

Invariant theory to study symmetry breakings in the adjoint representation of E_8

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Abstract. To study analytically the possible symmetry breakings that may take place in the adjoint representation of the compact Lie group E_8 one may take advantage of the knowledge of the basic polynomials of its Weyl group, that is of the finite reflection group E_8 , and of the (polynomial) equations and inequalities defining its orbit space. This orbit space is stratified and each of its strata represents a possible symmetry breaking, both for the adjoint representation of the Lie group E_8 and for the defining representation of the finite reflection group E_8 . The concrete determination of the equations and inequalities defining the strata is not yet done but it is possible in principle using rank and positive semi-definiteness conditions on a symmetric 8×8 matrix whose matrix elements are real polynomials in 8 variables. This matrix, called the \hat{P} -matrix, has already been determined completely. In this article it will be reviewed how one may determine the equations and inequalities defining the strata of the orbit space and how these equations and inequalities may help to study spontaneous symmetry breakings. Here the focus is on E_8 , because some of the results for E_8 are new, but all what it is here said for E_8 may be repeated for any other compact simple Lie group. The \hat{P} -matrices of all irreducible reflection groups have in fact already been determined.

1. Basic invariant polynomials and orbit spaces

Orbit spaces may be useful to study with clarity and simplicity functions that are invariant under transformations of a compact symmetry group, in particular in problems concerning spontaneous symmetry breakings or structural phase transitions. The invariant functions are in fact constant on the orbits of the group, and may be studied by means of functions defined on the orbit space of the group action.

It may happen that different groups share the same orbit space structure. For example, this may happen for the adjoint representation of a compact connected Lie group G and the defining representation of the Weyl group W of G . When different groups share the same orbit space, the study of invariant functions in the orbit space requires exactly the same calculations and yields, in a certain sense, exactly the same results, for all these groups. Moreover, one may consider the simplest of all these groups to determine the algebraic and geometric structure of the orbit space.

Let's assume that the symmetry group G is a compact group (finite or continuous) acting linearly on a finite dimensional Euclidean space V . Without loss of generality, with respect to a orthonormal basis in V , one may identify V with \mathbb{R}^n , for a certain $n \in \mathbb{N}$, and G with a group of real orthogonal $n \times n$ matrices acting on \mathbb{R}^n with the matrix multiplication. In addition, one

may assume that this G -action be effective, that is, with no non-zero vector left invariant by all the transformations of G .

Let $f(x)$, $x \in \mathbb{R}^n$, be a real function of n real variables. f is said to be G -*invariant* if $f(gx) = f(x)$, $\forall g \in G$ and $x \in \mathbb{R}^n$ in the domain of f (and this implies that in the domain of a G -invariant function there are complete orbits of the G -action). The polynomial functions defined in \mathbb{R}^n form an algebra, indicated with $\mathbb{R}[\mathbb{R}^n]$. G acts naturally on $\mathbb{R}[\mathbb{R}^n]$ by defining $(gp)(x) = p(g^{-1}x)$, $\forall p \in \mathbb{R}[\mathbb{R}^n]$ and $x \in \mathbb{R}^n$. $p \in \mathbb{R}[\mathbb{R}^n]$ is then G -invariant if $gp = p \forall g \in G$. The G -invariant real polynomial functions on \mathbb{R}^n form an algebra, indicated by $\mathbb{R}[\mathbb{R}^n]^G$. For the linearity of the G -action, if $p \in \mathbb{R}[\mathbb{R}^n]$ is homogeneous of degree d , then $gp \in \mathbb{R}[\mathbb{R}^n]$ is homogeneous of degree d .

By Hilbert's theorem on invariants, $\mathbb{R}[\mathbb{R}^n]^G$ is finitely generated, that is, there exist $p_1, \dots, p_q \in \mathbb{R}[\mathbb{R}^n]^G$, such that $\forall p \in \mathbb{R}[\mathbb{R}^n]^G$, there exists a unique polynomial \hat{p} in q indeterminates such that $p(x) = \hat{p}(p_1(x), \dots, p_q(x))$, $\forall x \in \mathbb{R}^n$. The invariant polynomials p_1, \dots, p_q , are usually called *basic invariants*, *basic (invariant) polynomials* or *basic generators*, and they can in all generality be chosen real and homogeneous. Especially in the physical literature, a set of basic polynomials is sometimes called an *integrity basis* of the G -action. I will usually write MIB for a *minimal integrity basis* p_1, \dots, p_q . For the minimality condition, no proper subset of the basic polynomials in a MIB can itself be a MIB. The choice of a set p_1, \dots, p_q of basic polynomials is not unique, but the number q and the degrees d_1, \dots, d_q of the basic polynomials p_1, \dots, p_q , do not depend on the MIB chosen and are uniquely determined by the group G . Any other MIB, formed by a set p'_1, \dots, p'_q , of q homogeneous invariant polynomials of the same degrees d_1, \dots, d_q , may differ from the MIB p_1, \dots, p_q either for a different choice of the (orthonormal) coordinate system in \mathbb{R}^n , or because the p'_a can be expressed as polynomials in the p_a , with the jacobian matrix $J(p) = \|\partial p'(p)/\partial p\|$ not singular.

It is usual to order the basic polynomials according to their degrees, for example in such a way that $2 = d_1 \leq d_2 \leq \dots \leq d_q$. One may also put $p_1(x) = (x, x) = \sum_{i=1}^n x_i^2$, where (\cdot, \cdot) is the canonical scalar product in \mathbb{R}^n (a natural choice for real orthogonal actions).

A MIB allows one to determine the algebraic equations and inequalities defining a semi-algebraic set $\mathcal{S} \subset \mathbb{R}^q$, that can be considered as a concrete representation of the abstract orbit space \mathbb{R}^n/G of the G -action. Moreover, a MIB allows one to determine a one to one correspondence between the smooth G -invariant functions $f(x)$, $x \in \mathbb{R}^n$ and the smooth functions $\hat{f}(p)$, $p \in \mathcal{S}$, in such a way that $f(x) = \hat{f}(p_1(x), \dots, p_q(x))$, $\forall x \in \mathbb{R}^n$ in the domain of f . Then, one may study the G -invariant functions on the set \mathcal{S} , and this may sometimes be convenient and/or enlightening.

The rest of this Section reports some known definitions and results, concerning orbit spaces of effective actions on \mathbb{R}^n of a general compact group $G \subset O(n)$.

- (i) The *orbit* of $x \in \mathbb{R}^n$ is the set $\Omega(x) = \{gx, \forall g \in G\} \subset \mathbb{R}^n$.
- (ii) The *orbit space* of the G -action is the quotient space \mathbb{R}^n/G , in which each orbit is reduced to a single point.
- (iii) The *isotropy subgroup* of $x \in \mathbb{R}^n$ is the subgroup $G_x = \{g \in G \mid gx = x\}$ of G .
- (iv) The *orbit type* $[G_x]$ of an orbit $\Omega(x)$ is the conjugacy class of the isotropy subgroups of the points $x \in \Omega(x)$: $[G_x] = \{gG_xg^{-1}, \forall g \in G\}$. One has in fact $G_{gx} = gG_xg^{-1}$, $\forall g \in G$.
- (v) The *stratum* of type $[G_x]$ is the set $\Sigma_{[G_x]} = \{\bar{x} \in \mathbb{R}^n \mid G_{\bar{x}} \in [G_x]\}$, containing all orbits of orbit type $[G_x]$. The set of strata form a partition of \mathbb{R}^n , called *stratification*. The number of different orbit types and strata is finite.
- (vi) An orbit type is *smaller* (or *greater*) of another orbit type, $[H] < [K]$ (or $[K] > [H]$), if $H' \subset K'$ for some $H' \in [H]$ and $K' \in [K]$. Note that this is a partial ordering in the set of orbit types, given in fact two arbitrary orbit types, it may happen that neither one of the two orbit types is smaller or greater than the other one. There is a unique smallest

orbit type $[G_p]$, the corresponding stratum $\Sigma_{[G_p]}$ is called the *principal stratum* and is dense in \mathbb{R}^n , all other strata are called *singular*. There is a unique greatest orbit type $[G]$, the corresponding stratum $\Sigma_{[G]}$, for all effective actions, is formed by just one point, the origin of \mathbb{R}^n : $\Sigma_{[G]} \equiv 0 \in \mathbb{R}^n$.

- (vii) The basic polynomials *separate the orbits*. This means that, given two different orbits, at least one of the q basic polynomials takes a different value on the two orbits. As a consequence, the *orbit map* $\bar{p} : \mathbb{R}^n \rightarrow \mathbb{R}^q \mid x \rightarrow \bar{p}(x) = (p_1(x), \dots, p_q(x))$ maps \mathbb{R}^n into a subset $\mathcal{S} \subset \mathbb{R}^q$ in such a way that there is a one to one correspondence between orbits in \mathbb{R}^n and points in \mathcal{S} . The orbit map induces a diffeomorphism between the orbit space \mathbb{R}^n/G and the set \mathcal{S} , and for this reason \mathcal{S} can be identified with the orbit space. \mathcal{S} is connected, even if G is not connected or is finite. If Σ is a stratum of \mathbb{R}^n then $\bar{p}(\Sigma)$ is said to be a *stratum* of \mathcal{S} . If $[K] > [H]$, then $\bar{p}(\Sigma_{[K]})$ lies in the boundary of $\bar{p}(\Sigma_{[H]})$. The interior of \mathcal{S} coincides with the principal stratum $\bar{p}(\Sigma_{[G_p]})$ and the boundary of \mathcal{S} contains all singular strata. The origin of \mathbb{R}^q is the image through the orbit map of the origin of \mathbb{R}^n , in fact $\bar{p}(\Sigma_{[G]}) = \bar{p}(0) = 0 \in \mathbb{R}^q$, and is the only stratum of \mathcal{S} that lies in the boundary of all other strata of \mathcal{S} . The orbit map determines a natural grading of the coordinates p_1, \dots, p_q of \mathbb{R}^q , such that $\deg(p_a) = \deg(p_a(x))$, $\forall a = 1, \dots, q$. Both the orbit map and \mathcal{S} are defined through a MIB and depend on the MIB chosen.
- (viii) Every polynomial (or C^∞) G -invariant function $f(x)$ is constant along the orbits of the G -action and may be expressed in a unique way as a polynomial (or a C^∞) function of the MIB: $f(x) = \hat{f}(\bar{p}(x)) = \hat{f}(p_1(x), \dots, p_q(x))$, $\forall x \in \mathbb{R}^n$ in the domain of f . The domain of $\hat{f}(p)$ in \mathbb{R}^q may be extended also outside \mathcal{S} , possibly in many ways, but to study $f(x)$ one is only interested in the restriction of $\hat{f}(p)$ to \mathcal{S} . If $f(x)$ is a G -invariant homogeneous polynomial function, then $\hat{f}(p)$ is a homogeneous polynomial function, and $\deg(\hat{f}(p)) = \deg(f(x))$, taking into account the degrees of the graded variables p_1, \dots, p_q . The correspondence $\pi : \mathbb{R}[\mathbb{R}^n]^G \rightarrow \mathbb{R}[\mathbb{R}^q] : f \rightarrow \hat{f}$ is then one to one and degree preserving.
- (ix) For the linearity of the G -action, the points x and λx , $\forall \lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$, belong to the same stratum. Then each ball centered in the origin of \mathbb{R}^q intersects all the strata of \mathcal{S} and each hyperplane of \mathbb{R}^q of equation $p_1 = \text{constant} > 0$ intersects all strata of \mathcal{S} different than the origin. The intersection of \mathcal{S} or of each connected component of its strata with each one of these spheres or hyperplanes is compact and connected. Roughly, \mathcal{S} appears like a pyramid in \mathbb{R}^q with vertex at the origin, curved faces, and infinite extension just towards the positive direction of the p_1 axis.
- (x) For any stratum $\Sigma \subset \mathbb{R}^n$, and a general point $x \in \Sigma$, the dimension of the stratum $\bar{p}(\Sigma) \subset \mathcal{S}$ coincides with the rank of the jacobian matrix $j(x) = \|\partial p(x)/\partial x\|$ and, consequently, with the rank of the matrix $P(x) = j(x)^T j(x)$, evaluated in x . The explicit expressions of the matrix elements of $j(x)$ and $P(x)$ are the following:

$$j_{ai}(x) = \frac{\partial p_a(x)}{\partial x_i}, \quad P_{ab}(x) = \sum_{i=1}^n \frac{\partial p_a(x)}{\partial x_i} \frac{\partial p_b(x)}{\partial x_i}, \quad a, b = 1, \dots, q, \quad i = 1, \dots, n.$$

$P(x)$ is a real symmetric positive semi-definite matrix whose matrix elements are homogeneous G -invariant polynomial functions of degrees $\deg(P_{ab}(x)) = d_a + d_b - 2$.

- (xi) The matrix elements of $P(x)$ may be expressed as polynomials of the MIB: $P_{ab}(x) = \hat{P}_{ab}(\bar{p}(x))$, $\forall a, b = 1, \dots, q$, $x \in \mathbb{R}^n$. One so obtains a real symmetric $q \times q$ matrix $\hat{P}(p)$, that is positive semi-definite in $\mathcal{S} \subset \mathbb{R}^q$. The matrix $\hat{P}(p)$ is called the *\hat{P} -matrix* (associated to the given MIB) and it is a matrix function of the points $p \in \mathbb{R}^q$. The \hat{P} -matrix clearly depends on the choice of the MIB. Other names that have been used in the literature for the \hat{P} -matrix are *displacement matrix* and *discriminant matrix*.

(xii) If the basic polynomials are algebraically independent (this will be the only case considered in this article because it is the case of the adjoint representations of the compact simple Lie groups), the set \mathcal{S} coincides with the subset of \mathbb{R}^q where the \widehat{P} -matrix is positive semi-definite:

$$\mathcal{S} = \{p \in \mathbb{R}^q \mid \widehat{P}(p) \geq 0\}.$$

Given $x \in \Sigma \subset \mathbb{R}^n$ and $p = \bar{p}(x)$ its image in \mathcal{S} , one has $\dim(\bar{p}(\Sigma)) = \text{rank}(\widehat{P}(p))$, $\forall p \in \bar{p}(\Sigma)$. The union \mathcal{S}_k of all k -dimensional strata of \mathcal{S} ($k \leq q$) is then the following set:

$$\mathcal{S}_k = \{p \in \mathbb{R}^q \mid \widehat{P}(p) \geq 0, \text{ rank}(\widehat{P}(p)) = k\}.$$

Clearly, \mathcal{S}_q coincides with the principal stratum and \mathcal{S}_0 with the origin of \mathbb{R}^q , and there are strata of \mathcal{S} of all dimensions k such that $0 \leq k \leq q$. To find practically the equations and inequalities defining \mathcal{S}_k , one may require that: 1) all the principal minors of $\widehat{P}(p)$ of order $k+1$ are 0; 2) at least one of those of order k is > 0 ; 3) all principal minors of order $\leq k$ are ≥ 0 (remind that the non negativity of just the leading minors of a real symmetric matrix is not a sufficient condition for its positive semi-definiteness). To determine \mathcal{S}_k one has then to solve a system of algebraic equations and inequalities that may be quite big.

(xiii) The set \mathcal{S} is a q -dimensional *semi-algebraic set* because it is defined just using polynomial equations and inequalities. As all semi-algebraic sets, \mathcal{S} is stratified in *primary strata*, that is, it has a connected q -dimensional interior, bordered by connected $(q-1)$ -dimensional faces, that are bordered by connected $(q-2)$ -dimensional faces, and so on, down to its unique 0-dimensional vertex at the origin of \mathbb{R}^q . \mathcal{S}_k is then the union of all the k -dimensional primary strata of \mathcal{S} . There is a one to one correspondence among the connected components of the strata determined by the orbit types of the group action and the primary strata of \mathcal{S} .

(xiv) A *MIB transformation* is a change of the MIB: $\{p_a(x)\} \rightarrow \{p'_a(x)\}$, such that $\forall a = 1, \dots, q$, $p'_a(x) = \widehat{p}'_a(\bar{p}(x))$, with the $\widehat{p}'_a(p)$ homogeneous polynomial functions of the variables $p \in \mathbb{R}^q$ of degrees $d_a = \deg(p_a)$, and the jacobian matrix $J(p) = \|\partial \widehat{p}'(p) / \partial p\|$ not singular. A MIB transformation always implies a change of the system of coordinates in \mathbb{R}^q : $p \rightarrow p'$, such that $\forall a = 1, \dots, q$, $p'_a = \widehat{p}'_a(p)$. A MIB transformation usually implies a change of the \widehat{P} -matrix and a consequent change of the set $\mathcal{S} \subset \mathbb{R}^q$. A change of the MIB consequent just to a change of the orthonormal system of coordinates in \mathbb{R}^n is not necessarily a MIB transformation and never implies a change of the \widehat{P} -matrix and of the set \mathcal{S} .

Let us write $\widehat{P}(p')$ and $\widehat{P}(p)$ for the \widehat{P} -matrices determined by the bases $\{p'_a(x)\}$ and $\{p_a(x)\}$ related by a MIB transformation. The relation between the \widehat{P} -matrices $\widehat{P}(p')$ and $\widehat{P}(p)$ is expressed by the following *\widehat{P} -matrix transformation formula*:

$$\widehat{P}(p') = J^T(p) \widehat{P}(p) J(p) \Big|_{p \rightarrow \widehat{p}(p')}.$$

This formula is very useful to determine the matrix $\widehat{P}(p')$ if one knows the matrix $\widehat{P}(p)$, corresponding to some other basis $\{p_a(x)\}$. In fact, using the \widehat{P} -matrix transformation formula, there is no need to calculate the matrix $P(x)$ corresponding to the basis $\{p'_a(x)\}$, a calculation that is often very long and tedious. The equations and inequalities defining the strata of \mathcal{S} in the new variables p' may be obtained either through the coordinate transformation $p \rightarrow p'$ applied to the equations and inequalities defining the strata of \mathcal{S} in the old variables p , or by solving the system of algebraic equations and inequalities expressing the rank and positive semi-definiteness conditions on the new \widehat{P} -matrix $\widehat{P}(p')$.

Details and proofs of most of the statements here recalled may be found in [2, 14, 1] and in the references therein.

2. The Chevalley isomorphism of the algebras of polynomial invariants

Let G be a compact connected simple Lie group and \mathfrak{g} its Lie algebra. Let l be the rank of G and \mathfrak{g} . The Weyl group W of G is an irreducible finite reflection group of rank l . We are here interested only in the defining representation of W , so $W \subset O(l)$. It is known that W has a MIB with l algebraically independent elements of known degrees $2 = d_1 \leq \dots \leq d_l$ [6, 5].

There is an isomorphism, found by Chevalley [4], between the algebra $\mathbb{R}[\mathfrak{g}]^{\text{Ad}(G)}$ of the real polynomial functions on \mathfrak{g} , invariant for the adjoint representation of G in \mathfrak{g} , and the algebra $\mathbb{R}[\mathbb{R}^l]^W$ of the real polynomial functions on \mathbb{R}^l , invariant for the action of W in \mathbb{R}^l . Independently, Harish-Chandra, in [8], Part III, proved the existence of an isomorphism between $\mathbb{R}[\mathbb{R}^l]^W$ and the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ of G (the elements of $Z(\mathfrak{g})$ are often called *Casimir elements*). Because of the Chevalley and Harish-Chandra isomorphisms, both $\mathbb{R}[\mathfrak{g}]^{\text{Ad}(G)}$ and $Z(\mathfrak{g})$ are generated by l algebraically independent elements of degrees d_1, \dots, d_l . In this Section I will review how Chevalley found the isomorphism between $\mathbb{R}[\mathfrak{g}]^{\text{Ad}(G)}$ and $\mathbb{R}[\mathbb{R}^l]^W$.

The Lie algebra \mathfrak{g} is a real vector space of dimension n . In \mathfrak{g} one can consider a basis $\{T_i\}$, $i = 1, \dots, n$, $T_i \in \mathfrak{g}$, such that any element $X \in \mathfrak{g}$ can be written in the following way:

$$X = \sum_{i=1}^n x_i T_i, \quad x_i \in \mathbb{R}.$$

The basic elements $\{T_i\}$, $i = 1, \dots, n$, are often called the *generators* of \mathfrak{g} . A one to one correspondence between \mathfrak{g} and \mathbb{R}^n is obtained by identifying $X \in \mathfrak{g}$ with the vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$.

The group elements $g \in G$ are obtained by the exponential law $g = \exp(X)$ from the elements $X \in \mathfrak{g}$. In what follows we think of $X \in \mathfrak{g}$ and $g \in G$ to belong to a given representation (of \mathfrak{g} and of G). More precisely, we suppose that the elements of \mathfrak{g} are represented by anti-hermitian matrices (so $X^\dagger = -X$) and that the group elements are represented by unitary matrices (so $g^\dagger = g^{-1}$). The order m of these matrices is the dimension of the representation.

In a representation of \mathfrak{g} realized by anti-hermitian matrices the Killing form $K(T_i, T_j) = -\text{Tr}(T_i^\dagger T_j)$ is a real symmetric $n \times n$ matrix that can be diagonalized and reduced to the identity matrix by choosing a proper basis $\{T_i\}$. In that case one has $K(T_i, T_j) = \delta_{ij}$ and the Killing form defines a scalar product in \mathfrak{g} for which the basis $\{T_i\}$ is orthonormal.

The *adjoint representation* $\text{Ad}(G)$ of the group G is given by the following linear action of G in the given representation space of \mathfrak{g} :

$$\text{Ad}(g) X = g X g^{-1} = g X g^\dagger \quad \forall g \in G, X \in \mathfrak{g},$$

where the g in the formula, except that one in $\text{Ad}(g)$, means the $m \times m$ unitary matrix that represents the abstract element $g \in G$ in the given representation of G . Sometimes the group $\text{Ad}(G)$, whose elements are $\text{Ad}(g)$, is called the *adjoint group* of G .

Using the Killing form as the scalar product in \mathfrak{g} , this action (of G on \mathfrak{g}) doesn't change the norms of the vectors $X \in \mathfrak{g}$, in fact, by the cyclic property of the trace and the unitarity of $g \in G$, one has:

$$\|\text{Ad}(g)X\|^2 = \|g X g^\dagger\|^2 = -\text{Tr}((g X g^\dagger)^\dagger g X g^\dagger) = -\text{Tr}(X^\dagger X) = \|X\|^2.$$

Hence, by a standard result of linear algebra, the linear action of the group $\text{Ad}(G)$ in \mathfrak{g} can be represented by a group Γ of real orthogonal $n \times n$ matrices acting in \mathbb{R}^n with the matrix multiplication. In fact, if $Y = \text{Ad}(g)X$, and $x, y \in \mathbb{R}^n$ represent $X, Y \in \mathfrak{g}$ with respect to the orthonormal basis $\{T_i\}$, and $y = O(g)x$, where $O(g) \in \Gamma$ is the real orthogonal $n \times n$ matrix that represents the linear transformation induced by $\text{Ad}(g)$ in \mathbb{R}^n , one has:

$$\|Y\|^2 = \|y\|^2 = (y, y) = (O(g)x, O(g)x) = (x, O(g)^T O(g)x) = (x, x) = \|x\|^2 = \|X\|^2.$$

Note that in the last formula there are two different, but equivalent, ways to evaluate the norms: $\|X\|^2 = K(X, X) = -\text{Tr}(X^\dagger X)$, if $X \in \mathfrak{g}$, and $\|x\|^2 = (x, x) = \sum_{i=1}^n x_i^2$, if $x \in \mathbb{R}^n$.

In the following we will always consider the action of Γ in \mathbb{R}^n instead of the action of $\text{Ad}(G)$ in \mathfrak{g} . For what has just been said, the two descriptions are equivalent and the use of Γ allows to employ all the results reviewed in Section 1.

It is known that, if $l = \text{rank}(\mathfrak{g})$, there exist an l -dimensional subalgebra \mathfrak{h} of \mathfrak{g} , called a Cartan subalgebra of \mathfrak{g} , such that all the elements of \mathfrak{h} commute among themselves. It is possible, and that will be here supposed, to choose the basic orthonormal elements $\{T_i\}$ in such a way that the first l of them generate \mathfrak{h} , so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{e}$, where \mathfrak{h} and \mathfrak{e} are subspaces of \mathfrak{g} generated by $\{T_1, \dots, T_l\}$ and by $\{T_{l+1}, \dots, T_n\}$, respectively. \mathfrak{h} may then be identified with the l -dimensional subspace of \mathbb{R}^n of the first l coordinates: for $x = (x_1, \dots, x_l, 0, \dots, 0) \in \mathbb{R}^n$, we set $\xi_i = x_i$, $\forall i = 1, \dots, l$, then $\xi = (\xi_1, \dots, \xi_l) \in \mathbb{R}^l$ is in a one to one correspondence with the element $X = \sum_{i=1}^l \xi_i T_i \in \mathfrak{h}$. For this reason we will sometimes write \mathfrak{h} and \mathfrak{g} also to mean the vector spaces \mathbb{R}^l and \mathbb{R}^n of the coefficients ξ_i and x_i , respectively. The l one-parameter subgroups of G with elements of the form $\exp(\xi_i T_i)$, $i = 1, \dots, l$, (in which no sum over i has to be understood), generate an l -dimensional Abelian subgroup H of G , the Cartan subgroup of G that has Lie algebra \mathfrak{h} .

For the compactness of G , every $g \in G$ is conjugated to an element $h \in H$. This means that all the orbits of the action of the group Γ in \mathfrak{g} intersect \mathfrak{h} . As every $\xi \in \mathfrak{h}$ completely specifies the orbit $\Omega(\xi)$ of the action of Γ in \mathfrak{g} that contains ξ , all the orbits of the action of Γ in \mathfrak{g} can be specified by points $\xi \in \mathfrak{h}$.

Let $N(H) = \{g \in G \mid gHg^{-1} \subseteq H\}$ be the normalizer of H in G . If $X \in \mathfrak{h}$ and $\epsilon \in \mathbb{R}$, $h = \exp(\epsilon X) = \mathbb{1}_m + \epsilon X + O(\epsilon^2) \in H$, and $ghg^{-1} = \mathbb{1}_m + \epsilon(gXg^{-1}) + O(\epsilon^2)$. If $g \in N(H)$, $ghg^{-1} \in H$ and then $gXg^{-1} \in \mathfrak{h}$. This implies that the restriction of Γ to the elements $g \in N(H)$, is a group Γ_N of inner automorphisms of \mathfrak{h} , under which \mathfrak{h} is invariant. The matrices of the linear group Γ_N are then block diagonal matrices, formed by the direct sum of an $l \times l$ orthogonal matrix, in correspondence to \mathfrak{h} , and an $(n-l) \times (n-l)$ orthogonal matrix, in correspondence to \mathfrak{e} . An invariant polynomial $p(x) \in \mathbb{R}[\mathfrak{g}]^\Gamma$ must also be an element of $\mathbb{R}[\mathfrak{g}]^{\Gamma_N}$, because Γ_N is a subgroup of Γ . Moreover, also the restriction $\tilde{p}(\xi)$ of $p(x)$ to \mathfrak{h} is in $\mathbb{R}[\mathfrak{g}]^{\Gamma_N}$, because \mathfrak{h} is Γ_N -invariant. Vice versa, a polynomial $\tilde{p}(\xi)$, $\xi \in \mathfrak{h}$, invariant for the action of Γ_N in \mathfrak{g} , defines a unique polynomial $p(x) \in \mathbb{R}[\mathfrak{g}]^\Gamma$, such that $\tilde{p}(\xi)$ is its restriction to \mathfrak{h} . In fact, $\forall x \in \mathfrak{g}$, $\exists g \in G \mid O(g)x \in \mathfrak{h}$, so that one may define $p(x) = \tilde{p}(O(g)x)$. Of course, it may exist a $g_1 \in G$, $g_1 \neq g$, such that $O(g_1)x \in \mathfrak{h}$, but then $\exists g' \in N(H) \mid g'g = g_1$, and both $O(g)x$ and $O(g_1)x$ must lie in the same orbit of Γ_N , so that $\tilde{p}(O(g)x) = \tilde{p}(O(g_1)x)$ and this means that $p(x)$ is well defined. The polynomial $p(x)$ so defined is Γ -invariant, in fact, the value of p in all the points of the orbit $\Omega(x)$ of the Γ -action is the same as in the points ξ of the intersection $\Omega(x) \cap \mathfrak{h}$. All this proves the existence of an isomorphism between the algebra $\mathbb{R}[\mathfrak{g}]^\Gamma$ and the restriction $\mathbb{R}[\mathfrak{g}]^{\Gamma_N}|_{\mathfrak{h}}$ of the algebra $\mathbb{R}[\mathfrak{g}]^{\Gamma_N}$ to \mathfrak{h} .

The image of the Abelian subgroup H of G in $\text{Ad}(G)$ is represented by an Abelian subgroup Γ_H of Γ that acts trivially on \mathfrak{h} (that is, the elements of Γ_H have the identity matrix on the first l -dimensional block). Then the quotient group Γ_N/Γ_H acts naturally on \mathfrak{h} . The *Weyl group* W of G is defined as the quotient group $N(H)/H$, a definition that does not depend on the Cartan subgroup $H \subset G$. W is in any case a finite group, even if G and H are not. The action of the group Γ_N/Γ_H in \mathfrak{h} is then isomorphic to the action of the group W in \mathfrak{h} and this also implies $\mathbb{R}[\mathfrak{g}]^{\Gamma_N}|_{\mathfrak{h}} \equiv \mathbb{R}[\mathfrak{h}]^W$.

All these facts prove that there is an isomorphism between $\mathbb{R}[\mathfrak{g}]^\Gamma$ and $\mathbb{R}[\mathfrak{h}]^W$ and that the restriction to \mathfrak{h} of a MIB for $\mathbb{R}[\mathfrak{g}]^\Gamma$ gives a MIB for $\mathbb{R}[\mathfrak{h}]^W$ and vice versa, a MIB for $\mathbb{R}[\mathfrak{h}]^W$ uniquely specifies a MIB for $\mathbb{R}[\mathfrak{g}]^\Gamma$.

The relation between the action of Γ in \mathbb{R}^n ($\simeq \mathfrak{g}$) and of the action of W in \mathbb{R}^l ($\simeq \mathfrak{h}$), may

be further described using a Cartan Weyl basis for the complexification $\mathfrak{g}_c = \mathfrak{g} + i\mathfrak{g}$ of \mathfrak{g} .

With a basis for \mathfrak{h} one may determine the roots and a Cartan Weyl basis for \mathfrak{g}_c , in which it is usual to write the basic elements as $\{H_1, \dots, H_l, E_{\alpha_i}, E_{-\alpha_i}, \dots\}$, with $H_i \in \mathfrak{h}$, $\forall i = 1, \dots, l$, and $E_{\alpha_i}, E_{-\alpha_i} \in \mathfrak{e}_c$, where α_i is ranging in the set of the positive roots of \mathfrak{h} in \mathfrak{g} . This basis may be expressed in terms of the basis $\{T_i\}$ in the following way: $H_i = T_i$, $\forall i = 1, \dots, l$, $E_{\alpha_i} = \frac{1}{\sqrt{2}}(T_{l+2i} - iT_{l+2i-1})$, $E_{-\alpha_i} = E_{\alpha_i}^\dagger$, $\forall i = 1, \dots, (n-l)/2$, if the $\{T_i\}$ are ordered properly.

Every root β is a non vanishing vector of \mathbb{R}^l and defines a reflection S_β in \mathbb{R}^l ,

$$S_\beta \xi = \xi - \frac{2(\xi, \beta)}{(\beta, \beta)}\beta, \quad \forall \xi \in \mathbb{R}^l,$$

sending each point of $\xi \in \mathbb{R}^l$ to its symmetric with respect to the hyperplane through the origin of \mathbb{R}^l orthogonal to β . The *Weyl group* W of G (and \mathfrak{g}) may be defined as the group generated by all the reflections S_β , with β a root. All the elements $w \in W$ may then be expressed as a product of a finite number of these reflections. W is then a group of real orthogonal $l \times l$ matrices, irreducible if G is simple. This definition of the Weyl group turns out to be equivalent to the former, given as the quotient group $N(H)/H$. All elements $w \in W$ send the set of the roots to itself and the set of the reflecting hyperplanes to itself. The action of w in \mathbb{R}^l corresponds then to a permutation of the roots: $\alpha_i \rightarrow w\alpha_i$, and induces a permutation of the generators of \mathfrak{e}_c according to the rule: $E_{\alpha_i} \rightarrow E_{w\alpha_i}$, for every root α_i .

3. Basic invariant polynomials and orbit spaces of the adjoint representations

In this Section I will review how one may determine the equations and inequalities defining the orbit spaces of the adjoint representations of the compact simple Lie groups. I will continue to use the notation introduced in the previous Section.

Elements of $\mathbb{R}[\mathbb{R}^n]^\Gamma$, that is Γ -invariant real polynomial functions defined on \mathbb{R}^n , are obtained by writing the explicit expressions of the Casimir elements in $Z(\mathfrak{g})$ [10] in terms of the n real coefficients x_i of the expansions of the elements $X \in \mathfrak{g}$ in terms of the generators T_i , $i = 1, \dots, n$. The explicit expressions of these *Casimir polynomials* are the following:

$$I^{(k)}(x) = \text{Tr}(X^k), \quad X = \sum_{i=1}^n x_i T_i, \quad x_i \in \mathbb{R}, \quad k \in \mathbb{N}.$$

The $I^{(k)}(x)$ so defined are Γ -invariant polynomials, one in fact has:

$$I^{(k)}(O(g)x) = \text{Tr}((gXg^\dagger)^k) = \text{Tr}(gX^k g^\dagger) = \text{Tr}(X^k) = I^{(k)}(x), \quad \forall g \in G, \quad x \in \mathbb{R}^n.$$

For the results in the previous Section, the restriction of a Casimir polynomial $I^{(k)}(x)$ to the l dimensional subspace corresponding to the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, gives an invariant polynomial $\tilde{I}^{(k)}(\xi)$ for the action of the Weyl group W in \mathbb{R}^l :

$$\tilde{I}^{(k)}(\xi) = \text{Tr}(X^k), \quad X = \sum_{i=1}^l \xi_i T_i, \quad \xi_i \in \mathbb{R}, \quad k \in \mathbb{N}.$$

Vice versa, $\tilde{I}^{(k)}(\xi)$ uniquely extends and defines the Casimir polynomial $I^{(k)}(x)$.

For a certain set of degrees d_1, \dots, d_l , the l Casimir polynomials $I^{(d_a)}(x)$, $a = 1, \dots, l$, are algebraically independent and form a MIB of $\mathbb{R}[\mathfrak{g}]^\Gamma$. Their restrictions to \mathfrak{h} , $\tilde{I}^{(d_a)}(\xi)$, form then a MIB of l algebraically independent polynomials for $\mathbb{R}[\mathfrak{h}]^W$ with the same set of degrees.

An important result is that both the MIB $I^{(d_a)}(x)$, $a = 1, \dots, l$, of $\mathbb{R}[\mathbb{R}^n]^\Gamma$, and the MIB $\tilde{I}^{(d_a)}(\xi)$, $a = 1, \dots, l$, of $\mathbb{R}[\mathbb{R}^l]^W$, determine the same \hat{P} -matrix. The proof is in [12], Section

III. For the results in Section 1, this implies that, both the actions of Γ in \mathbb{R}^n and of W in \mathbb{R}^l have the same orbit space, with the same geometric stratification. The strata of \mathcal{S} correspond in the two cases of Γ or of W to different orbit types, that is to the conjugacy classes of the isotropy subgroups of the actions of Γ or of W . Obviously, to a MIB transformation for $\mathbb{R}[\mathbb{R}^n]^\Gamma$ (for $\mathbb{R}[\mathbb{R}^l]^W$) it corresponds a MIB transformation for $\mathbb{R}[\mathbb{R}^l]^W$ (for $\mathbb{R}[\mathbb{R}^n]^\Gamma$), given by the same transforming equations, if these are written in terms of the basic invariants. This MIB transformation implies a change of the \widehat{P} -matrix (according to the \widehat{P} -matrix-transformation formula) and a consequent change of the equations and inequalities defining the orbit space \mathcal{S} and its strata. It happens then that if the MIB $\{\tilde{p}_a(\xi)\}$ of $\mathbb{R}[\mathbb{R}^l]^W$ is obtained by restricting to \mathfrak{h} a MIB $\{p_a(x)\}$ of $\mathbb{R}[\mathbb{R}^n]^\Gamma$, then the MIB's $\{\tilde{p}_a(\xi)\}$ and $\{p_a(x)\}$ determine the same \widehat{P} -matrix and the same orbit space $\mathcal{S} \subset \mathbb{R}^l$ for the actions of W and Γ . In fact, there exists a unique MIB transformation that transforms the MIB $\{\tilde{I}^{(d_a)}(\xi)\}$ to the MIB $\{\tilde{p}_a(\xi)\}$ and the MIB $\{I^{(d_a)}(x)\}$ to the MIB $\{p_a(x)\}$, this last one differs from the first one just for the substitutions $\tilde{I}^{(d_a)} \rightarrow I^{(d_a)}$, and $\tilde{p}_a \rightarrow p_a$, $\forall a = 1, \dots, l$.

The arguments described in this Section suggest that to determine the orbit space of the adjoint representation of a simple compact Lie group G , it is possible, and more convenient, to determine the orbit space of the Weyl group W of G . The main reasons of this convenience are the following: 1) W is finite while Γ is not; 2) W acts in \mathbb{R}^l , while Γ acts in \mathbb{R}^n , and $l < n$; 3) both W and Γ have the same \widehat{P} -matrix, the same orbit space \mathcal{S} and the same equations and inequalities defining all strata of \mathcal{S} , if these equations and inequalities are expressed in terms of the MIB, and if the MIB of $\mathbb{R}[\mathbb{R}^l]^W$ is obtained by restricting to \mathfrak{h} a MIB of $\mathbb{R}[\mathbb{R}^n]^\Gamma$.

If the rank l is large, given a general MIB $\{p_a(\xi)\}$ of $\mathbb{R}[\mathbb{R}^l]^W$, it may be very hard to determine the \widehat{P} -matrix directly from the MIB $\{p_a(\xi)\}$ (that is, through the determination of the matrix $P(\xi)$). It is often convenient to use a known \widehat{P} -matrix, corresponding, in general, to another MIB $\{p'_a(\xi)\}$ of $\mathbb{R}[\mathbb{R}^l]^W$, determine the MIB transformation relating the MIB $\{p'_a(\xi)\}$ to the MIB $\{p_a(\xi)\}$ and make use of the \widehat{P} -matrix transformation formula. For example, if one knows the \widehat{P} -matrix corresponding to the MIB $\{p'_a(\xi)\}$ of $\mathbb{R}[\mathbb{R}^l]^W$, and one wishes to determine the \widehat{P} -matrix corresponding to the MIB $\{I^{(d_a)}(x)\}$ of $\mathbb{R}[\mathbb{R}^n]^\Gamma$, it is convenient to determine the MIB transformation relating the MIB's $\{p'_a(\xi)\}$ and $\{\tilde{I}^{(d_a)}(\xi)\}$ and use the \widehat{P} -matrix transformation formula. The \widehat{P} -matrices of the finite reflection groups E_7 and E_8 have been calculated in [16]. The \widehat{P} -matrices of all the other irreducible reflection groups have been calculated in [7, 11, 17, 13].

4. Symmetry breakings in the orbit spaces

We continue here to use the notation of the previous Section and write Γ for the real orthogonal group acting on \mathbb{R}^n that realizes the adjoint representation $\text{Ad}(G)$, and W for the Weyl group of G , acting on \mathbb{R}^l as a reflection group. Clearly, $n = \dim(\mathfrak{g})$ and $l = \text{rank}(\mathfrak{g}) = \text{rank}(W)$.

Let f be a function invariant for the group Γ (or for the group W). f may be expressed in a unique way in terms of a given MIB of Γ (or of W), so f uniquely defines a function \widehat{f} of l variables such that $f(x) = \widehat{f}(p_1(x), \dots, p_l(x))$, $\forall x$ in the domain of f . The function \widehat{f} represents both a Γ -invariant function and a W -invariant function, the only difference is in considering the l variables in \widehat{f} as the l basic polynomials of Γ or of W . When the MIB of $\mathbb{R}[\mathbb{R}^l]^W$ is the restriction to \mathfrak{h} of a MIB of $\mathbb{R}[\mathbb{R}^n]^\Gamma$, the actions of W and Γ determine the same orbit space $\mathcal{S} \subset \mathbb{R}^l$, with the same stratification. Only this case will be considered in the following. In this case, when one studies the function \widehat{f} in \mathcal{S} , there is no difference between the problem in which the symmetry group is Γ and the problem in which the symmetry group is W . In both cases one has in fact to study the function \widehat{f} in the same semi-algebraic connected subset $\mathcal{S} \subset \mathbb{R}^l$ that represents the orbit space, of Γ and of W .

\widehat{f} depends in general on the l variables corresponding to the l basic polynomials (of Γ or of W) and to a certain number of other parameters related to the physical problem one is dealing

with. If f is an invariant potential function, and one wants to study spontaneous symmetry breakings or phase transitions, one is interested in the location of the minima of \hat{f} and in the orbit type of the stratum hosting the minimum point. Let the minimum point of \hat{f} be in a given point of \mathcal{S} , belonging to a given stratum that corresponds to a given orbit type. If for some reasons the parameters in \hat{f} change with continuity, the minimum point of \hat{f} , in general, changes its position with continuity, and it may happen that it changes stratum, changing in this way the orbit type of the orbit of minimum points (and consequently the residual symmetry), realizing in this way what is called a spontaneous symmetry breaking or a second order structural phase transition.

This changing may take place only between neighboring strata of \mathcal{S} , because of the continuity of the changing of the location of the minimum point of a continuous function.

What one has to do to study the spontaneous symmetry breakings in the orbit space \mathcal{S} is summarized in the following points.

1) Determine the equations and inequalities defining the various strata of \mathcal{S} . 2) Determine the orbit types (of the Γ - or of the W -action) corresponding to the various strata of \mathcal{S} . 3) Express the invariant potential function f in terms of the MIB, obtaining in this way a function \hat{f} of l variables. 4) Determine the constrained minima of \hat{f} in the various strata of \mathcal{S} , and the range of the parameters in \hat{f} for which the absolute minimum of \hat{f} falls into one or another stratum. Point 1) is solved using rank and positive semi-definiteness conditions of the \hat{P} -matrix. Point 2) is solved by determining the isotropy subgroups in convenient points $\xi \in \mathbb{R}^l$ (or $x \in \mathbb{R}^n$), where \mathbb{R}^l (or \mathbb{R}^n) is the space on which W (or Γ) acts, chosen in such a way to have images through the orbit map in each one of the primary strata of \mathcal{S} . It may be useful to remind that points lying in the boundary of \mathcal{S} are images through the orbit map of orbits lying in the set of the reflecting hyperplanes of W . To each stratum of \mathcal{S} it corresponds both an orbit type of W and an orbit type of Γ , depending on which group action one considers. The two sets of orbit types are clearly related. Point 4) is solved for example with the method of Lagrange multipliers to find constrained extrema on the various primary strata of \mathcal{S} .

The study of minimization of invariant functions here described can be done analytically, that is with full precision, because from the \hat{P} -matrices one may determine all the equations and inequalities defining the various strata of the orbit space. The calculations are not straightforward but do not present conceptual difficulties. In practice, one has to work with systems of algebraic equations and inequalities that may be quite complicated to handle.

5. Results for the adjoint representation of E_8

During the last 30 years, the adjoint representation of the compact Lie group E_8 has been many times proposed in models of grand unification theories and of string theories and an exact computational method to employ in the study of spontaneous symmetry breakings in these models may be of a certain interest.

As before, let's write $W \subset O(8)$ for the finite reflection group E_8 and $\Gamma \subset O(248)$ for the linear group realizing the adjoint representation of the Lie group E_8 .

The degrees of a set of basic polynomials p_1, \dots, p_8 , both for $\mathbb{R}[\mathbb{R}^8]^W$ and for $\mathbb{R}[\mathbb{R}^{248}]^\Gamma$, are known to be 2, 8, 12, 14, 18, 20, 24, 30. The center $Z(\mathfrak{e}_8)$ of the universal enveloping algebra $U(\mathfrak{e}_8)$ of E_8 is also generated by 8 algebraically independent Casimir elements of the same set of degrees. Besides the quadratic and the octic (of degree 8) Casimir element [3], to my knowledge, no other generator of $Z(\mathfrak{e}_8)$ have been explicitly calculated so far. Many authors described how to construct a set of basic polynomials for $\mathbb{R}[\mathbb{R}^8]^W$. A simple description was given for example by Mehta in [9], in terms of the linear functions defining the invariant hyperplanes of W . These basic polynomials were calculated explicitly in [16] and their expressions may be found in the supplementary material of that article (Ref. 30 of [16]). Their expressions are quite large, for example, $p_8(x)$ is a real homogeneous polynomial function of 8 variables of degree 30 that has

10592 terms when expanded. The explicit expression of the corresponding \widehat{P} -matrix is also reported in Ref. 30 of [16]. This \widehat{P} -matrix, using the \widehat{P} -matrix transformation formula, allows to determine the \widehat{P} -matrices corresponding to any other MIB of $\mathbb{R}[\mathbb{R}^8]^W$ (in [16], for example, this formula was used to calculate the \widehat{P} -matrix corresponding to a flat basis). The geometric structure of the orbit space \mathcal{S} , corresponding both to the group W and to the group Γ can then be determined explicitly, as described in Section 1 above, and in particular it is possible to determine explicitly the equations and inequalities of all the strata of \mathcal{S} . The calculations to do that are not yet done and are not trivial because the \widehat{P} -matrix is an 8×8 matrix with elements that are real homogeneous polynomials in 8 variables with degrees ranging from 2 to 58 (by considering the degrees of the graded variables p_1, \dots, p_8). Anyway, these calculations are possible because one knows completely the explicit expression of the \widehat{P} -matrix.

The study of the minima of a Γ - (or W -) invariant potential function and of the related spontaneous symmetry breakings may be performed following the method outlined in Section 4, however, because of the high degrees of the equations that are involved, one encounters quite complicated systems of algebraic equations and inequalities.

Acknowledgments

The author wishes to thank prof. Pietro Corvaja for the numerous enlightening discussions on Lie group theory and for the reading of the manuscript. Many thanks also to the referee that helped to improve both the English and the format of the article.

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