

Kerr-CFT and gravitational perturbations

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ABSTRACT: Motivated by the Kerr-CFT conjecture, we investigate perturbations of the near-horizon extreme Kerr spacetime. The Teukolsky equation for a massless field of arbitrary spin is solved. Solutions fall into two classes: normal modes and traveling waves. Imposing suitable (outgoing) boundary conditions, we find that there are no unstable modes. The explicit form of metric perturbations is obtained using the Hertz potential formalism, and compared with the Kerr-CFT boundary conditions. The energy and angular momentum associated with scalar field and gravitational normal modes are calculated. The energy is positive in all cases. The behaviour of second order perturbations is discussed.

KEYWORDS: Gauge-gravity correspondence, Black Holes

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1 Introduction

Some time ago, Bardeen and Horowitz (BH) showed that one can take a near-horizon limit of the extreme Kerr geometry to obtain a spacetime similar to $AdS_2 \times S^2$ [1]. This near-horizon extreme Kerr (NHEK) geometry has an $SL(2, \mathbb{R}) \times U(1)$ isometry group, where the

$U(1)$ is inherited from the axisymmetry of the Kerr solution and the $SL(2, \mathbb{R})$ extends the Kerr time-translation symmetry. Recently, Guica, Hartman, Song and Strominger (GHSS) have conjectured that quantum gravity in the NHEK geometry with certain boundary conditions is equivalent to a chiral conformal field theory (CFT) in 1+1 dimensions [2]. Using this, they gave a statistical calculation of the entropy of an extreme Kerr black hole.

More precisely, GHSS showed that there exist boundary conditions on the asymptotic behaviour of the metric such that the asymptotic symmetry group is generated by time translations plus a single copy of the Virasoro algebra, the latter extending the $U(1)$ symmetry of the background. Hence, if a consistent theory of quantum gravity can be defined in NHEK with these boundary conditions then it must be a chiral CFT. There has been considerable interest in extending the Kerr-CFT conjecture, and entropy calculation, to other extremal black holes [3].

The GHSS boundary conditions are unusual in two respects. First, they specify the rate at which components of $h_{\mu\nu}$ (the deviation of the metric from the NHEK geometry) should behave asymptotically. We shall refer to these as the “fall-off” conditions. Most components decay relative to the background but some are allowed to be $\mathcal{O}(1)$ relative to the background. Secondly, GHSS impose a supplementary boundary condition, namely that the energy (the conserved charge associated with the generator L_0 of $SL(2, \mathbb{R})$) should vanish.

One motivation for this paper is that the GHSS fall-off conditions are motivated entirely by considerations of the asymptotic symmetry group. However, boundary conditions are also required for classical physics to be predictable from initial data in a non-globally hyperbolic spacetime such as NHEK (or anti-de Sitter). It is not clear whether these boundary conditions will be compatible with the unusual GHSS boundary conditions. Indeed, it is not even clear whether the GHSS boundary conditions allow propagating gravitational degrees of freedom, or whether they lead to physics similar to Einstein gravity in AdS_3 , where non-trivial physics is associated with large gauge transformations (i.e., non-trivial elements of the asymptotic symmetry group) and black holes that are locally, but not globally, gauge [4]. We shall investigate these issues by studying linearized gravitational perturbations of NHEK.

Another motivation for studying perturbations of NHEK is associated with positivity of the energy. The GHSS “zero energy” condition arises from the desire to consider only the ground states corresponding to an extreme Kerr black hole, rather non-extremal excitations. However, this presupposes that the energy must be non-negative. The NHEK geometry possesses an ergoregion, inherited from the ergoregion of the Kerr black hole. It is well-known that, in the presence of an ergoregion, one can construct initial data for test matter fields for which the energy of these fields is negative [5]. For a Kerr black hole, this is not a problem because the positive energy theorem [6] ensures that the total energy of the spacetime (black hole plus matter) is non-negative. This is a non-trivial result, which may not extend to NHEK.¹ Furthermore, in a spacetime with an ergoregion but no event horizon, e.g. NHEK (adopting the global perspective), if one imposes boundary conditions

¹If one wanted to prove such a theorem using spinorial methods then NHEK would have to admit a spinor field covariantly constant with respect to some connection. As far as we know, no such spinor field has been constructed.

such that there is no energy in the matter fields entering from infinity, then the total energy of these fields can only decrease. If it is initially negative then it will become more negative, suggesting an instability [5].

It should be noted that the issue of NHEK stability is subtle: BH pointed out that the singularity theorems imply that there exist small perturbations of NHEK that will lead to the formation of a singularity. In this sense, NHEK is unstable. However, as BH also observed, such a singularity might be hidden inside a tiny black hole.² If this has positive mass then there would not be a problem. However, if the energy is negative, or the singularity is naked, then it would be difficult to make sense of NHEK.

The NHEK geometry shares many similarities with AdS_3 : indeed, it is foliated by warped AdS_3 submanifolds, which have been discussed extensively in recent work on topologically massive gravity (TMG) [7]. In TMG, there are propagating gravitational degrees of freedom but some of these turn out to have negative energy, signaling a potential instability of AdS_3 [8]. In the chiral limit, the propagating modes are eliminated by boundary conditions at infinity [8, 9], leaving only pure gauge modes and BTZ black holes, just as in Einstein gravity. Away from the chiral limit, AdS_3 is unstable but there exist warped AdS_3 solutions that might provide an alternative ground state [10]. The stability of some of these has been investigated recently [11]. Again, there are propagating modes with negative energy but these are excluded by boundary conditions.

We now describe the approach we shall take. NHEK is a type D vacuum spacetime so one can obtain decoupled equations describing gravitational perturbations using Teukolsky's method [12, 13]. The Teukolsky equation turns out to be very similar to the equation governing a massless scalar field in NHEK, which was discussed by BH, and the qualitative features of our solutions closely resemble theirs.

By expanding in (spin-weighted, spheroidal) harmonics on the S^2 of the NHEK geometry, we reduce the Teukolsky equation to the equation of a charged massive scalar in AdS_2 with a homogeneous electric field. This equation can be solved in terms of hypergeometric functions. Depending on the labels (l, m) of the spheroidal harmonics, the solutions either grow or decay as powers of the AdS_2 radial coordinate, or they are oscillating at infinity. In the former case, the natural “normalizable” boundary conditions lead to quantized frequencies: we shall refer to these as normal modes. These modes fill out highest-weight representations of a Virasoro algebra which extends the $SL(2, \mathbb{R})$ isometry group of AdS_2 , indeed such modes have been obtained previously in the context of a charged scalar in AdS_2 with electric field [14]. A particularly important set of normalizable modes are those arising from axisymmetric ($m = 0$) perturbations of NHEK.

The other set of modes are those that oscillate at infinity. Following BH, we refer to these as traveling waves. These modes typically have large m for given l : $|m| \approx l$. From the AdS_2 perspective, these correspond to modes that have complex weight with respect to the generator L_0 of $SL(2, \mathbb{R})$ and so would not normally be considered. However, in NHEK it

²The same might be true in AdS_d for $d \geq 4$: AdS_d is like a confining box, and a small gravitational perturbation in a box might be expected to evolve ergodically. If so, eventually sufficient energy will be concentrated into a small enough region to produce a tiny black hole. We thank G. Horowitz for discussion of this point.

would be very restrictive to discard these modes since that would correspond to a restriction on the allowed values of (l, m) . Even if such a restriction were imposed at the linearized level, it would be violated at the nonlinear level through interactions between modes.

The traveling waves carry energy and angular momentum to infinity. BH showed that such modes are associated with superradiant scattering in the NHEK geometry. However, rather than considering scattering, we are interested in the question of what happens to localized initial data. We therefore impose purely outgoing boundary conditions at infinity. We find that the modes corresponding to traveling waves become exponentially damped, i.e., they are quasinormal modes of NHEK, describing the decay of a small perturbation via radiation to infinity. Therefore NHEK is stable against linearized gravitational perturbations. The reason that the above argument for instability based on the energy in matter (or linearized gravitational) fields fails is that some outgoing waves carry negative energy to infinity. Hence the energy flux through infinity need not be positive and so the energy need not decrease with time.

So far, our discussion of gravitational perturbations has been based entirely on the Teukolsky equation. However, in order to calculate the energy, or discuss fall-off conditions on the metric, we need to know the perturbed metric tensor rather than just the Teukolsky scalars. Fortunately, there exists a method for determining the metric perturbation in terms of a scalar potential, called the Hertz potential [15]–[19]. This satisfies an equation closely related to the Teukolsky equation. Using this, we obtain explicit results for the form of the metric perturbation.

We find that most (but not quite all) normal modes satisfy the GHSS fall-off conditions but traveling waves violate these conditions. Although one can construct localized wavepackets involving the latter, they will eventually propagate to infinity and violate the fall-off conditions. Therefore, at the linearized level, they should be excluded, leaving just the normal modes.

Next, we consider the energy of the normal modes. To warm-up, we start by considering a massless scalar field. We are able to show that an arbitrary superposition of normal modes has positive energy. Then we turn to gravitational perturbations. We define the energy of the latter in the usual way using the Landau-Lifshitz “pseudotensor”. Since the metric perturbation involves second derivatives of the Hertz potential, the energy involves an integral of a complicated quantity sixth order in derivatives. Nevertheless, using a combination of analytical and numerical methods, we find that the energy of gravitational normal modes is positive, thus supporting the validity of the GHSS zero-energy condition.

This positive energy result is satisfying but the exclusion of the traveling waves is worrying. First, it is worrying that we can construct initial data that satisfy the fall-off conditions, but violate these conditions when evolved. It suggests that the initial value problem, at least for linearized fields, may not be well-posed. Furthermore, if one goes beyond linearized theory then interactions between modes will excite traveling waves even if they are not present initially.³ So one might worry about well-posedness of the nonlinear theory too. It is possible that these problems are cured by backreaction, i.e. going beyond the linearized approximation. We shall discuss this further at the end of the paper.

³The only way to escape this conclusion is to consider only axisymmetric ($m = 0$) modes, which form a consistent truncation of the full set of modes.

This paper is organized as follows. In section 2, we derive and solve the Teukolsky equation in the NHEK background, obtaining the spectrum of normal, and quasinormal modes. In section 3 we introduce the Hertz potential and use it to obtain the explicit form of linearized perturbations. We compare the asymptotic behaviour of these with the GHSS boundary conditions. We then calculate the energy of scalar field and gravitational normal modes. Finally, section 4 discusses how going beyond the linearized approximation may solve some of the problems just discussed.

Note added. As this work was nearing completion, we learned that another group is exploring similar issues [20].

2 Massless fields of arbitrary spin in NHEK

2.1 NHEK and its Newman-Penrose tetrad

In global coordinates the NHEK metric is [1] (we use the notation of ref. [2] and, because we shall employ the Newman-Penrose formalism, a negative signature metric)

$$ds^2 = -2GJ\Omega^2(\theta) \left(-(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + d\theta^2 + \Lambda^2(\theta)(d\phi + rdt)^2 \right), \quad (2.1)$$

with

$$\Omega^2(\theta) \equiv \frac{1}{2}(1 + \cos^2 \theta), \quad \Lambda(\theta) = \frac{2 \sin \theta}{1 + \cos^2 \theta}, \quad GJ = G^2 M_{\text{ADM}}^2 \equiv M^2. \quad (2.2)$$

Surfaces of constant θ are warped AdS_3 geometries, i.e., a circle fibred over AdS_2 with warping parameter $\Lambda^2(\theta)$. The isometry group is $SL(2, \mathbb{R}) \times U(1)$. BH showed that the solution is geodesically complete, with timelike infinities at $r = \pm\infty$. There is an ergoregion (where $\partial/\partial t$ is spacelike) which extends to $r = \pm\infty$.

In the next subsection we study perturbations in the NHEK using the Teukolsky formulation. For that we need the Newman-Penrose (NP) tetrad, spin coefficients and directional derivatives. In appendix A we obtain the shear-free null geodesics of this background and use them to construct the associated NP null tetrad [21], $\mathbf{e}_{(1)} = \boldsymbol{\ell}$, $\mathbf{e}_{(2)} = \mathbf{n}$, $\mathbf{e}_{(3)} = \mathbf{m}$, $\mathbf{e}_{(4)} = \mathbf{m}^*$, where (coordinates are listed in the order $\{t, r, \theta, \phi\}$)

$$\begin{aligned} \ell^\mu &= \frac{1}{1+r^2} (1, 1+r^2, 0, -r), \\ n^\mu &= \frac{1}{4M^2\Omega^2(\theta)} (1, -(1+r^2), 0, -r), \\ m^\mu &= \frac{1}{\sqrt{2}M(1+i\cos\theta)} (0, 0, 1, i\Lambda^{-1}(\theta)), \end{aligned} \quad (2.3)$$

and $\mathbf{e}^{(a)} = \eta^{(a)(b)}\mathbf{e}_{(b)}$ with non-vanishing symmetric $\eta^{(a)(b)} = \eta_{(a)(b)}$ given by $\eta^{(1)(2)} = -\eta^{(3)(4)} = 1$. This NP tetrad satisfies the normalization and orthogonality conditions (A.16), and the null vector $\boldsymbol{\ell}$ is tangent to affinely parametrized geodesics: $\ell^\mu \nabla_\mu \ell_\nu = 0$.

$\Psi^{(s)}$	$(-\Psi_2)^{\frac{4}{3}} \Psi_4$	Ψ_0	$(-\Psi_2)^{\frac{2}{3}} \phi_2$	ϕ_0	$(-\Psi_2)^{\frac{1}{3}} \chi_1$	χ_0	Φ
s	-2	2	-1	1	$-\frac{1}{2}$	$\frac{1}{2}$	0
$T_{(s)}$	$2(-\Psi_2)^{\frac{4}{3}} T_4$	$2T_0$	$(-\Psi_2)^{\frac{2}{3}} J_2$	J_0	$(-\Psi_2)^{\frac{1}{3}} T_{\chi_1}$	T_{χ_0}	T_Φ

Table 1. Teukolsky fields $\Psi^{(s)}$, spin s and source terms for the master equation (2.5).

The unperturbed Weyl scalars in the NHEK geometry are computed using (A.20), yielding

$$\begin{aligned} \Psi_0 &= \Psi_1 = \Psi_3 = \Psi_4 \equiv 0, \\ \Psi_2 &= -[M^2(1 - i \cos \theta)^3]^{-1}. \end{aligned} \quad (2.4)$$

The first line confirms that this solution is indeed Petrov type D.

2.2 Teukolsky master equation

Teukolsky has shown how, for type D spacetimes, one can use the NP formalism to derive a system of decoupled equations, that furthermore separate into an angular and radial part, for the perturbations of several NP scalars [12, 13]. For gravitational perturbations, the relevant quantities are the perturbed Weyl scalars $\Psi_0^{(1)}$ (spin $s = +2$) and $\Psi_4^{(1)}$ ($s = -2$); the complex NP scalars $\phi_{0,1}$ for spin $s = \pm 1$ Maxwell perturbations; the Weyl fermionic scalars $\chi_{0,1}$ for massless spin $s = \pm \frac{1}{2}$ perturbations; and the scalar field Φ for massless spin $s = 0$ perturbations. Teukolsky's master equation encompasses all of these cases [13].

Using the NP quantities listed in appendix A, we find that the Teukolsky master equation for spin s field perturbations $\Psi^{(s)}$ in the NHEK geometry is

$$\begin{aligned} & \frac{1}{(1+r^2)} \partial_t^2 \Psi^{(s)} - \frac{2r}{(1+r^2)} \partial_t \partial_\phi \Psi^{(s)} + \left(\frac{r^2}{1+r^2} - \frac{(1+\cos^2 \theta)^2}{4 \sin^2 \theta} \right) \partial_\phi^2 \Psi^{(s)} \\ & - (1+r^2)^{-s} \partial_r \left((1+r^2)^{s+1} \partial_r \Psi^{(s)} \right) - \frac{1}{\sin \theta} \partial_\theta \left(\sin \theta \partial_\theta \Psi^{(s)} \right) - 2s \frac{r}{(1+r^2)} \partial_t \Psi^{(s)} \\ & - 2s \left(\frac{1}{(1+r^2)} + i \frac{\cos \theta}{\sin^2 \theta} + i \frac{1}{2} \cos \theta \right) \partial_\phi \Psi^{(s)} + (s^2 \cot^2 \theta - s) \Psi^{(s)} = T_{(s)}. \end{aligned} \quad (2.5)$$

We have allowed for the possibility of a source term on the RHS (see appendix C). The relation between the nomenclature used here and the original notation of Teukolsky [13] is $\{\Psi^{(2)}, \Psi^{(1)}, \Psi^{(1/2)}\} = \{\Psi_0, \phi_0, \chi_0\}$ and $\{T_{(2)}, T_{(1)}, T_{(1/2)}\} = \{T_0, J_0, T_{\chi_0}\}$ for positive spin. For negative spin the map is $\{\Psi^{(-2)}, \Psi^{(-1)}, \Psi^{(-1/2)}\} = \{(-\Psi_2)^{\frac{4}{3}} \Psi_4, (-\Psi_2)^{\frac{2}{3}} \phi_2, (-\Psi_2)^{\frac{1}{3}} \chi_1\}$. Here, the powers of the unperturbed Weyl scalar Ψ_2 are those that allow for the separation of the master equation, when we further assume an ansatz for the perturbation that is a radial function times the spin-weighted spheroidal harmonic; see (2.6). For the source term one has the map $\{T_{(-2)}, T_{(-1)}, T_{(-1/2)}\} = \{T_4, J_2, T_{\chi_1}\}$. These relations are summarized in table 1.

2.3 Separation of variables

We shall solve the Teukolsky equation in the NHEK geometry by separation of variables. Assuming

$$\Psi^{(s)} = \begin{cases} e^{-i\omega t} e^{im\phi} R_{lm\omega}^{(s)}(r) S_{lm}^{(s)}(\theta) (-\Psi_2)^{-\frac{2s}{3}}, & s \leq 0, \\ e^{-i\omega t} e^{im\phi} R_{lm\omega}^{(s)}(r) S_{lm}^{(s)}(\theta), & s \geq 0, \end{cases} \quad (2.6)$$

equation (2.5) separates into an angular and radial equations. The angular equation is

$$\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d}{d\theta} S_{lm}^{(s)}(\theta) \right) + \left[(C \cos\theta)^2 - 2sC \cos\theta + s + \Lambda_{lm}^{(s)} - \frac{(m + s \cos\theta)^2}{\sin^2\theta} \right] S_{lm}^{(s)}(\theta) = 0, \quad (2.7)$$

for $C = m/2$ and where $\Lambda_{lm}^{(s)}$ is the separation constant. Its eigenfunctions are the spin-weighted spheroidal harmonics $e^{im\phi} S_{lm}^{(s)}(\theta)$ (the nomenclature usually includes an appropriate normalization factor; see e.g., [22]), with positive integer l specifying the number of zeros, $l - \max\{|m|, |s|\}$, of the eigenfunction. The associated eigenvalues $\Lambda_{lm}^{(s)}$ can be computed numerically with very good accuracy and are specified by s, l, m subject to the regularity constraints that $-l \leq m \leq l$ must be an integer and $l \geq |s|$. The transformation $\theta \rightarrow \pi - \theta$ can be used to show that

$$\Lambda_{lm}^{(s)} = \Lambda_{l(-m)}^{(s)}, \quad \Lambda_{lm}^{(-s)} = \Lambda_{lm}^{(s)} + 2s. \quad (2.8)$$

We also note that, to leading order in C , $\Lambda_{lm}^{(s)} = (l - s)(l + s + 1) + \mathcal{O}(C)$. This is useful when $|m| \ll l$.

Equation (2.7) represents the most standard way to write the spin-weighted spheroidal harmonic equation. However, it will be convenient here to work with shifted eigenvalues $\tilde{\Lambda}_{lm}^{(s)}$ defined by

$$\tilde{\Lambda}_{lm}^{(s)} \equiv \Lambda_{lm}^{(s)} + s^2 + s - 7C^2. \quad (2.9)$$

The advantage of using these quantities is that they have the symmetry

$$\tilde{\Lambda}_{lm}^{(-s)} = \tilde{\Lambda}_{lm}^{(s)}. \quad (2.10)$$

Notice that in the Kerr background with mass M and angular velocity Ω_H the angular equation for spin s perturbations is also (2.7) but with $C_{\text{Kerr}} = a\tilde{\omega}$, where $a = 2Mr_+ \Omega_H$ is Kerr's rotation parameter and $\tilde{\omega}$ the wave's frequency in this geometry. As observed in [1], in the near-horizon limit of extreme Kerr, all finite frequencies ω in the NHEK throat correspond to the single frequency $\tilde{\omega} = m\Omega_H^{\text{ext}} = \frac{m}{2M}$ in the extreme Kerr geometry. This $\tilde{\omega}$ corresponds precisely to the marginally unstable superradiant frequency, and in the NH limit one finds $C_{\text{Kerr}} = M\tilde{\omega} \rightarrow C = m/2$.

Writing for any spin,

$$R_{lm\omega}^{(s)}(r) = (1 + r^2)^{-s/2} \Phi_{lm\omega}^{(s)}(r), \quad (2.11)$$

we find that the radial equation associated with (2.5) can be written also in a unified way as

$$\frac{d}{dr} \left[(1 + r^2) \frac{d}{dr} \Phi_{lm\omega}^{(s)}(r) \right] - \left[\mu^2 - \frac{(\omega + qr)^2}{1 + r^2} \right] \Phi_{lm\omega}^{(s)}(r) = 0, \quad (2.12)$$

with

$$\begin{aligned}
 q &= m - is, \\
 \mu^2 &= q^2 + \tilde{\Lambda}_{lm}^{(s)} = \Lambda_{lm}^{(s)} + s - 2ism - \frac{3m^2}{4}.
 \end{aligned}
 \tag{2.13}$$

This is exactly the equation for a charged massive scalar field in AdS_2 with a homogeneous electric field: take the AdS_2 metric in global coordinates,

$$ds_2^2 = (1 + r^2)dt^2 - \frac{dr^2}{1 + r^2},
 \tag{2.14}$$

and the electric field to arise from the potential

$$A = rdt.
 \tag{2.15}$$

Define the covariant derivative for a field of charge q as

$$\mathcal{D} = \nabla - iqA,
 \tag{2.16}$$

where ∇ is the Levi-Civita connection in AdS_2 . The equation for a charged scalar field $\Phi(t, r)$ with mass μ is then

$$\mathcal{D}^2\Phi + \mu^2\Phi = 0.
 \tag{2.17}$$

Assuming

$$\Phi(t, r) = e^{-i\omega t} \Phi(r),
 \tag{2.18}$$

the equation of motion reduces to (2.12). Therefore, a general spin s perturbation with angular momentum m in NHEK obeys the wave equation for a massive charged scalar field in AdS_2 with a homogeneous electric field. However, note that the charge q is complex, as is the squared mass μ^2 , although $\mu^2 - q^2$ is real. The problem of a massive charge scalar field in AdS_2 with homogeneous electric field was studied in ref. [14], where solutions corresponding to highest weight representations of a Virasoro algebra extending $SL(2, \mathbb{R})$ were obtained. We shall recover the same solutions in the next section.

2.4 Solving the radial equation

Asymptotically, the solutions of (2.12) behave as

$$\Phi(r) \sim |r|^{-1/2 \pm \eta/2},
 \tag{2.19}$$

where

$$\eta = \sqrt{1 + 4(\mu^2 - q^2)} = \sqrt{1 + 4\tilde{\Lambda}_{lm}^{(s)}}, \quad \text{Im}(\eta) \geq 0.
 \tag{2.20}$$

Note that $\eta(s, l, m) = \eta(-s, l, m) = \eta(s, l, -m)$. We can now see that the modes can exhibit qualitatively different behaviour, depending on the value of (l, m) , as first noticed by BH (for $s = 0$). Some modes have real η and others have imaginary η . For example, axisymmetric modes ($m = 0$), have, for general s ,

$$\eta = 2l + 1 \quad (m = 0).
 \tag{2.21}$$

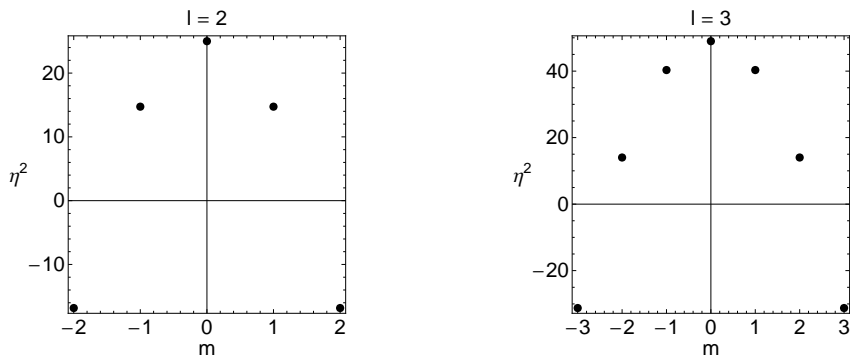


Figure 1. Values of η^2 , defined in (2.20), for $|s| = 2$, and a) $l = 2$ and b) $l = 3$.

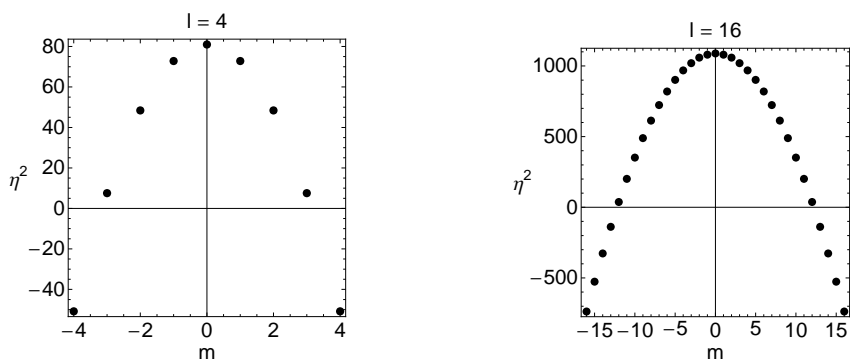


Figure 2. Values of η^2 , defined in (2.20), for $|s| = 2$, and a) $l = 4$ and b) $l = 16$.

i.e., such modes exhibit power-law behaviour at infinity. However, for certain other modes, specifically those with $|m| \approx l$, η is imaginary and hence the solutions oscillate at infinity. In figures 1 and 2 we show how η^2 depends on m for gravitational perturbations with some different values of l .

It is interesting to ask which modes have the smallest real value for η since these will give the normal modes that decay most slowly at infinity. For gravitational perturbations ($|s| = 2$) we have calculated η for all (l, m) with $l \leq 30$ and find that the mode with the smallest real value for η occurs for $l = 4$, $|m| = 3$, which gives $\eta = 2.74$.

Equation (2.12) can be solved exactly. This is not a surprise since in the Kerr geometry, Teukolsky and Press [23] found that the corresponding Teukolsky radial equation can also be analytically solved in the particular case where we have extreme Kerr and a wave frequency that saturates the superradiant bound, $\tilde{\omega} = m\Omega_H^{\text{ext}}$. As discussed after (2.10), all frequencies in NHEK correspond to the single superradiant threshold frequency in the extreme Kerr. So we indeed expect this property for the radial equation in NHEK.

Introducing the new radial coordinate,

$$z = \frac{1}{2}(1 - ir) , \tag{2.22}$$

and redefining the radial wavefunction as

$$\Phi_{lm\omega}^{(s)}(r) = z^\alpha(1-z)^\beta F, \quad \text{with} \quad \alpha \equiv \frac{1}{2}(\omega - iq), \quad \beta \equiv \frac{1}{2}(\omega + iq), \quad (2.23)$$

the radial equation (2.12) can be rewritten as

$$z(1-z)\partial_z^2 F + [2\alpha - 2(\alpha + \beta)z]\partial_z F - [(\alpha + \beta + 1)(\alpha + \beta - 2) + (q^2 - \mu^2 + 2)]F = 0. \quad (2.24)$$

This wave equation is a standard hypergeometric equation [24], $z(1-z)\partial_z^2 F + [c - (a+b+1)z]\partial_z F - abF = 0$, with

$$a = \frac{1}{2}(1 + \eta + 2\omega), \quad b = \frac{1}{2}(1 - \eta + 2\omega), \quad c = 1 + \omega - iq, \quad (2.25)$$

and hence the most general solution in the neighborhood of $z = 0$ is [24]

$$\Phi_{lm\omega}^{(s)} = Az^\alpha(1-z)^\beta F(a, b, c, z) + Bz^{\alpha+1-c}(1-z)^\beta F(a-c+1, b-c+1, 2-c, z). \quad (2.26)$$

We render this function single valued in the complex z plane by taking branch cuts to run from $-\infty$ to 0 and from 1 to $+\infty$, corresponding to taking $|\arg(z)| < \pi$, $|\arg(1-z)| < \pi$. Note that the branch cuts do not intersect the line $\text{Re}(z) = 1/2$, which corresponds to real r .

2.5 Boundary conditions

The above solution of the radial equation is regular for all finite r . Using standard properties of the hypergeometric function, we find that it exhibits the following behaviour as $r \rightarrow \pm\infty$:

$$\Phi_{lm\omega}^{(s)} \approx \Gamma(b-a)C^\pm e^{\pm i\pi(\beta-\alpha-a)/2} \left(\frac{|r|}{2}\right)^{-(1+\eta)/2} + \Gamma(a-b)D^\pm e^{\pm i\pi(\beta-\alpha-b)/2} \left(\frac{|r|}{2}\right)^{-(1-\eta)/2}, \quad (2.27)$$

where

$$\begin{aligned} C^\pm &= A \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-a)} - B e^{\pm i\pi c} \frac{\Gamma(2-c)}{\Gamma(b-c+1)\Gamma(1-a)}, \\ D^\pm &= A \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-b)} - B e^{\pm i\pi c} \frac{\Gamma(2-c)}{\Gamma(a-c+1)\Gamma(1-b)}. \end{aligned} \quad (2.28)$$

The boundary conditions now depend on whether η is real or imaginary.

2.5.1 Normal modes

Assume that η is real. In this case, we impose normalizable boundary conditions, corresponding to demanding that $D^+ = D^- = 0$, a pair of simultaneous equations for A, B . Non-zero solutions exist only if the determinant of this system vanishes. Using $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, this gives

$$\frac{(1-c)\pi}{\Gamma(a)\Gamma(1-b)\Gamma(c-b)\Gamma(a-c+1)} = 0. \quad (2.29)$$

This imposes a quantization condition on ω , corresponding to the two solutions $a = -n$ and $1 - b = -n$ where $n = 0, 1, 2, \dots$ ⁴ The former solution gives

$$\omega = -(n + 1/2 + \eta/2), \quad n = 0, 1, 2, \dots, \quad B = 0, \quad (2.30)$$

and the latter gives

$$\omega = n + 1/2 + \eta/2, \quad n = 0, 1, 2, \dots, \quad A = 0. \quad (2.31)$$

We can summarize the normal mode spectrum as

$$\omega = \pm(n + 1/2 + \eta/2), \quad n = 0, 1, 2, \dots \quad (2.32)$$

This is precisely the spectrum of normal modes found for a massive charged scalar in AdS_2 with a homogeneous electric field in ref. [14].

Note that we have allowed ω to be positive or negative. This is because the Teukolsky equation for $s \neq 0$ is *not* invariant under complex conjugation, so negative frequency solutions are not simply related to positive frequency solutions by complex conjugation, they have to be considered separately. The two possible signs correspond to the two different helicities of the field. The radial equation is invariant under $\omega \rightarrow -\omega$, $r \rightarrow -r$ hence $\Phi_{lm(-\omega)}^{(s)}(r) \propto \Phi_{lm\omega}^{(s)}(-r)$.

For $n = 0$, the positive frequency solution of the radial equation is

$$\Phi_{lm(n=0)}^{(s)} \propto z^{-(1+\eta)/4+iq/2}(1-z)^{-(1+\eta)/4-iq/2}, \quad (2.33)$$

and the negative frequency solution is obtained by $r \rightarrow -r$, i.e., $z \rightarrow 1-z$. The solutions with positive n are related to these $n = 0$ solutions by multiplication by a polynomial of degree n in z .

2.5.2 Traveling waves

Now consider the case of imaginary η . Define $\tilde{\eta} > 0$ by $\eta = i\tilde{\eta}$. The radial function oscillates at infinity, corresponding to incoming or outgoing waves (see appendix B for details). Rather than considering scattering in NHEK, we shall impose boundary conditions corresponding to purely outgoing waves at infinity, which will discretize the frequency ω and render it complex. A solution with positive imaginary part corresponds to an instability, and a solution with negative imaginary part is a quasinormal mode.

As discussed by BH, there are two inequivalent notions of “outgoing” that one can use in NHEK because the phase velocity and group velocity of wavepackets need not have the same sign, e.g. for positive ω and m , the group and phase velocities have the same sign at $r = +\infty$ but opposite sign at $r = -\infty$ (see table 3). Physical boundary conditions correspond to the notion of “outgoing” defined using the group velocity. However, it is easier to analyze the case of outgoing phase, so we shall consider this case first.

⁴At first sight, condition (2.29) could also be satisfied if we imposed $c = 1$, i.e., $\omega = iq$. However, a more careful analysis rules out this possibility because for $c = 1$, (2.26) is not a solution of the problem: one must allow for a logarithmic dependence in the second part. Redoing the analysis with the appropriate regular radial solution for this special case [24], we conclude that nothing physically special occurs for $c = 1$.

Assume that $\text{Re}(\omega) > 0$. Then the solutions with outgoing phase at $r \rightarrow \pm\infty$ are the solutions with $C^\pm = 0$. This leads to the quantization condition

$$\frac{(1-c)\pi}{\Gamma(b)\Gamma(1-a)\Gamma(c-a)\Gamma(b-c+1)} = 0, \quad (2.34)$$

with solution $1-a = -n$, $n = 0, 1, 2, \dots$ ($b = -n$ is inconsistent with $\text{Re}(\omega) > 0$), which gives $\omega = n + 1/2 - i\tilde{\eta}/2$. Repeating the exercise for $\text{Re}(\omega) < 0$ requires $D^\pm = 0$, and leads to $\omega = -(n + 1/2) - i\tilde{\eta}/2$. We can summarize the result as

$$\omega = n + 1/2 - i\tilde{\eta}/2, \quad n \in \mathbb{Z} \quad (2.35)$$

The imaginary part is negative, hence these are quasinormal modes. This is a little surprising. BH pointed out that the energy flux (for positive frequency modes) has the same sign as the phase velocity. Hence outgoing phase should correspond to outgoing energy at infinity. As discussed in the introduction, this is precisely the situation in which one expects an instability associated with the negative energy in matter fields within the ergoregion becoming increasingly negative. We have found that outgoing phase leads to stable quasinormal modes rather than an instability. However, these boundary conditions are unphysical: we are arranging that an initial wavepacket (at finite r) composed of modes with positive ω, m does not propagate to $r = -\infty$ by sending in an appropriate (finely tuned) wavepacket from $r = -\infty$ to scatter with it in such a way as to produce only a wavepacket propagating to $r = +\infty$. This is analogous to boundary conditions for a Kerr black hole in which one arranges that initial data leads to no waves crossing the future horizon by sending in appropriate waves from the past horizon. Presumably, the fine-tuning is the reason that we do not see an instability here.

Now consider the physical boundary conditions corresponding to “outgoing” defined with respect to the group velocity. Assume that $\text{Re}(\omega) > 0$ and $m > 0$. BH showed that, under these conditions, the phase and group velocities have the same sign for $r \rightarrow \infty$ but opposite sign for $r \rightarrow -\infty$. Hence the boundary conditions that we need are $C^+ = D^- = 0$. In fact, the same holds for $\text{Re}(\omega) < 0$ and $m > 0$ (see appendix B). Using the identity $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, we find that the quantization condition is

$$\sin(\pi b) \sin[\pi(c-a)]e^{-i\pi c} = \sin(\pi a) \sin[\pi(c-b)]e^{i\pi c}, \quad (2.36)$$

which gives⁵

$$\omega = n + \frac{1}{2} - \frac{i}{2\pi} \log \left[\frac{\cosh[\pi(\tilde{\eta}/2 + m)]}{\cosh[\pi(\tilde{\eta}/2 - m)]} \right], \quad n \in \mathbb{Z}, \quad (m > 0) \quad (2.37)$$

where we have specialized to scalar field ($s = 0$) or gravitational (± 2) perturbations for simplicity. Repeating the analysis for $m < 0$ requires $C^- = D^+ = 0$. The general result is

$$\omega = n + \frac{1}{2} - \frac{i}{2\pi} \log \left[\frac{\cosh[\pi(\tilde{\eta}/2 + |m|)]}{\cosh[\pi(\tilde{\eta}/2 - |m|)]} \right], \quad n \in \mathbb{Z} \quad (2.38)$$

We see that $\text{Im}(\omega) < 0$ hence these are stable quasinormal modes. So NHEK is stable against linearized gravitational (and scalar field) perturbations.

⁵A solution corresponding to c taking integer values is ruled out for the reason discussed in footnote 4.

3 Metric perturbations

3.1 Hertz potentials

The NP scalar perturbations $\Psi^{(s)}$ are useful because they are invariant under infinitesimal diffeomorphisms and under rotations of the NP tetrad. Many physically interesting quantities can be computed directly from the knowledge of these NP fields [12, 13, 21]. However, in some problems as is our case, we really need to know the perturbations of the metric itself, $h_{\mu\nu}$, or the perturbations of the Maxwell or Weyl fermionic vector fields, respectively A_μ and χ_μ . Cohen and Kegeles [15, 17], and Chrzanowski [16] have proposed a unique map that provides the $h_{\mu\nu}$, A_μ or χ_μ perturbations given the so-called Hertz potential $\Psi_H^{(s)}$ (see a good discussion also in [19]). Wald proved Cohen-Kegeles–Chrzanowski’s results [18]. See appendix C for a detailed discussion of these works. The main conclusion is that the Hertz potential also obeys a pair of decoupled equations, again one for positive and the other for negative s . These are written in equations (C.18) and (C.19).

For gravitational perturbations, this method yields the metric perturbation in a particular gauge: the ingoing (outgoing) radiation gauge IRG (ORG), specified by the conditions

$$\ell^\mu h_{\mu\nu} = g^{\mu\nu} h_{\mu\nu} = 0 \text{ (IRG)}, \quad n^\mu h_{\mu\nu} = g^{\mu\nu} h_{\mu\nu} = 0 \text{ (ORG)}. \quad (3.1)$$

At first sight, these gauge conditions appear overdetermined but it has been shown that, for perturbations of a type II vacuum spacetime, there is a residual gauge freedom that allows one to impose the IRG provided that $\ell^\mu \ell^\nu T_{\mu\nu} = 0$ where ℓ is the repeated principal null direction and $T_{\mu\nu}$ the stress-tensor of any matter perturbation present [25]. Similarly, for type D one can impose either the IRG or the ORG (if $n^\mu n^\nu T_{\mu\nu} = 0$). The spin of the Hertz potential corresponds to these two different gauges: the metric perturbation in the IRG (ORG) is obtained from the Hertz potential with $s = -2$ ($s = +2$). The two Hertz potentials contain exactly the same physical information, so one need only work with one of them.

For vacuum type D spacetimes, the Hertz potential itself satisfies a master equation. For the Kerr solution, this master equation turns out to be exactly the same as for the original NP scalars $\Psi^{(s)}$, equation (2.5), with no source term on the RHS [26]. We have checked that the same is true for NHEK. More concretely, $\Psi_H^{(s)} = \{\Psi_H^{(-2)}, \Psi_H^{(-1)}, \Psi_H^{(-1/2)}\}$ are the Hertz potentials conjugate to the positive spin Teukolsky perturbations $\{\Psi^{(2)}, \Psi^{(1)}, \Psi^{(1/2)}\}$ but satisfy exactly the same master equation (2.5) as $(-\Psi_2)^{-\frac{2s}{3}} \Psi^{(s)}$ for *negative* spin. Similarly, $\Psi_H^{(s)} = \{\Psi_H^{(2)}, \Psi_H^{(1)}, \Psi_H^{(1/2)}\}$ are the Hertz potentials conjugate to the negative spin Teukolsky perturbations $\{\Psi^{(-2)}, \Psi^{(-1)}, \Psi^{(-1/2)}\}$ but the positive spin Hertz potential $(-\Psi_2)^{-\frac{2s}{3}} \Psi_H^{(s)}$ obeys the same master equation as $\Psi^{(s)}$ for *positive* spin. In short, the Hertz potential obeys the same master equation as its conjugated Teukolsky field but with spin sign traded. This relation is better clarified if we use tables 1 and 2.

Assuming perturbations for the Hertz potentials of the form

$$\Psi_H^{(s)} = \begin{cases} e^{-i\omega t} e^{im\phi} R_{lm\omega}(r) S_{lm}(\theta), & s \leq 0, \\ e^{-i\omega t} e^{im\phi} R_{lm\omega}(r) S_{lm}(\theta) (-\Psi_2)^{-\frac{2s}{3}}, & s \geq 0, \end{cases} \quad (3.2)$$

$\Psi_{\text{H}}^{(s)}$	$\Psi_{\text{H}_0}^{(-2)}$	$(-\Psi_2)^{-\frac{4}{3}} \Psi_{\text{H}_4}^{(2)}$	$\Psi_{\text{H}_0}^{(-1)}$	$(-\Psi_2)^{-\frac{2}{3}} \Psi_{\text{H}_2}^{(1)}$	$\Psi_{\text{H}_0}^{(-1/2)}$	$(-\Psi_2)^{-\frac{1}{3}} \Psi_{\text{H}_1}^{(1/2)}$	Φ
s	-2	2	-1	1	$-\frac{1}{2}$	$\frac{1}{2}$	0

Table 2. Spin s Hertz fields $\Psi_{\text{H}}^{(s)}$ that satisfy the master equation (2.5) with no source term.

where $R_{lm\omega}(r)$ further satisfies (2.11), equation (2.5) separates into an angular and radial equations. The angular equation is (2.7) with $C = m/2$, and the radial equation is (2.12). Its solution is given by (2.26).

As stated above, given the Hertz potential for the gravitational field there is a unique map between it and the metric perturbations [15–18]. A similar map exists between the spin $s = \pm 1, \pm 1/2$ Hertz potentials and the Maxwell and Weyl fermionic vector perturbations, but we leave the discussion of these cases to appendix C. In the ingoing radiation gauge the metric perturbation in NP notation is given by (see appendix C)

$$h_{\mu\nu}^{\text{IRG}} = \left\{ \ell_{(\mu} m_{\nu)} [(D+3\epsilon+\bar{\epsilon}-\rho+\bar{\rho})(\delta+4\beta+3\tau) + (\delta+3\beta-\bar{\alpha}-\tau-\bar{\pi})(D+4\epsilon+3\rho)] - \ell_{\mu} \ell_{\nu} (\delta+3\beta+\bar{\alpha}-\bar{\tau})(\delta+4\beta+3\tau) - m_{\mu} m_{\nu} (D+3\epsilon-\bar{\epsilon}-\rho)(D+4\epsilon+3\rho) \right\} \Psi_{\text{H}} + \text{c.c.}, \quad (3.3)$$

and a similar correspondence exists between the Hertz potential and the metric perturbations $h_{\mu\nu}^{\text{ORG}}$ in the outgoing radiation gauge. (See the second relation of (C.10).) One can check that (3.3) indeed satisfies the linearized Einstein's equations for a traceless metric perturbation:

$$-\nabla_{\alpha} \nabla^{\alpha} h_{\mu\nu} - 2R_{\mu\alpha\nu\beta} h^{\alpha\beta} + 2g^{\alpha\beta} \nabla_{(\mu} \nabla_{|\alpha|} h_{\nu)\beta} = 0. \quad (3.4)$$

3.2 Behaviour of solutions

The basis vector fields ℓ and n are globally well-defined. However, the vector field m is singular at $\theta = 0, \pi$. Nevertheless, one can check that angular dependence of the Hertz potential contains a sufficiently high power of $\sin \theta$ to ensure that the above metric perturbation is smooth at $\theta = 0, \pi$.

The asymptotic behaviour of the Hertz potential $\Psi_{\text{H}}^{(\pm 2)}$ can be obtained using (2.11) and (2.19). Use of (3.3) yields then for the asymptotic $h_{\mu\nu}$ behaviour (rows and columns follow the order: $\{t, r, \theta, \phi\}$)

$$h_{\mu\nu}^{\text{IRG}} \sim r^{\frac{3}{2} \pm \frac{1}{2} \eta} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\frac{1}{r^2}) & \mathcal{O}(\frac{1}{r}) & \mathcal{O}(\frac{1}{r}) \\ & \mathcal{O}(\frac{1}{r^4}) & \mathcal{O}(\frac{1}{r^3}) & \mathcal{O}(\frac{1}{r^3}) \\ & & \mathcal{O}(\frac{1}{r^2}) & \mathcal{O}(\frac{1}{r^2}) \\ & & & \mathcal{O}(\frac{1}{r^2}) \end{pmatrix}, \quad (3.5)$$

where η is given by (2.20). Exactly the same result is obtained in the outgoing radiation gauge. In (3.5) we have not imposed any boundary condition. These were discussed in subsection 2.5; e.g., for $\eta^2 > 0$, the lower sign would correspond to normal modes.

We shall now compare the above asymptotic behaviour of metric perturbations with the GHSS fall-off conditions. The tr and $t\theta$ components are the most restrictive. For these to satisfy the fall-off conditions, η must be real, so traveling waves are excluded, we must use normalizable boundary conditions (i.e. the lower sign choice) and we need $\eta \geq 3$. Recall that there are normal modes with $\eta = 2.74$, so it appears that the GHSS fall-off conditions exclude some of the normal modes.⁶

As emphasized in the introduction, at the nonlinear level, we expect that interactions will lead to modes corresponding to traveling waves ($\eta^2 < 0$) being excited, which would lead to a violation of the GHSS fall-off conditions. The only modes that escape this conclusion are the axisymmetric ones (which have with $\eta = 2l + 1$), which always obey the GHSS boundary conditions. Axisymmetric modes form a consistent truncation of the full set of modes in the sense that linearized axisymmetric modes will not excite non-axisymmetric modes at next order in perturbation theory.

3.3 The energy

3.3.1 Massless scalar field

We want to compute the energy associated with the gravitational perturbations that we found in the previous subsection. Since this will involve a rather lengthy calculation, we shall start with the conceptually simpler case of a massless complex scalar field:

$$\square\Phi = 0. \tag{3.6}$$

The canonical energy momentum tensor is given by

$$T_{\mu\nu} = \nabla_{(\mu}\Phi\nabla_{\nu)}\Phi^* - \frac{1}{2}\bar{g}_{\mu\nu}\nabla_\alpha\Phi\nabla^\alpha\Phi^*. \tag{3.7}$$

Let Σ be a spacelike hypersurface with future-directed unit normal n^μ . Then, given any Killing vector ξ^μ , we can define the associated conserved charge

$$Q_\xi[\Phi] = \int_\Sigma d^3x\sqrt{-\gamma}T^{\mu\nu}n_\mu\xi_\nu, \tag{3.8}$$

where $\gamma_{\mu\nu} = \bar{g}_{\mu\nu} - n_\mu n_\nu$ is the induced metric on Σ . We shall choose Σ to be a surface of constant t in the NHEK geometry. The conserved charges of interest are the energy \mathcal{E} , for $\xi = \partial/\partial t$, and the angular momentum \mathcal{J} , for $\xi = -\partial/\partial\phi$. (The latter is the U(1) charge of GHSS.) Written out explicitly, these are

$$\begin{aligned} \mathcal{E} &= M^2 \int_0^{2\pi} d\phi \int_0^\pi d\theta \int_{-\infty}^\infty dr \left[\frac{|\partial_t\Phi|^2}{1+r^2} + (1+r^2)|\partial_r\Phi|^2 + |\partial_\theta\Phi|^2 + \left(\frac{1}{\Lambda(\theta)^2} - \frac{r^2}{1+r^2} \right) |\partial_\phi\Phi|^2 \right], \\ \mathcal{J} &= M^2 \int_0^{2\pi} d\phi \int_0^\pi d\theta \int_{-\infty}^\infty dr \frac{1}{1+r^2} \left[2r|\partial_\phi\Phi|^2 - (\partial_t\Phi\partial_\phi\Phi^* + \partial_t\Phi^*\partial_\phi\Phi) \right]. \end{aligned} \tag{3.9}$$

⁶It is conceivable that a gauge transformation could be used to bring a mode violating the fall-off conditions to one that satisfies these conditions but this seems unlikely, especially for traveling waves.

In the energy integrand, all the terms except the last are manifestly positive. The last is proportional to $|\partial_t|^2 = g_{tt}$ and thus is positive only outside the ergosphere where $1 + r^2 - r^2\Lambda(\theta)^2 > 0$. The energy can thus be negative (for a rigorous proof of this, see ref. [5]).

Consider a general superposition of normalizable modes (recall that n is defined by the frequency quantization (2.32)):

$$\Phi(x) = \sum_{nlm} (a_{nlm}\Phi_{nlm}(x) + b_{nlm}\Phi_{nlm}(x)^*). \quad (3.10)$$

First, we shall show that the conserved charges associated with such a solution can be decomposed into a sum of conserved charges of the individual modes.

A charge integral $Q_\xi[\Phi]$ can be regarded as defining a (typically indefinite) norm on the space of solutions of the wave equation. Given a norm $\| \cdot \|$, it can be “polarized” to obtain a Hermitian scalar product (\cdot, \cdot) : the real and imaginary parts of (u, v) are given by $(|u + v| - |u| - |v|)/2$ and $(|iu + v| - |u| - |v|)/2$ respectively. In our case, polarizing the charge integral defines a scalar product $(\Phi_1, \Phi_2)_\xi$, antilinear in Φ_1 and linear in Φ_2 . Since the norm is conserved, so will be the scalar product. Note that $(\Phi, \Phi)_\xi = Q_\xi[\Phi]$.

We shall now argue that modes with different (nlm) are orthogonal with respect to this scalar product. The scalar product has the form

$$(\Phi_1, \Phi_2)_\xi = \int_\Sigma d^3x Q^{\mu\nu}(x) \partial_\mu \Phi_1^* \partial_\nu \Phi_2, \quad (3.11)$$

where $Q^{\mu\nu}$ is preserved by any Killing vector field that commutes with ξ . Now let η be such a Killing field. We can then write

$$(\Phi_1, -i\mathcal{L}_\eta \Phi_2)_\xi - (-i\mathcal{L}_\eta \Phi_1, \Phi_2)_\xi = \int_\Sigma \mathcal{L}_\eta(Q^{\mu\nu}(x) \partial_\mu \Phi_1^* \partial_\nu \Phi_2). \quad (3.12)$$

If the RHS vanishes then this shows that $-i\mathcal{L}_\eta$ is self-adjoint with respect to this scalar product. For NHEK, we take $\xi = \partial/\partial t$ or $\partial/\partial \phi$. Taking $\eta = \partial/\partial t$, the RHS vanishes because the scalar product is conserved, and hence independent of t . Taking $\eta = \partial/\partial \phi$, the RHS vanishes because it is a total derivative on Σ . It follows that modes with different ω or different m will be orthogonal with respect to this scalar product. Hence, in calculating the charge associated with (3.10), there are no cross-terms in the charge arising from modes with different ω or m (in particular, there are no cross-terms between the positive and negative frequency parts of (3.10)).

Now consider the l -dependence. Since l is not associated with a Killing symmetry of the background, we cannot use the above argument. Instead, for separable solutions, the angular dependence will be given by (2.7) with $s = 0$. This equation is self-adjoint, so two solutions with different values of $\Lambda_{lm}^{(s)}$ will be orthogonal with respect to the measure $\sin \theta$, i.e.,

$$\int_0^\pi d\theta \sin \theta S_{l_1 m}^{(s)}(\theta) S_{l_2 m}^{(s)}(\theta)^* \propto \delta_{l_1 l_2}. \quad (3.13)$$

Fortunately, it turns out that $\sin \theta$ is precisely the measure that arises in the scalar products associated with the energy and angular momentum.

From these results, we see that no cross-terms between modes with different (nlm) contribute to the energy and angular momentum. Substituting (3.10) into (3.8) for $\xi = \partial_t$ gives the energy as a sum over contributions from individual modes:

$$\mathcal{E} = \sum_{nlm} \mathcal{E}_{nlm} (|a_{nlm}|^2 + |b_{nlm}|^2), \quad (3.14)$$

where

$$\mathcal{E}_{nlm} \equiv 4\pi M^2 \omega_{nlm} \int_0^\pi d\theta \sin\theta |S_{lm}^{(0)}(\theta)|^2 \int_{-\infty}^{+\infty} dr |R_{nlm}^{(0)}(r)|^2 \frac{\omega_{nlm} + mr}{1+r^2}. \quad (3.15)$$

Note that \mathcal{E}_{nlm} is manifestly positive only when $m = 0$. However, we have evaluated the radial integral above for many cases, namely for $0 \leq l \leq 10$, $-l \leq m \leq l$ and $0 \leq n \leq 10$. In all these cases, it is positive. Hence, for a massless complex scalar field in the NHEK geometry, the energy of an arbitrary superposition of normalizable modes is positive.

The angular momentum can be similarly decomposed:

$$\mathcal{J} = \sum_{nlm} \mathcal{J}_{nlm} (|a_{nlm}|^2 - |b_{nlm}|^2), \quad (3.16)$$

where we find the simple result

$$\frac{\mathcal{J}_{nlm}}{\mathcal{E}_{nlm}} = \frac{m}{\omega_{nlm}}. \quad (3.17)$$

3.3.2 Gravitational perturbations

The energy of gravitational perturbations is calculated from the Landau-Lifshitz “pseudotensor” defined as follows. Consider metric perturbations $h_{\mu\nu}$ around NHEK up to second order in the amplitude,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}^{(1)} + h_{\mu\nu}^{(2)} + \mathcal{O}(h^3), \quad (3.18)$$

The linearized Einstein equation is⁷

$$G_{\mu\nu}^{(1)}[h^{(1)}] = 0. \quad (3.19)$$

At second order, the Einstein equation relates terms linear in $h^{(2)}$ to terms quadratic in $h^{(1)}$:

$$G_{\mu\nu}^{(1)}[h^{(2)}] = -G_{\mu\nu}^{(2)}[h^{(1)}] \equiv 8\pi GT_{\mu\nu}[h^{(1)}], \quad (3.20)$$

where the RHS is quadratic in $h^{(1)}$. Written out explicitly, for traceless perturbations it reads (here, we use the notation $h_{\mu\nu} \equiv h_{\mu\nu}^{(1)}$)

$$\begin{aligned} 8\pi GT_{\mu\nu} = & -\frac{1}{2} \left[\frac{1}{2} (\nabla_\mu h_{\alpha\beta}) \nabla_\nu h^{\alpha\beta} + h^{\alpha\beta} (\nabla_\nu \nabla_\mu h_{\alpha\beta} + \nabla_\alpha \nabla_\beta h_{\mu\nu} - \nabla_\alpha \nabla_\mu h_{\nu\beta} - \nabla_\alpha \nabla_\nu h_{\mu\beta}) \right. \\ & \left. + \nabla_\alpha h_{\mu\beta} (\nabla^\alpha h_{\beta\nu} - \nabla_\beta h_{\nu}^\alpha) - \nabla_\alpha h^{\alpha\beta} (\nabla_\mu h_{\beta\nu} + \nabla_\nu h_{\mu\beta} - \nabla_\beta h_{\mu\nu}) \right] \\ & + \frac{1}{4} \bar{g}_{\mu\nu} \left[\frac{1}{2} (\nabla_\gamma h_{\alpha\beta}) \nabla^\gamma h^{\alpha\beta} + h^{\alpha\beta} (\nabla_\gamma \nabla^\gamma h_{\alpha\beta} - 2\nabla_\alpha \nabla^\gamma h_{\gamma\beta}) \right. \\ & \left. + \nabla_\alpha h^{\beta\gamma} (\nabla^\alpha h_{\beta\gamma} - \nabla_\beta h_{\gamma}^\alpha) - 2 (\nabla_\alpha h^{\alpha\beta}) \nabla^\gamma h_{\beta\gamma} \right]. \quad (3.21) \end{aligned}$$

⁷Written out, this takes the standard Lichnerowicz form. If we assume that $h^{(1)}$ is traceless then this equation reduces to (3.4).

We now define the conserved charges $Q_\xi[h^{(1)}]$ associated with the first order perturbation exactly as in (3.8) with $\xi = \partial/\partial t$, $-\partial/\partial\phi$ giving the energy and angular momentum respectively.

Recall that $h^{(1)}$ is related to the Hertz potential by equation (3.3), which is second order in derivatives. It follows that the conserved charges are given by integrals of quantities that are sixth order in derivatives. Hence calculating these charges involves very lengthy calculations, which we have performed using computer algebra.

We consider a Hertz potential corresponding to an arbitrary superposition of normal modes:

$$\Psi_H^{(s)}(x) = \sum_{nlm} \left(a_{nlm} \Psi_{nlm}^{(s)+}(x) + b_{nlm} \Psi_{nlm}^{(s)-}(x) \right), \quad (3.22)$$

where $s = \pm 2$ and the superscript \pm refers to positive and negative frequency respectively.

As in the scalar field, case the conserved charges can be used to define a scalar product $(\cdot)_\xi$ between solutions of the linearized Einstein equation. The only significant difference here is that the metric perturbation is real, so the scalar product will also be real.⁸ One can argue exactly as in the scalar field case that \mathcal{L}_η is anti-self-adjoint with respect to this scalar product if η is a Killing field that commutes with ξ : $(h, \mathcal{L}_\eta k) = (-\mathcal{L}_\eta k, h)$. It follows that the operator $-\mathcal{L}_\eta^2$ is self-adjoint and hence linearized metric perturbations with different ω^2 or different m^2 must be orthogonal.

Given the complexity of the charge integrals, we have not succeeded in demonstrating that modes with different l are orthogonal in the same way that we did for the scalar field. However, note that ω depends on l in a very complicated way (through the eigenvalues $\Lambda_{lm}^{(2)}$ which must be found numerically). Hence it seems very unlikely that modes with different l could have the same ω^2 . Therefore the orthogonality of modes with different ω^2 should ensure the orthogonality of modes with different l . An exception are the axisymmetric ($m = 0$) modes, which have $\omega = n + l + 1$ so modes with the same $n + l$ have the same ω . However, the axisymmetric modes are the “least dangerous” as far as the possibility of negative energy is concerned so we shall not worry about this further, and simply assume that all modes with different l will be orthogonal.

We now turn to our calculation of the conserved charges associated with individual modes. These charges are most easily computed in NP tetrad, since this is the basis in which the metric perturbation takes the relatively simple form (3.3), although the explicit expressions for the components are still too long to be written here. Using Mathematica, the separated equation of motion can be used to reduce the integrands of the charge integrals to expressions first order in derivatives, which were then calculated numerically using Mathematica’s NIntegrate function.

We have calculated the energy of normal modes with $l = 2, 3, 4, 5, 6$ for all allowed values of m , $n = 0, 1, 2, 3, 4, 5, 6$ and both positive and negative frequency. In all cases it comes out positive. This is the main result of this section.

⁸The polarization formula is $(u, v) = (1/2)(|u+v| - |u-v|)$. We could have chosen to work with complex modes $h_{\mu\nu}$, for which the negative frequency modes are complex conjugates of the positive frequency modes. However, then we would have had to take account of two different polarizations for the gravitational modes.

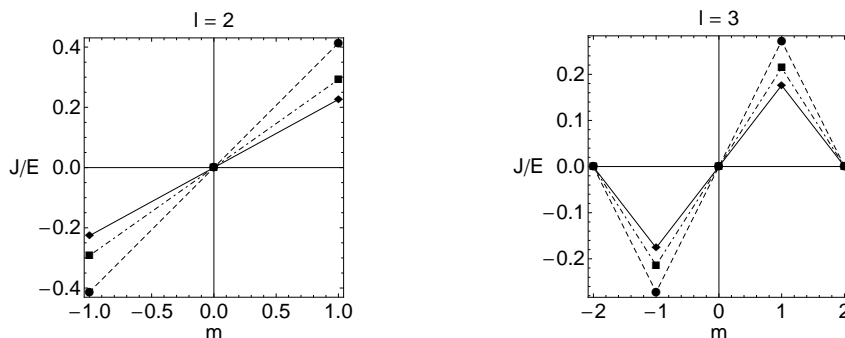


Figure 3. Ratio of the conserved charges \mathcal{J}/\mathcal{E} for spin $|s| = 2$ perturbations as a function of the azimuthal angular number m for a) $l = 2$ and b) $l = 3$. We only consider normal modes, i.e., values of m that yield $\eta^2 > 0$ as defined in (2.20). Data points corresponding to $n = 0, 1, 2$ are plotted, with the solid line representing $n = 2$ and the dashed line $n = 0$.

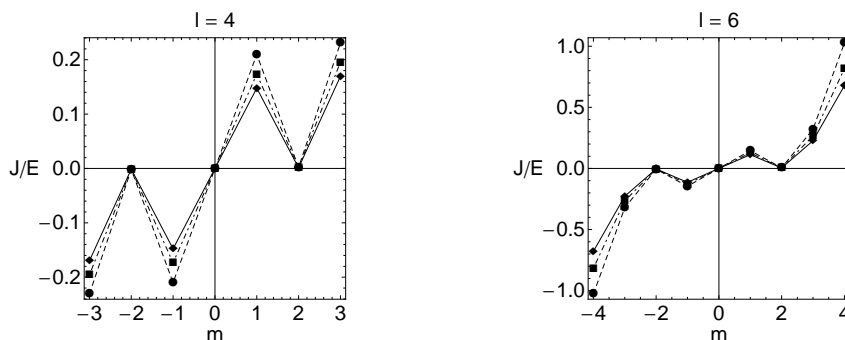


Figure 4. Ratio of the conserved charges \mathcal{J}/\mathcal{E} for spin $|s| = 2$ perturbations as a function of the azimuthal angular number m for a) $l = 4$ and b) $l = 6$.

The only difference between positive and negative frequency modes is the sign of \mathcal{J} so we focus on the positive frequency case. The numerical value of the energy depends on the normalization of the Hertz potential. However, the ratio of the conserved charges, $\mathcal{J}_{nlm}/\mathcal{E}_{nlm}$ is normalization independent. This ratio as a function of m and n for fixed l is displayed in figures 3–4. Note that $\mathcal{J}_{nl(-m)} = -\mathcal{J}_{nlm}$ and that $\mathcal{E}_{nl(-m)} = \mathcal{E}_{nlm}$. Moreover, for any given l the ratio $|\mathcal{J}|/\mathcal{E}$ always has a (non-vanishing) minimum at $|m| = |s| = 2$ (the ratio decreases with n but only slowly, so this is not apparent in the plots). The corresponding modes also exhibit special behaviour in the Kerr geometry: Teukolsky and Press [23] found that for a given black hole rotation and wave frequency, the modes whose energy is most absorbed or superradiantly amplified are precisely those with $l = |m| = |s|$.

4 Discussion: second order perturbations

We now turn to the question of what happens if we go beyond first order in perturbation theory. The second order metric perturbation $h^{(2)}$ is determined by solving (3.20).⁹ We are not going to attempt to solve this equation. Instead, following recent work on TMG [27] we consider the conserved charges.

So far, we have worked with conserved charges defined via bulk integrals quadratic in $h^{(1)}$. However, one can also define conserved charges via boundary integrals, indeed these are the charges discussed by GHSS. So first we shall explain how they are related to our bulk integrals. Consider a 1-parameter family of exact vacuum solutions $g(\lambda)$, where $g(0) \equiv \bar{g}$ is the NHEK metric. Let $h^{(1)} = g'(0)$ and $h^{(2)} = (1/2)g''(0)$. ($h^{(1)}$ is the linearized solution arising from the linearization of $g(\lambda)$, $h^{(2)}$ is the second order correction.) Owing to the unusual fall-off conditions, the conserved charge $Q_\xi(\lambda) \equiv Q_\xi[g(\lambda)]$ associated to a generator ξ of the asymptotic symmetry group are defined by integrating the following expression:

$$\frac{dQ_\xi}{d\lambda} = \mathcal{Q}_\xi[g'(\lambda), g(\lambda)] \tag{4.1}$$

where

$$\begin{aligned} \mathcal{Q}_\xi[h, g] \equiv & -\frac{1}{32\pi G} \int_{\partial\Sigma} \epsilon_{\alpha\beta\mu\nu} \left[\xi^\nu \nabla^\mu h - \xi^\nu \nabla_\sigma h^{\mu\sigma} + \xi_\sigma \nabla^\nu h^{\mu\sigma} + \frac{1}{2} h \nabla^\nu \xi^\mu - h^{\nu\sigma} \nabla_\sigma \xi^\mu \right. \\ & \left. + \frac{1}{2} h^{\sigma\nu} (\nabla^\mu \xi_\sigma + \nabla_\sigma \xi^\mu) \right] dx^\alpha \wedge dx^\beta. \end{aligned} \tag{4.2}$$

Now, our gravitational normal modes decay sufficiently fast that they give $\mathcal{Q}_\xi[h^{(1)}, \bar{g}] = 0$, hence $dQ_\xi/d\lambda = 0$ at $\lambda = 0$. This is no surprise since we know that the energy should be quadratic in $h^{(1)}$. Hence we have to go to next order, and calculate $(1/2)d^2Q_\xi/d\lambda^2$ at $\lambda = 0$. This can be done by differentiating (4.1), which gives a sum of a part linear in $h^{(2)}$, equal to $\mathcal{Q}_\xi[h^{(2)}, \bar{g}]$, and a part quadratic in $h^{(1)}$. However, the normal modes decay so fast that this second part vanishes. Hence, to second order in λ , we have that

$$Q_\xi(\lambda) = \lambda^2 \mathcal{Q}_\xi[h^{(2)}, \bar{g}]. \tag{4.3}$$

Now, assuming that ξ is a Killing field of the background, a standard manipulation [28–30] based on the second-order Einstein equation (3.20) enables one to rewrite this surface integral as the bulk integral quadratic in $h^{(1)}$ that we used in the previous section.

One subtlety is that the NHEK geometry has two boundaries (at $r = \pm\infty$). The bulk integral for the charge will be the sum of the two surface integrals. Hence, if the first order perturbation gives a non-zero conserved charge, then the second order perturbation $h^{(2)}$ must decay sufficiently slowly for these surface integrals to be non-zero.

Consider initial data (say at $t = 0$) for a first order perturbation $h^{(1)}$ that is of compact support (this will necessarily involve harmonics (l, m) corresponding to traveling waves). How will the second order perturbation sourced by this first order perturbation behave?

⁹If we considered scalar field perturbations then the gravitational backreaction of the perturbation would be governed by the same equation with $T_{\mu\nu}$ the scalar field stress tensor.

Near infinity (at least at early times), $h^{(2)}$ must satisfy the source free linearized Einstein equation, i.e., the same equation as the first order perturbation. Hence the behaviour of $h^{(2)}$ near infinity should be the same as that of a first order perturbation. However, none of the first order normal modes decays sufficiently slowly to make a non-vanishing contribution to the surface integrals for the charges, e.g., a non-vanishing contribution to the energy would require $\eta \leq 1$ in (3.5) (with the lower sign choice) whereas we have seen that normal modes have $\eta > 2.74$. A non-vanishing contribution to the angular momentum requires $\eta \leq -1$. Therefore $h^{(2)}$ does not behave like a normal mode at infinity. Furthermore, even the traveling waves decay too slowly to contribute to the surface integral for the angular momentum. So what linearized solution does $h^{(2)}$ behave like near infinity?

Precisely the same issue arises for a Kerr black hole. Gravitational perturbations with $l \geq 2$ decay too fast to contribute to the surface integrals for the energy or angular momentum. For Kerr, the resolution is that the Teukolsky or Hertz potential formalisms miss certain modes, specifically those modes that preserve the type D condition to first order. For Kerr, it has been shown that the only such perturbations correspond to deformations towards a nearby type D solution [31]. The nearby solutions are: the Kerr solution with different (M, J) , the Kerr-NUT solution, and the spinning C-metric. The latter perturbations are excluded by asymptotic boundary conditions or regularity. Hence, for Kerr, one must add by hand the non-dynamical modes corresponding to infinitesimal variations in the mass and angular momentum of the black hole, which we can regard as $l = 0$ and $l = 1$ perturbations respectively. Clearly these will decay at an appropriate rate to contribute to the surface integrals.

This suggests that, in our case, the fall-off of $h^{(2)}$ will be the same as that of linearized modes that preserve the type D property. There are two classes of such modes: (i) modes that are locally gauge, i.e., locally of the form $\nabla_{(\mu}\eta_{\nu)}$, and (ii) modes corresponding to a non-trivial deformation towards a type D solution continuously connected to NHEK.

Consider first the case that $h^{(2)}$ behaves asymptotically as a linearized mode that is locally gauge. By this we mean that, in a neighbourhood of the S^2 on which a boundary integral is computed, $h^{(2)}$ is locally, but not globally, of the form $\nabla_{(\mu}\eta_{\nu)}$.¹⁰ This is precisely what happens for Einstein gravity in AdS_3 , for example, where η_μ cannot be globally defined on the S^1 boundary. Could the same thing happen here? One might consider infinitesimal diffeomorphisms of the GHSS form $\epsilon(\phi)\partial/\partial\phi - \epsilon'(\phi)r\partial/\partial r$ and, instead of taking ϵ to be periodic in ϕ (which would be globally defined), take $\epsilon(\phi) = \phi$, which leads to a metric perturbation independent of t and ϕ . However, this has the effect of introducing a conical singularity into the metric near infinity (at the poles of the S^2), which does not seem appropriate.

This “locally gauge” behaviour would arise from solutions that are obtained by identifications of the NHEK background (in the same way that the BTZ black hole is obtained as an identification of AdS_3). Could one obtain a “NHEK black hole” by identifying the NHEK geometry in some way? Assuming any such identification acts only on the surfaces of constant θ , the possibilities have been well-studied [32], and there appears to be no candidate free of pathologies such as conical singularities or closed timelike curves.

¹⁰If it were globally a gauge transformation then it would give a vanishing boundary integral since, for a Killing field ξ , $\mathcal{Q}_\xi[h, \bar{g}]$ is invariant under $h_{\mu\nu} \rightarrow h_{\mu\nu} + \nabla_{(\mu}\eta_{\nu)}$ (even if η is a non-trivial element of the asymptotic symmetry group).

Consider then, the second possibility, that $h^{(2)}$ behaves asymptotically as a linearized mode corresponding to a deformation towards a nearby type D solution. What solutions are there? Using Kinnersley’s classification of type D solutions [33], the only such solutions appear to be: NHEK with a change in the angular momentum, the full (asymptotically flat) Kerr solution, or the near-horizon geometry of the extremal spinning C-metric. The latter has a conical singularity and so presumably must be excluded.

It appears that the only candidate for a “ $l = 1$ ” mode, i.e., a mode contributing to the surface integral for angular momentum, is the perturbation that corresponds to a change in the angular momentum of the NHEK geometry ($J \rightarrow J + \delta J$ in (2.1)). This violates the GHSS fall-off conditions. Hence it would appear that, at second order, any perturbation with non-vanishing angular momentum is excluded by the fall-off conditions.

What about the energy? One can attempt to obtain a solution with non-zero energy by taking a decoupling limit of the near-extremal Kerr solution at fixed temperature and angular momentum. An analogous decoupling limit of Reissner-Nordström was discussed in ref. [34]. However, in the latter case, it was shown that, even with non-zero temperature, the decoupling geometry is simply $AdS_2 \times S^2$. We find that the same is true for Kerr: in appendix D, we show that the decoupling limit at fixed non-zero temperature leads back to the NHEK geometry. The explanation is presumably the same as in ref. [34], namely that the extreme Kerr black hole has a mass gap.

It appears that the only regular modes with non-zero energy correspond to going to next order in the decoupling limit. This is probably equivalent (up to a $SL(2, \mathbb{R})$ transformation) to retaining the next to leading order term in the near-horizon limit leading from extreme Kerr to NHEK. This clearly gives a solution $k^{(1)}$ of the linearized Einstein equation. However, it violates the GHSS fall-off conditions, indeed at the fully nonlinear level it amounts to considering an asymptotically flat black hole rather than its near-horizon limit.

If correct, this implies that, if the first order perturbation has any non-zero energy (whether positive or negative) or angular momentum then at second order there will be a violation of the GHSS fall-off conditions.¹¹ This would be a satisfying conclusion: one does not have to worry about negative energy initial data, and the positive energy condition is redundant (at least in perturbation theory). What about initial data for a linearized gravitational field with *vanishing* energy and angular momentum? There certainly exists initial data with this property. We have seen that the normal modes have positive energy, so this data must involve traveling waves. With outgoing boundary conditions, the linearized theory predicts that these will disperse, leaving behind only normal modes, with positive energy. If this extends to the nonlinear theory then there still would be a problem since the final state would have to violate the fall-off conditions. It seems to us that the only solution is that, even though this initial data has vanishing energy, the two boundary integrals for the energy would be non-zero, but opposite in sign. Hence one would still obtain $h^{(2)}$ with the asymptotic behaviour just discussed, and thereby violate the fall-off conditions.

¹¹If one wanted to impose these fall-off conditions only at $r \rightarrow \infty$ but didn’t care what happened at $r \rightarrow -\infty$ then one could always add to $h^{(2)}$ an appropriate multiple of $k^{(1)}$ to arrange this, because $k^{(1)}$ satisfies the linearized Einstein equation, and one is free to add to $h^{(2)}$ (which satisfies the inhomogeneous equation (3.20)) any solution of the linearized Einstein equation. Note that the boundary integrals associated with any solution of the linearized Einstein equation, e.g. $k^{(1)}$, must sum to zero.

This reasoning suggests that, at the nonlinear level, there are no non-trivial (i.e. non-isometric to NHEK) solutions of the Einstein equation that are continuously connected to NHEK, and satisfy the GHSS fall-off conditions (see ref. [34] for a proof of a similar result for $AdS_2 \times S^2$). This may imply that the only solutions that satisfy the latter are related to NHEK by large gauge transformations. However, in this case, the dual CFT would consist purely of conformal descendants of the vacuum, which leads to a problem with modular invariance. Alternatively, there might be further solutions that are asymptotic to NHEK in the GHSS sense, but not continuously connected to it. If so, it would be interesting to find these solutions.

Acknowledgments

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A Shear-free null geodesics and NP tetrad for NHEK

A.1 NP quantities for NHEK in global coordinates

NHEK is a Petrov type D geometry and its NP null tetrad is found by looking into the congruence of shear-free null geodesics [21].

Geodesics are the paths that minimise the action associated with the Lagrangian

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = \frac{\delta}{2}, \quad (\text{A.1})$$

where λ is an affine parameter, and $\delta = 0, 1$, respectively, for null and time-like geodesics. Since the NHEK geometry (2.1) is stationary, the energy E and angular momentum L of the particle,

$$p_t = g_{tt}\dot{t} + g_{t\varphi}\dot{\varphi} \equiv E, \quad p_\varphi = g_{t\varphi}\dot{t} + g_{\varphi\varphi}\dot{\varphi} \equiv L, \quad (\text{A.2})$$

are conserved in a geodesic motion, where $p_\mu \equiv \frac{d\mathcal{L}}{dx^\mu}$ is the conjugated momentum. Equation (A.2) yields

$$\dot{t} = \frac{E - Lr}{M^2(1 + \cos^2 \theta)(1 + r^2)}, \quad \dot{\varphi} = -\frac{r(E - Lr)}{M^2(1 + \cos^2 \theta)(1 + r^2)} - \frac{L}{4M^2} \frac{1 + \cos^2 \theta}{\sin^2 \theta}. \quad (\text{A.3})$$

The Hamilton-Jacobi equation for the geodesic motion on a geometry $g_{\mu\nu}$ reads

$$\frac{\partial S}{\partial \lambda} = H\left(x^\mu, \frac{\partial S}{\partial x^\mu}, \lambda\right), \quad H\left(x^\mu, \frac{\partial S}{\partial x^\mu}, \lambda\right) = \frac{1}{2} g_{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu}. \quad (\text{A.4})$$

Assuming a separation *ansatz* of the form

$$S = \frac{1}{2} \delta\lambda + Et + L\varphi + S_r(r) + S_\theta(\theta), \quad (\text{A.5})$$

equation (A.4) for the NHEK boils down to

$$\begin{aligned} \left(\frac{\partial S_\theta}{\partial \theta}\right)^2 &= \Theta(\theta), & \Theta(\theta) &\equiv \Lambda - L^2 \frac{(1 + \cos^2 \theta)^2}{4 \sin^2 \theta} - M^2 \delta \cos^2 \theta; \\ (1 + r^2)^2 \left(\frac{\partial S_r}{\partial r}\right)^2 &= \mathcal{R}(r), & \mathcal{R}(r) &\equiv (E - rL)^2 - (\Lambda + M^2 \delta)(1 + r^2); \end{aligned} \quad (\text{A.6})$$

where Λ is the separation constant. Typically one has $\dot{r}^2 \propto \mathcal{R}(r)$ and $\dot{\theta}^2 \propto \Theta(\theta)$ and the conservation equations (A.3) give the remaining equations for \dot{t} and $\dot{\varphi}$.

The shear-free principal null geodesics are found by requiring $\dot{\theta} \propto \Theta(\theta) = 0$ for $\delta = 0$, which in our case requires

$$\Lambda = L^2 \frac{(1 + \cos^2 \theta_0)^2}{4 \sin^2 \theta_0}, \quad (\text{A.7})$$

for a constant $\theta = \theta_0$. Moreover, these geodesics must also keep \dot{r} , i.e.,

$$\mathcal{R} = (E - rL)^2 - (1 + r^2)L^2 \frac{(1 + \cos^2 \theta_0)^2}{4 \sin^2 \theta_0} \quad (\text{A.8})$$

constant along the motion. Clearly this is possible for any r and θ_0 only if $L = 0$. The energy and angular momentum conservation equations (A.3) then require

$$M^2(1 + \cos^2 \theta_0)\dot{t} = \frac{E}{1 + r^2}, \quad M^2(1 + \cos^2 \theta_0)\dot{\varphi} = -\frac{rE}{1 + r^2}. \quad (\text{A.9})$$

This Hamilton-Jacobi analysis then concludes that shear-free null geodesics have the tangent vectors

$$\begin{aligned} \ell^\mu \partial_\mu &= \frac{1}{1 + r^2} \partial_t + \partial_r - \frac{r}{1 + r^2} \partial_\varphi, \\ n^\mu \partial_\mu &= \frac{1}{4M^2 \Omega^2(\theta)} (\partial_t - (1 + r^2)\partial_r - r\partial_\varphi), \end{aligned} \quad (\text{A.10})$$

that we choose for the real vectors of the NP tetrad since they satisfy the appropriated relations in (A.16). In particular, the normalisation factor for n^μ was chosen to satisfy the normalisation condition $\ell \cdot n = 1$.

We can now check that (A.10) are indeed null geodesic generators and in our way we find Carter's constant of motion for the NHEK. With the NP tetrad choice (2.3) we find the Weyl scalars (2.4) and NHEK is then Petrov type D. In such a spacetime, if we take $k^\mu = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\varphi})$ to be an affinely parametrised geodesic, $k^\mu \nabla_\mu k_\nu = 0$, then [21]

$$\begin{aligned} K &= 2|\Psi_2|^{-2/3}(\mathbf{k} \cdot \boldsymbol{\ell})(\mathbf{k} \cdot \mathbf{n}) - Q|\mathbf{k}|^2 \\ &= 2|\Psi_2|^{-2/3}(\mathbf{k} \cdot \mathbf{m})(\mathbf{k} \cdot \bar{\mathbf{m}}) - \left(Q - |\Psi_2|^{-2/3}\right)|\mathbf{k}|^2, \end{aligned} \quad (\text{A.11})$$

is conserved along k if and only if a scalar Q exists which satisfies the equations

$$DQ = D|\Psi_2|^{-2/3}, \quad \Delta Q = \Delta|\Psi_2|^{-2/3}, \quad \delta Q = \delta^* Q = 0. \quad (\text{A.12})$$

In our case, from (2.4), one has $|\Psi_2|^{-2/3} = M^{4/3} (1 + \cos^2 \theta)$. The scalar $Q = M^{4/3}$ satisfies (A.12) and we can then construct the two conserved Carter quantities (A.11). This yields the pair of equations

$$\begin{aligned} M^2 (1 + \cos^2 \theta)^2 \dot{\theta}^2 &= -L^2 \frac{(1 + \cos^2 \theta)^2}{4 \sin^2 \theta} - M^2 (\delta \cos^2 \theta - K), \\ M^2 (1 + \cos^2 \theta)^2 \dot{r}^2 &= (E - rL)^2 - M^2 (\delta + K)(1 + r^2), \end{aligned} \quad (\text{A.13})$$

whose RHS is, respectively, $\Theta(\theta)$ and $\mathcal{R}(r)$ defined in (A.6) if we identify $\Lambda \equiv M^2 K$. Carter's equations (A.13), combined with the energy and angular momentum conservation equations (A.3), reduce the finding of geodesics in NHEK to a quadrature problem.

If we want shear-free null geodesics we demand $\delta = 0$ and $\dot{\theta} = 0$ which implies the relation (A.7). The radial equation then stays

$$M^2 (1 + \cos^2 \theta_0)^2 \dot{r}^2 = (E - rL)^2 - (1 + r^2)L^2 \frac{(1 + \cos^2 \theta_0)^2}{4 \sin^2 \theta_0}, \quad (\text{A.14})$$

which can be independent of r only for $L = 0$. Inserting this condition in (A.3) yields (A.9), and under the redefinition $E \rightarrow E (1 + \cos^2 \theta_0)$ we finally confirm that shear-free null geodesics are those that satisfy

$$\dot{t} = \frac{E}{1 + r^2}, \quad \dot{r} = \pm E, \quad \dot{\theta} = 0, \quad \dot{\phi} = -\frac{r}{1 + r^2} E. \quad (\text{A.15})$$

These two geodesics give us the null NP vectors ℓ and \mathbf{n} as well as the Eddington-Finkelstein coordinates for the NHEK. The NP tetrad is completed with the introduction of the complex conjugate pair of vectors m^μ and \bar{m}^μ as defined in (2.3). These are found requiring that the NP tetrad satisfies the normalization and orthogonality conditions

$$\begin{aligned} \ell \cdot \mathbf{m} &= \ell \cdot \bar{\mathbf{m}} = \mathbf{n} \cdot \mathbf{m} = \mathbf{n} \cdot \bar{\mathbf{m}} = 0, \\ \ell \cdot \ell &= \mathbf{n} \cdot \mathbf{n} = \mathbf{m} \cdot \mathbf{m} = \bar{\mathbf{m}} \cdot \bar{\mathbf{m}} = 0, \\ \ell \cdot \mathbf{n} &= 1, \quad \mathbf{m} \cdot \bar{\mathbf{m}} = -1. \end{aligned} \quad (\text{A.16})$$

In terms of the NP tetrad, the metric components read

$$g_{\mu\nu} = 2\ell_{(\mu} n_{\nu)} - 2m_{(\mu} \bar{m}_{\nu)}. \quad (\text{A.17})$$

The 12 complex spin coefficients are introduced through linear combinations of the 24 Ricci rotation connection coefficients $\gamma_{cab} = e_{(c)}^\mu e_{(b)}^\nu \nabla_\nu e_{(a)\mu}$,

$$\begin{aligned} \kappa &= \gamma_{311} = 0, \quad \sigma = \gamma_{313} = 0, \quad \nu = \gamma_{242} = 0, \quad \lambda = \gamma_{244} = 0, \quad \epsilon = \frac{1}{2}(\gamma_{211} + \gamma_{341}) = 0, \\ \mu &= \gamma_{243} = 0, \quad \rho = \gamma_{314} = 0, \quad \gamma = \frac{1}{2}(\gamma_{212} + \gamma_{342}) = \frac{r}{2M^2(1 + \cos^2 \theta)}, \\ \tau &= \gamma_{312} = -\frac{i \sin \theta}{\sqrt{2}M(1 + \cos^2 \theta)}, \quad \alpha = \frac{1}{2}(\gamma_{214} + \gamma_{344}) = -\frac{\cos \theta - i(2 - \cos^2 \theta)}{2\sqrt{2}M(1 - i \cos \theta)^2 \sin \theta}, \\ \pi &= \gamma_{241} = \frac{i \sin \theta}{\sqrt{2}M(1 - i \cos \theta)^2}, \quad \beta = \frac{1}{2}(\gamma_{213} + \gamma_{343}) = \frac{\cos \theta}{2\sqrt{2}M(1 + i \cos \theta) \sin \theta}. \end{aligned} \quad (\text{A.18})$$

Their complex conjugates are obtained through the replacement $3 \leftrightarrow 4$ in γ_{cab} . From the Goldberg-Sachs theorem, $\kappa = \sigma = \nu = \lambda = 0$ implies that the NHEK is Petrov type D (as it must be by construction). Moreover, $\epsilon = 0$ implies that ℓ is affinely parametrised as it is indeed the case.

The Weyl tensor

$$C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} - \frac{1}{2}(g_{\mu\alpha}R_{\nu\beta} + g_{\nu\beta}R_{\mu\alpha} - g_{\nu\alpha}R_{\mu\beta} - g_{\mu\beta}R_{\nu\alpha}) + \frac{1}{6}(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}), \quad (\text{A.19})$$

reduces to the Riemann tensor because (2.1) is Ricci flat. The 5 complex Weyl scalars Ψ_i in the NP formalism encode the information on the 10 independent components C_{abcd} of the Weyl tensor,

$$\begin{aligned} \Psi_0 &= -C_{1313} = -C_{\mu\nu\alpha\beta} \ell^\mu m^\nu \ell^\alpha m^\beta, \\ \Psi_1 &= -C_{1213} = -C_{\mu\nu\alpha\beta} \ell^\mu n^\nu \ell^\alpha m^\beta, \\ \Psi_2 &= -C_{1342} = -C_{\mu\nu\alpha\beta} \ell^\mu m^\nu \bar{m}^\alpha n^\beta, \\ \Psi_3 &= -C_{1242} = -C_{\mu\nu\alpha\beta} \ell^\mu n^\nu \bar{m}^\alpha n^\beta, \\ \Psi_4 &= -C_{2424} = -C_{\mu\nu\alpha\beta} n^\mu \bar{m}^\nu n^\alpha \bar{m}^\beta. \end{aligned} \quad (\text{A.20})$$

For the NHEK these Weyl scalars are listed in (2.4).

The fundamental quantities in the NP formalism needed to study perturbations are the spin coefficients listed in (A.18) and the directional derivative operators,

$$D = \ell^\mu \nabla_\mu, \quad \Delta = n^\mu \nabla_\mu, \quad \delta = m^\mu \nabla_\mu, \quad \delta^* = \bar{m}^\mu \nabla_\mu. \quad (\text{A.21})$$

A.2 Master equation for NHEK in Poincaré coordinates

For completeness we write here the master equation for NHEK in Poincaré coordinates. This is the counterpart of the global coordinate master equation (2.5).

Let quantities with tildes denote Boyer-Lindquist coordinates of the full black hole solution (D.1), and take $\{\tau, y, \theta, \varphi\}$ to be the Poincaré coordinates describing NHEK.

Bardeen and Horowitz define the near-horizon limit of the extreme Kerr solution by setting [1]

$$\tilde{r} = a + \lambda y, \quad \tilde{t} = \frac{\tau}{\lambda}, \quad \tilde{\phi} = \varphi + \frac{\tau}{2a\lambda}, \quad (\text{A.22})$$

where a is the extreme value for the Kerr rotation parameter, and taking the limit $\lambda \rightarrow 0$ with the untilded quantities held fixed. The limit yields the near-horizon solution in Poincaré coordinates. Taking this limit in the Kinnersley tetrad [13], one finds that

$$\begin{aligned} \lambda \ell &\rightarrow \frac{2a^2}{y} \frac{\partial}{\partial \tau} + \frac{\partial}{\partial y} - \frac{1}{y} \frac{\partial}{\partial \varphi}, \\ \lambda^{-1} n &\rightarrow \frac{1}{a^2(1 + \cos \theta)} \left(a^2 \frac{\partial}{\partial \tau} - \frac{y^2}{2} \frac{\partial}{\partial y} - \frac{y}{2} \frac{\partial}{\partial \varphi} \right), \\ m &\rightarrow \frac{1}{\sqrt{2}a(1 + i \cos \theta)} \left(\frac{\partial}{\partial \theta} + i \frac{(1 + \cos^2 \theta)}{2 \sin \theta} \frac{\partial}{\partial \varphi} \right). \end{aligned} \quad (\text{A.23})$$

Hence, by performing a boost before taking the limit, we can ensure that the tetrad remains well-defined and must therefore give a tetrad aligned with the principal null directions of the near-horizon geometry.

Consider the Teukolsky equation for a field ψ of spin s in the extreme Kerr geometry [13]. Let

$$\psi = f(\tau, y, \theta, \varphi) = f\left(\lambda\tilde{t}, \frac{\tilde{r}-a}{\lambda}, \theta, \tilde{\phi} - \frac{\tilde{t}}{2a}\right), \quad (\text{A.24})$$

Plugging this into the Teukolsky master equation [13], and taking $\lambda \rightarrow 0$, we find that it becomes

$$\begin{aligned} & \frac{4a^4}{y^2} \partial_\tau^2 f - \frac{4a^2}{y} \partial_\tau \partial_\varphi f + \left(2 - \frac{1}{4} \sin^2 \theta - \frac{1}{\sin^2 \theta}\right) \partial_\varphi^2 f - y^{-2s} \partial_y (y^{2s+2} \partial_y f) \\ & - \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta f) - is \left(\frac{2 \cos \theta}{\sin^2 \theta} + \cos \theta\right) \partial_\varphi f - \frac{4a^2 s}{y} \partial_\tau f + (s^2 \cot^2 \theta - s) f = 0 \end{aligned} \quad (\text{A.25})$$

This is the master equation governing perturbations of the near-horizon Kerr geometry written in Poincaré coordinates.

We separate variables by setting

$$f(\tau, y, \theta, \varphi) = F(\tau, y) S(\theta) e^{im\varphi}. \quad (\text{A.26})$$

Equation (A.25) separates into an angular equation given by (2.7) with $C = m/2$ and into

$$\frac{4a^4}{y^2} \partial_\tau^2 F - \frac{4a^2(s+im)}{y} \partial_\tau F - y^{-2s} \partial_y (y^{2s+2} \partial_y F) + \left(\Lambda_{lm}^{(s)} - \frac{7m^2}{4}\right) F = 0, \quad (\text{A.27})$$

where $\Lambda_{lm}^{(s)}$ is the constant of separation already discussed after (2.7).

B Phase and group velocities

In this appendix we give some details of the analysis done in subsection 2.5.2.

To discuss travelling waves and their phase and group velocities we need the next-to-leading contribution to the asymptotic behavior (2.27). This is obtained applying to (2.26) the transformation law $z \rightarrow 1/(1-z)$ of the hypergeometric function and its asymptotic expansion for large radial distances yielding,

$$\begin{aligned} \Phi_{lm\omega}^{(s)} & \approx 2^{(1+\eta)/2} \Gamma(b-a) C^\pm e^{\pm i\pi(\beta-\alpha-a)/2} e^{-\frac{1+\eta}{2} \ln|r| - \frac{2q\omega}{1+\eta} \frac{1}{r}} \\ & + 2^{(1-\eta)/2} \Gamma(a-b) D^\pm e^{\pm i\pi(\beta-\alpha-b)/2} e^{-\frac{1-\eta}{2} \ln|r| - \frac{2q\omega}{1-\eta} \frac{1}{r}}, \end{aligned} \quad (\text{B.1})$$

where the amplitudes C^\pm and D^\pm are defined in (2.28).

Introduce the quantities

$$\begin{aligned} S_\pm(r) & = \exp\left[i\left(\pm\frac{1}{2}\tilde{\eta}\ln|r| + \frac{2\omega(\mp m\tilde{\eta} + s)}{1+\tilde{\eta}^2}\frac{1}{r}\right)\right], \quad \eta = i\tilde{\eta}, \\ k_\pm(r) & = -i\frac{\partial_r S_\pm(r)}{S_\pm(r)}. \end{aligned} \quad (\text{B.2})$$

		C^-	D^-	C^+	D^+
v_{ph}	$\text{Re}(\omega) > 0$	> 0	< 0	< 0	> 0
v_{ph}	$\text{Re}(\omega) < 0$	< 0	> 0	> 0	< 0
v_{g}	$m > 0$	< 0	> 0	< 0	> 0
v_{g}	$m < 0$	> 0	< 0	> 0	< 0

Table 3. Phase and group velocities for the possible amplitudes choices in the asymptotic solution (2.27).

Here, $S_{\pm}(r)$ encodes the radial contribution to the wave propagation and, in a WKB approximation to the traveling waves, $k_{\pm}(r)$ is the effective wavenumber of the wave. The superscript ‘+’ (‘-’) is in correspondence with the amplitude C^{\pm} (D^{\pm}). The phase velocity of the traveling wave is then

$$v_{\text{ph}}^{\pm} = \frac{\omega}{k_{\pm}(r)} \simeq \pm \frac{2\omega r}{\tilde{\eta}}, \quad (\text{B.3})$$

while the group velocity is

$$v_{\text{g}}^{\pm} = \left(\frac{dk_{\pm}(r)}{d\omega} \right)^{-1} = \pm \frac{1}{2} \frac{(1 + \tilde{\eta}^2) r^2}{m \tilde{\eta} \mp s}. \quad (\text{B.4})$$

For $2 \leq l \leq 20$ and the values of $|m| \leq l$ that yield $\tilde{\eta} > 0$ we have checked that, in the denominator of v_{g}^{\pm} , $m \tilde{\eta} \mp s$ is positive for $m > 0$ and negative for $m < 0$.

Both at $r = \pm\infty$, depending on whether we choose the C^{\pm} or the D^{\pm} contributions in (B.1), we can have the combinations for the sign of the phase and group velocities displayed in table 3.

C Hertz map between Weyl scalar and metric perturbations

In this appendix we derive the map $h_{\mu\nu}(\Psi)$ between the Weyl scalars $\Psi_{0,4}$ (that satisfy the decoupled Teukolsky equations) and the metric perturbations $h_{\mu\nu}$, as well as the map $A_{\mu}(\phi)$ between the NP complex scalars perturbations $\phi_{0,2}$ and the Maxwell perturbed vector potential A_{μ} . We follow [18] and recover the results of [15–19] but we take the opportunity to emphasize the importance of these works (not so well-known in the community) and to give some details not presented in the original articles and to also pinpoint small typos in some of these works that have propagated in the literature. We start by reviewing Wald’s work [18] in the next subsection. Then we apply it to get $h_{\mu\nu}(\Psi)$ (subsection C.2), and to find $A_{\mu}(\phi)$ (subsection C.3). A similar analysis could be done to obtain the map for the Weyl fermionic perturbations.

C.1 Problem statement. The Hertz potential map

Let us start by briefly reviewing the seminal work of Teukolsky [12, 13]. Suppose we wish to solve the perturbation equation $\varepsilon(h) = 0$ where ε is a linear differential operator and h is the field perturbation on which ε acts. For example, ε can be the Maxwell operator

describing electromagnetic perturbations A_α , $[\Delta_E(A_\alpha)]_\mu = 0$, i.e., $\nabla_\nu F^{\mu\nu} = 0$; or it can be the gravitational operator describing gravitational perturbations $h_{\alpha\beta}$, $[\Delta_G(h_{\alpha\beta})]_{\mu\nu} = 0$ written in (3.4).

Suppose now that:

- a new variable $\Psi = L_\Psi(h)$, where L_Ψ is a linear differential operator, has been introduced (these are e.g., the NP complex electromagnetic scalars $\phi_{0,1,2}$ or the Weyl scalars $\Psi_{0,\dots,4}$);
- a linear partial differential operator \mathbb{D} has been found such that for all h one has

$$\mathbb{D}\varepsilon(h) = \mathcal{O}L_\Psi(h) = \mathcal{O}\Psi, \tag{C.1}$$

where \mathcal{O} is another partial differential operator.

Then

$$\varepsilon(h) = 0 \quad \Rightarrow \quad \mathcal{O}(\Psi) = 0. \tag{C.2}$$

This is the main result of [12, 13], who found the variable Ψ and operators L_Ψ , \mathbb{D} and \mathcal{O} , as well as the well-known decoupled Teukolsky equations (C.1), that describe the problem of electromagnetic, Weyl fermionic and gravitational perturbations in the Kerr black hole.

So assume that we have carried on the previous steps that lead to (C.2) and that furthermore we solved it and have a solution for Ψ . The next question is how to get the original perturbation h from the knowledge of the scalar perturbation Ψ , i.e., the unique map $h = h(\Psi)$. This issue has been addressed by Cohen and Kegeles [15, 17], and Chrzanowski [16] and later Wald [18] proved rigorously and in a few lines their results. In the sequel we review Wald’s proof [18].

Start by recalling the notion of adjoint of an operator. Let \mathcal{O} be a linear differential operator taking a scalar, vectorial or tensorial field into another similar field. Then there is an unique adjoint operator \mathcal{O}^\dagger such that

$$\Psi_H(\mathcal{O}\Psi) - (\mathcal{O}^\dagger\Psi_H)\Psi = \nabla_\mu s^\mu, \tag{C.3}$$

for arbitrary fields Ψ and Ψ_H , and where $\nabla_\mu s^\mu$ is a total divergence.

Wald’s theorem states the following. Assume that:

- the identity (C.1) is satisfied for the linear differential operators \mathbb{D} , ε , \mathcal{O} , L_Ψ ;
- Ψ_H satisfies $\mathcal{O}^\dagger\Psi_H = 0$, where Ψ_H is called the Hertz potential;

Then

- $\mathbb{D}^\dagger\Psi_H$ satisfies $\varepsilon^\dagger(\mathbb{D}^\dagger\Psi_H) = 0$.
- Moreover, this in particular also implies that if ε is self-adjoint, $\varepsilon^\dagger = \varepsilon$, then

$$h = \mathbb{D}^\dagger\Psi_H \text{ is a solution of } \varepsilon(h) = 0. \tag{C.4}$$

and provides the map we are looking for.

The proof of this result is short and simple [18]. Taking the adjoint of $\mathbb{D}\varepsilon = \mathcal{O}L_\Psi$ one has $\varepsilon^\dagger \mathbb{D}^\dagger = L_\Psi^\dagger \mathcal{O}^\dagger$. Applying these operators to Ψ_H one gets $\varepsilon^\dagger \mathbb{D}^\dagger \Psi_H = 0$ after using the assumption $\mathcal{O}^\dagger \Psi_H = 0$. Moreover, if $\varepsilon^\dagger = \varepsilon$, it trivially follows that $\varepsilon(\mathbb{D}^\dagger \Psi_H) = 0$, i.e., $h = \mathbb{D}^\dagger \Psi_H$ is a solution of the initial perturbation equation $\varepsilon(h) = 0$.

Finally note that the Hertz potential construction reviewed here applies to geometries where no energy-momentum tensor is present. For a discussion of the method when this is not the case as well as for the second order perturbation analysis we ask the reader to see [25].

C.2 Application: the map $h_{\mu\nu}(\Psi)$ for gravitational perturbations

Our starting point are the decoupled Teukolsky equations for the perturbed Weyl scalars $\Psi_0^{(1)}$ and $\Psi_4^{(1)}$, namely equations (2.12)–(2.15) of [13], which are written in (C.16) for $s = 2$ and (C.17) for $s = -2$. These can be written as (to make the connection with the nomenclature of the previous subsection straightforward)

$$\begin{aligned}\mathcal{O}_{G_0} \Psi_0^{(1)} &= \mathbb{D}_{G_0}^{\mu\nu} T_{\mu\nu}^{(1)}, \\ \mathcal{O}_{G_4} \Psi_4^{(1)} &= \mathbb{D}_{G_4}^{\mu\nu} T_{\mu\nu}^{(1)},\end{aligned}\tag{C.5}$$

where we use the superscript (1) to denote a perturbed quantity (otherwise it refers to an unperturbed quantity), and

$$\begin{aligned}\mathcal{O}_{G_0} &= (D - 3\epsilon + \bar{\epsilon} - 4\rho - \bar{\rho})(\Delta + \mu - 4\gamma) - (\delta + \bar{\pi} - \bar{\alpha} - 3\beta - 4\tau)(\bar{\delta} + \pi - 4\alpha) - 3\Psi_2, \\ \mathcal{O}_{G_4} &= (\Delta + 3\gamma - \bar{\gamma} + 4\mu + \bar{\mu})(D + 4\epsilon - \rho) - (\bar{\delta} - \bar{\tau} + \bar{\beta} + 3\alpha + 4\pi)(\delta - \tau + 4\beta) - 3\Psi_2,\end{aligned}\tag{C.6}$$

and

$$\begin{aligned}\mathbb{D}_{G_0}^{\mu\nu} &= (\delta + \bar{\pi} - \bar{\alpha} - 3\beta - 4\tau) \left[(D - 2\epsilon - 2\bar{\rho})\ell^{(\mu} m^{\nu)} - (\delta + \bar{\pi} - 2\bar{\alpha} - 2\beta)\ell^\mu \ell^\nu \right] \\ &\quad + (D - 3\epsilon + \bar{\epsilon} - 4\rho - \bar{\rho}) \left[(\delta + 2\bar{\pi} - 2\beta)\ell^{(\mu} m^{\nu)} - (D - 2\epsilon + 2\bar{\epsilon} - \bar{\rho})m^\mu m^\nu \right], \\ \mathbb{D}_{G_4}^{\mu\nu} &= (\Delta + 3\gamma - \bar{\gamma} + 4\mu + \bar{\mu}) \left[(\bar{\delta} - 2\bar{\tau} + 2\alpha)n^{(\mu} \bar{m}^{\nu)} - (\Delta + 2\gamma - 2\bar{\gamma} + \bar{\mu})\bar{m}^\mu \bar{m}^\nu \right] \\ &\quad + (\bar{\delta} - \bar{\tau} + \bar{\beta} + 3\alpha + 4\pi) \left[(\Delta + 2\gamma + 2\bar{\mu})n^{(\mu} \bar{m}^{\nu)} - (\bar{\delta} - \bar{\tau} + 2\bar{\beta} + 2\alpha)n^\mu n^\nu \right].\end{aligned}\tag{C.7}$$

Following Wald's procedure, the Hertz potential Ψ_H is introduced to be such that it satisfies the equation $\mathcal{O}_G^\dagger \Psi_H = 0$, that is

$$\begin{aligned}[(\Delta + 3\gamma - \bar{\gamma} + \bar{\mu})(D + 4\epsilon + 3\rho) - (\bar{\delta} + \bar{\beta} + 3\alpha - \bar{\tau})(\delta + 4\beta + 3\tau) - 3\Psi_2] \Psi_{H_0} &= 0, \\ [(D - 3\epsilon + \bar{\epsilon} - \bar{\rho})(\Delta - 4\gamma - 3\mu) - (\delta - 3\beta - \bar{\alpha} + \bar{\pi})(\bar{\delta} - 4\alpha - 3\pi) - 3\Psi_2] \Psi_{H_4} &= 0,\end{aligned}\tag{C.8}$$

where, to obtain the adjoint of (C.6), we used the relations¹²

$$\begin{aligned} D^\dagger &= -(D + \epsilon + \bar{\epsilon} - \rho - \bar{\rho}), & \Delta^\dagger &= -(\Delta - \gamma - \bar{\gamma} + \mu + \bar{\mu}), \\ \delta^\dagger &= -(\delta + \beta - \bar{\alpha} - \tau + \bar{\pi}), & \bar{\delta}^\dagger &= -(\bar{\delta} + \bar{\beta} - \alpha - \bar{\tau} + \pi), \end{aligned} \quad (\text{C.9})$$

and the well known property $(AB)^\dagger = B^\dagger A^\dagger$. Equations (C.8) can be written, respectively, as (C.18) with $s = -2$ and (C.19) with $s = 2$.

Since the gravitational perturbation operator $\varepsilon \equiv [\Delta_G(h_{\alpha\beta})]_{\mu\nu}$ is self-adjoint, Wald's theorem tell us that the map (C.4) between the Hertz potential Ψ_H and the metric perturbations $h_{\mu\nu}$ is given by $h_{\mu\nu} = 2\text{Re} \left[\mathbb{D}_{G\mu\nu}^\dagger \Psi_H \right]$, i.e.,

$$\begin{aligned} h_{\mu\nu}^{IRG} &= \left\{ \ell_{(\mu} m_{\nu)} [(D + 3\epsilon + \bar{\epsilon} - \rho + \bar{\rho})(\delta + 4\beta + 3\tau) + (\delta + 3\beta - \bar{\alpha} - \tau - \bar{\pi})(D + 4\epsilon + 3\rho)] \right. \\ &\quad \left. - \ell_\mu \ell_\nu (\delta + 3\beta + \bar{\alpha} - \bar{\tau})(\delta + 4\beta + 3\tau) - m_\mu m_\nu (D + 3\epsilon - \bar{\epsilon} - \rho)(D + 4\epsilon + 3\rho) \right\} \Psi_{H_0} + \text{c.c.}, \\ h_{\mu\nu}^{ORG} &= \left\{ n_{(\mu} \bar{m}_{\nu)} [(\bar{\delta} + \bar{\beta} - 3\alpha + \bar{\tau} + \pi)(\Delta - 4\gamma - 3\mu) + (\Delta - 3\gamma - \bar{\gamma} + \mu - \bar{\mu})(\bar{\delta} - 4\alpha - 3\pi)] \right. \\ &\quad \left. - n_\mu n_\nu (\bar{\delta} - \bar{\beta} - 3\alpha + \pi)(\bar{\delta} - 4\alpha - 3\pi) - \bar{m}_\mu \bar{m}_\nu (\Delta - 3\gamma + \bar{\gamma} + \mu)(\Delta - 4\gamma - 3\mu) \right\} \Psi_{H_4} + \text{c.c.}, \end{aligned} \quad (\text{C.10})$$

where to get the adjoint of (C.7) we used again (C.9), and c.c. stands for complex conjugate. The first of these relations gives the metric perturbations in the ingoing radiation gauge (*IRG*), while the second provides the map in the outgoing radiation gauge (*ORG*); see (3.1). The first relation agrees with the results of [15–18], and is equation (3.3) in the main body of the text. The outgoing radiation map corrects typos in the relation of table 1 of [16] that have propagated in the literature.

C.3 Application: the map $A_\mu(\phi)$ for electromagnetic perturbations

We begin our discussion with the decoupled Teukolsky equations for the perturbed electromagnetic NP scalars $\phi_0^{(1)}$ and $\phi_2^{(1)}$, namely equations (3.5)–(3.8) of [13], which are written in (C.16) for $s = 1$ and (C.17) for $s = -1$. These can be written as

$$\begin{aligned} \mathcal{O}_{E_0} \phi_0^{(1)} &= \mathbb{D}_{E_0}^\mu J_\mu^{(1)}, \\ \mathcal{O}_{E_2} \phi_2^{(1)} &= \mathbb{D}_{E_2}^\mu J_\mu^{(1)}, \end{aligned} \quad (\text{C.11})$$

where, again, we use the superscript (1) to denote a perturbed quantity, and

$$\begin{aligned} \mathcal{O}_{E_0} &= (D - \epsilon + \bar{\epsilon} - 2\rho - \bar{\rho})(\Delta + \mu - 2\gamma) - (\delta - \beta - \bar{\alpha} - 2\tau + \bar{\pi})(\bar{\delta} + \pi - 2\alpha), \\ \mathcal{O}_{E_2} &= (\Delta + \gamma - \bar{\gamma} + 2\mu + \bar{\mu})(D + 2\epsilon - \rho) - (\bar{\delta} + \alpha + \bar{\beta} + 2\pi - \bar{\tau})(\delta - \tau + 2\beta), \end{aligned} \quad (\text{C.12})$$

¹²To get (C.9), introduce the internal product $\langle \Psi_H, \mathcal{O}\psi \rangle = \int_a^b d^4x \sqrt{-g} \Psi_H \mathcal{O}\psi$. For the directional derivative operators $\mathcal{O} = e_{(a)}^\mu \nabla_\mu$ it then follows, after integrations by parts and use of $\nabla_\mu \sqrt{-g} = 0$, that $\langle \Psi_H, \mathcal{O}\psi \rangle = -\langle \psi, \mathcal{O}^\dagger \Psi_H \rangle + \int_a^b \nabla_\mu s^\mu$ with $\mathcal{O}^\dagger = -\mathcal{O} - \nabla_\mu e_{(a)}^\mu$, and $\int_a^b \nabla_\mu s^\mu$ being the short notation for a total divergence contribution. To compute $\nabla_\mu e_{(a)}^\mu$ use the well-known relation between covariant derivative of the NP tetrad and the spin coefficients [21], $\nabla_\mu e_{(a)\nu} = e^{(c)}{}_\nu \gamma_{cab} e^{(b)}{}_\mu$, and relations (A.18).

and

$$\begin{aligned}\mathbb{D}_{E_0}^\mu &= (\delta - \bar{\alpha} - \beta + \bar{\pi} - 2\tau)\ell^\mu - (D - \epsilon + \bar{\epsilon} - 2\rho - \bar{\rho})m^\mu, \\ \mathbb{D}_{E_2}^\mu &= (\Delta + \gamma - \bar{\gamma} + 2\mu + \bar{\mu})\bar{m}^\mu - (\bar{\delta} + \alpha + \bar{\beta} + 2\pi - \bar{\tau})n^\mu.\end{aligned}\quad (\text{C.13})$$

The Hertz potential Ψ_H is introduced to be such that it satisfies the equation $\mathcal{O}_E^\dagger \Psi_H = 0$, that is

$$\begin{aligned}[(\Delta + \gamma - \bar{\gamma} + \bar{\mu})(D + 2\epsilon + \rho) - (\bar{\delta} + \bar{\beta} + \alpha - \bar{\tau})(\delta + 2\beta + \tau)]\Psi_{H_0} &= 0, \\ [(D - \epsilon + \bar{\epsilon} - \bar{\rho})(\Delta - 2\gamma - \mu) - (\delta - \beta - \bar{\alpha} + \bar{\pi})(\bar{\delta} - 2\alpha - \pi)]\Psi_{H_2} &= 0,\end{aligned}\quad (\text{C.14})$$

where, to obtain the adjoint of (C.12), we used (C.9) and $(AB)^\dagger = B^\dagger A^\dagger$. Equations C.14 are, respectively, equations (C.18) for $s = -1$ and (C.19) for $s = 1$.

Since the Maxwell perturbation operator $\varepsilon \equiv [\Delta_E(A_\alpha)]_\mu$ is self-adjoint, Wald's theorem tell us that the map (C.4) between the Hertz potential Ψ_H and the vector potential perturbations A_μ is given by $A_\mu = 2\text{Re} \left[\mathbb{D}_{E\mu}^\dagger \Psi_H \right]$, i.e.,

$$\begin{aligned}A_\mu^{IRG} &= [-\ell_\mu(\delta + 2\beta + \tau) + m_\mu(D + 2\epsilon + \rho)]\Psi_{H_0} + \text{c.c.}, \\ A_\mu^{ORG} &= [-\bar{m}_\mu(\Delta - 2\gamma - \mu) + n_\mu(\bar{\delta} - 2\alpha - \pi)]\Psi_{H_2} + \text{c.c.},\end{aligned}\quad (\text{C.15})$$

where to get the adjoint of (C.13) we used again (C.9). The first of these relations gives the Maxwell perturbations in the ingoing radiation gauge, $\ell^\mu A_\mu = 0$, while the second provides the map in the outgoing radiation gauge, $n^\mu A_\mu = 0$. The first relation agrees with the results of [15–18]. The outgoing radiation map agrees with [17] and corrects typos in the relation of table 1 of [16].

C.4 Hertz potential for spin s in a Ricci flat Petrov type D geometry

For an arbitrary Petrov type D solution we can write the decoupled Teukolsky equations in a single equation that depends on the spin of the field [13], and the same happens for the decoupled equations for the Hertz potential if the the solution is furthermore Ricci flat.

For positive spin and negative spin, Teukolsky's decoupled equations are, respectively, [13]

$$\begin{aligned}\left[[D - (2s - 1)\epsilon + \bar{\epsilon} - 2s\rho - \bar{\rho}](\Delta + \mu - 2s\gamma) - [\delta + \bar{\pi} - \bar{\alpha} - (2s - 1)\beta - 2s\tau](\bar{\delta} + \pi - 2s\alpha) \right. \\ \left. - 3s \left(s - \frac{1}{2} \right) (s - 1)\Psi_2 \right] \Psi^{(s)} = 4\pi T_{(s)}, \quad \text{for } s = +2, +1, +\frac{1}{2}, 0,\end{aligned}\quad (\text{C.16})$$

$$\begin{aligned}\left[[\Delta - (2s + 1)\gamma - \bar{\gamma} - 2s\mu + \bar{\mu}](D - 2s\epsilon - \rho) - [\bar{\delta} - \bar{\tau} + \bar{\beta} - (2s + 1)\alpha - 2s\pi](\delta - \tau - 2s\beta) \right. \\ \left. + 3s \left(s + \frac{1}{2} \right) (s + 1)\Psi_2 \right] \Psi^{(s)} = 4\pi T_{(s)}, \quad \text{for } s = -2, -1, -\frac{1}{2}, 0.\end{aligned}\quad (\text{C.17})$$

The relation between the nomenclature used here and the original notation of Teukolsky [13] was already displayed in table 1 and associated discussion.

The Hertz potential $\Psi_{\text{H}}^{(s)}$ in the ingoing radiation gauge describes negative spin $s = -2, -1, -\frac{1}{2}$ field perturbations, while in the outgoing radiation gauge $\Psi_{\text{H}}^{(s)}$ describes positive spin $s = +2, +1, +\frac{1}{2}$ perturbations. They are, respectively, the solutions of the scalar equations

$$\left[\Delta - (2s+1)\gamma - \bar{\gamma} + \bar{\mu} \right] [D - 2s\epsilon - (2s+1)\rho] - [\bar{\delta} + \bar{\beta} - (2s+1)\alpha - \bar{\tau}] [\delta - 2s\beta - (2s+1)\tau] + 3s \left(s + \frac{1}{2} \right) (s+1) \Psi_2 \Psi_{\text{H}}^{(s)} = 0, \quad \text{for } s = -2, -1, -\frac{1}{2}, 0, \quad (\text{C.18})$$

$$\left[D - (2s-1)\epsilon + \bar{\epsilon} - \bar{\rho} \right] [\Delta - 2s\gamma - (2s-1)\mu] - [\delta - (2s-1)\beta - \bar{\alpha} + \bar{\pi}] [\bar{\delta} - 2s\alpha - (2s-1)\pi] - 3s \left(s - \frac{1}{2} \right) (s-1) \Psi_2 \Psi_{\text{H}}^{(s)} = 0, \quad \text{for } s = +2, +1, +\frac{1}{2}, 0. \quad (\text{C.19})$$

The conjugate Teukolsky perturbations $\Psi^{(s)}$ to these Hertz potentials $\Psi_{\text{H}}^{(s)}$ can be read from tables 1 and 2.

For the NHEK geometry written in global coordinates the NP tetrad is written in (2.3), the spin coefficients are listed in (A.18), and the directional derivative operators can be read from (A.21). Using this information in (C.16)–(C.17), and (C.18)–(C.19) we get the master equation (2.5).

D Decoupling limit of near-extreme Kerr. Mass changing modes

In this appendix we show that a decoupling limit of the near-extremal Kerr black hole yields the NHEK geometry. We follow [34] where decoupling limits like the one we take were first discussed for charged non-rotating solutions.

The Kerr black hole solution in Boyer-Lindquist form reads,

$$ds^2 = \frac{\Sigma \Delta}{(\tilde{r}^2 + a^2)^2 - \Delta a^2 \sin^2 \theta} d\tilde{t}^2 - \frac{\Sigma}{\Delta} d\tilde{r}^2 - \Sigma d\theta^2 - \sin^2 \theta \frac{(\tilde{r}^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma} \left(d\tilde{\phi} - \frac{a(a^2 + \tilde{r}^2 - \Delta)}{(\tilde{r}^2 + a^2)^2 - \Delta a^2 \sin^2 \theta} d\tilde{t} \right)^2, \quad (\text{D.1})$$

with

$$\Delta = (\tilde{r} - \tilde{r}_-)(\tilde{r} - \tilde{r}_+), \quad \Sigma = \tilde{r}^2 + a^2 \cos^2 \theta, \quad \tilde{r}_{\pm} = \frac{1}{2} \left(\ell_P^2 \Delta E + \ell_P \sqrt{\ell_P^2 \Delta E^2 + 4J} \right) \pm \sqrt{\ell_P^3 \Delta E \sqrt{\ell_P^2 \Delta E^2 + 4J}}. \quad (\text{D.2})$$

We used the fact that in four dimensions the Planck length ℓ_P is related to Newton's constant G by $\ell_P^2 = G$, and we defined the excitation energy above extremality as

$$\Delta E = \frac{m - a}{\ell_P^2}, \quad \text{with } a = \ell_P^2 \frac{J}{m}, \quad (\text{D.3})$$

where J and $M = m/\ell_P^2$ are the ADM angular momentum and mass of the Kerr black hole.

The black hole temperature is

$$T_H = \frac{r_+^2 - a^2}{4\pi r_+ (r_+^2 + a^2)}, \quad (\text{D.4})$$

from which follows that near-extremality the relation between the excitation energy and the temperature is

$$\Delta E \simeq 8\pi^2 J^{\frac{3}{2}} T_H^2 \ell_P. \quad (\text{D.5})$$

Typically, the energy of a quantum of Hawking emission is of order T_H . When this energy is of order or greater than the available energy above extremality, $T_H \gtrsim \Delta E$, the semiclassical analysis of the black hole thermodynamics breaks down. This occurs at an excitation energy of order

$$E_{\text{gap}} \simeq \frac{1}{8\pi^2 J^{\frac{3}{2}} \ell_P}. \quad (\text{D.6})$$

We now want to take a decoupling limit where

$$\ell_P \rightarrow 0, \quad \text{with } (T_H, J) \text{ fixed.} \quad (\text{D.7})$$

In this limit the excitation energy ΔE vanishes and the gap energy E_{gap} goes to infinity.

In the decoupling limit, and after introducing the new radial and azimuthal coordinates (U, ψ) :

$$\tilde{r} = \tilde{r}_+ + 2\ell_P^2 U, \quad \tilde{\phi} = \psi + \frac{\tilde{t}}{2m}, \quad (\text{D.8})$$

the Kerr geometry (D.1) reduces to

$$\frac{ds^2}{\ell_P^2} = -2J\Omega^2(\theta) \left[-\frac{U(U+4\pi JT_H)}{J^2} d\tilde{t}^2 + \frac{dU^2}{U(U+4\pi JT_H)} + d\theta^2 + \Lambda^2(\theta) \left(d\psi + \left(2\pi T_H + \frac{U}{J} \right) d\tilde{t} \right)^2 \right], \quad (\text{D.9})$$

with $\Omega^2(\theta)$ and $\Lambda^2(\theta)$ defined in (2.2).

Finally if we introduce the new time, radial and azimuthal coordinates (τ, y, φ) through the transformations,

$$\begin{aligned} \tilde{t} &= \frac{1}{2\pi T_H} [\arctan(\tau - y) + \arctan(\tau + y)], \\ U &= \frac{\pi T_H J}{y} [(y-1)^2 - \tau^2], \\ \psi &= \varphi + \arctan\left(\frac{\tau-1}{y}\right) + \arctan\left(\frac{\tau+1}{y}\right), \end{aligned} \quad (\text{D.10})$$

the decoupling geometry (D.9) reduces to

$$\frac{ds^2}{\ell_P^2} = -2J\Omega^2(\theta) \left[\frac{-d\tau^2 + dy^2}{y^2} + d\theta^2 + \Lambda^2(\theta) \left(d\varphi + \frac{d\tau}{y} \right)^2 \right]. \quad (\text{D.11})$$

We recognize this geometry as the NHEK solution written in Poincaré coordinates. A final transformation between the Poincaré coordinates $(\tau, y, \theta, \varphi)$ and the global coordinates

(t, r, θ, ϕ) [1],

$$\begin{aligned}\tau &= \frac{\sin t \sqrt{1+r^2}}{\cos t \sqrt{1+r^2} + r}, \\ y &= \left(\cos t \sqrt{1+r^2} + r \right)^{-1}, \\ \varphi &= \phi + \ln \left(\frac{\cos t + r \sin t}{1 + \sin t \sqrt{1+r^2}} \right),\end{aligned}\tag{D.12}$$

takes (D.11) into the NHEK (2.1) written in global coordinates. Therefore, at the classical level, when we take the decoupling limit (D.7) of the near-extremal Kerr geometry, one gets a geometry that is independent of the temperature T_H and that is precisely the NHEK geometry.

An important observation is that going to the next-to-leading order in the ℓ_P expansion, a path similar to (D.7)–(D.12) yields the next-to-leading order contribution to (D.11) or (2.1), that we call $h_{\mu\nu}$. This $h_{\mu\nu}$ can be considered as a perturbation to the NHEK since it satisfies the linearized Einstein equations (with $h \neq 0$) by construction. Asymptotically, $h_{\mu\nu}$ goes as a power of r^2 higher (in all components) than the GHSS boundary conditions [2].

To interpret physically these perturbations we compute the relevant conserved charges associated to them as defined in (4.2). One finds that the energy of these perturbations is finite, $Q_{\partial t}[h, g] \neq 0$, while the U(1) charge vanishes, $Q_{\partial \phi}[h, g] = 0$. These perturbations therefore increase the energy of the solution while leaving its angular momentum unchanged. They correspond thus to perturbations that take NHEK away from the extremality state. Since the full series expansion in ℓ_P reconstructs near-extreme Kerr, these perturbations actually take the NHEK geometry into near-extreme Kerr black hole.

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