PTEP

Prog. Theor. Exp. Phys. **2013**, 073B03 (32 pages)
DOI: 10.1093/ptep/ptt052

Localization method for volume of domain-wall moduli spaces

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Received April 10, 2013; Revised May 12, 2013; Accepted May 15, 2013; Published July 1, 2013

The volume of the moduli space of non-Abelian Bogomol'nyi–Prasad–Sommerfield (BPS) domain walls is obtained exactly in $U(N_c)$ gauge theory with N_f matters. The volume of the moduli space is formulated, without an explicit metric, by a path integral under constraints on BPS equations. The path integral over fields reduces to a finite-dimensional contour integral by a localization mechanism. Our volume formula satisfies a Seiberg-like duality between moduli spaces of the $U(N_c)$ and $U(N_f - N_c)$ non-Abelian BPS domain walls in a strong coupling region. We also find a T-duality between domain walls and vortices on a cylinder. The moduli space volume of non-Abelian local ($N_c = N_f$) vortices on the cylinder agrees exactly with that on a sphere. The volume formula reveals various geometrical properties of the moduli space.

Subject Index A11, B34, B35

1. Introduction

A moduli space of Bogomol'nyi–Prasad–Sommerfield (BPS) solitons, which is a space of parameters describing positions, orientations, and sizes, is important in understanding the properties of BPS solitons themselves. For example, the metric of the moduli space is important for seeing scatterings among BPS solitons.

The volume of the moduli space is essentially obtained from an integral of a volume form, which is constructed by the metric, over the moduli space. A local structure of the moduli space is smeared out by the volume integration, but the volume of the moduli space still has significant information on the dynamics of BPS solitons. The volume of the moduli space is directly proportional to a thermodynamical partition function of a many-body system of BPS solitons. The thermodynamics of vortices is investigated by evaluation of the volume of the moduli space [1–5].

The volume of the moduli space of BPS solitons also tells us non-perturbative dynamics in supersymmetric gauge field theories. Nekrasov has shown that one of the non-perturbative corrections in $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions can be obtained from a volume of the moduli space of self-dual Yang–Mills instantons [6] by using a *localization method*, developed in [7,8]. The localization method has recently become more important in investigating the non-perturbative dynamics of supersymmetric gauge theories through exact partition functions. The exact partition function of supersymmetric gauge theory is essentially proportional to the volume of the moduli space of the BPS solitons, which produce the non-perturbative corrections.

It is very difficult to construct an explicit metric of the moduli space of BPS solitons in general [5], so the calculation of the volume of the moduli space is also difficult. However, we do not need an explicit metric on the moduli space to evaluate the volume in the localization method. This fact comes from the integrability and supersymmetry behind the BPS solitons. Indeed, the supersymmetry is closely related to *equivariant cohomology*, which plays an important role in the mathematical formulation of the localization method. Then, the localization method is very useful in calculating the volume of the moduli space and extends a range of applicable cases in the volume calculation of the BPS soliton moduli space.

The advantage of the localization method in calculating the volume has been shown in the calculation of the volume of the instanton moduli space, which gives the non-perturbative corrections in four-dimensional supersymmetric gauge theory [6]. The localization method has then been applied to evaluate the volume of the moduli space of non-Abelian BPS vortices [9,10]. The results from the localization method agree perfectly with the previous results using the other method, and we could extend to more complicated systems, where the metric of the moduli space is not explicitly known.

In this paper, we calculate the volume of the moduli space of the non-Abelian BPS domain walls, which is described by first-order differential equations for matrix- and vector-valued variables, where the matrices are in adjoint representations of $U(N_c)$ and N_f sets of the vectors are in fundamental representations of $U(N_c)$. We consider the BPS equations of the domain walls on a finite line interval with boundaries. Solutions of the BPS domain-wall equations depend on the boundary conditions. So we need to carefully treat the boundary conditions to consider the moduli spaces of the BPS domain walls. The differential equations of the domain walls can be regarded as BPS equations in supersymmetric gauge theory with $U(N_c)$ gauge group and N_f flavors (matters) in the fundamental representation. The domain walls are soliton-like objects with co-dimension one in supersymmetric gauge theory. We are interested in the moduli space of the BPS equations only, so we do not assume an explicit supersymmetric system in the calculation of the volume.

We utilize the localization method associated with the equivariant cohomology in mathematics in order to evaluate the volume of the moduli space of the BPS domain walls. The localization method is essentially equivalent to an evaluation of a field-theoretical partition function of some constrained system. A path integral of the partition function is restricted on the moduli space of the domain walls. We again emphasize that we need the constraints of the BPS equations, but do not need an explicit metric of the moduli space in this localization method.

The path integral which gives the volume of the moduli is localized at the fixed points of a symmetry, which is a part of the supersymmetry. This symmetry is called a Becchi–Rouet–Stora–Tyutin (BRST) symmetry and is related to the equivariant cohomology. In the evaluation of the path integral, it is necessary to know the number of zero modes of the fields. We find that the number of zero modes is determined by the boundary conditions, and is given by a Callias-like index theorem with boundary. After counting the zero modes explicitly, we find that the path integral reduces to a usual contour integral and a simple formula is obtained for the volume of the moduli space of the BPS domain walls. For non-Abelian gauge theories, we find that the contour integral reduces to a sum of products of the Abelian gauge theories with non-trivial signs. The sign of each product in the sum could not be determined by the localization method itself. We assume that the signs are determined by a topological index (intersection number) of the profile of the solution. Then, the sum of products is expressed by the determinant of a simple matrix depending on the boundary conditions.

In order to check our volume formula for the moduli space of the BPS domain walls, we discuss dualities between various systems of the domain walls. First of all, we investigate the duality between the moduli spaces of the non-Abelian BPS domain walls in the strong coupling (asymptotic) region. We find that the moduli spaces of the domain walls of $U(N_c)$ and $U(\tilde{N}_c)$ differ from each other in general, but if $\tilde{N}_c$ is given by $N_f - N_c$, then we expect that the moduli spaces (and its volume) coincide with each other in the strong coupling region [11,12]. We can conclude that our results agree with the expected dualities. Secondly, we show that there exists a T-dual relation between the domain walls and vortices on a cylinder [13]. The domain walls and vortices have different co-dimensions, but if we consider the domain walls winding along a circle direction of the cylinder, the volume of the moduli space can be regarded as that of the moduli space of the vortices on the cylinder [14]. The winding number of the domain walls corresponds to a vortex charge. We find that the volume of the moduli space of the vortices on the cylinder coincides with that of the vortices on the sphere if $N_c = N_f$ (non-Abelian local vortex). These non-trivial duality relations support that our volume formula for the moduli space of the BPS domain walls works correctly.

This paper is organized as follows: In the next section, we explain a general argument on the volume calculation of the moduli space of the BPS equations. We introduce a path integral over the constrained system to evaluate the volume without the explicit metric. In Sect. 3, we evaluate the path integral to see that it is localized at fixed points of the BRST symmetry, and reduces to a simple contour integral. In Sect. 4, we explicitly evaluate the contour integral for various examples of domain walls in Abelian and non-Abelian gauge theories. In order to check our results for the volume of the moduli space of the BPS domain walls, we consider two kinds of dualities of the moduli spaces in Sects. 5 and 6. The last section is devoted to conclusion and discussion.

2. Volume of moduli space

We take the $U(N_c)$ gauge theory with the gauge field A_μ, together with a real scalar field Σ in the adjoint representation and N_f complex scalar field H_r^A, $r = 1, \dots, N_c$, $A = 1, \dots, N_f$ in the fundamental representation. The gauge coupling and the Fayet–Iliopoulos (FI) parameter are denoted as g and ζ, respectively. Since we are interested in domain walls, we consider our theory in one spatial dimension. Let us consider the BPS equations for domain walls [11,15,16] in a finite interval $y \in [-\frac{L}{2}, \frac{L}{2}]$:

$$\mu_r \equiv \mathcal{D}_y \Sigma - \frac{g^2}{2}(\zeta \mathbf{1}_{N_c} - HH^\dagger) = 0, \tag{1}$$

$$\mu_c \equiv \mathcal{D}_y H + \Sigma H - HM = 0, \tag{2}$$

$$\mu_c^\dagger \equiv \mathcal{D}_y H^\dagger + H^\dagger \Sigma - MH^\dagger = 0, \tag{3}$$

where Σ, H, and $H^\dagger$ are $N_c \times N_c$, $N_c \times N_f$, and $N_f \times N_c$ matrix-valued functions of y, respectively, and the covariant derivatives are defined by $\mathcal{D}_y \Sigma = \partial_y \Sigma + \mathrm{i}[A_y, \Sigma]$, $\mathcal{D}_y H = \partial_y H + \mathrm{i} A_y H$ and $\mathcal{D}_y H^\dagger = \partial_y H^\dagger - \mathrm{i} H^\dagger A_y$. The mass matrix M is taken to be diagonal as $M = \mathrm{diag}(m_1, m_2, \dots, m_{N_f})$ and ordered as $m_1 < m_2 < \cdots < m_{N_f}$ without loss of generality.

Domain-wall solutions are defined by specifying vacuum at the left and right boundaries. Vacua of the system are labeled by choosing N_c out of N_f flavors [11,15,16], such as $(A_1, \dots, A_{N_c})$, with $A_1 < A_2 < \cdots A_{N_c}$. Let us consider domain-wall solutions connecting the vacuum $(A_1, \dots, A_{N_c})$ at the left boundary $y = -L/2$ and the vacuum $(B_1, \dots, B_{N_c})$ at the right boundary $y = L/2$. For finite intervals, we demand the following boundary condition at the left boundary $y = -L/2$:

$$\Sigma\left(-\frac{L}{2}\right) = \mathrm{diag}(m_{A_1}, m_{A_2}, \ldots, m_{A_{N_c}}), \tag{4}$$

$$H^A_{r=A_r} = 0, \quad A < A_r. \tag{5}$$

Similarly, at the right boundary $y = L/2$, we demand

$$\Sigma\left(\frac{L}{2}\right) = \mathrm{diag}(m_{B_1}, m_{B_2}, \ldots, m_{B_{N_c}}), \tag{6}$$

$$H^A_{r=B_r} = 0, \quad A > B_r. \tag{7}$$

Since Weyl permutations are a part of gauge invariance, we need to combine possible Weyl permutations of these boundary conditions.

The BPS equations (1), (2), and (3) with the above boundary conditions produce soliton-like objects which are localized on the one-dimensional interval and connect field configurations specified by the label of indices $\vec{A} = (A_1, \ldots, A_{N_c})$ and $\vec{B} = (B_1, \ldots, B_{N_c})$. Since these BPS solitons have unit co-dimension and are constructed using a non-Abelian gauge theory, these BPS solitons are called *non-Abelian domain walls*.

The moduli space of domain walls is defined by a space of parameters of solutions of the BPS equations with identification up to gauge transformations. Hence, the moduli space is represented by a quotient space by the $U(N_c)$ gauge identification

$$\mathcal{M}^{N_c,N_f}_{\vec{A}\to\vec{B}} = \frac{\mu_r^{-1}(0) \cap \mu_c^{-1}(0) \cap \mu_c^{\dagger -1}(0)}{U(N_c)}, \tag{8}$$

where $\mu_r^{-1}(0)$, $\mu_c^{-1}(0)$, and $\mu_c^{\dagger -1}(0)$ stand for the space of solutions of the BPS equations $\mu_r = \mu_c = \mu_c^\dagger = 0$ with the boundary conditions labeled by $\vec{A}$ and $\vec{B}$ at $y = -L/2$ and $y = L/2$, respectively. This quotient space is known to be a Kähler quotient space, and μ_r, μ_c, and $\mu_c^\dagger$ are called moment maps in this sense.

The volume of the moduli space is usually defined by an integral of the volume form over the whole moduli space with $2n$-dimensional coordinates x,

$$\mathrm{Vol}\left(\mathcal{M}^{N_c,N_f}_{\vec{A}\to\vec{B}}\right) = \int_{\mathcal{M}^{N_c,N_f}_{\vec{A}\to\vec{B}}} \mathrm{d}^{2n}x\sqrt{\det g_{ij}}, \tag{9}$$

if we know a metric of the moduli space g_{ij}. However, it is difficult to find the metric of the moduli space explicitly in general.

To avoid a direct integration of the volume form on the moduli space, we note that the Kähler manifold admits the Kähler form Ω and the volume form on the Kähler quotient space can be written in terms of Ω as $\mathrm{d}^{2n}x\sqrt{\det g_{ij}} = \frac{1}{n!}\Omega^n$. On the moduli space, the volume is expressed by

$$\mathrm{Vol}\left(\mathcal{M}^{N_c,N_f}_{\vec{A}\to\vec{B}}\right) = \int_{\mathcal{M}^{N_c,N_f}_{\vec{A}\to\vec{B}}} \mathrm{e}^{\Omega}, \tag{10}$$

under the condition that the integral exists only on the $2n$-form.

We can also express the volume integral (10) by a path integral over all field configurations with suitable constraints onto the moduli space $\mathcal{M}^{N_c,N_f}_{\vec{A}\to\vec{B}}$,

$$\mathrm{Vol}\left(\mathcal{M}^{N_c,N_f}_{\vec{A}\to\vec{B}}\right) = \frac{1}{\mathrm{Vol}(\mathcal{G})}\int \mathcal{D}\Phi\,\mathcal{D}\vec{\mathcal{B}}_v\,\mathcal{D}\vec{\mathcal{F}}_v\,\mathcal{D}^2\vec{\mathcal{B}}_m\,\mathcal{D}^2\vec{\mathcal{F}}_m\,\mathrm{e}^{-S_0}, \tag{11}$$

where $\vec{\mathcal{B}}_v = (A_y, \Sigma)$ and $\vec{\mathcal{F}}_v = (\lambda_y, \xi)$ are vectors of bosonic and fermonic fields in the adjoint representation, $\vec{\mathcal{B}}_m = (H, Y_c)$ and $\vec{\mathcal{F}}_m = (\psi, \chi_c)$ are vectors of bosonic and fermonic fields in the fundamental representation, and $\mathrm{Vol}(\mathcal{G})$ is the volume of the $U(N_c)$ gauge transformation group $\mathcal{G}$. Precisely speaking, the definition of the volume of the moduli space via the path integral has an ambiguity corresponding to an ambiguity in the definition of the normalization of the metric (Kähler form) of the moduli space. We will discuss this point later.

We choose an "action" S_0 to give constraints on the moduli space, which are achieved by integrating over Lagrange multiplier fields Φ, Y_c, and $Y_c^\dagger$, and introduce fermions λ_y, ξ, ψ, $\psi^\dagger$, χ_c, and $\chi_c^\dagger$ to give a suitable Kähler form on the moduli space and Jacobians for the constraints. Inspired by the general discussion in [9,10], we take the following action S_0,

$$\begin{aligned} S_0 = \mathrm{i}\beta\int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}y\,\mathrm{Tr}\Bigg[&\Phi\mu_r - \lambda_y\xi + \mathrm{i}\frac{g^2}{2}\psi\psi^\dagger + Y_c^\dagger\mu_c \\ &- \chi_c^\dagger\left(\frac{\delta\mu_c}{\delta A_y}\lambda_y + \frac{\delta\mu_c}{\delta\Sigma}\xi + \frac{\delta\mu_c}{\delta H}\psi\right) + (\mathrm{h.c.})\Bigg], \end{aligned} \tag{12}$$

in order to impose the constraints, and to give the Kähler form and Jacobians in the path integral over the field configurations. We also introduced a parameter β with a dimension of length. Thus, the volume of the domain-wall moduli space is evaluated by the path integral over fields like a partition function of a gauge field theory. The role of the Lagrange multiplier field Φ is rather special compared to other fields. We treat the path integral over Φ separately from other fields.

We can evaluate the integral (11) directly by using an established field theoretical procedure as performed in [9,10]. However, once we notice that the action S_0 possesses an extra symmetry (BRST symmetry), we can evaluate the path integral (11) via the so-called localization method (cohomological field theory) much more easily than the direct evaluation of the path integral. We will see that the path integral (11) is localized at the fixed-point sets of the BRST symmetry and is reduced to a finite-dimensional integral.

3. Localization in field theory

To proceed with the evaluation of the path integral (11), we introduce the following fermionic transformations (BRST transformations) for the vector fields (fields in the adjoint representation),

$$\begin{aligned} &QA_y = \lambda_y, \quad Q\lambda_y = -\mathcal{D}_y\Phi, \\ &Q\Sigma = \xi, \quad Q\xi = \mathrm{i}[\Phi, \Sigma], \\ &Q\Phi = 0, \end{aligned} \tag{13}$$

and for the matter fields (fields in the fundamental representation),

$$\begin{aligned} &QH = \psi, \quad Q\psi = \mathrm{i}\Phi H, \\ &QY_c = \mathrm{i}\Phi\chi_c, \quad Q\chi_c = Y_c, \end{aligned} \tag{14}$$

and for their hermitian conjugates. We see that a square of this transformation generates a gauge transformation $\delta_G(\Phi)$ with Φ as the transformation parameter: $Q^2 = \delta_G(\Phi)$. This means that Q^2

is nilpotent on gauge-invariant operators $\mathcal{O}$. If we restrict a space of operator fields to the gauge-invariant ones, Q gives a cohomology by the identification

$$\mathcal{O} \sim \mathcal{O} + Q(\text{gauge inv. op.}), \tag{15}$$

which is called the *equivariant cohomology*.

Under this transformation, we find that the action S_0 is invariant (Q-closed),

$$QS_0 = 0. \tag{16}$$

We also find that the action S_0 can be written by

$$S_0 = \mathrm{i}\beta \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}y\, \mathrm{Tr}\left[\Phi\left(\mathcal{D}_y\Sigma - \frac{g^2\zeta}{2}\mathbf{1}_{N_c}\right) - \lambda_y\xi\right] + \mathrm{i}\beta Q \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}y\, \mathrm{Tr}\left[-\frac{g^2}{2}\psi H^\dagger + \mu_c\chi_c^\dagger + \chi_c\mu_c^\dagger\right]. \tag{17}$$

Here we imposed the periodic boundary condition for the product $\Phi\xi$ in order to preserve the BRST invariance for the action. So an essential cohomological part (Q-closed but not Q-exact) of the action S_0 is

$$S_{\mathrm{coh}} = \mathrm{i}\beta \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}y\, \mathrm{Tr}\left[\Phi\left(\mathcal{D}_y\Sigma - \frac{g^2\zeta}{2}\mathbf{1}_{N_c}\right) - \lambda_y\xi\right], \tag{18}$$

in terms of the equivariant cohomology.

Using the nature of the BRST symmetry, we can add an extra Q-exact action $Q\Xi$ to S_0 without changing the path integral; that is, the deformed path integral

$$\mathrm{Vol}\left(\mathcal{M}_{\vec{A}\to\vec{B}}^{N_c,N_f}\right) = \frac{1}{\mathrm{Vol}(\mathcal{G})}\int \mathcal{D}\Phi\,\mathcal{D}\vec{\mathcal{B}}_v\,\mathcal{D}\vec{\mathcal{F}}_v\,\mathcal{D}^2\vec{\mathcal{B}}_m\,\mathcal{D}^2\vec{\mathcal{F}}_m\,\mathrm{e}^{-S_0 - tQ\Xi}, \tag{19}$$

is independent of a deformation parameter (coupling) t since the path integral measure is constructed to be Q-invariant. In the $t \to 0$ limit, the path integral (19) reduces to the original one which gives the volume of the moduli space. When we choose the deformation parameter t appropriately, we can evaluate the path integral exactly.

To evaluate the path integral, we choose Ξ to be the following form:

$$tQ\Xi = t_1Q\Xi_1 + t_2Q\Xi_2, \tag{20}$$

where

$$\Xi_1 = \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}y\, \mathrm{Tr}\left[\mu_c\chi_c^\dagger + \chi_c\mu_c^\dagger\right], \tag{21}$$

$$\Xi_2 = \frac{1}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}y\mathrm{Tr}\left[\vec{\mathcal{F}}_m\cdot(Q\vec{\mathcal{F}}_m^\dagger) + \vec{\mathcal{F}}_m^\dagger\cdot(Q\vec{\mathcal{F}}_m)\right]. \tag{22}$$

The former $Q\Xi_1$ is already included in the original action S_0 and gives a δ-functional constraint on $\mu_c = \mu_c^\dagger = 0$ by integrating out the Y_c and $Y_c^\dagger$. This constraint means that the field configuration must satisfy

$$\mathcal{D}_y H + \Sigma H - HM = 0 \tag{23}$$

for the bosonic field H. The fermionic fields in the fundamental representation must strictly obey the equation of motion

$$\mathcal{D}_y \psi + \Sigma\psi - \psi M = 0, \tag{24}$$

$$\mathcal{D}_y \chi_c - \chi_c \Sigma + M\chi_c = 0. \tag{25}$$

As we will see later, the above constraints for the fields in the fundamental representation are important for counting the number of zero modes of the fields at the localization point.

First of all, we introduce the Cartan–Weyl basis $(H_a, E_\alpha, E_{-\alpha})$ of the Lie algebra $\mathfrak{u}(N_c)$ and decompose the fields in the adjoint representation as follows:

$$\Phi = \sum_{a=1}^{N_c} \Phi^a \mathtt{H}_a + \sum_{\alpha>0} \Phi^\alpha \mathtt{E}_\alpha + \sum_{\alpha>0} \Phi^{-\alpha} \mathtt{E}_{-\alpha}, \tag{26}$$

$$A_y = \sum_{a=1}^{N_c} A_y^a \mathtt{H}_a + \sum_{\alpha>0} A_y^\alpha \mathtt{E}_\alpha + \sum_{\alpha>0} A_y^{-\alpha} \mathtt{E}_{-\alpha}, \tag{27}$$

$$\Sigma = \sum_{a=1}^{N_c} \Sigma^a \mathtt{H}_a + \sum_{\alpha>0} \Sigma^\alpha \mathtt{E}_\alpha + \sum_{\alpha>0} \Sigma^{-\alpha} \mathtt{E}_{-\alpha}, \tag{28}$$

where $\mathtt{H}_a$, $\mathtt{E}_\alpha$, and $\mathtt{E}_{-\alpha}$ satisfy the following commutation relations:

$$[\mathtt{H}_a, \mathtt{H}_b] = 0, \tag{29}$$

$$[\mathtt{H}_a, \mathtt{E}_\alpha] = \alpha_a \mathtt{E}_\alpha, \tag{30}$$

$$\mathrm{Tr}\, \mathtt{E}_\alpha \mathtt{E}_\beta = \delta_{\alpha+\beta,0}, \tag{31}$$

and $\mathtt{E}_{-\alpha} = \mathtt{E}_\alpha^\dagger$.

To perform the path integral, we introduce the ghosts c and $\bar{c}$ for the diagonal gauge-fixing condition ($\Phi^\alpha = 0$). The ghosts induce the action

$$S_{\text{ghost}} = \mathrm{i}\beta \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}y\, \mathrm{Tr}\, c[\Phi, \bar{c}], \tag{32}$$

which gives a one-loop determinant

$$\prod_{\alpha>0} (\beta\alpha(\Phi))^{2f_\alpha^{\text{adj}}}. \tag{33}$$

where f_α^{adj} represents the degree of freedom for each off-diagonal component of a real fermion in one dimension. In this gauge choice, the bosonic term for the Q-closed action (18) can be written as

$$S_{\text{coh}}|_{\text{bosonic}} = \mathrm{i}\beta \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}y \left[\sum_{a=1}^{N_c} \Phi^a \left(\partial_y \Sigma^a - \frac{g^2\zeta}{2} \right) + \mathrm{i} \sum_\alpha \alpha(\Phi) A_y^{-\alpha} \Sigma^\alpha \right]. \tag{34}$$

The path integral over off-diagonal elements $(A_y^\alpha, \Sigma^\alpha)$ leads to the one-loop determinant for bosonic fields in the vector multiplet,

$$\prod_{\alpha>0} |\beta\alpha(\Phi)|^{-2b_\alpha^{\text{adj}}}. \tag{35}$$

From (33) and (35), we obtain the one-loop determinant for off-diagonal elements in the adjoint representation

$$\prod_{\alpha>0} \frac{(\beta\alpha(\Phi))^{2f_\alpha^{\text{adj}}}}{|\beta\alpha(\Phi)|^{2b_\alpha^{\text{adj}}}}. \tag{36}$$

Naively, scalar fields and vector fields carry the same degrees of freedom in one dimension, so we can conclude that $b_\alpha^{\text{adj}} = f_\alpha^{\text{adj}}$, that is, the one-loop determinants for the adjoint fields are canceled out up to a sign ± 1. It is difficult to determine the sign of the one-loop determinant at this stage, but we will assume later that this sign depends on permutations of the boundary conditions. We can non-trivially check that this assumption is consistent and leads to correct answers to the volume and dualities of the domain walls.

Next, we evaluate the one-loop determinant of the fields in the fundamental representation. The matter fields enter into the action through the Q-exact term:

$$Q\,\Xi_2 = \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}y \left[\mathrm{i} \sum_{a=1}^{N_c} (H_a^\dagger \Phi^a H_a + \chi_{c,a}^\dagger \Phi^a \chi_{c,a}) + \mathrm{i}\psi^\dagger \psi + \mathrm{i} Y_c^\dagger Y_c \right]. \tag{37}$$

The matter action is quadratic with respect to the field in the fundamental representation, so we can perform the path integral and obtain the one-loop determinant:

$$\prod_{a=1}^{N_c} (\mathrm{i}\Phi^a)^{f_a^{\text{fund}} - b_a^{\text{fund}}}. \tag{38}$$

Here, f_a^{fund} and b_a^{fund} are the degrees of freedom for the fundamental fields $\chi_{c,a}$ and H_a, respectively.

On the other hand, since fields in the fundamental representation originally obey the constraints (23)–(25), when we define the differential operator for the general fields Ψ_a and $\tilde{\Psi}_a$ in the fundamental representation by

$$P_a \Psi_a \equiv \mathcal{D}_y \Psi_a + \Sigma \Psi_a - \Psi_a M, \tag{39}$$

and

$$\tilde{P}_a \tilde{\Psi}_a \equiv \mathcal{D}_y \tilde{\Psi}_a - \Sigma \tilde{\Psi}_a + \tilde{\Psi}_a M, \tag{40}$$

the fields $(H_a, H_a^\dagger)$ and $(\chi_{c,a}, \chi_{c,a}^\dagger)$ should be expanded by the eigenmodes for the operators P_a and $\tilde{P}_a$, respectively. Since P_a and $\tilde{P}_a$ are adjoint for each other, their eigenmodes coincide, including the degeneracy, and their difference in Eq. (38) is canceled out except for the zero modes. Thus, we find that the difference of the number of modes for the fields in the fundamental representation is characterized by the dimensions of the zero modes, i.e. the index

$$\text{ind}\, P_a \equiv \dim\ker P_a - \dim\ker \tilde{P}_a. \tag{41}$$

The one-loop determinant of the matter fields becomes

$$\prod_{a=1}^{N_c} \frac{1}{(\mathrm{i}\Phi^a)^{\text{ind}\, P_a}}. \tag{42}$$

Note that the index of P_a depends only on the boundary condition of Σ^a, similarly to the Callias index theorem [18]. We will show how to compute this index for various examples in the next section.

Thus, the path integral (19) reduces to that of a direct product $U(1)^{N_c}$ of Abelian gauge theories after the off-diagonal components of the fields are integrated out,

$$\mathrm{Vol}\left(\mathcal{M}^{N_c,N_f}_{\vec{A}\to\vec{B}}\right) = \frac{1}{\mathrm{Vol}(\mathcal{H})N_c!}\prod_{a=1}^{N_c}\int \mathcal{D}\Phi^a\mathcal{D}A^a_y\mathcal{D}\Sigma^a\mathcal{D}\lambda^a_y\mathcal{D}\xi^a\frac{1}{(\mathrm{i}\Phi^a)^{\mathrm{ind}\,P_a}}\mathrm{e}^{-S^a_{\mathrm{coh}}[\Phi^a,\Sigma^a]}, \tag{43}$$

where

$$S^a_{\mathrm{coh}}[\Phi^a,\Sigma^a] = \mathrm{i}\beta\int_{-\frac{L}{2}}^{\frac{L}{2}}\mathrm{d}y\left[\Phi^a\left(\partial_y\Sigma^a - \frac{g^2\zeta}{2}\right) - \lambda^a_y\xi^a\right], \tag{44}$$

and $\mathrm{Vol}(\mathcal{H})$ is the volume of the gauge transformation group of $\mathcal{H} = U(1)^{N_c}$. The pre-factor $1/N_c!$ comes from the order of the Weyl permutation group in the original $U(N_c)$ gauge group.

To perform the path integral (43) of the $U(1)^{N_c}$ gauge theory, we choose a gauge $A^a_y = 0$ and expand Σ^a around a specific profile function Σ^a_0 by

$$\Sigma^a(y) = \Sigma^a_0(y) + \tilde{\Sigma}^a(y), \tag{45}$$

where Σ^a_0 satisfies the given boundary condition at $y = -\frac{L}{2}$ and $y = \frac{L}{2}$. We note that there still exists a degree of freedom of the Weyl permutation group after fixing the gauge and the "classical" background profile Σ^a_0 satisfying the boundary condition.

A partial integration over the fluctuations $\tilde{\Sigma}^a$ of the action (44) gives the constraint $\partial_y\Phi^a = 0$, as expected from the localization. So the path integral over $\Phi^a(y)$ reduces to an integration over constant modes[1] ϕ_a.

In the original non-Abelian gauge theory, the boundary condition is chosen to be $\Sigma(-\frac{L}{2}) = \mathrm{diag}(m_{A_1}, m_{A_2}, \ldots, m_{A_{N_c}})$ and $\Sigma(\frac{L}{2}) = \mathrm{diag}(m_{B_1}, m_{B_2}, \ldots, m_{B_{N_c}})$ up to the Weyl permutations. For a given Weyl permutation σ, the boundary condition for the background profile Σ^a_0 becomes $\Sigma^a_0(-\frac{L}{2}) = m_{A_{\sigma(a)}}$ and $\Sigma^a_0(\frac{L}{2}) = m_{B_{\tilde{\sigma}(a)}}$, where $\sigma(a)$ and $\tilde{\sigma}(a)$ are elements of the permutation group $\mathfrak{S}_{N_c}$. The above choice of boundary condition gives the classical value of the action at the fixed points as

$$\begin{aligned} S^a_{\mathrm{coh}}[\phi_a,\Sigma^a_0] &= \mathrm{i}\beta\phi_a\int_{-\frac{L}{2}}^{\frac{L}{2}}\mathrm{d}y\left\{\partial_y\Sigma^a_0 - \frac{g^2\zeta}{2}\right\} \\ &= \mathrm{i}\beta\phi_a\left\{(m_{B_{\tilde{\sigma}(a)}} - m_{A_{\sigma(a)}}) - \frac{g^2\zeta}{2}L\right\}, \end{aligned} \tag{46}$$

for the permutation σ and $\tilde{\sigma}$.

Using this evaluation of the cohomological action at the fixed points, we obtain the integral formula for the volume of moduli space of the domain walls after integrating out all fluctuations of the fields,

$$\mathrm{Vol}\left(\mathcal{M}^{N_c,N_f}_{\vec{A}\to\vec{B}}\right) = \frac{1}{N_c!}\sum_{(\sigma,\tilde{\sigma})\in(\mathfrak{S}_{N_c})^2}\prod_{a=1}^{N_c}\int_{-\infty}^{\infty}\frac{\mathrm{d}\phi_a}{2\pi}\frac{(-1)^{|\sigma||\tilde{\sigma}|}}{(\mathrm{i}\phi_a)^{\mathrm{ind}\,P_a}}\mathrm{e}^{\mathrm{i}\beta\phi_a\left\{\hat{L}-(m_{B_{\tilde{\sigma}(a)}}-m_{A_{\sigma(a)}})\right\}}, \tag{47}$$

where we define

$$\hat{L} \equiv \frac{g^2\zeta}{2}L, \tag{48}$$

and introduce the sign dependence which is determined by the order of the permutations $|\sigma|$ and $|\tilde{\sigma}|$. As explained before, the sign dependence coming from the one-loop determinant is not obvious, but we will see that this assumption works well and passes non-trivial checks in the later discussions.

[1] We use a subscript of the Cartan indices a for the constant modes to simplify later equations.

Since Eq. (47) depends only on the relative permutation between σ and $\tilde{\sigma}$, a sum over one permutation simply cancels $1/N_c!$ and only a sum over the relative permutation remains:

$$\mathrm{Vol}\left(\mathcal{M}^{N_c,N_f}_{\vec{A}\to\vec{B}}\right) = \sum_{\sigma\in\mathfrak{S}_{N_c}} \prod_{a=1}^{N_c} \int_{-\infty}^{\infty} \frac{\mathrm{d}\phi_a}{2\pi} \frac{(-1)^{|\sigma|}}{(\mathrm{i}\phi_a)^{\mathrm{ind}\,P_a}} \mathrm{e}^{\mathrm{i}\beta\phi_a\left\{\hat{L}-(m_{B_{\sigma(a)}}-m_{A_a})\right\}}. \tag{49}$$

We will apply this formula, which is written by an integral over the constant modes of Φ^a and a summation over the Weyl permutation group of the boundary conditions, to explicitly evaluate various examples of domain walls in the next section.

4. Various examples

4.1. Abelian domain walls

In this section, we give some examples of the volume of the moduli space of domain walls following the general formula (49). A key to evaluating the volume concretely is computation of the index of the operator P_a. We will see that the index is obtained from the (topological) profile of the function $\Sigma(y)$.

We first show how to evaluate the volume of the domain-wall moduli space for Abelian gauge theories. The integral formula (49) for non-Abelian gauge theories is essentially a direct product of Abelian gauge theories, except for the existence of the permutations, thanks to the localization. Then, if we obtain the volume of moduli space of the Abelian domain walls, we can easily extend it to the non-Abelian case. So we would like to explain carefully here a detail of the Abelian case.

To make the example more explicit, we consider the case $N_c = 1$ (Abelian) and 4 flavors $N_f = 4$. The mass for H and $H^\dagger$ can be set as $M = \mathrm{diag}(m_1, m_2, m_3, m_4)$ with $m_1 < m_2 < m_3 < m_4$ without loss of generality. We also impose the boundary condition $\Sigma(-\frac{L}{2}) = m_1$ and $\Sigma(\frac{L}{2}) = m_4$ as the first example.

Applying the integral formula in Eq. (49) to the case of $N_c = 1$ and $N_f = 4$, we obtain, for this example,

$$\mathrm{Vol}\left(\mathcal{M}^{1,4}_{1\to4}\right) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\phi}{2\pi} \frac{1}{(\mathrm{i}\phi)^{\mathrm{ind}\,P}} \mathrm{e}^{\mathrm{i}\phi\beta\left(\hat{L}-(m_4-m_1)\right)}, \tag{50}$$

where we suppressed the suffix a in ϕ_a, P_a and so on, since $a = 1$ for the $N_c = 1$ case. To perform this integral, we have to determine the index of P defined in Eq. (41).

4.1.1. Counting of zero modes.

Let us first consider a differential equation

$$P\Psi_i = \partial_y\Psi_i + \sum_{j=1}^{4} A_{ij}(y)\Psi_j = 0, \tag{51}$$

where $A_{ij}(y) \equiv (\Sigma(y) - m_i)\delta_{ij}$. We define the kernel of P as "normalizable" modes of the solution of the above differential equation $\Psi_i(y)$. Although the term "normalizable" is used here, it is actually not determined by the convergent normalization of the mode function, but is determined by physical considerations as described below. We will also give a mathematically more precise definition later.

In order to find these normalizable or non-normalizable modes concretely, let us assume simply that a profile of $\Sigma(y)$ is a straight line,

$$\Sigma(y) = \frac{d}{L}y + \overline{m}, \tag{52}$$

where $d \equiv m_4 - m_1$ and $\overline{m} \equiv \frac{m_1+m_4}{2}$. Using this profile, we can solve the differential equation (51) and (55). The result is

$$\Psi(y) = \left(C_1 \exp^{-\frac{d}{2L}(y+\frac{L}{2})^2}, C_2 \exp^{-\frac{d}{2L}\left(y-\frac{L(m_2-\overline{m})}{d}\right)^2}, C_3 \exp^{-\frac{d}{2L}\left(y-\frac{L(m_3-\overline{m})}{d}\right)^2}, C_4 \exp^{-\frac{d}{2L}(y-\frac{L}{2})^2} \right),$$

$$\tilde{\Psi}(y) = \left(\tilde{C}_1 \exp^{\frac{d}{2L}(y+\frac{L}{2})^2}, \tilde{C}_2 \exp^{\frac{d}{2L}\left(y-\frac{L(m_2-\overline{m})}{d}\right)^2}, \tilde{C}_3 \exp^{\frac{d}{2L}\left(y-\frac{L(m_3-\overline{m})}{d}\right)^2}, \tilde{C}_4 \exp^{\frac{d}{2L}(y-\frac{L}{2})^2} \right), \quad (53)$$

with integration constants $C_i, \tilde{C}_i, i = 1, \ldots, 4$. Since $d > 0$, all the solutions of $\tilde{\Psi}$ rapidly diverge at the boundary of the interval when L is sufficiently large. We call these divergent modes for large L "non-normalizable". On the other hand, the functions in $\Psi(y)$ are Gaussian and damp well at the boundary. We classify these modes as "normalizable". The number of normalizable modes is four in Ψ for any size L of the interval. These observations imply that $\dim \ker P = 4$ and $\dim \ker \tilde{P} = 0$. So we find $\text{ind}\, P = \dim \ker P - \dim \ker \tilde{P} = +4$.

We need to be careful when we consider other boundary conditions where the profile of $\Sigma(y)$ does not reach some of the values of the masses. For instance, if we consider a different boundary condition $\Sigma(-\frac{L}{2}) = m_2$ and $\Sigma(\frac{L}{2}) = m_4$, the profile of $\Sigma(y)$ does not reach at $\Sigma = m_1$ and $A_{11}(y) = \Sigma(y) - m_1$ is always positive. In this case, the function $\Psi_1(y)$ behaves as

$$\Psi_1(y) = C \setminus \exp^{-\frac{d'}{2L}\left(y+\frac{L(\overline{m}'-m_1)}{d'}\right)^2}, \quad (54)$$

where $d' = m_4 - m_2$, $\overline{m}' = (m_2 + m_4)/2$, and C is an integration constant. This mode should be normalizable in the previous sense that the function damps at the boundaries for large L. However, this kind of function, which is monotonically decreasing or increasing in the interval, is localized outside the interval since there are no zeros of $A_{11}(y)$ within the interval. The localized position of $\Psi_i(y)$ corresponds to the position moduli of walls. We should not include the position moduli made outside of the interval. So we exclude the localized modes expressed like (54) by setting $C = 0$ as the boundary condition.

More generally, the sign of the function $\Sigma(y) - m_i$ can change between $y = -\frac{L}{2}$ and $y = \frac{L}{2}$. When the sign of $\Sigma(y) - m_i$ changes from negative to positive, a new normalizable mode appears for P. Since we have chosen the boundary condition as $\Sigma(y) - m_i = 0$ $(i = 1, 4)$, the sign change at the boundary is a little ambiguous. We regard the contribution of the sign change from 0 to positive as the same as the change from negative to positive. Namely, we assume the existence of the function $\Sigma(y)$ outside of the interval.

The kernel of $\tilde{P}$ is also evaluated in a similar way to $\ker P$. The differential equation for $\tilde{P}$ is now given by

$$\tilde{P}\tilde{\Psi}_i = \partial_y \tilde{\Psi}_i - \sum_{j=1}^{4} A_{ij}(y)\tilde{\Psi}_j = 0. \quad (55)$$

Since the sign in front of the matrix operator $A_{ij} = (\Sigma(y) - m_i)\delta_{ij}$ is opposite to the P case, the counting of the normalizable modes is completely opposite. The normalizable modes come from the change of the sign of $\Sigma(y) - m_i$ from positive to negative.

4.1.2. Index theorem. This counting of the index of P, by choosing a specific profile function of Σ and thinking physically whether the mode is normalizable or not, appears a little ambiguous.

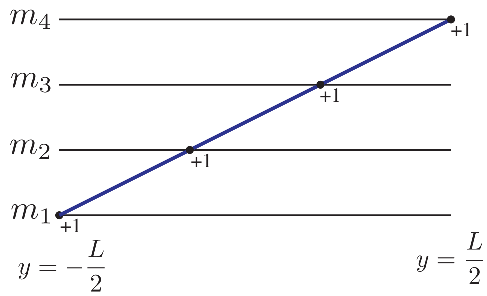

Fig. 1. The contribution to the index of P when the profile of $\Sigma(y)$ is a straight line. The straight profile crosses the mass level from negative to positive. Each crossing contributes to the index by $+1$, so the index is $+4$ in total.

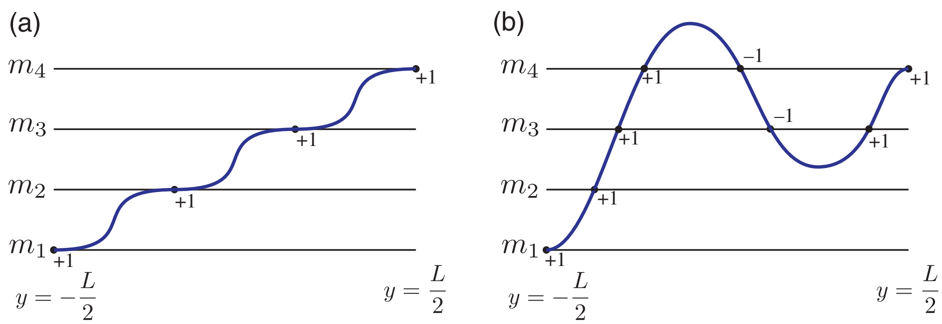

Fig. 2. The index does not change by continuous deformations of the profile of $\Sigma(y)$ with the fixed boundary condition. (a) The kink profile which may be obtained after solving the BPS equations gives the same index as the straight line. (b) Even if a continuous deformation produces negative contributions to the index, additional positive contributions are also produced and the total contribution to the index remains the same, at $+4$.

However, we can clearly define the index of P in a mathematical way which is similar to the Atiyah–Patodi–Singer [19] or Callias index theorem [18].

The profile of the function $\Sigma(y)$ is completely determined by the original BPS equations, especially by solving the equation $\mu_r = 0$. However, in our derivation of the integral formula we did not take account of one of the BPS equations $\mu_r = 0$ before integrating ϕ. So while the index is considered for a P with a specific $\Sigma(y)$, the index is actually independent of choices of the profile of $\Sigma(y)$.

To see this, let us consider a kink-like profile which may be realized by solving the full BPS equations including μ_r to examine the index (41) for the $N_f = 4$ case. At the boundary $y = -\frac{L}{2}$, the eigenvalues of $A_{ij}(y)$ are $(0, +, +, +)$ ($+$ means a positive eigenvalue). Since we consider extending the function $\Sigma(y)$ infinitesimally outside of the interval $y < -L/2$, the eigenvalues at $y = -\epsilon - \frac{L}{2}$ are $(-, +, +, +)$. Going through the boundary $y = -\frac{L}{2}$ we obtain a contribution to the index of $+1$. When $\Sigma(y)$ reaches m_2 at some y, the eigenvalues change from $(-, +, +, +)$ to $(-, -, +, +)$, that is, the index increases by $+1$. If we continue to $y = \frac{L}{2} + \epsilon$ in this way, we obtain the value of the index to be $\text{ind}\, P = +4$ (see Fig. 1).

When we choose the profile $\Sigma(y)$ freely, we always obtain the same index $\text{ind}\, P = +4$. So the index is invariant under a continuous deformation of $\Sigma(y)$ (see Fig. 2).

4.1.3. Evaluation of integral. Thus, we have the indices for the $N_f = 4$ case, and obtain the integration formula for the volume of the moduli space as

$$\text{Vol}\left(\mathcal{M}^{1,4}_{1\to 4}\right) = \int_{-\infty}^{\infty} \frac{d\phi}{2\pi} \frac{1}{(i\phi)^4} e^{i\phi\beta\left(\frac{g^2\zeta}{2}L - d\right)}. \tag{56}$$

This integral has a fourth-order pole at $\phi = 0$. We can perform this integral by using the following residue calculus with a suitable contour dictated by the convergence of the H, $H^\dagger$ path integral

$$\int_{-\infty-\mathrm{i}\epsilon}^{\infty-\mathrm{i}\epsilon} \frac{\mathrm{d}\phi}{2\pi\mathrm{i}} \frac{1}{\phi^{n+1}} \mathrm{e}^{\mathrm{i}\phi B} = \begin{cases} \frac{1}{n!}(\mathrm{i}B)^n & \text{if } B \geq 0 \text{ and } n \geq 0 \\ 0 & \text{otherwise} \end{cases}. \tag{57}$$

Thus, we obtain the volume of the moduli space,

$$\mathrm{Vol}\left(\mathcal{M}_{1\to 4}^{1,4}\right) = \frac{1}{3!}\left(\frac{g^2\zeta}{2}L - d\right)^3, \tag{58}$$

when $\frac{g^2\zeta}{2}L - d \geq 0$.

We next discuss the implications of the result (58). If we consider the case $L \gg \frac{2d}{g^2\zeta}$, where the size of the interval L is sufficiently large in comparison with the width [16,17,41] of the domain wall $\frac{2d}{g^2\zeta}$, then the volume is proportional to $\frac{L^3}{3!}$. This is nothing but the volume of the moduli space of three undistinguished points on the interval L. So we can regard the power of $(\frac{g^2\zeta}{2}L - d)$ as the number of BPS domain walls on the interval (the dimension of the domain-wall moduli space). This agrees with the number of kinks which is depicted in Fig. 2(a). Recalling that the order of the pole comes from the index of P, we can conclude that

$$\mathrm{ind}\, P = (\text{the number of BPS domain walls}) + 1. \tag{59}$$

We can also understand this fact from another point of view. The index of P is obtained from the equations $\mu_c = \mu_c^\dagger = 0$ without imposing the other BPS equation $\mu_r = 0$. Only after ϕ is integrated out is the equation $\mu_r = 0$ taken into account. The number of domain walls coincides with the dimension of the moduli space. An additional $+1$ of the index of P is removed by the contour integral and the number reduces to the dimension of the moduli space. The dimension of the moduli space is also calculated by an index theorem where all of the BPS equations are considered. We have finally obtained the dimension of the moduli space after imposing the condition $\mu_r = 0$. Thus, we have done a correct evaluation of the moduli space volume by the contour integral of ϕ.

Using similar arguments as above, we can easily extend our computation to the case where N_f and the boundary conditions are general:

$$\begin{aligned} \mathrm{Vol}\left(\mathcal{M}_{i\to j}^{1,N_f}\right) &= \int_{-\infty}^{\infty} \frac{\mathrm{d}\phi}{2\pi} \frac{1}{(\mathrm{i}\phi)^{j-i+1}} \mathrm{e}^{\mathrm{i}\phi\beta\left(\frac{g^2\zeta}{2}L - d_{ij}\right)} \\ &= \frac{\beta^{j-i}}{(j-i)!}\left(\frac{g^2\zeta}{2}L - d_{ij}\right)^{j-i}, \end{aligned} \tag{60}$$

where $d_{ij} \equiv m_j - m_i$.

4.2. *Non-Abelian domain walls*

The localization formula (43) says that the non-Abelian gauge group $U(N_c)$ reduces to a product of the Abelian groups $U(1)^{N_c}$ at the fixed-point set. So the localization formula for the non-Abelian gauge group is essentially a direct product of the formula for the Abelian group. In particular, the indices (number of walls) for each Abelian factor is determined by the boundary conditions as in Eq. (59). With this result for the indices ind$P$, Eq. (49) for the volume of the moduli space of the non-Abelian walls becomes

$$\mathrm{Vol}\left(\mathcal{M}^{N_c,N_f}_{\vec{A}\to\vec{B}}\right) = \sum_{\sigma\in\mathfrak{S}_{N_c}} \prod_{a=1}^{N_c} \int_{-\infty}^{\infty} \frac{\mathrm{d}\phi_a}{2\pi} \frac{(-1)^{|\sigma|}}{(\mathrm{i}\phi_a)^{B_{\sigma(a)}-A_a+1}} \mathrm{e}^{\mathrm{i}\phi_a\beta\left(\hat{L}-(m_{B_{\sigma(a)}}-m_{A_a})\right)}. \tag{61}$$

If some of the permutations of the boundary conditions satisfy $B_{\sigma(a)} - A_a < 0$, the corresponding ϕ_a integral does not have a pole and vanishes. These boundary conditions $B_{\sigma(a)} - A_a < 0$ correspond to non-BPS wall solutions. Although the non-BPS walls are, in general, contained in the ϕ_a integral (61), they give vanishing contributions. So the integral is finally restricted to a set of permutations $\mathfrak{S}'_{N_c}$ which satisfy $\forall(B_{\sigma(a)} - A_a) \geq 0$ (BPS wall conditions). The integral can be evaluated by

$$\mathrm{Vol}\left(\mathcal{M}^{N_c,N_f}_{\vec{A}\to\vec{B}}\right) = \beta^D \sum_{\sigma\in\mathfrak{S}'_{N_c}} \prod_{a=1}^{N_c} \frac{(-1)^{|\sigma|}}{(B_{\sigma(a)}-A_a)!} \left(\hat{L} - (m_{B_{\sigma(a)}} - m_{A_a})\right)^{B_{\sigma(a)}-A_a}, \tag{62}$$

where $D = \dim \mathcal{M}^{N_c,N_f}_{\vec{A}\to\vec{B}} = \sum_{a=1}^{N_c}(B_a - A_a)$ is the dimension of the moduli space.

It is interesting to note that the above volume formula of the non-Abelian domain wall can be expressed by a determinant of a matrix $\mathcal{T}$,

$$\mathrm{Vol}\left(\mathcal{M}^{N_c,N_f}_{\vec{A}\to\vec{B}}\right) = \beta^D \det \mathcal{T}^{N_c,N_f}_{\vec{A}\to\vec{B}}, \tag{63}$$

where

$$\left(\mathcal{T}^{N_c,N_f}_{\vec{A}\to\vec{B}}\right)_{ab} = \begin{cases} \dfrac{1}{(B_b - A_a)!}\left(\hat{L} - (m_{B_b} - m_{A_a})\right)^{B_b - A_a} & \text{if } B_b \geq A_a, \\ 0 & \text{if } B_b < A_a. \end{cases} \tag{64}$$

We will call this matrix $\mathcal{T}$ a transition matrix in the following.

Using the above formula, let us consider some concrete examples for the non-Abelian gauge group in order to understand the meaning of the volume formula (62). We first consider the case of $N_c = 2$ and $N_f = 4$ with the boundary condition $\Sigma(-L/2) = \mathrm{diag}(m_1, m_3)$ and $\Sigma(L/2) = \mathrm{diag}(m_2, m_4)$. The ϕ_a integral (62) for this boundary condition is given concretely by

$$\begin{aligned}\mathrm{Vol}\left(\mathcal{M}^{2,4}_{(1,3)\to(2,4)}\right) = &\int \frac{\mathrm{d}\phi_1}{2\pi}\frac{\mathrm{d}\phi_2}{2\pi} \frac{\mathrm{e}^{\mathrm{i}\beta\phi_1(\hat{L}-(m_2-m_1))}}{(\mathrm{i}\phi_1)^2} \frac{\mathrm{e}^{\mathrm{i}\beta\phi_2(\hat{L}-(m_4-m_3))}}{(\mathrm{i}\phi_2)^2} \\ &- \int \frac{\mathrm{d}\phi_1}{2\pi}\frac{\mathrm{d}\phi_2}{2\pi} \frac{\mathrm{e}^{\mathrm{i}\beta\phi_1(\hat{L}-(m_4-m_1))}}{(\mathrm{i}\phi_1)^4} \frac{\mathrm{e}^{\mathrm{i}\beta\phi_2(\hat{L}-(m_2-m_3))}}{(\mathrm{i}\phi_2)^0}. \end{aligned} \tag{65}$$

The second term contains the anti-BPS wall configuration and the ϕ_a integral vanishes. So only the first term contributes to the volume. Thus, we obtain

$$\mathrm{Vol}\left(\mathcal{M}^{2,4}_{(1,3)\to(2,4)}\right) = \beta^2(\hat{L} - (m_2 - m_1))(\hat{L} - (m_4 - m_3)). \tag{66}$$

This result is essentially a direct product of independent Abelian walls (see Fig. 3). In this case, the 2×2 transition matrix becomes

$$\mathcal{T}^{2,4}_{(1,3)\to(2,4)} = \begin{pmatrix} \hat{L} - (m_2 - m_1) & \dfrac{1}{3!}(\hat{L} - (m_4 - m_1))^3 \\ 0 & \hat{L} - (m_2 - m_1) \end{pmatrix}. \tag{67}$$

The determinant of this matrix (times β^2) gives precisely (66).

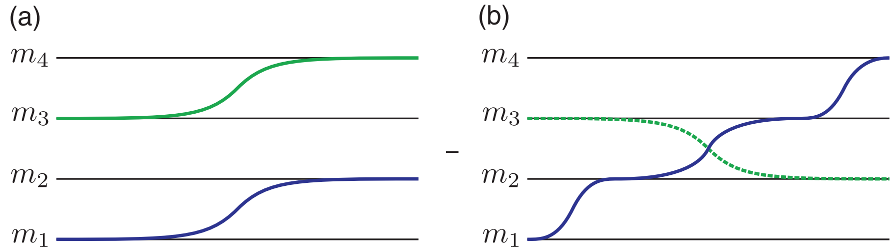

Fig. 3. The possible domain-wall profiles of the boundary condition $\Sigma(-L/2) = \mathrm{diag}(m_1, m_3)$ and $\Sigma(L/2) = \mathrm{diag}(m_2, m_4)$ with the permutations. The solid line represents the BPS profile, while the dashed line represents the anti-BPS one. The contribution from the non-BPS domain-wall profile which includes the anti-BPS domain wall disappears from the evaluation of the sum of the volume.

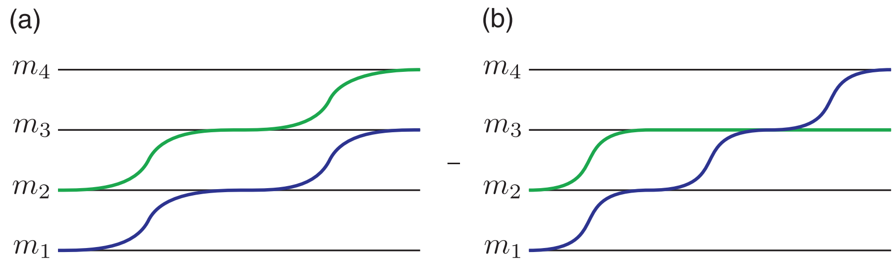

Fig. 4. The possible domain-wall profiles of the boundary condition $\Sigma(-L/2) = \mathrm{diag}(m_1, m_2)$ and $\Sigma(L/2) = \mathrm{diag}(m_3, m_4)$ with the permutations. (a) The contribution with no color line intersections. (b) The contribution with an intersection of two color lines, which carries a negative sign.

The next example is almost the same as the previous case except for the boundary condition: $\Sigma(-L/2) = \mathrm{diag}(m_1, m_2)$ and $\Sigma(L/2) = \mathrm{diag}(m_3, m_4)$ in the case of $N_c = 2$ and $N_f = 4$. Similarly to the previous case, permutations of boundary conditions provide two contributions, as shown in Fig. 4. Both of them now correspond to BPS configurations and are non-vanishing, unlike the previous case:

$$\mathrm{Vol}\left(\mathcal{M}^{2,4}_{(1,2)\to(3,4)}\right) = \int \frac{\mathrm{d}\phi_1}{2\pi}\frac{\mathrm{d}\phi_2}{2\pi}\frac{\mathrm{e}^{\mathrm{i}\beta\phi_1(\hat{L}-(m_3-m_1))}}{(\mathrm{i}\phi_1)^3}\frac{\mathrm{e}^{\mathrm{i}\beta\phi_2(\hat{L}-(m_4-m_2))}}{(\mathrm{i}\phi_2)^3} - \int \frac{\mathrm{d}\phi_1}{2\pi}\frac{\mathrm{d}\phi_2}{2\pi}\frac{\mathrm{e}^{\mathrm{i}\beta\phi_1(\hat{L}-(m_4-m_1))}}{(\mathrm{i}\phi_1)^4}\frac{\mathrm{e}^{\mathrm{i}\beta\phi_2(\hat{L}-(m_3-m_2))}}{(\mathrm{i}\phi_2)^2}. \quad (68)$$

The second term corresponds to the case of two color lines intersecting each other, as shown in Fig. 4(b). Evaluating the ϕ_a integral, we obtain

$$\mathrm{Vol}\left(\mathcal{M}^{2,4}_{(1,2)\to(3,4)}\right) = \beta^4\left\{\frac{1}{4}(\hat{L}-(m_3-m_1))^2(\hat{L}-(m_4-m_2))^2 - \frac{1}{6}(\hat{L}-(m_4-m_1))^3(\hat{L}-(m_3-m_2))\right\}. \quad (69)$$

This can be expressed by a determinant of the 2×2 transition matrix as in Eq. (63):

$$\mathcal{T}^{2,4}_{(1,2)\to(3,4)} = \begin{pmatrix} \frac{1}{2!}(\hat{L}-(m_3-m_1))^2 & \frac{1}{3!}(\hat{L}-(m_4-m_1))^3 \\ \hat{L}-(m_3-m_2) & \frac{1}{2!}(\hat{L}-(m_4-m_2))^2 \end{pmatrix}. \quad (70)$$

Let us examine the meaning of our result more closely. The kink profiles such as in Fig. 4 may be understood to represent $\Sigma(y)$ connecting vacuum values given by boundary conditions. Taking, for instance, the wall connecting m_3 and m_4 in the upper line of Fig. 4(a), its position can only go to the right up to the other wall connecting m_2 and m_3 in the lower line, namely they are non-penetrable of each other [15,16]. This type of restriction gives an interesting behavior of moduli space volume, as illustrated in a concrete example in Appendix A. On the other hand, our volume formula is given in terms of ϕ_a, the zero mode of Φ^a, which is canonically conjugate to the variable Σ^a. The ϕ_a integral counts the number of domain walls in the a-th color line without particular restrictions on the possible range of wall positions. Instead, our formula compensates for the over-counting of integration range by subtracting appropriate contributions in the form of permutations of boundary conditions carrying a sign given by the intersection number of color lines, as shown in Fig. 4(b). Combining all contributions from permutations of boundary conditions, the volume is finally given in terms of the determinant of the transition matrix (63).

5. Duality between non-Abelian domain walls

We have found a formula for evaluating the volume of the domain-wall moduli space. We give here a non-trivial check of our localization formula by examining duality relations between two different theories and boundary conditions. We take two different gauge theories: $G = U(N_c)$ gauge group with N_f flavors and $\tilde{G} = U(\tilde{N}_c)$ gauge group with N_f flavors, where $\tilde{N}_c \equiv N_f - N_c$. The boundary conditions of both theories should be chosen to connect complementary vacua as follows. If the boundary conditions of the original $G = U(N_c)$ theory are $\vec{A} \to \vec{B}$, then the corresponding boundary conditions of the other $\tilde{G} = U(\tilde{N}_c)$ theory should be $\overline{\vec{B}} \to \overline{\vec{A}}$, where $\overline{\vec{A}}$ and $\overline{\vec{B}}$ are the complement of $\vec{A}$ and $\vec{B}$, respectively. For example, the boundary condition $\vec{A} = (2, 4, 5)$ of $G = U(3)$ theory with $N_f = 5$ flavors is complementary to the boundary condition $\overline{\vec{A}} = (1, 3)$ of $\tilde{G} = U(2)$ with $N_f = 5$ flavors. Let us call both theories with the complementary boundary conditions *dual theories*.

In the strong coupling limit, the gauge theories become non-linear sigma models and the two dual theories become identical [11,12]. It has been demonstrated explicitly that the moduli spaces of domain walls (in the infinite interval) have a one-to-one correspondence and become identical in the dual theories in the strong coupling limit [11]. Even in the finite gauge coupling, the moduli spaces of the domain walls of these dual theories are topologically the same,

$$\mathcal{M}^{N_c, N_f}_{\vec{A} \to \vec{B}} \simeq \mathcal{M}^{\tilde{N}_c, N_f}_{\overline{\vec{B}} \to \overline{\vec{A}}}, \tag{71}$$

but their metric and other properties are different [11]. Consequently, the moduli space of domain walls in these two theories are different at finite gauge coupling, but should become identical in the strong coupling limit. We need to specify the boundary condition for dual theories. The explicit formula of one-to-one correspondence of dual theories [11] suggests that the color–flavor locking of vacua should be chosen in such a way that those vacua occupied in dual theories should be the complement of each other. Namely, among N_f flavors, N_c should be selected to specify a vacuum in $U(N_c)$ gauge theory, whereas the remaining $\tilde{N}_c$ flavors should be selected in the dual $U(\tilde{N}_c)$ gauge theory to give the dual boundary condition.

We will see that our results for the volume of moduli spaces for these two theories differ for finite gauge coupling, but become identical for the strong coupling limit $g^2 \to \infty$.

5.1. *Abelian versus non-Abelian duality*

First of all, let us consider duality between Abelian gauge group $G = U(1)$ and non-Abelian gauge group $\tilde{G} = U(N_f - 1)$ with the N_f flavors of the same ordered masses $m_1, m_2, \ldots, m_{N_f}$.

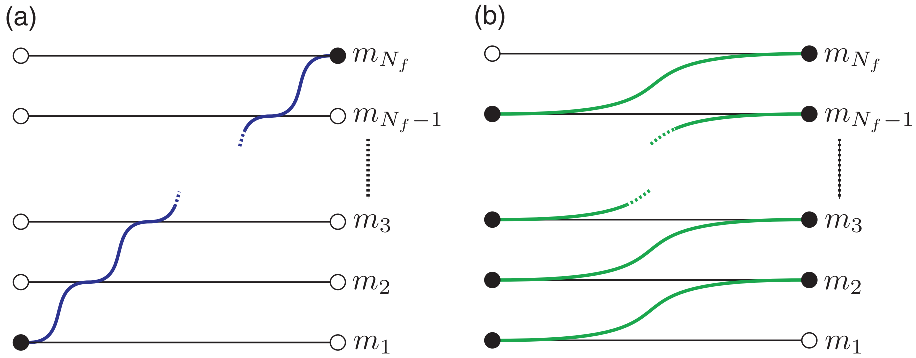

Fig. 5. The duality between Abelian and non-Abelian theories. (a) Abelian theory with N_f flavors and the boundary condition of $\Sigma(-L/2) = m_1$ and $\Sigma(L/2) = m_{N_f}$. (b) $G = U(N_f - 1)$ non-Abelian theory with N_f flavors and the boundary condition of $\Sigma(-L/2) = \text{diag}(m_1, m_2, m_3, \ldots, m_{N_f-1})$ and $\Sigma(L/2) = \text{diag}(m_2, m_3, \ldots, m_{N_f-1}, m_{N_f})$. Black and white circles represent vacua specified by the boundary conditions and their complements, respectively.

For the Abelian model, we take a boundary condition to be $\Sigma(-L/2) = m_1$ and $\Sigma(L/2) = m_{N_f}$ to obtain the maximal dimensions of the moduli space. Using the localization formula of the volume for this boundary condition, we obtain

$$\text{Vol}\left(\mathcal{M}^{1,N_f}_{1\to N_f}\right) = \frac{\beta^{N_f-1}}{(N_f-1)!}\left(\hat{L} - d_{1,N_f}\right)^{N_f-1}. \tag{72}$$

The dual of the above Abelian model is the $\tilde{G} = U(N_f - 1)$ gauge group with N_f flavors. The dual boundary condition also corresponds to the maximal dimensions of the moduli space and is given by $\Sigma^a(-L/2) = m_a$ and $\Sigma^a(L/2) = m_{a+1}$ $(a = 1, \ldots, N_f - 1)$. The transition matrix of this model becomes

$$\mathcal{T}^{N_f-1,N_f}_{(1,2,\ldots,N_f-1)\to(2,3,\ldots,N_f)}$$
$$= \begin{pmatrix} \hat{L} - d_{12} & \frac{1}{2!}(\hat{L} - d_{13})^2 & \frac{1}{3!}(\hat{L} - d_{14})^3 & \cdots & \frac{1}{(N_f-1)!}(\hat{L} - d_{1,N_f})^{N_f-1} \\ 1 & \hat{L} - d_{23} & \frac{1}{2!}(\hat{L} - d_{24})^2 & \cdots & \frac{1}{(N_f-2)!}(\hat{L} - d_{2,N_f})^{N_f-2} \\ 0 & 1 & \hat{L} - d_{34} & \cdots & \frac{1}{(N_f-3)!}(\hat{L} - d_{3,N_f})^{N_f-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \hat{L} - d_{N_f,N_f-1} \end{pmatrix}. \tag{73}$$

The volume of the moduli space can be evaluated from the determinant of the above transition matrix as

$$\text{Vol}\left(\mathcal{M}^{N_f-1,N_f}_{(1,2,\ldots,N_f-1)\to(2,3,\ldots,N_f)}\right) = \beta^{N_f-1}\text{Det}\,\mathcal{T}^{N_f-1,N_f}_{(1,2,\ldots,N_f-1)\to(2,3,\ldots,N_f)}. \tag{74}$$

The boundary conditions and typical kink profiles of Abelian and non-Abelian theories with N_f flavors are depicted in Fig. 5.

The volumes (72) and (74) are different from each other in a general coupling region. In Appendix A we will explicitly demonstrate this difference in the simplest case of $N_f = 3$ with

$N_c = 1$ and $N_c = 2$ as a concrete example. We will also show there that the results agree with those of a direct calculation using the rigid-rod approximation [14].

On the other hand, in the strong coupling limit $g^2 \to \infty$ where $\hat{L} = \frac{g^2 \zeta}{2} L$ becomes large, we find

$$\mathrm{Vol}\left(\mathcal{M}^{1,N_f}_{1\to N_f}\right) \approx \frac{\beta^{N_f-1}}{(N_f-1)!}\hat{L}^{N_f-1}, \tag{75}$$

and

$$\mathrm{Vol}\left(\mathcal{M}^{N_f-1,N_f}_{(1,2,\ldots,N_f-1)\to(2,3,\ldots,N_f)}\right) \approx \frac{\beta^{N_f-1}}{(N_f-1)!}\hat{L}^{N_f-1} \tag{76}$$

(see Appendix B). Thus, the volumes agree with each other in the strong coupling limit as expected by the duality. This result means that the leading terms of the volume in $\hat{L}$ coincide in two different models, including a combinatorial coefficient. This fact is highly non-trivial and suggests that our localization formula for the volume expressed by the determinant works correctly.

In this case, the moduli spaces of both theories are topologically isomorphic to a complex projective space $\mathbb{C}P^{N_f-1}$,

$$\mathcal{M}^{1,N_f}_{1\to N_f} \simeq \mathcal{M}^{N_f-1,N_f}_{(1,2,\ldots,N_f-1)\to(2,3,\ldots,N_f)} \simeq \mathbb{C}P^{N_f-1}. \tag{77}$$

Indeed, the volumes (75) and (76) represent a rigid volume[2] of $\mathbb{C}P^{N_f-1}$ with a "size" $\frac{\beta\hat{L}}{2\pi}$.

5.2. *Non-Abelian versus non-Abelian duality*

To give another non-trivial check, let us consider a duality between two non-Abelian gauge groups. One model is $G = U(3)$ with $N_f = 5$, and the other dual model is $\tilde{G} = U(2)$ with $N_f = 5$. First, we consider the complementary boundary conditions: $\Sigma(-L/2) = \mathrm{diag}(m_1, m_2, m_3)$ and $\Sigma(L/2) = \mathrm{diag}(m_3, m_4, m_5)$ for $G = U(3)$ theory, and $\Sigma(-L/2) = \mathrm{diag}(m_1, m_2)$ and $\Sigma(L/2) = \mathrm{diag}(m_4, m_5)$ for $\tilde{G} = U(2)$ theory. These boundary conditions maximize the dimension of the moduli space for these dual theories.

The transition matrices of both theories are

$$\mathcal{T}^{3,5}_{(1,2,3)\to(3,4,5)} = \begin{pmatrix} \frac{1}{2!}(\hat{L}-d_{13})^2 & \frac{1}{3!}(\hat{L}-d_{14})^3 & \frac{1}{4!}(\hat{L}-d_{15})^4 \\ \hat{L}-d_{23} & \frac{1}{2!}(\hat{L}-d_{24})^2 & \frac{1}{3!}(\hat{L}-d_{25})^3 \\ 1 & \hat{L}-d_{34} & \frac{1}{2!}(\hat{L}-d_{35})^2 \end{pmatrix}, \tag{78}$$

and

$$\mathcal{T}^{2,5}_{(1,2)\to(4,5)} = \begin{pmatrix} \frac{1}{3!}(\hat{L}-d_{14})^3 & \frac{1}{4!}(\hat{L}-d_{15})^4 \\ \frac{1}{2!}(\hat{L}-d_{24})^2 & \frac{1}{3!}(\hat{L}-d_{25})^3 \end{pmatrix}. \tag{79}$$

The boundary conditions and typical kink profiles of both non-Abelian theories are shown in Fig. 6.

[2] The power of $\frac{\beta\hat{L}}{2\pi}$ represents a *complex* dimension of $\mathbb{C}P^{N_f-1}$ even though it is a real parameter. The volume at the unit "size" is obtained by setting $\frac{\beta\hat{L}}{2\pi} = 1$.

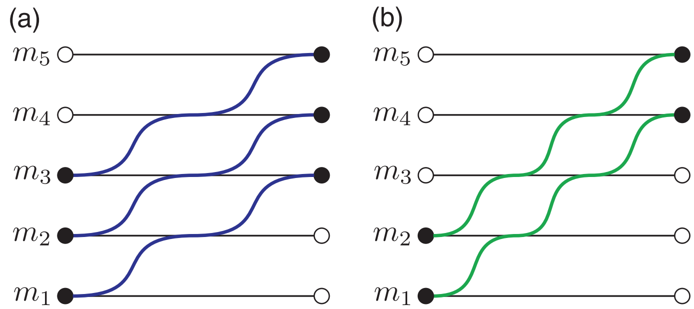

Fig. 6. The duality between non-Abelian theories. (a) $G = U(3)$ non-Abelian theory with $N_f = 5$ and the boundary condition of $\Sigma(-L/2) = \mathrm{diag}(m_1, m_2, m_3)$ and $\Sigma(L/2) = \mathrm{diag}(m_3, m_4, m_5)$. (b) $G = U(2)$ non-Abelian theory with $N_f = 5$ and the boundary condition of $\Sigma(-L/2) = \mathrm{diag}(m_1, m_2)$ and $\Sigma(L/2) = \mathrm{diag}(m_4, m_5)$. The boundary conditions are given to connect complementary vacua (exchanging the role of black and white circles).

The volumes of both theories coincide with each other in the strong coupling limit:

$$\mathrm{Vol}\left(\mathcal{M}^{3,5}_{(1,2,3)\to(3,4,5)}\right) = \beta^6 \mathrm{Det}\,\mathcal{T}^{3,5}_{(1,2,3)\to(3,4,5)} = \frac{\beta^6}{144}\hat{L}^6 + \mathcal{O}(\hat{L}^5), \tag{80}$$

and

$$\mathrm{Vol}\left(\mathcal{M}^{2,5}_{(1,2)\to(4,5)}\right) = \beta^6 \mathrm{Det}\,\mathcal{T}^{2,5}_{(1,2)\to(4,5)} = \frac{\beta^6}{144}\hat{L}^6 + \mathcal{O}(\hat{L}^5). \tag{81}$$

This result shows that the (complex) dimension[3] of the moduli space is 6.

In this maximal dimension case, the moduli spaces are isomorphic to a complex Grassmann manifold (Grassmannian),

$$\mathcal{M}^{3,5}_{(1,2,3)\to(3,4,5)} \simeq G_{3,5} \simeq G_{2,5} \simeq \mathcal{M}^{2,5}_{(1,2)\to(4,5)}, \tag{82}$$

where G_{N_c,N_f} is expressed by a coset space

$$G_{N_c,N_f} \equiv \frac{U(N_f)}{U(N_c)\times U(\tilde{N}_c)}. \tag{83}$$

The volume of the Grassmannian of unit "size" is obtained from a quotient of unitary group volumes [20–22] (see also the appendix in [9,10]):

$$\mathrm{Vol}(G_{N_c,N_f}) = \frac{\prod_{j=1}^{N_c}(j-1)! \times \prod_{k=1}^{\tilde{N}_c}(k-1)!}{\prod_{i=1}^{N_f}(i-1)!}(2\pi)^{N_c\tilde{N}_c}. \tag{84}$$

The volume of the Grassmannian is invariant under exchanging N_c and $\tilde{N}_c$.

Using this formula, we notice that the leading term of the volumes (80) and (81) are nothing but the volume of the Grassmannian $G_{3,5}$ or $G_{2,5}$ with "size" $\frac{\beta\hat{L}}{2\pi}$, since

$$\mathrm{Vol}(G_{3,5}) = \mathrm{Vol}(G_{2,5}) = \frac{2!1! \times 1!}{4!3!2!1!}(2\pi)^6 = \frac{1}{144}(2\pi)^6. \tag{85}$$

Therefore, our results are consistent with the notion that the moduli spaces of the domain walls in dual theories asymptotically coincide with the Grassmannian $G_{3,5}$ or $G_{2,5}$ with the standard metric,

[3] Half of the moduli are compact corresponding to relative phases of adjacent vacua separated by the domain wall.

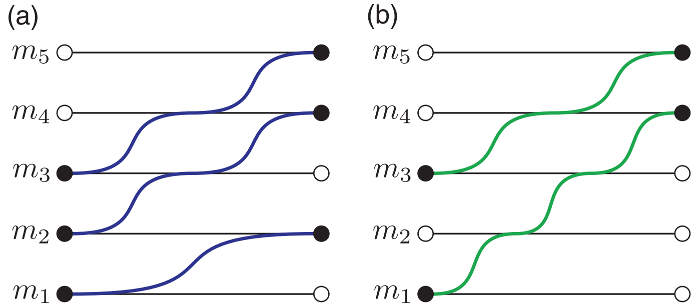

Fig. 7. The duality between non-Abelian theories with a non-maximal boundary condition. (a) $G = U(3)$ non-Abelian theory with $N_f = 5$ and the boundary condition of $\Sigma(-L/2) = \text{diag}(m_1, m_2, m_3)$ and $\Sigma(L/2) = \text{diag}(m_2, m_4, m_5)$. (b) $G = U(2)$ non-Abelian theory with $N_f = 5$ and the boundary condition of $\Sigma(-L/2) = \text{diag}(m_1, m_3)$ and $\Sigma(L/2) = \text{diag}(m_4, m_5)$.

but the differential structure (metric) is deformed by the sub-leading terms in $\hat{L}$. These non-trivial agreements strongly suggest that the duality between different non-Abelian gauge theories is valid in the strong coupling region.

As another example, let us next consider a different boundary condition with non-maximal dimensions of moduli space: $\Sigma(-L/2) = \text{diag}(m_1, m_2, m_3)$ and $\Sigma(L/2) = \text{diag}(m_2, m_4, m_5)$ for $G = U(3)$ theory as the boundary condition in one theory, and $\Sigma(-L/2) = \text{diag}(m_1, m_3)$ and $\Sigma(L/2) = \text{diag}(m_4, m_5)$ for $\tilde{G} = U(2)$ theory as the corresponding dual. The boundary conditions and typical kink profiles of both non-Abelian theories are shown in Fig. 7.

The transition matrices of both theories with these boundary conditions are

$$\mathcal{T}^{3,5}_{(1,2,3)\to(2,4,5)} = \begin{pmatrix} \hat{L} - d_{12} & \frac{1}{3!}(\hat{L} - d_{14})^3 & \frac{1}{4!}(\hat{L} - d_{15})^4 \\ 1 & \frac{1}{2!}(\hat{L} - d_{24})^2 & \frac{1}{3!}(\hat{L} - d_{25})^3 \\ 0 & \hat{L} - d_{34} & \frac{1}{2!}(\hat{L} - d_{35})^2 \end{pmatrix}, \tag{86}$$

and

$$\mathcal{T}^{2,5}_{(1,3)\to(4,5)} = \begin{pmatrix} \frac{1}{3!}(\hat{L} - d_{14})^3 & \frac{1}{4!}(\hat{L} - d_{15})^4 \\ \hat{L} - d_{34} & \frac{1}{2!}(\hat{L} - d_{35})^2 \end{pmatrix}. \tag{87}$$

The volumes of both theories coincide with each other in the strong coupling limit:

$$\text{Vol}\left(\mathcal{M}^{3,5}_{(1,2,3)\to(2,4,5)}\right) = \beta^5 \text{Det}\, \mathcal{T}^{3,5}_{(1,2,3)\to(2,4,5)} = \frac{\beta^5}{24}\hat{L}^5 + \mathcal{O}(\hat{L}^4), \tag{88}$$

and

$$\text{Vol}\left(\mathcal{M}^{2,5}_{(1,3)\to(4,5)}\right) = \beta^5 \text{Det}\, \mathcal{T}^{2,5}_{(1,3)\to(4,5)} = \frac{\beta^5}{24}\hat{L}^5 + \mathcal{O}(\hat{L}^4). \tag{89}$$

This result shows that the (complex) dimension of the moduli space is 5, which is smaller than the maximal dimension 6, as expected. So the moduli space for the present boundary conditions should be a complex sub-manifold of the Grassmannian $G_{3,5} \simeq G_{2,5}$.

In Appendix B, we evaluate the asymptotic behavior of the volume of the moduli space in the case of maximal dimensions for the general N_c gauge theories with N_f flavors to obtain

$$\mathrm{Vol}\left(\mathcal{M}^{N_c,N_f}_{(1,2,\ldots,N_c)\to(\tilde{N}_c+1,\ldots,N_f-1,N_f)}\right) = \frac{\prod_{j=1}^{N_c}(j-1)! \times \prod_{k=1}^{\tilde{N}_c}(k-1)!}{\prod_{i=1}^{N_f}(i-1)!}\left(\beta\hat{L}\right)^{N_c\tilde{N}_c} + \cdots,$$
$$\mathrm{Vol}\left(\mathcal{M}^{\tilde{N}_c,N_f}_{(1,2,\ldots,\tilde{N}_c)\to(N_c+1,\ldots,N_f-1,N_f)}\right) = \frac{\prod_{j=1}^{N_c}(j-1)! \times \prod_{k=1}^{\tilde{N}_c}(k-1)!}{\prod_{i=1}^{N_f}(i-1)!}\left(\beta\hat{L}\right)^{N_c\tilde{N}_c} + \cdots. \tag{90}$$

This result shows that there exists a duality relation between two different domain-wall theories in the strong coupling region.

6. T-duality to vortex on a cylinder

In this section, we discuss another kind of duality between the domain walls and vortices. As discussed in [14,23], there exists a T-duality relation between vortices on a cylinder and domain walls on the interval. We would like to show here that the volume of the moduli space exhibits this T-duality. As a base space we consider a cylinder, which is a two-dimensional surface of a circle S^1 with radius β times an interval I with the length L.

To see this duality, we start with the simplest case: vortices in $U(1)$ gauge theory with a single charged matter, which are called Abelian local vortices, or Abrikosov–Nielsen–Olesen (ANO) vortices [24,25]. If there are k vortices on the cylinder, the vortices are mapped to k domain walls (kinks) on the interval with the length L by the T-duality. The charged matters are mapped to the matter branes [13] and we can regard mass differences for each kink to be $1/\beta$, which is the radius of the dual circle in the domain-wall picture (see Fig. 8).

The total mass difference between the boundary conditions at $y = -L/2$ and at $y = L/2$ is k/β. So we can derive the integral formula for the volume of this domain-wall moduli space as

$$\mathrm{Vol}\left(\mathcal{M}^{1,1}_k\right) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\phi}{2\pi}\frac{1}{(\mathrm{i}\phi)^{k+1}}\mathrm{e}^{\mathrm{i}\phi\beta\left(\hat{L}-\frac{k}{\beta}\right)}$$
$$= \frac{1}{k!}\left(\beta\hat{L}-k\right)^k. \tag{91}$$

Recalling that the area of the cylinder in the vortex picture is given by $\mathcal{A} = 2\pi\beta L$ and that $\hat{L} = g^2\zeta L/2$ in Eq. (48), the volume (91) is equivalent to the volume of the ANO vortices on the cylinder

$$\mathrm{Vol}\left(\mathcal{M}^{1,1}_k\right) = \frac{1}{k!}\left(\frac{g^2\zeta}{4\pi}\mathcal{A}-k\right)^k, \tag{92}$$

where we can regard g and ζ as the gauge coupling and the FI parameter in the two-dimensional vortex system, respectively, since the combination $g^2\zeta$ is invariant under the T-duality. In the large area limit $\mathcal{A}\to\infty$, the volume is proportional to $\mathcal{A}^k/k!$, which is the volume of the symmetric product space of the cylinder $(S^1\times I)^k/\mathfrak{S}_k$. This is consistent with the point-like behavior of the vortex in the large area limit.

Now let us consider the Abelian k vortices with N_f matter fields of identical charges. These are called Abelian semi-local vortices. In the vortex side, the masses of the charged matters are degenerate and they are T-dual to degenerate vacua in the domain-wall picture. Since it is subtle to treat the degenerate masses in the domain-wall side [26], we split the masses of the N_f matters by giving small mass differences ε.

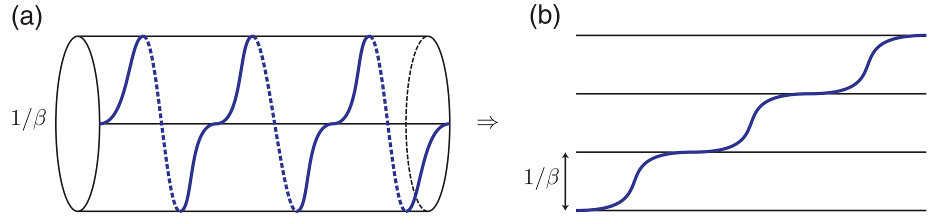

Fig. 8. T-dual picture of the vortex on a cylinder. The k vortices on the cylinder are dual to the domain walls wrapping k times around the circle. (a) A covering space of the k domain walls on the cylinder. (b) The k domain walls in the infinite number of flavors of the mass difference $1/\beta$. They are equivalent.

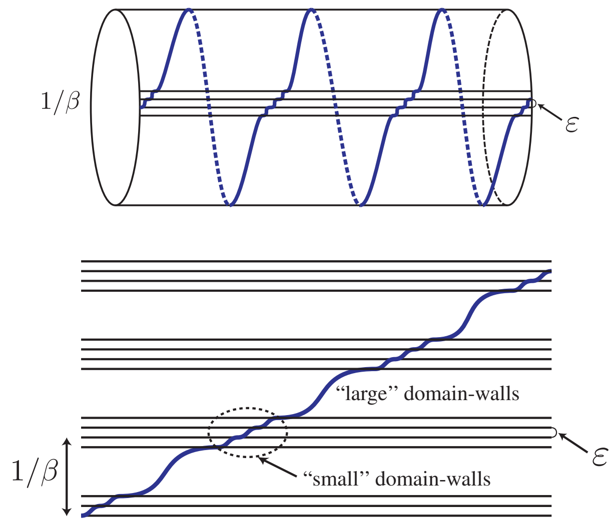

Fig. 9. T-dual picture of the vortex on the cylinder with $N_f > 1$ (Abelian semi-local vortex). The k vortices on the cylinder are dual to the large domain walls wrapping k times around the circle, and the multiple N_f flavors produce the small domain walls with the small mass differences ε. We depict the case of $N_f = 4$ and $k = 3$. The number of small domain walls is 10 and the total number of domain walls is $10 + 3 = 3 \times 4 + 1$, that is, $n = 1$ in this example.

There are two different types of domain wall in this N_f-flavor case. One type comes from the k vortices, which become "large" domain walls with the mass difference $1/\beta$. The other type are "small" domain walls connecting the small mass differences ε. The number of large domain walls is always k, since there are k winding domain walls around the cylinder. The number of small domain walls varies from $(k-1) \times (N_f - 1)$ to $(k+1) \times (N_f - 1)$, depending on the boundary conditions at $y = -L/2$ and $y = L/2$. So the total number of domain walls with N_f charged matters varies from $kN_f - (N_f - 1)$ to $kN_f + (N_f - 1)$, where we have assumed $k > 0$. We denote this number by $kN_f + n$, where $n = -(N_f - 1), \ldots, +(N_f - 1)$ if $k > 0$.[4] This means that the index for the domain walls is $kN_f + n + 1$. Note here that n is determined by the number of small domain walls adjacent to the boundaries (boundary condition of the domain walls). An example of the domain wall configuration is shown in Fig. 9.

Noting that the mass difference of each one of the large and small domain walls is $1/\beta - (N_f - 1)\varepsilon$ and ε, respectively, we find the total mass difference of k large domain walls and $k(N_f - 1) + n$

[4] When $k = 0$, n runs from 0 to $+(N_f - 1)$.

small domain walls is $k \times (1/\beta - (N_f - 1)\varepsilon) + (k(N_f - 1) + n) \times \varepsilon = k/\beta + n\varepsilon$. Then, applying the localization formula to the above domain wall configuration, we obtain the volume formula

$$\begin{aligned}
\mathrm{Vol}\left(\mathcal{M}_k^{1,N_f}(S^1 \times I)\right) &= \int_{-\infty}^{\infty} \frac{\mathrm{d}\phi}{2\pi} \frac{1}{(\mathrm{i}\phi)^{kN_f+n+1}} \mathrm{e}^{\mathrm{i}\phi\beta\left(\hat{L} - \frac{k}{\beta} - n\varepsilon\right)} \\
&= \frac{1}{(kN_f + n)!}\left(\beta\hat{L} - k - n\beta\varepsilon\right)^{kN_f+n} \\
&= \frac{1}{(kN_f + n)!}\left(\hat{\mathcal{A}} - k - n\beta\varepsilon\right)^{kN_f+n},
\end{aligned} \tag{93}$$

where we have defined $\hat{\mathcal{A}} \equiv \beta\hat{L} = \frac{g^2\zeta}{4\pi}\mathcal{A}$. In the $\varepsilon \to 0$ limit, we find

$$\mathrm{Vol}\left(\mathcal{M}_k^{1,N_f}(S^1 \times I)\right) = \frac{1}{(kN_f + n)!}\left(\hat{\mathcal{A}} - k\right)^{kN_f+n}. \tag{94}$$

We can see that the above volume is the same as the volume of the moduli space of Abelian semi-local vortices with N_f flavors on the sphere [9,10] if $n = N_f - 1$.

In the large area limit $\mathcal{A} \to \infty$, the volume of the moduli space of the vortices on the cylinder (dual to the large and small domain walls) is proportional to $\mathcal{A}^{kN_f+n}$. We do not know an explicit formula for the volume of the moduli space of the vortices on the cylinder, but this large-area behavior suggests that the dimension of the moduli space of the vortex is $N_f + n$ and the index of the operator $\mathcal{D}_{\bar{z}}$ on the cylinder with the appropriate boundary condition, which counts the number of zero modes of the Higgs fields obeying $\mathcal{D}_{\bar{z}}H = 0$ and determines the power of $\mathcal{A}$ via the contour integral, is $N_f + n + 1$. So we expect that the index of the operator $\mathcal{D}_{\bar{z}}$ on the cylinder may be given by the Atiyah–Patodi–Singer index theorem [19],

$$\begin{aligned}
\mathrm{ind}\,\mathcal{D}_{\bar{z}} &= N_f \int_{S^1\times I} F - \frac{N_f}{2}\left[\eta(S_R^1) - \eta(S_L^1)\right] \\
&= kN_f + \left\lfloor N_f\left(\oint_{S_R^1} A - \oint_{S_L^1} A\right)\right\rfloor + 1 \\
&= kN_f + n + 1,
\end{aligned} \tag{95}$$

where S_R^1 and S_L^1 are the right and left boundaries of the cylinder, respectively, η is the eta-invariant at the boundaries, and $\lfloor x \rfloor$ stands for the floor function which gives the largest integer not greater than x. The index theorem implies that the value of n in Eq. (95) for vortices on the cylinder is also limited to $-(N_f - 1) \leq n \leq +(N_f - 1)$ because of the T-duality. We expect that n is determined by the holonomies at the boundaries of the cylinder. To see a precise correspondence between n and holonomies, we need further investigation of the moduli of the vortex on the cylinder.

Finally, we discuss an extension of the above observations in the Abelian case to the non-Abelian case. As explained in the previous sections, the evaluation of the volume of the non-Abelian domain-wall moduli space can be reduced to a sum of products of the Abelian ones. So the T-dualized domain walls of the non-Abelian vortex can also be decomposed into the Abelian ones. In this decomposition, we have to take into account the permutations of the boundary conditions for each Abelian component. The boundary condition is labeled by the integer n, which reflects the different number of the small domain walls, as explained above. Thus, each Abelian part of the domain walls is labeled by the decomposed vortex charge k_a and the integer n_a associated with the boundary conditions, where a runs over the rank of $U(N_c)$, namely $a = 1, \ldots, N_c$ and k_a satisfies $k = \sum_{a=1}^{N_c} k_a$.

Thus, using the decomposition, we obtain the localization formula for the volume of the moduli space of the T-dualized non-Abelian vortex on the cylinder:

$$\begin{aligned}\mathrm{Vol}\left(\mathcal{M}_k^{N_c,N_f}(S^1\times I)\right) &= \sum_{\vec{k},\vec{n}}(-1)^{|\sigma(\vec{k},\vec{n})|}\prod_{a=1}^{N_c}\int_{-\infty}^{\infty}\frac{\mathrm{d}\phi_a}{2\pi}\frac{1}{(\mathrm{i}\phi_a)^{k_aN_f+n_a+1}}\mathrm{e}^{\mathrm{i}\phi_a\beta\left(\hat{L}-\frac{k_a}{\beta}-n_a\varepsilon\right)}\\ &= \sum_{\vec{k},\vec{n}}(-1)^{|\sigma(\vec{k},\vec{n})|}\prod_{a=1}^{N_c}\frac{1}{(k_aN_f+n_a)!}\left(\beta\hat{L}-k_a-n_a\beta\varepsilon\right)^{k_aN_f+n_a}\\ &= \sum_{\vec{k},\vec{n}}(-1)^{|\sigma(\vec{k},\vec{n})|}\prod_{a=1}^{N_c}\frac{1}{(k_aN_f+n_a)!}\left(\hat{\mathcal{A}}-k_a-n_a\beta\varepsilon\right)^{k_aN_f+n_a},\end{aligned}\tag{96}$$

where $\vec{k}$ are all possible N_c-component integer vectors satisfying $k=\sum_{a=1}^{N_c}k_a$ and with an ordering of $k_1\geq k_2\geq\cdots\geq k_{N_c}$, and $\vec{n}$ are varied in the given boundary conditions. The sign of each term depends on the order of the permutations $\sigma(\vec{k},\vec{n})$ of the boundary conditions determined by $\vec{k}$ and $\vec{n}$. The sign $(-1)^{|\sigma(\vec{k},\vec{n})|}$ is given by the parity of the intersection number of the N_c color lines. The volume of the moduli space of the non-Abelian vortex on the cylinder may be obtained in the limit of $\varepsilon\to 0$. This formula should also be directly checked from the localization theorem for the vortex on the cylinder with the boundary conditions of the various holonomies.

To see the above construction concretely, let us consider only an example of $N_c=2$ and general N_f for simplicity in the following, since the number of charge partitions increases rapidly for large N_c. We also take a trivial boundary condition, namely, $(1,2)\to(1,2)$.

First, for $k=0$, there is no partition of the charges, namely $(k_1,k_2)=(0,0)$. There is also no choice of the boundary conditions. In the T-dual picture of domain walls, this means that there exists no domain wall, but the volume of the moduli space gives a finite contribution

$$\mathrm{Vol}\left(\mathcal{M}_0^{2,N_f}(S^1\times I)\right)=1.\tag{97}$$

This should provide the relative normalization of the volume.

For $k=1$, there are two partitions of the charges, which are $\vec{k}=(1,0)$. For this partition, there are two permutations of the boundary conditions, that give $\vec{n}=\{(-1,+1),(0,0)\}$. Thus, the summation over all possible combinations of the charges and the boundary conditions gives the volume of the moduli space of the non-Abelian domain walls:

$$\mathrm{Vol}\left(\mathcal{M}_1^{2,N_f}(S^1\times I)\right)=-\frac{2}{N_f!}(\hat{\mathcal{A}}-1)^{N_f}+\frac{1}{(N_f-1)!}\left(\hat{\mathcal{A}}-1+\hat{\varepsilon}\right)^{N_f-1}\left(\hat{\mathcal{A}}-\hat{\varepsilon}\right)\tag{98}$$

if $\hat{\mathcal{A}}>1$, where we define $\hat{\varepsilon}\equiv\beta\varepsilon$. The volume of the moduli space of the vortices is obtained in the limit of $\hat{\varepsilon}\to 0$.

For $k=2$, we have the choices of the charges and boundary conditions as $\vec{k}=\{(2,0),(1,1)\}$ and $\vec{n}=\{(0,0),(-1,1)\}$. Then the volume becomes

$$\begin{aligned}&\mathrm{Vol}\left(\mathcal{M}_2^{2,N_f}(S^1\times I)\right)\\ &=\frac{2}{(2N_f)!}(\hat{\mathcal{A}}-2)^{2N_f}-\frac{1}{(2N_f-1)!}\left(\hat{\mathcal{A}}-2+\hat{\varepsilon}\right)^{2N_f-1}\left(\hat{\mathcal{A}}-\hat{\varepsilon}\right)\\ &\quad+\frac{1}{N_f!N_f!}(\hat{\mathcal{A}}-1)^{2N_f}-\frac{1}{(N_f+1)!(N_f-1)!}\left(\hat{\mathcal{A}}-1-\hat{\varepsilon}\right)^{N_f+1}\left(\hat{\mathcal{A}}-1+\hat{\varepsilon}\right)^{N_f-1},\end{aligned}\tag{99}$$

if $\hat{\mathcal{A}}>2$.

Similarly, we obtain

$$\begin{aligned}
&\mathrm{Vol}\left(\mathcal{M}_3^{2,N_f}(S^1\times I)\right)\\
&= -\frac{2}{(3N_f)!}\left(\hat{\mathcal{A}}-3\right)^{3N_f} + \frac{1}{(3N_f-1)!}\left(\hat{\mathcal{A}}-3+\hat{\varepsilon}\right)^{3N_f-1}\left(\hat{\mathcal{A}}-\hat{\varepsilon}\right)\\
&\quad -\frac{2}{(2N_f)!N_f!}\left(\hat{\mathcal{A}}-2\right)^{2N_f}\left(\hat{\mathcal{A}}-1\right)^{N_f}\\
&\quad +\frac{1}{(2N_f+1)!(N_f-1)!}\left(\hat{\mathcal{A}}-2-\hat{\varepsilon}\right)^{2N_f+1}\left(\hat{\mathcal{A}}-1+\hat{\varepsilon}\right)^{N_f-1}\\
&\quad +\frac{1}{(2N_f-1)!(N_f+1)!}\left(\hat{\mathcal{A}}-2+\hat{\varepsilon}\right)^{2N_f-1}\left(\hat{\mathcal{A}}-1-\hat{\varepsilon}\right)^{N_f+1}
\end{aligned} \tag{100}$$

if $\hat{\mathcal{A}} > 3$ for $k = 3$, and

$$\begin{aligned}
&\mathrm{Vol}\left(\mathcal{M}_4^{2,N_f}(S^1\times I)\right)\\
&= \frac{2}{(4N_f)!}\left(\hat{\mathcal{A}}-4\right)^{4N_f} - \frac{1}{(4N_f-1)!}\left(\hat{\mathcal{A}}-4+\hat{\varepsilon}\right)^{4N_f-1}\left(\hat{\mathcal{A}}-\hat{\varepsilon}\right)\\
&\quad +\frac{2}{(3N_f)!N_f!}\left(\hat{\mathcal{A}}-3\right)^{3N_f}\left(\hat{\mathcal{A}}-1\right)^{N_f}\\
&\quad -\frac{1}{(3N_f+1)!(N_f-1)!}\left(\hat{\mathcal{A}}-3-\hat{\varepsilon}\right)^{3N_f+1}\left(\hat{\mathcal{A}}-1+\hat{\varepsilon}\right)^{N_f-1}\\
&\quad -\frac{1}{(3N_f-1)!(N_f+1)!}\left(\hat{\mathcal{A}}-3+\hat{\varepsilon}\right)^{3N_f-1}\left(\hat{\mathcal{A}}-1-\hat{\varepsilon}\right)^{N_f+1} + \frac{1}{(2N_f)!(2N_f)!}\left(\hat{\mathcal{A}}-2\right)^{4N_f}\\
&\quad -\frac{1}{(2N_f+1)!(2N_f-1)!}\left(\hat{\mathcal{A}}-2-\hat{\varepsilon}\right)^{2N_f+1}\left(\hat{\mathcal{A}}-2+\hat{\varepsilon}\right)^{2N_f-1},
\end{aligned} \tag{101}$$

if $\hat{\mathcal{A}} > 4$ for $k = 4$.

So far, we have considered the general number of flavors N_f with the $U(2)$ gauge group. The volume of the moduli space of k vortices is a complicated kN_fth-order polynomial in $\hat{\mathcal{A}}$. However, setting $N_f = N_c = 2$, we see that the order of polynomial of the volume reduces remarkably. The vortices in this situation ($N_f = N_c$) are called non-Abelian local vortices.

Putting $N_f = N_c = 2$ into the results (97)–(101) for general N_f, we find the volume of the moduli space of non-Abelian local vortices on the cylinder as

$$\mathrm{Vol}\left(\mathcal{M}_0^{2,2}(S^1\times I)\right) = 1,$$

$$\mathrm{Vol}\left(\mathcal{M}_1^{2,2}(S^1\times I)\right) = \hat{\mathcal{A}} - 1 + \hat{\varepsilon} - \hat{\varepsilon}^2,$$

$$\mathrm{Vol}\left(\mathcal{M}_2^{2,2}(S^1\times I)\right) = \frac{1}{2}\hat{\mathcal{A}}^2 - \left(\frac{5}{3} - \hat{\varepsilon} + \hat{\varepsilon}^2\right)\hat{\mathcal{A}} + \frac{17}{12} - \frac{5}{3}\hat{\varepsilon} + 2\hat{\varepsilon}^2 - \frac{2}{3}\hat{\varepsilon}^3 + \frac{1}{3}\hat{\varepsilon}^4,$$

$$\begin{aligned}
\mathrm{Vol}\left(\mathcal{M}_3^{2,2}(S^1\times I)\right) &= \frac{1}{6}\hat{\mathcal{A}}^3 - \frac{1}{2}\left(\frac{7}{3} - \hat{\varepsilon} + \hat{\varepsilon}^2\right)\hat{\mathcal{A}}^2 + \left(\frac{331}{120} - \frac{7}{3}\hat{\varepsilon} + \frac{8}{3}\hat{\varepsilon}^2 - \frac{2}{3}\hat{\varepsilon}^3 + \frac{1}{3}\hat{\varepsilon}^4\right)\hat{\mathcal{A}}\\
&\quad - \frac{793}{360} + \frac{331}{120}\hat{\varepsilon} - \frac{85}{24}\hat{\varepsilon}^2 + \frac{29}{18}\hat{\varepsilon}^3 - \frac{11}{12}\hat{\varepsilon}^4 + \frac{2}{15}\hat{\varepsilon}^5 - \frac{2}{45}\hat{\varepsilon}^6,
\end{aligned}$$

$$\begin{aligned}\mathrm{Vol}\left(\mathcal{M}_4^{2,2}(S^1\times I)\right) &= \frac{1}{24}\hat{\mathcal{A}}^4 - \frac{1}{6}\big(3-\hat{\varepsilon}+\hat{\varepsilon}^2\big)\hat{\mathcal{A}}^3 + \frac{1}{2}\left(\frac{409}{90} - 3\hat{\varepsilon} + \frac{10}{3}\hat{\varepsilon}^2 - \frac{2}{3}\hat{\varepsilon}^3 + \frac{1}{3}\hat{\varepsilon}^4\right)\hat{\mathcal{A}}^2 \\ &\quad - \left(\frac{292}{63} - \frac{409}{90}\hat{\varepsilon} + \frac{111}{20}\hat{\varepsilon}^2 - \frac{37}{18}\hat{\varepsilon}^3 + \frac{41}{36}\hat{\varepsilon}^4 - \frac{2}{15}\hat{\varepsilon}^5 + \frac{2}{45}\hat{\varepsilon}^6\right)\hat{\mathcal{A}} \\ &\quad + \frac{18047}{5040} - \frac{292}{63}\hat{\varepsilon} + \frac{37}{6}\hat{\varepsilon}^2 - \frac{16}{5}\hat{\varepsilon}^3 + \frac{35}{18}\hat{\varepsilon}^4 - \frac{19}{45}\hat{\varepsilon}^5 \\ &\quad + \frac{7}{45}\hat{\varepsilon}^6 - \frac{4}{315}\hat{\varepsilon}^7 + \frac{1}{315}\hat{\varepsilon}^8, \end{aligned} \tag{102}$$

at the finite $\hat{\varepsilon}$.

Taking the limit of $\hat{\varepsilon} \to 0$ in the above results of (102) for $k = 1, 2, 3, 4$, we finally obtain the moduli space volume of vortices with $N_f = N_c = 2$ on the cylinder $S^1 \times I$:

$$\begin{aligned} \mathrm{Vol}\left(\mathcal{M}_0^{2,2}(S^1\times I)\right) &= 1, \\ \mathrm{Vol}\left(\mathcal{M}_1^{2,2}(S^1\times I)\right) &= \hat{\mathcal{A}} - 1, \\ \mathrm{Vol}\left(\mathcal{M}_2^{2,2}(S^1\times I)\right) &= \frac{1}{2}\hat{\mathcal{A}}^2 - \frac{5}{3}\hat{\mathcal{A}} + \frac{17}{12}, \\ \mathrm{Vol}\left(\mathcal{M}_3^{2,2}(S^1\times I)\right) &= \frac{1}{6}\hat{\mathcal{A}}^3 - \frac{7}{6}\hat{\mathcal{A}}^2 + \frac{331}{120}\hat{\mathcal{A}} - \frac{793}{360}, \\ \mathrm{Vol}\left(\mathcal{M}_4^{2,2}(S^1\times I)\right) &= \frac{1}{24}\hat{\mathcal{A}}^4 - \frac{1}{2}\hat{\mathcal{A}}^3 + \frac{409}{180}\hat{\mathcal{A}}^2 - \frac{292}{63}\hat{\mathcal{A}} + \frac{18047}{5040}. \end{aligned} \tag{103}$$

Surprisingly, they completely agree with the volume of the moduli space of the local vortices on the sphere S^2, derived in [9,10], up to an overall normalization and a rescaling to define the moduli space. The computation of the volume of the moduli space of vortices on sphere S^2 has given the asymptotic behavior at large area $\hat{\mathcal{A}}$ which reduces drastically when $N_c = N_f$, and has suggested a formula [9,10],[5]

$$\mathrm{Vol}\left(\mathcal{M}_k^{N,N}(S^2)\right) \sim \frac{\hat{\mathcal{A}}^k}{k!}. \tag{104}$$

The physical reason for the reduction of asymptotic power of the volume is the following. When $N_c = N_f$, the non-Abelian vortices are called non-Abelian local vortices, since the field configuration approaches the (unique) vacuum exponentially [27] outside of the local vortices of the intrinsic size $1/(g^2\zeta)$. Their position moduli (k complex dimensions) can extend to the entire area, whereas all the other moduli ($k(N-1)$ complex dimensions) correspond to orientations in internal flavor symmetry and can spread only up to the size $1/(g^2\zeta)$ around the local vortex [28,29]. Therefore, the asymptotic power of $\hat{\mathcal{A}}$ for local vortices is just k, corresponding only to the number of position moduli. When $N_c < N_f$, on the other hand, vortices are called semi-local vortices, since the field configuration approaches to (non-unique) vacua only in some powers of the distance away from the vortices. Not only the position moduli (k complex dimensions) but also all the other moduli ($k(N_f - 1)$ complex dimensions) can now extend to the entire area. These $k(N_f - 1)$-dimensional moduli are called the size moduli instead of the orientational moduli [30]. This is the reason why the asymptotic power of $\hat{\mathcal{A}}$ becomes kN_f for the semi-local vortices.

[5] Eq. (4.52) of Ref. [9,10] has an additional factor of $N!$ which we have forgotten to divide out, apart from a rescaling by $(2\pi)^N$ to define the moduli space.

From this physical consideration, it is interesting and gratifying to see that the volume (103) of the moduli space of the local vortices $N_c = N_f$ on the cylinder agrees exactly with that on the sphere S^2. We also note that the volume on the cylinder (102) before taking the limit $\hat{\varepsilon} \to 0$ can depend on the mass difference $\hat{\varepsilon}$, but only at non-leading powers of $\hat{\mathcal{A}}$. Since the mass differences are originated from holonomies at the boundaries of the cylinder [14,23], this result is also consistent with the notion that the effect of holonomy only extends up to a finite distance from the boundary for local vortices with the intrinsic size $1/(g^2\zeta)$. So these non-trivial results, including the coefficients of the polynomial, suggest that our localization formula and T-duality between the domain walls and vortices work correctly.

So far, we have assumed that the area $\hat{\mathcal{A}}$ is sufficiently larger than the vortex charge k. However, for the fixed vortex charge k, there exists an exact lower bound of the area, which is called the Bradlow bound [31]. The Bradlow bound of the volume essentially comes from the integral formula (57), where the integral vanishes if the exponent is negative. So the behavior of the volume changes depending on whether the area is larger than the charge or not. As a result, the functional dependence of the volume on $\hat{\mathcal{A}}$ changes as $\hat{\mathcal{A}}$ decreases towards the Bradlow bound. For example, let us consider again the case that $N_c = N_f = 2$ (local vortex) and $k = 4$ in the limit of $\hat{\varepsilon} \to 0$. As explained, if $\hat{\mathcal{A}}$ is larger than 4, the volume is given in (103). If $3 < \hat{\mathcal{A}} \leq 4$, then all the terms containing the factor $(\hat{\mathcal{A}} - 4)$ (in the limit of $\hat{\varepsilon} \to 0$) in (101) drop out because of the formula (57). Then the volume becomes

$$\mathrm{Vol}\left(\mathcal{M}_4^{2,2}(S^1 \times I)\right) = \frac{\hat{\mathcal{A}}^8}{6720} - \frac{\hat{\mathcal{A}}^7}{252} + \frac{2\hat{\mathcal{A}}^6}{45} - \frac{4\hat{\mathcal{A}}^5}{15} + \frac{67\hat{\mathcal{A}}^4}{72} - \frac{173\hat{\mathcal{A}}^3}{90} + \frac{409\hat{\mathcal{A}}^2}{180} - \frac{436\hat{\mathcal{A}}}{315} + \frac{1663}{5040}. \quad (105)$$

If $2 < \hat{\mathcal{A}} \leq 3$, we similarly find

$$\mathrm{Vol}\left(\mathcal{M}_4^{2,2}(S^1 \times I)\right) = \frac{\hat{\mathcal{A}}^8}{2880} - \frac{\hat{\mathcal{A}}^7}{180} + \frac{7\hat{\mathcal{A}}^6}{180} - \frac{7\hat{\mathcal{A}}^5}{45} + \frac{7\hat{\mathcal{A}}^4}{18} - \frac{28\hat{\mathcal{A}}^3}{45} + \frac{28\hat{\mathcal{A}}^2}{45} - \frac{16\hat{\mathcal{A}}}{45} + \frac{4}{45}. \quad (106)$$

The volume vanishes if $\hat{\mathcal{A}} \leq 2$. We plot the volume as a function of $\hat{\mathcal{A}}$ for the above regions in Fig. 10. We note that the functions are smoothly connected at each boundary ($\hat{\mathcal{A}} = 3$ and 4), since the derivatives coincide with each other up to high orders.

7. Conclusion and Discussion

In this paper, we have formulated a path integral to obtain the volume of the moduli space of the domain walls. We have seen that the localization method is a powerful tool to calculate the volume of the moduli space without the explicit metric. We have also noticed that the localization method is useful in understanding not only the global structure of the moduli space like the volume, but also the detailed and interesting properties of the moduli space through the dualities.

So far, we have not assumed that supersymmetry is behind the BPS domain-wall system. However, the BRST symmetry, which plays important roles in the localization method, is known to be regarded as a part of supersymmetry. Actually, our BRST transformations (13) and (14) are the dimensional reduction of the two-dimensional A-twisted supersymmetric transformation to one dimension. So we can expect that our volume formula is closely related to a partition function or vacuum expectation

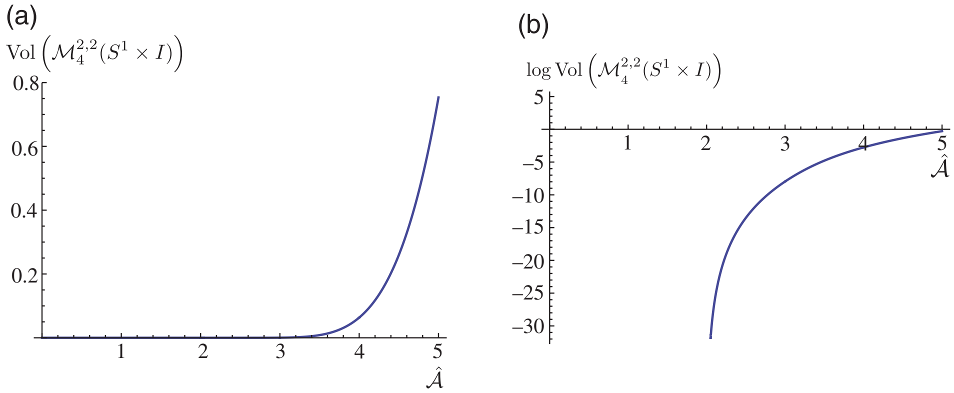

Fig. 10. (a) The volume of the moduli space of the non-Abelian local vortices on the cylinder $S^1 \times I$ as a function of the area $\hat{\mathcal{A}}$, for $N_c = N_f = 2$ and $k = 4$. (b) A logarithmic plot of the volume, showing that the volume vanishes at $\hat{\mathcal{A}} = 2$ (the Bradlow bound). The volume as a function of $\hat{\mathcal{A}}$ differs in the regions $\hat{\mathcal{A}} > 4$, $3 < \hat{\mathcal{A}} \leq 4, 2 < \hat{\mathcal{A}} \leq 3$, and $\hat{\mathcal{A}} < 2$ as one approaches the Bradlow bound. The different functions are smoothly connected with each other at the boundaries of each region up to high derivatives.

value (vev) in supersymmetric gauge theories. It is interesting to explore non-perturbative corrections in supersymmetric gauge theories from the viewpoint of the volume formula of the moduli space of the BPS domain walls. When boundaries are present, in particular, not much is known about the non-perturbative corrections in supersymmetric gauge theories. We have found that the boundary conditions are important and determine various interesting properties of the volume calculation. The volume calculation of the moduli space in supersymmetric gauge theories with the boundaries may shed light on the non-perturbative dynamics and dualities.

We have obtained exact results for the volume of the moduli space by using the localization method, but more directly we can also obtain the volume from an integral of a volume form, constructed by the explicit metric, over the moduli space. The volume is an integral result, where the local information is smeared out, but we can expect that information on the local metric can be reconstructed from the various uses of the localization method.

We sometimes encounter a mysterious relationship between the BPS solitons and (quantum mechanical) integrable systems like spin chains. The partition functions and vevs in supersymmetric gauge theories often become important quantities in the integrable systems. Our integral formula for the volume of the moduli space, which is expressed in terms of the determinant of the transition matrix, is also reminiscent of the integrable systems. We would like to investigate the relationship between the volume calculation of the BPS solitons and integrable systems in the future.

The volume of the moduli space is also mathematically interesting since the localization method says that the volume is almost determined by a topological nature of the moduli space. The volume of the moduli space may express topological invariants of the moduli spaces. Recently, localization of the $\mathcal{N} = (2, 2)$ supersymmetric gauge theories on S^2 have been performed [32,33]. The partition function of the $\mathcal{N} = (2, 2)$ supersymmetric gauge theories has two alternative expressions. One uses the localization around the Higgs branch, where the partition function reduces to the (anti-)vortex moduli zero-modes theory known as the (anti-)vortex partition function [9,10,34–38]. The other uses the localization around the Coulomb branch, where the path integral reduces to the multi-contour integrals. These two expressions turn out to be identical. Moreover, it is conjectured in [39] (see also [40]) that the free energy of the $\mathcal{N} = (2, 2)$ supersymmetric gauge theories is the quantum (world sheet instanton) corrected Kähler potential of Kähler moduli for the Higgs branch and actually reproduces the genus-zero Gromov–Witten invariant which counts holomorphic maps from the sphere to the target space manifold.

We have investigated the volume of the moduli space of the vortices on the cylinder via the T-duality. So we can expect that our vortex results on the cylinder may produce the moduli space of novel holomorphic maps from the cylinder to the target manifold.

The width of domain walls in an infinite interval has been studied in detail. If the mass difference of scalar fields H taking non-vanishing values in the two adjacent vacua is denoted as Δm, the width of the domain wall is given by $|\Delta m|/(g^2\zeta)$ in the weak coupling region ($g\sqrt{\zeta} \ll |\Delta m|$), but by $1/g\sqrt{\zeta}$ in the strong coupling region ($g\sqrt{\zeta} \gg |\Delta m|$) [16,17,41]. Our results from the localization formula are consistent with the weak coupling result for the infinite interval. Therefore, our results suggest that the width of the domain wall for finite intervals does not change significantly as we move from weak coupling toward the strong coupling region. Since the effect of the boundary is stronger as the length of interval decreases, it is quite possible that the intuition gained from the infinite interval case is not valid for domain walls in short intervals. It is an interesting future problem to work out the domain-wall solution at finite (short) intervals carefully.

We had to guess the sign factors associated with the intersection number of color lines. We can guess that the sign factor may originate from the fact that our diagonal gauge-fixing condition $\Phi^\alpha = 0$ is ambiguous and ill-defined when eigenvalues ϕ_a of the matrix Φ are degenerate. The color line connecting the boundary conditions at left and right boundaries are usually formulated in terms of eigenvalues of the matrix Σ, which is canonically conjugate to Φ. This complication is one of the reasons that prevented us deriving more explicitly the sign factors from the precise treatment of the path integral. We leave this question for a future study.

Acknowledgment

This work is supported in part by Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology, Japan No.21540279 (N.S.), No.21244036 (N.S.).

Appendix A. Explicit computation of $N_c = 1$ and $N_c = 2$ with $N_f = 3$

Using Eqs. (72) and (74) for $N_f = 3$, we find that our localization formula gives

$$\mathrm{Vol}\left(\mathcal{M}^{1,3}_{1\to 3}\right) = \frac{\beta^2}{2}(\hat{L} - d_{13})^2, \tag{A1}$$

$$\mathrm{Vol}\left(\mathcal{M}^{2,3}_{(1,2)\to(2,3)}\right) = \frac{\beta^2}{2}(\hat{L}^2 - d_{13}^2 + 2d_{12}d_{23}). \tag{A2}$$

They differ already at the next-to-leading order in $\hat{L}$.

To check our results for the localization formula at non-leading powers of $\hat{L}$, let us compute the volume using the rigid-rod approximation [14] where the domain wall connecting masses m_i and m_j have width d_{ij}. Let us denote the position of the first (second) wall as y_1 (y_2). For the Abelian gauge theory $N_c = 1$, two walls are non-penetrable [15,16]. Therefore we obtain

$$\begin{aligned}\mathrm{Vol}\left(\mathcal{M}^{1,3}_{1\to 3}\right) &= \beta^2 \int_{\frac{d_{12}}{2}}^{\hat{L}-d_{23}-\frac{d_{12}}{2}} \mathrm{d}y_1 \int_{y_1+\frac{d_{13}}{2}}^{\hat{L}-\frac{d_{23}}{2}} \mathrm{d}y_2 \\ &= \frac{\beta^2}{2}(\hat{L} - d_{13})^2, \end{aligned}\tag{A3}$$

giving an identical result as our localization formula (A1). For non-Abelian gauge theory $N_c = 2$, two domain walls are also non-penetrable, but the allowed region of positions is different. We separate the integration region into two and obtain

$$\begin{aligned}\mathrm{Vol}\left(\mathcal{M}^{2,3}_{(1,2)\to(2,3)}\right) &= \beta^2\left(\int_{\frac{d_{23}}{2}+d_{12}}^{\hat{L}-\frac{d_{23}}{2}} \mathrm{d}y_1 \int_{y_1-\frac{d_{13}}{2}}^{\hat{L}-\frac{d_{12}}{2}} \mathrm{d}y_2 + \int_{\frac{d_{23}}{2}}^{\frac{d_{23}}{2}+d_{12}} \mathrm{d}y_1 \int_{\frac{d_{12}}{2}}^{\hat{L}-\frac{d_{12}}{2}} \mathrm{d}y_2\right)\\ &= \frac{\beta^2}{2}(\hat{L}-2d_{12}+d_{13})(\hat{L}-d_{13}) + \beta^2(L-d_{12})d_{12}\\ &= \frac{\beta^2}{2}(\hat{L}^2 - d_{13}^2 + 2d_{12}d_{23}), \end{aligned} \tag{A4}$$

giving an identical result to our localization formula (A2).

Appendix B. Volume of moduli space of dual non-Abelian domain walls

We consider the topological sector with the maximal number of domain walls in $U(N_c)$ gauge theories with N_f flavors of scalar fields in the fundamental representation. The volume of the moduli space of domain walls is given by the determinant of the transition matrix $\mathcal{T}^{N_c,N_f}_{(1,\ldots,N_c)\to(\tilde{N}_c,\ldots,N_f)}$ as

$$\mathrm{Vol}\left(\mathcal{M}^{N_c,N_f}_{(1,\ldots,N_c)\to(\tilde{N}_c,\ldots,N_f)}\right) = \beta^{N_f}\,\mathrm{Det}\,\mathcal{T}^{N_c,N_f}_{(1,\ldots,N_c)\to(\tilde{N}_c,\ldots,N_f)}. \tag{B1}$$

The leading behavior at large volume is given by the largest powers in $\hat{L}$ as

$$\lim_{\hat{L}\to\infty}\frac{\mathcal{T}^{N_c,N_f}_{(1,\ldots,N_c)\to(\tilde{N}_c,\ldots,N_f)}}{\hat{L}^{N_f}} = \begin{pmatrix} \frac{1}{\tilde{N}_c!} & \cdots & \frac{1}{(N_c-2)!} & \frac{1}{(N_c-1)!}\\ \vdots & \ddots & \vdots & \vdots\\ \frac{1}{(\tilde{N}_c-N_c+2)!} & \cdots & \frac{1}{\tilde{N}_c!} & \frac{1}{(\tilde{N}_c+1)!}\\ \frac{1}{(\tilde{N}_c-N_c+1)!} & \cdots & \frac{1}{(\tilde{N}_c-1)!} & \frac{1}{\tilde{N}_c!} \end{pmatrix}. \tag{B2}$$

Let us define the determinant of the matrix on the right-hand side as Δ^{N_c,N_f}. For $N_c > \tilde{N}_c$, the above formula contains factorials of negative integers at the lower left corner. These factorials should be interpreted as zeros:

$$\frac{1}{(-n)!} = \frac{1}{\Gamma(-n+1)} = 0, \quad n\in\mathbb{Z}_+. \tag{B3}$$

In order to obtain the determinant, we subtract the (N_c-1)th row multiplied by $\tilde{N}_c+1$ from the N_cth (last) row of the right-hand side in order to eliminate the right-most entry of the N_cth row:

$$\Delta^{N_c,N_f} = \mathrm{Det}\begin{pmatrix} \frac{1}{\tilde{N}_c!} & \cdots & \frac{1}{(N_c-2)!} & \frac{1}{(N_c-1)!}\\ \vdots & \ddots & \vdots & \vdots\\ \frac{1}{(\tilde{N}_c-N_c+2)!} & \cdots & \frac{1}{\tilde{N}_c!} & \frac{1}{(\tilde{N}_c+1)!}\\ \frac{-(N_c-1)}{(\tilde{N}_c-N_c+1)!} & \cdots & \frac{-1}{(\tilde{N}_c-1)!} & 0 \end{pmatrix}. \tag{B4}$$

Similarly, subtracting the (N_c-2)th row multiplied by $\tilde{N}_c+2$ from the (N_c-1)th row and continuing the procedure, we can eliminate all the entries of the N_cth column except for the first row. Thus, we find

$$\Delta^{N_c,N_f} = \mathrm{Det}\begin{pmatrix} \frac{1}{\tilde{N}_c!} & \cdots & \frac{1}{(N_c-2)!} & \frac{1}{(N_c-1)!} \\ \frac{-(N_c-1)}{(\tilde{N}_c-1)!} & \cdots & \frac{-1}{(N_f-2)!} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \frac{-(N_c-1)}{(\tilde{N}_c-N_c+1)!} & \cdots & \frac{-1}{(\tilde{N}_c-1)!} & 0 \end{pmatrix}. \tag{B5}$$

Therefore, we obtain the recursion relation

$$\Delta^{N_c,N_f} = \frac{(N_c-1)!}{(N_f-1)!}\Delta^{N_c-1,N_f-1}. \tag{B6}$$

The recursion relation is solved with the initial condition $\Delta^{1,N_f} = 1/(N_f-1)!$ to give

$$\Delta^{N_c,N_f} = \frac{\prod_{j=1}^{N_c}(j-1)! \times \prod_{k=1}^{\tilde{N}_c}(k-1)!}{\prod_{i=1}^{N_f}(i-1)!}. \tag{B7}$$

Thus, we find the duality (90) is valid. Moreover, the coefficient of the leading term is given by the volume of the Grassmann manifold (84) apart from the intrinsically ambiguous overall normalization factor to define the moduli space. The proof here is valid also for the leading behavior of the equivalence of Abelian and non-Abelian domain walls, namely agreement between Eqs. (75) and (76).

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