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RADIATION FROM A RING CHARGE PASSING THROUGH A RESONATOR

J. D. LAWSON

Rutherford Appleton Laboratory, Chilton, Oxon OX11 0QX, United Kingdom

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An order-of-magnitude estimate is made of the power lost by a charge Q traveling along a drift tube that opens out to form a resonator of finite length G. Both point and ring-shaped charges are considered, the latter configuration being of importance for the Electron Ring Accelerator (ERA). For highly relativistic charges ($\gamma \gg 1$) the radiated power is found to be considerably larger than that previously quoted. The reason for this is that in previous calculations frequency components with wavelength less than the drift-tube circumference were not included. If the present estimate is correct, then a severe limitation on the parameters available for a high-energy ERA is implied.

1. INTRODUCTION

In assessing the potentialities of the Electron Ring Accelerator[1,2] it is essential to know the energy lost by radiation when a ring passes through one of the accelerating cavities. An accurate calculation of this quantity is difficult; the following order-of-magnitude estimate is intended as a guide until someone completes a formal calculation.

We consider first the special case of a point charge passing through a resonator of cylindrical form, as shown in Fig. 1. This problem has been treated by Kopalkov and Kotov[3] using a modal analysis. However, they only calculated the contribution to the energy loss arising from modes trapped in the cavity; energy in the continuum of modes that can propagate down the drift tube was not included.‡ A cutoff in the sum over modes was introduced at $\lambda_{\min} = 2\pi R$.

2. OUTLINE OF METHOD

In the present paper we make an estimate of the energy lost to modes for which $\lambda < 2\pi R$. First, the charge is represented as a δ-function of current, which is then Fourier analysed into a spectrum of sinusoidal current distributions $\cos \omega(t - x/\beta c)$. In free space the E and B fields associated with each component may be represented in terms of Hankel functions with argument $i\omega r/c\gamma$; such functions

† See postscript at end of paper.

‡ These two components are analogous to excitation and ionization of atoms by a passing charged particle.

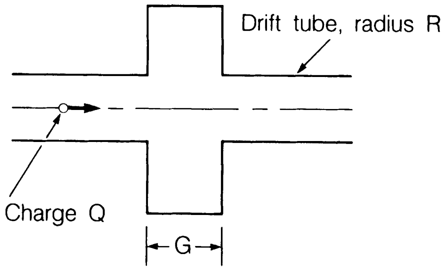

FIGURE 1

extend radially to a distance of order $c\gamma/\omega = \gamma\lambda\!\!\!^{-}$, where λ is the free-space wavelength associated with the frequency ω. For a charge moving in a drift tube of radius R we should therefore expect wavelengths down to about $2\pi R/\gamma$ to "notice" the effect of the drift tube and the discontinuity caused when the cavity is entered. Our analysis includes all frequencies, but it will be seen that the contribution of the ultrahigh-frequency components is unimportant (cf. the cutoff in the calculation of the synchrotron radiation spectrum). Because of this cutoff a knowledge of the axial extent of the ring (Lorentz contracted by a factor $1/\gamma$ in the laboratory frame) is unimportant.

Let us fix attention for the moment on a single frequency component for which $R \gg \lambda\!\!\!^{-} \gg R/\gamma$. The field in the drift tube will be very little different from what it would be in free space, since the main field components are E_r and B_θ, which satisfy the boundary conditions at a metal wall. Locally, when $\beta \sim 1$, the field looks very like a plane wave; indeed in the limit of $\beta = 1$ it becomes a plane wave similar to that in a coaxial transmission line. Considering now this pseudo "plane wave," we ask what happens when the drift tube opens out to form a cavity. It seems reasonable to assume that in the region $r \sim R$ Fresnel diffraction occurs, and the energy associated with a particular mode spreads in the manner given by the standard theory of "diffraction at an edge."[4] Having crossed the cavity, energy that has diffracted outside the cylinder $r = R$ is reflected and lost as sketched in Fig. 2. The essence of the calculation is to estimate the energy in this ring and integrate over ω.

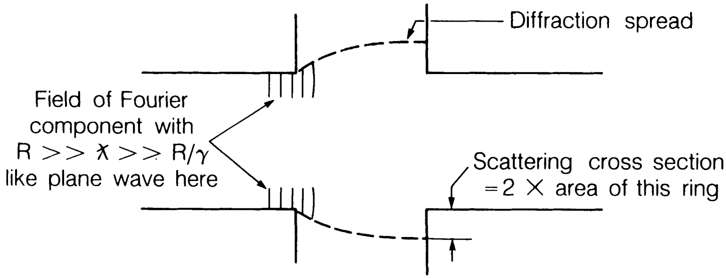

FIGURE 2

3. DETAILS OF CALCULATION

We first establish the energy diffracted into the shadow of a straight edge. An elementary calculation shows that the field amplitude in the shadow region at a distance D behind a screen is given by

$$\frac{E}{E_0}=\left(\frac{1}{2\lambda D}\right)^{1/2}\int_x^\infty \exp\left(\frac{i\pi x^2}{\lambda D}\right)dx, \tag{1}$$

whence

$$\left|\frac{E}{E_0}\right|=\frac{1}{\sqrt{2}}\left\{\left[C\left(\frac{\pi x^2}{\lambda D}\right)-\frac{1}{2}\right]^2+\left[S\left(\frac{\pi x^2}{\lambda D}\right)-\frac{1}{2}\right]^2\right\}^{1/2}, \tag{2}$$

where x is the distance measured into the shadow region, E_0 the amplitude in the absence of the screen, and C and S the Fresnel integrals[5] (Fig. 3). The total energy flux in the shadow region per unit length of screen, in terms of the flux per unit area in the absence of the screen, is given by $\int (E/E_c)^2\,dx$. This may be written

$$Y=\left(\frac{\lambda D}{8}\right)^{1/2}\int_0^\infty\left\{\left[C(u)-\frac{1}{2}\right]^2+\left[S(u)-\frac{1}{2}\right]^2\right\}du\approx 0.11\sqrt{\lambda D}. \tag{3}$$

Having established this lemma we now proceed to the main calculation.

The spectral analysis of the field of a point charge Q in terms of the Poynting flux density W of each component as a function of r has been given, for example, by Heitler[6]:

$$W(r)\,d\omega=\frac{Q^2\omega^2\,d\omega}{4c^3\gamma^2}\left[H_1^{(1)}\left(\frac{i\omega r}{c\gamma}\right)\right]^2. \tag{4}$$

The energy scattered by the second wall of the cavity is therefore

$$\Delta U=2\int_{c/R}^\infty W_{r=R}\;Y\cdot 2\pi R\,d\omega. \tag{5}$$

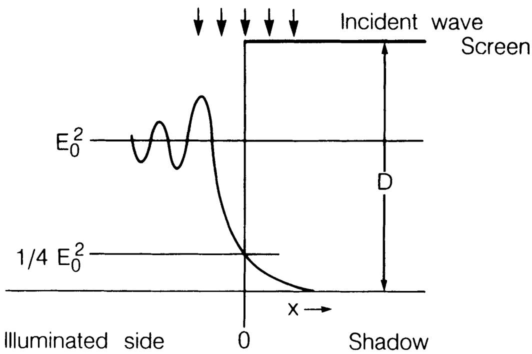

FIGURE 3

The lower limit in the integration has been put equal to c/R. At this limit the condition $R \gg \lambda\!\!\!^{-}$ is not satisfied. For small γ most of the energy is in this region, and the approximation is poor. For large γ, however, it should be reasonable. The factor 2 in front of the equation occurs because the scattering cross section of the ring is twice the geometrical area.

Insertion of Eqs. (3) and (4) into Eq. (5) and setting $D = G$ yields the explicit form

$$\Delta U = 0.11 \times \sqrt{2}\pi^{3/2} \frac{Q^2 G^{1/2} \gamma^{1/2}}{R^{3/2}} \int_{1/\gamma}^{\infty} y^{3/2} H_1^{(1)}(iy)\, dy. \tag{6}$$

We may further approximate for large γ by taking the asymptotic form of the Hankel function for small x, $H_1^{(1)}(ix) = 2/\pi x$. This is a good approximation for $x = 0.1$, so the integral becomes (for $\gamma > q10$)

$$\begin{aligned}\int_{1/\gamma}^{\infty} y^{3/2} H_1^{(1)}(iy)\, dy &= \int_{1/\gamma}^{0.1} \frac{4}{\pi^2} y^{-1/2}\, dy + \int_{0.1}^{\infty} y^{3/2} H_1^{(1)}(iy)\, dy \\ &\approx \frac{8}{\pi^2}\left[(0.1)^{1/2} - \left(\frac{1}{\gamma}\right)^{1/2}\right] + 0.47 \\ &\approx 0.72,\end{aligned} \tag{7}$$

whence

$$\Delta U \approx 0.11 \times \sqrt{2} \times \pi^{3/2} \times 0.72 \frac{Q^2 G^{1/2} \gamma^{1/2}}{R^{3/2}},$$

and

$$\Delta U \approx 0.6 \frac{Q^2 G^{1/2} \gamma^{1/2}}{R^{3/2}} \quad \text{for} \quad \gamma \gg 1. \tag{8}$$

4. RADIATION FROM A SINGLE HOLE IN A SCREEN

Clearly, if G is so large that the diffraction spreading is not small compared with R, the formula will overestimate the loss. As G increases indefinitely it must disappear from the formula for ΔU, which then tends to the value for passage through a single hole in a screen. To find this we calculate $\int_0^\infty \int_R^\infty 2\pi R W\, dr\, d\omega$ from Eq. (4). The integration is easier if done before the Fourier decomposition (Faltens, Ref. 2, p. 363†). Changing the limits of his first integral to R to ∞ yields

$$\Delta U = \frac{3\pi}{8} \frac{Q^2 \gamma}{R}, \tag{9}$$

which, apart from a factor of $3\pi/8$, agrees with Eq. (18) of Ref. 3. Note that γ

† Although his expression for the fields is correct, the assumption that the radiation goes in a cone with angle $\theta = 1/\gamma$ is not valid.

enters here to the first power. This might be expected on physical grounds by analogy with bremsstrahlung.

5. EXTENSION TO A RING CHARGE

We now repeat the calculation for a ring charge where the radius of the ring is almost as great as that of the drift tube. The distance measured outward from the ring will be denoted by ρ. For $\beta \sim 1$ the radial field components associated with such a current in free space would be

$$E_r = B_\theta = \frac{2Q}{A}\left(\frac{2}{\pi}\right)^{1/2} d\omega \exp\left(\frac{-\omega\rho}{c\gamma}\right). \tag{10}$$

As before, we now assume the current to be surrounded by a tube of radius R.

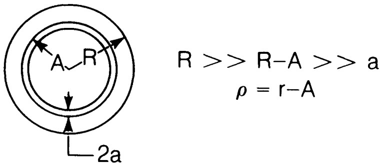

FIGURE 4

The Poynting flux density at $\rho = R - A$ is given by

$$W(\rho)\, d\omega = \frac{1}{c}\int \frac{Q^2}{\pi^2 A^2} \exp\left[\frac{-2\omega(R-A)}{c\gamma}\right] d\omega. \tag{11}$$

Combining with Eqs. (3) and (5), we have, analogous to Eq. (6),

$$\Delta U = 0.11 \times \frac{4}{\pi^{1/2}} \times \frac{Q^2 G^{1/2}\gamma^{1/2}}{A^2(R-A)^{1/2}} \int_{2/\gamma}^{\infty} [\exp(-y)\, dy/y^{1/2}], \quad \text{and}$$

$$\Delta U \approx 0.44 \frac{Q^2 G^{1/2} R\gamma^{1/2}}{R(R-A)^{1/2}} \quad \text{for} \quad \gamma \gg 1,\ R \gg R - A. \tag{12}$$

6. IMPLICATIONS FOR THE ERA

The energy gained by the ring passing through the gap is of the form

$$\Delta U = QEG - Q^2/L, \tag{13}$$

where L is a characteristic length. Obviously ΔU must be positive; this implies that if $E_{\max}$ is the maximum field that can be supported at the gap there is a limit to the value of $N = Q/e$, the number of electrons in the ring, given by

$$Ne = LE_{\max}G. \tag{14}$$

For a workable machine, therefore,

$$L > Ne/E_{\max}G. \tag{15}$$

If E is in kV/cm this becomes

$$L > 1.44 \times 10^{-10} N/GE_{\max}\,\text{cm}. \tag{16}$$

For $N = 10^{13}$, $G = 2$ cm, $E_{\max} = 300$ kV/cm; this gives a lower limit to L of 2.4 cm. From Eq. (8), $L = R^{3/2}/0.6\gamma^{1/2}G^{1/2}$; for $R = 5$ cm this gives $\gamma_{\max} = 30$. The numbers in this example have been inserted somewhat arbitrarily; it is evident, however, that the radiation would appear to impose a limit on the machine well within the region of parameters of interest.

It is stated in the Russian report,[1] however, that "an accurate numerical calculation of the beam energy growth in the resonator shows that for the presently available resonator electric intensities it is quite possible to accelerate a bunch having a charge of $10^{14}\,e$ in the region of β from about 0.1 to any value close to unity." This would imply either that the estimates in the present paper are in error or that the Russian parameters are very different from those given above. (Perhaps R is much larger for example.) This needs further investigation.

CONCLUSION

If the results in this paper are correct, they imply severe constraints on the parameters of a high-energy ERA. An independent accurate calculation is urgently needed.

REFERENCES

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POSTSCRIPT

The work described in this paper was done in 1968 in connection with the Electron Ring Accelerator concept described in Ref. 1. Remarks in Section 6 and the Conclusion should be treated with caution, since for a series of sufficiently closely spaced cavities the radiation is reduced. The steady state loss for infinite systems with different cavity spacings has been calculated by E. Keil, (Nuclear Inst. and Meth. **100,** 419, 1972), and the results compared with the theory given here and a multi cavity model due to A. M. Sessler.