ATYPICAL MODULES OF THE LIE SUPERALGEBRA $gl(m/n)$

J. Van der Jeugt[1] (University of Ghent, Belgium), J.W.B. Hughes (Queen Mary and Westfield College, U.K.), R.C. King (University of Southampton, U.K.) and J. Thierry-Mieg (University of Montpellier, France)[2]

Let $G = G_{\bar{0}} \oplus G_{\bar{1}}$ be the *general linear* Lie superalgebra $gl(m/n)$ [2] consisting of complex matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ of size $(m+n)^2$. The *even* subspace $G_{\bar{0}}$ of G consists of the matrices $\left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$ and the *odd* subspace $G_{\bar{1}}$ consists of the matrices $\left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right)$. The *bracket* between homogeneous elements is defined by $[a,b] = ab - (-1)^{\alpha\beta} ba$ for $a \in G_\alpha, b \in G_\beta$ $(\alpha, \beta \in \{\bar{0}, \bar{1}\} = \mathbb{Z}_2)$. Thus the even subalgebra is isomorphic to $gl(m) \oplus gl(n)$. G admits a consistent $\mathbb{Z}$-grading $G = G_{-1} \oplus G_0 \oplus G_{+1}$ where $G_0 = G_{\bar{0}}$, G_{+1} is the space of matrices of the form $\left(\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right)$ and G_{-1} is the space of matrices of the form $\left(\begin{smallmatrix} 0 & 0 \\ C & 0 \end{smallmatrix}\right)$. The *special linear* Lie superalgebra $sl(m/n)$ is the subalgebra of $gl(m/n)$ consisting of matrices with vanishing supertrace. In what follows we put $G = gl(m/n)$, but all of the results can be reformulated for $sl(m/n)$ as well.

The *Cartan subalgebra* H of G consists of the subspace of diagonal matrices. The *root* or *weight space* H^* is the dual space of H and is spanned by the forms ϵ_i $(i = 1,\ldots,m)$ and δ_j $(j = 1,\ldots,n)$. The inner product on the weight space H^* is given by [5] $\langle \epsilon_i | \epsilon_j \rangle = \delta_{ij}$, $\langle \epsilon_i | \delta_j \rangle = 0$, $\langle \delta_i | \delta_j \rangle = -\delta_{ij}$, where δ_{ij} is the usual Kronecker symbol. In this $\epsilon\delta$-basis the even roots of G are of the form $\epsilon_i - \epsilon_j$ or $\delta_i - \delta_j$, and the odd roots are of the form $\pm(\epsilon_i - \delta_j)$. Let Δ denote the set of all roots, Δ_0 the set of even roots and Δ_1 the set of odd roots. As a system of *simple roots* one takes the so-called *distinguished set* [3] $\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_m - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n$. Then the set Δ^+ of *positive* roots consists of the elements $\epsilon_i - \epsilon_j$ $(i < j)$, $\delta_i - \delta_j$ $(i < j)$ and $\epsilon_i - \delta_j$. Now the notations Δ_0^+ and Δ_1^+ are obvious; in particular :

$$\Delta_1^+ = \{\beta_{ij} = \epsilon_i - \delta_j, \quad i = 1,\ldots,m, \quad j = 1,\ldots,n\}. \tag{1}$$

All simple modules (i.e. irreducible representations) of the classical simple Lie superalgebras were classified by Kac [3]. Kac's result specified to $sl(m/n)$ implies that every finite-dimensional simple G-module V is a highest weight module $V(\Lambda)$ specified by an *integral dominant* weight Λ. A weight $\Lambda \in H^*$ is said to be integral dominant if and only if its so-called *Kac-Dynkin labels* $\Lambda = [a_1, a_2, \ldots, a_{m-1}; a_m; a_{m+1}, \ldots, a_{m+n-1}]$ are such that $a_i \in \mathbb{N}$ for $i \neq m$ whereas a_m can be any complex number. For our purpose it is sufficient to consider only those Λ for which $a_m \in \mathbb{Z}$. If Λ is expressed in terms of the $\epsilon\delta$-basis as $\Lambda = \sum \mu_i \epsilon_i + \sum \nu_j \delta_j$, then the Kac-Dynkin labels of Λ are given by $a_i = \mu_i - \mu_{i+1}$ $(i < m)$, $a_m = \mu_m + \nu_1$, $a_{m+j} = \nu_j - \nu_{j+1}$ $(j < n)$. Note that the coordinates in the $\epsilon\delta$-basis represent a unique weight of $gl(m/n)$ whereas the Kac-Dynkin labels represent a unique weight of $sl(m/n)$ rather than $gl(m/n)$. Often it will be useful to represent a weight Λ by a composite Young diagram, consisting of

[1]Research Associate of the NFWO (National Funds for Scientific Research of Belgium)
[2]Talk presented by J. Van der Jeugt

the diagrams of $\{\mu\}$ and $\{\overline{\nu}\}$ in appropriate positions [5]. For example, for $gl(4/6)$ and $\Lambda = (7,6,6,3|\bar{1},\bar{1},\bar{3},\bar{3},\bar{5},\bar{5})$ in the $\epsilon\delta$-basis (where $\bar{k}$ stands for $-k$), the composite Young diagram is shown in (5). In this case, for example, the Kac-Dynkin labels of Λ are [1,0,3;2;0,2,0,2,0].

The basic problem we are concerned with is the determination of the weights and weight multiplicities of $V(\Lambda)$. Such information is contained in the so-called *character* of $V(\Lambda)$, which is by definition equal to $\text{ch}V(\Lambda) = \sum_\eta (\text{dim}V_\eta)\, e^\eta$, where $\text{dim}V_\eta$ is the multiplicity of the weight η appearing in the weight space decomposition $V(\Lambda) = \bigoplus_\eta V_\eta$. Recall that for a (reductive) Lie algebra G_0 (in the present case we can think of G_0 as the even part $gl(m) \oplus gl(n)$ of $gl(m/n)$) the character formula of a G_0-module $V_0(\Lambda)$ with highest weight Λ is given by Weyl's character formula :

$$\text{ch}V_0(\Lambda) = L_0^{-1} \sum_{w \in W} \varepsilon(w)\, w\left(e^{\Lambda+\rho_0}\right), \qquad L_0 = \prod_{\alpha\in\Delta_0^+} (e^{\alpha/2} - e^{-\alpha/2}), \tag{2}$$

where W is the Weyl group of G_0, $\varepsilon(w)$ is the *signature* of $w \in W$ and $\rho_0 = \frac{1}{2}\sum_{\alpha\in\Delta_0^+} \alpha$.

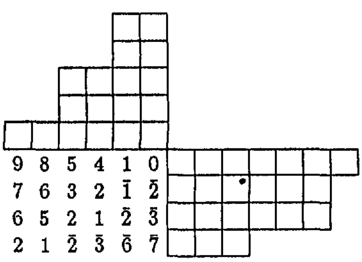
A very important finite-dimensional highest weight module $\overline{V}(\Lambda)$, the so-called Kac-module, was introduced in [3]. For given integral dominant weight Λ, the G_0-module $V_0(\Lambda)$ is uniquely determined (up to isomorphism), and can be extended to a $G_0 \oplus G_{+1}$-module by putting $G_{+1}V_0(\Lambda) = 0$. Then one defines the induced module

$$\overline{V}(\Lambda) = \text{Ind}_{G_0\oplus G_{+1}}^{G} V_0(\Lambda) \cong U(G_{-1}) \otimes V_0(\Lambda). \tag{3}$$

It follows from the structure of $U(G_{-1})$ that

$$\text{ch}\overline{V}(\Lambda) = \chi_K(\Lambda) = L_0^{-1} \sum_{w\in W} \varepsilon(w)\, w\Big(e^{\Lambda+\rho_0} \prod_{\beta\in\Delta_1^+} (1 + e^{-\beta})\Big). \tag{4}$$

When is $\overline{V}(\Lambda)$ a *simple* module? The answer to this question was given by Kac [3] : $\overline{V}(\Lambda)$ is simple if and only if $\langle\Lambda + \rho|\beta\rangle \neq 0$ for all β in Δ_1^+. Herein $\rho = \rho_0 - \rho_1$, where ρ_0 has been defined previously and $\rho_1 = \frac{1}{2}\sum_{\beta\in\Delta_1^+}\beta$. In the case that all $\langle\Lambda + \rho|\beta\rangle \neq 0$, Λ and $V(\Lambda) = \overline{V}(\Lambda)$ are said to be *typical*, otherwise Λ and $V(\Lambda) \neq \overline{V}(\Lambda)$ are said to be *atypical*. If Λ is atypical, $\overline{V}(\Lambda)$ contains a unique maximal submodule $M(\Lambda)$ and $V(\Lambda) \cong \overline{V}(\Lambda)/M(\Lambda)$. Our main aim is to determine characters for such atypical modules. One of the useful tools in studying atypical modules is the so-called *atypicality matrix* $A(\Lambda)$ consisting of the mn integers $A(\Lambda)_{ij} = \langle\Lambda + \rho|\beta_{ij}\rangle$ [4], where β_{ij} has been defined in (1). In terms of its components in the $\epsilon\delta$-basis, $A(\Lambda)_{ij}$ is given by $\mu_i + \nu_j + m - i - j + 1$. The $m \times n$ atypicality matrix fits nicely into the compositie Young diagram, as is illustrated here for our example, $\Lambda = (7,6,6,3|\bar{1},\bar{1},\bar{3},\bar{3},\bar{5},\bar{5})$ for $gl(4/6)$:



(5)

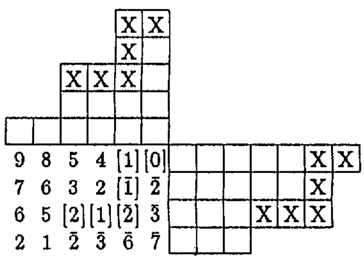
This Λ is atypical of type $\beta_{1,6}$, and is *singly atypical.*

Various character formulae for atypical $V(\Lambda)$ have been proposed, most of which are of the following form (see [4,5] and references therein) :

$$\chi_{\Delta(\Lambda)}(\Lambda) = L_0^{-1} \sum_{w \in W} \varepsilon(w)\, w\Big(e^{\Lambda+\rho_0} \prod_{\beta \in \Delta(\Lambda)} (1 + e^{-\beta})\Big), \tag{6}$$

where $\Delta(\Lambda)$ is some subset of Δ_1^+. In particular, Bernstein and Leites [1] proposed $\Delta(\Lambda) = \{\beta \in \Delta_1^+ \mid \langle\Lambda + \rho|\beta\rangle \neq 0\}$, in which case $\chi_{\Delta(\Lambda)}(\Lambda)$ in (6) is replaced by $\chi_L(\Lambda)$. However, counterexamples were found to their formula. Similarly, counterexamples were found to other formulae of the type (6) [5], and in particular we were able to prove that for $G = gl(3/4)$ and $\Lambda = [1,1;0;0,1,0]$ no set $\Delta(\Lambda)$ exists yielding the correct character of $V(\Lambda)$. Hence no formula of the type (6) can give correctly the characters of all simple modules $V(\Lambda)$ of $gl(m/n)$.

There is, however, the important class of singly atypical modules where the problem of finding character formulae has been solved [4]. When there is only one γ in Δ_1^+ with $\langle\Lambda + \rho|\gamma\rangle = 0$ (and $\langle\Lambda + \rho|\beta\rangle \neq 0$ for all $\beta \neq \gamma$), Λ is singly atypical. In this case, we proved that the maximal submodule $M(\Lambda)$ is itself a simple G-module, and that $M(\Lambda) \cong V(\Phi)$, where $\Phi = w \cdot (\Lambda - k\gamma) = w(\Lambda - k\gamma + \rho) - \rho$ and $\Lambda - k\gamma$ is the first element of the sequence $\Lambda - \gamma, \Lambda - 2\gamma, \ldots$ that can be mapped into an integral dominant weight Φ by means of a $w\cdot$ action. In terms of the composite Young diagram, with the zero in the atypicality matrix at position (i,j), we move to the end of row i in the μ-part of the diagram and to the end of column j in the ν-part of the diagram, and perform a strip removal of length k in both parts of the diagram, removing one box at a time until the composite diagram is *standard* [5]. In our example (5) this leads to the following strip removal :



(7)

Note that only after 6 box removals, the remaining Young diagrams are standard. Thus $\Phi = w \cdot (\Lambda - 6\beta_{1,6})$ for some $w \in W$, and it follows that $\Phi = \Lambda - (\beta_{1,6} + \beta_{1,5} + \beta_{2,5} + \beta_{3,5} + \beta_{3,4} + \beta_{3,3}) = (5,5,3,3|\bar{1},\bar{1},\bar{2},\bar{2},\bar{2},\bar{4})$. Both strip removals (indicated by X's) are necessarily of the same shape, and the positions of the brackets [ ] in the atypicality matrix (which constitute the same shape again) determine the β_{ij} one has to subtract from Λ in order to obtain Φ. Also, Φ is atypical of type $\beta_{3,3}$, which corresponds to the "tail" of the removal strip. Making use of these properties, of combinatorial properties of the atypicality matrix, and of recursion, one is able to prove [4] that for the singly atypical case $\mathrm{ch}V(\Lambda) = \chi_L(\Lambda)$. Then, making a formal expansion, one can rewrite the character as an infinite alternating series of Kac-characters $\chi_K(\lambda)$:

$$\mathrm{ch}V(\Lambda) = \chi_L(\Lambda) = \sum_{t=0}^{\infty}(-1)^t \chi_K(\Lambda - t\gamma). \tag{8}$$

Let us now return to the more general case of *multiply atypical modules*. For reasons of presentation we shall illustrate here the case of doubly atypical modules. Thus $\langle \Lambda + \rho|\beta_1\rangle = 0$, $\langle \Lambda + \rho|\beta_2\rangle = 0$, and $\langle \Lambda + \rho|\beta\rangle \neq 0$ for every $\beta \neq \beta_1, \beta_2$. Similarly as in (8) one can formally expand the Bernstein-Leites formula as an infinite alternating series of Kac-characters :

$$\chi_L(\Lambda) = \sum_{t_1,t_2=0}^{\infty} (-1)^{t_1+t_2} \chi_K(\Lambda - t_1\beta_1 - t_2\beta_2) = \sum_{C_\Lambda}(-1)^{|\Lambda-\lambda|}\chi_K(\lambda), \tag{9}$$

where $C_\Lambda = \{\lambda = \Lambda - t_1\beta_1 - t_2\beta_2\}$ is the "cone" with vertex Λ and $(-1)^{|\Lambda-\lambda|} = (-1)^{t_1+t_2}$. Let $\beta_1 = \epsilon_i - \delta_j$ and $\beta_2 = \epsilon_k - \delta_l$ with $i > k$ and $j < l$. Then there is a unique w_{12} in W which permutes the components i and k, $m + j$ and $m + l$, and leaves all the other components of a weight in the $\epsilon\delta$-basis invariant. Let $H_{12} = \{\eta \in H^*|w_{12} \cdot (\eta) = \eta\}$. Clearly, such a hyperplane splits the weight space H^* into two half-spaces. The *truncated cone* C_Λ^+ is defined to be the set of weights of C_Λ that are in the same half-space as Λ. Then we conjecture : $\mathrm{ch}V(\Lambda) = \chi_L(\Lambda) = \sum_{C_\Lambda}(-1)^{|\Lambda-\lambda|}\chi_K(\lambda)$ if Λ is not critical, and $\mathrm{ch}V(\Lambda) = \sum_{C_\Lambda^+}(-1)^{|\Lambda-\lambda|}\chi_K(\lambda)$ if Λ is critical, where Λ is *critical* if and only if the entry $A(\Lambda)_{k,j}$ in the atypicality matrix is equal to the "hook length" connecting the two zeros (at positions (i,j) and (k,l)) in the atypicality matrix, i.e. equal to $i - k + l - j - 1$ [5]. The ways in which this conjecture has been tested, and how it works for atypical modules with degree of atypicality > 2 is described in [5].

Let us emphasize that the given formulae are expansions of $\mathrm{ch}V(\Lambda)$ in terms of the formal characters $\chi_K(\lambda)$, which are characters of Kac-modules when λ is dominant integral. One may also consider the inverse problem : given the Kac-module $\overline{V}(\Lambda)$, how can $\mathrm{ch}\overline{V}(\Lambda)$ be expressed as a (necessarily finite) sum of characters of simple modules $\mathrm{ch}V(\sigma)$? In other words, what are the non-zero multiplicities n_σ in the expression $\mathrm{ch}\overline{V}(\Lambda) = \sum_\sigma n_\sigma \mathrm{ch}V(\sigma)$? This is known as the problem of the determination of the composition series of $\overline{V}(\Lambda)$. Recently, we have made a lot of progress in solving this question. Our results concerning the determination of the composition factors of the Kac-module $\overline{V}(\Lambda)$ were presented at this Colloquium by R.C. King, who reports on it elsewhere in this Volume.

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