GROUP-THEORETIC APPROACH TO SCATTERING:

THE DIRAC-COULOMB PROBLEM AND RELATIVISTIC SUPERSYMMETRY

Alfons Stahlhofen* and L. C. Biedenharn**

Department of Physics
University of Texas at Austin
Austin, Texas 78712 USA

ABSTRACT

Recently developed techniques for calculating the S-matrix in noncompact Lie-groups are extended to supersymmetry. The Dirac-Coulomb problem is shown to provide a natural realization of supersymmetry. The treatment given here may be useful in the discussion of nucleus-nucleus interactions and resulting general questions.

I. INTRODUCTION

The fundamental importance of group-theoretic methods in physics is widely recognized. Despite the extensive literature, a group-theoretic approach to scattering theory is relatively recent[1], primarily for the S-matrix of nonrelativistic problems. An extension to relativistic scattering problems has been given.[2] The present discussion develops a further generalization of the group-theoretic approach: to relativistic supersymmetry. We first review the general principles in the abstract setting of a non-compact symmetry group and then sketch the group-theoretic approach to the relativistic scattering of a spin 1/2 particle (Dirac-Coulomb problem). We then re-analyze the treatment and show that, in fact, the whole discussion is based on the supersymmetry of the Dirac-Coulomb problem. This symmetry turns out to be very natural and general, a demonstration of the close connection between supersymmetric quantum mechanics (Susy QM) and the factorization method. The last section contains our conclusions and suggestions for further research.

II. THE GROUP-THEORETIC APPROACH TO SCATTERING

Let us consider a Hamiltonian dynamical system which realizes in Minkowski space a non-compact symmetry group G with Lie algebra L. The group-theoretic approach to scattering identifies the (time-independent) scattering process in terms of asymptotics (in/out-going waves of sharp energy prepared/measured at spatial infinity), with the S-matrix defined as transforming an (asymptotic) in-state into an (asymptotic) out-state. More precisely we abstract the asymptotics by defining the factor space of

---

* Supported in part by the Alexander von Humboldt Stiftung (Feodor-Lynen Program).

** Supported in part by the National Science Foundation.

scattering states (or equivalently the scattering space) to be the sphere bundle $S^2 \times S^0$ obtained by deleting the origin and contracting the radial space ($\mathbb{R}^3 - \{0\} \cong S^2$) such that the radial derivation (describing the in/out motion) limits to the sphere $S^0 (\cong \mathbb{Z}_2)$ with generator R, $R^2 = E$. The generators L of the symmetry group G (which necessarily realize the commutation relations of the Lie algebra in all of Minkowski space) when restricted (asymptotically) to the factor space of scattering states realize two abstractly equivalent sets of generators distinguished by the eigenvalue of $R \to \varepsilon = \pm 1$. That is: $\{L\} \to \{L^\varepsilon\}$ realized on $S^2 \times S^0$.

This construction, viewed abstractly, *implies that the original Lie algebra can be split into two parts:* $L = K + P$, such that under C: $K \to K$, $P \to -P$. Consistency implies that C is abstractly a Cartan involution, which we identify as R when restricted to the scattering space. The introduction of the scattering space leads to an asymptotic equivalence relation defined on the original Hamiltonian eigenstates[3]:

$$< \mathbb{R}^3 | \; E\ell m > \sim \sum_{\varepsilon = \pm 1} A_\varepsilon < S^2 \times \mathbb{Z}_2 \; | \; E\ell m; \;\; \varepsilon = \pm 1 > \tag{1}$$

where the A_ε are numerical constants. Applying ladder operators constructed out of generators of G (that is, L on the left hand side and L^ε on the right hand side of eq.(1)) leads to a recursion relation for the A_ε, which, in turn, implies the S-matrix.

III. THE DIRAC-COULOMB PROBLEM

The Dirac-Coulomb problem is treated most easily in terms of the solutions of an associated second order equation (iterated Dirac-equation), where an especially appropriate basis is chosen.[4] The procedure clearly exhibits the symmetry of the problem.[5] The Dirac equation for a central potential V(r) (using Dirac's notation and setting $\hbar = c = 1$):

$$H_D \psi = (\rho_1 \vec{\sigma} \cdot \vec{p} + \rho_3 m + V(r))\psi = E_D \psi \tag{2}$$

is iterated by using projection operators for positive and negative mass states, written as

$$Q_\pm = \rho_3 (H_D - E_D - \rho_3 m) \pm m \tag{3}$$

and becomes

$$Q_+ Q_- \psi \equiv (H - E)\psi = 0 \; . \tag{4}$$

This construction has the following algebraic properties:

$$\{Q_+, Q_-\} = 2(H - E); \qquad [H, Q_\pm] = 0, \tag{5a}$$

$$Q_\pm^2 = 2(H - E) \pm 2m \, Q_\pm \; . \tag{5b}$$

These equations have a close resemblance to the defining relations of supersymmetric quantum mechanics (see Sect. IV below).

A repulsive Coulomb potential leads in the iterated Dirac equation to a nondiagonal expression of the form $\Gamma(\Gamma + 1)$, where Γ is given by

$$\Gamma = \rho_3 K - i \alpha \rho_1 \vec{\sigma} \cdot \hat{r} \; . \tag{6}$$

(K is Dirac's operator $K \equiv -\rho_3(\vec{\sigma} \cdot \vec{L} + 1)$ with the eigenvalues $K \to \kappa = \pm 1, \pm 2 \ldots$.) Diagonalizing Γ via

$$\tilde{\Gamma} \;\equiv\; S\Gamma S^{-1} = \rho_3 K\,|\,(1-(\alpha/K)^2)|^{\frac{1}{2}} \quad , \tag{7}$$

where S is given by:

$$S \;\equiv\; \exp\,(-\tfrac{1}{2}\rho_2\,\vec{\sigma}\cdot\hat{r}\,\tanh^{-1}\alpha/K) \quad , \tag{8}$$

we obtain the eigenvalues of $\tilde{\Gamma}$:

$$\tilde{\Gamma} \to \gamma \;=\; \pm\,|(\kappa^2-\alpha^2)|^{\frac{1}{2}} \tag{7'}$$

with $\mathrm{sign}(\gamma) \equiv \mathrm{sign}(\kappa)$.

This brings the iterated Dirac-equation to the form:

$$\left(1/r^2\,\partial_r r^2\,\partial_r - \frac{\ell(\gamma)(\ell(\gamma)+1)}{r^2} - \frac{2k\,\alpha_r}{r} + k^2\right) S\,\phi \;=\; 0 \quad , \tag{9}$$

which has the *form* of the radial equation of the *nonrelativistic* Coulomb problem, but *relativistic parameters*, that is, $k^2 = E^2 - m^2$, $\alpha_r = \alpha E/k$, and with *irrational* (!) angular momenta $\ell(\gamma) = |\gamma| + \frac{1}{2}(\mathrm{sign}(\gamma)-1)$. The second order Dirac-Coulomb Hamiltonian commutes with K *and with the relativistic analogue of the Laplace-Runge-Lenz vector*, which has in the frame S the form:

$$B \;=\; k^{-1}\vec{\sigma}\cdot\hat{r}\left(i\hat{r}\cdot\vec{p} + \frac{1+\tilde{\Gamma}}{r}\right)\tilde{\Gamma} \;+\; \alpha_r\,\vec{\sigma}\cdot\hat{r} \quad . \tag{10}$$

The operator B *anticommutes* with K and fulfills the identities

$$B^2 \;=\; \gamma^2 + \alpha_r^2 \quad , \tag{11a}$$

$$B\,|\,E,\kappa,\mu,\rho_3'\,\rangle \;=\; -\,(\gamma^2+\alpha_r^2)^{\frac{1}{2}}\,|E,-\kappa,\mu,\rho_3'\,\rangle \tag{11b}$$

where $|E,\kappa,\mu,\rho_3'\rangle$ denotes a solution of the iterated Dirac equation with a fixed eigenvalue of ρ_3 (the minus sign is a phase convention in the action of $\vec{\sigma}\cdot\hat{r}$ on the eigenkets of $-(\vec{\sigma}\cdot\vec{L}+1)$).

These results show, that B is (when normalized) not only the generator of the symmetry group $\mathbb{Z}_2$ for the (second order) Hamiltonian, but is, furthermore, a *raising/lowering operator* for the radial eigenfunctions, changing $\ell(\gamma)$ to $\ell(\gamma)\pm 1$.

Restricting now to the factor space of scattering states, $S^2\times S^0$ (in the energy subspace E), we see, that B takes the simpler form:

$$B \longrightarrow B^{\mathrm{scatt}} \;=\; i\,\vec{\sigma}\cdot\hat{r}\,R\,\tilde{\Gamma} + \alpha_r\,\vec{\sigma}\cdot\hat{r} \quad , \tag{12}$$

and we have the asymptotic equivalence relations:

$$|E,\kappa,\mu,\rho_3'\rangle \sim A_{-1}\,|E,\kappa,\mu,\rho_3';\;\varepsilon=-1\rangle + A_{+1}|E,\kappa,\mu,\rho_3';\;\varepsilon=+1\rangle \;. \tag{13}$$

Taking definite values for κ and ρ_3 in eq. (13) and operating with B (B^{scatt}) on the left hand side (right hand side) of eq. (13), leads to the scattering matrix for the second order Hamiltonian (that is, one calculates the ratio A_{+1}/A_{-1} of the coefficients).

The asymptotic solutions for the first order (Dirac) equation have the form:

$$|E,\kappa,\mu\rangle \sim Q_-^{\infty}\,S^{-1}\,|\,E,\kappa,\mu,\rho_3'\,\rangle^{\infty} \quad , \tag{14}$$

where Q_{-}^{∞} denotes the limit of Q_{-} (in the "scattering space") and $|E,\kappa,\mu,\rho_3^{\prime}>^{\infty}$ denotes the right hand side of eq. (13). Calculating this expression, one obtains the desired phase shifts δ_{κ}:

(for $\ell \geq 1$)

$$\exp(2i\,\delta_{\kappa=\ell}) = \exp\{i(\ell-\gamma)\pi\}\,\frac{\Gamma(\gamma+1+i\alpha_r)}{\Gamma(\gamma+1-i\alpha_r)}\exp(2i\phi_{\kappa})\Delta(k)\,, \tag{15a}$$

(for $\ell \geq 0$)

$$\exp(2i\delta_{\kappa=-\ell-1}) = \exp\{i(\ell+1-\gamma)\pi\}\,\frac{\Gamma(\gamma+i\,\alpha_r)}{\Gamma(\gamma-i\,\alpha_r)}\,\exp(-2i\phi_{\kappa})\,\Delta(k)\,, \tag{15b}$$

where

$$\phi_{\kappa} = \tan^{-1}\left(\frac{k}{E+m}\,\frac{|\kappa|-\gamma}{\alpha}\right) \tag{15c}$$

and $\Delta(k)$ denotes an energy dependent constant. It can be shown, that this result is identical to the standard expression[6].

IV. THE SUPERSYMMETRY OF THE DIRAC-COULOMB PROBLEM

The symmetry of the nonrelativistic Coulomb problem (SO(3,1)) is broken in the case of a Dirac electron (by fine structure splitting) to the obvious symmetry $\overline{SO(3)\times\mathbb{Z}_2}$, where (see Sect. III) $\mathbb{Z}_2$ is generated by the (normalized) operator

$$B = k^{-1}\left(\gamma^2 + \frac{\alpha^2E^2}{k^2}\right)^{-\frac{1}{2}}\vec{\sigma}\cdot\hat{r}\,\left|\left(i\,\hat{r}\cdot\vec{p} + \frac{1+\tilde{\Gamma}}{r}\right)\tilde{\Gamma} + \alpha E\right|\,. \tag{10'}$$

The operator B (also commuting with the first order Hamiltonian) was first given by Johnson and Lippmann (in a different representation) as:

$$B = \alpha\,\vec{\sigma}\cdot\hat{r} - i\,K\rho_1(H - \rho_3 m)/m\,, \tag{16}$$

which can be written in the compact form:

$$B = i\,\rho_1\,(KH/m - \Gamma)\,. \tag{16'}$$

Remark: Normalizing B as

$$B' = (m^2 - H_D^2)^{-\frac{1}{2}}\,B \tag{17}$$

allows one--just as in the nonrelativistic case--to give an algebraic expression in the frame S for the eigenvalues of the Hamiltonian:

$$H_D = m\left(1 + \alpha^2/(\tilde{B}' + \tilde{\Gamma})^2\right)^{-\frac{1}{2}}\,, \tag{18}$$

(where $\tilde{B}' = SB'S^{-1}$; this result is valid in higher dimensions, too.[7])

As seen from the explicit construction in Section III, the derivation of the iterated equation can be written in terms of commutators and anti-commutators[20]. This is a characteristic feature of Susy QM, which is defined by:

$$\{O_+,\,O_-\} = H\,, \tag{19a}$$

$$[H,\,O_+] = [H,\,O_-] = 0\,, \tag{19b}$$

$$O_+^2 = O_-^2 = 0\,. \tag{19c}$$

A typical and well known representation of this algebra has the form:

$$O_\pm = (p \pm i\,\phi(x))\, a^\pm \quad , \tag{20a}$$

where the fermionic operators $a^\pm$ satisfy

$$\{a^+, a\} = 1; \quad (a^+)^2 = (a^-)^2 = 0 \quad . \tag{20b}$$

The charges $O_\pm$ act on the eigenstates of

$$H\psi = H \begin{pmatrix} \psi_-^n \\ \psi_+^n \end{pmatrix} = E_n \psi, \tag{20c}$$

as

$$O_\pm\, \psi_\mp^n = \pm i \sqrt{E_n}\; \psi_\pm^n \quad ,$$

where the index n is *not* confined to discrete values[8].

It is now easy to show that the components of the operator B in eq. (10') are the building blocks needed for the construction of the second order Hamiltonian and at the same time constitute a realization of the algebra (19). This establishes the supersymmetry of the second order Dirac-Coulomb Hamiltonian.

The Coulomb 'helicity operator' B can be written as[4]:

$$B = k^{-1} \left(\gamma^2 + \frac{\alpha^2 E^2}{k^2} \right)^{-\frac{1}{2}} \rho_3 \vec{\sigma}\cdot\hat{r}\,\hat{\Gamma} \left(i\, \hat{r}\cdot\vec{p} + \frac{1+\hat{\Gamma}}{r} + \frac{\alpha E}{\hat{\Gamma}} \right) \quad . \tag{21}$$

This notation uses the fact, that--as indicated by the explicit structure of the iterated Dirac equation (eq. (9))--the explicit solutions in the frame S have the form:

$$\phi \sim \begin{pmatrix} F_{\ell(\gamma),E}(r) \;\; \chi_{+\kappa}^{\mu} \\ F_{\ell(-\gamma),E}(r) \;\; \chi_{-\kappa}^{\mu} \end{pmatrix} \quad . \tag{22}$$

Since the operator B in (21) is a product of operators acting separately on the angular part (i.e. $\vec{\sigma}\cdot\hat{r}$) and the radial part of the wavefunction, the (unnormalized) Susy-generators $O_\pm$ can be defined as:

$$O_\pm \equiv \left(i\, \hat{r}\cdot\vec{p} + \frac{1+\hat{\Gamma}}{r} + \frac{\alpha E}{\hat{\Gamma}} \right) \left(\frac{1}{2} (\rho_1 \pm i\rho_2) \right) \quad , \tag{23}$$

which leads to (using the two signs of γ):

$$\{O_+, O_-\} = \left(\frac{1}{r^2}\,\partial_r\, r^2\, \partial_r - \frac{\ell(\gamma)(\ell(\gamma)+1)}{r^2} - \frac{\alpha^2 E^2}{\gamma^2} - \frac{2\alpha E}{r} \right) \quad . \tag{24}$$

But this is already the desired result: Using the normalization of the helicity operator B, too, leads immediately to the iterated Dirac-Coulomb equation

$$\begin{aligned} \{O_+, O_-\}\phi &= \left(\frac{1}{r^2}\,\partial_r\, r^2 \partial_r - \frac{\ell(\gamma)(\ell(\gamma)+1)}{r^2} - \frac{\alpha^2 E^2}{\gamma^2} - \frac{2\alpha E}{r} \right) \phi \\ &= \left(E^2 - m^2 - \frac{\alpha^2 E^2}{\gamma^2} \right) \phi \quad . \end{aligned} \tag{25}$$

Furthermore, the recursion relations

$$\left(\partial_r + \frac{1+\gamma}{r} + \frac{\alpha E}{\gamma}\right) F_{\ell(\gamma)}(r) = \left(k^2 - \frac{\alpha^2 E^2}{\gamma^2}\right)^{\frac{1}{2}} F_{\ell(-\gamma)}(r) \quad , \qquad (26)$$

include both typical ladder operator relations of the algebra (19) (see eq. (20)).

In this way the Dirac-Coulomb problem becomes a natural example for the hidden supersymmetry of the Dirac operator.[9] The supersymmetry of the Dirac-Coulomb problem is in the case of the discrete eigenvalues well established[10]; the present discussion shows there exists an extension to scattering. The construction of the supercharges $O_{\pm}$ used the fact, that the algebra (19) is exactly the algebraic structure realized in the factorization method[11,12,13]. The recursion relations (26)--which established the ladder operator property of the Susy generators $O_{\pm}$--are nothing else than the radial equations of the Dirac-Coulomb problem in the factorization scheme[14]. Traditionally, the factorization method was restricted to a solution of the eigenvalue problem of second order, determining the discrete eigenvalues. But it is easy to show, that the whole scheme of the factorization method remains valid when considering continuous eigenvalue problems. This result has been independently proven by many authors (see ref. 15, 16; ref. 13 contains an extensive literature list).

The construction of Sect. III, to conclude, provides a solution of the Dirac-Coulomb scattering problem which is entirely based on the symmetry properties of the problem.

V. CONCLUDING REMARKS

We have shown, by example, that recently developed techniques for a group-theoretical determination of the S-matrix can, in a natural way, be extended from ordinary Lie-groups to a (simple) superalgebra, generated by the conserved quantities of the system. The close connection between this supersymmetric quantum mechanics and the factorization method establishes this extension, we feel, as a natural construction. Such a connection may be useful to prove an old conjecture of Mc Intosh[17] on the existence of an analogue to the Coulomb helicity operator B for all exactly solvable Dirac equations with central potentials. (This is known for special cases; an interesting example is the 'relativistic equivalent oscillator'[18].) There are hints, that supersymmetry might be useful in the algebraic construction and discussion of nucleus-nucleus potentials and the general discussion of scattering problems.[19] Our method might be useful in the treatment of such models and in the discussion of general questions (Levinson theorems, phase equivalent potentials, ...).

References:

1. F. Iachello in B. Gruber and R. Lenczewski (eds.): "Symmetries in Science II", Plenum Press (1986); 1, 247.

2. J. Wu, et al, Journal of Physics A, in press.

3. L.C. Biedenharn, A. Stahlhofen in: Proc. XV International Coll. on Group-Theoretical Meth. in Physics, Drexel Univ., Philadelphia, PA, 1986 (to appear).

4. L.C. Biedenharn, Phys. Rev. 126 (1962) 845.

5. S. Kim, Journ. Math. Phys. 21 (1980) 2286.

6. N.F. Mott and H.S.W. Massey, "Theory of Atomic Collisions,", Clarendon Press (1956).

7. L.A. Coulson and A. Joseph, Rev. Mod. Phys. 39 (1967) 838.

8. R. Akhoury, A. Comtet, Nucl. Phys. B246 (1984) 255.

9. L. O'Raifeartaigh in B. Gruber and R. Lenczewski (eds.): "Symmetry in Science II", Plenum Press (1986); 591.

10. P.D. Jarvis, G.E. Stedman, Journ. Phys. 19A (1986) 1373.

11. A.A. Andrianov, et al, Phys. Lett. 105A (1984) 19; M.M. Nieto, Phys. Lett. 145B (1984) 208.

12. C.V. Sukumar, Journ. of Phys. 18A (1985) 657.

13. A. Stahlhofen, Ph.D. Thesis, Bonn (1986), unpublished.

14. L. Infeld, T.E. Hull, Rev. Mod. Phys. 23 (1951) 21(section 8).

15. M. Humi, Journ. Math. Phys., 9 (1968) 1258.

16. J.J. Duistermaat, F.A. Gruenbaum, Comm. Math. Phys. 103 (1986) 177.

17. H.V. McIntosh in E. Loebl (ed.): "Group Theory and it's Applications II", Academic Press (1972).

18. E. Chaffin, Journ. Math. Phys. 14 (1973) 977.

19. J.C. Olivio, et al, Phys. Lett. 159B (1985) 307; D. Boye, Phys. Rev. Lett. 58 (1987) 2738; A. Bohm, M. Kmiecik, L. Boya, Austin preprint (1987), Center for Particle Theory, Univ. of Texas at Austin, Austin, TX 78712.

For earlier work see the review by A.O. Barut, to appear in: A. Bohm, Y. Ne'eman, A.O. Barut (eds.): "Dynamical Groups and Spectrum Generating Algebras", World Scientific (December 1987).

20. This algebra is a special case in the general classification of V.G. Kac (Adv. Math. 26 (1977)8).