)}\lambda_-^2 - \lambda_+^2)\lambda_+^2],\\ b_3^{(n)} &= \frac{1+\sqrt{1-\gamma_{n-1}^2}}{2}(b_3^{(n-1)}\lambda_-^2 - \lambda_+^2),\\ t_3^{(2)} &= \frac{1+\sqrt{1-\gamma_1^2}}{2}(2\lambda_-^4 - 2\lambda_-^2 + 1),\\ b_3^{(2)} &= \frac{1+\sqrt{1-\gamma_1^2}}{2}(\lambda_-^2 - 1).\end{aligned} \tag{32}$$

It follows from Sect. 2.1 that the maximal CHSH value of $\rho^{(n)}_{A_I B_I}$ is

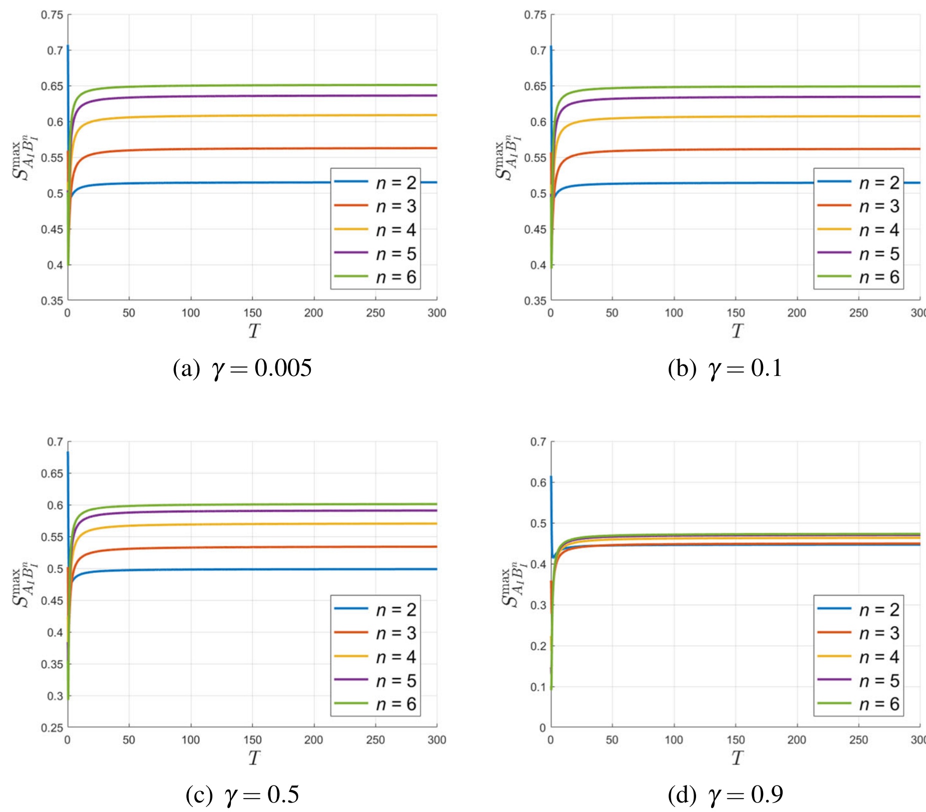


Fig. 8 The x-axis represents the Hawking temperature T, and the y-axis represents the value of $S^{\max}_{A_I B_I^n}$

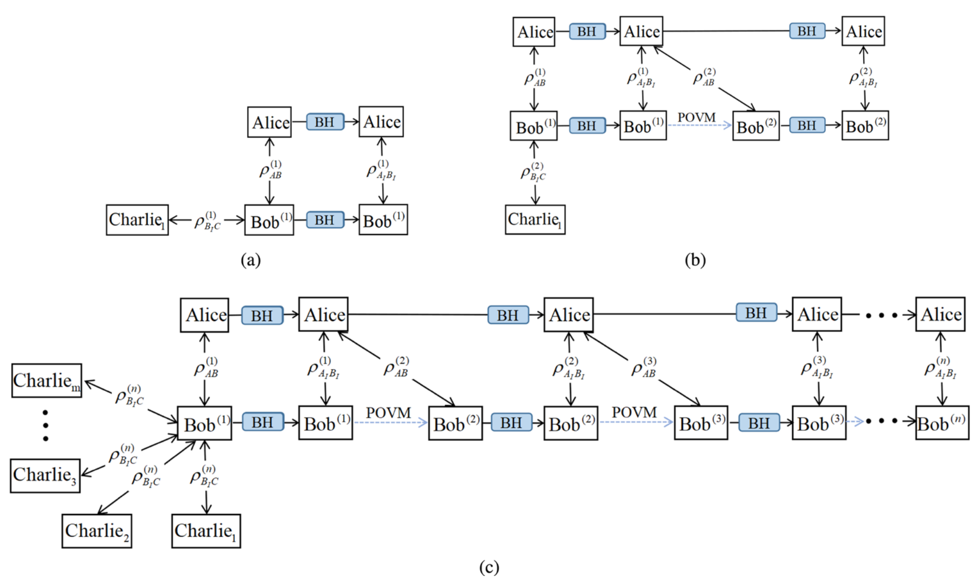


Fig. 9 Schematic diagram of sharing, where BH denotes black hole. **a** Represents Alice and Bob near the black hole horizon, with an initial state of $\rho_{\text{net}} = \rho^{(1)}_{AB} \otimes \rho^{(1)}_{B_I C}$. **b** Represents Alice and Bob near the black hole horizon, with an initial state of $\rho_{\text{net}} = \rho^{(1)}_{AB} \otimes \rho^{(1)}_{B_I C}$. After Bob performs measurements, the corresponding network state becomes $\rho^{(2)}_{\text{net}} = \rho^{(2)}_{A_I B_I} \otimes \rho^{(2)}_{B_I C}$. **c** Represents Alice and Bob near the black hole horizon, with an initial state of $\rho_{\text{net}} = \rho^{(1)}_{AB} \otimes \rho^{(1)}_{BC} \otimes \cdots \otimes \rho^{(1)}_{BC}$. After Bob performs $n-1$ measurements, the corresponding network state become $\rho^{(n)}_{\text{net}} = \rho^{(n)}_{A_I B_I} \otimes \rho^{(n)}_{B_I C} \otimes \cdots \otimes \rho^{(n)}_{B_I C}$

$$S^{\max}_{A_I B_I^n} = \sqrt{\tau_1^{A_I B_I^n} + \tau_2^{A_I B_I^n}} \tag{33}$$

where $\tau_1^{A_I B_I^n}$ and $\tau_2^{A_I B_I^n}$ denote the two larger non-negative eigenvalues of $(T(\rho^{(n)}_{A_I B_I}))^{\dagger} T(\rho^{(n)}_{A_I B_I})$. Assuming that Bob performs identical measurements sequentially, i.e., $\gamma_1 = \gamma_2 = \cdots = \gamma_{n-1} = \gamma$ in Eqs. (6) and (7), we can compute the value of $S^{\max}_{A_I B_I^n}$. Figure 8 shows the corresponding values of $S^{\max}_{A_I B_I^n}$ when the number of measurements is 1, 2, 3, 4 and 5, respectively, for different γ values. By analyzing the results in Fig. 8, it is evident that regardless of the number of parameters chosen for Bob's measurements, $S^{\max}_{A_I B_I^n} < 1$ when $n \geq 2$. This indicates that near a black hole's event horizon, with low Hawking temperature, Alice and Bob can share nonlocal correlations at most once.

To increase the nonlocality sharing times, we expand the original state $\rho^{(1)}_{AB}$ by adding a party Charlie$_1$ and a source, forming a star network with 3 parties and 2 sources. The

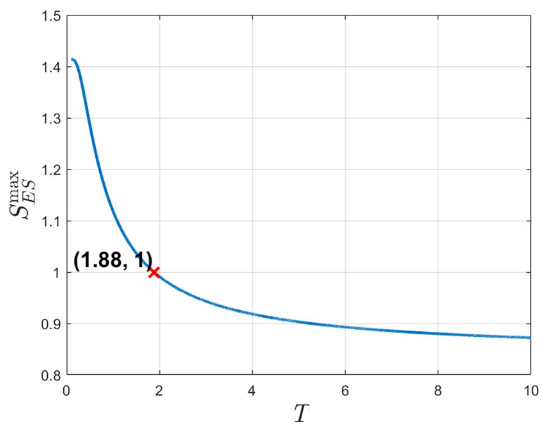


Fig. 10 The x-axis represents the Hawking temperature T, and the y-axis represents the value of $S_{ES}^{\max}$

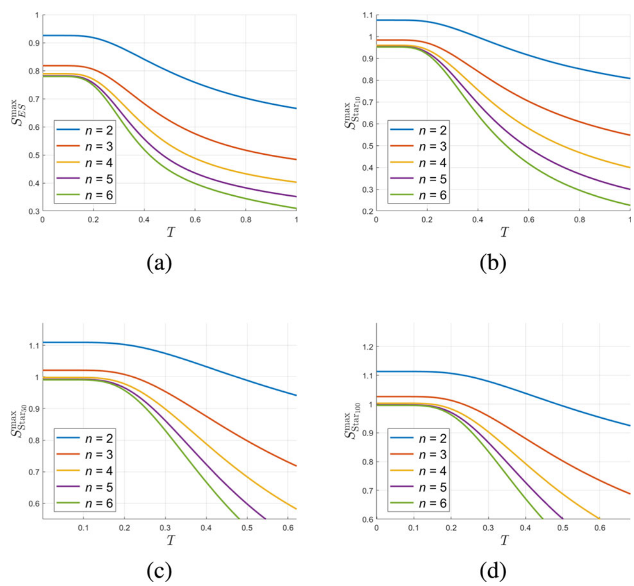


Fig. 11 The x-axis represents the Hawking temperature T, and the y-axis represents the value of $S_{\text{net}}^{\max}$. The values of $S_{\text{ES}}^{\max}$, $S_{\text{star10}}^{\max}$, $S_{\text{star50}}^{\max}$ and $S_{\text{star100}}^{\max}$ correspond to the $S_{\text{net}}^{\max}$ values of the postmeasurement state in a star network with 2, 10, 50, and 100 sources, respectively, after Bob has performed $n-1$ measurements

shared initial state is $\rho_{ES} = \rho_{AB}^{(1)} \otimes \rho_{BC}^{(1)}$ with $\rho_{BC}^{(1)} = \rho_{AB}^{(1)}$. The network state influenced by the black hole horizon is $\rho_{ES}^{(1)} = \rho_{A_I B_I}^{(1)} \otimes \rho_{B_I C}^{(1)}$ (see Fig. 9a). Using the form of $S_{\text{starnet}}^{\max}$, we get the value of $S_{\text{ES}}^{\max}$ for $\rho_{ES}^{(1)}$. Figure 10 shows the value of $S_{\text{ES}}^{\max}$ at different Hawking temperatures T when $\gamma = 0.005$. Compared with Fig. 7, this scenario allows nonlocality to persist when Hawking temperature $T \in [1.14, 1.88]$.

After Bob performs a measurement (see Fig. 9b), the network state takes the form $\rho_{ES}^{(2)} = \rho_{A_I B_I}^{(2)} \otimes \rho_{B_I C}^{(2)}$. Eqs. (30) and (17) denote the forms of states $\rho_{A_I B_I}^{(2)}$ and $\rho_{B_I C}^{(2)}$ respectively. We can obtain the value of $S_{\text{ES}}^{\max}$ corresponding to network state $\rho_{ES}^{(2)}$. After Bob performs multiple measurements, for each postmeasurement state, we can analogously calculate $S_{\text{ES}}^{\max}$, and the results are presented in Fig. 11a. The analysis of Fig. 11a indicates that despite the addition of a source, the network nonlocality vanishes immediately following a measurement performed by Bob. This suggests that the network nonlocality can only be shared once at maximum.

To address this issue, we attempt to add $m-1$ source states $\rho_{BC}^{(1)} = \rho_{AB}^{(1)} = |\phi^+\rangle\langle\phi^+|$ to transform the original network into a star network with m sources. After Bob performs $n-1$ measurements, the network state is denoted as $\rho_{\text{net}}^{(n)} = \rho_{A_I B_I}^{(n)} \otimes \rho_{B_I C}^{(n)} \otimes \cdots \otimes \rho_{B_I C}^{(n)}$ (as shown in Fig. 9c). For $m = 10, 50, 100$, we calculate the $S_{\text{starnet}}^{\max}$ of the network state $\rho_{\text{net}}^{(n)}$ after Bob performs $n-1$ measurements, and present the results in Fig. 11. From the Fig. 11, it can be seen that if the number of sources in the star network is increased, Bob and Alice can share network nonlocality more times when the Hawking temperature T approaches 0.

4 Conclusion and discussion

This paper delves into nonlocality sharing scenarios impacted by a Schwarzschild black hole, focusing on cases where either Alice or Bob is near the horizon. Overall, the influence of black holes increases the difficulty of nonlocality sharing. Especially, when both are near the horizon, even a single measurement by Bob causes the nonlocal correlation to disappear. This highlights the black hole significant impact on quantum nonlocal correlations, particularly their extreme fragility when both parties are near the horizon. We find that adding extra sources can sustain nonlocality sharing. This finding demonstrates that star networks remain a more robust architecture even when Hawking radiation is present. Moreover, our results can be leveraged to further examine quantum-information tasks under the influence of black holes, such as randomness certification and random encoding.

Over the past extended period, projective measurement was regarded as unsuitable for nonlocality sharing due to its tendency to cause state collapse. However, recent studies have demonstrated that observers can employ projective measurement to achieve nonlocality sharing. An interesting subsequent question is to analyze the impact of Hawking radiation on nonlocality sharing when projective measurement is utilized.

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Data Availability Statement Data will be made available on reasonable request. [Authors' comment: The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.]

Code Availability Statement Code will be made available on reasonable request. [Author' comment: The code generated during and analysed during the current study is available from the corresponding author on reasonable request.]

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