symmetry

MDPI

Article

Symmetries for Nonconservative Field Theories on Time Scale

Octavian Postavaru *,† and **Antonela Toma** †

Center for Research and Training in Innovative Techniques of Applied Mathematics in Engineering, University Politehnica of Bucharest, 060042 Bucharest, Romania; antonela.toma@upb.ro
* Correspondence: opostavaru@linuxmail.org; Tel.: +40-0770-241-912
† These authors contributed equally to this work.

Abstract: Symmetries and their associated conserved quantities are of great importance in the study of dynamic systems. In this paper, we describe nonconservative field theories on time scales—a model that brings together, in a single theory, discrete and continuous cases. After defining Hamilton's principle for nonconservative field theories on time scales, we obtain the associated Lagrange equations. Next, based on the Hamilton's action invariance for nonconservative field theories on time scales under the action of some infinitesimal transformations, we establish symmetric and quasi-symmetric Noether transformations, as well as generalized quasi-symmetric Noether transformations. Once the Noether symmetry selection criteria are defined, the conserved quantities for the nonconservative field theories on time scales are identified. We conclude with two examples to illustrate the applicability of the theory.

Keywords: time scales; Noether theory; conserved quantity

Citation: Postavaru, O.; Toma, A. Symmetries for Nonconservative Field Theories on Time Scale. *Symmetry* **2021**, *13*, 552. https://doi.org/10.3390/sym13040552

Academic Editors: Marin Marin and Dumitru Baleanu

Received: 22 February 2021
Accepted: 24 March 2021
Published: 26 March 2021

1. Introduction

Noether's theorem, considered the most beautiful theorem in mathematical physics, attributes a conservation law to each symmetry [1]. In fact, Noether's theorem states that if the Lagrangian of a physical system is not affected by a continuous transformation in its proper coordinate system, then there is a conservation law, more precisely a quantity that remains constant. There is also an extension of Noether's theorem, called Noether's second theorem, which states that a discretization that maintains a variational symmetry creates a proper conservation law.

In general, the studied systems are dissipative, regardless of whether they have applicability in physics or engineering. This field of research is of particular interest due to the few existing contributions in the literature. A consistent formalism for describing the interaction of certain systems with the environment is described in [2]. This problem is a particular case of a class of problems called the inverse problem of mechanics and occurs when a Lagrangian is constructed from the equations of motion of a mechanical system. Since the end of the last century, this topic has long been studied by many mathematicians and theoretical physicists [3–6], and the interest of physicists in this subject has recently increased with the quantification of dissipative systems [7–10].

The study of real-time dissipative processes, such as radiative processes, is not possible without a technique that is capable of introducing nonconservative interactions into action. In [11], we find a series of effective field theory techniques that can be seen as modern theoretical tools to find the existence of hierarchies of scale for a physical problem. Power-counting results are presented for several situations of practical interest, and several applications of quantum electrodynamics are also illustrated. In the course notes [12], some of the basic ideas of effective field theories are introduced, such as relevant and irrelevant operators and scaling, renormalization in effective field theories, the decoupling of heavy particles, power counting and naive dimensional analysis. A number of introductions to some of the latest techniques and applications in the field can be found in [13].

To unify continuous analysis with discrete analysis, Hilger [14] introduced calculation on time scales. The paper presented in [15] developed the concept of the discrete calculus of variations and showed how it can be applied to the optimization problems of discrete systems. In [16], the authors investigate the invariant properties of the discrete Lagrangian for conservative systems, the discrete analog of the calculus of variations and the Noether theorem, and another work [17] extends this theory to nonconservative systems. Noether conservation symmetries and laws for nonconservative discrete systems with irregular lattices were developed in [18], first by finding the discrete analog of Noether identities and then by introducing the generalized quasi-external equations and their properties. In [19,20], the Noether symmetries and the conservation laws of nonconservative and nonholonomic mechanical systems on time scales were analyzed—a theory that unifies the two cases of continuous and discrete theories. For the first time, a relationship between isochronous variations and delta derivatives was established, as well as a link between isochronous variations and total variations on time scale. Q-versions of some basic concepts of continuous variational calculus such as the Euler–Lagrange equation and its applications to the isoperimetric Lagrange and optimal control problems were introduced in [21].

In the present paper, we extend the formalism of the nonconservative field theory presented in [22] to time scales. The field theory on discrete time is very different from classical dynamics except for the fact that it is a multi-freedom system, because every point of discrete time has a dynamical freedom. Apart from this fundamental character, the field theory has engineering applications in gravitation, electrostatics, magnetism, electric current flow, conductive heat transfer, fluid flow and seepage [23].

The paper is structured as follows: In Section 2, we examine delta differentiation and integration on time scales and review some properties that we use later. In Section 3, we derive the Lagrange equation for nonconservative field theories on time scales. Further, in Section 4 we discuss the criteria of symmetric and quasi-symmetric Noether transformations for two types of transformations, and then in Section 5, we define and identify the quantities that are conserved. We end the paper with some illustrative examples.

2. Preliminaries and Notations

In this section, we give some basic knowledge about the calculus on time scales introduced and developed by Hilger [14]. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. From the multitude of examples, we would like to mention the following: the real numbers $\mathbb{R}$, integers $\mathbb{Z}$, natural numbers $\mathbb{N}$, a sequence of points of $\mathbb{R}$ with a varying step size $\mathbb{S}$, the Cantor set $\mathbb{D}$ and sequence of closed intervals $\mathbb{P}$. In general, $\mathbb{P}$ is understood as a time scale which underpins differential equations with pulses. The theory presented in this section is valid for any type of time scale.

Definition 1. *The mapping $\sigma : \mathbb{T} \to \mathbb{T}$ defined by $\sigma(\tau) \equiv \inf\{s \in \mathbb{T} | \, s > \tau\}$ is called the jump operator. Accordingly, we may define the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ to be the mapping $\rho(\tau) \equiv \sup\{s \in \mathbb{T} | \, s < \tau\}$. Via these two operators, we may classify the points $\tau \in \mathbb{T}$ in terms of their right and left neighborhood as follows: τ is called right-dense, right-scattered, left-dense or left-scattered if $\sigma(\tau) = \tau$, $\sigma(\tau) > \tau$, $\rho(\tau) = \tau$ or $\rho(\tau) < \tau$, respectively.*

Definition 2. *If $\mathbb{T}$ has a left-scattered maximum M, then we define $\mathbb{T}^k = \mathbb{T} - \{M\}$; thus, this k-operator cuts off an eventually existing isolated maximum of $\mathbb{T}$. The graininess function $\mu : \mathbb{T}^k \to [0, \infty)$ is defined by $\mu(\tau) = \sigma(\tau) - \tau$.*

Remark 1. *If $\mathbb{T} = \mathbb{R}$, then $\sigma(\tau) = \rho(\tau) = \tau$. If $\mathbb{T} = \mathbb{Z}$, $\sigma(\tau) = \tau + 1$, $\rho(\tau) = \tau - 1$, and $\mu(\tau) = 1$.*

Definition 3. *A function $f\,:\,\mathbb{T}\to\mathbb{R}$ is called differentiable at $\tau\in\mathbb{T}$, with derivative $f^{\Delta}(\tau)\in\mathbb{T}$, if for each $\epsilon>0$ there is a neighborhood U of τ such that*

$$|f(\sigma(\tau))-f(s)-f^{\Delta}(\tau)(\sigma(\tau)-s)|\le\epsilon|\sigma(\tau)-s|\,,\quad \text{for all}\quad s\in U\,.$$

The derivative can also be defined in terms of a limit as follows:

$$f^{\Delta}(\tau)\equiv\frac{\Delta f}{\Delta\tau}=\lim_{s\to\tau}\frac{f(\sigma(\tau))-f(s)}{\sigma(\tau)-s}\,.$$

If $f\,,g:\,\mathbb{T}\to\mathbb{R}$ are differentiable in τ, then one has the following properties:

$$\begin{aligned} f(\sigma(t))=f^{\sigma}(\tau) &= f(\tau)+\mu(\tau)f^{\Delta}(\tau)\,,\\ (af+bg)^{\Delta}(\tau) &= af^{\Delta}(\tau)+bg^{\Delta}(\tau)\,,\\ (fg)^{\Delta}(\tau) &= f^{\Delta}(\tau)g^{\sigma}(\tau)+f(\tau)g^{\Delta}(\tau)=f^{\Delta}(\tau)g(\tau)+f^{\sigma}(\tau)g^{\Delta}(\tau)\,. \end{aligned} \tag{1}$$

Remark 2. *If $\mathbb{T}=\mathbb{R}$, then $f^{\Delta}(\tau)=f'(\tau)$. If $\mathbb{T}=\mathbb{Z}$, then $f^{\Delta}(\tau)=f(\tau+1)-f(\tau)$.*

Definition 4. *A function $f\;:\;\mathbb{T}\to\mathbb{R}$ is called rd-continuous if it is continuous at right-dense points and its left-sided limits exist (finite) at left-dense points. The set of rd-continuous functions is denoted by $C_{\rm rd}$, and the set of differentiable functions with rd-continuous derivative is denoted by $C^1_{\rm rd}$.*

Remark 3. *Every rd-continuous functions have an antiderivative. In particular, if $\tau_1\,,\tau_2\in\mathbb{T}$, then the antiderivative function F is defined as*

$$F(\tau)\equiv\int_{\tau_1}^{\tau_2}f(\tau)\,\Delta\tau\,,$$

with $\Delta\tau$ the measure on $\mathbb{T}$ [14].

Theorem 1. *For a strictly increasing function $\nu\,:\,\mathbb{T}\to\mathbb{R}$, we have the following properties:*
(1) (Chain rule) Let $\omega\,:\,\mathbb{T}^*\to\mathbb{R}$, where $\mathbb{T}^*\equiv\nu(\mathbb{T})$. If $\nu^{\Delta}(\tau)$ and $\omega^{\Delta^*}(\nu(\tau))$ exists, then

$$(\omega\circ\nu)^{\Delta}=\left(\omega^{\Delta^*}\circ\nu\right)\nu^{\Delta}\,, \tag{2}$$

(2) (Derivative of the inverse) For $\nu^{\Delta}(\tau)\neq0$, we have the following relation:

$$\left(\nu^{-1}\right)^{\Delta^*}(\nu(\tau))=\frac{1}{\nu^{\Delta}(\tau)}\,,$$

(3) (Substitution) If $f\,:\,\mathbb{T}\to\mathbb{R}$ is a $C_{\rm rd}$ function and ν is a $C^1_{\rm rd}$ function, then

$$\int_a^b f(\nu(\tau))\nu^{\Delta}(\tau)\Delta\tau=\int_{\nu(a)}^{\nu(b)}\left(f\circ\nu^{-1}\right)(s)\Delta^*s\,, \tag{3}$$

with $a\,,b\in\mathbb{T}$.

Lemma 1. *(Dubois–Reymond) Let $g\,:\,[a,b]\to\mathbb{R}^n$, and $g\in C_{\rm rd}$, then*

$$\int_a^b g(\tau)\,\eta^{\Delta}(\tau)\Delta\tau=0\,,$$

for all $\eta \in C^1_{\mathrm{rd}}$ with $\eta(a) = \eta(b) = 0$, if and only if $g(\tau) = c$, with $c \in \mathbb{R}^n$.

3. Lagrange Equation for Nonconservative Field Theories on Proper Time Scales

We consider N fields $\Phi^I(x^\mu)$, with $I = \overline{1,N}$ and $x^\mu = (x^0, x^1, x^2, x^3) \in \{T \times V\}$ with the time segment $T = [t_1, t_2]$ with $t \in \mathbb{T}$ and space volume V. To capture the nonconservative (e.q., dissipative) effects of a field that is subject to nonconservative interactions, we must double its degrees of freedom $\Phi^I \to (\Phi^I_1, \Phi^I_2)$. Thus, we can explain the correct causal evolution of the dynamics of the open system.

We denote the nonconservative Lagrange density with Ω, and it is generally an arbitrary function of two variables, their derivatives and their space-time coordinates x_μ:

$$\Omega\left[\Phi^I_a\right] = \Omega\left(x^\mu, \left(\Phi^I_a\right)^\sigma, \left(\Phi^I_a\right)^\Delta, \partial_i\left(\Phi^I_a\right)^\sigma\right), \tag{4}$$

with $i = \overline{1,3}$ and $\left(\Phi^I_a\right)^\sigma(x^\mu) = \Phi^I_a\left(\sigma(t), x^1, x^2, x^3\right)$. Additionally, Φ^I_a satisfies Definition 1 in the sense that

$$\frac{\Delta}{\Delta t}\Phi^I_a(t, x^1, x^2, x^3) = \lim_{s \to t} \frac{\Phi^I_a(\sigma(t), x^1, x^2, x^3) - \Phi^I_a(s, x^1, x^2, x^3)}{\sigma(t) - s}.$$

In the case of nonconservative field theories, we state Hamilton's principle on a time scale as

$$\int_{t_1}^{t_2} \Delta t \int_V d^3x\, \delta\Omega\left[\Phi^I_a\right] = 0, \tag{5}$$

where the isochronous variations are represented by δ.

Proposition 1. *For isochronous variations, we have the following relations:*

$$\delta\left(\Phi^I_a\right)^\Delta = \left(\delta\Phi^I_a\right)^\Delta, \quad \text{and} \quad \delta\left(\Phi^I_a\right)^\sigma = \left(\delta\Phi^I_a\right)^\sigma. \tag{6}$$

Proof. For the first relationship, we have

$$\begin{aligned}\left(\delta\Phi^I_a(t)\right)^\Delta &= \frac{\Delta}{\Delta t}\left(\delta\Phi^I_a\right) = \frac{\Delta}{\Delta t}\left(\frac{\partial\Phi^I_a}{\partial x^\mu}\delta x^\mu\right) = \frac{\Delta}{\Delta t}\left(\frac{\partial\Phi^I_a}{\partial x^i}\delta x^i\right) \\ &= \frac{\partial}{\partial x^i}\left(\frac{\Delta\Phi^I_a}{\Delta t}\right)\delta x^i = \frac{\partial\left(\Phi^I_a\right)^\Delta}{\partial x^i}\delta x^i = \delta\left(\Phi^I_a\right)^\Delta,\end{aligned}$$

where $\mu = \overline{0,3}$, $i = \overline{1,3}$ and we used Einstein's summation convention.

For the second relationship, we have

$$\left(\delta\Phi^I_a\right)^\sigma = \left(\frac{\partial\Phi^I_a}{\partial x^i}\delta x^i\right)^\sigma = \frac{\partial\left(\Phi^I_a\right)^\sigma}{\partial x^i}\delta x^i = \delta\left(\Phi^I_a\right)^\sigma.$$

□

Proposition 2. *The relationship between the total variation and the isochronous variation is given by the formula*

$$\Delta\Phi^I_a = \left(\Phi^I_a\right)^\Delta \Delta t + \delta\Phi^I_a. \tag{7}$$

Proof. The total variation of Φ_a^I is written as follows:

$$\Delta\Phi_a^I = \frac{\partial\Phi_a^I}{\partial t}\delta t + \frac{\partial\Phi_a^I}{\partial x^i}\delta x^i\,, \tag{8}$$

and the isochronous one is as follows:

$$\delta\Phi_a^I = \frac{\partial\Phi_a^I}{\partial x^i}\delta x^i\,. \tag{9}$$

Next, we consider $t = t(\alpha)$, where α is a generalized parameter that has convenient interpretations for the classical and relativistic case. We can write

$$\Delta t = \frac{\partial t}{\partial\alpha}\delta\alpha\,.$$

From Equations (8) and (9), we get

$$\Delta\Phi_a^I = \frac{\partial\Phi_a^I}{\partial t}\frac{\partial t}{\partial\alpha}\delta\alpha + \delta\Phi_a^I\,,$$

and the demonstration is done. □

We can write Hamilton's principle (5) in the following form:

$$\int_{t_1}^{t_2}\Delta t\int_V d^3x\left(\frac{\partial\Omega}{\partial x^\mu}\delta x^\mu + \frac{\partial\Omega}{\partial(\Phi_a^I)^\sigma}\delta\left(\Phi_a^I\right)^\sigma + \frac{\partial\Omega}{\partial(\Phi_a^I)^\Delta}\delta\left(\Phi_a^I\right)^\Delta + \frac{\partial\Omega}{\partial\partial_i(\Phi_a^I)^\sigma}\delta\partial_i\left(\Phi_a^I\right)^\sigma\right) = 0\,, \tag{10}$$

where the I indices satisfy the Einstein summation convention.

Next, we use the following relation:

$$\int d^3x\frac{\partial\Omega}{\partial(\partial_i\Phi)}\delta(\partial_i\Phi) = \int d^3x\left[-\partial_i\left(\frac{\partial\Omega}{\partial(\partial_i\Phi)}\right)\delta\Phi + \partial_i\left(\frac{\partial\Omega}{\partial(\partial_i\Phi)}\delta\Phi\right)\right]. \tag{11}$$

Because the edges of the volume are considered fixed, the second term of this expression does not contribute to the result. The first term in Equation (10) is a constant that does not influence the final result. Thus, Equation (10) together with Equation (11) reduce the Hamilton problem to

$$\int_{t_1}^{t_2}\Delta t\int_V d^3x\left(\frac{\partial\Omega}{\partial(\Phi_a^I)^\sigma}\delta\left(\Phi_a^I\right)^\sigma + \frac{\partial\Omega}{\partial(\Phi_a^I)^\Delta}\delta\left(\Phi_a^I\right)^\Delta - \partial_i\left(\frac{\partial\Omega}{\partial\partial_i(\Phi_a^I)^\sigma}\right)\delta\left(\Phi_a^I\right)^\sigma\right) = 0\,,$$

or

$$\int_{t_1}^{t_2}\Delta t\int_V d^3x\left[\left(\frac{\partial\Omega}{\partial(\Phi_a^I)^\sigma} - \partial_i\left(\frac{\partial\Omega}{\partial\partial_i(\Phi_a^I)^\sigma}\right)\right)\delta\left(\Phi_a^I\right)^\sigma + \frac{\partial\Omega}{\partial(\Phi_a^I)^\Delta}\delta\left(\Phi_a^I\right)^\Delta\right] = 0\,. \tag{12}$$

Proposition 3. *To transform Φ^σ in Φ^Δ, we use the following relation:*

$$\int_{t_1}^{t_2} f(t)(\delta\Phi(t))^\sigma\Delta t = -\int_{t_1}^{t_2}\int_{t_1}^{t} f(\tau)\Delta\tau(\delta\Phi(t))^\Delta\Delta t\,. \tag{13}$$

Proof. We use Equation (1) to achieve

$$\begin{aligned}\left[\int_{t_1}^{t} f(\tau)\Delta\tau\,\delta\Phi(t)\right]^{\Delta} &= \frac{\Delta}{\Delta t}\left[\int_{t_1}^{t} f(\tau)\Delta\tau\,\delta\Phi(t)\right] \\ &= \left[\int_{t_1}^{t} f(\tau)\Delta\tau\right]^{\Delta}(\delta\Phi(t))^{\sigma} + \int_{t_1}^{t} f(\tau)\Delta\tau\,(\delta\Phi(t))^{\Delta},\end{aligned}$$

or

$$\left[\int_{t_1}^{t} f(\tau)\Delta\tau\,\delta\Phi(t)\right]^{\Delta} = f(t)(\delta\Phi(t))^{\sigma} + \int_{t_1}^{t} f(\tau)\Delta\tau\,(\delta\Phi(t))^{\Delta}.$$

Integrating on both sides over Δt and applying Lemma 1, on the left side of the equality, we get zero. In Lemma 1, $g(t) = 1$ and

$$\eta(t) = \int_{t_1}^{t} f(\tau)\Delta\tau\,\delta\Phi(t) = (F(t) - F(t_1))\,\delta\Phi(t).$$

The four-volume is fixed at the boundary; therefore, $\eta(t_1) = \eta(t_2) = 0$. □

Theorem 2. *Lagrange equations for nonconservative theories on time scale are written*

$$\frac{\partial\Omega}{\partial(\Phi_a^I)^{\sigma}} - \partial_i\left(\frac{\partial\Omega}{\partial\partial_i(\Phi_a^I)^{\sigma}}\right) - \frac{\Delta}{\Delta t}\frac{\partial\Omega}{\partial(\Phi_a^I)^{\Delta}} = 0.$$

Proof. Combining Proposition 3 with Equation (12), we obtain

$$\begin{aligned}\int_{t_1}^{t_2}\Delta t\int_V d^3x\Bigg[-\int_{t_1}^{t}\left(\frac{\partial\Omega}{\partial(\Phi_a^I)^{\sigma}} - \partial_i\left(\frac{\partial\Omega}{\partial\partial_i(\Phi_a^I)^{\sigma}}\right)\right)\Delta\tau(\delta\Phi(\tau))^{\Delta} \\ + \frac{\partial\Omega}{\partial(\Phi_a^I)^{\Delta}}\delta\left(\Phi_a^I\right)^{\Delta}\Bigg] = 0.\end{aligned}$$

Because $\delta(\Phi_a^I)$ are independent of each other, the following expression is obtained with the help of Lemma 1:

$$-\int_{t_1}^{t}\left(\frac{\partial\Omega}{\partial(\Phi_a^I)^{\sigma}} - \partial_i\left(\frac{\partial\Omega}{\partial\partial_i(\Phi_a^I)^{\sigma}}\right)\right)\Delta\tau + \frac{\partial\Omega}{\partial(\Phi_a^I)^{\Delta}} = \text{const.}$$

Taking the delta derivatives of the above formula, the demonstration is finished. □

We define the current densities on time scale as follows:

$$\Pi_{Ia}^{0} = \frac{\partial\Omega}{\partial(\Phi_a^I)^{\Delta}}, \quad \Pi_{Ia}^{i} = \frac{\partial\Omega}{\partial\partial_i(\Phi_a^I)^{\sigma}}.$$

For historical reasons, it is advisable to separate the Lagrange density in conservative and nonconservative terms; i.e.,

$$\begin{aligned}\Omega\left[\Phi_1^I,\Phi_2^I\right] &= \mathcal{L}\left(x^{\mu},\left(\Phi_1^I\right)^{\sigma},\left(\Phi_2^I\right)^{\sigma},\left(\Phi_1^I\right)^{\Delta},\left(\Phi_2^I\right)^{\Delta},\partial_i\left(\Phi_1^I\right)^{\sigma},\partial_i\left(\Phi_2^I\right)^{\sigma}\right) \\ &+\mathcal{K}\left(x^{\mu},\left(\Phi_1^I\right)^{\sigma},\left(\Phi_2^I\right)^{\sigma},\left(\Phi_1^I\right)^{\Delta},\left(\Phi_2^I\right)^{\Delta},\partial_i\left(\Phi_1^I\right)^{\sigma},\partial_i\left(\Phi_2^I\right)^{\sigma}\right), \quad (14)\end{aligned}$$

where $\mathcal{L}$ is a sum of independent non-interacting field Lagrangians for each of the fields to be considered to be relevant within the interaction Lagrangian density $\mathcal{K}$.

A more correct parametrization is obtained using the $\pm$ base, defined as [22]

$$\Phi^I_+ = \frac{\Phi^I_1 + \Phi^I_2}{2}, \quad \Phi^I_- = \Phi^I_1 - \Phi^I_2. \tag{15}$$

The physical limit (PL) is given by the equations

$$\Phi^I_+ \to \Phi^I, \quad \Phi^I_- \to 0. \tag{16}$$

Because only the variable "-" contributes to the physical result, we get

$$\left[\frac{\partial\Omega}{\partial(\Phi^I_-)^\sigma} - \partial_i\left(\frac{\partial\Omega}{\partial\partial_i(\Phi^I_-)^\sigma}\right) - \frac{\Delta}{\Delta t}\frac{\partial\Omega}{\partial(\Phi^I_-)^\Delta}\right]_{PL} = 0,$$

or using Equation (14),

$$\begin{aligned}&\frac{\partial\mathcal{L}}{\partial(\Phi^I)^\sigma} - \partial_i\left(\frac{\partial\mathcal{L}}{\partial\partial_i(\Phi^I)^\sigma}\right) - \frac{\Delta}{\Delta t}\frac{\partial\mathcal{L}}{\partial(\Phi^I)^\Delta}\\ &= \left[\partial_i\left(\frac{\partial\mathcal{K}}{\partial\partial_i(\Phi^I_-)^\sigma}\right) + \frac{\Delta}{\Delta t}\frac{\partial\mathcal{K}}{\partial(\Phi^I_-)^\Delta} - \frac{\partial\mathcal{K}}{\partial(\Phi^I_-)^\sigma}\right]_{PL} = \mathcal{Q}_I.\end{aligned}$$

If $\mathcal{Q}_I = 0$, the Lagrange equations for the nonconservative field theory turn into the usual differential equations of motion on time scales.

4. Noether Symmetries on Time Scales

In this section, we discuss some criteria for symmetric Noether transformations for nonconservative field theories on time scales. There is no concept of a complete classification of the conserved currents. In the following, all Φ functions will be read as Φ^I_a.

On time scale, the action has the following definition:

$$S[\Phi] = \int_{t_1}^{t_2}\Delta t\int_V d^3x\,\Omega[\Phi]. \tag{17}$$

Definition 5. *We say that the Hamilton action (17) is invariant to the following infinitesimal transformations, which depend on a parameter r*

$$t^* = t, \quad (\Phi(x^\mu))^* = \Phi(x^\mu) + \epsilon_\alpha\xi^\alpha(x^\mu), \tag{18}$$

with infinitesimal parameters ϵ_α $(\alpha = \overline{1,r})$ and with the generators of the infinitesimal transformation $\xi^\alpha \equiv \xi^{I\alpha}_a$, if and only if

$$\begin{aligned}&\int_{t_1}^{t_2}\Delta t\int_V d^3x\,\Omega\left(x^\mu, \Phi^\sigma, \Phi^\Delta, \partial_i\Phi^\sigma\right)\\ &= \int_{t_1}^{t_2}\Delta t\int_V d^3x\,\Omega\left(x^{*\mu}, \Phi^{*\sigma}, \Phi^{*\Delta}, \partial_i\Phi^{*\sigma}\right).\end{aligned}$$

This transformation is called a symmetrical Noether transformation.

Theorem 3. *Noether symmetric transformations corresponding to infinitesimal transformations (18) are subject to the following conditions:*

$$\frac{\partial\Omega}{\partial\Phi^\sigma}(\xi^\alpha)^\sigma + \frac{\partial\Omega}{\partial\Phi^\Delta}(\xi^\alpha)^\Delta + \frac{\partial\Omega}{\partial\partial_i\Phi^\sigma}(\partial_i\xi^\alpha)^\sigma = 0.$$

Proof. Equation (18) together with Equation (19) give

$$\int_{t_1}^{t_2} \Delta t \int_V d^3x\, \Omega\Big(x^\mu, \Phi^\sigma, \Phi^\Delta, \partial_i \Phi^\sigma\Big)$$
$$= \int_{t_1}^{t_2} \Delta t \int_V d^3x \Omega\Big(x^{*\mu}, \Phi^\sigma + \epsilon_\alpha (\xi^\alpha)^\sigma, \Phi^\Delta + \epsilon_\alpha (\xi^\alpha)^\Delta, \partial_i \Phi^\sigma + \epsilon_\alpha (\partial_i \xi^\alpha)^\sigma\Big).$$

Deriving both sides of the equals at ϵ_α and doing $\epsilon_\alpha = 0$, we obtained the desired result. □

Definition 6. *We say that the Hamiltonian action (17) is invariant to infinitesimal transformations (18) if and only if*

$$\int_{t_1}^{t_2} \Delta t \int_V d^3x\, \Omega\Big(x^\mu, \Phi^\sigma, \Phi^\Delta, \partial_i \Phi^\sigma\Big)$$
$$= \int_{t_1}^{t_2} \Delta t \int_V d^3x \left[\Omega\Big(x^{*\mu}, \Phi^{*\sigma}, \Phi^{*\Delta}, \partial_i \Phi^{*\sigma}\Big) + \frac{\Delta}{\Delta t}(\tilde{\Delta} G)\right],$$

where $\tilde{\Delta} G = \epsilon_\alpha G^\alpha$, $G^\alpha = G^\alpha\left(x^\mu, \Phi^\sigma, \Phi^\Delta, \partial_i \Phi^\sigma\right)$. *This transformation is called a quasi-symmetric Noether transformation.*

Theorem 4. *Noether quasi-symmetric transformations corresponding to infinitesimal transformations (18) are subject to the following conditions:*

$$\frac{\partial \Omega}{\partial \Phi^\sigma}(\xi^\alpha)^\sigma + \frac{\partial \Omega}{\partial \Phi^\Delta}(\xi^\alpha)^\Delta + \frac{\partial \Omega}{\partial \partial_i \Phi^\sigma}(\partial_i \xi^\alpha)^\sigma = -\frac{\Delta}{\Delta t} G^\alpha .$$

Proof. Demonstration similar to that of the Theorem 3. □

Next, we consider the following sets of transformations:

$$\begin{aligned} t^* &= T(x^\mu, \epsilon^\alpha) = t + \epsilon_\alpha \xi_0^\alpha(x^\mu), \\ (\Phi(x^\mu))^* &= P_k(x^\mu, \epsilon^\alpha) = \Phi(x^\mu) + \epsilon_\alpha \xi^\alpha(x^\mu), \end{aligned} \tag{19}$$

$k = \overline{1, a\, N}$, and $\xi_0^\alpha \equiv \xi_0^{I\alpha}$.

Consider the function $t \in [t_1, t_2] \to \beta(t) = T(x^\mu, \epsilon^\alpha) \in \mathbb{R}$ to be a strictly increasing of class C^1_{rd} whose image is a new time scale. We denote the jump operator and delta derivatives with σ^* and Δ^*. Between σ and σ^*, we have the following relation:

$$\sigma^* \circ \beta = \beta \circ \sigma . \tag{20}$$

Definition 7. *We say that the Hamilton action (17) is invariant to the infinitesimal transformations (19) if and only if*

$$\int_{t_1}^{t_2} \Delta t \int_V d^3x\, \Omega\Big(x^\mu, \Phi^\sigma, \Phi^\Delta, \partial_i \Phi^\sigma\Big)$$
$$= \int_{\beta(t_1)}^{\beta(t_2)} \Delta^* t^* \int_V d^3x\, \Omega\Big(x^{*\mu}, \Phi^{*\sigma^*}, \Phi^{*\Delta^*}, \partial_i \Phi^{*\sigma^*}\Big) + \int_{t_1}^{t_2} \Delta t \int_V d^3x\, \frac{\Delta}{\Delta t}(\tilde{\Delta} G),$$

where $\tilde{\Delta} G = \epsilon_\alpha G^\alpha$, $G^\alpha = G^\alpha\left(x^\mu, \Phi^\sigma, \Phi^\Delta, \partial_i \Phi^\sigma\right)$. *This transformation is called a generalized quasi-symmetric Noether transformation.*

Theorem 5. *Noether generalized quasi-symmetric transformations corresponding to infinitesimal transformations (18) are subject to the following conditions:*

$$\frac{\partial\Omega}{\partial t}\xi_0^{I\alpha} + \Omega\,(\xi_0^{\alpha})^{\Delta} + \frac{\partial\Omega}{\partial\Phi^{\sigma}}(\xi^{\alpha})^{\sigma} + \frac{\partial\Omega}{\partial\Phi^{\Delta}}\Big((\xi^{\alpha})^{\Delta} - \Phi^{\Delta}(\xi_0^{\alpha})^{\Delta}\Big) + \frac{\partial\Omega}{\partial\partial_i\Phi^{\sigma}}(\partial_i\xi^{\alpha})^{\sigma} = -\frac{\Delta}{\Delta t}G^{\alpha}\,.$$

Proof.

$$\begin{aligned} &\int_{\beta(t_1)}^{\beta(t_2)} \Delta^* t^* \int_V d^3x\,\Omega\Big(x^{*\mu}, \Phi^{*\sigma^*}, \Phi^{*\Delta^*}, \partial_i\Phi^{*\sigma^*}\Big) \\ = &\int_{t_1}^{t_2} \Delta t \beta^{\Delta}(t) \int_V d^3x\,\Omega\left(\beta(t), x^{*i}, \Phi^*\circ\sigma^*\circ\beta, \frac{(\Phi^*\circ\beta)^{\Delta}}{\beta^{\Delta}(t)}, \partial_i\Phi^*\circ\sigma^*\circ\beta\right), \end{aligned} \tag{21}$$

where $(\Phi\circ f)(t) = \Phi(f(t), x^i)$, $i = \overline{1,3}$. To obtain the formula (21), we used the following property:

$$\Big(\Phi^{*\Delta^*}\circ f\Big)(t) = \frac{(\Phi^*\circ f)^{\Delta}}{f^{\Delta}(t)}(t)\,.$$

Using Equations (20), Equation (21) becomes

$$\int_{t_1}^{t_2} \Delta t \beta^{\Delta}(t) \int_V d^3x\,\Omega\left(\beta(t), x^{*i}, \Phi^*\circ\beta\circ\sigma, \frac{(\Phi^*\circ\beta)^{\Delta}}{\beta^{\Delta}(t)}, \partial_i\Phi^*\circ\beta\circ\sigma\right).$$

This equation together with the transformations (20) gives

$$\int_{t_1}^{t_2} \Delta t\, T^{\Delta} \int_V d^3x\,\Omega\left(T, x^{*i}, P_k^{\sigma}, \frac{P_k^{\Delta}}{T^{\Delta}}, \partial_i P_k^{\sigma}\right).$$

Differentiating this formula at ϵ_α and performing $\epsilon_\alpha = 0$ together with Equations (2) and (3), we get

$$\begin{aligned} \int_{t_1}^{t_2} \Delta t \int_V d^3x \Bigg[&\frac{\partial\Omega}{\partial t}\xi_0^{\alpha} + \Omega\,(\xi_0^{\alpha})^{\Delta} + \frac{\partial\Omega}{\partial\Phi^{\sigma}}(\xi^{\alpha})^{\sigma} \\ &+ \frac{\partial\Omega}{\partial\Phi^{\Delta}}\Big((\xi^{\alpha})^{\Delta} - \Phi^{\Delta}(\xi_0^{\alpha})^{\Delta}\Big) + \frac{\partial\Omega}{\partial\partial_i\Phi^{\sigma}}(\partial_i\xi^{\alpha})^{\sigma} + \frac{\Delta}{\Delta t}G^{\alpha}\Bigg] = 0\,. \end{aligned}$$

Given the arbitrariness of the integration interval, we obtain the desired result. □

5. Noether's Theorems on a Time Scale

In this section, we define conserved quantities in nonconservative field theories on time scales.

Definition 8. *We say a function of type*

$$I\Big(x^{\mu}, \Phi^{\sigma}, \Phi^{\Delta}, \partial_i\Phi^{\sigma}\Big),$$

is a conserved quantity in nonconservative field theories on time scales if and only if

$$\frac{\Delta}{\Delta t} I\Big(x^{\mu}, \Phi^{\sigma}, \Phi^{\Delta}, \partial_i\Phi^{\sigma}\Big) = 0\,.$$

Below, we present a series of conserved quantities that are obtained from Noether symmetries.

Theorem 6. *The infinitesimal transformations given by the Theorem 3 have the conserved quantities of the form*

$$I^\alpha = \frac{\partial \Omega}{\partial \Phi^\Delta} \xi^\alpha = c^\alpha \,. \tag{22}$$

Proof. Applying $\Delta/\Delta t$ to the right side of Equation (22), and after the operator $\int_{t_1}^{t_2} \Delta t \int_V d^3x$; i.e.,

$$\int_{t_1}^{t_2} \Delta t \int_V d^3x \left[\frac{\Delta}{\Delta t} \frac{\partial \Omega}{\partial \Phi^\Delta} (\xi^\alpha)^\sigma + \frac{\partial \Omega}{\partial \Phi^\Delta} (\xi^\alpha)^\Delta \right].$$

Using Theorem 2, we obtain

$$\int_{t_1}^{t_2} \Delta t \int_V d^3x \left[\frac{\partial \Omega}{\partial \Phi^\sigma} (\xi^\alpha)^\sigma - \partial_i \frac{\partial \Omega}{\partial \partial_i \Phi^\sigma} (\xi^\alpha)^\sigma + \frac{\partial \Omega}{\partial \Phi^\Delta} (\xi^\alpha)^\Delta \right],$$

and integrating by parts, we obtain

$$\int_{t_1}^{t_2} \Delta t \int_V d^3x \left[\frac{\partial \Omega}{\partial \Phi^\sigma} (\xi^\alpha)^\sigma + \frac{\partial \Omega}{\partial \Phi^\Delta} (\xi^\alpha)^\Delta + \frac{\partial \Omega}{\partial \partial_i \Phi^\sigma} (\partial_i \xi^\alpha)^\sigma \right] = 0,$$

where we used Theorem 3. □

Theorem 7. *The infinitesimal transformations given by Theorem 4 have conserved quantities of the form*

$$I^\alpha = \frac{\partial \Omega}{\partial \Phi^\Delta} \xi^\alpha + G^\alpha = c^\alpha \,.$$

Proof. Similar to the proof of Theorem 6. □

Theorem 8. *Infinitesimal transformations (19) together with Theorem 5 have conserved quantities of the form*

$$I^\alpha = \left(\Omega - \frac{\partial \Omega}{\partial t} \mu(t) \right) \xi_0^\alpha + \frac{\partial \Omega}{\partial \Phi^\Delta} \left(\xi^\alpha - \Phi^\Delta \xi_0^\alpha \right) + G^\alpha = c^\alpha \,.$$

Proof. We start from the following two definitions:

$$S[\Phi] = \int_{t_1}^{t_2} \Delta t \int_V d^3x \Omega \left(x^\mu, \Phi^\sigma, \Phi^\Delta, \partial_i \Phi^\sigma \right),$$

and

$$\overline{S}[\Phi, \Psi] = \int_{t_1}^{t_2} \Delta t \int_V d^3x \overline{\Omega} \left(x^\mu, \Psi^\sigma, \Phi^\sigma, \Psi^\Delta, \Phi^\Delta, \partial_i \Psi^\sigma, \partial_i \Phi^\sigma \right).$$

Let

$$\overline{\Omega} \left(x^\mu, \Psi^\sigma, \Phi^\sigma, \Psi^\Delta, \Phi^\Delta, \partial_i \Psi^\sigma, \partial_i \Phi^\sigma \right) = \Omega \left(\Psi^\sigma - \mu(t) \Psi^\Delta, x^i, \Phi^\sigma, \frac{\Phi^\Delta}{\Psi^\Delta}, \partial_i \Phi^\sigma \right).$$

When $\Psi(t) = t$, $S[\Phi] = \overline{S}[\Phi, \Psi]$. According to Definition 7, we have

$$S\left[\Phi_a^I\right] = \int_{t_1}^{t_2} \Delta t \int_V d^3x \frac{\Delta}{\Delta t}(\tilde{\Delta} G) + \int_{\beta(t_1)}^{\beta(t_2)} \Delta^* t^* \int_V d^3x \Omega\left(x^{*\mu}, \Phi^{*\sigma^*}, \Phi^{*\Delta^*}, \partial_i \Phi^{*\sigma^*}\right)$$

$$\begin{aligned} &= \int_{t_1}^{t_2} \Delta t \beta^{\Delta}(t) \int_V d^3x \Omega\left(\beta(t), x^{*i}, \Phi^* \circ \sigma^* \circ \beta, \frac{(\Phi^* \circ \beta)^{\Delta}}{\beta^{\Delta}(t)}, \partial_i \Phi^* \circ \sigma^* \circ \beta\right) \\ &\quad + \int_{t_1}^{t_2} \Delta t \int_V d^3x \frac{\Delta}{\Delta t}(\tilde{\Delta} G) \\ &= \overline{S}[\Phi \circ \beta, \beta(t)] + \int_{t_1}^{t_2} \Delta t \int_V d^3x \frac{\Delta}{\Delta t}(\tilde{\Delta} G)\,. \end{aligned}$$

It follows that the action $\overline{S}[\Phi \circ \beta, \beta(t)]$ corresponds to the Noether symmetry in Definition 7. Applying Theorem 7, the Lagrange equation given in Theorem 2 has the following conserved quantities:

$$I^\alpha = \frac{\partial \overline{\Omega}}{\partial \Psi^{\Delta}} \xi_0^\alpha + \frac{\partial \overline{\Omega}}{\partial \Phi^{\Delta}} \xi^\alpha + G^\alpha = c^\alpha\,.$$

This equation, along with the relationships

$$\begin{aligned} \frac{\partial \overline{\Omega}}{\partial \Psi^{\Delta}} &= -\frac{\partial \Omega}{\partial t} \mu(t) - \frac{\partial \Omega}{\partial \Phi^{\Delta}} \Phi^{\Delta} + \Omega\,, \\ \frac{\partial \overline{\Omega}}{\partial \Phi^{\Delta}} &= \frac{\partial \Omega}{\partial \Phi^{\Delta}}\,, \end{aligned}$$

ends the demonstration. □

6. Illustrative Examples

In this section, we discuss two particular cases of this theory; namely, a continuous case and a discrete case.

6.1. Example 1

If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$ and $\mu(t) = 0$, and the Euler–Lagrange equations in Theorem 2 are written

$$\frac{\partial \Omega}{\partial \Phi} - \partial_\mu \left(\frac{\partial \Omega}{\partial \partial_\mu \Phi} \right) = 0\,.$$

Further, the Theorem 5 turns into

$$\dot{\Omega}\, \xi_0^\alpha + \Omega\, \dot{\xi}_0^\alpha + \frac{\partial \Omega}{\partial \Phi} \xi^\alpha - \frac{\partial \Omega}{\partial \dot{\Phi}} \dot{\Phi} \dot{\xi}_0^\alpha + \frac{\partial \Omega}{\partial \partial_\mu \Phi} \partial_\mu \xi^\alpha = -\dot{G}^\alpha\,,$$

and the conserved quantity in Theorem 8 becomes

$$I^\alpha = \Omega\, \xi_0^\alpha + \frac{\partial \Omega}{\partial \dot{\Phi}} (\xi^\alpha - \dot{\Phi} \xi_0^\alpha) + G^\alpha = c^\alpha\,.$$

6.2. Example 2

Next, we consider the discrete case $\mathbb{T} = h\mathbb{Z}$. Consequently, $\sigma(t) = t + h$ and $\mu(t) = h$, and the Ω function becomes

$$\Omega\left(x^{\mu}, \Phi(t+h, x^{i}), \frac{\Phi(t+h, x^{i}) - \Phi(t, x^{i})}{h}, \partial_{i}\Phi(t+h, x^{i})\right).$$

Theorems 5 and 8 are transformed accordingly, taking into account the fact that

$$\frac{\Delta}{\Delta t} f(t, x^{i}) = \frac{f(t+h, x^{i}) - f(t, x^{i})}{h}, \quad \text{and} \quad f(t, x^{i})^{\sigma} = f(t+h, x^{i}).$$

7. Conclusions and Outlook

In this paper, we studied the Noether theorem for nonconservative field theories on time scales. After establishing Hamilton's principle, we extracted from it the Lagrange equations on time scales. For two different types of infinitesimal transformations, we established the selection criteria of Noether symmetries and based on these, we found the conserved quantities in the nonconservative field theories on time scales. In conclusion, we illustrated two examples for the discrete case and the continuous case to show that they were particular cases of the theory presented in this paper.

Continuous cases are common in the literature and underpin, for example, quantum field theory [24]. For future investigations, we want to extend our analysis to the following physical equations: Klein–Gordon, Maxwell and Dirac, from which it would be possible to extract the principles of time scale mechanics. Continuous theory is elegant but badly defined mathematically in many places, whilst the discrete time analogues are perhaps better defined mathematically. We would like to mention the success of the lattice field theory in the Yang–Mills problem. If no empirical test could distinguish between their predictions, we would rather choose the discrete theory. Possibly, using the time scale expression, we can highlight certain intervals of interest for which we can use discrete calculation.

The study of Maxwell's equations allows us to address this theory to particular problems. We can discuss the case of a simple relativistic engine made of two current loops of arbitrary geometry, in which we shall consider the mechanical momentum and energy gained by the engine [25]. Extending this theory to the gravitational field on a time scale, we could calculate the interaction field Lagrangians for the electromagnetic and gravitational interactions [26]. Next, we can look for analogies between the Fibonacci sequence and certain spatially homogeneous and isotropic universes in Friedmann–Lemaitre–Robertson–Walker cosmology on time scales [27].

Beyond the Lagrangian and Hamiltonian dynamics of typical nonconservative field theory investigated in this work, a range of other phenomena such as spontaneous breaking symmetry can be theoretically studied using the same framework.

The standard case of the Nambu–Goldstone theorem remains unchanged on this time scale, and as a consequence, the numbers of broken generators are definitely related to the existence of the Nambu–Goldstone bosons.

Using the pioneering work of [28,29], in a subsequent work, we could calculate both relativistically and non-relativistically the appropriate dispersion relation and the appropriate counting for the Nambu–Goldstone bosons if we study systems in which the symmetry is spontaneously broken. As benchmarks, within the same scenario, in future we can use the method of operators used in [30] in non-relativistic theory, or more generally, for both relativistic and non relativistic cases, we can check if the dynamics of Nambu–Goldstone bosons are somehow governed by the quantum Yang–Baxter equations [31].

Author Contributions: Conceptualization, O.P. and A.T.; methodology, O.P. and A.T.; software, O.P. and A.T.; validation, O.P. and A.T.; formal analysis, O.P. and A.T.; investigation, O.P. and A.T.; resources, O.P. and A.T.; data curation, O.P. and A.T.; writing—original draft preparation, O.P. and A.T.; writing—review and editing, O.P.; visualization, O.P. and A.T. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Noether, E. Invariante Variationsprobleme. *Math. Phys. Kl.* **1918**, *1918*, 235–257.
2. Sebens, C.T.; Carroll, S.M. Self-Locating Uncertainty and the Origin of Probability in Everettian Quantum Mechanics. *arXiv* **2014**, arXiv:1405.7577.
3. Darboux, G. *Lecons Sur la Theorie General des Surfaces et les Applications Geometriques du Calcul Infinitesimal*; Gauthier-Villars: Paris, France, 1984.
4. Douglas, J. Solution of the inverse problem of the calculus of variations. *Trans. Am. Math. Soc.* **1941**, *50*, 71–128. [CrossRef]
5. Havas, P. The range of application of the lagrange formalism–I. *Il Nuovo Cimento* **1957**, *5*, 363–388. [CrossRef]
6. Santilli, R.M. *Foundations of Theoretical Physics I*; Springer: New York, NY, USA, 1978.
7. Dekker, H. *Classical and Quantum Mechanics of the Damped Harmonic Oscillator*; North-Holland: Amsterdam, The Netherlands, 1981.
8. Glauber, R.; Man'ko, V.I. Damping and fluctuations in coupled quantum oscillator systems. *Zh. Eksp. Teor. Fiz.* **1984**, *87*, 790–804.
9. Dodonov, V.V.; Man'ko, V.I.; Skarzhinsky. Inverse variational problems and ambiguities of quantization for a particle in a magnetic field. *Hadron. J.* **1983**, *6*, 159–179.
10. Okubo, S. Canonical quantization of some dissipative systems and nonuniqueness of Lagrangians. *Phys. Rev. A* **1981**, *23*, 2776–1784. [CrossRef]
11. Burgess, C.P. An Introduction to Effective Field Theory. *Ann. Rev. Nucl. Part. Sci.* **2007**, *57*, 329–362. [CrossRef]
12. Manohar, A.V. Effective field theories. *Lect. Notes Phys.* **1997**, *479*, 311–362.
13. Rothstein, I.Z. Effective field theories. In *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 2002): Particle Physics and Cosmology: The Quest for Physics Beyond the Standard Model(s)*; World Scientific: Singapore, 2002; pp. 101–189.
14. Hilger, S. *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*; Universtät Würzburg: Würzburg, Germany, 1988.
15. Cadzow, J.A. Discrete calculus of variations. *Int. J. Control.* **1970**, *11*, 393–407. [CrossRef]
16. Logan, J.D. First integrals in the discrete variational calculus. *Aequationes Math.* **1973**, *9*, 210–220. [CrossRef]
17. Chen, L.Q.; Zhang, H.B.; Liu, R.W. First integrals of the discrete nonconservative and nonholonomic systems. *Chin. Phys.* **2005**, *14*, 238–243.
18. Fu, J.L.; Chen, L.Q.; Chen, B.Y. Noether-type theorem for discrete nonconservative dynamical systems with nonregular lattices. *Sci. China-Phys. Mech. Astron.* **2010**, *53*, 545–554. [CrossRef]
19. Fu, J.L.; Chen, L.Q.; Chen, B.Y. Noether-type theory for discrete mechanico-electrical dynamical systems with nonregular lattices. *Sci. China-Phys. Mech. Astron.* **2010**, *53*, 1687–1698. [CrossRef]
20. Cai, P.P.; Fu, J.L.; Guo, Y.X. Noether symmetries of the nonconservative and nonholonomic systems on time scales. *Sci. China-Phys. Mech. Astron.* **2013**, *56*, 1017–1028. [CrossRef]
21. Bangerezako, G. Variational q-calculus. *J. Math. Anal. Appl.* **2004**, *289*, 650–665. [CrossRef]
22. Galley, C.R. Classical Mechanics of Nonconservative Systems. *Phys. Rev. Lett.* **2013**, *110*, 174301. [CrossRef]
23. Fuller, A.J.B. *Engineering Field Theory*; Pergamon: Oxford, UK, 1973.
24. Peskin, M.E.; Schroeder, D.V. *An Introduction to Quantum Field Theory*; Perseus Books: Reading, MA, USA, 1995.
25. Rajput, S.; Yahalom, A.; Qin, H. Lorentz Symmetry Group, Retardation and Energy Transformations in a Relativistic Engine. *Symmetry* **2021**, *13*, 420. [CrossRef]
26. Gueorguiev, V.G.; Maeder, A. Geometric Justification of the Fundamental Interaction Fields for the Classical Long-Range Forces. *Symmetry* **2021**, *13*, 379. [CrossRef]
27. Faraoni, V.; Atieh, F. Generalized Fibonacci Numbers, Cosmological Analogies, and an Invariant. *Symmetry* **2021**, *13*, 200. [CrossRef]
28. Goldstone, J. Field Theories with Superconductor Solutions. *Nuovo Cim.* **1961**, *19*, 154. [CrossRef]
29. Nambu, Y.; Jona-Lasinio, G. Dynamical Model of Elementary Particles Based on an Analogy with Superconductivity. I. *Phys. Rev.* **1961**, *122*, 345. [CrossRef]
30. Arraut, I. The Nambu Goldstone theorem in nonrelativistic systems. *Int. J. Mod. Phys. A* **2017**, *32*, 1750127. [CrossRef]
31. Arraut, I. The Quantum Yang-Baxter Conditions: The Fundamental Relations behind the Nambu-Goldstone Theorem. *Symmetry* **2019**, *11*, 803. [CrossRef]