# Simultaneous baldness and cosmic baldness and the Kottler spacetime

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The Schwarzschild-de Sitter/Kottler geometry is the unique spherical solution of the vacuum Einstein equations with positive cosmological constant. Putative alternatives in the literature are shown to either solve different equations or to be the SdSK solution in disguise. No-hair and cosmic no-hair come together in a new simultaneous theorem for SdSK in the presence of an imperfect fluid.

*Keywords*: Schwarzschild-de Sitter, no-hair, cosmic no-hair.



## 1. Introduction

The Jebsen-Birkhoff theorem of general relativity (GR) states that the Schwarzschild geometry is the unique vacuum, spherical, asymptotically flat solution of the Einstein equations

$$G_{ab} \equiv R_{ab} - \frac{1}{2}\, g_{ab} R = 0\,. \tag{1}$$

The theorem is no longer true if matter is present, but what if a cosmological constant $\Lambda$ is included? The Einstein equations change to $G_{ab} = -\Lambda g_{ab}$. It is straightforward to extend the proof of the Jebsen-Birkhoff theorem to this case. In particular, *for $\Lambda > 0$, the Schwarzschild-de Sitter/Kottler (SdSK) geometry is the unique solution,*

$$ds^2 = -\left(1 - \frac{2m}{R} - H^2 R^2\right) dT^2 + \frac{dR^2}{1 - \frac{2m}{R} - H^2 R^2} + R^2 d\Omega_{(2)}^2 \tag{2}$$

*(where $H = \sqrt{\Lambda/3}$).*

This fact is not mentioned in modern GR textbooks, even though de Sitter space is extremely important for inflationary and dark energy cosmology[1] and, for $\Lambda < 0$, anti-de Sitter space is fundamental for string theories and the AdS/CFT correspondence. A proof of the generalized Jebsen-Birkhoff theorem for $\Lambda \neq 0$ can be found in Synge's 1960 textbook.[2] A recent proof in null coordinates has appeared in[3]. Complicated proofs appear in[4–7]. The proof in spherical coordinates is a very simple extension of the standard textbook proof of the Jebsen-Birkhoff theorem.[11] However, there is confusion in the literature: purported alternatives to SdSK have been reported, which would violate the theorem. More general solutions with FLRW "background" in the presence of matter do not seem to reduce to SdSK when the "background" becomes de Sitter. Enter modified gravity: there are claims that polytropic stars in $f(R)$ and scalar-tensor gravity cannot be matched to SdSK. If they are interesting for cosmology, these theories contain a time-dependent $\Lambda$. Perhaps SdSK is not the correct solution to match internal solutions to—the situation is unclear. It does not help if the situation is unclear even in GR, which is what we clarify here.

## 2. Putative Alternatives to SdSK

Here we compare putative alternative geometries to SdSK, using the areal radius as the radial coordinate.

### 2.1. *Abbassi-Meissner proposal*

It is claimed that an alternative to SdSK is the Abbassi-Meissner solution[8–10]

$$ds^2 = -f(t,r)dt^2 + \frac{\mathrm{e}^{2Ht}}{f(t,r)}\,dr^2 + \mathrm{e}^{2Ht}r^2 d\Omega_{(2)}^2\,, \tag{3}$$

where $H = \sqrt{\Lambda/3}$, $m = $ const., and

$$f(t,r) = h(t,r) + \sqrt{h^2(t,r) + H^2 r^2 \mathrm{e}^{2Ht}}\,, \tag{4}$$

$$h(t,r) = \frac{1}{2}\left(1 - H^2 r^2 \mathrm{e}^{2Ht} - \frac{2m}{r}\,\mathrm{e}^{-Ht}\right). \tag{5}$$

Let us change to the areal radius $R = \mathrm{e}^{Ht}r$ instead of $r$, and to the new time $T$ defined by

$$dT = dt + \frac{2HRdR}{A_0\left(A_0 + \sqrt{A_0^2 + 4H^2R^2}\right)}\,, \tag{6}$$

where

$$A_0(R) = 1 - \frac{2m}{R} - H^2R^2 = 2h(0,R)\,. \tag{7}$$

The line element becomes diagonal and locally static

$$ds^2 = -A_0(R)dT^2 + \frac{2}{A_0(R) + \sqrt{A_0^2(R) + 4H^2R^2}} \cdot \left[1 + \frac{2H^2R^2}{A_0\left(A_0 + \sqrt{A_0^2 + 4H^2R^2}\right)}\right] dR^2 + R^2 d\Omega_{(2)}^2\,, \tag{8}$$

and it is clearly not SdSK. It solves the field equations $G_{ab} = -\Lambda g_{ab} + T_{ab}$ with a radial flow.[11]

### 2.2. *Non-rotating Thakurta solution*

The Thakurta solution of GR[12] is conformal to Kerr and describes a rotating black hole embedded in a FLRW universe. The non-rotating, spherical subcase is the late time limit of generalized McVittie solutions[13] and it is also the $\omega \to \infty$ limit of a class of Brans-Dicke perfect fluid solutions describing inhomogeneous universes found by Ref. 14. The non-rotating Thakurta line element is

$$\begin{aligned} ds^2 &= a^2(\eta)\left[-\left(1-\frac{2m}{r}\right)d\eta^2 + \frac{dr^2}{1-2m/r} + r^2 d\Omega^2_{(2)}\right] \\ &= -\left(1-\frac{2m}{r}\right)dt^2 + \frac{a^2dr^2}{1-2m/r} + a^2r^2d\Omega^2_{(2)}\,, \end{aligned} \tag{9}$$

where $a(\eta)$ is the scale factor of the FLRW "background". Changing coordinates to $(t,r) \to (T, a(t)r)$ with

$$dT = \frac{1}{F}\left(dt + \frac{HRdR}{A^2 - H^2R^2}\right)\,, \tag{10}$$

$$A(t,R) = 1 - \frac{2m}{r} = 1 - \frac{2M}{R}\,, \qquad M \equiv ma(t)\,, \tag{11}$$

the line element becomes

$$ds^2 = -\left(1 - \frac{2M}{R} - \frac{H^2R^2}{1-\frac{2M}{R}}\right)F^2dT^2 + \frac{dR^2}{1-\frac{2M}{R} - \frac{H^2R^2}{1-2M/R}} + R^2d\Omega^2_{(2)} \tag{12}$$

(where $F$ is an integrating factor) and it solves the field equations $G_{ab} = -\Lambda g_{ab} + q_a u_b + q_b u_a$, where $q^c$ (with $q^c u_c = 0$) describes a purely spatial radial flow.

### 2.3. *Castelo Ferreira metric*

The Castelo Ferreira line element[15] is

$$\begin{aligned} ds^2 = &-\left[1-\frac{2m}{R} - H^2R^2\left(1-\frac{2m}{R}\right)^{\alpha}\right]dt^2 + \frac{dR^2}{1-\frac{2m}{R}} - 2HR\left(1-\frac{2m}{R}\right)^{\frac{\alpha-1}{2}}dtdR \\ &+ R^2d\Omega^2_{(2)} \end{aligned} \tag{13}$$

with $\alpha, m$ constants and $H = H(t)$. The coordinate change

$$dT = \frac{1}{F}\left(dt + \frac{HRdR\left(1-\frac{2m}{R}\right)^{\frac{\alpha-1}{2}}}{1-\frac{2m}{R} - H^2R^2\left(1-\frac{2m}{R}\right)^{\alpha}}\right) \tag{14}$$

turns the line element into

$$\begin{aligned} ds^2 = &-\left[1-\frac{2m}{R} - H^2R^2\left(1-\frac{2m}{R}\right)^{\alpha}\right]F^2dT^2 \\ &+ \frac{dR^2}{1-\frac{2m}{R} - H^2R^2\left(1-\frac{2m}{R}\right)^{\alpha}} + R^2d\Omega^2_{(2)}\,. \end{aligned} \tag{15}$$

One can choose the background to be de Sitter space, then one obtains the non-rotating Thakurta solution if $\alpha = -1$ and the SdSK solution if $\alpha = 0$. However, the general Castelo Ferreira metric is not SdSK.



## 3. A Simultaneous No-Hair/Cosmic No Hair Theorem

The Jebsen-Birkhoff no-hair theorem states that the Schwarzschild solution is generic; the cosmic no-hair theorem states that de Sitter space is generic. Then, does a simultaneous no-hair/cosmic no-hair theorem exist, stating that SdSK is generic in some sense?

One can write any spherical metric as

$$ds^2 = -A^2(t,R)dt^2 + B^2(t,r)dR^2 + R^2 d\Omega^2_{(2)} \tag{16}$$

without loss of generality. Now one needs some assumptions about the matter content. In the previous examples of putative SdSK solutions, a common ingredient was the imperfect fluid stress-energy tensor

$$T_{ab} = (P+\rho)\, u_a u_b + P g_{ab} + q_a u_b + q_b u_a\,, \quad u_c u^c = -1\,, \quad q_c u^c = 0\,, \tag{17}$$

so we assume 1) this $T_{ab}$ with barotropic and constant equation of state $P = w\rho$, with $w =$ const.; 2) spherical symmetry; 3) the solution of $G_{ab} = -\Lambda g_{ab} + 8\pi T_{ab}$ is asymptotically de Sitter: there is a de Sitter-like cosmological horizon of radius $R_H$ and the solution reduces to SdSK as $R \to R_H$.

The Einstein equations become

$$\frac{\dot{B}}{BR} = 4\pi T_{01}\,, \tag{18}$$

$$A^2\left(\frac{2B'}{B^3 R} - \frac{1}{B^2R^2} + \frac{1}{R^2}\right) = \Lambda A^2 + 8\pi T_{00}\,, \tag{19}$$

$$\frac{2A'}{AR} - \frac{B^2}{R^2} + \frac{1}{R^2} = -\Lambda B^2 + 8\pi T_{11}\,, \tag{20}$$

$$\frac{A'B}{A} - B' - \frac{RB^2\ddot{B}}{A^2} + \frac{R\dot{A}\dot{B}B^2}{A^3} - \frac{RA'B'}{A} + \frac{RA''B}{A} = \left(-\Lambda R^2 + 8\pi T_{22}\right)\frac{B^3}{R}\,. \tag{21}$$

The fluid 4-velocity and energy flux density are

$$u_\mu = (-|A|, 0, 0, 0)\,, \qquad q_\mu = \left(0, B^2 q, 0, 0\right)\,, \tag{22}$$

while the stress-energy tensor is

$$T_{00} = A^2\rho\,, \qquad T_{01} = -|A|B^2 q\,, \tag{23}$$

$$T_{11} = B^2 P\,, \qquad T_{22} = \frac{T_{33}}{\sin^2\theta} = R^2 P\,, \tag{24}$$

where $T_{01} > 0$ and $q < 0$ correspond to radial inflow.

In the case of inflow $q < 0$ we have

$$(B^2)^{\cdot} = -8\pi|A|B^4 Rq > 0\,, \tag{25}$$

therefore $B^2 = g_{11}$ increases with time. Assuming the metric coefficients to be continuous and differentiable, there are then two possibilities:

- either $B^2(t,R) \to +\infty$ for any fixed $R$ as $t \to +\infty$, or
- $B^2(t,R)$ has an horizontal asymptote as $t \to +\infty$.

Consider the first case. The apparent horizons are located by

$$\nabla^c R \nabla_c R = 0 \quad \leftrightarrow \quad 1/B^2 = 0. \tag{26}$$

If $B^2 \to +\infty$ as $t \to +\infty$ or as $t \to t_{max}$, then at late times all points of space (at any $R < R_H$) lie arbitrarily close to an apparent horizon. This situation is familiar in cosmology: it corresponds to a phantom universe ending in a Big Rip singularity at $t_{max}$ and the apparent horizon shrinks around a comoving observer as the cosmic expansion super-accelerates. This phantom asymptotics contradict the assumption of de Sitter asymptotics and we discard this case.

In the other case in which $B^2(t,R) \to B_0^2(R)$ as $t \to +\infty$, we have $\dot{B} \to 0$ as $t \to +\infty$. Then the radial flow $q \to 0$ as $t \to +\infty$. Differentiate the (0,0) Einstein equation to obtain

$$8\pi\dot{\rho} = \frac{2}{R}\left(\frac{B'}{B^3}\right)^{\cdot} - \frac{1}{R^2}\left(\frac{1}{B^2}\right)^{\cdot} \to 0 \qquad \text{as } \; t \to +\infty\,. \tag{27}$$

Then the equation of state $P = w\rho$ yields $\dot{P} \to 0$ as $t \to +\infty$. The (2,2) (or (3,3)) equation gives

$$8\pi\dot{P} = \frac{2}{R}\left(\frac{A'}{AB^2}\right)^{\cdot} + \frac{1}{R^2}\left(\frac{1}{B^2}\right)^{\cdot} \approx \frac{2}{RB^2}\left(\frac{A'}{A}\right)^{\cdot} \to 0 \tag{28}$$

as $t \to +\infty$, then $A^2$ also becomes time-independent, and the metric becomes static. To make progress, use the covariant conservation equation $\nabla^b T_{ab} = 0$ for the imperfect fluid, obtaining

$$\begin{aligned} & u_a u^b \nabla_b \left(P+\rho\right) + \left[\left(P+\rho\right) u_a + q_a\right] \nabla^b u_b \\ & + \left[\left(P+\rho\right) u_b + q_b\right] \nabla^b u_a + \nabla_a P + u^b \nabla_b q_a + u_a \nabla^b q_b = 0\,. \end{aligned} \tag{29}$$

Projecting onto the time direction $u^a$ and using $u^a \nabla_b u_a = 0$ leads to

$$-\dot{\rho} - \left(P+\rho\right) \nabla^b u_b + u^a q^b \nabla_b u_a + u^a u^b \nabla_b q_a - \nabla^b q_b = 0\,. \tag{30}$$

At late times $q^c$ and $\dot{\rho}$ disappear from this equation, leaving $\left(P+\rho\right) \nabla^b u_b \simeq 0$. In general $\nabla^b u_b \neq 0$ (this quantity reduces to $3H > 0$ for large $r$) and we are left with $P + \rho \to 0$ as $t \to +\infty$. Either the fluid reduces to a pure $\Lambda$ (then the vacuum uniqueness theorem for SdSK holds trivially), or else both $\rho$ and $P = w\rho$ become subdominant and $\Lambda$ dominates. In this case the solution also reduces to SdSK.

In the case of outflow $q > 0$, one has instead $(B^2)^{\cdot} = -8\pi|A|B^4Rq < 0$; since $B^2$ is bounded from below by zero and it decreases as $t \to +\infty$, it must have an horizontal asymptote with $B^2(t,R) \to B_0^2(R)^+$ as $t \to +\infty$. Then $\dot{B} \to 0$ and $q \to 0$ and we repeat the reasoning done for $q < 0$ from here.

A special case is that of a perfect fluid $q^a = 0$; then $T_{01} = 0$ and the (0,1) equation gives $B = B(R)$. It is then straightforward to prove that it must be $P = -\rho$ and that SdSK is the unique solution.[16]

## 4. Conclusions

A generalized Jebsen-Birkhoff theorem holds for $\Lambda \neq 0$ but, although its proof is straightforward, it does not appear in modern GR textbooks. In particular, for $\Lambda > 0$, SdSK is the unique spherical vacuum solution. Putative alternatives to SdSK in this situation are either non-vacuum solutions or SdSK in disguise. Going beyond the vacuum case, we have proved a simultaneous no-hair and cosmic no-hair theorem in the presence of $\Lambda$ and a radial purely spatial heat flow. Further generalization to other forms of matter will be pursued in the future.

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