Quark-gluon thermodynamics with the $\mathbb{Z}_3$ symmetry

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Abstract. We propose a simple model with the $\mathbb{Z}_3$ symmetry in order to answer whether the symmetry is a good concept in QCD with light quark mass. The model is constructed by imposing the flavor-dependent twisted boundary condition (TBC) on the three-flavor Polyakov-loop extended Nambu-Jona-Lasinio model. In the model, the $\mathbb{Z}_3$ symmetry is preserved below some temperature T_c, but spontaneously broken above T_c. Dynamics of the simple model is similar to that of the original PNJL model without the TBC, indicating that the $\mathbb{Z}_3$ symmetry is a good concept. We also investigate the interplay between the $\mathbb{Z}_3$ symmetry and the emergence of the quarkyonic phase.

1. Introduction

In the limit of zero current quark mass, the chiral condensate is an exact order parameter for the chiral restoration. In the limit of infinite current quark mass, on the contrary, the Polyakov loop becomes an exact order parameter for the deconfinement transition, since the $\mathbb{Z}_3$ symmetry is exact there. For the real world in which u and d quarks have small current quark masses, the chiral condensate is considered to be a good order parameter, but it is not clear whether the Polyakov loop is a good order parameter. In this paper, we approach this problem by proposing a simple model with the $\mathbb{Z}_3$ symmetry. This paper is based on our recent papers [1, 2].

We start with the SU(3) gauge theory with three degenerate flavor quarks. The partition function Z in Euclidean spacetime is

$$Z = \int DqD\bar{q}DA \exp\left(-S_0\right), \quad S_0 = \int d^4x[\sum_f \bar{q}_f(\gamma_\nu D_\nu + m_f)q_f + \frac{1}{4g^2}{F^a_{\mu\nu}}^2] \tag{1}$$

with the temporal boundary condition $q_f(x, \beta = 1/T) = -q_f(x, 0)$ where T is the temperature. The $\mathbb{Z}_3$ transformation changes the ferimon boundary condition as [3, 4]

$$q_f(x, \beta) = -\exp\left(-i2\pi k/3\right)q_f(x, 0) \tag{2}$$

for integer $k = 0, 1, 2$, while the action keeps the original form. The $\mathbb{Z}_3$ symmetry thus breaks down by the fermion boundary condition. Now we assume the twisted boundary conditions (TBC)

$$q_f(x, \beta) \quad = \quad -\exp\left(i\theta_f\right)q_f(x, 0) \equiv -\exp\left[i2\pi(f-1)/3\right]q_f(x, 0) \tag{3}$$

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for flavors $f = 1, 2, 3$. QCD with the TBC has the $\mathbb{Z}_3$ symmetry. Actually the $\mathbb{Z}_3$ transformation changes f into $f - k$, but $f - k$ can be relabeled by f since the action is invariant under the relabeling. The TBC is useful to understand the color confinement.

When the fermion field q_f is transformed as $q_f = \exp(-i\theta_f T\tau)q'_f$ for Euclidean time τ, the action S_0 is changed into

$$S(\theta_f) = \int d^4x[\sum_f \bar{q}'_f(\gamma_\nu D_\nu - \mu_f\gamma_4 + m_f)q'_f + \frac{1}{4g^2}F_{\mu\nu}^2] \tag{4}$$

with the imaginary quark number chemical potential $\mu_f = iT\theta_f$, while the TBC is transformed back to the standard boundary condition. The action S_0 with the TBC is thus equivalent to the action $S(\theta_f)$ with the standard one. The partition function $Z(T,\theta)$ has the Roberge-Weiss (RW) periodicity [3]: $Z(T,\theta) = Z(T,\theta+2\pi k/3)$ for any integer k. The Polyakov-loop extended Nambu-Jona-Lasinio (PNJL) model is a good model to understand QCD at finite imaginary chemical potential [4].

In this paper, we consider QCD with the TBC in order to answer whether the $\mathbb{Z}_3$ symmetry is a good concept in QCD with light quark mass. Dynamics of the theory is studied concretely by imposing the TBC on the PNJL model, i.e., the TBC model.

2. PNJL and TBC models

The three-flavor PNJL Lagrangian is

$$\begin{aligned}\mathcal{L} &= \sum_f \bar{q}_f(\gamma_\nu D_\nu - \mu_f\gamma_4 + m_f)q_f - G_\mathrm{S}\sum_f\sum_{a=0}^{8}[(\bar{q}_f\lambda_a q_f)^2 + (\bar{q}_f i\gamma_5\lambda_a q_f)^2] \\ &\quad + G_\mathrm{D}\Big[\det_{ij}\bar{q}_i(1+\gamma_5)q_j + \det_{ij}\bar{q}_i(1-\gamma_5)q_j\Big] + \mathcal{U}(\Phi[A],\Phi^*[A],T).\end{aligned} \tag{5}$$

For the Polyakov potential $\mathcal{U}$ as a function of the Polyakov-loop Φ and its conjugate Φ^*, we take the potential of Ref. [5]. Now we consider the PNJL model with the TBC (3), that is, the TBC model. The thermodynamic potential of the TBC model is nothing but that of the PNJL model with the flavor-dependent imaginary chemical potential $\mu_f = i\theta_f T$. In the mean-field approximation, the thermodynamic potential Ω of the TBC model is obtained as

$$\Omega = -2\sum_f\int\frac{d^3p}{(2\pi)^3}\Big[3E_f + \frac{1}{\beta}\left(\ln\,\mathcal{F}_f + \ln\mathcal{F}_{\bar{f}}\right)\Big] + \sum_f 2G_\mathrm{S}\sigma_f^2 - 4G_\mathrm{D}\sigma_u\sigma_d\sigma_s + \mathcal{U}(\Phi,\Phi^*,T), \tag{6}$$

where $\sigma_f \equiv \langle\bar{q}_f q_f\rangle$, $E_f \equiv \sqrt{\mathbf{p}^3 + M_f{}^2}$, $M_f = m_f - 4G_\mathrm{S}\sigma_f + 2G_\mathrm{D}|\epsilon_{fgh}|\sigma_g\sigma_h$ with the antisymmetric symbol ϵ_{fgh}. The function $\mathcal{F}_f$ and $\mathcal{F}_{\bar{f}}$ are obtained as

$$\mathcal{F}_f = 1 + 3\Phi e^{-\beta E_f^-} + 3\Phi^* e^{-\beta 2E_f^-} + e^{-3\beta E_f^-}, \tag{7}$$

$$\mathcal{F}_{\bar{f}} = 1 + 3\Phi^* e^{-\beta E_f^+} + 3\Phi e^{-\beta 2E_f^+} + e^{-3\beta E_f^+}. \tag{8}$$

When $\Phi = 0$, Ω has no flavor dependence, since the factors $e^{\pm 3i\theta_f}$ do not depend on flavor. The flavor symmetry is thus preserved in the confinement phase with $\Phi = 0$.

The vacuum term in (6) is regularized with the three-dimensional cutoff Λ. For the parameter set (G_S, G_D, m_l, m_s, Λ), we take the set of Ref. [6], except that the s-quark mass m_s is taken to be the same as the light quark mass $m_l \equiv m_u = m_d$.

3. Numerical results

Figure 1 shows T dependence of order parameters σ, Φ and $a_0 \equiv \sigma_u - \sigma_d = \sigma_u - \sigma_s$ in (a) the PNJL model with $(\theta_u, \theta_d, \theta_s) = (0, 0, 0)$ and (b) the TBC model with $(\theta_u, \theta_d, \theta_s) = (0, 2\pi/3, 4\pi/3)$; note that a_0 is an order parameter of the flavor symmetry. In the PNJL model, both the chiral restoration and the deconfinement transition are of a crossover type. In the TBC model, the first-order deconfinement transition takes place at $T = T_c \approx 195$ MeV. Below T_c, a_0 and Φ are zero, as expected. The flavor symmetry is thus preserved by the color confinement. Above T_c, a_0 and Φ become finite, indicating that the flavor and $\mathbb{Z}_3$ symmetries break simultaneously. The chiral restoration is very slow in the TBC model because of the breaking of flavor symmetry.

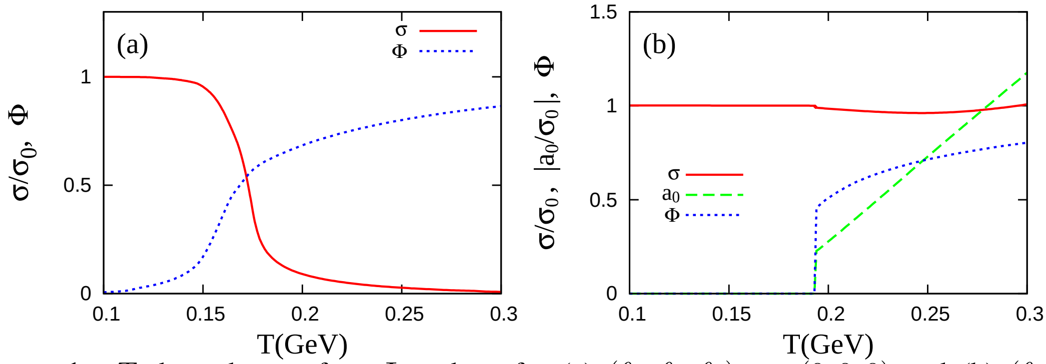

Figure 1. T dependence of σ, Φ and a_0 for (a) $(\theta_u, \theta_d, \theta_s) = (0, 0, 0)$ and (b) $(\theta_u, \theta_d, \theta_s) = (0, 2\pi/3, 4\pi/3)$. σ is normalized by the value σ_0 at $T = 0$.

Next we consider the entanglement-PNJL (EPNJL) model [7, 8]. The four-quark vertex $G_{\rm S}$ is originated in a gluon exchange between quarks and its higher-order diagrams. If the gluon field A_ν has a vacuum expectation value $\langle A_0 \rangle$, A_ν is coupled to $\langle A_0 \rangle$ and hence to Φ [9]. This effect allows $G_{\rm S}$ to depend on Φ: $G_{\rm S} = G_{\rm S}(\Phi)$ [9]. In this paper, we simply assume the following $G_{\rm S}(\Phi)$ by respecting the chiral symmetry, the charge-conjugation symmetry [10] and the extended $\mathbb{Z}_3$ symmetry [4]:

$$G_{\rm S}(\Phi) = G_{\rm S}[1 - \alpha_1 \Phi\Phi^* - \alpha_2(\Phi^3 + \Phi^{*3})]. \tag{9}$$

In principle, $G_{\rm D}$ can also depend on Φ. However, Φ-dependence of $G_{\rm D}$ yields qualitatively the same effect on the phase diagram as that of $G_{\rm S}$ [8]. We can then neglect Φ-dependence of $G_{\rm D}$. The parameters α_1 and α_2 in (9) are so determined as to reproduce two results of LQCD at finite T; one is the result of 2+1 flavor LQCD at $\mu = 0$ [11] that the chiral transition is crossover at the physical point and another is the result of degenerate three-flavor LQCD at $\theta = \pi$ [12] that the order of the RW endpoint is first-order for small and large quark masses but second-order for intermediate quark masses. The parameter set (α_1, α_2) satisfying these conditions is located in the triangle region [8]

$$\{-1.5\alpha_1 + 0.3 < \alpha_2 < -0.86\alpha_1 + 0.32,\ \alpha_2 > 0\}. \tag{10}$$

As a typical example, we take $\alpha_1 = 0.25$ and $\alpha_2 = 0.1$, following Ref. [8].

In Fig. 2, σ, a_0 and Φ are calculated as a function of T with the EPNJL model for (a) $(\theta_u, \theta_d, \theta_s) = (0, 0, 0)$ and (b) $(\theta_u, \theta_d, \theta_s) = (0, 2\pi/3, 4\pi/3)$. In panel (a), the chiral restoration and the deconfinement transition are first-order, because the current quark mass (5.5MeV) is small and the correlation between σ_f and Φ is strong [8]. In panel (b), one can see that $\Phi = a_0 = 0$ in the confinement phase. The EPNJL model with the TBC yields similar T dependence to that without the TBC for both the chiral restoration and the deconfinement transition, since the flavor-symmetry breaking above T_c is weakened by the strong correlation between σ_f and Φ.

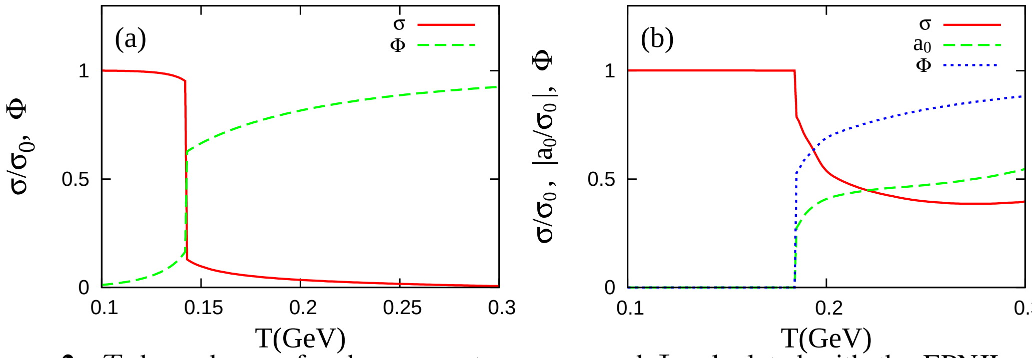

Figure 2. T-dependence of order parameters σ, a_0 and Φ calculated with the EPNJL model for (a) $(\theta_u, \theta_d, \theta_s) = (0, 0, 0)$ and (b) $(\theta_u, \theta_d, \theta_s) = (0, 2\pi/3, 4\pi/3)$. Here σ and a_0 are normalized by σ_0. Note that $a_0 = 0$ in panel (a) and $a_0 \geq 0$ in panel (b), while $\sigma < 0$ in both panels.

Now we consider the PNJL model with the flavor-dependent complex chemical potentials $\mu_f = \mu + iT\theta_f$ in which the θ_f are defined by

$$(\theta_f) = (0, \theta, -\theta). \tag{11}$$

The present model with the μ_f is reduced to the standard PNJL model with the flavor-independent real chemical potential μ when $\theta = 0$ and to the TBC model with real μ when $\theta = 2\pi/3$. Varying θ from 0 to $\theta = 2\pi/3$, one can see how the phase diagram changes between the approximate color-confinement in the standard PNJL model and the exact color-confinement in the TBC model.

Figure 3 shows the phase diagram in the μ-T plane. Panels (a)-(c) correspond to three cases of $\theta = 0$, $8\pi/15$ and $2\pi/3$, respectively. When $\theta = 0$, both the chiral and deconfinement transitions are of a crossover type at smaller μ, but the chiral transition becomes first-order at larger μ. When $\theta = 2\pi/3$, the deconfinement transition is the first-order at any μ, whereas the chiral transition line becomes first-order only at $\mu \approx M_f = 323$ MeV. The region labeled by "Qy" at $\mu \gtrsim M_f$ and small T is the quarkyonic phase [13], since $\Phi = 0$ but the quark number density n is finite there. The region labeled by "Had" is the hadron phase, because the chiral symmetry is broken there and thereby the equation of state is dominated by the pion gas [14]. The region labeled by "QGP" corresponds to the quark gluon plasma (QGP) phase, although the flavor symmetry is broken there by the TBC. As θ decreases from $2\pi/3$ to zero, the first-order chiral transition line declines toward smaller μ and the critical endpoint moves to smaller μ. Once θ varies from $2\pi/3$, the quarkyonic phase defined by $\Phi = 0$ and $n \neq 0$ shrinks on a line with $T = 0$ and $\mu \gtrsim M_f$ and a region at small T and $\mu \gtrsim M_f$ becomes a quarkyonic-like phase with small but finite Φ and $n \neq 0$; the latter region is labeled by "Qy-like".

4. Summary

We have proposed a simple model with the $\mathbb{Z}_N$ symmetry in order to answer whether the $\mathbb{Z}_N$ symmetry is a good concept in QCD with light quark mass. The model called the TBC model is constructed by imposing the flavor-dependent twisted boundary condition (3) on the PNJL model.

In the TBC model, the $\mathbb{Z}_3$ symmetry is preserved below T_c, but spontaneously broken above T_c. Below T_c, the color confinement preserves the flavor symmetry. Above T_c, meanwhile, the flavor symmetry is broken explicitly by the TBC. The flavor-symmetry breaking makes the chiral restoration slower, but the entanglement interaction between σ and Φ makes the restoration faster. Consequently, we can except that QCD with the TBC is similar to original QCD with the standard quark boundary condition.

We have also investigated the interplay between the $\mathbb{Z}_N$ symmetry and the emergence of the quarkyonic phase, considering the complex chemical potentials $\mu_f = \mu + iT\theta_f$ with $(\theta_f) = (0, \theta, -\theta)$

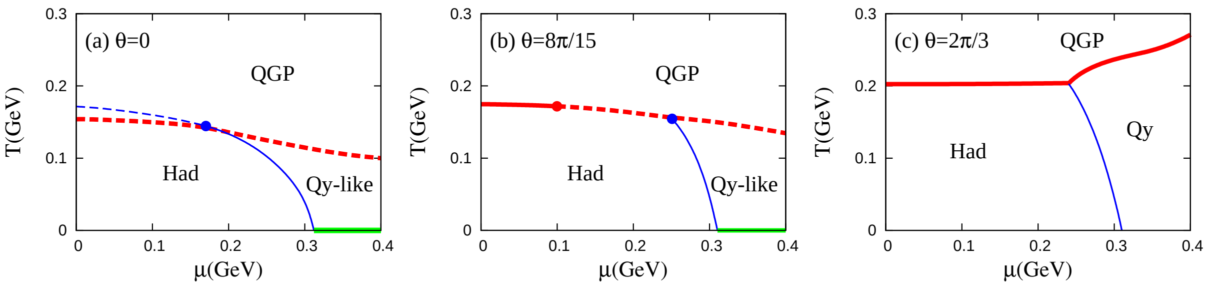

Figure 3. Phase diagram in the μ-T plane. Panels (a)-(c) correspond to three cases of $\theta = 0,\ 8\pi/15$ and $2\pi/3$, respectively. The thick (thin) solid curve denotes the first-order deconfinement (chiral) phase transition line, while the thick (thin) dashed curve the deconfinement (chiral) crossover line. The closed circles stand for the endpoints of the first-order deconfinement and chiral phase transition lines. In panels (a) and (b), the thick-solid line at $T = 0$ and $\mu \gtrsim M_f = 323$ MeV represents the quarkyonic phase.

in the PNJL model. The PNJL model with the μ_f is reduced to the PNJL model with real μ for $\theta = 0$, but to the TBC model with real μ for $\theta = 2\pi/3$. When $\theta = 2\pi/3$, the quarkyonic phase defined by $\Phi = 0$ and $n > 0$ exists at small T and large μ. Once θ varies from $2\pi/3$ to zero, the $\mathbb{Z}_{N_c}$ symmetry is broken and thereby the quarkyonic phase exists only on a line of $T = 0$ and $\mu \gtrsim M_f$. The region at small T and large μ is dominated by the quarkyonic-like phase characterized by small but finite Φ and $n > 0$. The quarkyonic-like phase at $\theta = 0$ is thus a remnant of the quarkyonic phase at $\theta = 2\pi/3$. Since the $\mathbb{Z}_N$ symmetry is explicitly broken at $\theta = 0$, it is natural to expand the concept of the quarkyonic phase and redefine it by a phase with small Φ and finite n. For this reason, the quarkyonic-like phase is often called the quarkyonic phase. The gross structure of the phase diagram thus has no qualitative difference between $\theta = 2\pi/3$ and zero, if the concept of the quarkyonic phase is properly expanded. In this sense, the $\mathbb{Z}_3$ symmetry is a good approximate concept for the case of $\theta = 0$, even if the current quark mass is small.

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