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Complete Intersection Calabi-Yau Six-folds

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Abstract

In paper a complete intersection Calabi-Yau six-folds are considered. Their Hodge diamond and Gromov-Witten invariants are calculated using the mirror symmetry methods. Several Wolfram Mathematica algorithms are proposed.

Mathematics Subject Classification: 14A25, 14J33, 14J40, 14N35, 14Q15

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1 Introduction

Intensive study of compact Ricci-flat manifolds started after Yau proved the Calabi conjecture that these spaces always admit a Kahler metric with $SU(3)$ holonomy group. Phenomenologically these spaces should be used to justification of $U(1) \otimes SU(2) \otimes SU(3)$ minimal model, which linking electromagnetic - $U(1)$, weak - $SU(2)$ and strong - $SU(3)$ interactions in string theory. From an algebraic geometry point of view the Calabi-Yau space be an elementary generalization of the well-studied K3 surfaces and before 1991 were not attracted mathematicians. In 1991 the computational experiments of physicists began to produce results [1], which were previously obtained by mathematicians in another theory [2]. Explaination of coincidence these results was suggested by [3]. After this, line of research Calabi-Yau manifolds acquired the status of "mainstream" both for physicists as for mathematicians. These spaces are included in the minimal model through 6-dimension group $SU(3)$. In this article we consider a 6-folds Calabi-Yau that are a complete intersection hypersurfaces in ordinary projective spaces (complete intersection Calabi-Yau). Also we calculate Hodge diamond and Gromov-Witten invariants using the methods of the theory of mirror symmetry.

2 Hodge Diamond

All definitions of this article are conventional [4]. We work in projective space $\mathbb{P}^n$ over an algebraically closed field of arbitrary characteristic. Let x be divisor degree of hypersurface $X \in \mathbb{P}^n$. We denote $X_m = \bigcap_{i=1}^{k} S_i^{s_i} \in \mathbb{P}^n$ as m-fold, which is a complete intersection of k hypersurfaces S_i degree s_i. Then X is a Calabi-Yau if $x = \sum s_i = n+1$. For a sheaf of differential forms $\Omega_X^i = \Lambda^i \Omega_X$ we introduce Hodge numbers $h^{ij} = \dim \mathrm{H^i}(\Omega_\mathrm{X}^\mathrm{j})$, which not only are symmetrical: $h^{ij} = h^{ji}$, but Serre symmetrical also: $h^{ij} = h^{n-i,n-j}$. If we have Hodge numbers then Betti numbers may be calculated as

$$b_k = \sum_{i+j=k} h^{ij}.$$

We can also define the Euler characteristic of X as the alternating sum of the Betti numbers:

$$\chi(X) = \sum_k (-)^k b_k.$$

For clarity, we rotate the matrix h^{ij} on 45° and call it as the Hodge diamond. So, for $n = 3$, we will have:

$$\begin{array}{ccccccc} & & & h^{00} & & & \\ & & h^{10} & & h^{01} & & \\ & h^{20} & & h^{11} & & h^{02} & \\ h^{30} & & h^{21} & & h^{12} & & h^{03} \\ & h^{31} & & h^{22} & & h^{13} & \\ & & h^{32} & & h^{23} & & \\ & & & h^{33} & & & \end{array}$$

$$\begin{aligned} b_0 &= h^{00} \\ b_1 &= h^{10} + h^{01} \\ b_2 &= h^{20} + h^{11} + h^{02} \\ b_3 &= h^{30} + h^{21} + h^{12} + h^{03} \\ b_4 &= h^{31} + h^{22} + h^{13} \\ b_5 &= h^{32} + h^{23} \\ b_6 &= h^{33} \end{aligned}$$

At the initial stage for construction of the Hodge diamond we will use standard tool of diagramatic search, which described in classical manuals on algebraic topology. If $X \in \mathbb{P}^n$ be hypersurface, then vector of cohomology $\mathbf{h}(\mathcal{O}_X) = h^{i,0}$ can be found from the exact sequence of sheaves

$$0-\mathcal{O}_{\mathbb{P}^n}(-x)-\mathcal{O}_{\mathbb{P}^n}-\mathcal{O}_X-0. \tag{1}$$

Since $h^k(\mathcal{O}_{\mathbb{P}^n}(-x)) = \delta_{k0}\delta_{x0}+\delta_{kn}C^{\,x-n-1}_{\,x-1}$, then $\mathbf{h}(\mathcal{O}_X) = (\overbrace{1,0,...,0,1}^{n})$ and it is sufficient for the construction of the Hodge diamond for curve in $\mathbb{P}^2$: $h^{i,j}=1$ $(i,j=0..1)$.

To calculate the cohomology of sheaf $\mathcal{O}_{\mathbb{P}^n}(-k)$ we will use the Wolfram Mathematica (Algorithm 1).

Algorithm 1: Bott formula for $\mathbf{h}(\mathcal{O}_{\mathbb{P}^n}(k))$

```
In[ ]:   OO[n_, k_] := Block[{Oo}, Oo = Array[0&, {n + 1}];
         Oo[[1]] = If[k >= 0, Binomial[k + n, k], 0];
         Oo[[n + 1]] = If[k < 0, Binomial[-k - 1, -k - 1 - n], 0]; Oo ];

         (*check*)
In[ ]:   n=5; k=−6; OO[n,k]
Out[ ]:  {0,0,0,0,0,1}
In[ ]:   OO[5,6]
Out[ ]:  {462,0,0,0,0,0}
```

To find the cotangent bundle Ω_X we take sequence dual to

$$0-T_X-T_{\mathbb{P}^n}-N_{X|\mathbb{P}^n}-0,$$

i.e.

$$0-N^*_{X|\mathbb{P}^n}-\Omega_{\mathbb{P}^n}-\Omega_X-0$$

and tensoring with $\mathcal{O}_X$:

$$0-N_X-\Omega_{\mathbb{P}^n|X}-\Omega_X-0, \tag{2}$$

where $N_X=N^*_{X|\mathbb{P}^n}=\mathcal{O}_X(-x)$. Further, if one tensors Euler sequence

$$0-\Omega_{\mathbb{P}^n}-\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)}-\mathcal{O}_{\mathbb{P}^n}-0 \tag{3}$$

with $\mathcal{O}_X$, then one gets restriction $\Omega_{\mathbb{P}^4|X}=\Omega_{\mathbb{P}^4}\otimes\mathcal{O}_X$:

$$0-\Omega_{\mathbb{P}^n|X}-\mathcal{O}_X(-1)^{\oplus(n+1)}-\mathcal{O}_X-0. \tag{4}$$

These equations (1), (2), (4) are sufficient to build a Hodge diamond of hypersurfaces $X\in\mathbb{P}^3$ and $X\in\mathbb{P}^4$.

With using Algorithm 2. we can calculate the cohomology sheaf C (0-A-B-C-0), if we know cohomology of A and B.

Algorithm 2.

```
In[ ]:   O3[A_, B_] := Block[{n, Oc}, n = Length[A]; Oc = Array[0&,{n}];
         Oc[[1]] = B[[1]] - A[[1]];
         Do[If[B[[i]] > A[[i]], Oc[[i]] = B[[i]] - A[[i]],
              Oc[[i - 1]] = Oc[[i - 1]] + A[[i]] - B[[i]]], {i, 2, n}]; Oc];

         (*check*)
In[ ]:   A={0,3,0,0,4};   B={0,5,0,0,1};  O3[A, B]
Out[ ]:  {0,2,0,3,0}
In[ ]:   O3[{1, 0, 3}, {2, 0, 1}]
Out[ ]:  {1,2,0}
```

If $X = S_1 \cap S_2$ is complete intersection of hypersurfaces of degree s_1 and s_2, then $N_X = \mathcal{O}_X(-s_1) \oplus \mathcal{O}_X(-s_2)$ and the sheaf $\mathcal{O}_X$ can be determined from the scheme:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & | & & | & & | & & \\
0 & - & \mathcal{O}_{S_2}(-s_1) & - & \mathcal{O}_R & - & \mathcal{O}_X & - & 0 \\
 & & | & & | & & | & & \\
0 & - & \mathcal{O}_{\mathbb{P}^n}(-s_1) & - & \mathcal{O}_{\mathbb{P}^n} & - & \mathcal{O}_{S_1} & - & 0 \\
 & & | & & | & & | & & \\
0 & - & \mathcal{O}_{\mathbb{P}^n}(-s_1-s_2) & - & \mathcal{O}_{\mathbb{P}^n}(-s_2) & - & \mathcal{O}_{S_1}(-s_2) & - & 0 \\
 & & | & & | & & | & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \qquad (5)
$$

Example 1. We will construct a Hodge diamond for three-fold Calabi-Yau $X_3 \in \mathbb{P}^5$, which is a complete intersection of a quadric and a quartics: $X = S^2 \cap S^4$.

We will use Alg.1. and Alg.2. Knowing that $\mathbf{h}(\mathcal{O}_{\mathbb{P}^5}(-2-4)) = (0,0,0,0,0,1)$ and $\mathbf{h}(\mathcal{O}_{\mathbb{P}^5}(-4)) = (0,0,0,0,0,0)$ from the bottom row (5) we get $\mathbf{h}(\mathcal{O}_{S_1}(-4)) = (0,0,0,0,1)$. Knowing that $\mathbf{h}(\mathcal{O}_{\mathbb{P}^5}) = (1,0,0,0,0,0)$ and $\mathbf{h}(\mathcal{O}_{\mathbb{P}^5}(-2)) = (0,0,0,0,0,0)$ from the middle row (5) we get $\mathbf{h}(\mathcal{O}_{S_1}) = (1,0,0,0,0,0)$. Substituting these cohomology in the right hand column we obtain $\mathbf{h}(\mathcal{O}_X) = (1,0,0,1)$. Since components of the Hodge diamond are $h^{i,0} = h^{i,3} = (1,0,0,1)$. Tensoring (5) with $\mathcal{O}_{\mathbb{P}^n}(-1)$ and repeated calculations we find $\mathbf{h}(\mathcal{O}_X(-1)) = (0,0,0,6,0,0)$. Knowing $\mathbf{h}(\mathcal{O}_X(-1))$ and $\mathbf{h}(\mathcal{O}_X)$ from (4) we can find $\mathbf{h}(\Omega_{\mathbb{P}^n|X}) = (0,1,35,0,0,0)$. Now, tensoring (5) with $\mathcal{O}_{\mathbb{P}^n}(-2)$ and with $\mathcal{O}_{\mathbb{P}^n}(-4)$ we find $\mathbf{h}(\mathcal{O}_X(-2)) = (0,0,0,20,0,0)$ and $\mathbf{h}(\mathcal{O}_X(-4)) = (0,0,0,104,0,0)$. Also, from $N_X = \mathcal{O}_X(-2) \oplus \mathcal{O}_X(-4)$ we get $\mathbf{h}(N_X) = (0,0,0,124,0,0)$. Substituting $\mathbf{h}(N_X)$ and $\mathbf{h}(\Omega_{\mathbb{P}^n|X})$ in (2) we obtain $\mathbf{h}(\Omega_X) = (0,1,89,0,0,0)$ or

$h^{i,1} = (0, 1, 89, 0)$ and $h^{i,2} = (0, 89, 1, 0)$. Therefore the Hodge diamond of complete intersectin of a quadric and a quartic in $\mathbb{P}^5$ has the form

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 89 & & 89 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \qquad \triangle$$

This result is easily verified using the theory of characteristic classes. For any morphism $X \xrightarrow{f} \mathbb{P}^n$ with uses the bundle

$$0 - T_X - f^*T_{\mathbb{P}^n} - N_f - 0$$

we find Chern class $c(T_X)$. Defining $c(\mathcal{O}_{\mathbb{P}^n}(d)) = 1 + dt$, for a complete intersection k hypersurfaces $X = \bigcap\limits_{i=1}^{k} S_i$ of degree s_i we obtain $c(N_f) = \prod\limits_{i=1}^{k}(1 + s_i t)$. The Euler sequence dual to (3) has the form

$$0 - \mathcal{O}_{\mathbb{P}^n} - \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} - T_{\mathbb{P}^n} - 0. \qquad (6)$$

From this $c(T_{\mathbb{P}^n}) = (1+t)^{n+1}$, hense

$$c(T_X) = \frac{(1+t)^{n+1}}{\prod\limits_{i=1}^{k}(1+s_i t)}.$$

The Euler characteristic of m-fold is

$$\chi = \int_X c_m(X) = \int_{\mathbb{P}^n} c_k(N_f) \wedge c_m(T_X) = c_k(N_f) \cdot c_m(T_X), \quad (k+m=n). \quad (7)$$

Example 2. For the intersection of a quadric and a quartic in $\mathbb{P}^5$ we have

$$c(N_f) = (1+2t)(1+4t) = 1 + 6t + 8t^2, \quad \text{i.e.} \quad c_2(N_f) = 8$$

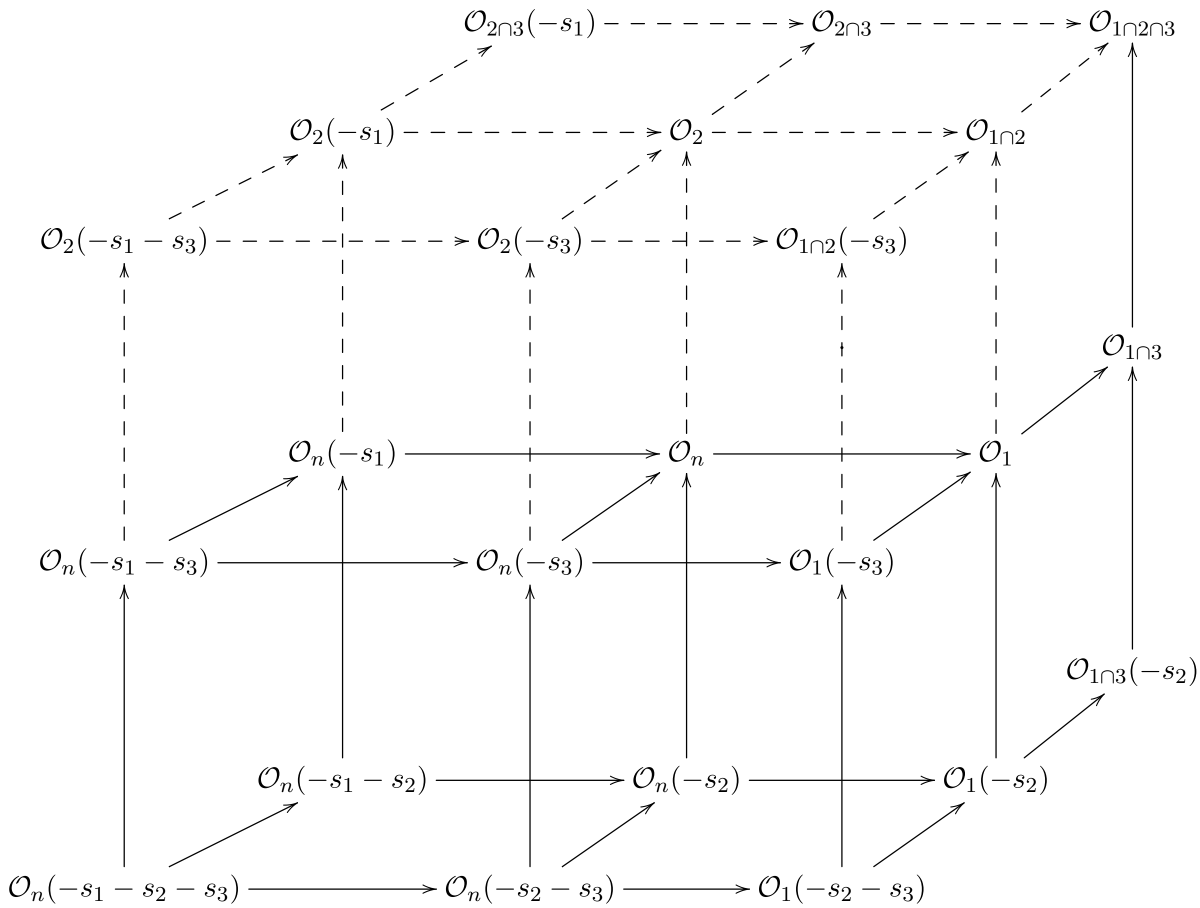
and

$$c(T_X) = \frac{(1+t)^6}{(1+2t)(1+4t)} \approx 1 + 7t^2 - 22t^3, \quad \Rightarrow \quad c_3(T_X) = -22.$$

Hence the Euler characteristic is $\chi = -176$. Comparing this with Example 1, we see that $\mathbf{b} = (1, 0, 1, 180, 1, 0, 1)$ and from formulae $\chi = \sum(-)^k b_k$ we obtain the same value χ. $\triangle$

If $X = S_1 \cap S_2 \cap S_3$ is complete intersection of 3 hypersurfaces of degree s_1, s_2 and s_3, then $N_X = \mathcal{O}_X(-s_1) \oplus \mathcal{O}_X(-s_2) \oplus \mathcal{O}_X(-s_3)$, and the sheaf

$\mathcal{O}_X$ is determined from the 3d-commutative diagrams (thus $\mathcal{O}_{\mathbb{P}^n} = \mathcal{O}_n$ and $\mathcal{O}_{S_1 \cap S_2 \cap S_3} = \mathcal{O}_{1\cap 2\cap 3}$ for simplification).



It is obvious that analysis of the intersection of k hyperplane will require building of k-dimensional commutative cube. In essence, for determination of $\mathcal{O}_{S_1 \cap S_2 \cap \ldots \cap S_k}$ we need use the following recurrence relations

$$\begin{gathered} 0 - \mathcal{O}_{S_1 \cap S_2 \cap \ldots \cap S_{k-1}}(-s_k) - \mathcal{O}_{S_1 \cap S_2 \cap \ldots \cap S_{k-1}} - \mathcal{O}_{S_1 \cap S_2 \cap \ldots \cap S_k} - 0 \\ 0 - \mathcal{O}_{S_1 \cap S_2 \cap \ldots \cap S_{k-2}}(-s_{k-1}) - \mathcal{O}_{S_1 \cap S_2 \cap \ldots \cap S_{k-2}} - \mathcal{O}_{S_1 \cap S_2 \cap \ldots \cap S_{k-1}} - 0 \\ \ldots \\ 0 - \mathcal{O}_{S_1}(-s_2) - \mathcal{O}_{S_1} - \mathcal{O}_{S_1 \cap S_2} - 0 \\ 0 - \mathcal{O}_{\mathbb{P}^n}(-s_1) - \mathcal{O}_{\mathbb{P}^n} - \mathcal{O}_{S_1} - 0 \end{gathered} \tag{8}$$

twisting them with $\mathcal{O}_{\mathbb{P}^n}(-s_i)$ if it is necessary. Further, we will use the Mathematical Algorithm 3. to calculate $\mathcal{O}_X(-j)$. Note that Algorithm 3. includes Algorithm 2. and also Algorithm 1.

	Algorithm 3: $\mathbf{h}(\mathcal{O}_X(j))$

```
In[ ]:   Ox[n_, k_, j_] := Block[{Y}, Y = Array[0&, n + 1];
         If[Length[k] == 1, Y = O3[OO[n, k[[1]] + j], OO[n, j]]];
         If[Length[k] >= 2,
         Y = O3[Ox[n, Delete[k, -1], k[[-1]] + j], Ox[n, Delete[k, -1], j]]];Y];

         (*check*)
In[ ]:   n = 7; k = {-2, -3, -3}; j = 0; Ox[n, k, j]
Out[ ]:  {1, 0, 0, 0, 1, 0, 0, 0}
In[ ]:   Ox[9, {-2, -2, -3, -3}, -10]
Out[ ]:  {0, 0, 0, 0, 0, 34 057, 0, 0, 0, 0}
```

We denote by X_m the m-fold, which is a complete intersection k hypersurfaces S_i of degree s_i. If $X_m = \bigcap_{i=1}^{k} S_i^{s_i} \in \mathbb{P}^n$ is complete intersection Calabi-Yau, then consistent application of the (8) yields $\mathbf{h}(\mathcal{O}_X) = (\overbrace{1, 0, ..., 0, 1}^{m+1})$.

Now we will find the $h^{i,2} = dim(H^i(\mathbb{P}^n, \Omega_X^2)$. Taking the symmetric square of (2), we get:

$$0 - Sym^2 N_X - N_X \otimes \Omega_{\mathbb{P}^n|X} - \Omega^2_{\mathbb{P}^n|X} - \Omega_X^2 - 0. \tag{9}$$

Here $Sym^2 N_X = \mathcal{O}_X(-2x)$, and $\Omega^2_{\mathbb{P}^n|X}$ we find using an external degree of the Euler sequence (3):

$$0 - \Omega^2_{\mathbb{P}^n|X} - \mathcal{O}_X(-2)^{\oplus C^2_{n+1}} - \Omega_{\mathbb{P}^n|X} - 0. \tag{10}$$

To determine the Ω_X^2 we split the equation (9) into two parts:

$$0 - Sym^2 N_X - Sym\ N_X \otimes \Omega_{\mathbb{P}^n|X} - E - 0 \tag{9.1}$$

and

$$0 - E - \Omega^2_{\mathbb{P}^n|X} - \Omega_X^2 - 0. \tag{9.2}$$

It is easy to show that $E \cong \Omega_X(-x)$. Indeed, multiplying (2) on $\mathcal{O}_X(-x)$ we have (9.1):

$$0 - \mathcal{O}_X(-2x) - N_X \otimes \Omega_{\mathbb{P}^n|X}(-x) - \Omega_X(-x) - 0.$$

On the other hand, the vedge power (2) is

$$0 - \Omega_X(-x) - \Omega^2_{\mathbb{P}^n|X} - \Omega_X^2 - 0,$$

i.e. (9.2).

In general the k-st symmetric power of (2)

$$0 - Sym^k N_X - Sym^{k-1} N_X \otimes \Omega_{\mathbb{P}^n|X} - Sym^{k-2} N_X \otimes \Omega^2_{\mathbb{P}^n|X} - \ldots - \Omega^k_{\mathbb{P}^n|X} - \Omega^k_X - 0. \tag{11}$$

allows us to calculate the Ω^k_X.

Example 3 We will construct the Hodge diamond of 4-fold Calabi-Yau $X_4 \in \mathbb{P}^7$, which is a complete intersection of a quadric and two cubics: $X = S^2 \cap S^3 \cap S^3$.

For calculations, we need to use Algorithm 2. and Algorithm 3. From (8) we have $\mathbf{h}(\mathcal{O}_X) = (1,0,0,0,1,0,0,0)$, so $h^{i0} = h^{i4} = (1,0,0,0,1)$. Substituting $\mathbf{h}(\mathcal{O}_X)$ and $\mathbf{h}(\mathcal{O}_X)(-1) = (0,0,0,0,8,0,0,0)$ in (4), we have $\mathbf{h}(\Omega_{\mathbb{P}^7|X}) = (0,1,0,0,63,0,0,0)$. Now, knowing $\mathbf{h}(N_X) = \mathbf{h}(\mathcal{O}_X(-2) \oplus \mathcal{O}_X(-3) \oplus \mathcal{O}_X(-3)) = (0,0,0,0,255,0,0,0)$ from (2) we have $\mathbf{h}(\Omega_X) = (0,1,0,192,0,0,0,0)$. Therefore $h^{i1} = (0,1,0,192,0)$, $h^{i3} = (0,192,0,1,0)$. Cohomology Ω^2_X we obtained from expanded scheme (9):

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Sym^2 N_X & \longrightarrow & \Omega_{\mathbb{P}^7|X}(-2) \oplus \Omega_{\mathbb{P}^7|X}(-3)^{\oplus 2} & \longrightarrow & \Omega^2_{\mathbb{P}^7|X} & \longrightarrow & \Omega^2_X \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_X(-3)^{\oplus 8} \oplus \mathcal{O}_X(-4)^{\oplus 8\cdot 2} & & \mathcal{O}_X(-2)^{\oplus 28} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_X(-2) \oplus \mathcal{O}_X(-3)^{\oplus 2} & & \Omega_{\mathbb{P}^7|X} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Knowing the $\mathbf{h}(\Omega_{\mathbb{P}^7|X})$ and $\mathbf{h}(\mathcal{O}_X)(-2) = (0,0,0,0,35,0,0,0)$ we obtain $\mathbf{h}(\Omega^2_{\mathbb{P}^7|X}) = (0,0,1,0,917,0,0,0)$. From $\mathbf{h}(Sym^2 N_X) = \mathbf{h}(\mathcal{O}_X(-4) \oplus \mathcal{O}_X(-5)^{\oplus 2} \oplus \mathcal{O}_X(-6)^{\oplus 3}) = (0,0,0,0,4971,0,0,0)$ and $\mathbf{h}(N_X \otimes \Omega_{\mathbb{P}^n|X}) = (0,0,0,0,5073,0,0,0)$ we have $\mathbf{h}(\Omega_X(-x)) = (0,0,0,815,917,0,0,0)$, $\mathbf{h}(\Omega^2_X) = (0,0,0,816,0,0,0,0)$. I.e. $h^{22} = (0,0,816,0,0)$ and Hodge diamond takes the form

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
1 & & 192 & & 816 & & 192 & & 1 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

Summing the Betti numbers $\mathbf{b} = (1, 0, 1, 0, 1202, 0, 1, 0, 1)$ we obtain $\chi = 1206$. With another hand, knowing

$$c(N_f) = (1+2t)(1+3t)^2 = 1 + 8t + 21t^2 + 18t^3, \quad \text{i.e.} \quad c_3(N_f) = 18$$

and

$$c(T_X) = \frac{(1+t)^8}{(1+2t)(1+3t)^2} \approx 1 + 7t^2 - 18t^3 + 67t^4, \quad \Rightarrow \quad c_4(T_X) = 67$$

for the Euler characteristic $X_4 \in \mathbb{P}^7$ from (7) we obtain

$$\chi = \int_{\mathbb{P}^7} c_3(N_f) \wedge c_4(T_X) = c_3(N_f) \cdot c_4(T_X) = 18 \cdot 67 = 1206. \qquad \triangle$$

Example 4. Find the Hodge diamond of 5-fold Calabi-Yau $X_5 \in \mathbb{P}^9$, which is a complete intersection of two quadrics and two dice: $X = S^2 \cap S^2 \cap S^3 \cap S^3$.

Using recurrent sequences (8) and the scheme (11)

$$0 - Sym^3 N_X - Sym^2 N_X \otimes \Omega_{\mathbb{P}^9|X} - N_X \otimes \Omega^2_{\mathbb{P}^9|X} - \Omega^3_{\mathbb{P}^9|X} - \Omega^3_X - 0,$$

with help the Algorithm 3 and Algorithm 2 we obtain

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & 1 & & 0 & & 0 & \\
1 & & 403 & & 4423 & & 4423 & & 403 & & 1 \\
 & 0 & & 0 & & 1 & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & & & 1 & & & & &
\end{array}$$

and $\chi = c_4(N_f) \cdot c_5(T_X) = 36 \cdot (-268) = -9648. \qquad \triangle$

Here is a table of the currently known complete intersections Calabi-Yau in ordinary projective spaces. In this table, m-fold $X_m = S^i \cap S^j \cap ... \cap S^k \in \mathbb{P}^n$ is indicated by $[n_m|ij...k]_\chi$. Spaces X_3, have been described in [5]. All 4-dimensional full intersection Calabi-Yau are described in [6] and 5-dimensional in work [7].

X_1	h^{11}	X_3	h^{12}	X_4	(h^{13}, h^{22})	X_5	(h^{14}, h^{23})
[2|3]	1	[4|5]	101	[5|6]	$(426, 1752)$	[6|7]	$(1667, 18327)$
		[5|33]	73	[6|52]	$(356, 1472)$	[7|62]	$(1357, 14917)$
		[5|42]	89	[6|43]	$(237, 996)$	[7|53]	$(811, 8911)$
		[6|322]	73	[7|422]	$(263, 1100)$	[7|44]	$(593, 6513)$
		[7|2222]	65	[7|332]	$(192, 816)$	[8|522]	$(971, 10671)$
				[8|3222]	$(183, 780)$	[8|432]	$(559, 6139)$
X_2	h^{11}			[9|2...2]	$(151, 652$	[8|333]	$(406, 4456)$
[3|4]	20					[9|4222]	$(609, 6689)$
[4|32]	20					[9|3322]	$(403, 4423)$
[5|222]	20					[10|32222]	$(369, 4049)$
						[11|2...2]	$(289, 3169)$

The aim of all previous detailed calculations was to show that for any rational complete intersection Calabi-Yau X_m all non-zero entries of the Hodge diamond always lying on its equator or on the central column. Also $h^{ii} = 1$ if $i \neq m/2$. Therefore, we can simplify all calculation with use of characteristic classes theory.

We take the Riemann-Roch-Hirzebruch equation

$$\chi(E, X) = \int_X ch(E) \wedge td(T_X), \tag{12}$$

attach it to $E = \bigwedge^q \Omega_X = \Omega_X^q$ and rewrite over Chern classes of tangent bundle $c(T_X) = \sum_i c_i(T_X) = \prod_i (1 + \alpha_i)$:

$$td(T_X) = \prod_i \frac{\alpha_i}{1 - e^{-\alpha_i}}, \tag{13}$$

$$ch(\Omega_X^q) = \sum_{i_1 < i_2 < ... < i_q} e^{\alpha_{i_1}} ... e^{\alpha_{i_q}}. \tag{14}$$

Here α_i are Chern roots of $T_{\mathbb{P}^n}$. It is not difficult to formalize a computation using Wolfram Mathematica.

According to (13) we find $Td(T_{\mathbb{P}^n})$.

Algorithm 4: $Td(T_{\mathbb{P}^n})$

In[ ]: **n=5**; f=$\dfrac{x}{1 - \text{Exp}[-x]}$; ff=Series[f,{x,0,n+1}];
Do $\left[E_k = \sum_{i=0}^{n} \left(x_k^i * \text{SeriesCoefficient[ff}, i]\right), \{k, 1, n\}\right]$;
$F = \prod_{\xi=1}^{n} E_\xi$; Xo=Table $[x_k, \{k, 1, n\}]$;Cc=Table$\left[c_k * h^k, \{k, 1, n\}\right]$;
Fc=SymmetricReduction [F,Xo,Cc];
th=Together [CoefficientList [Fc[[1]],h]];
Do[Print [td_{k-1},"=",th[[k]]],{k,1,n+1}];

Out[ ]: $td_0 = 1$ $\quad td_1 = \frac{c_1}{2}$ ...

I.e.

$\mathrm{td}_0 = 1;\ \mathrm{td}_1 = \frac{c_1}{2};\ \mathrm{td}_2 = \frac{c_1^2+c_2}{12};\ \mathrm{td}_3 = \frac{c_1c_2}{24};\ \mathrm{td}_4 = \frac{-c_1^4+4c_1^2c_2+3c_2^2+c_1c_3-c_4}{720};$

$\mathrm{td}_5 = \frac{-c_1^3c_2+3c_1c_2^2+c_1^2c_3-c_1c_4}{1440};$

$\mathrm{td}_6 = \frac{2c_1^6-12c_1^4c_2+11c_1^2c_2^2+10c_2^3+5c_1^3c_3+11c_1c_2c_3-c_3^2-5c_1^2c_4-9c_2c_4-2c_1c_5+2c_6}{60480};$

$\mathrm{td}_7 = \frac{2c_1^5c_2-10c_1^3c_2^2+10c_1c_2^3-2c_1^4c_3+11c_1^2c_2c_3-c_1c_3^2+2c_1^3c_4-9c_1c_2c_4-2c_1^2c_5+2c_1c_6}{120960};$

$$\mathrm{td}_8 = \frac{\begin{pmatrix} -3c_1^8+24c_1^6c_2-50c_1^4c_2^2+8c_1^2c_2^3+21c_2^4-14c_1^5c_3+26c_1^3c_2c_3 \\ +50c_1c_2^2c_3+3c_1^2c_3^2-8c_2c_3^2+14c_1^4c_4-19c_1^2c_2c_4-34c_2^2c_4-13c_1c_3c_4 \\ +5c_4^2-7c_1^3c_5-16c_1c_2c_5+3c_3c_5+7c_1^2c_6+13c_2c_6+3c_1c_7-3c_8 \end{pmatrix}}{3628800}$$

$$\mathrm{td}_9 = \frac{\begin{pmatrix} -3c_1^7c_2+21c_1^5c_2^2-42c_1^3c_2^3+21c_1c_2^4+3c_1^6c_3-29c_1^4c_2c_3+50c_1^2c_2^2c_3 \\ +8c_1^3c_3^2-8c_1c_2c_3^2-3c_1^5c_4+26c_1^3c_2c_4-34c_1c_2^2c_4-13c_1^2c_3c_4+5c_1c_4^2 \\ +3c_1^4c_5-16c_1^2c_2c_5+3c_1c_3c_5-3c_1^3c_6+13c_1c_2c_6+3c_1^2c_7-3c_1c_8 \end{pmatrix}}{7257600}$$

$$\mathrm{td}_{10} = \frac{\begin{pmatrix} 10c_1^{10}-100c_1^8c_2+317c_1^6c_2^2-302c_1^4c_2^3-69c_1^2c_2^4+90c_2^5+67c_1^7c_3 \\ -303c_1^5c_2c_3+98c_1^3c_2^2c_3+381c_1c_2^3c_3+52c_1^4c_3^2+115c_1^2c_2c_3^2 \\ -67c_1^6c_4+258c_1^4c_2c_4-17c_1^2c_2^2c_4-219c_2^3c_4-59c_1^3c_3c_4 \\ +21c_3^2c_4+18c_1^2c_4^2+87c_2c_4^2+45c_1^5c_5-81c_1^3c_2c_5-162c_1c_2^2c_5 \\ +53c_2c_3c_5+42c_1c_4c_5-5c_5^2-45c_1^4c_6+58c_1^2c_2c_6+109c_2^2c_6 \\ -32c_4c_6+23c_1^3c_7+53c_1c_2c_7-10c_3c_7-23c_1^2c_8-43c_2c_8-10c_1c_9 \\ -21c_1c_3^3-81c_2^2c_3^2-269c_1c_2c_3c_4-19c_1^2c_3c_5+42+10c_{10}c_1c_3c_6 \end{pmatrix}}{479001600}$$

Similarly, according to (14) we calculate $Ch(\Omega^p(X_m))$.

Algorithm 5: $Ch(\Omega^p(X_m))$.

In[ ]: **p=2; m=4**; If[p==0, p=m]; A=Do[A=Append[A,1],{i,1,p}];
Do[A=Append[A,0],{i,p,m-1}]; pr=Permutations[A];
Do$\left[E_k = \sum_{i=0}^{m} \frac{x_k^i*(-1)^i}{i!}, \{k,1,m\}\right]$;
$F = \sum_{\xi=1}^{\text{Length[pr]}} \prod_{k=1}^{m} \left(E_k^{\text{pr}[[\xi,k]]}\right)$;
Xo=Table[x_k,{k,1,m}]; Cc=Table $\left[c_k * t^k, \{k,1,m\}\right]$;
Fc=SymmetricReduction[F,Xo,Cc];
chh=Together[CoefficientList[Fc[[1]],t]];
Do[Print[ch_{k-1},"=",chh[[k]]],{k,1,m+1}]

Out[ ]: $ch_0 = 6 \qquad ch_1 = -3c_1 \qquad ch_2 = \frac{1}{2}\left(3c_1^2-4c_2\right)$

$ch_3 = \frac{1}{2}\left(-c_1^3+2c_1c_2\right)$

$ch_4 = \frac{1}{24}\left(3c_1^4-8c_1^2c_2+4c_2^2-4c_1c_3+16c_4\right)$

Using the Riemann-Roch-Hirzebruch formula (12), we obtain the Euler characteristic: $\chi(\mathbb{P}^n, \Omega^p(X_m)$.

	Algorithm 6: $\chi(\mathbb{P}^n, \Omega^p(X_m)$
In[ ]:	Do$[td_{k-1} = th[[k]], \{k, 1, m+1\}]$; Do $[ch_{k-1}$=chh[[k]],{k,1,m+1}] Td=$\sum_{i=0}^{m} td_i * x^i$; Ch=$\sum_{i=0}^{m} ch_i * x^i$; χ=CoefficientList[Series[Td*Ch,{x,0,m+2}],x]; Print["$\chi = td \cdot ch =$",χ [[m+1]]]
Out[ ]:	$\chi = td \cdot ch = \frac{1}{120}\left(-c_1^4 + 4c_1^2c_2 + 3c_2^2 - 19c_1c_3 + 79c_4\right)$

For example if $X_4 = S^6 \in \mathbb{P}^5$, then for $c(\Omega^2(X_4))$ we will have

	Algorithm 7: $\chi(\mathbb{P}^n, \Omega^p(S)$
In[ ]:	n=5; S={6,0,0}; ; ct=$\frac{(1+x)^{n+1}}{\prod_{i=1}^{\text{Length[S]}}(1+\text{S}[[i]]*x)}$; ctc=RotateLeft[CoefficientList[Series[ct,{x,0,n+2}],x]]; Do[c_i=ctc[[i]],{i,1,Length[ctc]}]; Td=$\sum_{i=0}^{m} td_i * x^i$; Ch=$\sum_{i=0}^{m} ch_i * x^i$; χ =CoefficientList[Series[Td*Ch,{x,0,m+2}],x]; χ [[m+1]]*(Times@@Select[S,##!=0&])
Out[ ]:	1752

Example 5. Find the Hodge diamond of 6-fold $X_6 \in \mathbb{P}^{10}$, which is a complete intersection of two quadrics, a cubic and a quartic: $X = S^2 \cap S^2 \cap S^3 \cap S^4$.

Substituting (13)-(14) into (12), taking into account that $c_1 = 0$, we obtain

$$\chi(\mathbb{P}^{10}, \mathcal{O}_X) = \chi(\mathbb{P}^{10}, \Omega_X^6) = \int_X \frac{10c_2^3 - c_3^2 - 9c_2c_4 + 2c_6}{60480};$$
$$\chi(\mathbb{P}^{10}, \Omega_X) = \chi(\mathbb{P}^{10}, \Omega_X^5) = \int_X \frac{10c_2^3 + 41c_3^2 - 51c_2c_4 - 82c_6}{10080};$$
$$\chi(\mathbb{P}^{10}, \Omega_X^2) = \chi(\mathbb{P}^{10}, \Omega_X^4) = \int_X \frac{50c_2^3 + 331c_3^2 - 381c_2c_4 + 4378c_6}{20160};$$
$$\chi(\mathbb{P}^{10}, \Omega_X^3) = \int_X \frac{50c_2^3 + 373c_3^2 - 423c_2c_4 - 8306c_6}{15120}.$$

Symbolic computations:
Alg.4:(n=6; ...) $\rightarrow$ Alg.5:(p=0; m=6; ...) $\rightarrow$ Alg.6:($c_1 = 0$; ...)

Alg.4:(n=6; ...) $\rightarrow$ Alg.5:(p=1; m=6; ...) $\rightarrow$ Alg.6:($c_1 = 0$; ...)
Alg.4:(n=6; ...) $\rightarrow$ Alg.5:(p=2; m=6; ...) $\rightarrow$ Alg.6:($c_1 = 0$; ...)
...
Since

$$c(T_X) = \frac{(1+t)^{11}}{(1+2t)^2(1+3t)(1+4t)}, \quad \Rightarrow \quad \mathbf{c} = (1, 0, 11, -32, 150, -616, 2542),$$

with use Alg.7:(n=10; S={2,2,3,4}; ...) we have

$$\chi(\mathbb{P}^{10}, \Omega^i_X) = (2, -1130, 25966, -67820, 25966, -1130, 2)$$

and this yields

$$h^{15} = 1129,\ h^{24} = 25965,\ h^{33} = 67820. \qquad \triangle$$

In the following table we show the Hodge numbers of the 6-fold Calabi-Yay that are complete intersections in ordinary projective spaces.

X_6	(h^{15}, h^{24}, h^{33})	X_6	(h^{15}, h^{24}, h^{33})
[7|8]	$(6371, 154645, 398568)$	[10|5222]	$(2246, 53093, 137714)$
[8|72]	$(5111, 123502, 318642)$	[10|4322]	$(1129, 25965, 67820)$
[8|63]	$(2921, 69590, 180158)$	[10|3332]	$(761, 17156, 45050)$
[8|54]	$(1691, 39493, 102744)$	[11|42222]	$(1211, 27965, 72968)$
[9|622]	$(3636, 87157, 225302)$	[11|33222]	$(734, 16517, 43394)$
[9|532]	$(1891, 44418, 115394)$	[12|322222]	$(651, 14557, 38312)$
[9|442]	$(1271, 29341, 76560)$	[13|2222222]	$(491, 10781, 28520)$
[9|433]	$(1031, 23609, 61736)$		

3 Gromov-Witten invariants

The increased interest of mathematicians in the theory of mirror symmetry is associated with the fact that with use of physical theory instruments were obtained results, (which were) previously considered exclusively mathematician. Furthermore, physical calculations turned out to be easier and faster then mathematical ones. For example, one problem is to count the number of curves of degree d, lying on arbitrary m-folde in $\mathbb{P}^n$. Mathematically, these type of problems solved by Schubert calculus. In some special cases the solution is expressed by analytically. For example the formula

$$N(d) = \sum_{k+l=d} N(k)N(l)k^2 l \left[l C^{3k-2}_{3d-4} - k C^{3k-1}_{3d-4} \right], \quad d \geq 2$$

it allows you to calculate the number of rational curves of degree d in $\mathbb{P}^2$ and through $3d-1$ point:

$$N(1)=1,\ N(2)=1,\ N(3)=12,\ N(4)=620,$$
$$N(5)=87304,\ N(6)=26312976,\ N(7)=14616808192,...$$

In this section we give a simple "physical" [8] algorithm and it software implementation and use them to calculate the invariants Gromov-Witten of 6-fold complete intersection Calabi-Yau.

If $X_k = \bigcap_{i=1}^{m} S_i^{s_i} \in \mathbb{P}^n$ is complete intersection Calabi-Yau manifolds, then comparison of series

$$W = \frac{\prod_{i=1}^m s_i}{\left(1 - z\prod_{i=1}^m (s_i)^{s_i}\right)(w_0)^2}\left(\frac{q}{z}\partial_q z\right)^3 \tag{15}$$

and

$$W = \sum_k n_k \frac{k^3 q^k}{1-q^k} \tag{16}$$

after q-expansion gives n_d, i.e. number rational curves of degree d, lying on X. Thus $z = z(q)$ is inverse series to $q = z\ \exp\left(\frac{w_1}{w_0}\right)$,

$$w_0 = \sum_{j=0}^{\infty} \frac{\prod_{i=1}^m (j\cdot s_i)!}{(j!)^{n+1}} z^j,$$
$$w_1 = \sum_{j=0}^{\infty}\left(\frac{\prod_{i=1}^m (j\cdot s_i)!}{(j!)^{n+1}}\left(\sum_{i=1}^m s_i\cdot\psi[s_i\cdot j+1] - (n+1)\cdot\psi[j+1]\right)\right) z^j,$$

$\psi(x)$ - polygamma function.

Example 6. We find the number of lines on a 6-fold Calabi-Yau $X_6 \in \mathbb{P}^9$, which is a complete intersection of a quadric, cubic and quintic: $X = S^2 \cap S^3 \cap S^5$.

Since

$$w_0 = \sum_{j=0}^{\infty} \frac{(2j)!(3j)!(5j)!}{(j!)^{10}} z^j = 1 + 1440z + 61236000z^2 + 5650444800000z^3 + ...,$$
$$w_1 = \sum_{j=0}^{\infty}\left(\frac{(2j)!(3j)!(5j)!}{(j!)^{10}} z^j \left(\begin{array}{c} 2\psi[2j+1] + 3\psi[3j+1] + \\ 5\psi[5j+1] - 10\psi[j+1] \end{array}\right)\right) z^j$$
$$= 14280z + 683486100z^2 + 65797827200000z^3 + O(z^4),$$

then inverse series to $q = z\ \exp(w_1/w_0)$ is

$$z = q - 14280q^2 - 357045300q^3 - 33867824768000q^4 + O(q^5).$$

Substituting the function $z(q)$ into (15)

$$W = \frac{2\cdot 3\cdot 5}{(1-2^2 3^3 5^5 z)\,(w_0)^2}\left(\frac{q}{z}\partial_q z\right)^3$$
$$= 30 + 8753400q + 2746866978000q^2 + 872843372113500000q^3 + O(q^4)$$

and comparing with (16)

$$W = \sum_k n_k \frac{k^3 q^k}{1-q^k}$$
$$= n_0 + n_1 q + (n_1 + 2^3 n_2)q^2 + (n_1 + 3^3 n_3)q^3 + (n_1 + 2^3 n_2 + 4^3 n_4)q^4 + O(q^4)$$

we obtain $n_1 = 8753400$ is the number of lines on X_6, $n_2 = 343357278075$ is the number of quadrics on X_6 ... etc. It may be calculated with use Algorithm 8.

Algorithm 8.

In[ ]: n=9; S={2,3,5}; S=Select[S,##!=0&];

$\text{w0[z\_]}=\sum_{m=0}^{8}\frac{\text{Times @@}((S*m)!)}{(m!)^{n+1}}z^m;$

$\text{w1[z\_]=Expand}\left[\sum_{m=0}^{8}\left(\frac{\text{Times @@}((S*m)!)}{(m!)^{n+1}}\ z^m\times\right.\right.$

$\times\left(\sum_{i=1}^{\text{Length}[S]} S[[i]]\text{PolyGamma}[S[[i]]m+1]\right.$

$\left.\left.\left.-(n+1)\text{PolyGamma}[m+1]\right)\right)\right];$

$\text{Q[z\_]=Series}\left[\text{Exp}\left[\text{Log}[z]+\frac{\text{w1}[z]}{\text{w0}[z]}\right],\{z,0,7\}\right];$

Z=InverseSeries[Q[q],q];

$\text{W=}\frac{\text{Times @@}S}{(1-Z\ \text{Times @@}(S^S))\,(\text{w0}[Z])^2}\left(\frac{q}{Z}\partial_q Z\right)^3;$

WL=CoefficientList[W,q];

$\{\text{nn}_1=\text{Coefficient}[W,q],\left(\text{Coefficient}\left[W,q^2\right]-\text{nn}_1\right)/2^3,$
$\left(\text{Coefficient}\left[W,q^3\right]-\text{nn}_1\right)/3^3\}$

Out[ ]: {8753400, 343357278075, 32327532300175800}

In next table we show the number of rational curves of given degree for the

6-folds complete intersection Calabi-Yau.

X_6	n_1	n_2	n_3
[7|8]	120273920	238475640766464	–
[8|72]	40692344	15726480104979	14564655481706503096
[8|63]	19906560	2935121327856	1037509900910125056
[8|54]	14027200	1312611137600	–
[9|622]	15558912	1352675813184	282158440157963520
[9|532]	8753400	343357278075	32327532300175800
[9|442]	7245824	220670905344	–
[9|433]	5785344	125260349184	6511867479177984
[10|5222]	6838400	158150815200	–
[10|4322]	4520448	57710641536	1770479072924160
[10|3332]	3610008	32768476371	714945700680984
[11|42222]	3530752	26576715776	–
[11|33222]	2820096	15094123200	194346853853184
[12|322222]	2202624	6951624192	52822439144448
[13|2222222]	1720320	3201773568	–

For the weighted projective spaces [9] $\mathbb{P}(\lambda_0, \lambda_1, ..., \lambda_n)$ formula (15) rewritten as

$$W = \frac{\sum \lambda / \prod \lambda}{\left(1 - z \frac{(\sum \lambda)^{\sum \lambda}}{\prod \lambda^{\lambda}}\right) (w_0)^2} \left(\frac{q}{z} \partial_q z\right)^3, \qquad (17)$$

where

$$w_0 = \sum_{j=0}^{\infty} \frac{(j \sum \lambda)!}{\prod (j\lambda)!} z^j,$$

$$w_1 = \sum_{j=0}^{\infty} \left(\frac{(j \sum \lambda)!}{\prod (j\lambda)!} \left(\sum \lambda \cdot \psi \left[\sum \lambda \cdot j + 1 \right] - \sum (\lambda \psi [\lambda j + 1]) \right) \right) z^j.$$

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