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Theory

Master Thesis

Kaluza Klein Spectra from Compactified and Warped Extra Dimensions

by

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Abstract

Extra dimensions provide a very useful tool for physics beyond the standard model, particularly in the quest for unification of the forces. In this thesis we explore several interesting models and aspects of extra dimensions that could help in the formulation of a viable theory. Motivated by String/ M-theory, we are especially interested in models that can describe both the observable 4-dimensional (4D) world and an extra dimensional space, consisting of 6 or 7 compactified dimensions. We discuss the basics of extra dimensions, including compactification, dimensional reduction and the general calculation of the Kaluza Klein mass spectrum. We then specialize to three interesting models of extra dimensions and calculate the energy scale at which the Kaluza Klein modes should become visible. We show how the ADD scenario and the Randall Sundrum scenario describe a universe containing branes to confine our observable world and solve the hierarchy problem between the fundamental scales. Within the Randall Sundrum scenarios, we cover the subjects of localization of gravity, the KK spectrum and stability issues. In the last chapter we consider solutions to Freund-Rubin universes in which the extra dimensions are naturally compactified by an extra-dimensional flux field. Solutions to a similar setup have led to interesting models like the Kinoshita ansatz. We will discuss these solutions and their stability, showing that a stable solution could emerge from a warping of the extra dimensional space.

Contents

1 Layman summary

Extra dimensions may form the basis of some fundamental theories in physics. You may wonder why we would believe in extra dimensions, how they are described in physics and how we search for them. In this thesis, our aim is to understand the most important aspects of extra dimensions and to lay out some interesting models that could lead to a realistic theory in the future.

Why extra dimensions?

The quest for unification Our observable world consists of 3 spatial dimensions $\{x_1, x_2, x_3\}$ and 1 dimension describing time $\{t\}$. According to relativity, these apparently separate coordinates are related by the speed of light c and they can be combined into one set called the *space-time coordinates*, $\{ct, x_1, x_2, x_3\}$. [1],

Space-time coordinates form the basis of Einstein's theory of *General Relativity* (GR), which describes the geometry of the universe, the behavior of gravity and in that sense, physics at large length scales. For physics at small length scales (e.g. particle physics), we use another theory called the *Standard Model* (SM). The SM describes electromagnetism, the weak and the strong nuclear forces and it is build upon the framework of *quantum mechanics*. Now what physicists would like, is to find one *unifying theory* that includes both GR and quantum mechanics: one formal mathematical structure to describes all of reality as we know it exists. Unfortunately though, it turns out to be extremely difficult, if not impossible, to reconcile GR with quantum mechanics and in particular to unify gravity with the other forces in nature.

The quest for unification may be considered the holy grail of modern physics and it turns out that extra dimensions provide a very useful tool in addressing problems with this unification. In fact, the most promising unifying theories, including String theory, are only consistently written in a universe that consists of 10 or 11 space-time dimensions. This implies the existence of an extra 6 or 7 spatial dimensions!

Obviously we see only 3 spatial dimensions, so we should ask ourselves how and where these extra dimensions are hiding and how we could possibly observe them in the near future.

What will extra dimensions look like?

Kaluza Klein theory One of the first models to describe extra dimensions was introduced by Theodor Kaluza and Oscar Klein in 1921. According to Kaluza and Klein (KK), extra dimensions could be curled up on tiny circles, too small to be observed by current observational methods. This mechanism is called *compactification*.

A good way to understand compactification, is by imagining a garden hose. From a large distance, the garden hose looks like a line, a one-dimensional object. When you come closer

[1] Don't worry, this is the only formula! I only put it in here, because it shows that time has been multiplied by the speed of light to become a length-like quantity: $[time] \times [speed] = sec. \times \frac{meter}{sec.} = meter = [length]$.

however, you observe that the line is actually a tube, a curled up 2-dimensional surface. In a way, the second dimension was this hidden from us when we were too far away, or in other words, when the hose was too small to be observed completely. Now compare the long direction of the garden hose to the regular (visible) spatial dimension and the curled up direction to the hidden (extra) dimension. We can only see the extra dimension if we come close enough! As an extension to the garden hose, we could add another curled up extra dimension, to obtain a "donut" or a torus of higher dimension. Extra dimensional toruses could exist at every point in space, but be too small to be observed with current methods.

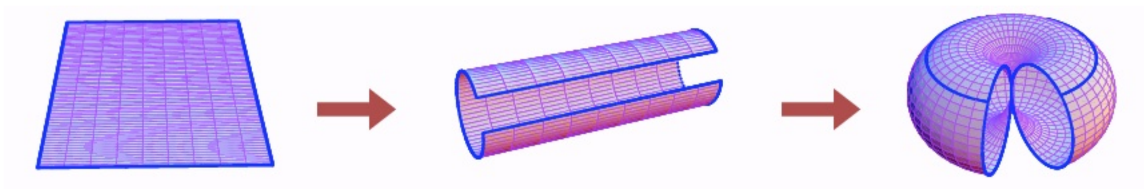


Figure 1: compactification of 1 and 2 extra dimensions respectively [2].

In the original KK theory, the extra dimensions are expected to have a radius of the order of the Planck length, $\ell_{Pl} \sim 10^{-33}\ cm$. The chance of observing extra dimensions at this scale is practically zero, even in the future.

Large extra dimensions In 1998 three scientists named Arkani-Hamed, Dimopoulos and Dvali (ADD) came up with a new model based on the idea of Kaluza and Klein, only with much larger extra dimensions. In this model our world is confined to a (3+1)-dimensional membrane. Nothing, except for gravity, can move into the extra dimensional space.

The purpose of ADD was to solve the *hierarchy problem*, which corresponds to the question: *"Why is gravity so much weaker than the other forces in nature?"*. Their explanation lies in the assumption that gravity dilutes into the extra dimensional volume, for any distance smaller than the radius of compactification. At larger distances, the usual behavior is recovered, but with an already weakened strength. The other forces are stuck on the brane and just spread out over 3 spatial dimensions, thereby seeming stronger than gravity.

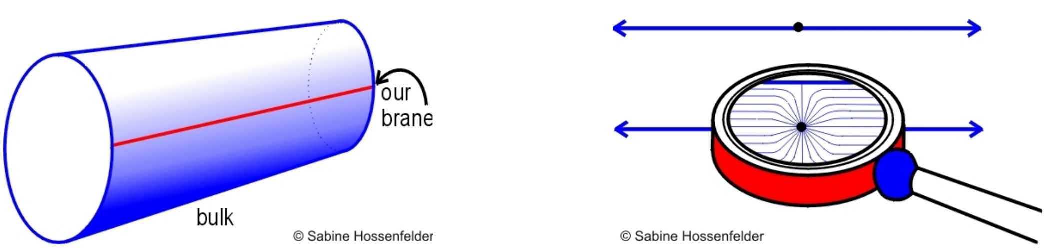


Figure 2: ADD Braneworlds *http://backreaction.blogspot.nl/2006/07/extra-dimensions.html*

According to ADD theory, we could observe a change in the gravitational potential for distances smaller than the compactification scale. Currently, gravity has been measured to behave normal up to distances of the about $0,1\ \mu m$. So according to ADD, the 'large' extra dimensions could not be larger than a μm.

Warped braneworlds In 1999 Lisa Randall and Raman Sundrum proposed an alternative solution to the hierarchy problem, that is referred to as the *warped braneworlds* scenario.

In the Randall Sundrum scenarios (RS), the universe consists of two parallel $(3+1)$-branes, called the "Planck-brane" and the "weak-brane". Our world is constrained to live on the weak brane, but the forces are unified on the Planck Brane. Unlike the ADD model, RS take into account the mass of the branes which leads to a deformation of the space-time in between the branes. This deformation is called *warping* and it affects the strength of gravity as we measure it at different points in space. RS state that all forces have equal strength at the Planck-brane, but due to the warping of space-time, gravity seems much weaker on the weak brane than at the Planck brane.

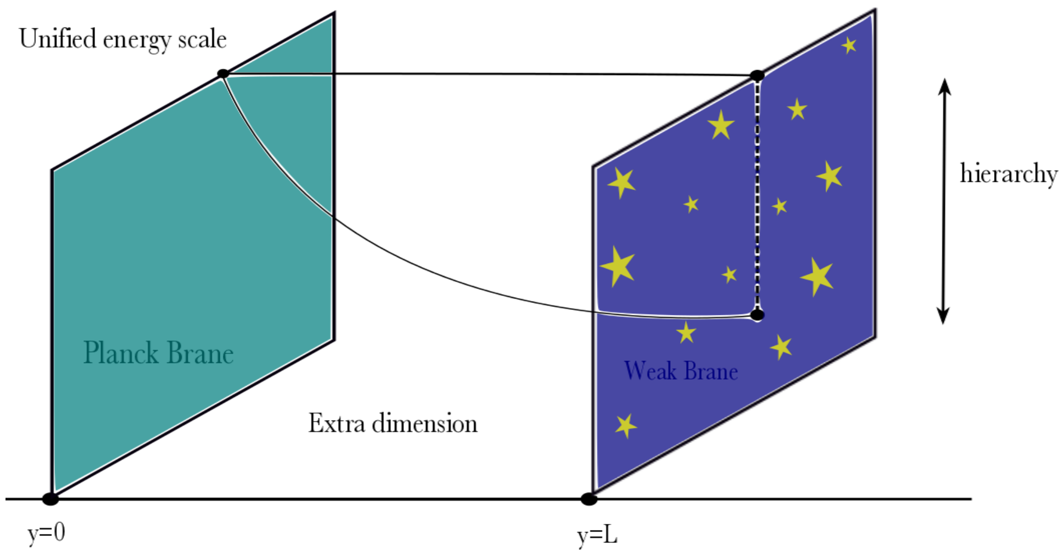


Figure 3: Warping of the extra dimension, localizes gravity to the Planck brane.

Both the ADD model and the RS scenario predict massive particles coming from extra dimensions. These would become observable in particle collider experiments at high enough energies. No such particles have been observed thus far.

Alternative scenarios In the ADD and RS scenarios the extra dimensions are assumed to be quite large, but unless evidence predicts differently, the most natural size for extra dimensions is still the Planck length, ℓ_{Pl}. It turns out that the form and shape of the extra dimensional manifolds can be chosen such that it fits important quantities in the standard model. We should imagine a universe with a complicated manifold of tiny extra dimensions at every single point in space.

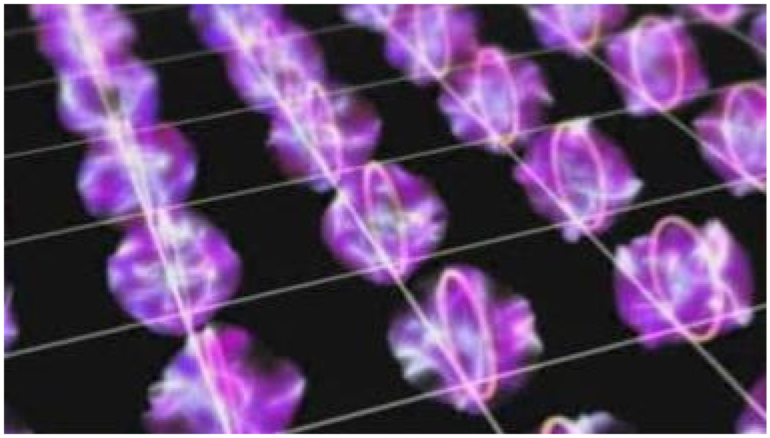


Figure 4: Manifolds of tiny extra dimensions at every point in space. *http://www.speed-light.info*

Unlike the large extra dimension-scenarios, these theories do not predict low energy features of extra dimensions. Therefore it will be difficult to find evidence of similar models.

Besides the models described above, many more exotic theories exist, including infinitely large extra dimensions and theories in which our observable world is only a *projection* of a higher dimensional reality. Until we find any signatures of extra dimensions, it is hard to tell which theory is right. The most fundamental questions thus remain unanswered, but hopefully in the future, new high energy experiments can tell us more about the size and shape of extra dimensions, if they exist at all. It is only a matter of time until we will find out.

2 Introduction

It seems quite likely that extra dimensions of space will play an important role in the eventual unified theory of interactions. Currently the most promising candidate for a complete unifying theory is String-theory (or M-theory) and it is only consistently written in a universe with 10 (or 11) space-time dimensions, thus requiring the existence of 6 or 7 extra spatial dimensions. These extra dimensions are usually expected to be compactified onto tiny circles of size comparable to the Planck length $\ell_{Pl} \sim 10^{-33}$ cm, or equivalently, the inverse *Planck scale* M_{Pl}^{-1}. [2]

The original idea of compactified extra dimensions springs from *Kaluza Klein theory*. In 1921 Theodor Kaluza tried to unify gravity and electromagnetism by introducing *one* extra spatial dimension. His approach was to apply general relativity to a *five*- rather than *four*-dimensional space-time manifold and show that the photon originates from extra components of the metric. In order to explain the unobserved extra dimension, Oscar Klein suggested that the extra dimension could be compactified at a size $R \sim M_{Pl}^{-1}$. In that case the observable world becomes effectively 4 dimensional.

An important side-effect of compactification is a hypothetical phenomenon called the *Kaluza Klein tower*. Due to the circular topology of the extra dimension, all fields would have quantized momenta with respect to the extra dimension. In 4D this could become observable as a series of higher dimensional particles with masses inversely proportional to the compactification scale, R, i.e.: $m_n \sim |n|/R$. This series is called the Kaluza Klein (KK) mass spectrum or the KK tower. Obviously for a compactification scale $R \sim M_{Pl}^{-1}$, the KK masses would be too high to be probed by current observational methods.

The Planck scale is obtained from the fundamental constants in physics. It is the energy scale at which quantum gravity effects should become important,

$$M_{Pl}c^2 = \left(\frac{\hbar c^5}{G_N}\right)^{1/2} = 1.22 \times 10^{18} \text{ GeV}. \qquad (1)$$

It is the most natural scale in physics and therefore, the Planck length $\ell_{Pl} \sim M_{Pl}^{-1}$, is a logic choice for the compactification scale. However, developments in string theory and studies by Horava and Witten showed, that the extra dimensions might be much larger than ℓ_{Pl} [13]. This was the start of a revival of extra dimensional theories and over the past decades, all kinds of models were developed. We can distinct large extra dimensions and infinitely large (non-compactified) extra dimensions, models with flat internal geometry and simple compactifications, but also very complex manifolds. Models that include higher-dimensional hyper surfaces to constrain our observable world to 4D and parallel universes and many more complex and exotic scenarios. Don't even get us started on projective theories or models with massive gravity.

Triggered by the variety and possibilities of different and fantastic scenarios, we aim to understand the physics required and caused by extra dimensions. We are motivated by the search for a viable theory of extra dimensions, that is both consistent with our observable

[2]Note that we work in natural units, where $c = \hbar \equiv 1$. See appendix A

world and could fit into a unifying theory like string theory. Obviously we do not aim to find such a complete and perfect theory, but we would like to make some comparisons between existing models and describe their strengths and weaknesses.

Since we are particularly interested in theories that could describe a universe with $(3+1)$ external dimensions and 6 or 7 extra spatial dimensions, our general focus will be on Kaluza Klein theories with compactified extra dimensions. We will consider several different models that have become important over the years and discuss their formalism, consistency and stability. We will also pay attention to the observability, by calculating or estimating the corresponding KK mass spectrum, to make a statement on whether it is reasonable to search for extra dimensions and if so, at what energy scale we should expect to find KK masses?

The setup of these notes is as follows. We will start by understanding the basics of extra dimensions and Kaluza Klein theory, in order to get familiar with the concept of compactification and dimensional reduction. Dimensional reduction is a method that is used to describe our 4D world within the higher dimensional picture. We will also show how the Kaluza Klein tower follows from this.

In the chapters that follow, we will specialize to two families of extra dimensional models. In chapter three we will focus on the so-called *brane world scenarios*, which were developed at the end of the 20^{th} century. These include the theory of large extra dimensions [16], and the Randall Sundrum scenarios [17], [18]. Within these models, our observable world is constrained to live on a *four*-dimensional hyper surface (called a *brane*), within a higher dimensional (bulk) space-time and only gravity propagates through the bulk. These models became particularly popular because they solve the long-standing *hierarchy problem* between the Planck scale and the electroweak scale, $M_{Pl} : M_{EW} \sim 10^{16}$ and they predict signatures of quantum gravity at the TeV-scale. Nevertheless, each of these models suffers theoretical weaknesses. We will discuss the formalism of both models, how they solve the hierarchy problem and discuss their most important features.

In the last chapter, we consider a completely different family of extra dimensional models. Within these models, there are no branes and a higher dimensional flux field leads to the spontaneous compactification of the extra dimensional space. This mechanism is referred to as *flux-compactification* and it was first described by Freund and Rubin in 1980 [24]. A series of solutions to this setup has been developed over the past decades. We will consider two types of solutions. One is a basic solution, referred to as a (regular) de Sitter solution and the other one is a generalization to the first type, in which the extra dimensional space is internally deformed. The latter is based on an Ansatz by Kinoshita [30].

We will try to compare the consistency, stability and observability (in terms of the KK spectrum) of the different scenarios. Throughout these notes we will work in natural (Planck) units, setting $c = \hbar \equiv 1$, such that all quantities can be expressed in terms of energy dimensions.[3]

[3] See Appendix A

3 Preliminaries and basics of extra dimensions

In the following we will briefly summarize the most important aspects of extra dimensional theories, focussing on GR and field theory. It is quite straight forward to extend the theories to higher dimensions, but it will become useful to have all the basics at hand before we go to more complicated calculations within higher dimensional space times. To fix conventions and notations I will start with a short overview of GR in arbitrary dimensions and an introduction to the techniques that we will use later on.

3.1 GR in arbitrary dimensions

We define the total number of dimensions to be $D \equiv 4+n$, where 4 refers to the (3+1) space-time dimensions of our observable world and n is the number of extra *spatial* dimensions. We will use the coordinates x^μ to denote the 'regular' space-time dimensions and any higher dimensional coordinate system will be denoted by a capital Roman index, M, which runs over both the 'normal' and the extra dimensions. We will use lowercase Roman indices to denote the extra dimensional coordinates explicitly. It is convention to skip the index 4, to express the difference between the regular and extra dimensions. Extra dimensions will thus start counting from 5 up, so

$$x^M = (x^\mu, x^5, ..., x^{4+n}). \tag{2}$$

The infinitesimal distance is related to the coordinates and the bulk metric tensor, g_{MN} by

$$ds^2 = g_{MN} dx^M dx^N, \tag{3}$$

just like in 4D GR. We use the sign convention (-,+,+,...,+) for the metric. Now, starting from a given metric tensor g_{MN} and its inverse g^{MN}, the Christoffel symbols are defined by

$$\Gamma^P_{MN} = \frac{1}{2} g^{PQ} \left(\partial_M g_{NQ} + \partial_N g_{QM} - \partial_Q g_{MN} \right). \tag{4}$$

From the Christoffel symbols, we calculate the Riemann tensor $R^P{}_{QMN}$, the Ricci tensor R_{MN} and the Ricci scalar R, which embody the geometry of space-time curvature. They are respectively defined:

$$R^P{}_{QMN} = \partial_M \Gamma^P_{NQ} - \partial_N \Gamma^P_{MQ} + \Gamma^P_{ML} \Gamma^L_{NQ} - \Gamma^P_{NL} \Gamma^L_{MQ}$$
$$R_{MN} = R^L{}_{MLN}$$
$$R = g^{MN} R_{MN}.$$

Note that the metric tensor is dimensionless, $[g] = 0$. Therefore the *Christoffel symbols*,

$$\Gamma^B_{MN} \propto g^{AB} \partial_M g_{NB} \tag{5}$$

carry dimension $[\Gamma] = 1$, the Ricci tensor, $R_{MN} \propto \Gamma^2$ will always carry dimension $[R_{MN}] = 2$ and the curvature scalar R also has dimension $[R] = 2$. It turns out that the dimension of these quantities are independent of the number of extra dimensions!

We can use this to generalize the Einstein-Hilbert action to higher dimensions. Assuming it takes the same form as the 4D version, for n extra dimensions we have

$$S_{4+n} \sim \int \sqrt{-g_{4+n}} R_{4+n} \, d^{4+n}x. \tag{6}$$

Now in natural units, the action should be dimensionless. This implies that we have to equilibrate the extra spatial (lenghtlike) dimensions by the appropriate power of the *fundamental Planck scale* in $(4+n)$ dimensions, $M_{(4+n)}$. The fundamental Planck scale is the generalization of the 4D Planck scale, M_{Pl} to higher dimensions:

$$M_{4+n}c^2 = \left(\frac{\hbar^{n+1} c^{n+5}}{G_{4+n}} \right)^{\frac{1}{(n+2)}}, \tag{7}$$

or in natural units, $c = \hbar = 1$, it becomes $M_{n+4}^{n+2} \sim G_{4+n}^{-1}$, where G_{4+n} is the higher dimensional gravitational constant. The key note, is that gravity is a property of space-time, thus it is related to the complete space-time volume and therefore the higher dimensional gravitational constant does not have to be equal to the 4D gravitational constant. In fact we can define $G_N = G_{4+n}/V_n$, where V_n is the extra dimensional volume.[4]

In a higher dimensional space-time, our 4D Newtonian constant, could thus be considered an *effective quantity*, that is related to the fundamental quantity by a volumetric scaling. We will get back to this subject in chapter 3 and 4. Consequently, the 4D fundamental Planck scale M_{Pl}, is not so fundamental at all. The higher dimensional fundamental scale is usually defined as $M_{4+n} \equiv M_*$. For convenience we will use this definition for general space times, throughout the rest of these notes.

The fundamental Planck scale is an energy scale by definition, so it carries dimension $[M_*]$=1. Earlier we found that $[R] = 2$ and $[g] = 0$ in any number of dimensions and dx^{4+n} carries dimension $-(4+n)$. We thus need an extra factor of M_*^{n+2} to make the action dimensionless, i.e.

$$S_{4+n} = -M_*^{2+n} \int \sqrt{-g_{4+n}} R_{4+n} \, d^{4+n}x. \tag{8}$$

Now that we have found our higher dimensional action, we can apply the *Lagrangian formalism*, to obtain the general Einstein equations by varying the action S_{4+n} with respect to the metric tensor g_{MN}, setting the variation $\delta S = 0$. Note that the given action, corresponds to empty space-time, thus leading to the Einstein equations in vacuum: $G_M N = 0$.

To describe non-vacuum space-times we should add extra terms to the action, describing the matter fields, i.e.

$$S = \frac{1}{16\pi G_{(D)}} S_H + S_M, \tag{9}$$

where S_M is the action for the matter fields and the Hilbert-term is normalized by the Planck scale as above. Following through the same procedure of varying the action, we will obtain

[4]See Appendix B

the general Einstein equations in non-vacuum:

$$R_{MN} - \frac{1}{2} g_{MN} R + g_{MN} \Lambda = 8\pi G_{(D)} T_{MN}, \tag{10}$$

where the energy momentum tensor is defined by

$$T_{MN} = \frac{-2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{MN}} \tag{11}$$

and we have added a higher dimensional cosmological constant Λ for completeness.

We are usually interested in the 4D phenomena that follow from a higher-dimensional theory as formulated above. The effective 4D theory is obtained from the higher dimensional one, by integrating over the extra dimensions. In order to do so, we need more information about the extra dimensions, in particular the general metric.

3.2 Form fields

In the discussion of extra-dimensional models, we will encounter several different matter fields in the action. One type of field we will encounter in section 5, are vector-fields like the electromagnetic field and more generally *form fields* giving rise to higher rank flux-fields in the equations of motion. To understand these fields I will briefly review the basics of differential forms.

3.2.1 Differential forms

Differential forms are a special class of tensors. In general a differential p-form is a completely anti-symmetric $(0, p)$ tensor. Note that a scalar is a 0-form, a vector is a 1-form and the Levi-Civita symbol $\epsilon_{\mu\nu\rho\sigma}$ is an example of a 4-form. Without getting into the theory of p-forms to far, I will discuss some basic useful properties that we will encounter later.

Given a p-form A and a q-form B, we can take the anti-symmetrized tensor product, better known as the *wedge product*, to form a $(p+q)$-form:

$$(A \wedge B)_{\mu_1 \dots \mu_{p+q}} = \frac{(p+q)!}{p!q!} A_{[\mu_1 \dots \mu_p} B_{\mu_{p+1} \dots \mu_{p+q}]}. \tag{12}$$

Another useful feature is the *exterior derivative* d, which is defined as an appropriately normalized and anti-symmetrized partial derivative. It differentiates p-form fields to obtain $(p+1)$-form fields.

$$(\mathrm{d}A)_{\mu_1 \dots \mu_{p+1}} = (p+1)\partial_{[\mu_1} A_{\mu_2 \dots \mu_{p+1}]}. \tag{13}$$

An important property of exterior differentiation is that for any form, A,

$$\mathrm{d}(\mathrm{d}A) = 0. \tag{14}$$

Finally, one last useful operation on differential forms is the *Hodge Duality*. The *Hodge star operator* is defined on an n-dimensional manifold as a map from p-forms to $(n-p)$-forms,

$$(*A)_{\mu_1 \dots \mu_{n-p}} = \frac{1}{p!} \epsilon^{\nu_1 \dots \nu_p}{}_{\mu_1 \dots \mu_{n-p}} A_{\nu_1 \dots \mu_p}, \tag{15}$$

mapping A to “A dual”.

3.2.2 Flux fields

The operations on forms, defined above, allow us to write the properties of the write the properties of electromagnetism in a very convenient form. First note that the electromagnetic field strength tensor, $F_{\mu\nu}$ can be written in terms of the wedge product between two one-forms, namely the partial derivative of the vector field A_μ:

$$F_{\mu\nu} = \partial_\mu \wedge A_\nu = \partial_{[\mu} A_{\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{16}$$

From that it follows immediately that

$$\mathrm{d}F = 0, \tag{17}$$

which is in fact just another way of writing the third and fourth Maxwell equations. The first and second Maxwell equations can be expressed as an equation between 3-forms:

$$\mathrm{d}(*F) = *J \tag{18}$$

where the 1-form current J is just the current four-vector with index lowered. Including electromagnetism in the theory of GR, we obtain the Einstein-Maxwell action:

$$S = \frac{1}{16\pi G}\int \sqrt{-g}\left(R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) d^4\,x \tag{19}$$

and the energy-momentum tensor, corresponding to this action using (11), is:

$$T^{EM}_{\mu\nu} = F^{\lambda}{}_{\mu}F_{\lambda\nu} - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g_{\mu\nu} \tag{20}$$

Note that electromagnetism is described by the *two*-form (flux) field-strength tensor $F_{\mu\nu}$, corresponding to the *one*-form (vector) field A_{μ}. Generally the flux tensor is always one rank higher than the propagating field it describes.

Extension to higher dimensions Generalizing the above theory to arbitrary dimensions, D, the higher dimensional action for an abelian vector field, $A_M(x^P)$ is the Maxwell action

$$S^{EM}_{(D)} = -\frac{1}{4}\int d^Dx\sqrt{-g_{(D)}}F_{MN}F^{MN}, \tag{21}$$

where $F_{MN} \equiv \nabla_M A_N - \nabla_N A_M$ is the field stregth tensor as usual. We can generalize the above to a p-form field $A_{M_1\ldots M_p}$ in arbitrary dimensions. It is described by a rank $(p+1)$ field strength tensor $F_{M_1M_2\ldots Mp+1} = (p+1)\nabla_{[M_1}A_{M_2\ldots M_{p+1}]}$ and its action is

$$S_{(D)} = -\frac{1}{2(p+1)!}\int d^Dx\sqrt{-g_{(D)}}F_{M_1M_2\ldots Mp+1}F^{M_1M_2\ldots Mp+1}, \tag{22}$$

which corresponds to the energy momentum tensor

$$T_{MN} = F^{L}{}_{M_1\ldots M_p}F_{LN_1\ldots N_p} - \frac{1}{(2p!)}F_{M_1\ldots M_{p+1}}F^{M_1\ldots M_{p+1}}g_{MN}. \tag{23}$$

3.3 Graviton dynamics

Gravitons are hypothetical particles that we think of as the mediator of gravitation and in the quantum field-theoretical sense they are expected to be massless helicity-2-bosons. We will be interested in the dynamics of higher dimensional gravitons and the the observability of their spectrum in 4-dimensional space-time. We will briefly review the most common methods for examining the dynamics of these hypothetical particles.

3.3.1 Perturbation theory

In GR graviton dynamics are often studied by introducing a small perturbation to the background space-time

$$g^{(0)}_{\mu\nu} \to g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}, \tag{24}$$

where $g^{(0)}_{\mu\nu}$ can be any curved space-time metric. For a sufficiently small perturbation, $|h_{\mu\nu}|^2 \ll 1$, this decomposition will lead to a linearized version of general relativity, where effects higher than first order in $h_{\mu\nu}$ are being ignored. The linearized Einstein equations will effectively describe the propagation of a symmetric tensor field $h_{\mu\nu}$ on a background space-time $g^{(0)}_{\mu\nu}$ and from examining these, we obtain the equations of motion obeyed by the perturbation.

Following the procedure of [1] we find the general *linearized Einstein tensor*

$$G_{\mu\nu} = {}^1\!/_2 \left(\partial_\sigma \partial_\nu h^\sigma_\mu = \partial_\sigma \partial_\mu h^\sigma_\nu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu} - g^{(0)}_{\mu\nu} \partial_\rho \partial_\lambda h^{\rho\lambda} + g^{(0)}_{\mu\nu} \Box h\right). \tag{25}$$

Generally we can decompose the metric perturbation into 'irreducible representations' of the rotation group. $h_{\mu\nu}$ is a $(0, 2)$ tensor, but under rotations the 00 component transforms as a scalar, the $0i$ components form a 3-vector and the ij components form a two-index symmetric spatial tensor, which can be further decomposed into a *trace* and a *trace-free* part. In this way, the perturbation $h_{\mu\nu}$ can be written as

$$h_{00} = -2\Phi \tag{26}$$
$$h_{0i} = w_i \tag{27}$$
$$h_{ij} = 2s_{ij} - 2\Psi_{ij}, \tag{28}$$

where s_{ij} encodes the traceless part of h_{ij} and Ψ_{ij} is the trace.

To proceed, we will generally use the degrees of freedom to pick a convenient gauge. Picking a gauge can simplify the Einstein equations, and we may try to solve them in order to understand the dynamics of the perturbation, or as we would like to see it: the 'propagating graviton'.

Higher dimensional gravitons The gravitational field in $D = 4 + n$ dimensions is described by the symmetric metric tensor $g_{MN} = \eta_{MN} + h_{MN}$, where awe have assumed the general background metric to be globally flat Minkowski space-time η_{MN}. To pick a convenient gauge, we are interested in the number of degrees of freedom.

In D dimensions, there are $D(D+1)/2$ independent components of a symmetric tensor, but many degrees of freedom can be removed by the general coordinate transformation (D-dimensional) $h_{MN} \to h_{MN} + \partial_M \xi_N + \partial_N \xi_M$.

We can impose D conditions to fix the gauge. If we choose for example the *harmonic gauge*, $\partial_M h^M_N = {}^1\!/_2 \partial_N h^M_M$, then any transformations that satisfy $\Box \xi_M = 0$ are still allowed. Again D conditions can be imposed. In general the number of independent degrees of freedom becomes

$$\frac{D(D+1)}{2} - 2D = \frac{D(D-3)}{2}. \tag{29}$$

The choice for a convenient gauge depends on the geometry of the model.

3.3.2 Scalar fields in curved spacetime

The method of perturbation theory is an extensive procedure, that we may want to avoid. A less precise, but easier approach to study gravitons in a curved background, is by examining the equations of motion of a *massless scalar field*. It turns out that the dynamics of a free massless scalar field give a reasonable approximation to the behavior of gravitons.

In the classical theory, the equations of motion for a real scalar field $\phi(x^\mu)$ in flat (*Minkowski*) space-time are derived from the action

$$S = \frac{1}{2} \int (\eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2)\, d^4x. \tag{30}$$

From the variational principle we find the equations of motion for the free scalar field, better known as the the *Klein Gordon equations*

$$\Box \phi - m^2\, \phi. \tag{31}$$

Generalizing this procedure to higher dimensional and curved space-time, requires replacing all terms in the action by their covariant form in $4+n$ dimensions, i.e.

- $\eta^{\mu\nu} \to g^{MN}$
- $d^4x \to d^{4+n}x\sqrt{-g^{4+n}}$, i.e. the invariant volume element
- $\partial \to D$.

Note that for the scalar field, the covariant derivative, D, is just the regular partial derivative, so we get:

$$S = \frac{1}{2} \int \sqrt{-g_{4+n}} (g^{MN} \partial_M \phi \partial_N \phi - m^2 \phi^2)\, d^D x \tag{32}$$

varying the action and requiring the variation to be zero, $\delta S = 0$, leads to the *Euler-Lagrange* equations for the free scalar field in non-Euclidean space-time

$$\partial_M \frac{\partial \mathscr{L}}{\partial(\partial_M \Phi)} = \frac{\partial \mathscr{L}}{\partial \Phi}. \tag{33}$$

The Euler-Lagrange equations give us the equations of motion for the field, and by applying the appropriate boundary conditions, we can calculate the spectrum of higher dimensional gravitons in several models.

4 Kaluza Klein theory and compactification

Rewinding back to the late 1910's, classical Maxwell's theory was well-established and Einstein had just developed his theory of General Relativity. Little was yet known or understood about the weak and the strong interactions, so in the search for a unifying theory the time seemed ripe to merge electromagnetism and gravity. As we mentioned in the introduction, one of the first attempts at such a formalism was put forth by the German mathematician Theodor Kaluza in 1921. In his paper "Zum Unittsproblem in der Physik" [4], he succesfully demonstrated that by choosing the right metric ansatz, five-dimensional general relativity in vacuum, $G_{AB} = 0$, *contains* four-dimensional general relativity in the presence of a (4-dimensionsal) electromagnetic field, satisfying Maxwell's laws ($G_{\alpha\beta} = T^{(EM)}_{\alpha\beta}$).

Various modifications of Kaluza's theory were suggested in the years after. Among which the proposal of Oscar Klein in 1926, suggesting compactification of the extra dimension. Eventually Kaluza Klein (KK) theory failed due to several inconsistencies with the standard model [5], but the idea was never completely abandoned. It gained renewed interest decades later, due to developments in supergravity and string theory and it forms the basis of most of the theories that we will consider later on.

Kaluza Klein theory has come to dominate higher dimensional unified physics. Therefore it is illustrative to follow the steps and understand the formalism behind it.

4.1 Original formalism

Kaluza's idea was to prove that all matter forces are just a manifestation of pure geometry. He therefore assumed that the universe in higher dimensions is empty and that all matter in 4-dimensional space-time springs from extra components of the higher dimensional metric. This suggests that the 5-dimensional energy momentum tensor, $T_{AB} = 0$, so we start from the Einstein equations in 5-dimensional empty space-time:

$$G_{AB} = 0, \tag{34}$$

where $G_{AB} \equiv R_{AB} - \frac{1}{2}Rg_{AB}$, or equivalently

$$R_{AB} = 0. \tag{35}$$

As we have seen in the previous section, the 5D Einstein equations can be derived from the 5D gravitational action

$$S_5 = \frac{1}{16\pi G_5}\int \sqrt{-g_5}\; R_5\; dx^4 dy, \tag{36}$$

with respect to the 5D metric tensor g_5. G_5 is the 5D gravitational constant and $y \equiv x^5$ is the coordinate of the extra dimension. The 5-dimensional Ricci tensor and the Christoffel

[5] KK theory encountered several difficulties. One of them was the deviation of predicted electron mass and electric ratio from experimental data. Moreover according to the *Witten no go theorem*, KK theories have severe difficulties obtaining massless fermions, chirally coupled to the KK gauge fields in 4D, as required by the SM. [9]

symbols are related to the 5-dimensional metric tensor just like in the 4-dimensional theory:

$$R_{AB} = \partial_C \Gamma^C_{AB} - \partial_B \Gamma^C_{AC} + \Gamma^C_{AB}\Gamma^D_{CD} - \Gamma^C_{AD}\Gamma^D_{BD} \tag{37}$$

$$\Gamma^C_{AB} = \frac{1}{2} g^{CD}(\partial_A g_{BD} + \partial_B g_{DA} - \partial_D g_{AB}). \tag{38}$$

Thus, aside from the indices running over one extra value, all is exactly as in 4D and the choice of the metric tensor determines everything. Kaluza chose to parametrize the metric in the following form

$$(g_{AB}) = \left(\begin{array}{c|c} g_{\alpha\beta} + \kappa^2\phi^2 A_\alpha A_\beta & \kappa\phi^2 A_\alpha \\ \hline \kappa\phi^2 A_\beta & \phi^2 \end{array}\right), \tag{39}$$

where we can identify the four-dimensional metric tensor, $g_{\alpha\beta}$, the electromagnetic four-potetial, A_α and some scalar field, ϕ. The electromagnetic potential is scaled by a constant, $\kappa = 4\sqrt{\pi G_{(4)}}$, in order to get the right multiplicative factors in the action later on. The next step would be, plugging in the metric, the Ricci tensor and the Christoffel symbols, and applying the principle of least action to find the equations of motion. In order to do so, Kaluza applied the so-called *cylinder condition*, implying that we drop all derivatives with respect to the fifth coordinate. Varying the action then leads to the following equations of motion:

$$\begin{array}{lcl} G_{\alpha\beta} & = & \frac{\kappa^2\phi^2}{2} T^{EM}_{\alpha\beta} - \frac{1}{\phi}[\nabla_\alpha(\partial_\alpha\phi) - g_{\alpha\beta}\Box\phi] \\ \nabla^\alpha F_{\alpha\beta} & = & -3\frac{\partial^\alpha\phi}{\phi} F_{\alpha\beta}, \\ \Box\phi & = & \frac{\kappa^2\phi^3}{4} F_{\alpha\beta}F^{\alpha\beta}, \end{array} \tag{40}$$

where $T^{EM}_{\alpha\beta} = \frac{1}{4} g_{\alpha\beta} F_{\gamma\delta} F\gamma\delta - F^\gamma_\alpha F_{\beta\gamma}$ is the electromagnetic energy-momentum tensor and $F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha$. There are $10 + 4 + 1 = 15$ equations, which is to be expected since there must be 15 independent elements in the 5-dimensional metric.

Choosing the scalar field ϕ to be constant throughout spacetime, the third equation drops out and the first two of equations are exactly the Einstein Maxwell equations! Kaluza chose to set $\phi = 1$, and obtained the following result in 1921:

$$\begin{array}{c} G_{\alpha\beta} = 8\pi G_{(4)}\, T^{EM}_{\alpha\beta}, \\ \nabla^\alpha F_{\alpha\beta} = 0, \end{array} \tag{41}$$

The reason for setting $\phi = 1$, was that at the time of writing the appearance of the scalar field was considered a problem. It was only acknowledged much later, that the condition $\phi =$ constant, is only consistent with the third of the field equations when $F_{\alpha\beta}F^{\alpha\beta} = 0$. This was first pointed out by Jordan and Thiry [5], [6]. Nowadays, the field ϕ is associated with the so-called *radion*, a hypothetical particle related to the size of the extra dimension.

4.2 Compactification

Kaluza introduced the extra dimension as a mathematical tool, but the physical interpretation of this unobserved extra dimension came from Oscar Klein in 1926. Klein suggested that the extra dimension could be compactified on a circle of size $2\pi R \sim M_{Pl}^{-1}$, thus identifying the points $y = 0$ and $y = 2\pi R$ of the extra dimension. This circular topology makes it possible to Fourier expand the metric and all fields with respect to the extra dimension, such that we can write

$$g_{AB}(x^\mu, y) = \sum_{n=-\infty}^{n=\infty} g_{\alpha\beta}^{(n)}(x^\mu)\, e^{iny/R}, \tag{42}$$

$$A_\alpha(x^\mu, y) = \sum_{n=-\infty}^{n=\infty} A_\alpha^{(n)}(x^\mu)\, e^{iny/R}, \tag{43}$$

$$\phi(x^\mu, y) = \sum_{n=-\infty}^{n=\infty} \phi^{(n)}(x^\mu)\, e^{iny/R}, \tag{44}$$

where the superscript $^{(n)}$ refers to the n^{th} Fourier mode. Note that we can interpret each mode as having a momentum in the y-direction of size $|n|/R$. Klein assumed R to be extremely small, such that all modes $n > 0$ would have stayed out of reach for experiments. Observable physics then depends only on the zero mode $n = 0$, which is independent of y and this could explain how physics is effectively *four*-dimensional at 'low' energies.

An important question that one should ask is how this compactification arises. What mechanism leads to this distinction in the characteristics between the normal- and extra dimension and moreover, how is such a setup stabilized?

Several theories exist, that have tried to explain the occurance of compactification. An elaborate discussion of different compactification mechanisms is given by Bailin and Love [10]. In general, we should be able to recover a 'ground state' solution corresponding to the *four*-dimensional Minkowski space-time plus a compactified d-dimensional manifold. Such a coaxing of space-time, generally goes at the cost of altering the higher-dimensional vacuum Einstein equations. Several approaches have been considered, including the addition of torsion by or higher derivative terms onto the Einstein action.

A more common way to achieve the requested setup though, is by adding an explicit higher dimensional energy-momentum tensor to the theory, which may lead to the *spontaneous compactification* of the extra dimensions. Such an approach sacrifices Kaluza's original idea of a purely geometrical unified theory. Still, spontaneous compactification has become a common method to reconcile extra dimensions with the observed 4-dimensional reality. One example of this is the spontaneous compactification of the extra dimensions, induced by a higher dimensional flux field. This was first shown by Freund and Rubin in 1980 [24] and for obvious reasons it is referred to as *flux* compactification. We will explicitly rederive the results in the last chapter.

4.3 Dimensional reduction

To understand how the effective 4 dimensional theory is obtained from the higher dimensional model, we use the method of dimensional reduction. In the following we will use Kaluza Klein theory as an example, to explicitly show how an effective 4D theory is derived from the 5D theory with one *flat and compactified* extra dimension. Assuming that the extra dimension is flat, the effective space is just the product space of our four-dimensional Minkowski space-time, M^4 and a circle S^1: denoted $M^4 \otimes S^1$. I will closely follow the notation and derivation of [7].

4.3.1 Scalar fields

Consider a massless scalar field extending over the complete bulk space-time. Due to the circular topology in the y-direction, it obeys

$$\Phi(x^\mu, y) = \Phi(x^\mu, y + 2\pi R), \tag{45}$$

and we can express it as a Fourier decomposition:

$$\Phi(x^\mu, y) = \frac{1}{\sqrt{2\pi R}} \sum_{n=-\infty}^{\infty} \phi_n(x^\mu) \cdot e^{i\frac{n}{R}y}. \tag{46}$$

The expansion coëfficients ϕ_n are referred to as the 'modes' of the field and they only depend on the 'ordinary' space-time coordinates x^μ. Note that the field is real, which implies that $\phi^{(-n)} = \phi^{(n)\dagger}$. Plugging this decomposition into the 5D scalar action, we obtain

$$\begin{aligned}
S &= \int d^5x \frac{1}{2} \partial_M \Phi(x^\mu, y) \partial^M \Phi(x^\mu, y) \\
&= \int d^4x \sum_{m,n} \left(dy \frac{1}{2\pi R} e^{i\frac{m+n}{R}y} \right) \frac{1}{2} \partial_\mu \phi^{(m)}(x^\mu) \partial^\mu \phi(n)(x^\mu) + \frac{mn}{R} \phi^{(m)}(x^\mu) \phi^{(n)}(x^\mu) \\
&= \int d^4x \frac{1}{2} \left(\sum_n \partial_\mu \phi^{(-n)} \partial^\mu \phi^{(n)} - \frac{n^2}{R^2} \phi^{(-n)} \phi^{(n)} \right) \\
&= \int d^4x \left(\frac{1}{2} \partial_\mu \phi^{(0)} \partial^\mu \phi^{(0)} + \sum_{n=1}^{\infty} \left(\partial_\mu \phi^{(n)\dagger} \partial^\mu \phi^{(n)} - \frac{n^2}{R^2} \phi^{(n)\dagger} \phi^{(n)} \right) \right).
\end{aligned}$$

It turns out that the the zero mode ($n = 0$) obeys the Klein-Gordon equation for a *massless scalar field* and the higher modes form a series obeying the 4-dimensional Klein-Gordon equation for a *massive scalarfield* with mass $m^2 = \frac{n^2}{R^2}$.

From the four-dimensional point of view thus, the spectrum of the 5-dimensional massless scalar field consists of

- One real massless scalar field, ϕ_0, called the *zero-mode*, which corresponds to the 4D particle.

- A series of 4-dimensional massive scalar fields $\sum_n \phi_n$ with masses $m_n = |\frac{n}{R}|$, called the *Kaluza Klein tower*. The massive particles are referred to as *Kaluza-Klein states*.

At energies small compared to R^{-1}, only the x^5-independent massless zero-mode is important, so the physics is effectively 4-dimensional. At energies above R^{-1}, the KK-states come into play and forces would behave 5-dimensional.

We can easily generalize this to massive scalar fields and higher numbers of extra dimensions. If the scalar field has a 5D mass m_0, then the 4D KK modes will have a mass $m_n^2 = m_0^2 + \frac{n^2}{R^2}$. For extra compactified dimensions, with radius $R_5, R_6..etc$ we add an extra term for each dimension. The general formula for the KK masses is then given by

$$m_n^2 = m_0^2 + \sum_{i=1}^{n} \frac{j_i^2}{R_i^2} \tag{47}$$

where j corresponds to the j^th mode in the KK tower and i sums over the number of extra dimensions.

4.3.2 Gauge fields

As a next step, we consider a vector field in 5D, $A_M(x^\mu, y)$, with one dimension compactified on a circle. The action for the 5D vector field is

$$S = \int dx^4\, dy \left(-\frac{1}{4}F_{MN}F^{MN}\right) \tag{48}$$

$$= \int dx^4\, dy \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu A_5 - \partial^5 A_\mu)(\partial^\mu A_5 - \partial_5 A^\mu)\right). \tag{49}$$

Again we can perform a Fourier decomposition along the compact dimension

$$A_M(x^\mu, y) = \frac{1}{\sqrt{2\pi R}} \sum_n A_M^{(n)}(x^\mu) e^{i\frac{n}{R}y}. \tag{50}$$

Note that under a Fourier decomposition, the derivative can be replaced by $\partial \to i(n/R)$, so the action can be written like

$$S = \int d^4x \sum_n \left(F_{\mu\nu}^{(n)} F^{(n)}\mu\nu + \frac{1}{2}(\partial_\mu A_5^{(-n)} + i\frac{n}{R}A_\mu^{(n)})(\partial^\mu A_5^{(n)} - i\frac{n}{R}A^{(n)\mu})\right). \tag{51}$$

We can remove the mixed terms in this expression, by performing a gauge transformation that makes A_5, constant along the extra dimension:

$$A_\mu^{(n)} \to A_\mu^{(n)} - i\frac{R}{n}\partial_\mu A_5^{(n)}, \tag{52}$$

$$A_5^{(n)} \to 0 \qquad \text{for } n \neq 0. \tag{53}$$

In this gauge, the action becomes

$$S = \int dx^4 \left(-\frac{1}{4} F_{\mu\nu}^{(0)} F^{(0)\mu\nu} + \frac{1}{2} \partial_\mu A_5^{(0)} \frac{1}{2} \partial^\mu A_5^{(0)} \right)$$
$$+2 \sum_{n \geq 1} \left(-\frac{1}{4} F_{\mu\nu}^{(-n)} F^{(n)\mu\nu} + \frac{1}{2} \frac{n^2}{R^2} A_\mu^{(-n)} A_\mu^{(n)} \right).$$

From the 4D point of view, the 5D Maxwell action thus describes a 4D gauge field and a real scalar field in the zero mode. The nonzero modes describe a massive vector field.

In general, starting with a $(4+n)$-dimensional gauge theory with n dimensions compactified on a torus, the zero modes will contain a 4D gauge field together with n adjoint scalars. Each higher KK mode will have a 4D massive vector field and $(n-1)$ massive adjoints.

4.3.3 Gravitons

Finally, let's consider the gravitational field. It is a bit more complicated than the scalar field and the vector field described above. We consider the graviton as the higher dimensional fluctuation of the general metric in a flat background (for now).

$$g_{MN} = \eta_{MN} + h_{MN}. \tag{54}$$

From the effective four-dimensional point of view the fluctuations h_{MN} would have several different 4D Lorentz components. An explicit decomposition of the higher dimensional graviton is given in [11] and [12]. I will summarize the most important findings. All the effective fields will have a KK decomposition of the form:

$$h_{MN}(x^\mu, y) = \sum_{\bar{n}} h_{MN}^{\bar{n}}(x^\mu) e^{i\bar{n}\bar{y}/R} \tag{55}$$

where the n-dimensional vector $\bar{n}$, corresponds to the Kaluza Klein numbers along the various extra dimensions.

A 5D graviton with one dimension compactified decomposes into

$$h_{MN} = h_{\mu\nu} \oplus h_{\mu 5} \oplus h_{55}. \tag{56}$$

The zero modes, $h^{(0)}$, contain a 4D graviton, a massless vector and a real scalar. The nonzero modes $h_{\mu 5}^{(n)}$ and $h_{55}^{(n)}$ are absorbed into $h_{\mu\nu}^{(n)}$ to form massive spin-2 fields.

As a generalization to $(4+n)$ dimensions: the zero modes consist of a 4D graviton, n massless vectors and $n(n+1)/2$ scalars. The nonzero modes have a massive spin-2 tensor, $(n-1)$ massive vector fields and $n(n-1)/2$ massive scalars. We can depict the different elements of the higher dimensional graviton in a $(4+n) \times (4+n)$ matrix as:

$$\left(\begin{array}{c|c} h_{\mu\nu}^{\bar{k}} & h_{\mu a}^{\bar{k}} \\ \hline h_{\mu a}^{k} & h_{ab}^{k} \end{array} \right), \tag{57}$$

The 4D graviton and its KK modes $h^{\bar{k}}_{\mu\nu}$ live in the upper left 4×4 part of the matrix. The 4D vectors and their KK modes, $h^{\bar{k}}_{\mu a}$, live in the off-diagonal blocks (the *graviphotons*). The 4D scalar fields and their KK modes, $h^{\bar{k}}_{ab}$ live in the lower right $n \times n$ block of the graviton matrix (the *graviscalar fields*). One of these graviscalars corresponds to the partial trace of h: $h^{a}_{\ a}$ and is called the *radion*.

Notice that the 4D graviton $h^{(0)}_{\mu\nu}$ is massless, because the higher dimensional graviton h_{MN} is massless itself. It turns out that only the 4D graviton, the radion and their KK modes couple to matter fields. Other fields do not couple to directly. For the purpose of observing extra dimensional phenomena in 4D, most articles focus only on the graviton and the radion.

4.4 Matching 4D observables to the higher dimensional theory

In order to understand how big the extra dimensions could be without being observed, we should understand how the fundamental scales match the higher dimensional theory. Using the method of dimensional reduction again, we can derive how the coupling constants of gravity and the gauge fields are contained into the higher dimensional theory. This will give us a bound on the compactification scale. Note that we will assume that all fields freely propagate through the bulk for now.

4.4.1 Gravitational coupling

We will start with the calculation of the Planck scale in $4+n$ dimensions. Remember that the relation between the coupling constant $G_{(4+n)}$ and the Planck scale in $(4+n)$ dimensions is $M_{(4+n)}^{2+n} \sim G_{(4+n)}^{-1}$. We are interested in how the effective 4D action is contained into the higher dimensional one. Therefore we perturb the 4D part of the metric and calculate the relations between the 4D Planck scale, M_{Pl} and the $(4+n)$-dimensional Planck scale $M_{(4+n)}$. For now we assume that the extra dimensions are flat and compact, so our perturbed metric should be of the form

$$ds^2 = (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu - r^2 d\Omega_{(n)}^2, \tag{58}$$

where r is the compactification scale of the n-dimensional torus and $d\Omega_{(n)}^2$ corresponds to the spherically symmetric line element of the extra dimensional space (Note that we use r instead of R here, to not get confused with the Ricci scalar). $\eta_{\mu\nu}$ is the flat Minkowski metric of the 4D universe and $h_{\mu\nu}$ is the perturbation, corresponding to the way the 4D graviton is contained in the higher dimensional metric. From the definitions for the determinant and the Ricci scalar we find the relations

$$\sqrt{g_{(4+n)}} = r^n \sqrt{g_{(4)}}, \tag{59}$$
$$R_{(4+n)} = R_{(4)}. \tag{60}$$

Plugging evertythig into the $4+n$-dimensional action we obtain

$$S_{4+n} = -M_*^{n+2} \int r^n \, d\Omega_{(n)} \int d^4x \sqrt{g_{(4)}} R_{(4)} \tag{61}$$
$$= -M_*^{n+2} \, V_n \int d^4x \sqrt{g_{(4)}} R_{(4)} \tag{62}$$

Note that integrating over $d\Omega_{(n)}$ gives the volume of the extra dimensional space V_n, which in the case of equally sized, compact extra dimension would be $V_n = (2\pi r)^n$. Comparing this to the 4D action, we find the relation between the 4D Planck scale and the fundamental Planckscale:

$$M_{Pl}^2 = (2\pi r)^n \, M_*^{n+2}. \tag{63}$$

4.4.2 Gauge coupling

Repeating the same procedure for gauge couplings, we start from the gauge action

$$S^{EM}_{(4+n)} = -\frac{1}{4}\int d^{4+n}x \frac{1}{g_*^2}\sqrt{-g_{(4+n)}}F_{MN}F^{MN}, \tag{64}$$

where g_* is defined as the fundamental gauge coupling. The 4D field strength tensor $F_{\mu\nu}$ is simply contained in the higher dimensional tensor F_{MN}, so using (58), we can integrate over the extra dimensional space like before

$$S^{EM}_{(4)} = -\frac{1}{4}\int d^4x \frac{V_n}{g_*^2}\sqrt{g_{(4)}}F_{\mu\nu}F^{\mu\nu}. \tag{65}$$

The relation between the gauge couplings is thus

$$\frac{1}{g_{eff}^2} = \frac{V_n}{g_*^2}. \tag{66}$$

The “fundamental” gauge coupling g_* is not dimensionless. In fact it has energy dimension $[g_*] = -\frac{n}{2}$. We should ask what its natural size would be. Assuming that its strength is set by the fundamental Planck scale, just like the gravitational coupling $g_* \sim \frac{1}{M_*^{n/2}}$, we find from equating (67) and (66), that the compactification scale $r \sim \frac{1}{M_{Pl}}$.

This implies that in a higher dimensional theory, a natural size for the compactification scale would be of the order of the inverse Planck scale. This is what Kaluza and Klein proposed in their theory too. Unfortunately, the chance of observing such small scales, is practically zero. Current experimental methods go up to energies of about 10^2 TeV in particle colliders. This is a factor 10^{14} smaller than the energy we needed to directly observe the extra dimensions.

5 Braneworlds

In the previous section we discussed how a small compactification scale $R \sim 10^{-33}$ cm, as proposed by Klein, explains why we do not observe any signatures of the extra dimensions. However, as we mentioned in the introduction, in the 90's Witten and Horava [13], suggested that the extra dimensions could possibly be larger than was originally assumed. The question to be answered then was how large these extra dimensions could actually be, without getting into conflict with current observations.

Now, while the SM fields had been accurately measured up to weak scale energies $\sim 10^{-17}$ cm, gravity had only been probed directly up to distances of about a *mm*. This implies that *gravity-only* dimensions could be hiding from us at sub millimeter scales!

This was the original idea of Arkani-Hamed, Dimopoulos and Dvali (ADD) in 1998. They suggested that extra dimensions could be as large as a *mm* and yet remain hidden from experiment [16]. Such a formalism is only realizable in a universe, where our observable world and the SM fields are constrained to live on a $(3+1)$-dimensional hyper surface within the higher dimensional (bulk) space-time. This way, the extra dimensions could only be probed with gravity and constraints from particle physics, do not apply.

Such hypersurfaces are called *branes* (from membrane) and the idea is very similar to D-brane models [14], which are a fundamental aspect of string theory. String theory D-branes are surfaces on which open strings can end. The open strings give rise to all kinds of fields, like the gauge fields. Gravitons on the other hand are represented by closed strings and they can not be bound to the branes [13], [15]. D-branes are usually characterized by the number of spatial dimensions of their surface. A *p-brane*, thus describes a $(p+1)$-dimensional hyper surface.

Basically there are two good reasons for confining the standard model fields to a brane in extra dimensional theories. First of all, it opens up new ways of addressing the large hierarchy between the Planck scale and the electroweak scale, $M_{Pl} : M_{EW} \sim 10^{16}$. [6] Second, if extra dimensions are large, we would be able to find them in the near future and we could observe effects of extra dimensions and/ or quantum gravity at relatively low energy scales.

In the following we will explore two models that both address the hierarchy problem and predict (low-energy) observable signatures of extra dimensions. In both models the SM fields are confined to a 3-brane within a higher dimensional bulk space-time, therefore they are referred to as *brane-world scenarios*. The first one is the ADD scenario, which we have briefly introduced just here. Second are the Randall Sundrum scenarios, which describe a universe with parallel universes and warped extra dimensions.

[6] As we mentioned before, the Planck scale is related to the strength of gravity and considered a fundamental scale in physics. However, its size causes a theoretical puzzle, because it differs so much from the electroweak scale. The electroweak scale is the energy scale at which the electroweak forces are unified. It is fixed by the Higgs vacuum expectation value at $M_{EW} \sim 1\ TeV$. The problem arises when one tries to calculate the physical Higgs mass, from one-loop order corrections, using a cut-off regularization. The natural cut-off is usually believed to be the Planck scale, but that implies an adjustment of order 10^{16} in order to get a Higgs mass $m_H \sim M_{EW}$. This large fine-tuning is knows as the *hierarchy problem.*

5.1 Large extra dimensions

In the previous section we found how the effective 4D Planck scale is related to the $(4+n)$ dimensional fundamental Planckscale, by the volume of the extra dimensional space, V_n:

$$M_{Pl}^2 = V_n \; M_*^{n+2}, \tag{67}$$

where $V_n = (2\pi R)^n$ for equally sized compactified extra dimensions. Remember that this relation implicitly tells us, how the strength of gravity in higher dimensions G_* is related to its 4D effectivel value, G_N. Now ADD suggested that if V_n is large enough, the fundamental Planck scale could be as small as the electroweak scale. In that sense the only fundamental scale in nature would be the electroweak scale and the hierarchy problem would be solved!

Obviously the ADD scenario led to great excitement among physicists. Not in the least place, because it suggests signatures of quantum gravity at weak scale energies. It seemed like physics was on the verge of observing the unified fundaments of nature directly!

To calculate the size that the extra dimensions should have in order to get $M_* = M_{EW}$, we use the relation above and choose $M_* \sim 10^3\ GeV$ and $M_{Pl} \sim 10^{19}\ GeV$. From this we can derive a relation between the compactification scale R and the number of extra dimensions n:

$$\frac{1}{R} = M_* \left(\frac{M_{Pl}}{M_*}\right)^{\frac{2}{n}} \simeq 10^{-\frac{32}{n}} \ \text{TeV}, \tag{68}$$

and using a conversion factor $1\ GeV^{-1} = 2 \cdot 10^{-14}$ cm, we obtain:

$$R \sim 2 \cdot 10^{-17} \cdot 10^{\frac{32}{n}} \ \text{cm}. \tag{69}$$

This is the constraint equation for the compactification scale R, such that the hierarchy problem can be solved.

Another interpretation of the ADD scenario and the way it solves the hierarchy problem, is expressed in terms of the weakness of gravity compared to the other forces in nature. The key of the solution lies in the assumption that higher dimensional gravity dilutes into the extra dimensional volume, for any distance smaller than the radius of compactification. At larger distances, the usual behavior is recovered, but with an already weakened strength. The other forces are stuck on the 3-brane and spread their power over just 3 spatial dimensions. Therefore they seem stronger than gravity. We can explicitly calculate how gravity behaves in a higher dimensional universe, for small and large r. In both limits, Newton's force law would become

$$F(r) \sim \frac{1}{M_*^{n+2}} \frac{1}{r^{n+2}} \qquad \text{for} \quad r << 2\pi R, \tag{70}$$

$$F(r) \sim \frac{1}{M_*^{n+2}} \frac{1}{(2\pi R)^n r^2} \qquad \text{for} \quad r >> 2\pi R. \tag{71}$$

This suggests that we could observe deviations of Newtonian gravity at distances smaller than the compactification scale, $r << 2\pi R$. Using the formula (69), we can calculate the corresponding scale at which these deviations should become visible, for every value of n.

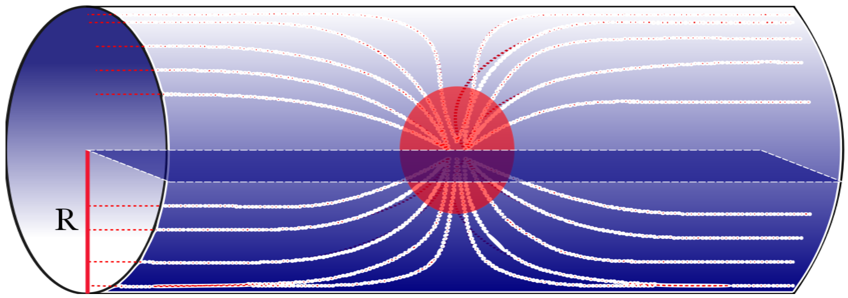


Figure 5: gravity diluting in extra dimensional volume

n	1	2	3	4	5	6	7
R (cm)	$2 \cdot 10^{13}$	10^{-3}	10^{-8}	10^{-11}	10^{-14}	10^{-15}	10^{-15}

Obviously, $n = 1$ is ruled out, because it implies the extra dimension to be the size of an astronomical unit. That would have made it quite hard to go unnoticed for so long. $n = 2$, implies extra dimensions at the size of a millimeter, [7] but this has been ruled out by direct measurements of the gravitational potential.

According to the original theory, the hierarchy problem can thus only be solved within the ADD model, for $n \geq 3$, but since we are interested in finding a scenario that describes 6 or 7 extra dimensions, that is not the biggest problem.

Although very popular and exciting for a long time, there are some weaknesses to the ADD scenario. First of all, solving the hierarchy the way ADD does it, goes at the cost of a new hierarchy problem. As we have calculated above, the inverse compactification scale $1/R$ can reach values between a $\mu m.$ and $10^{-15}cm. \sim$. Although this seems appealing, it is still a factor 10^3 larger than the electroweak scale. We are thus left with a hierarchy between the inverse compactification scale and the electroweak scale. No logical explanation has been given for the large compactification scale.

Second, if the SM fields are confined to a brane, this brane could lead to a deformation of the bulk. In the ADD scenario, this is not being taken into account.

[7] At the time of writing this was about the distance that could be probed for gravity, so you can imagine the feast of excitement among gravity specialists.

5.2 Warped braneworlds

The Randall Sundrum braneworld scenarios were introduced in 1999 as an alternative solution to the hierarchy problem. They extend the idea of the ADD scenario, by taking into account the *brane tension* and they overcomes the new hierarchy between the weak scale and the inverse compactification scale R.

Brane tension can be compared to the mass of the brane and it leads to a bending of the extra-dimensional geometry, referred to as *warping*. To work with branes and brane-tension we need a theory that describes the interactions on the brane and those in the bulk. For a general setup, with n extra dimensions, $\bar{y}^n$, branes labeled by an index i, and fixed at the extra dimensional coordinates $\bar{y}_i$, the total action is just the sum of the bulk and brane terms:

$$\begin{aligned} S &= S_{bulk} + \sum_i S_{brane_i} \\ S_{bulk} &= \int \sqrt{-g_{(4+n)}}(M_*^3 R_{(4+n)} - \Lambda) d^4x\, d^n y \\ S_{brane_i} &= -\lambda_i \int \sqrt{-g_i}\delta(\bar{y} - \bar{y}_i)\, d^4 x d^n y. \end{aligned}$$

Note that the higher dimensional vacuum energy density Λ does not have to be zero or even small. The original Randall Sundrum scenarios existed of just a 5D bulk space-time, with a negative cosmological constant. The extra dimensional space-time is bounded by *two* 3-branes at fixed points. The bulk is in fact just a slice of AdS_5 space-time. To obtain a static Einstein solution, a fine-tuning between the tension of the branes and the cosmological constant is necessary. [8].

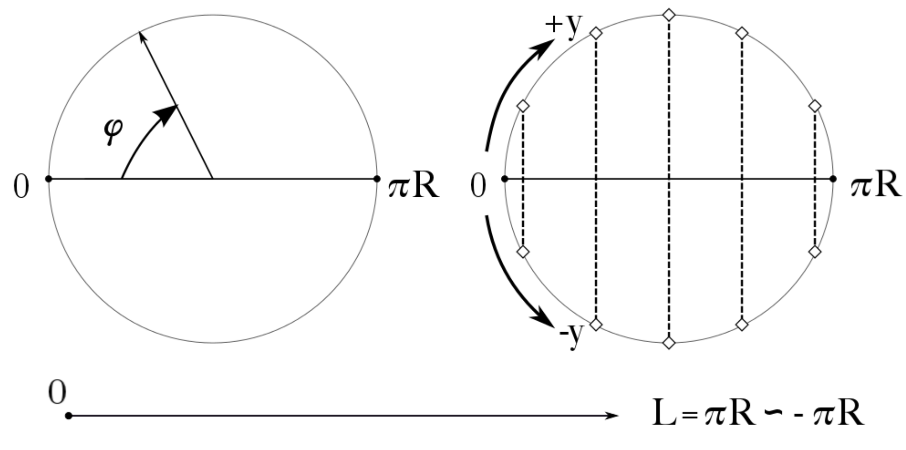
In the following we will derive the main aspects of the RS scenarios. There are two scenarios, referred to as *RS1* and *RS2* respectively. RS1 addresses the hierarchy problem as an alternative to the ADD scenario and RS2 proves the possibility of an infinitely large extra dimension. We will derive the metric, the fine-tuning condition and address the hierarchy problem as is done in the original theory. Finally we will calculate the KK mass spectrum for both models.

5.2.1 RS1 formalism

The first Randall Sundrum scenario describes a five-dimensional bulk space-time enclosed by *two* 3-branes. To specify conditions at the boundaries of the bulk, the extra dimension is compactified on a circle of which the upper and lower half are identified. Mathematically speaking, we consider an $S^1/\mathbf{Z}_2$ orbifold, where S^1 is the circle group and $\mathbf{Z}_2$ is the multiplicative group $(-1, 1)$.

The two 3-branes are located at the orbifold fixed points $y_1 = 0$ and $y_2 = \pi R \equiv L$ and our world is confined to the brane at $y_2 = L$. Taking y to be periodic with period 2L, it is

[8] This fine-tuning was considered one of the weakness of the model, but it could be stabilized by an extra massive scalar field in the bulk creating potential minima at the positions of the branes [19]



enough to consider only the space between y_1 and y_2.

The metric The metric should be a solution to the 5-dimensional Einstein equations, preserving Poincaré invariance on the 4D brane. This leads to the metric ansatz

$$ds^2 = e^{-2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2, \tag{72}$$

where $x^5 \equiv y$ and $e^{-2A(y)}$ is called the *warp factor*. The warp factor is some function of the fifth coordinate only, which is to be derived from the Einstein equations. Note that due to this factor, the metric is non-factorizable, but by a simple coordinate transformation, we can change to a conformally invariant metric.

To determine the function $A(y)$ we use the 5D Einstein equations within the bulk (forgetting about the branes for now):

$$G_{MN} = R_{MN} - \frac{1}{2}g_{MN}R = \kappa^2 T_{MN}, \tag{73}$$

where κ is defined for convenience by

$$\kappa^2 = \frac{1}{2M_{Pl}^3} \tag{74}$$

and the energy-momentum tensor is defined from the 5D action

$$T_{MN} = \frac{-2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{MN}} = -g_{MN}\Lambda. \tag{75}$$

Working out the Einstein tensors for the given metric we obtain

$$G_{MN} = \left\{ \begin{array}{l} G_{\mu\nu} = (6A'^2 - 3A''); \\ G_{55} = 6A'^2. \end{array} \right\} \tag{76}$$

and plugging this back into the Einstein equations, we obtain a definition for $A'(y)$ from the 55-component of the Einstein tensor :

$$A'^2 = \frac{-\Lambda}{12M_*^3}. \tag{77}$$

Note: In order to get real solutions for A, we need $\Lambda < 0$. If Λ is positive, $A \in \mathbb{C}$ and we would get an oscillating warp factor. This is not a relevant scenario for our purposes, thus we require the 5D cosmological constant, Λ to be negative, i.e. the bulk space is an *anti-de Sitter space*.

Now defining

$$\begin{array}{ll} A'^2 & = \frac{-\Lambda}{12M_*^3} \equiv k^2 \\ A' & = \pm k \\ A(y) & = \pm ky \end{array}$$

Since we have assumed an orbifold symmetry in the y-direction, $A(y)$ should be invariant under the transformation $y \to -y$, and we can choose

$$A(y) = k|y|. \tag{78}$$

Plugging this into the metric ansatz, we have arrived at the Randall Sundrum metric:

$$ds^2 = e^{-2k|y|}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2, \tag{79}$$

with $k \equiv \frac{-\Lambda}{12M_*^3}$ and $-L \leq y \leq L$.

Next, looking at the $\mu\nu$-component of the Einstein tensor

$$G_{\mu\nu} = 6(A'^2 - 3A'')g_{\mu\nu}. \tag{80}$$

We can plug in the values for $A(y)$, $A'(y)$ and $A''(y)$ derived from the 55-component, namely

$$\begin{array}{rcl} A(y) & = & k|y| \\ A'(y) & = & k \times sgn(y) \\ & = & k(\Theta(y) - \Theta(-y)) \end{array}$$

but for the second order derivative of a Θ function, the boundaries become important and we have to add some information. If we were to use the function $A(y)$ as given above, we would consider only one boundary at $y = 0$ and the second derivative would just be

$$\begin{aligned} A''(y) &= \frac{\delta}{\delta y}(k(\Theta(y) - \Theta(-y)) \\ &= k(\frac{\delta}{\delta y}\Theta(y) - \frac{\delta}{\delta y}\Theta(-y)) \\ &= k(\delta(y) - -\delta(y)) \\ &= -2k\delta(y). \end{aligned}$$

Note that this function gives a spike at the position of the brane at $y = 0$. For the second brane, we should introduce another boundary and thus another delta-function at $y = L$, giving

$$A''(y) = -2k(\delta(y) - \delta(y)) \tag{81}$$

Plugging all this into the $\mu\nu$-component of the Einstein tensor, we obtain

$$G_{\mu\nu} = 6k^2 g_{\mu\nu} - 6k(\delta(y) - \delta(y - L))g_{\mu\nu}. \tag{82}$$

Now since

$$\kappa^2 T_{\mu\nu} = \frac{-\Lambda}{2M_{Pl}^3} g_{\mu\nu} = 6k^2 g_{\mu\nu}, \tag{83}$$

we immediately see that the first term of the Einstein tensor is equal to the $\mu\nu$ component of the energy momentum tensor times the 5D Newton constant. However, the other two terms have to be resolved in another way. This is where the *brane tensions* come in.

Defining the tensions of the 2 branes; λ_1 and λ_2, we can solve the inequality in the Einstein equation, by adding the tension terms to the action

$$S_i = -\lambda_i \int \sqrt{-g_i}\delta(y - y_i)\, d^4x\, dy, \tag{84}$$

where $i \in (1, 2)$ and g_i are the induced metrics of the branes, defined by

$$\begin{aligned} ds^2 &= g^i_{\mu\nu} dx^\mu dx^\nu \\ &= g_{\mu\nu}(x, y_i) dx^\mu dx^\nu, \end{aligned}$$

with that $y_1 = 0$ and $y_2 = L$. The induced metrics $g_{(i)\mu\nu}$ define the distances along the 3-branes and $g_{55} = 1$, so the metric determinants are just $g_1 = g\delta(y)g_{55} = g\delta(y)$ and $g_2 = g\delta(y - L)$.

The extra terms in the energy-momentum tensor that follow from this are just

$$T^i_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_i}{\delta g^{\mu\nu}} = \lambda_i g_{\mu\nu}\delta(y - y_i). \tag{85}$$

We can now solve for the brane tensions to satisfy the Einstein equations:

$$\begin{aligned} G_{\mu\nu} &= T_{\mu\nu} + T^1_{\mu\nu} + T^2_{\mu\nu} \\ &= 6k^2 g_{\mu\nu} - 6k(\delta(y) - \delta(y - L))g_{\mu\nu}. \end{aligned}$$

So the brane tensions have to satisfy

$$-12kM_*^3(\delta(y)-\delta(y-L))=\lambda_1\delta(y)-\lambda_2(y-L) \tag{86}$$

$$\lambda_1=-\lambda_2=12kM_{Pl}^3. \tag{87}$$

Since we have expressed the brane tensions in terms of k and we have earlier defined

$$k^2=\frac{-\Lambda}{12M_*^3}, \tag{88}$$

we can also express the 5D cosmological constant, Λ in terms of the brane tensions:

$$\Lambda=\frac{-\lambda_1^2}{12M_*^3}. \tag{89}$$

These conditions are called the *fine-tuning* conditions of the RS model. There is a static solution to the Einstein equations if and *only* if the 2 fine-tuning conditions are satisfied.

5.2.2 Solving the hierarchy problem in RS

How does gravity behave with respect to the warped extra dimension? The answer is obtained from the way the 5D action S_5 contains the 4D action S_4 at the weak-brane. The effective 4D action follows from integrating over the y-coordinate within the 5D action, using the background metric parametrized by $ds^2=e^{-2k|y|}\eta_{\mu\nu}dx^\mu dx^\nu+dy^2$. This produces a term with the schematic form

$$\begin{aligned} S_{eff} &\ni M^3\int d^4x\int_{-L}^{+L}e^{-4k|y|}\sqrt{g^{(4)}}e^{2k|y|}R^{(4)}\,dy \\ &= \left[\frac{M^3}{k}(1-e^{-2k|y_c|})\right]\int\sqrt{-g^{(4)}}R^{(4)}\,d^4x. \end{aligned}$$

Comparing this to the 4D action, S_4, we see that the relation between the 4D Planck-scale M_{Pl} and the Fundamental Planck-scale, M_* is given by

$$M_{Pl}^2=\frac{M^3}{k}(1-e^{-2k|y_c|}). \tag{90}$$

Assuming y_c is quite large, it turns out that the size of the Planck-scale hardly depends on the size of the extra dimensions. We can choose $M_*\sim k\sim M_{Pl}$ and still solve the hierarchy problem.

Solving the hierarchy problem Assuming that the SM fields are trapped on the negative tension (weak-)brane and considering the Higgs scalar field, H, one can use a similar method as the one above, to show that any fundamental mass parameter is red-shifted on the negative tension brane according to the warp-factor. We consider the Higgs scalar field action:

$$S_{Higgs} = \int d^4x\sqrt{-g}\left[g^{\mu\nu}D_\mu H^\dagger D_\nu H - \lambda(H^\dagger H - v_0^2)^2\right] \quad (91)$$

$$= \int d^4x e^{-4k|y_c|}\left[e^{2k|y_c|}\eta^{\mu\nu}D_\mu H^\dagger D_\nu H - \lambda(H^\dagger H - v_0^2)^2\right], \quad (92)$$

where v is the vacuum expectation value of the Higgs field. To get a canonically normalized action, one should redefine the Higgs field like $H = e^{ky_c}\bar{H}$. The action in terms of this new definition becomes

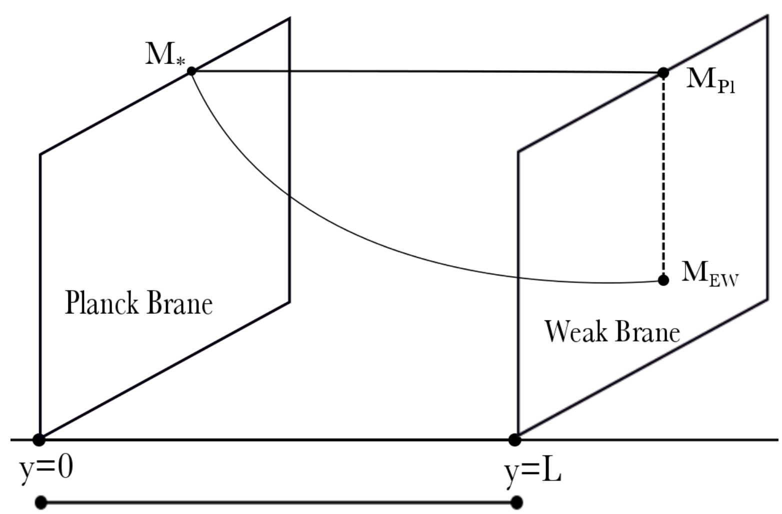
$$S_{Higgs} = \int d^4x\left[\eta^{\mu\nu}D_\mu\bar{H}^\dagger D_\nu\bar{H} - \lambda(\bar{H}^\dagger\bar{H} - (e^{-k|y_c|}v_0^2)^2\right]. \quad (93)$$

From this we see that the Higgs scalar is exponentially suppressed over the extra dimensional space. The effective vev, that we observe is thus much lower than the real value:

$$v = e^{-k|y_c|}v_0 \quad (94)$$

The bare Higgs mass could thus be of the order of the Planck scale, while the physical Higgs mass is redshifted down to the weak scale. To generate a mass parameter of order 1TeV with $M_* \sim k \sim M_{Pl}$ one only needs $ky_c \sim \ln(10^{16}) \sim 30$. This is how the size of the extra dimension is determined in the RS scenario.

Comparing the two parameters in the fifth dimension, we see that the Planck-scale is more or less constant, while the mass-parameters for the SM fields are redshifted to lower scales away from the (positive tension) Planck-brane. Since we measure the scales on the (negative tension) TeV-brane, this solves the hierarchy problem.



5.2.3 Calculation of the KK spectrum

We will estimate the mass spectrum of the KK modes in the RS model, by calculating the spectrum of a massless scalar field in the 5D RS space-time. We start from the Lagrangian density for a massless scalar field in the 5D RS background is

$$\mathscr{L} = \frac{1}{2}\sqrt{|g|}(g^{MN}\partial_M \Phi \partial_N \Phi), \tag{95}$$

and derive the wave-equations in the form of the *Euler-Lagrange* equations, by the variational principle

$$\partial_M \frac{\partial \mathscr{L}}{\partial(\partial_M \Phi)} = \frac{\partial \mathscr{L}}{\partial \Phi}. \tag{96}$$

The RS metric was given by

$$ds^2 = e^{-2k|y|}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2, \tag{97}$$

but it is more convenient to write it in a conformally invariant form, defining:

$$\begin{aligned} dz^2 &= e^{2k|y|}dy^2 \\ e^{-2k|y|} &= \frac{1}{(k|z|+1)^2} \\ k|z| &= e^{k|y|} - 1, \end{aligned}$$

where the last definition is chosen to identify the zero-value of z with the zero-value of y. In terms of our new variable z, the metric is then

$$ds^2 = \frac{1}{(k|z|+1)^2}\eta_{MN}dx^M dx^N \tag{98}$$

or

$$ds^2 = e^{-2A(z)}\eta_{MN}dx^M dx^N, \tag{99}$$

where I have defined the function $A(z) = ln(k|z|+1)$. In tensor notation then

$$\begin{aligned} \{g_{MN}\} &= e^{-2k|z|}\{\eta_{MN}\} \\ \{g^{MN}\} &= e^{2k|z|}\{\eta^{MN}\} \end{aligned}$$

so the square root of the metric determinant becomes

$$\sqrt{-g} = \sqrt{\begin{vmatrix} -e^{2A(z)} & 0 & 0 & 0 & 0 \\ 0 & e^{2A(z)} & 0 & 0 & 0 \\ 0 & 0 & e^{2A(z)} & 0 & 0 \\ 0 & 0 & 0 & e^{2A(z)} & 0 \\ 0 & 0 & 0 & 0 & e^{2A(z)} \end{vmatrix}} = e^{5A(z)} \tag{100}$$

Plugging all quantities into the Lagrangian, we obtain

$$\mathscr{L} = \frac{1}{2} e^{3A(z)} (\eta^{MN} \partial_M \Phi \partial_N \Phi), \tag{101}$$

and the Euler-Lagrange equations give:

$$\begin{aligned} \partial_M \frac{\partial \mathscr{L}}{\partial(\partial_M \Phi)} &= \partial_M (e^{3A(z)} \eta^{MN} \partial_N \Phi) \\ \frac{\partial \mathscr{L}}{\partial \Phi} &= 0. \end{aligned}$$

Splitting off the z-dependent parts, we obtain the wave-equation

$$e^{3A(z)} (\partial_\mu \partial^\mu \Phi - \partial_z \partial^z \Phi - 3(\partial_z A(z)) \partial_z \Phi) = 0. \tag{102}$$

Assuming that the full particle field in 4D Minkowski space-time, M_4, is the holographic picture of the 5D field $\Phi(x^\mu, z)$, we would look for solutions to the wave-equation that are the product of a free field in 4D Minkowski space-time multiplied by a function depending on the fifth variable, z, i.e.

$$\Phi(x^\mu, z) \sim e^{-ip\cdot x} \phi(z), \tag{103}$$

where $p \cdot x = p^\mu x_\mu$. Plugging this into the wave-equation above, we obtain the wave-equation for the z-dependent part of the field, $\phi(z)$

$$\partial_z^2 \phi(z) - 3(\partial_z A(z)) \partial_z \phi(z) - p^2 \phi(z) = 0. \tag{104}$$

We hope to find discrete masses of the scalar particle in 4 dimensions, with values $m^2 = p^2$. Therefore it would be convenient to have an equation of the form of a Schrödinger equation

$$-\partial_z^2 \psi + V(z) \psi = m^2 \psi. \tag{105}$$

To do so we need to get rid of the single-derivative term $(-3\partial_z A(z) \partial_z \phi)$ and introduce the relation

$$\phi(z) = e^{f(z)} \psi(z), \tag{106}$$

where $f(z)$ is just a test function, that we can choose, such that the linear derivative terms cancel. We obtain from the definition above

$$\begin{aligned} \partial_z \phi &= e^{f(z)} (\partial_z f \psi + \partial_z \psi) \\ \partial_z^2 \phi &= e^{f(z)} ((\partial_z f)^2 \psi + (\partial_z^2 f) \psi + 2\partial_z f \partial \psi + \partial_z^2 \psi) \end{aligned}$$

Inserting this into the z-dependent wave-equation and dividing by $e^{f(z)}$, we obtain

$$-\partial_z^2 \psi - (2\partial_z f + 3\partial_z A) \partial_z \psi - (\partial_z^2 + (\partial_z f)^2 + 3(\partial_z A) \partial_z f) \psi = m^2 \psi \tag{107}$$

So choosing $f(z)$ such that $\partial_z f = -\frac{3}{2} \partial_z A$, the linear terms cancel and we have our wave-equation in the Schrödinger form

$$-\partial_z^2 \psi + (\frac{3}{2} \partial_z^2 A + (\frac{3}{2} \partial_z A)^2) \psi = m^2 \psi, \tag{108}$$

where compared to the general Schrödinger equation, the potential $V(z)$ is given by

$$V(z) \sim \frac{3}{2}\partial_z^2 A + (\frac{3}{2}\partial_z A)^2. \tag{109}$$

We defined $A(z) = ln(k|z|+1)$, so its derivatives are

$$\begin{aligned}\partial_z A(z) &= \frac{k \cdot sgn(z)}{k|z|+1}\\ \partial_z^2 A(z) &= \frac{-2k\delta(z)}{k|z|+1} + \frac{k^2}{(k|z|+1)^2}.\end{aligned}$$

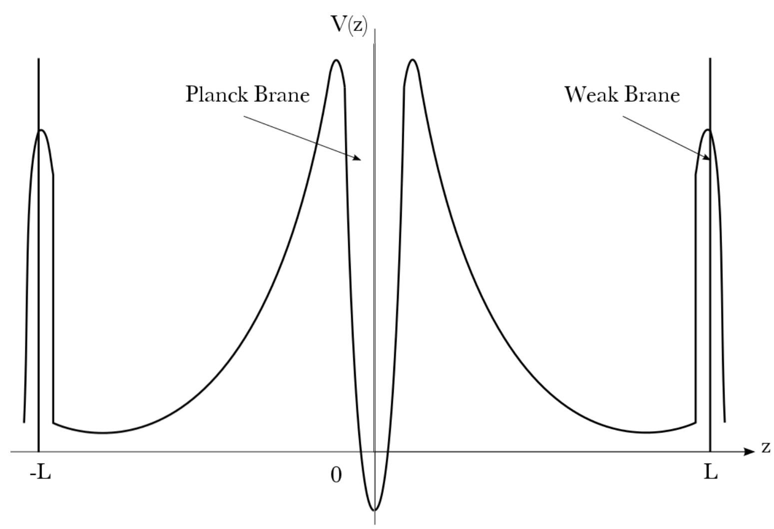
Note that squaring the $sgn(z)$ function, just gives 1 everywhere. Plugging it all into the wave equation, gives

$$-\partial_z^2\psi + (\frac{15}{4}\frac{k^2}{(k|z|+1)^2} - 3\frac{k\delta(z)}{k|z|+1})\psi = m^2\psi. \tag{110}$$

Note that the introduction of a second brane would introduce an extra delta-function in the second derivative of the the function $A(z)$, i.e. an extra brane at $z+L$ leads to the following potential

$$V(z) = \frac{15}{4}\frac{k^2}{(k|z|+1)^2} - \frac{3k(\delta(z)-\delta(z-L)}{k|z|+1} \tag{111}$$

For obvious reasons, this potential is called the *vulcano* potential. It is a typical aspect of the RS models because it is important for the localization of gravity.



Solving the wave-equation We have found an equation that looks like the Schrödinger equation with a vulcano-shaped potential and we need to solve it to find the mass-spectrum of the KK modes. We will find a series of solutions ψ_n, with corresponding mass m_n, where the zero-mode ψ_0 is the solution to the equation

$$-\partial_z^2\psi_0 + \left(\frac{15}{4}\frac{k^2}{(k|z|+1)^2} - \frac{3k(\delta(z)-\delta(z-L))}{k|z|+1}\right)\psi_0 = 0. \tag{112}$$

Rewriting this in terms of the function $A(z)$ we get the easier equation

$$-\partial_z^2\psi_0 + \left((\frac{3}{2}\partial_z^2 A)^2 - \frac{3}{2}\partial_z^2 A(z)\right)\psi_0 = 0, \tag{113}$$

which is solved by the wave-function

$$\psi_0 = e^{-\frac{3}{2}A} = (k|z|+1)^{-\frac{3}{2}}. \tag{114}$$

This equation corresponds to the way gravitation behaves; it falls off exponentially in the z-direction away from the brane.

Between the Boundaries, the massive KK gravitons solve the wave-equation

$$-\partial_z^2\psi_n + \left(\frac{15}{4}\frac{k^2}{(k|z|+1)^2}\right)\psi_n = m^2\psi_n. \tag{115}$$

This equation can be brought into the form of the general *Bessel's equation*

$$\left(-\partial_y^2 - \frac{1}{y}\partial_y + \frac{(\nu^2-1)}{y^2}\right)J_\nu = 0, \tag{116}$$

of which the solution is a Bessel function J_ν of order ν. To bring the wave-equation into the right form, we first introduce $J_\nu(y) = y^{-\frac{1}{2}}\psi_\nu(y)$. Inserting this into the Bessel-equation, gives

$$\left(-\partial_y^2 + \frac{4\nu^2-1}{4y^2}\right)\psi(y) = \psi(y). \tag{117}$$

Now put $y \equiv m(\frac{k|z|+1}{k} = |z| + \frac{1}{k})$, and multiply by m^2 to obtain:

$$\left(-\partial_z + \frac{4\nu^2-1}{4}\frac{k^2}{(k|z|+1)^2}\right)\psi = m^2\psi. \tag{118}$$

Comparing this to our original wave-equation for the KK gravitons in the space between the boundaries (115), we see that the equations are equal for $\nu = 2$, thus the solutions to (115) are second-order Bessel functions of the first kind, J_2. For integer ν, the Bessel functions $J_{+\nu}$ and $J_{-\nu}$ are linearly dependent, so to form a basis of solutions, we also need Bessel functions of the second kind, referred to as Y_ν. The general solution to the wave-equation for the massive KK modes, is thus given by linear combinations of second order Bessel functions of the first- and second kind, i.e.:

$$\psi_n = (|z| + 1/k)^{\frac{1}{2}} \left[a_n J_2(m_n(|z| + 1/k)) + b_n Y_2(m_n(|z| + 1/k)) \right], \tag{119}$$

with coefficients, a_n and b_n.

To better understand the KK modes, we can study the small and large argument limits of the Bessel-functions. For small $m_n(|z| + \frac{1}{k})$ the second order Bessel functions can be approximated by

$$J_2(m_n(|z| + 1/k)) \sim \frac{m_n^2(|z| + 1/k)^2}{8}, \tag{120}$$

$$Y_2(m_n(|z| + 1/k)) \sim -\frac{4}{\pi m^2(|z| + 1/k)^2} - \frac{1}{\pi}. \tag{121}$$

Note that the zero-mode wave function ψ_0 is the limit of $(m(|z| + \frac{1}{k}))Y_2(m(|z| + \frac{1}{k})$ when $m \to 0$. For large mz the Bessel-functions are approximated by

$$\sqrt{z} J_2(m_n z) \sim \sqrt{\frac{4}{\pi m}} cos \left(m_n z - \frac{5}{4}\pi \right) \tag{122}$$

$$\sqrt{z} Y_2(m_n z) \sim \sqrt{\frac{2}{\pi m}} sin \left(m_n z - \frac{5}{4}\pi) \right) \tag{123}$$

These solutions allow any value of m, so the spectrum is continuous. To obtain a discrete spectrum we need to add some boundary conditions.

Discrete KK spectrum In the RS1 model, the z-coordinate is confined to the values $0 \leq z \leq L_z$, by the 2 branes. The presence of these branes induces the quantization of the KK masses.

We can demand the functions $\psi_n(z)$ to vanish at the boundaries. That means taking $\psi_n(0) = \psi_n(L_z) \equiv 0$, which turns out to be a good approximation. This boundary condition selects discrete values of m, determined by the zeros of the Bessel functions, because $\psi \sim z^{\frac{1}{2}} J_2$. We work in the large distance limit $mz >> 1$, so we can use the approximations to the Bessel functions (123); $J(m(|z| + 1/k)) \to J(mz)$.

Defining the coordinate j_2^n, to correspond to the n^{th} zero-point of the second order Bessel function, we find the KK mass spectrum by equating

$$J_2(m_n L_z) = J_2(j_0^m) \equiv 0. \tag{124}$$

By approximation, the spectrum is thus quantized in units of L_z, i.e. the distance between the two branes defines the quantization of the KK masses; $m_n \sim j_0^n / L_z$.

For small values of ν the zeros of the Bessel functions can be approximated by

$$j_\nu^n \simeq (n + \frac{\nu}{2} - \frac{1}{4})\pi - \frac{4\nu^2 - 1}{8\pi(n + \frac{\nu}{2} - \frac{1}{4})} - ... \tag{125}$$

So for $\nu = 2$, we get

$$j_2^n \simeq (n + \frac{2}{2} - \frac{1}{4})\pi - \frac{4 \cdot 2^2 - 1}{8\pi(n + \frac{2}{2} - \frac{1}{4})} = (n + \frac{3}{4})\pi - \frac{15}{8\pi(n + \frac{3}{4})}. \tag{126}$$

We can conclude that the KK modes have a small probability to tunnel to $z = 0$ due to the potential wall. The solutions described by the Bessel functions, correspond to a KK tower with a mass splitting of $\Delta m \sim L_z^{-1} \sim k(e^{-k|y_c|})$, where y_c is the distance of between the Planck- and the weak brane in the y-coordinates. Now with $k \sim M_{Pl}$ and $ky_c \sim 30$, one achieves a mass splitting of the order of $\Delta m \sim 1TeV$.

5.2.4 Radius stabilization

The RS1 model solves the hierarchy problem with a fine-tuning between the brane-tensions and the 5D cosmological constant, and a static interbrane spacing $L = \pi R$. To stabilize the setup we would need a mechanism that fixes the space between the branes. Some early discussions on this subject and related issues can be found [19], [20].

Goldberger and Wise [19] proposed to introduce a bulk scalar field to the theory, to stabilize the positions of the branes by breaking translational invariance along the extra dimension and creating potential minima at the brane-positions.

If the scalar field has localized interaction terms on the branes, which develop non-zero vacuum expectation values for ϕ, one can obtain non-trivial minima for a potential that stabilizes the branes' positions.

One may wonder whether the vacuum energy might disturb the background and in fact it does. However this correction is negligible. This follows from the Einstein-scalar field calculations as done in the references mentioned above.

Other proposals for stabilizing y_c exist and can be found in [21].

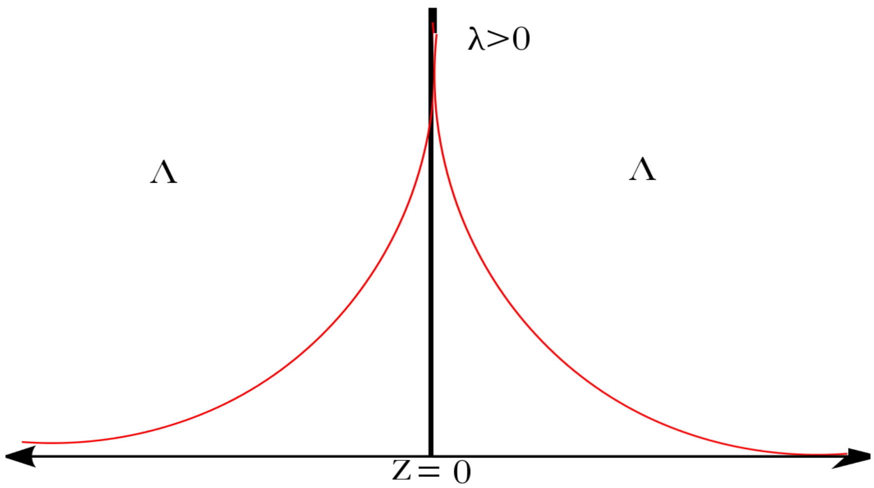
5.3 RS II

In the previous scenario, Randall and Sundrum described a space with one compact extra dimension, bounded by two 3-branes. Their aim was to solve the hierarchy problem. In the next model, the so-called RS II scenario, the set-up is basically the same, except that the second boundary at $z = L_z$ is taken to infinity and the SM fields are moved to the brane at $z = 0$. Taking the second brane to infinity is in fact the same as completely removing it from the space-time.

From equation (90) we see that removing the effect of removing the second brane on the Planck scale is

$$M_{Pl}^2 = \frac{M^3}{k}. \tag{127}$$

Thus, the Planck scale remains finite, which is a good hint that gravity might remain effectively four-dimensional on the brane. The solution for the metric in the two brane context



remains valid, as long as we stick to the fine-tuning between brane tension and bulk cosmological constant. The KK modes can therefore be approximated by the continuum modes, since there is no second boundary

$$\psi_n = (|z| + 1/k)^{\frac{1}{2}} \left[a_n J_2(m_n(|z| + 1/k)) + b_n Y_2(m_n(|z| + 1/k)) \right]. \tag{128}$$

Having found the complete KK spectrum and pretending all SM fields are trapped on the positive brane at $z = 0$, we can calculate the gravitational potential between two point masses m_1 and m_2. This potential is the result of the exchange of the zero-mode and KK modes. The zero modes gives us the expected Newtonian potential, while the continuum KK modes produce a potential of the Yukawa type, constituting corrections to the Newtonian potential

$$\begin{aligned} V(r) &\sim \frac{G_N m_1 m_2}{r} + \frac{1}{M^3} \int_0^\infty \frac{m_1 m_2 e^{-mr}}{r} \psi^2(0)\, dm \\ V(r) &\sim G_N m_1 m_2 \left(\frac{1}{r} + \int_0^\infty \frac{m}{k^2} \frac{e^{-mr}}{r}\, dm \right) \\ V(r) &\sim \frac{G_N m_1 m_2}{r} \left(1 + \frac{1}{r^2 k^2} \right). \end{aligned}$$

Note that I used $\frac{G_N}{k} = \frac{1}{M^3}$ and $\psi^2(0) \sim \frac{m}{k}$ in the limit of large z. We see from this that the potential uncovers corrections to the Newtonian potential, only at distances of the order of the inverse fundamental gravity scale: M_*^3.

5.3.1 Higher dimensional warped braneworlds

The RS2 model does not solve the hierarchy problem, like RS1 does, but it provides an alternative to compactification. What makes it even more interesting is that it could be extended to higher numbers of extra dimensions, e.g. 6 or 7. Following the approach of [22], our world could be confined to the intersection of n $(2+n)$-branes in a $(4+n)$-dimensional AdS_n space-time. The bulk cosmological constant would thus be negative and all the branes would have positive tension, λ_i. Consequently the graviton would be localized on the intersection of all branes, which can be confirmed by solving the Einstein equations for this particular setup with n $(2+n)$-branes and a cosmological constant, Λ

$$S = \int d^4x\, d^n y \sqrt{-g^{(4+n)}} \left(M_*^{2+n} R^{(4+n)} - \Lambda \right) - \sum_i \lambda_i \int d^4x\, d^{n-1} y \sqrt{-g^{(3+n)}}. \tag{129}$$

If the branes are all orthogonal to each other, the metric would be conformally flat and by using the appropriate bulk coordinates, it could be written as

$$ds^2 = \Omega(z) \left(\eta_{\mu\nu} d^\mu x d^\nu x - \delta_{ij} dz^i dz^j \right), \tag{130}$$

where the warp factor is $\Omega(z) = (k \sum_i |z_i| + 1)^{-1}$ and the curvature parameter k is defined by

$$k^2 = \frac{2\Lambda}{M_*^{2+n} n(n+2)(n+3)}. \tag{131}$$

Note that this is a generalization of the conformally invariant 5D RS metric (98). From this the generalized fine-tuning conditions become

$$\Lambda = \frac{\lambda^2}{M_*^{2+n}} \frac{n(n+3)}{(n+2)} \tag{132}$$

and the effective Planck scale would be related to the fundamental Planck scale by

$$M_{Pl}^2 = M_*^{2+n} \int d^n z \Omega^{(2+n)} = M_*^{2+n} \frac{2^n n^{\frac{n}{2}}}{(n+1)!} L^n. \tag{133}$$

Similar to the 5D case we can find a Schrodinger like wave-equatuation for the graviton, which would be of the form

$$\left[-\frac{1}{2} m^2 + \left(-\frac{1}{2} \nabla_z^2 + V(z) \right) \right] \psi = 0, \tag{134}$$

with the potential being

$$V(z) = \frac{n(n+2)(n+4)k^2}{8} \Omega - \frac{(n+2)k}{2} \Omega \sum_i \delta(z^i). \tag{135}$$

Note that the massless bound state solution is of the form $\Psi \sim \Omega^{(n+2)/2}(z)$ localized at the brane intersection $(z_i = 0)$. The potential falls off to zero for large z. Just like the 5D RS2 case, the higher modes with small masses are suppressed.

For a more extensive and complete discussion on the extended RS2 model, including a calculation of the KK spectrum see for instance [23].

6 Solutions to flux compactification

In the following section we will discuss an alternative family of extra dimensional models, which is based on the mechanism of flux-compactification. Flux compactification was first introduced by Peter Freund and Mark Rubin in 1980 and was originally proposed as a building block for 11D Supergravity. It shows that in a $(p+q)$-dimensional gravitational theory, the presence of a q-form flux field $F_{(q)}$ can lead to the natural compactification of either p or q spatial dimensions [24]. In a similar setup, several forms of Einstein solutions may exist. We will discuss the origin and stability of such solutions.

6.1 Flux compactification

To understand the mechanism of flux compactification, we will start by deriving the original results from Freund and Rubin.

6.1.1 Freund Rubin compactication

The easiest example of Freund-Rubin compactification starts from d-dimensional Einstein-Maxwell theory and shows that a 2-form flux field F_{MN} can lead to the compactification of either 2 or $(d-2)$ spatial dimensions if we choose a specific form fro the metric and the flux field. The d-dimensional Einstein-Maxwell equations read

$$R^{MN} - \tfrac{1}{2}Rg^{MN} = -8\pi G\left(F^M_P F^{PN} - \tfrac{1}{4}F_{AB}F^{AB}g^{MN}\right) \tag{136}$$
$$\nabla_M F^{MN} = 0, \tag{137}$$

Now we choose solutions such that

- the d-dimenional space-time is the product of a p-dimensional manifold $\mathcal{M}_p$ and a q-dimensional manifold $\mathcal{N}_q$. I will use capital Latin letters (M,N) to indicate the complete set of spacetime coordinates on $\mathcal{M}\times\mathcal{N}$, the Greek indices correspond to the coordinates on $\mathcal{M}_p$ and the lower-case Latin indices correspond to the coordinates on $\mathcal{N}_q$.i.e.:

$$g_{MN} = \begin{pmatrix} g_{\mu\nu}(x^\rho) & 0 \\ 0 & g_{mn}(x^p) \end{pmatrix}, \tag{138}$$

  Note that the capital Latin letters (M,N) indicate the complete set of spacetime coordinates on $\mathcal{M}\times\mathcal{N}$, the Greek indices correspond to the coordinates on $\mathcal{M}_p$ and the lower-case Latin indices correspond to the coordinates on $\mathcal{N}_q$.

- and the flux field is proportional to the levi-civita tensor:

$$F^{MN} = (\epsilon^{MN}/\sqrt{-g})f \tag{139}$$
$$\epsilon^{MN} = \begin{matrix} ccc\epsilon^{\mu\nu} & : & M=\mu, N=\nu \\ 0 & : & otherwise \end{matrix} \tag{140}$$

  where f has dimensions of mass squared.

Note that the scalar term $F_{AB}F^{AB}$ in the Einstein equations gives an overall cosmological constant term:

$$\begin{aligned} F_{AB}F^{AB} &= (\epsilon_{AB}\cdot\sqrt{-g})f\cdot(\epsilon^{AB}/\sqrt{-g})f \\ &= 2f^2 sgn(g), \end{aligned}$$

and after plugging everything into the Einstein equations, it follows that the first term on the RHS of the Einstein equations is only non-zero for $M=\mu$ and $N=\nu$, when it is proportional to $g^{\mu\nu}$. Everything thus reduces to cosmological terms with different constants on the two manifolds. Now defining the scalar curvatures $R_{d-2}=g_{mn}R^{mn}$ and $R_2=g_{\mu\nu}R^{\mu\nu}$, and the total scalar curvature $R=R_2+R_{d-2}$, we find from the Einstein equations that

$$R_{d-2}=\lambda\ , \qquad R_2=-2\frac{d-3}{d-2}\lambda, \tag{141}$$

where $\lambda \equiv 8\pi G\ f^2\ sgn(g_2)$.

Proof: From contracting the complete Einstein equations with the inverse metric, we obtain an expression for the complete Ricci scalar R:

$$\begin{aligned} \left(1-\frac{d}{2}\right)R &= -8\pi G[F_{PN}F^{PN}-\frac{1}{4}F_{AB}F^{AB}\cdot d] \\ &= -8\pi G[(2-\frac{d}{2})f^2\cdot sgn(g_2)], \end{aligned}$$

Doing the same thing for the two submanifolds, by contracting with the appropriate submetric, we get two equations in which we can plug in the above expression for R.

$$\begin{aligned} R_{(d-2)}-\frac{1}{2}(d-2)R &= -8\pi G[-\frac{1}{4}F_{AB}F^{AB}\cdot(d-2)] \\ R_{(d-2)}+(1-\frac{d}{2})R &= 8\pi G[\frac{d-2}{2}f^2\cdot sgn(g_2)] \\ R_{(d-2)} &= 8\pi G\,f^2\,sgn(g_2)\left[\frac{d-2}{2}+(2-\frac{d}{2})\right] \\ &= 8\pi G\,f^2\,sgn(g_2)\ \equiv\ \lambda. \end{aligned}$$

Similarly we find for $\mathcal{M}_2$:

$$\begin{aligned} R_2+R &= -8\pi G f^2 sgn(g_2) \\ R_2 &= -8\pi G f^2 sgn(g_2)\left[1+\frac{2-\frac{d}{2}}{1-\frac{d}{2}}\right] \\ &= -2\frac{(d-3)}{(d-2)}\lambda. \end{aligned}$$

Note that since $d\geq 4$, it follows from $Gf^2>0$ that R_2 and R_{d-2} have opposite signs, while R_{d-2} and g_2 have the same sign. This implies that when the time-dimension is in $\mathcal{M}_2$ then $\mathcal{N}_{d-2}$ is compact and vice versa.

6.1.2 General flux compactification

In a similar but more general way, we can derive that a p-form flux field in a $(p+q)$-dimensional gravitational theory can stabilize the compactification of either p or q dimensions.

The action for a $(p+q)$-dimensional Einstein-Maxwell theory is

$$S = \frac{1}{16\pi G}\int dx^d\sqrt{-g}\left(R - 2\Lambda - \frac{1}{q!}F_{(q)}^2\right), \tag{142}$$

where R is the total scalar curvature, $F_{(q)}$ is the q-form flux field and $g = det(g_{MN})$, the determinant of the total metric. For generality I have included a cosmological term Λ, which can later be chosen to have any positive, negative or null-value. The notation of the indices is as before. Varying the action with respect to the metric leads to the Einstein equations:

$$R^{MN} - \frac{1}{2}Rg^{MN} = -8\pi G T^{MN}, \tag{143}$$

$$T^{MN} = F^{M}_{L_1\ldots L_{q-1}}F^{L_1\ldots Lq-1N} - \frac{1}{2q!}F^2 g^{MN} - \Lambda g^{MN}, \tag{144}$$

$$\nabla_M F^{ML_1\ldots L_{q-1}} = 0, \tag{145}$$

Contracting the Einstein tensor with the inverse metric g_{MN} gives an expression for the Ricci scalar R of the complete space

$$(p+q-2)R = (p+q)\Lambda - \frac{p-q}{q!}, \tag{146}$$

which we can substitute back into into (143) to obtain a general expression for the Riemann tensor:

$$R^{MN} = \frac{1}{(q-1)!}F^{M}_{L_1\ldots L_{q-1}}F^{L_1\ldots Lq-1N} - \frac{1}{q!}\frac{q-1}{p+q-2}F^2 g^{MN} + \frac{2}{p+q-2}\Lambda g_{MN}. \tag{147}$$

Solutions Again we look for solutions of the form $\mathcal{M}_p \times \mathcal{N}_q$, such that the metric is given by

$$g_{MN} = \begin{pmatrix} g_{\mu\nu}(x^\rho) & 0 \\ 0 & g_{mn}(x^p) \end{pmatrix} \tag{148}$$

and the curvature scalar can be written as the sum of the two scalars on the separate manifolds $R = R^{\mathcal{M}_p} + R^{\mathcal{N}_q}$ and the scalars $R^{\mathcal{M}_p}$ and $R^{\mathcal{N}_q}$ are given by

$$R^{\mathcal{M}_p}_{\mu\nu} = \frac{1}{(q-1)!}F^{\mu}_{L_1\ldots L_{q-1}}F^{L_1\ldots Lq-1\nu} - \frac{1}{q!}\frac{q-1}{p+q-2}F^2 g^{\mu\nu} + \frac{2}{p+q-2}\Lambda g_{\mu\nu} \tag{149}$$

$$R^{\mathcal{N}_q}_{mn} = \frac{1}{(q-1)!}F^{m}_{L_1\ldots L_{q-1}}F^{L_1\ldots Lq-1n} - \frac{1}{q!}\frac{q-1}{p+q-2}F^2 g^{mn} + \frac{2}{p+q-2}\Lambda g_{mn} \tag{150}$$

Now again choosing the q-form Flux field proportional to the completely anti-symmetric tensor on $\mathcal{N}_q$ with strength b (following the notation of [bron])

$$F^{M_1 \dots M_q} = \begin{cases} b\epsilon^{m_1 \dots m_q} & : \quad M_i = m_i \\ 0 & : \quad otherwise \end{cases} \tag{151}$$

we obtain the Ricci scalars on the two manifolds after contracting with the inverse metric

$$R^{\mathcal{M}_p} = \frac{1}{p+q-2}(-b^2(q-1) + 2\Lambda), \tag{152}$$

$$R^{\mathcal{N}_q} = \frac{1}{p+q-2}(b^2(p-1) + 2\Lambda). \tag{153}$$

Note that setting the cosmological constant equal to zero returns the original result of Freund and Rubin. Again, the scalars have opposite sign and the Euclidean signature (+...+) of the manifold $\mathcal{N}_q$ implies that it is compact.

6.2 Solutions

We showed that in a $p+q$-dimensional bulk space-time, with factorized geometry, $\mathcal{M}_p \times \mathcal{N}_q$, the presence of a q-form flux field can naturally compactify either p or q spatial dimensions. An interesting scenario would be a setup with 7 compactified spatial dimensions and $(3+1)$ external dimensions, which correspond to the universe we observe. In the search for a viable theory we should also ensure the stability of such a setup.

We are interested in Einstein solutions within a similar setup. If the space-time is maximally symmetric, the solution is of the form $(A)dS_p \times \mathcal{M}_q$, where p and q are the number of macroscopic and internal dimensions, respectively. Including a cosmological constant in the theory, we can also use the above theory to describe cosmological models. The stability of these solutions has been investigated by Wolfe et al. [25], Bousso [27] and Martin [26]. It turns out that for dS-models, no stable solutions exist for $q > 4$. To write a viable theory in 11 dimensions, we thus need another solution.

Unstable solutions suggest the existence of a different class of solutions at the points of marginal stability. Kinoshita [30] [31] suggested a generalization of the $dS_q \times S^q$ solutions that corresponds to a warping of the internal sphere. In a second paper, the dynamical stability of this new branch of solutions has been related to the thermodynamic stability of the model.

In the following we will study both type of solutions and discuss their stability.

6.3 de Sitter solutions

In chapter 3, we found the Ricci scalars for the two manifolds $\mathcal{M}_p$ and $\mathcal{N}_q$, from the general flux compactification (153):

$$\begin{aligned} R^{\mathcal{M}_p} &= \frac{1}{p+q-2}(-b^2(q-1)+2\Lambda), && (154) \\ R^{\mathcal{N}_q} &= \frac{1}{p+q-2}(b^2(p-1)+2\Lambda). && (155) \end{aligned}$$

Choosing a positive cosmological constant in the bulk, $\Lambda > 0$, we can find background solutions of the form $dS_p \times S^q$, i.e. a p-dimensional de Sitter space-time times a q-dimensional sphere. Solutions of this form are interesting, because de Sitter or quasi-de Sitter spacetimes describe the inflationary epoch of the universe at its early stages and also the present universe in its period of accelerated expansion. The constraints and stability of such configurations, have been investigated by [28], [27] and [26]. We will see that solutions of this form are unstable for any $q \geq 5$.

6.3.1 Constraint equations

De Sitter space-time is parametrized by the Hubble parameter H and the q-sphere by its radius ρ. The line-element can thus be written as

$$ds^2 = -dt^2 + e^{2Ht}d\bar{x}^2_{(p-1)} + \rho^2 d\Omega^2_q. \qquad (156)$$

The curvature scalars of the separate manifolds are thus given by

$$\begin{aligned} R^{dS_p} &= p(p-1)H^2 && (157) \\ R^{S_q} &= \frac{q(q-1)}{\rho^2}. && (158) \end{aligned}$$

Equating these to the above Ricci scalars from the Eistein equations (155), we obtain relations between the parameters H, b and ρ:

$$\begin{aligned} \frac{1}{p+q-2}(-b^2(q-1)+2\Lambda) &= (p-1)H^2 && (159) \\ \frac{1}{p+q-2}(b^2(p-1)+2\Lambda) &= \tfrac{(q-1)}{\rho^2}. && (160) \end{aligned}$$

This can be easily rearranged into the form given by Kinoshita:

$$\begin{aligned} (q-1)\rho^{-2} - (p-1)H^2 &= b^2 && (161) \\ (q-1)^2\rho^{-2} + (p-1)H^2 &= 2\Lambda. && (162) \end{aligned}$$

and we obtain the constraint equation for the parameter space in which solutions exist:

$$(p-1)(p+q-2)H^2 + b^2(q-1) = 2\Lambda. \qquad (163)$$

6.3.2 Stability of $dS_p \times S^q$

Analysis of the above solution under small perturbations, shows the instabilities that may occur among the family of de Sitter solutions. At points of marginal stability, new solutions may occur. The analysis is done by perturbing the metric (156) and searching for solutions that satisfy the linearized Einstein-Maxwell equations (see appendix).

According to the work done by Martin [26] and Bousso [27], linear perturbations in this background show two channels of instabilities in the scalar sector with respect to the p-dimensional de Sitter symmetry [9]. To summarize the most important findings, for $q = 2$ and $q = 3$ there is one channel of instability and an additional one for $q \geq 4$. Both channels lead to a stability condition, implying that the mass squared of this particular mode will not be tachyonic. For $q = 2$ and $q = 3$ this implies

$$H^2 \leq \frac{2\Lambda(p-2)}{(p-1)^2(p+q-2)}, \quad \text{or} \quad b^2 \geq \frac{2\Lambda}{(p-1)(q-1)}. \tag{164}$$

Solutions for fluxes higher than this critical value are stable. For $q = 4$ there is an additional tachyonic mode, which is constrained by:

$$H^2 \geq \frac{2\Lambda[2+q-3pq+(p-1)q^2]}{q(q-3)(p-1)^2(p+q-2)}, \quad \text{or} \quad b^2 \leq \frac{4\Lambda}{q(p-1)(q-3)} \tag{165}$$

So solutions for $q = 4$ are only stable within the range of (164) and (165). For $q \geq 5$, there is no stable solution of the form $dS_p \times S^q$ for any flux value. this means that we should look for alternative solutions if we want to write a theory with 6 or 7 extra dimensions.

6.4 Warped solutions

At the parameter values (b, H), where the solutions of the form $dS_p \times S^q$ are unstable, other solutions may exist that are more stable. In [30] a generalized form of the $dS_p \times S^q$ class solutions is suggested, that includes a warp factor depending on the internal coordinates. In this scenario the q-dimensional internal space is a deformed sphere. The general geometry of such solutions is described by the general metric

$$ds^2 = A^2(y)[-dt^2 + e^{2Ht}d\bar{x}_{p-1}^2] + B^2(y)g_{mn}dy^m dy^n, \tag{166}$$

where both the internal and external space contain a warp factor depending on the internal coordinates y, and g_{mn} is the q-dimensional metric describing the compact internal manifold S^q. Assuming spherical symmetry and using coordinate freedom, to let the warp factors depend on only one internal coordinate, the metric reduces to a simpler form:

$$\begin{aligned} ds^2 &= A(y)^2[-dt^2 + e^{2Ht}d\bar{x}_{p-1}^2] + B(r)^2dr^2 + C(r)^2 d\Omega_{q-1}^2 & (167) \\ &= e^{2\Phi(r)}[-dt^2 + e^{2Ht}d\bar{x}_{p-1}^2] + e^{-\frac{2p}{q-2}}\Phi r[dr^2 + a^2(r)d\Omega_{q-1}^2], & (168) \end{aligned}$$

[9] the $dS_p \times S^q$ solution is stable against vector and tensor perturbations.

and the q-form flux field is again proprotional to the Levi-Civita tensor, but in order to satisfy the Maxwell equations, we need to take into account the specific powers of $A(y)$ and $B(r)$:

$$F_{L_1 \dots L_{q-1}} = \frac{b}{A^p} \cdot B^{q-1} \sqrt{g_{mn}} dr \wedge d\Omega_{q-1}, \tag{169}$$

So in the case described above this becomes:

$$F_{L_1 \dots L_q} = \begin{cases} b \cdot e^{-p\Phi(r)} e^{-\frac{2p(q-1)}{q-2}\Phi(r)} \epsilon_{l_1 \dots l_q} & \text{if} \quad L_i = l_i \\ 0 & \text{otherwise} \end{cases} \tag{170}$$

Setting $\Phi = 0$ returns the trivial solution.

Einstein Maxwell solutions From the Einstein equations we may derive constraints on the Φ and $a(r)$, so we would like to solve the field equations in terms of the functions Φ and $a(r)$. The first step is to plug in our definition of the flux field (170) and its contractions into the field equations (150),(150). The nonzero terms are then given by:

$$R_{tt} = \frac{-1}{p+q-2} e^{2\Phi} \left(b^2 e^{-2p\Phi} + 2\Lambda \right) \tag{171}$$

$$R_{x_i x_i} = \frac{-1}{p+q-2} e^{2\Phi} e^{2Ht} \left((q-1) b^2 e^{-2p\Phi} + 2\Lambda \right) \tag{172}$$

$$R_{rr} = \frac{1}{p+q-2} \left((p-1) b^2 e^{-\frac{2p(q-1)}{q-2} Phi} + 2\Lambda e^{-\frac{2p}{q-2}} \right) \tag{173}$$

$$R_{\varphi_j \varphi_j} = \frac{a^2}{p+q-2} \left((p-1) b^2 e^{-\frac{2p(q-1)}{q-2} Phi} + 2\Lambda e^{-\frac{2p}{q-2}} \right). \tag{174}$$

Note here that we use the fact that the fluxfield is zero within the external dimensions $(t, \bar{x})$ and the levi-civitá tensor has the property that $\epsilon_{\mu_1 \dots \mu_q} \epsilon^{\mu_1 \dots \mu_q} = q!$. To solve the above equations in terms of Φ and $a(r)$, we need to find appropriate expressions for the Christoffel symbols, Riemann- and Ricci tensors. The non-vanishing Christoffel symbols are

$$\begin{aligned}
\Gamma^t_{tr} &= \Phi' \\
\Gamma^t_{x_i x_i} &= H\ e^{2Ht} \\
\Gamma^{x_i}_{x_i t} &= H \\
\Gamma^{x_i}_{r x_i} &= \Phi' \\
\Gamma^r_{tt} &= \Phi'\ e^{(\frac{2p}{q-2}+2)\Phi} \\
\Gamma^r_{x_i x_i} &= -\Phi'\ e^{(\frac{2p}{q-2}+2)\Phi + 2Ht} \\
\Gamma^r_{rr} &= -\left(\frac{p}{q-2} \right) \Phi' \\
\Gamma^r_{\varphi_j \varphi_j} &= -a a' + \left(\frac{p}{q-2} \right) a^2 \Phi' \\
\Gamma^{\varphi_j}_{r \varphi_j} &= \frac{a'}{a} - \left(\frac{p}{q-2} \right) \Phi',
\end{aligned}$$

where the prime defines the derivative with respect to the coordinate r and φ is the variable of the q-shpere, Ω_{q-1}

$$d\Omega^2_{q-1} = \sum_{j=1}^{q-1}\prod_{l=1}^{j-1} \sin^2\varphi_l \mathrm{d}\varphi_j^2. \tag{175}$$

Note that the indices i and j on the x and φ coordinates run over $(p-1)$ and $(q-1)$, respectively. For simplicity we will forget about the Christoffel symbols coming from the internal spherical coordinates, because they do not add interesting information to the theory. From Christoffel symbols we obtain the Ricci tensors: [10]

$$R_{tt} = \left(\Phi'' + (q-1)\frac{a'}{a}\Phi'\right) e^{2\Phi(\frac{p}{q-2}+1)} - (p-1)H^2 \tag{176}$$

$$R_{x_ix_i} = -e^{(\frac{2p}{q-2}+2)\Phi+2Ht}\left(\Phi'' - (q-1)\frac{a'}{a}\Phi\right) + (p-1)H^2e^{2Ht} \tag{177}$$

$$R_{rr} = -(q-1)\frac{a''}{a} + \frac{p(q-1)}{q-2}\frac{a'}{a}\Phi' - \frac{p(p+q-2)}{q-2} + p\Phi'' \tag{178}$$

$$R_{\varphi_1\varphi_1} = \frac{p}{q-2}a^2\Phi'' + \frac{p(q-1)}{q-2}a^2a'\Phi' - (q-2)(a'^2-1)\frac{a''}{a}. \tag{179}$$

Note the implicit summation over the $(p-1)$ x and the $(q-1)$ φ coordinates. Now we equate (171-174) to (176-179) to find expressions for the Einstein equations in terms of the functions Φ and $a(r)$. We obtain a set of 4 equations.

From the R_{tt} equation we get:

$$\Phi'' + (q-1)\tfrac{a'}{a}\Phi' = \tag{180}$$

$$\tfrac{q-1}{p+q-2}b^2e^{-\frac{2p(q-1)}{q-2}\Phi} + (p-1)H^2e^{-\frac{2(p+q-2)}{q-2}\Phi} - \tfrac{2\Lambda}{p+q-2}e^{-\frac{2p}{q-2}\Phi}, \tag{181}$$

From the R_{xx} equations

$$\Phi'' + (q-1)\tfrac{a'}{a}\Phi' = \tag{182}$$

$$\tfrac{q-1}{p+q+2}b^2e^{-\frac{2p}{q-2}\Phi} + (p-1)H^2e^{-\frac{2(p+q-2)}{q-2}\Phi} - \tfrac{2\Lambda}{p+q-2}e^{-\frac{2p}{q-2}\Phi}, \tag{183}$$

From the R_{rr} equations

$$\tfrac{p}{q-2}\left(\Phi'' + (q-1)\tfrac{a'}{a}\Phi' - (p+q-2)\Phi'^2\right) - (q-1)\tfrac{a''}{a} = \tag{184}$$

$$\tfrac{p-1}{p+q+2}b^2e^{-\frac{2p(q-1)}{q-2}\Phi} + + \tfrac{2\Lambda}{p+q-2}e^{-\frac{2p}{q-2}\Phi}, \tag{185}$$

and from the $R_{\varphi\varphi}$ equation

$$\tfrac{p}{q-2}\left(\Phi'' + (q-1)\tfrac{a'}{a}\Phi'\right) - \tfrac{a''}{a} - (q-2)\tfrac{a'^2-1}{a^2} = \tag{186}$$

$$\tfrac{p-1}{p+q+2}b^2e^{-\frac{2p(q-1)}{q-2}\Phi} + \tfrac{2\Lambda}{p+q-2}e^{-\frac{2p}{q-2}\Phi}. \tag{187}$$

[10]The Ricci tensors for general coordinates φ_j include an extra function of spherical coordinates in the last term. For our purposes we will only need the $\varphi_1\varphi_1$ Ricci tensor, but for a derivation, see Appendix D.

We now have a coupled system of second order equations, which we can use to put constraints on the functions Φ and $a(r)$.

To reduce the set of equations, notice that the last two equations are equal on the right hand side. From this it follows that

$$\frac{a''}{a} - \frac{a'^2 - 1}{a^2} = -\frac{p(p+q-2)}{(q-2)^2}\Phi'^2 \tag{188}$$

Another expression for $\frac{a''}{a}$ is found by substation of the R_{xx} equation into the $R_{\varphi\varphi}$ equation. This way we can eliminate the first and second derivatives of Φ, to obtain:

$$\frac{a''}{a} = -(q-2)\frac{a'^2-1}{a^2} - \frac{2\Lambda}{q-2}e^{-\frac{2p}{q-2}\Phi} + \frac{2p(p-1)}{q-2}H^2 e^{-\frac{2(p+q-2)}{q-2}\Phi} + \frac{1}{q-2}b^2 e^{-\frac{2p(q-1)}{q-2}\Phi}. \tag{189}$$

Equating the above equations, we obtain an equation without second derivatives

$$\begin{aligned}(q-1)(q-2)\frac{a'^2-1}{a^2} &= \frac{p(p+q-2)}{q-2}\Phi'^2 + b^2 e^{-\frac{2p(q-1)}{q-2}\Phi} \\ &\quad + p(p-1)H^2 e^{-\frac{2(p+q-2)}{q-2}\Phi} - 2\Lambda e^{-\frac{2p}{q-2}\Phi}.\end{aligned}$$

Effectively there are two independent equations that determine the allowed values of the functions $\Phi(r)$ and $a(r)$, namely (188), (183). The constraint equation obtained above can be rewritten in terms of a normalized cosmological constant, following the notations of [30]. This is done by a rescaling $b \to b\Lambda^{1/2}$, $H \to H\Lambda^{1/2}$, $a \to a\Lambda^{-1/2}$ and $r \to r\Lambda^{-1/2}$.

6.4.1 Stability and spectrum of warped solutions

The above equations determine the values of the functions $\Phi(r)$ and $a(r)$. Numerical solutions have been calculated by the authors of the original article. It turns out that the points of marginal stability from the $dS_p \times S^q$ solutions correspond to the same values for the warped solutions. This is in agreement with the original idea, that the warped solutions emerge from these points [30].

We are interested in the stability and KK spectrum of solutions of this form. Following the same approach as in the previous subsection, we are looking for Einstein solutions to linear perturbations of the background metric. In [31], the authors assume that the both manifolds within the background solutions are Einstein manifolds. One can then decompose the $(p+q)$-dimensional space-time into scalar-type, vector-type and tensor-type components with respect to $g_{\mu\nu}$ and g_{mn}. With this decomposition, one obtains decoupled perturbation equations in each sector. We are only interesed in perturbations, which are scalar-type quantities with respect to both manifolds, $\mathcal{M}_p$ and $\mathcal{N}_q$.

Without getting into the lengthy and difficult calculation that is done by [31], we will use their results to estimate the mass-splitting of the Kaluza Klein spectrum. The masses of the KK modes are defined by the eigenvalues of the Laplacian operator on the scalar harmonics $Y(x)$within the de Sitter space with Hubble expansion rate, H:

$$\nabla^2 Y(x) = \mu^2 Y(x), \tag{190}$$

where μ^2 is the KK mass squared and ∇_μ is the covariant derivative with respect to the metric $g_{\mu\nu}$. If the spectrum of μ^2 is non-negative, we can conclude that the background space-time is dynamically stable. The mass eigenvalues are given by [?]

$$\mu_\pm^2 = -\lambda + \frac{(q-1)(p-2)}{p+q-2}b^2 - (p-1)H^2$$
$$\pm\sqrt{\left[\frac{(q-1)(p-2)}{p+q-2}b^2 - (p-1)H^2\right]^2 + \frac{4(q-1)(p-1)}{p+q-2}b^2\lambda},$$

where λ are the eigenvalues of the Laplacian D^2 on S^q with radius ρ, as defined in the appendix by $D^2Y(y) = -\lambda Y(y)$, and $\lambda = l(l+q-1)\rho^{-2}$. For each multipole moment l, the scalar perturbations thus have two independent modes corresponding to the eigenvalues μ^2. Only one of them is physical however. The other one is a gauge mode.

In the case $l = 0$ we have a physical mode with

$$\mu^2(l=0) = 2\frac{(q-1)(p-2)}{p+q-2}b^2 - 2(p-1)H^2 \tag{191}$$

We are interested in the mass eigenvalues for a scenario that fits our observable world. We will therefore use the relations between the parameters b, H and Λ, in the case $p = 4$, $q = 7$ to estimate the mass spectrum.

With the Hubble expansion rate in 4D, being $70\,\mathrm{kms}^{-1}\,\mathrm{Mpc}^{-1} \sim 10^{-42}\,\mathrm{GeV} << 1$, the parameter relations (161, 162) reduce to

$$6\rho^{-2} = b^2$$
$$6^2\rho^{-2} = 2\Lambda$$

and the KK masses are reduced to $\mu^2 \sim b^2$. Remember that in this particular scenario, there are no branes, so all fields propagate through the extra dimensional space. The compactification scale $R \sim \rho$ is thus bounded by the experimental observations, that have tested the SM fields up to distances of about 10^2 TeV, which corresponds to distances of abot $\sim 10^{-19}\,cm$. This would be the upper limit of the compactification scale. On the other hand, we could assume that $R \sim M_{Pl} \sim 10^{16}$ TeV, as KK supposed. Assuming that the extra dimensional space is more or less spherical and recognizing that $\mu^2 \sim b^2 \sim \rho^{-2}$, we can estimate the upper and lower limit for the KK masses,

$$10^{16}\ \text{ TeV} \geq \mu \geq 10^2\ \text{ TeV}, \tag{192}$$

Thus in this rough approximation, the KK masses should be somewhere between 10^2 and 10^{16} TeV. It should be mentioned though, that there is no particular reason why the extra dimensions should be as large as 10^{-19} cm, besides that it forms an upper bound by observation. Also we did not take into account the warping factor. A good reference and method for calculating KK spectra on general manifolds can be found in [33]. In general, negatively curved manifolds have higher KK masses.

In our opinion, the Planck scale, seems the most natural compactification scale for universal extra dimensions. In that case, the chance of probing the extra dimensions and finding KK gravitons directly is terribly small. At least in the near future.

7 Summary and conclusion

We have discussed several models of extra dimensions, focussing on extensions of Kaluza Klein theory. We started from the the ADD scenario and the Randall Sundrum scenarios and we ended with some solutions to flux compactification.

We have shown how models with large extra dimensions like the ADD scenario and the Randall Sundrum scenario, could help to solve the hierarchy problem between the fundamental scales in nature. Both scenarios introduce branes into the theory, but only the Randall Sundrum scenario takes into account the tension of the brane(s) and the effect of that tension on the internal geometry.

The fact that the ADD scenario does not consider this is one of to weaknesses of the model. Moreover the ADD scenario creates a new hierarchy problem, by using the compactification scale R as a variable to solve the hierarchy problem. The problem has thus been shifted and not solved. For these reasons the ADD scenario is not a very complete or realistic theory in itself.

The Randall Sundrum scenarios are more realistic and complete than the ADD model. However we should take into account that an extra massive scalar field is needed to stabilize the setup in RS1. Also, RS1 describes a universe with 5 space-time dimensions, but it is not extended to higher dimensional theories. The RS2 scenario does not solve the hierarchy problem. However, it does not have the problem of fine-tuning and we have seen that it is extendable to higher dimensions, possibly to D=10 or 11.

Both the ADD scenario and the Randall Sundrum scenarios predict (relatively) light KK modes, that could be observable in high energy experiments. no such particles have been found yet.

Solutions to flux compactification give a very different perspective on extra dimensions. Freund-Rubin compactification could account for the natural compactification of 6 or 7 spatial dimensions, while leaving $(3+1)$ untouched. The search for a stable solution that preserves our 4D observations, has led to the warped ansatz, which seems like an interesting model to do further research on. Other solutions could be possible as well, but have not been explicitly described yet. The fact that no branes are needed in these models, makes it easier to find a stable and solution. However, this implies that extra dimensions should be very small and Kaluza Klein modes will have high masses, that will not be easy to observe.

Within these notes we have focussed on the subject of stability and the Kaluza Klein spectrum of the considered extra dimensional scenarios. We have not been able to discuss the possibility of including the SM within the theory. Neither have we discussed the possibility to extend the theories to realistic string theoretical models. When it comes to realistic model building, the Randall Sundrum is probably the most developed and therefore currently the most interesting. The Kinoshita ansatz seems very interesting as well, but since it is quite young, we can not make a clear statement about it being very realistic or not, yet.

An interesting construction, reproducing the key features of the RS setup within a concrete string theory embedding is discussed in [36]. It shows that the original RS proposal can be realized by considering flux compactifications and branes, including a so-called *Klebanov-Strassler* throat [37]. Such a "warped throat" looks like a RS background plus 5 compact extra dimensions. The question on how to include the SM in this scenario is considered in [38]. Similar constructions, named “ubiquitous stringy throats” [11], have been related to dark matter scenarios [39]. It seems like there is no stopping the creativity in physics when it comes to extra dimensional models.

In the case of the Kinoshita model, further research should tell us, whether this scenario fits in to a string theoretical model as well. We should note however, that even if string theory appears to *not* be true, extra dimensions may still exist.

Obviously, we have only been able to cover just a glimpse of all the theory and scenarios of extra dimensions [12]. Therefore, we can not make a sure statement about what the extra dimensions should look like in general. However, since no (light) Kaluza Klein modes have been observed so far, we do not expect very large extra dimensions to exist. We assume that the Planck Scale is the most natural scale for extra dimensional compactification after all and consequently, we do not expect to see any light KK modes soon.

All in all, to summarize our findings into one catchy conclusion, our statement has become:

> “We ain’t seen nothing yet, and we probably will not for a while”.

Having said that, we do not mean to imply that the topic of extra dimensions should be put aside, due to a lack of observable evidence. On the contrary. We prefer a future scenario in which crazy models of extra dimensions become abundant and physicists create all kinds of excuses to look for extra dimensions of space in every experiment they build.

After all, the excitement of the possibilities is half the fun and what would physics be without something to fantasize about...? Very boring indeed, so let’s keep searching.

[11] The theory is about as exotic as the name itself. It is worth scaring yourself.

[12] Some interesting models, that we have not been able to discuss include ‘massive gravity’, (a good introduction is given by [34]) and infinite volume extra dimensions [35].

A Planckian system of units

We obtain the Planck scale from the *Planckian system of units*, which expresses all quantities in terms of the most important physical constants:

$$\begin{aligned} \hbar &= 1.05457266 \times 10^{-34} \;\; J \cdot s, \\ c &= 2.99792458 \times 10^{8} \;\; m/s, \\ G &= 6.67259 \times 10^{-11} \;\; N \cdot m^2/kg^2. \end{aligned}$$

Using dimensional analysis, we obtain the Planckian units for mass, length, time and energy:

The *Planck Mass*:

$$m_{Pl} \sim \left(\frac{\hbar c}{G}\right)^{1/2} = 2.18 \times 10^{-5} \; g, \tag{193}$$

The *Planck Length*:

$$\ell_{Pl} \sim \left(\frac{\hbar G}{c^3}\right)^{1/2} = 1.62 \times 10^{-33} \; cm, \tag{194}$$

The *Planck time*:

$$t_{Pl} \sim \left(\frac{\hbar G}{c^5}\right)^{1/2} = 5.39 \times 10^{-44} \; sec, \tag{195}$$

The *Planck Energy*:

$$M_{Pl} \sim m_{Pl} c^2 = \left(\frac{\hbar c^5}{G}\right)^{1/2} = 1.22 \times 10^{19} \;\; \text{GeV} \tag{196}$$

The reduced Planck mass is defined as $\bar{M}_{Pl}/\sqrt{8\pi} = 2.4 \times 10^{18}$ GeV.

In natural units we set $c = \hbar \equiv 1$. This way we can express any Planck quantity in terms of the energy scale $\ell_{Pl}^{-1} \sim t_{Pl}^{-1} \sim m_{Pl} \sim M_{Pl}$.

A useful conversion factor is: 1 GeV^{-1} $\sim$ 1,97 $\times 10^{-16}$ m.

B Spheres in arbitrary dimensions

We are interested in the flux of a field through a sphere in arbitrary dimensions, d. Therefore we want to know the volume of the higher dimensional spherical shell.[13] In a d dimensional space $\mathbb{R}^d$, with coordinates $\{x_1, x_2, ..., x_d\}$, we define a $(d-1)$-sphere, $S^{(d-1)}$ of radius R, by the mathematical region

$$S^{(d-1)}(R) = x_1^2 + x_2^2 + ... + x_d^2 = R^2. \tag{197}$$

The corresponding d-Ball, B^d is defined by the region it encloses:

$$B^d(R) = x_1^2 + x_2^2 + ... + x_d^2 \leq R^2, \tag{198}$$

but is not relevant for our purposes.

In arbitrary d, the volume of a sphere of radius R, is related to the volume of a sphere of *unit radius* by:

$$vol[S^{d-2}(R)] = R^{(d-1)} vol[S^{(d-1)}], \tag{199}$$

where the volume of a sphere of unit radius is defined by

$$vol[S^{(d-1)}] = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}, \tag{200}$$

and $\Gamma(\frac{d}{2})$, is the Gamma function:

$$\Gamma(x) = \int_0^\infty dt\ e^{-t} t^{(x-1)}, \qquad x > 0. \tag{201}$$

For d=1 and d=2, the Gamma function $\Gamma(\frac{d}{2})$ is easily calculated:

$$\Gamma(1/2) = \int_0^\infty dt\ e^{-t} t^{(-1/2)} = \sqrt{\pi}, \tag{202}$$

$$\Gamma(1) = \int_0^\infty dt\ e^{-t} t^0 = 1, \tag{203}$$

and for all other values of d, the Gamma function follows from the relation

$$\Gamma(x) = (x-1)\ \Gamma(x-1). \tag{204}$$

Finally, we have arrived at the general volume of a (d-1)-sphere $S^{(d-1)}$ in d-space:

$$vol[S^{(d-1)}(R)] = R^{(d-1)} \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}. \tag{205}$$

The flux of a field at a distance R form the origin, in $D = d+1$ space-time dimensions, is the strength of the field divided by the above volume.

[13]For generality, we use to mathematical definition, *volume*, for the shell. In $d = 3$, a spherical shell corresponds to a 2-dimensional volume, i.e. a surface.

C Perturbation theory

C.1 Linearized Einstein equations

The most general perturbation to the metric would be

$$ds^2 = (g_{\mu\nu} + \delta g_{\mu\nu})dx^\mu dx^\nu + \delta g_{\mu n} dx^\mu dy^n + (g_{mn} + \delta g_{mn})dy^m dy^n. \tag{206}$$

in Martin's analysis [26], the de Donder gauge is chosen, such that the metric perturbations are explicitly

$$\begin{aligned}
\delta g_{\mu\nu} &= \left(2\Psi - \frac{2q}{p-2}\Phi\right) g_{\mu\nu} + h_{\mu\nu}, && (207)\\
\delta g_{\mu\nu} &= 2\Phi g_{mn} + h_{mn}, && (208)\\
\delta g_{\mu n} &= V_{\mu n}. && (209)
\end{aligned}$$

Note that the tensor perturbations are traceless and symmetric.

The linearized Ricci tensor is given by the standard formula

$$\delta R_{MN} = \frac{1}{2}(\nabla_M \nabla_L h^L_N + \nabla_N \nabla_L h^L_M - \nabla_M \nabla_N h^L_L - \nabla_L \nabla_L h^M_N), \tag{210}$$

where the right hand side corresponds to the energy momentum tensor, which sources the curvature. The complete linearization of the source requires some manipulation. The complete calculation can be found in [26].

C.2 Linearized Maxwell equations

The fluctuations of the q-form flux field can be represented by the exterior derivative of a $(q-1)$-form potential $A_{(q-1)}$:

$$\delta F_{(q)} = \mathrm{d}A_{(q-1)}. \tag{211}$$

For the higher dimensional flux fields, the perturbation of $\nabla^N F_{NL_2 \ldots L_q}$ leads to

$$\begin{aligned}
0 &= \delta(g^{MN} \nabla_M F_{NL_2 \ldots L_q})\\
&= \delta[g^{MN}(\partial_M F_{NL_2 \ldots L_q} - \Gamma^K_{MN} F_{KL_2 \ldots L_q} - (q-1)\Gamma^K_{ML_2} F_{NKL_3 \ldots L_q})],
\end{aligned}$$

where the perturbed (linearized) Christoffel symbol is given by

$$\delta \Gamma^P_{MN} = \frac{1}{2} g^{PL}(\nabla_M h_{NL} + \nabla_N h_{ML} - h_{MN}). \tag{212}$$

C.3 Harmonic expansion

The metric fluctuations can be expanded in terms of summations over various modes of spherical harmonics:

$$\Phi(x,y) = \sum_\alpha \Phi^\alpha(x)Y^\alpha(y)$$
$$\Psi(x,y) = \sum_\alpha \Psi^\alpha(x)Y^\alpha(y)$$
$$V_{\mu\nu}(x,y) \sum_\alpha V_\mu^\alpha(x)Y_n^\alpha(y)$$
$$h_{mn}(x,y) = \sum_\alpha h^\alpha(x)Y_{mn}^\alpha(y).$$

Each mode should satisfy the Klein Gordon equation for a massless scalar field

$$[\Box - m_\alpha^2]\Phi^\alpha = 0. \tag{213}$$

Modes with a negative eigenvalue m_α are referred to as tachyonic modes. If such modes exist for some perturbation, we state that the background solution is unstable against that particular perturbation.

In the case of the Freund Rubin solutions, we only care about the scalar modes of the p-dimensional manifold $\mathcal{M}_P$. In this case we will denote the d'Alembetrian operator in the dS_p space by $\Box_x$, corresponding to eigenvalues m^2 and for the extra dimensional space S^q, we will use the Laplacian operate $\Box_y$, corresponding to eigenvalues λ. For q-spheres the eigenvalues of the Laplacian operator are given by $\lambda = l(l+q-1)$, with $l = 0, 1, 2, ...$ Thus l represents a particular mode of the perturbation. A summation over the eigenmodes, represented by l, is in fact the same as summing over the index α in the harmonic expansion.

On the q-sphere of radius ρ, the eigenvalues of the Lapacian are given by $\rho^2\Box_y Y = -\lambda Y$. In practice this amounts to replacing all $\rho^2\Box_y$ by λY. In the external space, we can explicitly write out the above scalar field equation on the internal space following [29]

$$\Box_x\Phi^\alpha - m_\alpha^2\Phi^\alpha = [-\partial_t^2 + (p-1)H\partial_t + e^{-2Ht}\bar{\nabla}^2 - m_\alpha^2]\Phi^\alpha - = 0, \tag{214}$$

where $\bar{\nabla}^2$ is the $(p-1)$-dimensional Laplacian. Taking the Fourier expansion of the scalar field Φ, we obtain a linear differential equation in terms of the Fourier modes. The form of the equation tells us that for $m^2 < 0$, solutions will diverge exponentially.

D Spherically symmetric Ricci tensors

We consider a spherically symmetric Euclidean manifold of q dimensions and radius $r = 1$:

$$ds^2 = d\Omega_q^2, \tag{215}$$

which can written out explicitly as

$$\begin{aligned} d\Omega_{q-1}^2 &= \sum_{j=1}^{q-1}\prod_{l=1}^{j-1} \sin^2\varphi_l \mathrm{d}\varphi_j^2 \\ &= \mathrm{d}\varphi_1^2 + \sin^2\varphi_1 \mathrm{d}\varphi_2^2 + \sin^2\varphi_1 \sin^2\varphi_2 \mathrm{d}\varphi_3^2 + ... + \sin^2\varphi_1 \sin^2\varphi_2 ... \sin^2\varphi_{q-1} \mathrm{d}\varphi_q^2. \end{aligned}$$

The non-zero Christoffel symbols are complicated functions of the different variables, but in the Ricci tensor everything reduces to the simple form

$$R_{\varphi_j\varphi_j} = \prod_{i=1}^{j-1}(q-1)\sin^2\varphi_i \tag{216}$$

or more explicitly, for the case $(q = 4)$

$$\begin{aligned} R_{\varphi_1\varphi_1} &= 3 \\ R_{\varphi_2\varphi_2} &= 3\ \sin^2\varphi_1 \\ R_{\varphi_3\varphi_3} &= 3\ \sin^2\varphi_1 \sin^2\varphi_2 \\ R_{\varphi_4\varphi_4} &= 3\ \sin^2\varphi_1 \sin^2\varphi_2 \sin^2\varphi_3. \end{aligned}$$

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