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Modified Hall–Littlewood polynomials and characters of affine Lie algebras

Nicholas Alexander Booth Bartlett

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Abstract

In the late 60's and early 70's V. Kac and R. Moody developed a theory of generalised Lie algebras which now bears their name. As part of this theory, Kac gave a beautiful generalisation of the famous Weyl character formula for the characters of integrable highest weight modules, raising the classical result to the level of Kac–Moody algebras. The Weyl–Kac character formula, as it is now known, is a powerful statement that preserves all of the desireable properties of Weyl's formula. However, there is one drawback that also remains. Kac's result formulates the characters of Kac–Moody algebras as an alternating sum over the Weyl group of the underlying affine root system. This inclusion-exclusion type representation obscures the natural positivity of these characters.

The purpose of this thesis is to provide manifestly positive (that is, combinatorial) representations for the characters of affine Kac–Moody algebras. In our pursuit of this task, we have been partially successful. For 1-parameter families of weights, we derive combinatorial formulas of so-called Littlewood type for the characters of affine Kac–Moody algebras of types $\mathrm{A}_{2n}^{(2)}$ and $\mathrm{C}_n^{(1)}$. Furthermore we obtain a similar result for $\mathrm{D}_{n+1}^{(2)}$, although this relies on an as-yet-unproven case of the key combinatorial q-hypergeometric identity underlying all of our character formulas.

Our approach employs the machinery of basic hypergeometric series to construct q-series identities on root systems. Upon specialisation, one side of these identities yields the above-mentioned characters of affine Kac–Moody algebras in their representation provided by the Weyl–Kac formula. The other side, however, leads to combinatorial sums of Littlewood type involving the modified Hall–Littlewood polynomials. These polynomials form an important family of Schur-positive symmetric functions.

This thesis is divided into two parts. The first part contains three chapters, each delivering a brief survey of essential classical material. The first of these chapters treats the theory of symmetric functions, with special emphasis on the modified Hall–Littlewood polynomials. The second chapter provides a short introduction to root sytems and the Weyl–Kac formula. The introductory sequence concludes with a chapter on basic hypergeometric series, highlighting the Bailey lemma.

All of our original work towards Littlewood-type character formulas is contained in Part II. This work is broken down into four chapters.

In the first chapter, we use Milne and Lilly's Bailey lemma for the C_n root system to derive a C_n analogue of Andrews' celebrated q-series transformation. It is from this transformation that we will ultimately extract our character formulas.

In the second chapter we develop a substantial amount of new material for the modified Hall–Littlewood polynomials Q'_λ. In order to transform one side of our C_n Andrews transformation into Littlewood-type combinatorial sums, we need to prove a novel q-hypergeometric series identity involving these polynomials. We (partially) achieve this by

first proving a new closed-form formula for the Q'_λ. For this proof in turn we rely heavily on earlier work by Jing and Garsia.

The highlight of our work is the third chapter, where we bring together all of our prior results to prove our new combinatorial character formulas. The most interesting part of the calculations carried out in this section is a bilateralisation procedure which transforms unilateral basic hypergeometric series on C_n into bilateral series which exhibit the full affine Weyl group symmetry of the Weyl–Kac character formula.

The fourth and final chapter explores specialisations of our character formulas, resulting in many generalisations of Macdonald's classical eta-function identities. Some of our formulas also generalise famous identities from partition theory due to Andrews, Bressoud, Göllnitz and Gordon.

Declaration by author

This thesis is composed of my original work, and contains no material previously published or written by another person except where due reference has been made in the text. I have clearly stated the contribution by others to jointly-authored works that I have included in my thesis.

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Publications during candidature

[1] N. Bartlett and S. O. Warnaar, *Hall–Littlewood polynomials and characters of affine Lie algebras*, arXiv:1304.1602.

Publications included in this thesis

Almost all of the findings of [1] are included this thesis. The breakdown of work undertaken in the production of [1] is as follows.

Contributor	Statement of Contribution	
N. Bartlett	Conception and design	50 %
	Analysis and interpretation	50 %
	Drafting and critical review	50 %
S. O. Warnaar	Conception and design	50 %
	Analysis and interpretation	50 %
	Drafting and critical review	50 %

Contributions by others to the thesis

No further contributions by others.

Statement of parts of the thesis submitted to qualify for the award of another degree

None.

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To my parents, Graeme L. and Elizabeth S. Bartlett

You are a reliable well-spring of courage and wisdom. I would call you up, deeply troubled, and you would wave your hands, sprinkle a little sage advice and in the span of a few minutes I would be feeling hale and hearty again. I now know that these are the rites of an ancient form of parental-witchcraft, in which the anxiety is drawn from the child and taken into oneself. I'm sorry you were called upon to suffer so often, even in the midst of your own substantial individual trials. I'm sorry further that I won't yet relieve you of your torment. You must be worried about my long-anticipated bicycle-touring trip, especially with our rather unfortunate family record for hospitalisation during overseas travel. I'd like you to bear in mind that you all survived, each with a complete recovery. Expect that I'll do the same (most likely, better). Now that this thesis comes to an end, recall your slight frustration with my lack of focus in highschool. I take this opportunity to point out that had I scored slightly better grades, I would have made the hideous error of entering a degree in Journalism.

Thanks again for your steady support. You are my loudest barrackers, my most trusted confidants, my battered shields to anxiety, and most importantly, my primary source of delicious morale-boosting goods delivered by post.

To my supervisor, Professor S. Ole Warnaar

I cannot hope to ever repay you for your kind instruction. Needless to say, I am ecstatic that this thesis will soon be behind me, but I have greatly enjoyed your tutelage and so it is with some sadness that I finish. There is also some trepidation, since I still have no clear idea of what I will do next. One observes your successes and wonders if they might be reproduced. In my case, I think not. In spite of your fine example and sharp mentoring, I will not be a professor of mathematics. Even with your veteran coaching and wise advice during my marathon training (perhaps too often brazenly ignored), a career as a professional athlete is not on the table either. It seems then that the determination of a worthy goal demands of me a little more creativity. Fortunately, under your direction I've learned that this is essentially a matter of persistent effort, and that's easy!

Although our paths soon diverge, know that your influence will not leave me. I know that whatever I do, I must find it in me to do it with all the boldness, exactitude and sheer joy that you do mathematics.

Keywords

Hall–Littlewood polynomials, symmetric functions, character identities

ANZSRC Classification

010101 Algebra and Number Theory 50%

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0101 Pure mathematics 100%

Contents

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Part I

Preliminary material

A brief introduction to symmetric functions, characters of affine Kac–Moody Lie algebras and basic hypergeometric series.

Chapter 1

Symmetric Functions

In later sections we develop a significant amount of new material concerning the modified Hall–Littlewood polynomials, an important family of symmetric functions. This section prepares the reader with a highly-selective introduction to the general theory of symmetric functions. To expert readers, the contents of this section may appear somewhat arbitrary. However, we have chosen to present those results most relevant to our later development of the modified Hall–Littlewood polynomials. A more comprehensive discussion of symmetric functions can be found in many places, see e.g., [Fu97, Ha08, La01, Macd95, Stan99]. We mainly follow the notation and conventions of Macdonald's monograph *Symmetric Functions and Hall Polynomials* [Macd95].

Preliminaries

The classical theory of symmetric functions begins with the classification of several important families of polynomials. Partitions and compositions are conventionally employed to index the members of these families; in this section we introduce these elementary notions.

Partitions and compositions

A sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is called a *composition* (sometimes a *weak composition*) of N on n entries if $|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_n = N$. A *partition* μ is a composition of N on an infinite number of entries, where the entries are in weakly decreasing order. The positive entries of a partition λ are called *parts* and the number of parts is called the *length*, denoted $\ell(\lambda)$. Sometimes it is useful to consider a partition to contain only a specified number of zeros. We do not distinguish between partitions whose entries differ only in the number of zeros and so we will usually record only the parts of a partition, unless it is convenient to do otherwise. The set of all compositions of N is denoted by $\mathcal{C}_N$ and we will use $\mathcal{P}_N$

to denote the set of all partitions of N. For example, there are 5 partitions in $\mathcal{P}_4$: $(4), (3,1), (2,2), (2,1,1), (1,1,1,1)$. The sets of all compositions and partitions are denoted $\mathcal{C}$ and $\mathcal{P}$ respectively. For the greater part of this thesis λ, μ and ν denote partitions, but occasionally these symbols are also used to denote compositions. The *symmetric group* S_n is the group of permutations on n symbols, which acts on a sequence $a = (a_1, \ldots, a_n)$ by permutation of the entries a_i. For any composition μ there is a unique partition μ^+ in the S_n orbit of μ. We should be careful to note that compositions with differing numbers of entries may correspond to the same partition, e.g., for $\mu = (0,4,1,1,0,3)$ and $\nu = (1,1,3,4)$,

$$\mu^+ = \nu^+ = (4,3,1,1).$$

Let the *multiplicity* $m_i := m_i(\lambda)$ be the number of times a part of size i occurs in the partition λ. Where convenient, we will employ exponent notation, so that $(1^{m_1}2^{m_2}3^{m_3}\cdots)$ is the partition where parts of size i occur m_i times. It is standard practice to omit the exponent of the part of size i when $m_i = 1$ and to omit the part itself when $m_i = 0$. With these conventions the partition $(4,3,3,1,1,1)$ can be compactly rendered $(1^3 3^2 4)$. Note that this introduces some slight ambiguity, e.g., $(13^2 45)$ might be $(5,4,3,3,1)$ or $(45,13,13)$, and so we will only use this notation when the intended part sizes are easily inferred.

The *dominance ordering* is a natural ordering on partitions, called *the natural ordering* by some authors. Given distinct partitions λ and μ such that $|\lambda| = |\mu|$ we say that λ *dominates* μ, denoted $\lambda > \mu$, if $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all $i \geq 1$. For example, $(432^2) > (3^3 1^2)$. Dominance order is a total order on partitions of size less than or equal to 5, but a partial ordering thereafter; e.g., $(31^3) \not> (2^3)$ and $(2^3) \not> (31^3)$. The distinct partitions λ and μ are said to be in *reverse lexicographical order*, denoted $\lambda >_R \mu$, if the first non-vanishing difference of $\lambda_i - \mu_i$ is positive. For example, $\mathcal{P}_5$ in reverse lexicographical order is

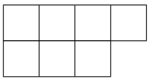
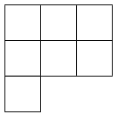
$$(5) >_R (41) >_R (32) >_R (31^2) >_R (2^2 1) >_R (21^3) >_R (1^5).$$

There is an important relationship between reverse lexicographical order and dominance order: if λ dominates μ, then λ has precedence over μ in reverse lexicographical order; i.e.,

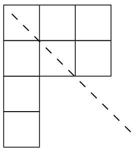
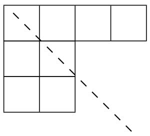
$$\lambda > \mu \implies \lambda >_R \mu. \tag{1.1}$$

Note that the converse of this statement is false; e.g., $(31^3) >_R (2^3)$ but $(31^3) \not> (2^3)$.

We now introduce a bijective graphical representation for partitions. Every partition has a *Young diagram*, which is a left-justified array of cells, where the number of cells in a row is weakly decreasing from top to bottom. French authors prefer to draw these diagrams with parts in increasing order from top to bottom. The partition $(\lambda_1, \lambda_2, \dots)$ corresponds to the Young diagram with λ_i cells in the ith row. For example:

(4,3)  (3,3,1) 

Young diagrams give rise to a natural involution on partitions, called *conjugation*. The partition *conjugate* to λ, denoted λ', is formed by reflecting the Young diagram of λ across the main diagonal:

(3,3,1,1)  $\longleftrightarrow$ (4,2,2) 

More formally, λ'_i is the number of parts of λ that are greater than or equal to i. From this definition it is easy to see that $m_i(\lambda) = \lambda'_i - \lambda'_{i+1}$ and that the length of λ is λ'_1.

The statistic $n(\lambda)$ is defined

$$n(\lambda) := \sum_{i \geq 1} (i-1)\lambda_i.$$

We can find an alternate form for this function with the following combinatorial interpretation: assign weight $i-1$ to each cell in row i of the Young diagram of λ; e.g.,

0	0	0	0	0	0
1	1	1	1	1	
2	2	2			
3	3	3			
4					

so that the ith row in the diagram corresponds to the ith term of the sum. If we calculate $n(\lambda)$ by summing over the entries of columns instead of rows we obtain

$$n(\lambda) = \sum_{i \geq 1} \binom{\lambda'_i}{2}. \tag{1.2}$$

Very occasionally we will consider partitions in which all of the parts are half-integers, i.e., the parts all lie in the set $\{n + \frac{1}{2} : n \in \mathbb{N}\}$, where $\mathbb{N} = \{0, 1, 2, \dots\}$. The objects will be referred to as *half-partitions*.

Young tableaux and Kostka numbers

Our later discussions of the Schur functions and the modified Hall–Littlwood polynomials use the notion of a Young tableaux. Here we briefly introduce these simple combinatorial objects.

Fix a positive integer n. A *semi-standard Young tableau* of *shape* λ is a Young diagram of λ filled with the numbers 1 to n such that the entries are weakly increasing from left to right along rows and strictly increasing down columns. The modifier *semi-standard* is to distinguish our Young tableaux from a *standard* Young tableaux which, for $|\lambda| = n$, we fill with the numbers 1 to n so that each number occurs exactly once. In this thesis we will discuss only semi-standard tableaux.

The *content* μ of a Young tableau is a composition $(\mu_1, \ldots, \mu_n)$, where μ_i is the number of times i occurs as an entry in the diagram. For example, the following distinct Young tableaux both have shape $(4, 3, 1, 1)$ and content $(2, 2, 3, 0, 2)$:

1	1	2	3
2	3	5	
3			
5			

1	1	3	3
2	2	5	
3			
5			

We remark that our terminology differs from that of Macdonald [Macd95]; in his text *content* refers to the function c that appears in the *hook-contents formula*, an elegant and elementary enumeration of Young tableaux.

Given a partition λ and composition μ, the *Kostka number* $K_{\lambda\mu}$ is the cardinality of the set of Young tableaux of shape λ and content μ, where we define $K_{\lambda\mu} = 0$ if $|\lambda| \neq |\mu|$. This last set we will denote by $\mathrm{Tab}(\lambda, \mu)$. We will see later that these numbers are invariant under permutation of the content composition μ; in particular

$$K_{\lambda\mu} = K_{\lambda\mu^+}. \tag{1.3}$$

Therefore, we need only consider $K_{\lambda\mu}$ where μ is a partition. We remark that $K_{\lambda\mu}$ vanishes if λ does not dominate μ.

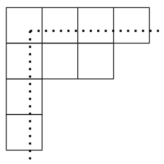
For example, Table 1.1 presents all $K_{\lambda\mu}$ for $|\lambda| = |\mu| = 4$.
Table 1.1 has some instructive properties. Since we have arranged the partitions in Table 1.1 in reverse lexicographical order, then by the contrapositive of (1.1), all the entries below the main diagonal are zero. The *Kostka matrix* is defined $K := (K_{\lambda\mu})$, where the rows and columns are in reverse lexicographical order. The entries of Table 1.1 are then exactly those of the Kostka matrix for $|\lambda| = |\mu| = 4$. The Kostka matrix is in general upper-triangular and moreover unipotent since $K_{\lambda\lambda} = 1$.

For later purposes we introduce the notion of the hook length of cells in a Young diagram. We will take this opportunity to state the *hook-contents formula* [Macd95],

Table 1.1

$\lambda \backslash \mu$	4	31	2^2	21^2	1^4
4	1	1	1	1	1
31		1	1	2	3
2^2			1	1	2
21^2				1	3
1^4					1

an elegant enumeration of Young tableaux. Let λ be a Young diagram and define $s = (i, j)$ to be a cell of λ, where $1 \leq i \leq \ell(\lambda)$ and $1 \leq j \leq \lambda_i$. The *hook length* $h(s)$ is the sum of the number of cells to the right of s and of those below s, plus 1 for s itself. For example, hook lengths of the cells of the Young diagram $(4, 3, 1, 1)$ are as follows



7	4	3	1
5	2	1	
2			
1			

We also need the simple statistic $c(s) := j - i$ which Macdonald calls the *content*. The following example shows the value of $c(s)$ for each cell for the Young diagram of $(5, 3, 3, 1, 1)$.

0	1	2	3	4
-1	0	1		
-2	-1	0		
-3				
-4				

The hook-contents formula is as follows. For a partition λ and integer n, the number of Young tableaux of shape λ with entries $1, \ldots, n$ is given by

$$\prod_{s \in \lambda} \frac{n + c(s)}{h(s)}. \tag{1.4}$$

Classical symmetric functions

The Hall–Littlewood polynomials are a q-analogue of the classical theory of symmetric functions. We now briefly introduce some of the standard polynomial bases of the ring of symmetric functions, giving particular attention to the Schur functions, and discuss their orthogonality properties under the Hall inner product.

Standard bases of the ring of symmetric functions

Let $\mathbb{Z}[x_1, \ldots, x_n] =: \mathbb{Z}[x]$ be the ring of polynomials in n independent variables $x_1, \ldots, x_n$ with integer coefficients. A polynomial $p(x_1, \ldots, x_n) =: p(x)$ in $\mathbb{Z}[x]$ is *symmetric* if it is invariant under permutation of its variables; i.e., for every $w \in S_n$,

$$p(x_1, \ldots, x_n) = p(x_{w(1)}, \ldots, x_{w(n)}).$$

The set of all symmetric polynomials in n variables, denoted Λ_n, is a graded subring of $\mathbb{Z}[x_1, \ldots, x_n]$:

$$\Lambda_n = \bigoplus_{k \geq 0} \Lambda_n^k,$$

where Λ_n^k is the ring formed of the set of homogeneous symmetric polynomials in n variables of degree k, together with the zero polynomial.

For $\alpha = (\alpha_1, \ldots, \alpha_n)$ a sequence of integers, let

$$x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

Given a partition λ of length at most n, the *monomial symmetric functions* $m_\lambda := m_\lambda(x)$ are defined by

$$m_\lambda := \sum x^\alpha,$$

where the sum is over all compositions on n entries in the S_n orbit of λ. In other words, for μ a composition,

$$m_\lambda = \sum_{\mu^+ = \lambda} x^\mu,$$

where $m_\lambda = 0$ when $\ell(\lambda) > n$. The set of polynomials $\{m_\lambda : \ell(\lambda) \leq n\}$ form an integer basis for Λ_n. Define Λ, the ring of symmetric functions in countably many variables. Clearly, the m_λ form a basis of Λ when λ ranges over all partitions. In the theory of symmetric functions it is often more convenient to work with a countably infinite number of variables, and so we adopt this convention except where indicated for the rest of this chapter. This convention may be given a rigorous grounding, but we will not include the details of these considerations here; see [Macd95, §I.2].

Using the m_λ we may proceed to define other special families of symmetric functions. The *complete symmetric functions*, denoted $h_r := h_r(x)$, are defined as

$$h_r = \sum_{|\lambda| = r} m_\lambda = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} \cdots x_{i_r},$$

with the conventions that $h_0 = 1$, and $h_r = 0$ when $r < 0$. The h_r admit a simple generating function $H_z(x) =: H_z$, which may be written as

$$H_z = \sum_{r \geq 0} h_r z^r = \prod_{i \geq 1} \frac{1}{1 - zx_i}. \tag{1.5}$$

We offer another explicit representation of the h_r, this time in a finite number of variables $x = (x_1, \ldots, x_n)$ to faciliate a later comparison with the Hall–Littlewood polynomials:

$$h_r(x) = \sum_{m=1}^{n} x_m^r \prod_{i \neq m} \frac{x_m}{x_m - x_i}. \tag{1.6}$$

This non-standard (and not manifestly polynomial) representation calls for a quick derivation.

Proof. Using (1.5), transform the product into a sum of partial fractions, introducing coefficients b_i so that

$$\prod_{i=1}^{n} \frac{1}{1 - zx_i} = \sum_{i=1}^{n} \frac{b_i}{1 - zx_i}. \tag{1.7}$$

Multiply both sides by $(1 - zx_m)$ and set $z = 1/x_m$ to obtain the general coefficient

$$b_m = \prod_{\substack{i=1 \\ i \neq m}}^{n} \frac{x_m}{x_m - x_i}.$$

Now, substitute this into (1.7) so that we can solve for h_r:

$$\sum_{r \geq 0} h_r z^r = \sum_{m=1}^{n} \frac{1}{1 - zx_m} \prod_{\substack{i=1 \\ i \neq m}}^{n} \frac{x_m}{x_m - x_i}.$$

Simply expand $1/(1 - zx_m)$ in a geometric series and equate the coefficients of z^r for the result. $\square$

The complete symmetric functions may be extended to compositions. For a composition λ, define h_λ by

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots . \tag{1.8}$$

We note that (1.6) amounts to a recursive form for h_λ:

$$h_\lambda = h_\mu \sum_{m=1}^{n} x_m^{\lambda_1} \prod_{i \neq m} \frac{x_m}{x_m - x_i}, \tag{1.9}$$

where $\mu = (\lambda_2, \lambda_3, \ldots)$. The set of partition-indexed polynomials $\{h_\lambda : \ell(\lambda) \leq n\}$ form a $\mathbb{Z}$-basis for Λ_n.

The *elementary symmetric functions* $e_r(x) =: e_r$ are defined by

$$e_r = m_{(1^r)} = \sum_{1<i_1<i_2<\cdots<i_r} x_{i_1} \cdots x_{i_r}.$$

Like the h_r, the e_r have a compact generating function $E_z := E_z(x)$:

$$E_z = \sum_{r\geq 0} e_r z^r = \prod_{i\geq 1}(1 + zx_i). \tag{1.10}$$

If we fix an alphabet of length n, observe that by the definition of m_λ, $e_r = 0$ for $r > n$. For a composition λ, the following product defines e_λ:

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots ,$$

where again we set $e_0 = 1$ and $e_r = 0$ where $r < 0$. The partition-indexed set $\{e_\lambda : \ell(\lambda') \leq n\}$ is then an integer basis for Λ_n. Observe that by the product sides of (1.5) and (1.10)

$$E_{-z}(x) H_z(x) = 1, \tag{1.11}$$

with the consequence that

$$\sum_{r=0}^{l} (-1)^r e_r h_{l-r} = 0,$$

for all $p \geq 1$. Later we will revisit (1.11) in the context of λ-ring notation.

For $r \geq 1$, the *power sums* $p_r(x) =: p_r$ are defined by

$$p_r = m_{(r)} = \sum_i x_i^r, \tag{1.12}$$

and have the generating function

$$P_z(x) = \sum_{r\geq 1} p_r z^{r-1} = \sum_i \sum_{r\geq 1} x_i^r z^{r-1} = \sum_i \frac{x_i}{1 - x_i z}$$

By comparison with (1.5) it is easy to see that

$$P_z = \frac{\mathrm{d}}{\mathrm{d}z} \log H_z. \tag{1.13}$$

The powers sums too may be extended to compositions. For a partition λ, p_λ is defined as the product

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots ,$$

where once more we set $p_0 = 1$ and $p_r = 0$ where $r < 0$. The set of p_λ for λ ranging over all partitions is a $\mathbb{Q}$-basis (and not a $\mathbb{Z}$-basis) for Λ.

The Schur functions

In the literature, the Schur functions appear across a vast breadth of fields and garner far more attention than the other classical bases of Λ. Correspondingly, we will discuss their properties to a greater depth, though our treatment is still very superficial overall. We begin with a derivation of the Schur functions from first principles, following Macdonald [Macd95].

For $\alpha = (\alpha_1, \ldots, \alpha_n)$ a composition, let $a_\alpha(x) =: a_\alpha$ be the anti-symmetrisation of the monomial $x^\alpha =: x_1^{\alpha_1} \cdots x_n^{\alpha_n}$; that is,

$$a_\alpha = \sum_{w \in S_n} \operatorname{sgn}(w) x^{w(\alpha)},$$

where $\operatorname{sgn}(w)$ is the *sign* (or *signature*) of the permutation w. Note that a_α is of homogeneous of total degree $|\alpha|$. For example,

$$a_{(1,3,2)} = x_1^1 x_2^2 x_3^3 + x_1^2 x_2^3 x_3^1 + x_1^3 x_2^1 x_3^2 - x_1^2 x_2^1 x_3^3 - x_1^3 x_2^2 x_3^1 - x_1^1 x_2^3 x_3^2.$$

By construction, the polynomial a_α is skew-symmetric, i.e.,

$$w(a_\alpha) = \operatorname{sgn}(w) a_\alpha.$$

Now, if $\alpha_i = \alpha_j$, the permutation $w = (ij)$ merely introduces a sign, so we have $a_\alpha = -a_\alpha$. As a consequence, $a_\alpha = 0$ unless $\alpha_1, \ldots, \alpha_n$ are all distinct. Therefore, up to a possible change of overall sign in a_α, it is harmless to reorder α so that under the new arrangement $\alpha_1 > \cdots > \alpha_n \geq 0$. Hence, we obtain $\alpha = \lambda + \delta$ for some partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ and where $\delta := (n-1, \ldots, 1, 0)$. The polynomial a_α naturally admits a determinant form, which arises immediately from the Leibniz formula for determinants,

$$a_{\lambda+\delta} = \det\left(x_i^{\lambda_j+n-j}\right).$$

Now, observe that by the skew-symmetry property $a_{\lambda+\delta} = 0$ when $x_i = x_j$ for $i \neq j$. Therefore, the factor $(x_i - x_j)$ divides $a_{\lambda+\delta}$ in $\mathbb{Z}[x_1, \ldots, x_n]$ and so the product of these factors over all $i < j$, which may be written as a_δ, is also a divisor of $a_{\lambda+\delta}$. The product a_δ is famously known as the *Vandermonde determinant* $\Delta(x)$ (2.8a), and so we will use the latter notation.

For a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, define the *Schur functions* $s_\lambda := s_\lambda(x)$ as the polynomial

$$s_\lambda(x) = a_{\lambda+\delta}(x)/\Delta(x), \tag{1.14}$$

which is homogeneous of total degree $|\lambda|$.

We will see the Schur functions appear in the study of characters of classical Lie algebras. The details of this relationship are given later, shortly after the presentation of the Weyl character formula (2.6).

Equation (1.14) gives rise to a recursive formulation of the Schur functions. By expanding the first column of the determinant in the numerator we obtain

$$s_\lambda(x) = \sum_{m=1}^{n} x_m^{\lambda_1} \Bigg(\prod_{\substack{i=1 \\ i\neq m}}^{n} \frac{x_m}{x_m - x_i} \Bigg) s_\mu\big(x^{(m)}\big), \tag{1.15}$$

where $\mu = (\lambda_2, \ldots, \lambda_n)$ and where we have used the notation

$$x^{(m)} = (x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_n).$$

We will see (1.15) again in the context of modified Hall–Littlewood polynomials. From (1.15) it is not difficult to see that the Schur functions are *stable*, that is

$$s_\lambda(x_1, \ldots, x_n) = s_\lambda(x_1, \ldots, x_n, 0).$$

Thanks to this property, the Schur functions may be extended to Λ. The Schur functions are symmetric and the coefficients of the monomials appearing are always positive. The symmetry of the Schur functions is easy to see (it is the ratio of two skew-symmetric polynomials), but proving that the coefficients are positive is not so straight-forward. We will not provide a proof but instead give another well-known representation of the Schur functions where the positivity is manifest, which comes from the the theory of Young tableaux.

The content $\mu = (\mu_1, \ldots, \mu_n)$ of any Young tableaux T defines a monomial as follows

$$x^T = x_1^{\mu_1} \cdots x_n^{\mu_n}.$$

Although we already possess notation that defines a monomial using a composition (i.e., the definition of x^α on page 9), in the context of Young tableaux the convention is to raise T itself instead of μ. By this assignment, the two Young tableaux given on page 4 which have content $(2, 2, 3, 0, 2)$ both correspond to $x_1^2 x_2^3 x_3^3 x_5^2$. Recalling the definition of $\mathrm{Tab}(\lambda, \mu)$ from page 4, we may then represent the Schur functions as

$$s_\lambda = \sum_{T \in \mathrm{Tab}(\lambda, \,\cdot\,)} x^T. \tag{1.16}$$

For example, we can compute $s_{(2,1,0)}(x_1, x_2, x_3)$ as follows:

1 1 / 2	1 1 / 3	1 2 / 2	1 2 / 3	1 3 / 3	2 2 / 3	2 3 / 3
$(2,1,0)$	$(2,0,1)$	$(1,2,0)$	$(1,1,1)$	$(1,0,2)$	$(0,2,1)$	$(0,1,2)$
$x_1^2 x_2$	$x_1^2 x_3$	$x_1 x_2^2$	$x_1 x_2 x_3$	$x_1 x_3^2$	$x_2^2 x_3$	$x_2 x_3^2$

and hence

$$s_{(2,1,0)} = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2.$$

It has been our practice to give each new symmetric function in terms of the m_λ. We may express s_λ as

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu, \tag{1.17}$$

where the coefficients $K_{\lambda\mu}$ are the Kostka numbers of page 4 and the sum is over all partitions μ. It is easy to show that (1.16) is equivalent to (1.17). Given some composition ν such that $|\nu| = |\lambda|$, the coefficent of the monomial x^ν in (1.16) is by definition $K_{\lambda\nu}$. Since s_λ is symmetric, s_λ can be rewritten as a sum of monomial symmetric functions. The monomial x^μ occurs in m_μ and therefore the coefficient of m_μ is $K_{\lambda\mu}$.

A change between any two of the $m_\lambda, h_\lambda, e_\lambda$ and s_λ bases admits in each case a transition matrix very similar to the Kostka matrix or, equivalently, a transition formula like (1.17). A table of these matrices can be found in [Macd95, pp. 101]. For later purposes, we wish to single out another transition formula that is of particular interest:

$$h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda, \tag{1.18}$$

where the sum is over all partitions λ. This last formula will appear again in the context of Hall–Littlewood polynomials.

The Hall inner product

There is a notion of duality amongst the bases of Λ that arises from consideration of the sum

$$\sum_{\lambda\in\mathcal{P}} h_\lambda(x) m_\lambda(y). \tag{1.19}$$

These considerations lead to important sum–product identities relating certain pairs of bases of symmetric functions, as follows.

By the definition of m_λ and the fact that $h_\lambda = h_\mu$ for all $\mu^+ = \lambda$,

$$\sum_{\lambda\in\mathcal{P}} h_\lambda(x) m_\lambda(y) = \sum_{\lambda\in\mathcal{P}} h_\lambda(x) \sum_{\mu^+=\lambda} y^\mu = \sum_{\lambda\in\mathcal{P}} \sum_{\mu^+=\lambda} h_\mu(x) y^\mu = \sum_{\mu\in\mathcal{C}} h_\mu(x) y^\mu,$$

This last sum can be rewritten as a product of geometric series, recognisable as the generating function for the h_r from (1.5)

$$\sum_{\mu\in\mathcal{C}} h_\mu(x) y^\mu = \prod_{j\geq 1} \sum_{k\geq 0} h_k(x) y_j^k = \prod_{j\geq 1} H_{y_j}(x) = \prod_{i,j\geq 1} \frac{1}{1 - x_i y_j},$$

and hence,

$$\sum_{\lambda} h_\lambda(x) m_\lambda(y) = \prod_{i,j\geq 1} \frac{1}{1 - x_i y_j}. \tag{1.20}$$

It is known that h_λ and m_λ are not the only pair of bases satisfying (1.20). We can capture this idea formally with the *Hall inner product* $\langle ., . \rangle$, defined on Λ by

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}. \tag{1.21}$$

Now, given two arbitrary bases $u_\lambda, v_\lambda \in \Lambda$, the following statements are equivalent [Macd95, pp. 63]:

$$\langle u_\lambda, v_\mu \rangle = \delta_{\lambda\mu}, \tag{1.22a}$$

$$\sum_{\lambda} u_\lambda(x) v_\lambda(y) = \prod_{i,j\geq 1} \frac{1}{1 - x_i y_j}, \tag{1.22b}$$

where the sum is over all partitions λ. Pairs of bases that satisfy (1.22) are said to be *dual* with respect to the Hall inner product. The existence of a dual partner is guaranteed by elementary linear algebra, so it is natural to seek the partner given a particular basis. Indeed, nothing prevents us from defining other bases in this way. For example, the *forgotten symmetric polynomials* f_λ are defined by

$$\langle e_\lambda, f_\mu \rangle = \delta_{\lambda\mu}.$$

The f_λ are called forgotten because no one has found a compelling reason to remember them.

The Schur functions are self-dual, and form an orthonormal basis for Λ,

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}. \tag{1.23}$$

The corresponding identity

$$\sum_{\lambda} s_\lambda(x) s_\lambda(y) = \prod_{i,j\geq 1} \frac{1}{1 - x_i y_j}, \tag{1.24}$$

is called the *Cauchy identity.* The Schur functions are in fact the *unique* orthonormal basis and so self-duality with respect to the Hall inner product identifies the s_λ.

Hall–Littlewood polynomials

We come now to the symmetric polynomials that play such a crucial role in this thesis. Where previously in this chapter we have been very brief, the importance of these polynomials warrants a more patient treatment.

We will need the following notation. The equations (1.25) below all define what are referred to as *q-shifted factorials*. We first define the infinite product

$$(a;q)_\infty = (a)_\infty = (1-a)(1-aq)(1-aq^2)\cdots. \tag{1.25a}$$

By means of a quotient of two q-shifted factorials, we may express a product with a finite number of terms

$$(a;q)_n = (a)_n = \frac{(a)_\infty}{(aq^n)_\infty}. \tag{1.25b}$$

Explicitly, we have $(a)_0 = 1$ and

$$(a)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \tag{1.25c}$$

and

$$(a)_{-n} = \frac{1}{(1-aq^{-n})(1-aq^{-n+1})\cdots(1-aq^{-1})}, \tag{1.25d}$$

when $n > 0$. Alternatively,

$$(a)_{-n} = \frac{1}{(aq^{-n})_n} = \frac{(-q/a)^n}{(q/a)_n}\, q^{\binom{n}{2}}, \tag{1.25e}$$

for all integers n. From (1.25d) it immediately follows that $1/(q)_{-n} = 0$ for $n > 0$. Where convenient, we will also employ the condensed notation

$$(a_1, \ldots, a_r)_n = (a_1, \ldots, a_r; q)_n = \prod_{i=1}^{r} (a_i)_n. \tag{1.25f}$$

For any $n \geq 0$, define

$$v_n = v_n(q) = \frac{(q)_n}{(1-q)^n},$$

where for a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, we use the notation

$$v_\lambda = v_{\lambda_1} \cdots v_{\lambda_n} = \prod_{i \geq 0} \frac{(q)_{m_i}}{(1-q)^{m_i}},$$

where we recall the multiplicity notation defined on page 2 and we use the convention that $m_0 = N - \ell(\lambda)$.

We may now define the *Hall–Littlewood polynomials* $P_\lambda := P_\lambda(x;q)$ by

$$P_\lambda = \frac{1}{v_\lambda} \sum w\Big(x^\lambda \prod_{i<j} \frac{x_i - qx_j}{x_i - x_j} \Big), \tag{1.26}$$

where the sum is over all permutations $w \in S_n$. For example, for an alphabet of three variables

$$P_{(2,2)} = x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 + (1-q)(x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2).$$

If the v_λ quotient is dropped from the definition (1.26), P_λ is not stable. This particular choice of v_λ is desirable since the leading coefficient (i.e., the coefficient of x^λ) is then 1. However, it is possible to eliminate v_λ and at the same time obtain a more computationally efficient representation of P_λ that is stable with leading coefficient 1. Observe that when λ contains repeated entries, there are several permutations that fix x^λ. We may take advantage of this fact by collapsing the sum over S_n down to a sum over S_n/S_n^λ, where S_n^λ is the set of permutations that fix λ. The idea is to dissect the product in (1.26) so that

$$\prod_{1\le i<j\le n} \frac{x_i - qx_j}{x_i - x_j} = \prod_{\lambda_i<\lambda_j} \frac{x_i - qx_j}{x_i - x_j} \prod_{k\ge 0} \prod_{\substack{i<j \\ \lambda_i=\lambda_j=k}} \frac{x_i - qx_j}{x_i - x_j}. \tag{1.27}$$

Now, $S_n^\lambda \simeq S_{m_{\lambda_1}} \times \cdots \times S_{m_0}$, and so

$$S_n \simeq \frac{S_n}{S_n^\lambda} \times S_{m_{\lambda_1}} \times \cdots \times S_{m_0}.$$

We can take care of these additional S_{λ_i} terms with the crucial identity

$$\sum_{w\in S_n} w\left(\prod_{1\le i<j\le n} \frac{x_i - qx_j}{x_i - x_j} \right) = v_n. \tag{1.28}$$

Using this identity, for each k in the final product of (1.27) we may extract a factor v_{m_k} and thereby eliminate the prefactor v_λ. Then we have

$$P_\lambda = \sum w\left(x^\lambda \prod_{\lambda_i>\lambda_j} \frac{x_i - qx_j}{x_i - x_j} \right), \tag{1.29}$$

where the sum is over representatives w taken from each coset of S_n/S_n^λ. To conclude the above considerations, we now briefly remark upon (1.28).

Arising from the study of finite reflection groups, the *Poincaré polynomial* [Bo02, Hu72] is defined as

$$W(q) = \sum_{w\in W} q^{\ell(w)}, \tag{1.30}$$

where W is a finite real reflection group and $\ell(w)$ is the length function on W. Thanks to the works of Chevalley [Ch55] and Solomon [So66], it is known that the Poincaré polynomial has the following product formula

$$W(q) = \prod_{i=1}^{n} \frac{1 - q^{d_i}}{1 - q}, \tag{1.31}$$

where the d_i are the *degrees of the fundamental invariants* of W. A later result due to Macdonald [Macd72b, Theorem 2.8] gives a further representation of the Poincaré polynomial for groups of crystallographic type (i.e., the Weyl groups that will be introduced in Chapter 2),

$$W(q) = \sum_{w \in W} \prod_{\alpha > 0} \frac{1 - q e^{-w(\alpha)}}{1 - e^{-w(\alpha)}}. \tag{1.32}$$

Macdonald states this result for the *multivariable Poincaré polynomial* $W(\boldsymbol{t})$, in which a variable t_α is attached to each of the positive roots α. Here we have specialised $t_\alpha \to q$ for all α.

Given also the introductory material on root systems in Chapter 2, we may understand (1.28) to be being nothing more than (1.31) equated with (1.32) for the root system A_{n-1}, under the assignment $\mathrm{e}^{-\epsilon_i} = x_i$, where the degrees of the fundamental invariants are $2, \ldots, n$.

From equations (1.26) and (1.29) it may not be clear that P_λ is always symmetric or even that it is polynomial. Like the Schur functions of Equation (1.14), P_λ is a homogeneous symmetric polynomial because it is the ratio of *a homogeneous skew-symmetric polynomial in* $x_1, \ldots, x_n$ and *the Vandermonde product* (2.8a), as can be seen in the following trivial reformulation of (1.26), rewritten so that the sum is an anti-symmetrisation:

$$P_\lambda = \frac{1}{v_\lambda \Delta(x)} \sum \operatorname{sgn}(w) w \Big(x^\lambda \prod_{i<j} (x_i - q x_j) \Big).$$

Together, definitions (1.26) and (1.29) make clear another important feature of the Hall–Littlewood polynomials: P_λ interpolates between s_λ and m_λ. From (1.26) it is immediately apparent that

$$P_\lambda(x; 0) = s_\lambda(x), \tag{1.33}$$

and from (1.29) we can observe that

$$P_\lambda(x; 1) = m_\lambda(x). \tag{1.34}$$

The set $\{P_\lambda : \ell(\lambda) \leq n\}$ is a $\mathbb{Z}[q]$ basis for $\Lambda_n[q]$ [Macd95, pp. 209].

The modified Hall–Littlewood polynomials Q'_λ

We now turn our attention to a particular family of Hall–Littlewood polynomials which are, for our purposes, the most important symmetric functions.

With respect to the Hall inner product (1.21), the polynomials dual to P_λ are known as the *modified Hall–Littlewood polynomials* $Q'_\lambda := Q'_\lambda(x; q)$, i.e.,

$$\langle P_\lambda, Q'_\mu \rangle = \delta_{\lambda\mu}. \tag{1.35}$$

As an immediate consequence of this definition, the polynomials Q'_λ interpolate between s_λ and h_λ. By (1.33) and (1.34), we have

$$Q'_\lambda(x;0) = s_\lambda(x),$$

and

$$Q'_\lambda(x;1) = h_\lambda(x).$$

The Q'_λ are polynomials whose coefficients are positive integer polynomials in q, although this not obvious from (1.35). Later, Q'_λ will feature in our new combinatorial representations for the characters of affine Lie algebras; it is the native positivity of Q'_λ that permits us to say that these representations are manifestly positive.

In the remainder of this chapter, we will make clear the positivity of Q'_λ. In particular we show that they are *Schur positive*, i.e., they can be represented as a linear combination of Schur functions where all coefficients are positive. Towards this end we introduce the Kostka–Foulkes polynomials $K_{\lambda\mu}(q)$, which are defined by [Macd95, pp. 239]

$$s_\lambda = \sum_\mu K_{\lambda\mu}(q) P_\mu. \tag{1.36}$$

Now, from this definition and (1.35), it is easy to see that

$$\langle s_\lambda, Q'_\mu\rangle = \sum_\nu K_{\lambda\nu}(q)\langle P_\nu, Q'_\mu\rangle = \sum_\nu K_{\lambda\nu}(q)\delta_{\nu\mu} = K_{\lambda\mu}(q),$$

which immediately implies that

$$Q'_\mu = \sum_\lambda K_{\lambda\mu}(q) s_\lambda. \tag{1.37}$$

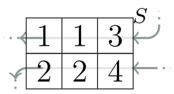
From this last equation it is clear that if $K_{\lambda\mu}(q)$ is positive, then Q'_μ is a Schur-positive polynomial. If one computes a few values of $K_{\lambda\mu}(q)$, there appears a strong suggestion that the Kostka–Foulkes polynomials are in general positive (and indeed, polynomial), a fact not clear from (1.36). Now, (1.18) and (1.37) together imply that $K_{\lambda\mu}(1) = K_{\lambda\mu}$, and so the Kostka–Foulkes polynomials are a generalisation of the Kostka numbers, and moreover, that (1.36) is equal to (1.17) when $q = 1$. In light of these facts, Foulkes [Fo74] conjectured that there must exist an interpretation of the $K_{\lambda\mu}(q)$ in terms of Young tableaux with an unknown nonnegative integer statistic $c(T)$:

$$K_{\lambda\mu}(q) = \sum_{T\in\mathrm{Tab}(\lambda,\mu)} q^{c(T)}. \tag{1.38}$$

The question of the existence of this combinatorial form was decided in the affirmative when Lascoux and Schützenberger [LaSchü78] provided a constructive proof, wherein $c(T)$ was named the *charge*. Using this result we may write (1.37) in a combinatorial form as

$$Q'_\mu = \sum_{T \in \mathrm{Tab}(\,\cdot\,,\mu)} q^{c(T)} s_{\mathrm{shape}(T)}. \tag{1.39}$$

To complete this section we provide a description of how to extract the charge from a given tableau, which was first published in full detail by Butler [Bu94]. This procedure requires that we establish a few conventions. A tableau T is read from right to left, top to bottom, starting at the upper-right-most cell S and following the path below, looping back to S as many times as necessary.



Each complete transit through all the cells of T is a *cycle*. The charge is extracted as follows: reading T as directed, remove the first 1 encountered, the first 2 thereafter, the first 3 following after that and so on until there is no larger number to be found in T. As each entry is removed, it is replaced with the number of cycles *completed* at the time of removal. Reset the cycle count to zero and begin another iteration, ignoring entries that have already changed, until all original entries of T have been replaced. The sum of the new entries is the charge of T. For example, there are two tableaux of shape $(3,3)$ and content $(2,2,1,1)$:

$$T_1 \quad \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&3&4\\\hline\end{array} \qquad\qquad T_2 \quad \begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&2&4\\\hline\end{array}$$

To compute $K_{(3,3),(2,2,1,1)}(q)$, we must find the charge of each tableaux:

Iteration	1	2	1	2
Cycle				
0	$\begin{array}{\|c\|c\|c\|}\hline 1&0&2\\\hline 0&3&4\\\hline\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 0&0&2\\\hline 0&1&2\\\hline\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&0&3\\\hline 2&0&4\\\hline\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 0&0&1\\\hline 0&0&1\\\hline\end{array}$
1	$\begin{array}{\|c\|c\|c\|}\hline 1&0&2\\\hline 0&1&4\\\hline\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 0&0&1\\\hline 0&1&2\\\hline\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&0&1\\\hline 2&0&1\\\hline\end{array}$	
2	$\begin{array}{\|c\|c\|c\|}\hline 1&0&2\\\hline 0&1&2\\\hline\end{array}$			
		$c(T_1) = 4$		$c(T_2) = 2$

Having calculated the charge for each tableaux, we can apply (1.38) and obtain

$$K_{(3,3),(2,2,1,1)}(q) = q^{c(T_1)} + q^{c(T_2)} = q^4 + q^2.$$

Other explanations of the charge statistic may be found in Macdonald [Macd95, pp. 129] and [Ha08, pp. 16].

The notation of λ-rings and the polynomials Q_λ

Due to their primary importance in this thesis, we have given a slightly non-standard treatment of Hall–Littlewood polynomials that emphasises the modified Hall–Littlewood polynomials Q'_λ. A more conventional approach first meets the Hall–Littlewood polynomials Q_λ. Recall the multiplicity notation for partitions from page 2 and let $\lambda = (1^{m_1}2^{m_2}3^{m_3}\dots)$. Then define the function $b_\lambda(q) =: b_\lambda$ by

$$b_\lambda = \prod_{i\geq 1} (q)_{m_i}. \tag{1.40}$$

The Q_λ are then usually defined in terms of P_λ as

$$Q_\lambda(x;q) = b_\lambda(q)P_\lambda(x;q).$$

Since $b_\lambda(0) = 1$, the Q_λ are another generalisation of the s_λ.

In this section we work in the opposite direction to a standard introduction and give a representation of Q_λ using Q'_λ and *λ-ring notation* (also called *plethystic notation*) [Ha08, La01], which we now revise. This notation will occasionally be employed to describe results in later chapters.

The notation of λ-rings is an extremely useful device that unifies much of the theory of symmetric functions. It is an abstraction that describes certain formal operations on alphabets of variables. These operations are defined below and give for each a simple example in terms of the rth power sum p_r (1.12), and another in terms of the generating function of the complete symmetric functions H_z (1.5). Most often we can easily define these operations at the level of the alphabet in simple language, but where this is not possible, the examples provided suffice to define the operation for all symmetric functions. Should the reader wish to compare these examples for consistency, the identity (1.13) will be useful.

Let X and Y be alphabets of variables. Addition of alphabets is achieved by merely joining alphabets together, i.e., $X + Y := X \cup Y$. The "subtraction" of alphabets is not easily described in such plain terms. However, observe that in terms of p_r and H_z the two operations are equally simple:

$$p_r(X \pm Y) = p_r(X) \pm p_r(Y) \tag{1.41a}$$

$$H_z(X \pm Y) = H_z(X)H_z^{\pm 1}(Y) \tag{1.41b}$$

Note that we have the agreeable property that

$$(X + Y) - Y = X.$$

More generally, we will see that λ-ring notation features many familiar arithmetic properties like this one.

Following from (1.41b) is the fact that

$$H_z(-X) = \frac{1}{H_z(X)}.$$

Recalling (1.11), we then have

$$E_z(X) = H_{-z}(-X),$$

and hence

$$e_r(X) = (-1)^r h_r(-X).$$

There exists many other relations between symmetric functions that are easily described by a change of alphabet.

Multiplication of alphabets X and Y produces another alphabet with the elements defined by

$$XY := \{xy : x \in X, y \in Y\}. \tag{1.42}$$

In terms of p_r and H_z:

$$p_r(XY) = p_r(X)p_r(Y), \tag{1.43a}$$

$$H_z(XY) = \prod_{\substack{x\in X \\ y \in Y}} \frac{1}{1 - zxy}. \tag{1.43b}$$

Our notation treats juxtaposition of an alphabet X with a scalar a as an instance of (1.42), so that $aX := \{a\}X$. Here there is important matter of interpretation. The notation $-X$ is to be read as $\{\} - X$, rather than $\{-1\}X$. As an aside, the latter is usually denoted ϵX.

Division of alphabets X/Y cannot be meaningfully defined in general, but there is a notion of division by the formal symbol $(1 - q)$, which amounts to taking each variable $x \in X$ and replacing it with an infinite number of variables x, qx, q^2x, and so on, i.e.,

$$\frac{X}{1-q} = \{xq^i : x \in X, i \in \mathbb{N}\}, \tag{1.44}$$

where $\mathbb{N}$ includes 0. For example:

$$p_r\Big(\frac{X}{1-q}\Big) = \frac{p_r(X)}{1-q^r}, \tag{1.45a}$$

$$H_z\Big(\frac{X}{1-q}\Big) = \prod_{x\in X} \frac{1}{(zx)_\infty}. \tag{1.45b}$$

We remark that

$$(1-q)X \cdot \frac{1}{1-q} = (1-q) \cdot \frac{X}{1-q} = X,$$

where we naturally interpret $(1-q)X$ as $X - qX$ and the order of operations follows ordinary arithmetic.

To close our discussion of λ-ring notation we remark that merely by reformulating statements using this notation, a proof concerning one symmetric function may be effortlessly distributed to many others.

The Hall–Littlewood polynomials Q_λ may then be represented as

$$Q_\lambda(X;q) := Q'_\lambda\big((1-q)X;q\big). \tag{1.46}$$

We remark that this relation is conventionally expressed as

$$Q'_\lambda(X;q) = Q_\lambda\Big(\frac{X}{(1-q)};q\Big).$$

We point out that the q-analogue of the Hall inner product may be defined using λ-ring notation. Observe that by the Hall inner product we have

$$\sum_\lambda P_\lambda(x;q)Q'_\lambda(y;q) = \prod_{i,j\geq 1}\frac{1}{1-x_iy_j} = H_1(xy). \tag{1.47}$$

By simply applying the definition of Q_λ (1.46) we obtain

$$\sum_\lambda P_\lambda(x;q)Q_\lambda(y;q) = H_1\big((1-q)xy\big) = \prod_{i,j\geq 1}\frac{1-qx_iy_j}{1-x_iy_j}.$$

Observe that when $q=0$, we obtain the Cauchy identity (1.24) in the first and last expressions. This then invites the definition of the *q-Hall inner product*:

$$\langle P_\lambda, Q_\mu\rangle_q = \delta_{\lambda\mu}.$$

Of course, generalisations of the statements equivalent to the Hall inner product (1.22) also follow. That is, given any two bases for $\Lambda[q]$, $u_\lambda := u_\lambda(x;q)$ and $v_\lambda := v_\lambda(x;q)$, the following are equivalent:

$$\langle u_\lambda, v_\mu\rangle_q = \delta_{\lambda\mu},$$

$$\sum_\lambda u_\lambda(x;q)v_\lambda(y;q) = \prod_{i,j\geq 1}\frac{1-qx_iy_j}{1-x_iy_j}.$$

Chapter 2

Characters of affine Kac–Moody Lie algebras

In this chapter we introduce the second of the main ingredients in this thesis, the *characters of Kac–moody Lie algebras*, specifically those corresponding to highest-weight modules of *affine* Lie algebras. The following chapter is a heavily truncated introduction that includes only slightly more material than what is required to be able to define the characters and understand our results concerning them.

Our interest in characters is of a purely combinatorial nature. We invoke the Kac–Moody Lie algebras, but do not define their algebraic structure, nor do we discuss their full representation theory. However, we occasionally use the language of this theory in accordance with convention, but all of our definitions are intended to stand free of these notions.

We begin by introducing the geometric structure underlying classical Lie algebras from the perspective of root systems. From these objects, we proceed to define Dynkin diagrams and Cartan matrices and give the Weyl character formula. For a reader new to these ideas, this is a slightly gentler approach than treating the finite and affine cases uniformly in the style of Kac [Kac90]. In the section on root systems our discussion follows Humphreys [Hu72], though the reader is cautioned that we make a significant departure by adopting conventions consistent with those of Kac. Consequently, our Cartan matrices are the transpose of those found in Humphreys' text. The classical section concludes with a brief discussion of character identities of Littlewood-type, and in particular some results due to Désarménien, Macdonald, Okada and Stembridge.

After the classical section, the discussion is raised to the affine case with the notion of generalised Cartan matrices, towards delivering the Weyl–Kac character formula. For the most part we follow Kac, which is well-complemented by Wakimoto [Wak01]. The reader is again warned in advance that we make some small departures that have subtle consequences for the intermediate (but not the final) results. These differences will be indicated where they appear.

We will conclude this chapter with a reformulation of the Weyl–Kac formula that more precisely suits our needs.

Finite root systems and the Weyl character formula

In this section we offer the reader some essential results from a standard treatment of root systems, before stating the Weyl character formula. We will also breifly touch on the notion of a Littlewood-type character formula. The following results and ideas are well-known and covered at length in many texts; for example, [Ja62, Hu72, Bo98, Bo02, Bo05].

Finite root systems

Let $\mathcal{E}$ be a Euclidean space, i.e., a finite dimensional vector space over $\mathbb{R}$ with a positive definite symmetric bilinear form $(.,.)$. For a nonzero vector $w \in \mathcal{E}$, the *dual* vector $w^\vee$ is defined by

$$w^\vee = 2w/\|w\|^2,$$

where $\|w\| = (w,w)^{1/2}$ is the *length* of w. The hyperplane P_w that is orthogonal to w forms a co-dimension 1 subspace of $\mathcal{E}$, such that $P_w = \{v \in \mathcal{E} : (w,v) = 0\}$. Let σ_w be the involution defined by reflection of $\mathcal{E}$ through the hyperplane P_w, that is,

$$\sigma_w(v) = v - (w^\vee, v)w.$$

Let Φ be a subset of the Euclidean space $\mathcal{E}$, with $\Phi^\vee$ the set of covectors. Φ is called a *reduced root system* in $\mathcal{E}$ if the following conditions are satisfied:

1. Φ spans $\mathcal{E}$ and does not contain the zero vector.

2. The only multiples of α in Φ are $\pm\alpha$.

3. For each $\alpha \in \Phi$, the reflection σ_α leaves Φ invariant.

4. If $\alpha, \beta \in \Phi$, then $(\alpha^\vee, \beta) \in \mathbb{Z}$.

The elements of Φ and $\Phi^\vee$ are then called *roots* and *coroots* and $\dim(\mathcal{E})$ is the *rank* of Φ.

It is worth noting that conditions 1–4 are not completely independent; (2) implies that the zero vector is not an element of Φ and both (2) and (3) have the consequence that $\Phi = -\Phi$, where $-\Phi = \{-\alpha \mid \alpha \in \Phi\}$.

The word *reduced* in *reduced root system* refers to the inclusion of condition (2). Our considerations concern only reduced root systems, and henceforth we suppress the word *reduced*. Condition (4) is commonly referred to as the *crystallographic condition* and amounts to a severe restriction on the admissible angles formed by pairs of roots, and also the relative lengths of roots. This is because

$$(\alpha^\vee, \beta)(\beta^\vee, \alpha) = 4\cos^2\theta,$$

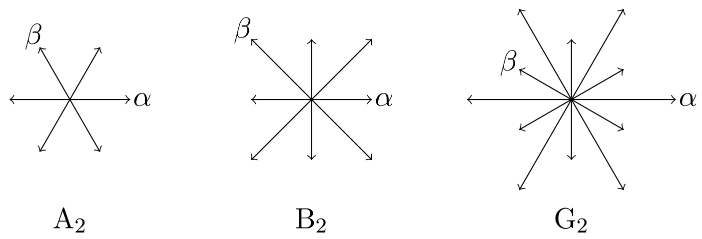


Figure 2.1: The root systems A_2, B_2 and G_2. The roots α and β mark the canonical bases.

and hence, θ is a multiple of $\pi/4$ or $\pi/6$. Furthermore, for $\|\alpha\| \geq \|\beta\|$, if $(\alpha, \beta) \neq 0$ then the only permissible ratios of root lengths are $\|\alpha\|^2/\|\beta\|^2 = 1, 2$ or 3.

A subset Δ of Φ is called a *base* if

1. Δ is a basis for $\mathcal{E}$,
2. Each root $\beta \in \Phi$ can be written as $\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha$, where every coefficient k_α is a nonnegative integer or every coefficient k_α is a nonpositive integer.

The elements of a base Δ are called *simple roots.* It is known that every root system has a base. Condition (2) means that, relative to Δ, we can partition the roots of Φ into the set of *positive* roots Φ^+, and *negative* roots Φ^-.

The *Weyl group* W of a root system Φ is defined as the group generated by the reflections $\{\sigma_\alpha : \alpha \in \Delta\}$, where W is independent of the particular choice of Δ. The elements of W permute the roots of Φ and hence W is isomorphic to a subgroup of the group of all permutations of the roots of Φ. From this it clearly follows that W is finite. Root systems that are dual (i.e., Φ and $\Phi^\vee$) share the same Weyl group.

A root system Φ is *irreducible** if Φ (or, equivalently Δ) cannot be partitioned into two orthogonal subsets. Irreducible root systems have been completely classified (up to rescaling) into 4 infinite families and 5 *exceptional* root systems. We will often use the notation X_r to refer to the irreducible root system of *type* X and rank r. For example, all irreducible rank-2 root systems correspond to one of the 3 diagrams in Figure 2.1.

Up to rescaling, root systems may be completely described by *Dynkin diagrams.* Given simple roots $\alpha_1, \ldots, \alpha_r$, the corresponding Dynkin diagram is a graph with r vertices labelled by $\alpha_1, \ldots, \alpha_r$, where the vertices α_i and α_j are connected by $(\alpha_i^\vee, \alpha_j)(\alpha_j^\vee, \alpha_i)$ edges. If $|(\alpha_i^\vee, \alpha_j)| > 1$ there is an arrow pointing to the vertex α_i.

Due to the fact that for irreducible root systems $(\alpha_i^\vee, \alpha_j)(\alpha_j^\vee, \alpha_i)$ may take only the values $0, 1, 2$ and 3, the corresponding Dynkin diagrams have only $0, 1, 2$ or 3

*Not to be confused with *reduced.*

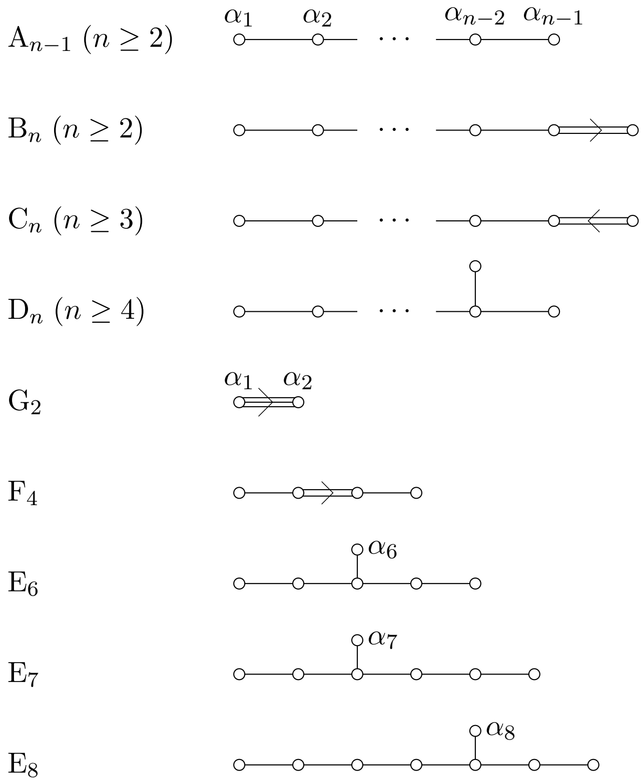


Figure 2.2: A classification of all irreducible root systems using Dynkin diagrams, with the standard ordering of roots. The rank restrictions on the root systems of type B and C are to avoid duplication.

edges between vertices. It can be shown that in any particular irreducible root system, the roots take at most two different lengths, and hence can be called either *long* or *short*. The sets of these roots are denoted Φ_ℓ and Φ_s respectively. It is easy to see that where we have 2 or 3 edges connecting 2 vertices in a Dynkin diagram, one of the vertices corresponds to a short root and the other to a long root. For Dynkin diagrams of irreducible root systems, the arrow points from the long root to the short root. Naturally, the simple roots connected by one edge have the same length. Figure 2.2 contains the complete classification of all irreducible root systems of positive rank.

If we were to omit condition (2) from the definition on page 22, this classification would contain an additional infinite family of root systems of type BC. If condition (4) were omitted, we must then include an infinite family associated with the *dihedral group*[*], and two extra exceptional root systems, H_3 and H_4.

[*] The group of symmetries of the regular m-gon. For the dihedral group of order $2m$, the associated root system may be constructed by taking as our roots the set of lines passing through the origin that are normal to the reflections that preserve the regular m-gon.

$$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

G_2 $\qquad$ A_3 $\qquad$ C_4

Figure 2.3: Cartan matrices of the root systems G_2, A_3 and C_4. The simple roots are ordered according to the classification in Figure 2.2.

Given a root system Φ with base Δ, where the simple roots have a fixed ordering $(\alpha_1, \ldots, \alpha_r)$, the entries of the corresponding *Cartan matrix* $A = [a_{ij}]_{1 \le i,j \le n}$ are defined as $a_{ij} = (\alpha_i^\vee, \alpha_j)$. For example, the root systems G_2, A_3 and C_4 have the Cartan matrices given in Figure 2.3. Given the Dynkin diagram of an irreducible root system it is easy to deduce the entries of the corresponding Cartan matrix: $a_{ij}a_{ji}$ is the number of edges between vertices α_i and α_j, and any ambiguity in the factorisation is resolved by the presence of an arrow. Up to simultaneous relabelling of the rows and columns, a Cartan matrix is independent of the choice of Δ and completely determines Φ. If A is the Cartan matrix of Φ, then the transpose matrix A^{T} is the Cartan matrix of the dual root system $\Phi^\vee$.

In most standard texts one may find explicit descriptions of all irreducible root systems in terms of $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$, the standard unit vectors in $\mathbb{R}^n$, but here we give only those descriptions that we will later employ. We adopt the normalisation of roots lengths found in [Hu72].

The following description of A_{n-1} uses unit vectors from $\mathbb{R}^n$, but the Euclidean space $\mathcal{E}$ spanned by A_{n-1} is in fact the $n-1$ dimensional hyperplane orthogonal to the vector $\epsilon_1 + \cdots + \epsilon_n$. For $n \ge 2$, the positive roots Φ_+ of the A_{n-1} root system are given by

$$\{\epsilon_i - \epsilon_j : 1 \le i < j \le n\}, \tag{2.1a}$$

where the canonical choice for Δ is

$$\{\epsilon_i - \epsilon_{i+1} : 1 \le i \le n-1\}. \tag{2.1b}$$

This description is quite convenient. Observe that in A_{n-1}, the reflection $\sigma_{\epsilon_i - \epsilon_{i+1}}$ sends each root $\epsilon_j - \epsilon_k$ to the root where the indices i and $i+1$ have been transposed, leaving all other indices unchanged. The corresponding Weyl group W is generated by the set of root transpositions $\{\sigma_{\epsilon_i - \epsilon_{i+1}} : 1 \le i \le n-1\}$, and so W is isomorphic to the symmetric group S_n.

For the B_n and C_n root systems, $\mathcal{E} = \mathbb{R}^n$. For $n \geq 1$, the positive roots of the B_n root system are given by

$$\Phi_+ = \{\epsilon_i \pm \epsilon_j : 1 \leq i < j \leq n\} \cup \{\epsilon_i : 1 \leq i \leq n\}, \tag{2.1c}$$

where the canonical choice for Δ is

$$\Delta = \{\epsilon_i - \epsilon_{i+1} : 1 \leq i \leq n-1\} \cup \{\epsilon_n\}. \tag{2.1d}$$

Similarly, the positive roots of the C_n root system are given by

$$\Phi_+ = \{\epsilon_i \pm \epsilon_j : 1 \leq i < j \leq n\} \cup \{2\epsilon_i : 1 \leq i \leq n\}, \tag{2.1e}$$

where the canonical choice for Δ is

$$\Delta = \{\epsilon_i - \epsilon_{i+1} : 1 \leq i \leq n-1\} \cup \{2\epsilon_n\}. \tag{2.1f}$$

Note that the simple roots of the B_n and C_n root systems are those of A_{n-1}, with one extra root. With this additional root, W is isomorphic to the group of signed permutations (or *hyperoctahedral group*) $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$, where $\ltimes$ is the semi-direct product.

Given the simple roots $\alpha_1, \ldots, \alpha_r$, the *fundamental weights* are the vectors

$$\Lambda_1, \ldots, \Lambda_r \in \mathcal{E},$$

such that $(\alpha_i^\vee, \Lambda_j) = \delta_{ij}$. A *weight* is any vector $\Lambda \in \mathcal{E}$ that can be written as $\Lambda = \sum_{i=1}^n \lambda_i \Lambda_i$, for integers λ_i. If all λ_i are nonnegative, the weight Λ is said to be *dominant*. The set of dominant weights relative to a basis Δ is denoted P_+. For the A_{n-1} root system, we provide the fundamental weights $\Lambda_1, \ldots, \Lambda_n$, first in terms of the simple roots as

$$\Lambda_i = \frac{1}{n}\big[(n-i)\sum_{k=1}^{i-1} k\alpha_k + i\sum_{k=i}^{n-1}(n-k)\alpha_k\big] \qquad \text{for } 1 \leq i < n, \tag{2.2a}$$

and then using the unit basis $\epsilon_1, \ldots, \epsilon_n$ as

$$\Lambda_i = \frac{1}{n}\big[(n-i)(\epsilon_1 + \cdots + \epsilon_i) - i(\epsilon_{i+1} + \cdots + \epsilon_n)\big] \qquad \text{for } 1 \leq i \leq n. \tag{2.2b}$$

Similarly, for the B_n root system the fundamental weights are given by

$$\Lambda_i = \begin{cases} \displaystyle\sum_{k=1}^{i-1} k\alpha_k + i\sum_{k=i}^{n-1}\alpha_k + \frac{i}{2}\alpha_n & \text{for } 1 \leq i < n, \\ \displaystyle\frac{1}{2}\sum_{k=1}^{n} k\alpha_k & \text{for } i = n, \end{cases} \tag{2.3a}$$

and

$$\Lambda_i = \begin{cases} \epsilon_1 + \cdots + \epsilon_i & \text{for } 1 \le i < n, \\ \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_n) & \text{for } i = n. \end{cases} \tag{2.3b}$$

Finally, for the C_n root system the fundamental weights may be expressed as

$$\Lambda_i = \sum_{k=1}^{i-1} k\alpha_k + i \sum_{k=i}^{n-1} \alpha_k + i\alpha_n/2 \qquad \text{for } 1 \le i \le n, \tag{2.4a}$$

and

$$\Lambda_i = \epsilon_1 + \cdots + \epsilon_i \quad \text{for } 1 \le i \le n. \tag{2.4b}$$

The Weyl character formula

We now introduce the Weyl character formula, which arises from the representation theory of *Lie algebras*. Before the statement of this formula, we provide the standard definition of a character in the following passage, using the language of representation theory.

Let $\mathfrak{g}$ be a semi-simple Lie algebra and $\mathfrak{h}^*$ the dual of the corresponding Cartan subalgebra. The *character* of an irreducible $\mathfrak{g}$-module $V(\Lambda)$ of highest weight $\Lambda \in P_+$ is defined as

$$\mathrm{ch}_\Lambda = \sum_{\mu \in \mathfrak{h}^*} \dim(V_\mu)\, \mathrm{e}^\mu \,. \tag{2.5}$$

Here e^μ is a formal exponential and $\dim(V_\mu)$ is the dimension of the weight space V_μ in the weight-space decomposition of $V(\Lambda)$.

In [We25,We26], Hermann Weyl showed that given a Lie algebra with underlying root system Φ and Weyl group W,

$$\mathrm{ch}_\Lambda = \frac{\sum_{w \in W} \mathrm{sgn}(w)\, \mathrm{e}^{w(\Lambda+\rho)-\rho}}{\prod_{\alpha \in \Phi^+}(1 - \mathrm{e}^{-\alpha})}, \tag{2.6}$$

where $\Lambda \in P_+$ and where the *Weyl vector* ρ is defined as either

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \qquad \text{or} \qquad \rho = \sum_{i=1}^{r} \Lambda_i.$$

The *Weyl character formula* (2.6) tells us that all of the information required to compute any particular character is contained in the corresponding root system.

The remainder of this section is devoted to the discussion of the Weyl chracter formula. We will order our remarks as follows. The first remark concerns an important result which follows immediately from (2.6). This result yields a list of product-determinant identities that will see frequent use in the remainder of this thesis. Next, we examine the natural positivity of characters and their connection to symmetric functions.

When Λ is the zero vector, corresponding to the trivial one-dimensional representation for which the character is 1, we obtain the *Weyl denominator formula*:

$$\sum_{w\in W} \operatorname{sgn}(w)\, \mathrm{e}^{w(\rho)-\rho} = \prod_{\alpha\in\Phi^+} (1-\mathrm{e}^{-\alpha}). \tag{2.7}$$

Later, we will need (2.7) specialised for each of the infinite families of irreducible root systems. We list the preferred forms of the Weyl denominator identities for types A_{n-1}, B_n, C_n and D_n as follows:

$$\det_{1\le i,j\le n}\left(x_i^{n-j}\right) = \prod_{1\le i<j\le n}(x_i-x_j) =: \Delta(x) \tag{2.8a}$$

$$\det_{1\le i,j\le n}\left(x_i^{j-1}-x_i^{2n-j}\right) = \prod_{i=1}^{n}(1-x_i)\prod_{1\le i<j\le n}(x_i-x_j)(x_ix_j-1) =: \Delta_{\mathrm{B}}(x) \tag{2.8b}$$

$$\det_{1\le i,j\le n}\left(x_i^{j-1}-x_i^{2n-j+1}\right) = \prod_{i=1}^{n}(1-x_i^2)\prod_{1\le i<j\le n}(x_i-x_j)(x_ix_j-1) =: \Delta_{\mathrm{C}}(x) \tag{2.8c}$$

$$\frac{1}{2}\det_{1\le i,j\le n}\left(x_i^{j-1}+x_i^{2n-j-1}\right) = \prod_{1\le i<j\le n}(x_i-x_j)(x_ix_j-1) =: \Delta_{\mathrm{D}}(x) \tag{2.8d}$$

We remarked earlier that equation (2.8a) is known as the *Vandermonde determinant*. Equations (2.8b) and (2.8c) are commonly regarded as its B and C type generalisations. For convenience we also provide here the corresponding Weyl vectors, which are as follows:

$$\rho_{\mathrm{A}} = \frac{1}{2}\sum_{i=1}^{n}(n-2i+1)\epsilon_i, \tag{2.9a}$$

$$\rho_{\mathrm{B}} = \sum_{i=1}^{n}\left(n-i+\tfrac{1}{2}\right)\epsilon_i, \tag{2.9b}$$

$$\rho_{\mathrm{C}} = \sum_{i=1}^{n}(n-i+1)\epsilon_i, \tag{2.9c}$$

$$\rho_{\mathrm{D}} = \sum_{i=1}^{n}(n-i)\epsilon_i. \tag{2.9d}$$

From (2.5), the positivity of characters is obvious, but this property is obscured by the inclusion-exclusion structure of Weyl's formulation. Naturally, one seeks forms where this positivity is manifest. In Chapter 1 we remarked that the character of the A_{n-1} root system is precisely the Schur function s_λ, which is a positive polynomial. This correspondence is achieved explicitly as follows.

We first compute the denominator of (2.6). The positive roots of A_{n-1} are given by (2.1) and so under the assignment $\mathrm{e}^{\epsilon_i} =: x_i$, we have

$$\prod_{\alpha\in\Phi^+}(1-\mathrm{e}^{-\alpha}) = \Delta(x)\prod_{i=1}^{n} x_i^{i-n}. \tag{2.10}$$

We now turn to the numerator of (2.6). Recall the fundamental weights $\Lambda_1,\dots,\Lambda_n$ for A_{n-1} (2.2b). For integers $c_1,\dots,c_{n-1}$ let the weight Λ be given by $\Lambda = \sum_{i=1}^{n-1} c_i\Lambda_i$ and define the partition $\lambda = (\lambda_1,\dots,\lambda_{n-1},0)$ by $\lambda_i = c_i+\cdots+c_{n-1}$. Then Λ may be written as

$$\Lambda = \sum_{i=1}^{n}\Big(\lambda_i - \frac{|\lambda|}{n}\Big)\epsilon_i.$$

For A_{n-1}, the corresponding Weyl vector ρ is given by (2.9a) and the Weyl group W is the symmetric group S_n. The numerator may then be expressed as

$$\sum_{w\in S_n}\operatorname{sgn}(w)\,\mathrm{e}^{w(\Lambda+\rho)-\rho} = \prod_{i=1}^{n} x_i^{i-n-\frac{|\lambda|}{n}} \det_{1\le i,j\le n}\Big(x_i^{\lambda_j+n-j}\Big).$$

Under the identification $x_1\cdots x_n = 1$, the denominator and numerator together yield the right-hand side of (1.14).

Littlewood-type sums

In this section we introduce the notion of a Littlewood-type sum, which are sums of the form

$$\sum_{\substack{\lambda\\ \text{suitable restrictions}}} c_\lambda f_\lambda(x),$$

where f_λ is a symmetric function, such as a Schur or Hall–Littlewood polynomial and c_λ is a combinatorially defined coefficient. In Chapter 6 we will derive combinatorial formulas for the characters of affine Kac–Moody algebras that are of this type.

To motivate our terminology, we revisit the Weyl character formula (2.6). This may be used as a means to produce analogues of the Schur function for other root systems. The analogues of types B, C and D are known as the *odd-orthogonal*, *symplectic* and *even-orthogonal* Schur functions respectively, denoted $\mathrm{so}_{2n+1,\lambda}(x) =: \mathrm{so}_{2n+1,\lambda}$, $\mathrm{sp}_{2n,\lambda}(x) =: \mathrm{sp}_{2n,\lambda}$ and $\mathrm{so}_{2n,\lambda}(x) =: \mathrm{so}_{2n,\lambda}$ [Li50]. By making the substitution

$\mathrm{e}^{-\epsilon_i} =: x_i$ in (2.6), we obtain the explicit forms

$$\mathrm{so}_{2n+1,\lambda}(x) = \frac{\det_{1\le i,j\le n}\left(x_i^{j-1-\lambda_j} - x_i^{2n-j+\lambda_j}\right)}{\Delta_{\mathrm{B}}(x)}, \tag{2.11a}$$

$$\mathrm{sp}_{2n,\lambda}(x) = \frac{\det_{1\le i,j\le n}\left(x_i^{j-1-\lambda_j} - x_i^{2n-j+1+\lambda_j}\right)}{\Delta_{\mathrm{C}}(x)}, \tag{2.11b}$$

$$\mathrm{so}_{2n,\lambda}(x) = f_\lambda \frac{\det_{1\le i,j\le n}\left(x_i^{j-1-\lambda_j} + x_i^{2n-j-1+\lambda_j}\right)}{\Delta_{\mathrm{D}}(x)}, \tag{2.11c}$$

where, in the last expression we have used the notation

$$f_\lambda = \begin{cases} 1/2 & \text{if } \ell(\lambda) < n, \\ 1 & \text{if } \ell(\lambda) = n. \end{cases}$$

We note that it will be convenient to allow λ to be a half-partition (see p. 3) in the case of (2.11a) and (2.11c). Note that the sign of ϵ_i in the substitution $\mathrm{e}^{-\epsilon_i} =: x_i$ differs from that used at the end of the previous section. The equations (2.11) are the same regardless of which sign we choose, our choice here is for consistency with later expressions in which the minus sign is convenient.

Each of the generalised Schur functions (2.11) are Laurent polynomials in x and display signed-permutation symmetry. It is easy to see that these polynomials have maximum degree $|\lambda|$. After rescaling by a monomial factor $(x_1 \cdots x_n)^{\lambda_1}$ the functions (2.11) become ordinary symmetric polynomials that may then be expanded in terms of the Schur basis. In [KoTe87], Koike and Terada obtained a general formula for the coefficients in this Schur function expansion in terms of alternating sums over Littlewood–Richardson coefficients. Such inclusion-exclusion representations are not combinatorial in nature and do not lend themselves to practical computations. However, there exist several manifestly positive expansions of Littlewood type for rectangular or near rectangular shapes. For example, in the case of rectangular partitions we have:

$$(x_1 \cdots x_n)^m \, \mathrm{sp}_{2n,(m^n)}(x) = \sum_{\substack{\lambda \text{ even} \\ \lambda_1 \le 2m}} s_\lambda(x) \tag{2.12a}$$

$$(x_1 \cdots x_n)^m \, \mathrm{so}_{2n+1,(m^n)}(x) = \sum_{\substack{\lambda \\ \lambda_1 \le 2m}} s_\lambda(x) \tag{2.12b}$$

$$(x_1 \cdots x_n)^m \, \mathrm{so}_{2n,(m^n)}(x) = \sum_{\substack{\lambda' \text{ even} \\ \lambda_1 \le 2m}} s_\lambda(x) \tag{2.12c}$$

The first identity is due to Désarménien [De86] and Stembridge [Stem90], the second to Macdonald [Macd95] and the last to Okada [Ok98]. Further examples for near-rectangular shapes may be found in [Kr98]. We point out that the second and third identities also allow for half-integer m and that the $m \to \infty$ case of all three identities was discovered by Littlewood in his classic text *The Theory of Group Characters and Matrix Representations of Groups* [Li50].

Apart from classical representation theory, Littlewood-type sums involving Schur functions have also played an important role in the theory of *plane partitions*. See e.g., [Br99, Macd95, Pro90, Stem90b].

Characters of affine Kac–Moody Lie algebras

In this section we introduce the Weyl–Kac character formula, a generalisation of the Weyl character formula for *affine Kac–Moody Lie algebras* [Mo67, Kac74]. In this generalised framework the picture is still essentially the same: underlying each affine Kac–Moody Lie algebra is a root system and a Weyl group, each now infinite dimensional, and contained within these objects is everything necessary to describe the corresponding characters of integrable highest-weight moduls. More thorough treatments of the representation theory of infinite-dimensional Lie algebras can be found in [Kac90, Wak01] and [KacPe84].

Preliminaries

There is quite a long list of definitions before we come to the Weyl–Kac character formula. Most importantly, we must generalise root systems and Weyl groups to the affine setting. Now, earlier we first described the classical root systems by way of their Dynkin diagrams and then from these objects defined each corresponding Cartan matrix. In the present generalised setting, it is useful to work in the opposite direction. Here we begin with generalised Cartan matrices and then proceed to describe the parts of affine root systems necessary to present the affine character formula.

The matrix $A = [a_{ij}]_{1\le i,j\le n}$ is a *generalised Cartan matrix* if it satisfies the following conditions

(C1) $a_{ii} = 2$ for $1 \le i \le n$,

(C2) a_{ij} are non-positive integers for $i \neq j$,

(C3) $a_{ij} = 0$ implies $a_{ji} = 0$.

Generalised Cartan matrices A and $A' = [a'_{ij}]_{1\le i,j\le n}$ are said to be *equivalent* if $a'_{ij} = a_{\sigma(i),\sigma(j)}$ for all $\sigma \in S_n$ and $1 \le i, j \le n$, that is, they are related by simultaneous permutation of rows and columns. If a generalised Cartan matrix is not equivalent to a matrix of the form

$$\left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$$

it is *indecomposable*. For our purposes, there are two types of indecomposable generalised Cartan matrices of interest: those of *finite type*, where all principal minors of A are positive, and those of *affine type*, where all proper principal minors of A are positive and $\det(A) = 0$. Indecomposable generalised Cartan matrices that are neither finite nor affine are *indefinite*. Naturally, the transpose of an indecomposable Cartan matrix is another indecomposable Cartan matrix of the same type.

The finite Cartan matrices are precisely classified by the Cartan matrices corresponding to the Dynkin diagrams in Figure 2.2.

We now proceed at quick pace through a long list of definitions leading up to the Weyl–Kac character formula.

Let $\mathfrak{g} = \mathrm{X}_N^{(r)}$ be a Kac–Moody Lie algebra with affine Cartan matrix $A = [a_{ij}]_{i,j\in I}$, where $I = \{0, 1, \ldots, n\}$, $N \ge 0$ is the *rank*, and $r = 1, 2$ or 3 is the *tier*. The relationship between n and the rank N is determined by type. Throughout this section, when given a group or set Y related to a Kac–Moody Lie algebra we will use the notation $\bar{Y}$ to represent the group or set corresponding to the Cartan matrix $\bar{A}$, which is obtained by deleting the zeroth row and column of A. The object $\bar{Y}$ is often called *the finite part of* Y.

Corresponding to $\mathfrak{g}$ is the $(n+2)$-dimensional Cartan subalgebra $\mathfrak{h}$ and its dual $\mathfrak{h}^*$, with pairing $\langle ., . \rangle$. Choose two sets of $n+1$ linearly independent elements: the *simple coroots* $\alpha_0^\vee, \ldots, \alpha_n^\vee \in \mathfrak{h}$ and the *simple roots* $\alpha_0, \ldots, \alpha_n \in \mathfrak{h}^*$, where $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$. From this choice arise the *labels* $a_0, \ldots, a_n$ and *colabels* $a_0^\vee, \ldots, a_n^\vee$, which are positive integers such that

$$(a_0^\vee, \ldots, a_n^\vee) \cdot A = A \cdot \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix} = 0,$$

subject to the condition $\gcd(a_0, \ldots, a_n) = \gcd(a_0^\vee, \ldots, a_n^\vee) = 1$. For all $\mathfrak{g}$ under consideration, $a_0^\vee = 1$. The sum $h = \sum_{i\in I} a_i$ is known as the *Coxeter number* and $h^\vee = \sum_{i\in I} a_i^\vee$ is the dual Coxeter number.

We point out that the affine Cartan matrix $A^{\mathrm{T}} = [a_{ji}]_{i,j\in I}$ corresponds to the Kac-Moody Lie algebra dual to $\mathrm{X}_N^{(r)}$, in which all objects and co-objects described above are interchanged.

The full classification of all indecomposable affine Cartan matrices (up to relabelling) may be expressed in the form of Dynkin diagrams. Those of *nontwisted* type are listed in Figure 2.4 and those of *twisted* type can be found in Figure 2.5. One may obtain the Dynkin diagram of the finite part of $\mathrm{X}_N^{(r)}$ by simply deleting the vertex α_0 and all its incident edges, however this results in the Dynkin diagram of the irreducible root system X_N only when $r = 1$. The Dynkin diagram of the dual algebra may be obtained by simply reversing the directions of the arrows.

We extend the simple roots and coroots to bases of $\mathfrak{h}^*$ and $\mathfrak{h}$ by introducing $\Lambda_0 \in \mathfrak{h}^*$ and $d \in \mathfrak{h}$, such that $\langle \alpha_i^\vee, \Lambda_0 \rangle = \langle d, \alpha_i \rangle = \delta_{i,0}$ and $\langle d, \Lambda_0 \rangle = 0$. The standard non-degenerate bilinear form $(.|.)$ on $\mathfrak{h}$ is defined by setting

$$(\alpha_i^\vee|\alpha_j^\vee) = \frac{a_j}{a_j^\vee}\, a_{ij}, \qquad (\alpha_i^\vee|d) = a_0\delta_{i,0}, \qquad (d|d) = 0.$$

The spaces $\mathfrak{h}$ and $\mathfrak{h}^*$ are identified by choosing $d = a_0\Lambda_0$ and $\alpha_i^\vee = a_i\alpha_i/a_i^\vee$. We can then see that

$$(\alpha_i|\alpha_j) = \frac{a_i^\vee}{a_i}\, a_{ij}, \qquad (\alpha_i|\Lambda_0) = \frac{1}{a_0}\delta_{i,0}, \qquad (\Lambda_0|\Lambda_0) = 0.$$

The *null root* (or *fundamental imaginary root*) δ is defined as the sum $\sum_{i\in I} a_i\alpha_i$. We wish to point out that under this construction $\mathfrak{h}^* = \mathbb{C}\Lambda_0 \oplus \bar{\mathfrak{h}}^* \oplus \mathbb{C}\delta$. The Weyl vector $\rho \in \mathfrak{h}^*$ is defined by $\langle \alpha_i^\vee, \rho \rangle = 1$ and $\langle d, \rho \rangle = 0$.

The *level* $\mathrm{lev}(\Lambda)$ of an element $\Lambda \in \mathfrak{h}^*$ is defined as $\mathrm{lev}(\Lambda) = \langle K, \Lambda \rangle$, in relation to the *canonical central element* $K = \sum_{i\in I} a_i^\vee \alpha_i^\vee$. Observe that $\mathrm{lev}(\rho) = h^\vee$. More generally for $\Lambda \in \mathfrak{h}^*$, we point out that $\mathrm{lev}(\Lambda)$ depends only on the Λ_0 component of the basis expansion of Λ in $\alpha_0, \ldots, \alpha_n, \Lambda_0$. For example, $\mathrm{lev}(\Lambda_0) = 1$. Extend Λ_0 to a complete set of fundamental weights $\Lambda_0, \ldots, \Lambda_n \in \mathfrak{h}^*$, where $\langle \alpha_i^\vee, \Lambda_j \rangle = \delta_{ij}$ and $\langle d, \Lambda_i \rangle = 0$. Let $P_+ = \{\Lambda : \langle \alpha_i^\vee, \Lambda \rangle \in \mathbb{N}\}$ be the set of *dominant integral weights* of $\mathfrak{g}$.

The integer span of the simple roots and coroots define Q and $Q^\vee$, the root and coroot *lattices* respectively. Similarly, $\bar{Q}$ and $\bar{Q}^\vee$ are the lattice subspaces with no α_0 and $\alpha_0^\vee$ component. To avoid confusion, we note explicitly that $\bar{Q}^\vee = \big\{ \sum_{i=1}^n r_i a_i \alpha_i / a_i^\vee : (r_1, \ldots, r_n) \in \mathbb{Z}^n, \alpha \in \bar{Q} \big\}$. Let the lattice M be defined by

$$M = \begin{cases} \bar{Q}^\vee & \text{for } \mathfrak{g} = \mathrm{X}_N^{(1)} \text{ or } \mathrm{A}_{2n}^{(2)}, \\ \bar{Q} & \text{otherwise.} \end{cases} \tag{2.13}$$

There are two important ways to naturally partition $\mathfrak{h}^*$. The set of *positive roots* $\Phi_+ \subset \mathfrak{h}^*$ is comprised of all roots that have a nonnegative expansion in the basis $\alpha_0, \ldots, \alpha_n, \Lambda_0$. The roots that are not positive are *negative*. The roots that satisfy

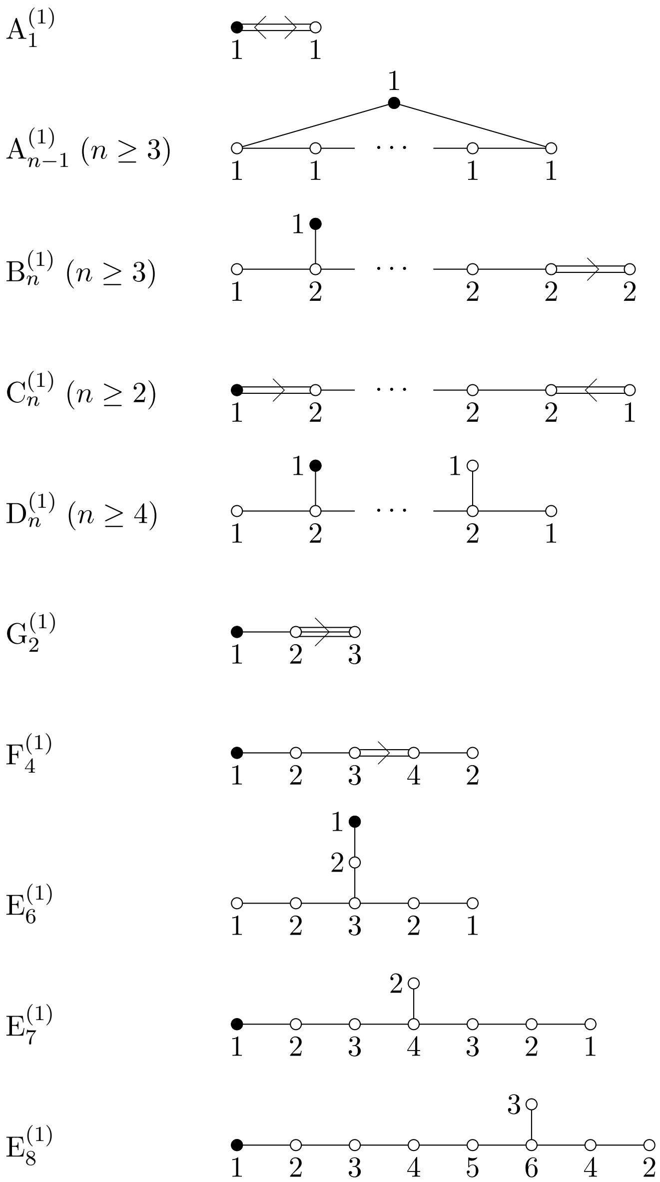


Figure 2.4: The Dynkin diagrams of affine Cartan matrices of nontwisted type, with the root α_0 marked black. The other vertices of $\mathrm{X}_N^{(1)}$ are indexed by α_i in a manner consistent with that of X_N in Figure 2.2. The corresponding labels a_i appear adjacent to each vertex.

$(\alpha|\alpha) > 0$ are called *real*, denoted Φ^{re}; otherwise they are *imaginary*, denoted Φ^{im}. In Figure 2.6 a complete description of the *positive imaginary roots* Φ_+^{im} and the *positive real roots* Φ_+^{re}, with their multiplicities, is provided for all $\mathrm{X}_N^{(r)}$. In this figure $\bar{\Phi}$ is the set of roots of the classical root system corresponding to $\bar{A}$, relative to the base $\bar{\Delta}$, and $\mathrm{mult}(\alpha)$ is the dimension of the rootspace [Hu72, pp. 35] corresponding

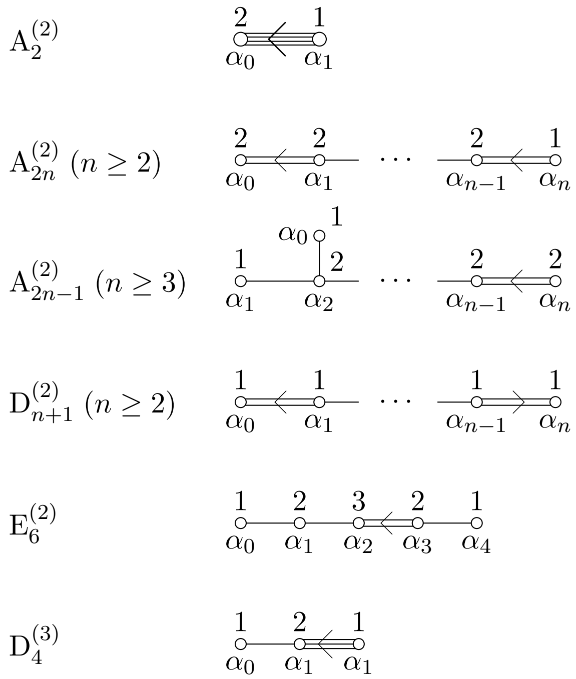


Figure 2.5: The Dynkin diagrams of affine Cartan matrices of twisted type. The number appearing above the vertex α_i is the corresponding label a_i.

to the root α.

$$\Phi_+^{\mathrm{im}} = \{m\delta : m \in \mathbb{Z}_+\} \qquad \mathrm{mult}(m\delta) = \begin{cases} n & \text{if } m \in r\mathbb{Z}_+ \\ \frac{N-n}{r-1} & \text{if } m \notin r\mathbb{Z}_+ \end{cases}$$

$$\Phi_+^{\mathrm{re}} = \bar{\Phi}_+ \cup \left\{ m\delta + \alpha : \alpha \in \bar{\Phi}, m \in \begin{cases} \mathbb{Z}_+ & \text{if } \alpha \in \bar{\Phi}_s \\ r\mathbb{Z}_+ & \text{otherwise} \end{cases} \right\}$$

$$\mathrm{mult}(\Phi_+^{\mathrm{re}}) = 1$$

Figure 2.6: The positive roots for each $X_N^{(r)}$. If $X_N^{(r)} = A_{2n}^{(2)}$, there are the additional positive real roots $\{m\delta \pm \frac{1}{2}(\delta - \sum_{i=k}^n a_i\alpha_i) : m \in \mathbb{Z}_+,\ 1 \leq k \leq n\}$ and $\{\frac{1}{2}(\delta - \sum_{i=k}^n a_i\alpha_i) : 1 \leq k \leq n\}$, all with multiplicity 1. Here $\mathbb{Z}_+$ is the set of positive integers. Recall the short roots $\bar{\Phi}_s$ from Section 2.

The Weyl–Kac Character formula

Let Λ be a dominant integral weight. The Weyl–Kac formula for the characters of the highest weight module $V(\Lambda)$ of a Kac–Moody algebra is given by

$$\mathrm{ch}_\Lambda = \frac{\sum_{w\in W} \mathrm{sgn}(w)\, \mathrm{e}^{w(\Lambda+\rho)-\rho}}{\prod_{\alpha\in\Phi^+}(1-\mathrm{e}^{-\alpha})^{\mathrm{mult}(\alpha)}}, \tag{2.14}$$

where W is the *affine Weyl group* of the root system underlying the Kac–Moody algebra. Note that the structure of this formula is exactly the same as the classical formula (2.6), up to the occurrence of $\mathrm{mult}(\alpha)$. In the setting of affine Kac–Moody algebras, it remains the case that the natural positivity of characters is hidden by the alternating-sum structure of the character formula. Again, a question presents itself: do there exist manifestly positive representations for affine characters? Our principle results in chapter 6 answer this question in the affirmative for types $\mathrm{A}_{2n}^{(2)}$ and $\mathrm{C}_n^{(1)}$, with a conditional affirmation for type $\mathrm{D}_{n+1}^{(2)}$. All of these results are presently restricted to 1-parameter families of weights. In Chapter 6 we will give a brief explanation for why our results concern only these types.

The methods that we employ to achieve these results require a modified representation of the Weyl–Kac character formula, due to Kac and Peterson [KacPe84]. We now prepare this reformulation.

Recall the root lattice M (2.13). By using the fact that $W = \bar{W} \ltimes M$, Kac and Peterson decompose the sum in (2.14) to yield an infinite sum over the lattice M and a finite sum over the classical Weyl group $\bar{W}$, so that

$$\begin{aligned}\mathrm{e}^{-\Lambda}\,\mathrm{ch}_\Lambda = \prod_{\alpha\in\Phi^+}&(1-\mathrm{e}^{-\alpha})^{-\,\mathrm{mult}(\alpha)}\\ &\times \sum_{\gamma\in M}\sum_{w\in\bar{W}} \mathrm{sgn}(w) q^{\frac{1}{2}\kappa(\gamma|\gamma)-(\gamma|w(\bar{\Lambda}+\bar{\rho}))}\, \mathrm{e}^{-\kappa\gamma+w(\bar{\Lambda}+\bar{\rho})-\bar{\Lambda}-\bar{\rho}}\end{aligned} \tag{2.15}$$

where $\kappa = \mathrm{lev}(\Lambda+\rho) = \mathrm{lev}(\Lambda) + h^\vee$ and $q = \exp(-\delta)$.

Equation (2.15) is the general character formula, but we require forms slightly more specialised, computed for each of $\mathrm{A}_{2n}^{(2)}$, $\mathrm{C}_n^{(1)}$ and $\mathrm{D}_{n+1}^{(2)}$. One may easily compute similar forms for all other affine types, but it is only for these three that our present methods have traction.

The Weyl groups of $\mathrm{A}_{2n}^{(2)}$, $\mathrm{C}_n^{(1)}$ and $\mathrm{D}_{n+1}^{(2)}$ are similar enough that we may proceed uniformly for a while longer. For each of these three types, $\bar{W} = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$. For $\mathrm{A}_{2n}^{(2)}$ and $\mathrm{D}_{n+1}^{(2)}$, M is precisely the B_n root lattice $\mathrm{span}(\Delta)$, where Δ is defined in (2.1d). Note that that M is sensitive to the scaling of root lengths. In the case of $\mathrm{C}_n^{(1)}$ (where our scaling in (2.1f) departs from Kac's normalisation), $M = \mathrm{span}(2\Delta)$.

Type	a	b
$\mathrm{A}_{2n}^{(2)}$	1	1
$\mathrm{C}_{n}^{(1)}$	2	$\frac{1}{2}$
$\mathrm{D}_{n+1}^{(2)}$	1	2

Figure 2.7: Values of a and b by type.

For these three affine types, we can then rewrite M in terms of the standard unit vectors $\epsilon_1, \ldots, \epsilon_n$ as

$$M = \left\{ \sum_{i=1}^{n} r_i \epsilon_i : (r_1, \ldots, r_n) \in \begin{cases} \mathbb{Z}^n & \text{for } \mathrm{A}_{2n}^{(2)} \text{ or } \mathrm{D}_{n+1}^{(2)} \\ 2\mathbb{Z}^n & \text{for } \mathrm{C}_{n}^{(1)} \end{cases} \right\} \tag{2.16}$$

Kac has the convention that the simple roots $\alpha_1, \ldots, \alpha_n$ are always scaled so that $(\alpha_i|\alpha_i) = 2$ for α_i a long root. We do not follow this convention. For our purposes, we prefer to scale the roots in a way consistent with the descriptions in (2.1b), which are given in terms of the standard basis vectors $\epsilon_1, \ldots, \epsilon_n$. The length of the long roots then depend on type. As a consequence, the value of $(\epsilon_i|\epsilon_j)$ varies according to type as follows:

$$(\epsilon_i|\epsilon_j) = \begin{cases} \delta_{ij} & \text{for } \mathrm{A}_{2n}^{(2)}, \\ \delta_{ij}/2 & \text{for } \mathrm{C}_{n}^{(1)}, \\ 2\delta_{ij} & \text{for } \mathrm{D}_{n+1}^{(2)}. \end{cases} \tag{2.17}$$

This consideration must be borne in mind during intermediate calculations, but the final results in this section are not affected by the scaling of roots. We define the numbers a and b, which are determined according to type by (2.16) and (2.17) respectively and tabulated in Figure 2.7.

Let $\upsilon_1, \ldots, \upsilon_n$ be defined by $\bar{\Lambda} + \bar{\rho} = \sum_{i=1}^{n} \upsilon_i \epsilon_i$ and, once again, let $\mathrm{e}^{-\epsilon_i} =: x_i$. Informed by the near-uniformity of W for types $\mathrm{A}_{2n}^{(2)}$, $\mathrm{C}_{n}^{(1)}$ and $\mathrm{D}_{n+1}^{(2)}$, the double sum in Equation (2.15) may be written as

$$\sum_{r \in \mathbb{Z}^n} \left(\prod_{i=1}^{n} q^{\frac{1}{2} a^2 b \kappa r_i^2} x_i^{a \kappa r_i + \upsilon_i} \right) \sum_{w \in \bar{W}} \mathrm{sgn}(w) \prod_{i=1}^{n} y_{w(i)}^{-\upsilon_i}. \tag{2.18}$$

where $y_i = q^{abr_i} x_i$ and $y_{w(i)} = q^{abr_{w(i)}} x_{w(i)}$. The sum over $\bar{W}$ in (2.18) may be rewritten two sums over $(\mathbb{Z}/2\mathbb{Z})^n$ and S_n, which leads to the following determinant formulation:

$$\det_{1 \le i,j \le n} \left(y_i^{-\upsilon_j} - y_i^{\upsilon_j} \right).$$

Type	h	$h^\vee$
$\mathrm{A}_{2n}^{(2)}$	$2n+1$	$2n+1$
$\mathrm{C}_n^{(1)}$	$2n$	$n+1$
$\mathrm{D}_{n+1}^{(2)}$	$n+1$	$2n$

Figure 2.8: The Coxeter and dual Coxeter numbers for $\mathrm{A}_{2n}^{(2)}$, $\mathrm{C}_n^{(1)}$ and $\mathrm{D}_{n+1}^{(2)}$.

Kac and Peterson's reformulation (2.15), specialised for $\mathrm{A}_{2n}^{(2)}$, $\mathrm{C}_n^{(1)}$ and $\mathrm{D}_{n+1}^{(2)}$, is then given by

$$\mathrm{e}^{-\Lambda}\,\mathrm{ch}_\Lambda = \prod_{\alpha\in\Phi^+}(1-\mathrm{e}^{-\alpha})^{-\operatorname{mult}(\alpha)} \times \sum_{r\in\mathbb{Z}^n}\bigg(\prod_{i=1}^n q^{\frac{1}{2}a^2b\kappa r_i^2} x_i^{a\kappa r_i+\upsilon_i}\bigg) \det_{1\le i,j\le n}\big(y_i^{-\upsilon_j}-y_i^{\upsilon_j}\big). \tag{2.19}$$

We will now provide the specialised details necessary to reach the desired final forms for each of the $\mathrm{A}_{2n}^{(2)}$, $\mathrm{C}_n^{(1)}$ and $\mathrm{D}_{n+1}^{(2)}$ characters.

We begin with $\mathrm{C}_n^{(1)}$. According to the Dynkin diagrams in Figure 2.4 and Figure 2.2, the finite part of $\mathrm{C}_n^{(1)} =: \Phi$ is the C_n root system, which has the positive roots given in (2.1e), with Weyl vector $\bar{\rho} = \rho_{\mathrm{C}}$ defined in (2.9c). Figure 2.6 describes the corresponding positive real and imaginary affine roots. The product in the denominator of (2.19) then yields

$$\prod_{\alpha\in\Phi_+}(1-\mathrm{e}^{-\alpha})^{\operatorname{mult}(\alpha)} = (q)_\infty^n\,\Delta_{\mathrm{C}}(x)\prod_{i=1}^n x_i^{1-i}(qx_i^{\pm 2})_\infty \prod_{1\le i<j\le n}(qx_i^{\pm}x_j^{\pm})_\infty, \tag{2.20}$$

where $(au^{\pm})_\infty = (au, au^{-1})_\infty$ and

$$(au^{\pm}v^{\pm})_\infty = (auv, auv^{-1}, au^{-1}v, au^{-1}v^{-1})_\infty.$$

In (2.20), the factor $(q)_\infty^n$ corresponds to the roots in Φ_+^{im}, with the rest arising from the positive real roots.

Next we consider the numerator of (2.19). For $\Lambda = c_0\Lambda_0+\cdots+c_n\Lambda_n \in P_+$, define the partition $\lambda = (\lambda_1,\dots,\lambda_n)$ by $\lambda_i = c_i+\cdots+c_n$. Hence, recalling that $\bar{\Lambda}_i = \epsilon_1+\cdots+\epsilon_i$ from (2.4b), we have $\bar{\Lambda}+\bar{\rho} = \sum_{i=1}^n(\lambda_i+\rho_i)\epsilon_i$, so that $\upsilon_i = \lambda_i+n-i+1$. Also note that

$$\kappa = \sum_{i=0}^n a_i^\vee(c_i+1) = h^\vee + c_0+\cdots+c_n = n+1+c_0+\lambda_1.$$

Up to an overall factor, the determinant in (2.19) is then the symplectic Schur function $\mathrm{sp}_{2n,\lambda}(y)$ (2.11b):

$$\det_{1\le i,j\le n}\left(y_i^{-\lambda_j-n+j-1}-y_i^{\lambda_j+n-j+1}\right)=\Delta_{\mathrm{C}}(y)\,\mathrm{sp}_{2n,\lambda}(y)\prod_{i=1}^{n}y_i^{-n}. \tag{2.21}$$

where $y_i := x_i q^{r_i}$. Combining equations (2.19), (2.20) and (2.21), we obtain the next lemma.

Lemma 2.1 ($\mathrm{C}_n^{(1)}$ character formula)**.** *For $q=\exp(-\delta)$, $\lambda=(\lambda_1,\dots,\lambda_n)$ a partition and*

$$\Lambda=c_0\Lambda_0+(\lambda_1-\lambda_2)\Lambda_1+\cdots+(\lambda_{n-1}-\lambda_n)\Lambda_{n-1}+\lambda_n\Lambda_n\in P_+, \tag{2.22a}$$

$$x_i=\mathrm{e}^{-\alpha_i-\cdots-\alpha_{n-1}-\alpha_n/2}, \tag{2.22b}$$

we have

$$\mathrm{e}^{-\Lambda}\,\mathrm{ch}V(\Lambda)=\frac{1}{(q)_\infty^n\prod_{i=1}^n(qx_i^{\pm2})_\infty\prod_{1\le i<j\le n}(qx_i^{\pm}x_j^{\pm})_\infty}\times\sum_{r\in\mathbb{Z}^n}\frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)}\prod_{i=1}^n q^{\kappa r_i^2-nr_i}x_i^{2\kappa r_i+\lambda_i}\,\mathrm{sp}_{2n,\lambda}\big(xq^r\big), \tag{2.23}$$

where $\kappa=n+1+c_0+\lambda_1$.

We remark that the reformulated assignment (2.22b) in terms of the roots $\alpha_1,\dots,\alpha_n$ makes clear the root scaling independence of Lemma 2.1.

For the $\mathrm{A}_{2n}^{(2)}$ algebra, the procedure is especially similar since $\mathrm{A}_{2n}^{(2)}$ and $\mathrm{C}_n^{(1)}$ share the same finite part, so that $\bar{\Phi}_+$ and $\upsilon_1,\dots,\upsilon_n$ are unchanged. We note that $h^\vee$ takes a new value, given in Figure 2.8. Compared to the other affine types, the positive roots of $\mathrm{A}_{2n}^{(2)}$ are a particularly complicated set, and are given special treatment in Figure 2.6. The denominator of (2.19) is then given by

$$\prod_{\alpha\in\Phi_+}(1-\mathrm{e}^{-\alpha})^{\mathrm{mult}(\alpha)}=(q)_\infty^n\,\Delta_{\mathrm{C}}(x)\prod_{i=1}^n x_i^{1-i}(q^{1/2}x_i^{\pm})_\infty(q^2x_i^{\pm2};q^2)_\infty\times\prod_{1\le i<j\le n}(qx_i^{\pm}x_j^{\pm})_\infty. \tag{2.24}$$

In (2.24), the *additional positive real roots* of Figure 2.6 correspond to the product

$$\prod_{i=1}^n(q^{1/2}x_i^{\pm})_\infty.$$

To obtain the next result, we then repeat in a straight-forward manner the steps that yield Lemma 2.1.

Lemma 2.2 ($\mathrm{A}_{2n}^{(2)}$ character formula, I)**.** *With the same assumptions as in Lemma 2.1,*

$$\mathrm{e}^{-\Lambda}\,\mathrm{ch}V(\Lambda) = \frac{1}{(q)_\infty^n \prod_{i=1}^n (q^{1/2}x_i^{\pm})_\infty (q^2 x_i^{\pm 2};q^2)_\infty \prod_{1\le i<j\le n}(qx_i^{\pm}x_j^{\pm})_\infty} \\ \times \sum_{r\in\mathbb{Z}^n} \frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)} \prod_{i=1}^n q^{\frac{1}{2}\kappa r_i^2 - nr_i} x_i^{\kappa r_i+\lambda_i}\,\mathrm{sp}_{2n,\lambda}(xq^r), \quad (2.25)$$

where $\kappa = 2n+1+c_0+2\lambda_1$.

Since $\mathrm{A}_{2n}^{(2)}$ is self-dual, the Dynkin diagram labelled $\mathrm{A}_{2n}^{(2)}$ in Figure 2.5 and its mirror-image correspond (up to relabeling) to the same algebra and hence, the same character. There is a choice in which diagram most naturally represents $\mathrm{A}_{2n}^{(2)}$ and this freedom leads us to a not-entirely-trivial reformulation of the character. If our choice is swapped, one is cautioned that the rule that determines M (i.e., (2.13)) must be adjusted so that for $\mathrm{A}_{2n}^{(2)}$ we have $M = \bar{Q}$. Recalling $\Delta_{\mathrm{B}}(x)$ from (2.8b), the alternate form of the $\mathrm{A}_{2n}^{(2)}$ character is as follows.

Lemma 2.3 ($\mathrm{A}_{2n}^{(2)}$ character formula, II)**.** *For* $q = \exp(-\delta)$, $\mu = (\mu_1,\dots,\mu_n)$ *a partition, and*

$$\Lambda = 2\mu_n\Lambda_0 + (\mu_{n-1}-\mu_n)\Lambda_1 + \cdots + (\mu_1-\mu_2)\Lambda_{n-1} + c_n\Lambda_n \in P_+,$$

$$y_i = \mathrm{e}^{-\alpha_0-\cdots-\alpha_{n-i}},$$

(*so that* $y_i = q^{1/2}x_{n-i+1}^{-1}$ *and* $\mu_i = c_0/2+\lambda_1-\lambda_{n-i+1}$ *compared to* (2.25)),

$$\mathrm{e}^{-\Lambda}\,\mathrm{ch}V(\Lambda) = \frac{1}{(q)_\infty^n \prod_{i=1}^n (qy_i^{\pm})_\infty (qy_i^{\pm 2};q^2)_\infty \prod_{1\le i<j\le n}(qy_i^{\pm}y_j^{\pm})_\infty} \\ \times \sum_{r\in\mathbb{Z}^n} \frac{\Delta_{\mathrm{B}}(yq^r)}{\Delta_{\mathrm{B}}(y)} \prod_{i=1}^n q^{\frac{1}{2}\kappa r_i^2 - (n-\frac{1}{2})r_i} y_i^{\kappa r_i+\mu_i}\,\mathrm{so}_{2n+1,\mu}(yq^r),$$

where $\kappa = 2n+1+2c_n+2\mu_1$.

Here we have employed the odd orthogonal Schur functions $\mathrm{so}_{2n+1,\lambda}(x)$, defined in (2.11a).

Finally, we prepare a similar formulation for type $\mathrm{D}_{n+1}^{(2)}$. As before, we first compute the denominator of (2.15). Observe that the finite part of $\mathrm{D}_{n+1}^{(2)}$ is the B_n root system. Once again, the positive affine roots are given in Figure 2.6 and from Figure 2.8 we can see that $h^\vee = 2n$. The denominator of (2.15) may then be expressed as

$$\prod_{\alpha\in\Phi_+}(1-\mathrm{e}^{-\alpha})^{\mathrm{mult}(\alpha)} = (q^2;q^2)_\infty^{n-1}(q)_\infty \Delta_{\mathrm{B}}(x)\prod_{i=1}^n x_i^{1-i}(qx_i^{\pm})_\infty \prod_{1\le i<j\le n}(q^2x_i^{\pm}x_j^{\pm};q^2)_\infty.$$

We turn our attention to the numerator of (2.15). For a dominant integral weight $\Lambda := c_0\Lambda + \cdots + c_n$, let $\lambda = (\lambda_1, \dots, \lambda_n)$ as $\lambda_i = c_i + \cdots + c_{n-1} + \frac{1}{2}c_n$. Note that for convenience we have allowed for the possibility that λ is a half-partition (see page 3). From this definition and (2.3b), it follows that $\bar{\Lambda} = \sum_{i=1}^{n}(\frac{1}{2}c_n + \sum_{j=i}^{n-1} c_j)\epsilon_i$. Given the corresponding Weyl vector $\bar{\rho} = \rho_{\mathrm{B}}$ (2.9b), we then have $\upsilon_i = \lambda_i + \rho_i = \lambda_i + n - i + \frac{1}{2}$, and so the determinant in (2.19) is given by

$$\det_{1\le i,j\le n}\big(y_i^{-\lambda_j-n+j-\frac{1}{2}} - y_i^{\lambda_j+n-j+\frac{1}{2}}\big) = \Delta_{\mathrm{B}}(y)\,\mathrm{so}_{2n+1,\lambda}(y)\prod_{i=1}^{n} y_i^{\frac{1}{2}-n}, \tag{2.26}$$

where $y_i := x_i q^{2r_i}$. We then obtain the following form for the $\mathrm{D}_{n+1}^{(2)}$ character.

Lemma 2.4 ($\mathrm{D}_{n+1}^{(2)}$ character formula, $c_n = 2\lambda_n$)**.** *For $q = \exp(-\delta)$, $\lambda = (\lambda_1, \dots, \lambda_n)$ a partition, and*

$$\Lambda = c_0\Lambda_0 + (\lambda_1 - \lambda_2)\Lambda_1 + \cdots + (\lambda_{n-1} - \lambda_n)\Lambda_{n-1} + 2\lambda_n\Lambda_n \in P_+, \tag{2.27a}$$

$$x_i = \mathrm{e}^{-\alpha_i - \cdots - \alpha_n}, \tag{2.27b}$$

we have

$$\mathrm{e}^{-\Lambda}\,\mathrm{ch}V(\Lambda) = \frac{1}{(q^2;q^2)_\infty^{n-1}(q)_\infty\prod_{i=1}^{n}(qx_i^{\pm})_\infty\prod_{1\le i<j\le n}(q^2x_i^{\pm}x_j^{\pm};q^2)_\infty}$$
$$\times\sum_{r\in\mathbb{Z}^n}\frac{\Delta_{\mathrm{B}}(xq^{2r})}{\Delta_{\mathrm{B}}(x)}\prod_{i=1}^{n} q^{\kappa r_i^2-(2n-1)r_i}x_i^{\kappa r_i+\lambda_i}\,\mathrm{so}_{2n+1,\lambda}\big(xq^{2r}\big), \tag{2.28}$$

where $\kappa = 2n + c_0 + 2\lambda_1$.

At this point we remark that for the purposes of presenting our results, these character formulas are more general than necessary. Later, we will exclusively study cases where λ is the empty partition.

Chapter 3

Basic hypergeometric series

The final elements in our introductory material concern *basic hypergeometric series.* In this chapter we introduce these objects and, in particular, study an important classical result known as the Bailey lemma. To conclude the chapter we will revise the celebrated Rogers–Ramanujan identities and their various generalisations due to Andrews, Bressoud, Göllnitz, and Gordon.

Once again our treatment of a vast subject will be brief, as we have chosen to convey from the literature only those notions and results most pertinent to our new work. Gaspar and Rahman's *Basic Hypergeometric Series* [GasRa90] provide a much more complete treatment of this topic. We remark that the appendices of this book compile a list of many important q-hypergeometric summation and transformation formulas, as well as a useful collection of elementary q-factorial identities.

Basic hypergeometric series

In this section we introduce the fundamental notions concerning very-well-poised basic hypergeometric series and give some illustrative classical results that will be useful in our discussion of the Bailey lemma.

A *basic hypergeometric series* (also called *q-hypergeometric series*) is a series $\sum_{k\geq 0} c_k$, such that the quotient c_{k+1}/c_k is a rational function of q^k. Without loss of generality, every basic hypergeometric series can be expressed with coefficients that are ratios of q-shifted factorials. We follow Gaspar and Rahman's convention and define an ${}_r\phi_s$ basic hypergeometric series by

$$ {}_r\phi_s\left[\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}; q, z\right] = \sum_{k=0}^{\infty} \frac{(a_1,\dots,a_r)_k}{(q,b_1,\dots,b_s)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^k, \qquad (3.1) $$

where sometimes the left-hand side is written as ${}_r\phi_s(a_1,\dots,a_r;b_1,\dots,b_s;q,z)$. Under this definition, basic hypergeometric series are normalised so that when $k=0$ the summand on the right-hand side is 1. Observe that if any $a_i = q^{-n}$, then for

$k > n$ the summand vanishes, and therefore, the sum has a finite number of nonzero terms. A series with this property is said to be *terminating*. In many works prior to [GasRa90], e.g. [Sl66, Ba35], the square-bracketed term does not appear in the definition of ${}_r\phi_s$ series. By including this term, (3.1) gains the desirable property that if we set $z \mapsto z/a_r$ and let $a_r \to \infty$, the result is again a series of the same functional form, but with $r \mapsto r-1$.

Though we have given a general definition of basic hypergeometric series, the summations and transformations we will encounter are (with only one exception) ${}_{r+1}\phi_r$ series. For example, the *q-binomial theorem* may be expressed as

$$ {}_1\phi_0(a;-;q,z) = \frac{(az)_\infty}{(z)_\infty} \qquad |z| < 1. \tag{3.2}$$

Further to our focus upon ${}_{r+1}\phi_r$ series, the scope of our discussion will be limited to those series that are *balanced* or *very-well-poised*. We will introduce these special requirements first by example. For $n \in \mathbb{N}$, consider the left-hand side of the following identity, known as the *q-Pfaff–Saalschütz* summation [GasRa90, (II.12)]:

$$ {}_3\phi_2\left[\begin{matrix} c/a, c/b, q^{-n} \\ c, cq^{1-n}/ab \end{matrix}; q, q\right] = \frac{(a,b)_n}{(c, ab/c)_n}. \tag{3.3}$$

Observe that the product of the arguments of the two q-factorial terms in the denominator is exactly q times the product of the three terms in the numerator. More generally, a ${}_{r+1}\phi_r$ series is called *balanced* if $b_1 b_2 \dots b_r = q a_1 a_2 \dots a_{r+1}$ and $z = q$.

Consider next the following terminating summation, known as *Jackson's* ${}_6\phi_5$ sum [GasRa90, II.20]:

$$ {}_6\phi_5\left[\begin{matrix} a, qa^{1/2}, -qa^{1/2}, b, c, q^{-n} \\ a^{1/2}, -a^{1/2}, aq/b, aq/c, aq^{n+1} \end{matrix}; q, \frac{aq^{n+1}}{bc}\right] = \frac{(aq, aq/bc)_n}{(aq/b, aq/c)_n}. \tag{3.4}$$

On the left-hand side, each numerator term may be paired with a corresponding denominator term so that the product of each pair is aq. This is the identifying property of a *well-poised* basic hypergeometric series. An ${}_{r+1}\phi_r$ series is called *very-well-poised* if it is of the form

$$ {}_{r+1}\phi_r\left[\begin{matrix} a_1, qa_1^{1/2}, -qa_1^{1/2}, a_4, \dots, a_{r+1} \\ a_1^{1/2}, -a_1^{1/2}, qa_1/a_4, \dots, qa_1/a_{r+1} \end{matrix}; q, z\right]. \tag{3.5}$$

The extra adjective *very* signals the presence of the factor

$$ \frac{(qa_1^{1/2}, -qa_1^{1/2})_k}{(a_1^{1/2}, -a_1^{1/2})_k} = \frac{1 - a_1 q^{2k}}{1 - a_1}, \tag{3.6}$$

where k is the summation index. It is important to note that under the assignment $a_1 := x_1^2$, (3.6) is precisely equal to $\Delta_{\mathrm{C}}(xq^k)/\Delta_{\mathrm{C}}(x)$ on a single variable alphabet

$x = (x_1)$, where $\Delta_{\mathrm{C}}(x)$ is from the type C Vandermonde determinant (2.8c). This apparently trivial relationship between very-well-poised series and root systems becomes much more significant in the setting of *multiple basic hypergeometric series*, which will be discussed later.

It is often convenient to suppress the very-well-poised terms in (3.5) using the notation

$$_{r+1}W_r(a_1; a_4, a_5, \ldots, a_{r+1}; q, z).$$

With this notation, Jackson's $_6\phi_5$ summation may be compactly written as

$$_6W_5\big(a; b, c, q^{-n}; q, aq^{n+1}/bc\big) = \frac{(aq, aq/bc)_n}{(aq/b, aq/c)_n}. \tag{3.7}$$

An $_{r+1}\phi_r$ very-well-poised series reduces to an $_{r-1}\phi_{r-2}$ very-well-poised series under the specialisation

$$_{r+1}W_r\big(a_1; a_4, a_5, \ldots, a_{r+1}; q, z\big)\Big|_{a_r a_{r+1} = a_1 q} = {}_{r-1}W_{r-2}(a_1; a_4, a_5, \ldots, a_{r-1}; q, z). \tag{3.8}$$

Observe that subject to (3.8) (i.e., the specialisation $bc = aq$) the right-hand side of Jackson's $_6W_5$ summation (3.7) vanishes unless $n = 0$, so that we have

$$_4W_3(a; q^{-n}; q^n) = \delta_{n,0}. \tag{3.9}$$

where $\delta_{r,s}$ is the *Kronecker delta*, which is 1 when $r = s$ and 0 otherwise.

Bailey's lemma

Bailey's lemma is a powerful tool that enables the recursive construction of infinite families of q-hypergeometric series identities. There exists several detailed accounts of the origins, applications and generalisations of Bailey's lemma, see e.g., [AnAsRoy99,An00,An86,War01]. Our intentions in this chapter are to acquaint the reader with Bailey's lemma in the classical setting, in preparation for a generalisation to the C_n root system that appears in a later chapter, and so what follows is a much-abridged history and introduction.

In [Ba48], Bailey presents his lemma in its earliest form, framed as a simplified strategy for finding, one-at-a-time, transformations of q-hypergeometric series: for sequences $\{\alpha_n\}_{n\geq 0}, \ldots, \{\delta_n\}_{n\geq 0}, \{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$, if

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r}, \tag{3.10a}$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n}, \tag{3.10b}$$

then

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \tag{3.10c}$$

subject to suitable convergence conditions. Bailey had particular success for the choice $u_n = 1/(q)_n$ and $v_n = 1/(aq)_n$, which lead to proofs of a number of identities of Rogers–Ramanujan type, old and new. Under this choice, two sequences α and β satisfying the condition (3.10a) are together called a *Bailey pair* relative to a:

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}}. \tag{3.11a}$$

Similarly, a pair of sequences (γ, δ) where

$$\gamma_n = \sum_{r=n}^{\infty} \frac{\delta_r}{(q)_{r-n}(aq)_{r+n}}, \tag{3.11b}$$

are a *conjugate Bailey pair* relative to a. Bailey's student Slater [Sl52] pushed the use of (3.10) further and compiled a list of 130 identities of Rogers–Ramanujan type using 96 Bailey pairs (polynomial versions of all 130 identites have been found by Sills in [Si03] and more identities of Rogers–Ramanujan type have been added and put in the context of contemporary results in [LauSiZi08]). Slater's list, which included many new q-series identities, was substantial evidence of the power of Bailey's result, but even then the full potential of the lemma had hardly begun to be realised. In [Ba48, §4], Bailey makes special mention of the conjugate pair:

$$\delta_r = \frac{(b, c, q^{-N})_r}{(bcq^{-N}/a)_r} q^r, \tag{3.12a}$$

$$\gamma_n = \frac{(aq/b, aq/c)_N (b, c, q^{-N})_n (aq/bc)^n}{(aq, aq/bc)_N (aq/b, aq/c, aq^{N+1})_n} (-1)^n q^{Nn-\binom{n}{2}}. \tag{3.12b}$$

To see that (3.12) is a conjugate Bailey pair as claimed, one needs only substitute (3.12a) into the right-hand side of (3.11b) and complete the sum. For this purpose we need the q-Pfaff–Saalschütz ${}_3\phi_2$ summation (3.3), applied under the simultaneous assignment $(a, b, c, n) \mapsto (bq^n, cq^n, aq^{2n+1}, N-n)$.

Substitution of (3.12) into (3.10c) yields, after some minor adjustments, the expression

$$\sum_{r=0}^{n} \frac{(b, c)_r (aq/bc)^r}{(aq/b, aq/c)_r} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}} = \sum_{r=0}^{n} \frac{(b, c)_r (aq/bc)^r (aq/bc)_{n-r}}{(aq/b, aq/c)_n (q)_{n-r}} \beta_r. \tag{3.13}$$

Unfortunately, Bailey only considered (3.13) as n tends to infinity and did not notice that this expression offers a second Bailey pair, building upon the first. Decades later, it was Andrews [An84] who first struck upon this fact and recast (3.10) in a new iterative form. To produce Andrews' iterative form, we do nothing more than pull apart (3.13).

Lemma 3.1 (Bailey's lemma)**.** *If (α, β) is a Bailey pair relative to a, then (α', β') is also a Bailey pair relative to a, where*

$$\alpha'_n = \frac{(b,c)_n (aq/bc)^n}{(aq/b, aq/c)_n} \alpha_n, \tag{3.14a}$$

and

$$\beta'_n = \sum_{r=0}^{n} \frac{(b,c)_r (aq/bc)^r (aq/bc)_{n-r}}{(aq/b, aq/c)_n (q)_{n-r}} \beta_r. \tag{3.14b}$$

We remark that certain special cases of Lemma 3.1 were discovered in prior work by Paule [Pa82].

Given a Bailey pair (α, β), the recursive formulation of Bailey's lemma allows one to generate an infinite sequence of Bailey pairs:

$$(\alpha, \beta) \mapsto (\alpha', \beta') \mapsto (\alpha'', \beta'') \mapsto \cdots$$

Such a sequence is known as a *Bailey chain.* There are many different Bailey chains of interest, and each Bailey chain is generated by a corresponding *seed* Bailey pair. There is a special seed that emerges naturally from inversion of the Bailey pair relation.

Lemma 3.2 (Bailey pair inversion [An79, Lemma 3])**.** *If (α, β) is a Bailey pair, then*

$$\alpha_n = \sum_{r=0}^{n} \frac{(1-aq^{2n})(-1)^{n-r} q^{\binom{n-r}{2}} (a)_{n+r}}{(1-a)(q)_{n-r}} \beta_r. \tag{3.15}$$

By choosing $\beta = \delta_{n,0}$, Lemma 3.2 yields the *unit Bailey pair*

$$\begin{aligned} \alpha_n &= (-1)^n q^{\binom{n}{2}} \frac{1-aq^{2n}}{1-a} \frac{(a)_n}{(q)_n}, \\ \beta_n &= \delta_{n,0}. \end{aligned} \tag{3.16}$$

By iterating the unit pair (3.16) with the Bailey lemma, we obtain the following infinite family of Bailey pairs for $k \in \mathbb{N}$:

$$\alpha_n^{(k)} = (-1)^n q^{\binom{n}{2}} \frac{1-aq^{2n}}{1-a} \frac{(a)_n}{(q)_n} \prod_{i=1}^{k} \frac{(b_i, c_i)_n}{(aq/b_i, aq/c_i)_n} \left(\frac{aq}{b_i c_i}\right)^n, \tag{3.17a}$$

and

$$\beta_n^{(k)} = \sum_{r_1,\dots,r_{k-1}\geq 0} \frac{(aq/b_kc_k)_{n-r_{k-1}}}{(aq/b_k, aq/c_k)_n (q)_{n-r_{k-1}}} \\ \times \prod_{i=1}^{k-1} \frac{(b_{i+1}, c_{i+1})_{r_i} (aq/b_ic_i)_{r_i - r_{i-1}}}{(aq/b_i, aq/c_i)_{r_i} (q)_{r_i - r_{i-1}}} \left(\frac{aq}{b_{i+1}c_{i+1}}\right)^{r_i}. \tag{3.17b}$$

We can substitute (3.17) into the Bailey pair relation (3.11a), to obtain the corresponding multiple-series q-hypergeometric identity called the *Andrews transformation* [An75]:

$$ {}_{2k+4}W_{2k+3}\left(a; b_1, c_1, \dots, b_k, c_k, q^{-n}; q, \frac{a^k q^{n+k}}{b_1c_1\dots b_kc_k}\right) \\ = \frac{(aq, aq/b_kc_k)_n}{(aq/b_k, aq/c_k)_n} \sum_{r_1,\dots,r_{k-1}\geq 0} \frac{(q^{-n})_{r_{k-1}}}{(b_kc_kq^{-n}/a)_{r_{k-1}}} \left(\frac{b_kc_k}{a}\right)^{r_{k-1}} \\ \times \prod_{i=1}^{k-1} \frac{(b_{i+1}, c_{i+1})_{r_i} (aq/b_ic_i)_{r_i - r_{i-1}}}{(aq/b_i, aq/c_i)_{r_i} (q)_{r_i - r_{i-1}}} \left(\frac{aq}{b_{i+1}c_{i+1}}\right)^{r_i}. \tag{3.18}$$

We point out that for $k = 1$, this result reduces to Jackson's ${}_6\phi_5$ summation (3.7).

Note that (3.15) effectively offers a second identity for every Bailey pair (α, β). By substituting (3.17) into (3.15) we obtain another very general identity. We do not reproduce these identities here and only remark that for $k = 1$ we recover the q-Pfaff–Saalschütz identity (3.3).

We provide here a short proof of Lemma 3.2, due to Andrews [An79].

Proof of Lemma 3.2. Let (α, β) be a Bailey pair and let $A_{n,r}$ and $A'_{n,r}$ be the summands of (3.11a) (without α_r) and (3.15) (without β_r), respectively. We may interpret $A_{n,r}$ and $A'_{n,r}$ as the entries of invertible infinite-dimensional lower-triangular matrices A and A'. The following calculations essentially show that $A' = A^{-1}$. First relabelling $(n, r) \mapsto (r, s)$, we substitute (3.15) into (3.11a) to obtain

$$\beta_n = \sum_{r=0}^{n} A_{n,r}\alpha_r = \sum_{r=0}^{n} A_{n,r} \sum_{s=0}^{r} A'_{r,s}\beta_s. \tag{3.19}$$

Next we interchange the order of the two sums and then shift $r \mapsto r + s$,

$$\beta_n = \sum_{s=0}^{n} \beta_s \sum_{r=0}^{n-s} A_{n,r+s}A'_{r+s,s}. \tag{3.20}$$

By carrying out some standard manipulations involving q-shifted factorials, this yields

$$\beta_n = \sum_{s=0}^{n} \beta_s \frac{(aq)_{2s}}{(q)_{n-s}(aq)_{n+s}} \, {}_4W_3(aq^{2s}; q^{s-n}; q^{n-s}). \tag{3.21}$$

By application of the identity (3.9) for a ${}_4W_3$ series we then obtain

$$\beta_n = \sum_{s=0}^{n} \beta_s \frac{(aq)_{2s}}{(q)_{n-s}(aq)_{n+s}}\, \delta_{n,s} = \beta_n. \qquad \square$$

Identities of Rogers–Ramanujan type

In this section we will revise several generalisations of the famous Rogers–Ramanujan identities due to Andrews, Bressoud, Göllnitz and Gordon. This revision is preparation for our work in chapter 7, where we will derive further generalisations for these identities in the setting of affine Kac–Moody algebras. Other relationships between partition identities of Rogers–Ramanujan type and the representation theory of affine Lie algebras have been known for some time. The interested reader may find many such connections in [Cap96, LepMi78a, LepMi78b, LepWi78, LepWi82, LepWi84, Kac90, MePri87, MePri99].

The Rogers–Ramanujan identities [Schu17, RogRa19, Rog1894] are often stated in analytic form as

$$\sum_{r=0}^{\infty} \frac{q^{r^2}}{(q)_r} = \frac{(q^2, q^3, q^5; q^5)_\infty}{(q)_\infty}, \tag{3.22a}$$

and

$$\sum_{r=0}^{\infty} \frac{q^{r^2+r}}{(q)_r} = \frac{(q, q^4, q^5; q^5)_\infty}{(q)_\infty}. \tag{3.22b}$$

These identities have an interpretation in terms of partition congruences, due to MacMahon [Macm16, pp. 33–36] and Schur [Schu17]. Equation (3.22a) may be understood as:

> The number of partitions of n such that consecutive parts differ by at least 2 is equal to the number of partitions of n into parts congruent to $\pm 1 \pmod 5$.

Similarly, (3.22b) is equivalent to:

> The number of partitions of n such that all parts are greater than 1 and consecutive parts differ by at least 2 is equal to the number of partitions of n into parts congruent to $\pm 2 \pmod 5$.

We now revise a proof of the Rogers–Ramanujan identities due to Watson [Wat29]. Many classical proofs of the Rogers–Ramanujan identities (3.22) start by establishing by some means the *Rogers–Selberg identity* [RogRa19, Se36] (see also [GasRa90, Eq. (2.7.6)]):

$$(aq)_\infty \sum_{r=0}^{\infty} \frac{a^r q^{r^2}}{(q)_r} = 1 + \sum_{r=1}^{\infty} \frac{(aq)_{r-1}(1-aq^{2r})}{(q)_r} (-a^2)^r q^{r(5r-1)/2}. \tag{3.23}$$

From (3.23), the identities (3.22) are obtained simply by first specialising a to 1 or q, and then subsequently applying the *Jacobi triple product identity* [J1829] to the right-hand side:

$$\sum_{r=-\infty}^{\infty} (-z)^r q^{\binom{r}{2}} = (z, q/z, q)_\infty. \tag{3.24}$$

We briefly remark that we will later encounter (3.24) again as the denominator identity for the character of the affine Kac–Moody Lie algebra $\mathrm{A}_1^{(1)}$. Watson's celebrated proof connects the Rogers–Ramanujan identities to q-hypergeometric series by demonstrating that the Rogers–Selberg identity is found in the limit as b, c, d, e and n tend to infinity in what is now known as *Watson's terminating* ${}_8W_7$ *transformation*:

$$\begin{aligned} &{}_8W_7\big(a; b, c, d, e, q^{-n}; q, a^2q^{n+2}/bcde\big) \\ &\qquad = \frac{(aq, aq/bc)_n}{(aq/b, aq/c)_n}\, {}_4\phi_3\left[\begin{matrix} q^{-n}, b, c, aq/de \\ aq/d, aq/e, bcq^{-n}/a \end{matrix}; q, q\right]. \end{aligned} \tag{3.25}$$

Note that (3.25) is the Andrews transformation for $k = 2$. We remark that the terminating condition is lifted as n tends to infinity, see [GasRa90, II.25].

Andrews discovered that an argument essentially identical to Watson's proof of the Rogers–Ramanjuan identities, but beginning with the Andrews transformation (3.18), yields a more general family of identities which form part of the *Andrews–Gordon identities* [An74, Go61].

Theorem 3.3 (The Andrews–Gordon identities)**.** *For* $1 \le i \le k$*, let* $M_i = m_i + m_{i+1} + \cdots + m_{k-1}$*,*

$$\sum_{m_1,\dots,m_{k-1}\ge 0}^{\infty} \frac{q^{M_1^2+\cdots+M_{k-1}^2+M_i+\cdots+M_{k-1}}}{(q)_{m_1}\cdots(q)_{m_{k-1}}} = \frac{(q^i, q^{2k+1-i}, q^{2k+1}; q^{2k+1})_\infty}{(q)_\infty}. \tag{3.26}$$

Note that the Rogers–Ramanujan identities appear when $k = 2$.

Recalling the multiplicity notation m_i, for completeness we include Gordon's partition-theoretic statement of Theorem 3.3:

> For all $k \ge 1$, $1 \le i \le k$, let $A_{k,i}(n)$ be the number of partitions of n into parts not congruent to $0, \pm i \pmod{2k+1}$ and let $B_{k,i}(n)$ be the number of partitions of n of the form $\lambda = (1^{m_1}2^{m_2}\dots)$, with $m_1 \le i-1$ and $m_j + m_{j+1} \le k-1$ for all $j \ge 1$. Then $A_{k,i}(n) = B_{k,i}(n)$.

When $k = 2$, this partition-congruence interpretation corresponds to that of the Rogers–Ramanujan identities. Comparing the above with MacMahon and Schur's statement, it is clear that $A_{2,i}(n)$ satisfies the required modular-arithmetic conditions. It is not difficult to see that the restrictions on $B_{2,i}(n)$ match the minimum-difference conditions of the other set. Observe that $m_j + m_{j+1} \leq 1$ implies that, for all j, at least one of the parts j or $j+1$ do not appear in λ, and so the entries of λ differ by at least two.

Bailey's lemma takes all the pain out of proving (3.26), which was previously demonstrable only with less-systematic methods involving q-difference equations. We include a short proof of Theorem 3.3 using the Andrews transformation.

Proof of Theorem 3.3 for $i = 1$ or $i = k$. Given (3.18) we let all $b_1, c_1, \ldots, b_k, c_k, n$ tend to infinity to arrive at a higher-level Rogers–Selberg identity:

$$(aq)_\infty \sum_{r_1,\ldots,r_{k-1}\geq 0} \prod_{i=1}^{k-1} \frac{a^{r_i} q^{r_i^2}}{(q)_{r_i - r_{i-1}}} = \sum_{r=0}^{\infty} \frac{(aq)_{r-1}(1 - aq^{2r})}{(q)_r} (-a^k)^r q^{r((2k+1)r-1)/2}. \quad (3.27)$$

We then set $a = 1$ or q to obtain (3.28a) and (3.28b) respectively:

$$\sum_{m_1,\ldots,m_{k-1}\geq 0} \prod_{i=1}^{k-1} \frac{q^{M_i^2}}{(q)_{m_i}} = \frac{1}{(q)_\infty} \sum_{r=-\infty}^{\infty} (-q^{k+1})^r q^{(2k+1)\binom{r}{2}}, \quad (3.28a)$$

$$\sum_{m_1,\ldots,m_{k-1}\geq 0} \prod_{i=1}^{k-1} \frac{q^{M_i^2+M_i}}{(q)_{m_i}} = \frac{1}{(q)_\infty} \sum_{r=-\infty}^{\infty} (-q)^r q^{(2k+1)\binom{r}{2}}, \quad (3.28b)$$

where we have defined $M_i = r_{k-i}$ and $m_i = r_{k-i} - r_{k-i-1}$ so that $M_i = m_i + \cdots + m_{k-1}$. The right side of each of the equations (3.28) may now be summed using the Jacobi triple product identity (3.24), to arrive at the $p = k$ and $p = 1$ instances of (3.26). □

The full set of the Andrews–Gordon identities in the range $1 \leq p \leq k$ may be recovered from the *Bailey lattice* [AgAnBr87], a multi-dimensional form of Bailey's Lemma. Later, it was revealed that this additional theory is not strictly necessary for the full complement of identites if one works hard enough; see [AnScWar99].

For later purposes we will introduce two more general families of Rogers–Ramanujan type identities. Each of the following results are similarly accessible using the classical Bailey machinery and each have well-known combinatorial intepretations as partition congruences. The first of these families of identities form an even modulus counterpart to the Andrews–Gordon identities, due to Bressoud [Br80].

Theorem 3.4. *For $1 \leq i \leq k$, let $M_i = m_i + m_{i+1} + \cdots + m_{k-1}$,*

$$\sum_{m_1,\dots,m_{k-1}\geq 0}^{\infty} \frac{q^{M_1^2+\cdots+M_{k-1}^2+M_i+\cdots+M_{k-1}}}{(q)_{m_1}\cdots(q)_{m_{k-2}}(q^2;q^2)_{m_{k-1}}} = \frac{(q^i,q^{2k-i},q^{2k};q^{2k})_\infty}{(q)_\infty}. \tag{3.29}$$

The last family of Rogers–Ramanujan-type identities we wish to introduce are the *generalised Göllnitz–Gordon identities.* The classical Göllnitz–Gordon identities [Gö60, Go65] are stated in analytic form as follows:

$$\sum_{r=0}^{\infty} \frac{q^{r^2}(-q;q^2)_r}{(q^2;q^2)_r} = \frac{(q^3,q^5,q^8;q^8)_\infty(q^2;q^4)_\infty}{(q)_\infty}, \tag{3.30a}$$

$$\sum_{r=0}^{\infty} \frac{q^{r(r+2)}(-q;q^2)_r}{(q^2;q^2)_r} = \frac{(q,q^7,q^8;q^8)_\infty(q^2;q^4)_\infty}{(q)_\infty}. \tag{3.30b}$$

The following generalised form is due to Andrews [An67, Equation 7.4.4] for $i = k$, and Bressoud [Br80b, Equation (3.8)] for $1 \leq i < k$.

Theorem 3.5 (The generalised Göllnitz–Gordon identities)**.** *For $1 \leq i \leq k$, let $M_i = m_i + m_{i+1} + \cdots + m_{k-1}$. Then*

$$\sum_{m_1,\dots,m_{k-1}\geq 0}^{\infty} \frac{q^{2(M_1^2+\cdots+M_{k-1}^2+M_i+\cdots+M_{k-1})}(-q^{1-2M_1};q^2)_{M_1}}{(q^2;q^2)_{m_1}\cdots(q^2;q^2)_{m_{k-1}}}$$
$$= \frac{(q^{2i-1},q^{4k-2i+1},q^{4k};q^{4k})_\infty(q^2;q^4)_\infty}{(q)_\infty}. \tag{3.31}$$

Note that by relabelling $M_1 = r$ and choosing $i = 2$ (3.31) yields (3.30a) for $k = 2$, up to the transformation $(-q^{1-2r})_r = (-q;q^2)_r q^{-r^2}$. Similarly, (3.30b) appears when $k = 2$, $i = 1$ and $M_1 = r$.

Part II

Combinatorial character formulas

New results including the C_n Andrews transformation, an explicit q-hypergeometric formulation for the modified Hall–Littlewood polynomials, combinatorial formulas for the characters of affine Lie algebras and generalisations of the Macdonald eta-functions.

Chapter 4

The C_n Andrews transformation

In this chapter we derive a C_n analogue of Andrews transformation (3.18). This C_n Andrews transformation is the foundation of our combinatorial character formulas. A key tool in our derivation is the C_n Bailey lemma developed over several papers by Milne and Lilly [MiLil92, MiLil95, LilMi93]. Unfortunately, the relevant statement of Milne and Lilly's main result contains a typographical error, which until now seems to have evaded notice. This has been corrected below.

C_n basic hypergeometric series

The notion of basic hypergeometric series as discussed in Chapter 3 can be generalised to the setting of root systems. Instead of giving the most general definition of such series, see e.g., [Gu87, Schl09, Mi87], and references therein, we restrict our attention to the C_n root system. Roughly, a C_n basic hypergeometric series is a multiple series containing the factor

$$\frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)} = \prod_{i=1}^{n} \frac{1-x_i^2 q^{2r_i}}{1-x_i^2} \prod_{1\le i<j\le n} \frac{x_i q^{r_i} - x_j q^{r_j}}{x_i - x_j} \cdot \frac{x_i x_j q^{r_i+r_j}-1}{x_i x_j - 1}, \tag{4.1}$$

where $r=(r_1,\dots,r_n)\in\mathbb{Z}^n$ is an n-dimensional summation index. Note that for $n=1$ and after replacing x_1^2 by a and r_1 by r, we recover the classical very-well-poised term

$$\frac{1-aq^{2r}}{1-a},$$

see (3.6).

Many of the classical identities for basic hypergeometric series admit generalisations to the C_n root system. For example, the C_n analogue of Jackson's ${}_6W_5$

summation (3.7), due to Milne and Lilly is given by

$$\sum_{0\subseteq r\subseteq N} \frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)} \prod_{i,j=1}^{n} \frac{(q^{-N_j}x_i/x_j, x_ix_j)_{r_i}}{(qx_i/x_j, q^{1+N_j}x_ix_j)_{r_i}} \prod_{i=1}^{n} \frac{(bx_i, cx_i)_{r_i}}{(qx_i/b, qx_i/c)_{r_i}} \left(\frac{q^{|N|+1}}{bc}\right)^{r_i}$$
$$= (q/bc)_{|N|} \prod_{i=1}^{n} \frac{(qx_i^2)_{N_i}}{(qx_i/b, qx_i/c)_{N_i}} \prod_{1\le i<j\le n} \frac{(qx_ix_j)_{N_i}}{(q^{1+N_j}x_ix_j)_{N_i}}, \quad (4.2)$$

where $N = (N_1, \ldots, N_n) \in \mathbb{N}^n$ is a sequence of nonnegative integers, $|N| := N_1 + \cdots + N_n$ and where $0 \subseteq r \subseteq N$ is shorthand for $r \in \mathbb{N}^n$ such that $r_i \le N_i$ for $1 \le i \le n$. More generally we will simply denote the empty sequence as 0, where the length is defined by context.

Note that the condition $0 \subseteq r \subseteq N$ is indeed the natural range of support for the above series, since

$$\frac{(q^{-N_j}x_i/x_j)_{r_i}}{(qx_i/x_j)_{r_i}}$$

simplifies to $(q^{-N_i})_{r_i}/(q)_{r_i}$ for $j = i$. When $r_i < 0$ or $r_i > N_i$ this terms clearly vanishes. In some of our later series we will use this observation and simply write $\sum_{r\in\mathbb{Z}^n}$.

For later reference we observe that by specialising $bc = q$, equation (4.2) yields a C_n analogue of the ${}_4W_3$ series identity (3.9)

$$\sum_{0\subseteq r\subseteq N} \frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)} \prod_{i,j=1}^{n} \frac{(q^{-N_j}x_i/x_j, x_ix_j)_{r_i}}{(qx_i/x_j, q^{N_j+1}x_ix_j)_{r_i}} q^{N_jr_i} = \delta_{N,0}, \quad (4.3)$$

where $\delta_{r,s}$ is the Kronecker delta, i.e., $\delta_{r,s}$ is 1 if r equals s and zero otherwise.

Not all series labelled by C_n necessarily contain all of the factors in (4.1). For example, subsequently we will need the C_n analogue of the q-Pfaff–Saalschütz summation (3.3) for a balanced ${}_3\phi_2$ series, which is given by

$$\sum_{0\subseteq r\subseteq N} \frac{q^{|r|}}{(bcq^{-|N|})_{|r|}} \prod_{1\le i<j\le n} \frac{x_iq^{r_i} - x_jq^{r_j}}{x_i - x_j} (qx_ix_j)_{r_i+r_j} \prod_{i,j=1}^{n} \frac{(q^{-N_j}x_i/x_j)_{r_i}}{(qx_i/x_j, qx_ix_j)_{r_i}}$$
$$\times \prod_{i=1}^{n} (bx_i, cx_i)_{r_i}$$
$$= \frac{1}{(q/bc)_{|N|}} \prod_{1\le i<j\le n} (qx_ix_j)_{N_i+N_j} \prod_{i,j=1}^{n} \frac{1}{(qx_ix_j)_{N_i}} \prod_{i=1}^{n} (qx_i/b, qx_i/c)_{N_i}. \quad (4.4)$$

For $n = 1$ the q-Pfaff–Saalschütz summation (3.3) is recovered under the simultaneous assignment $(b, c, x_1^2) \mapsto (q/a, q/b, c/q)$. The above form of (4.4) is due to Bhatnagar [Bh99, Theorem 1], although the result was first discovered by Milne and Lilly [MiLil95, Theorem 4.2] in a dual form corresponding to a reversed order of summation. Note that the C_n very-well-poised term (4.1) is not present in its entirety. This is not unexpected since the classical q-Pfaff-Saalschütz summation is not very-well-poised. In fact, attaching a root system to series of this nature is somewhat problematic and Bhatnagar and Schlosser refer to (4.4) as a D_n series [Bh99, BhSchl98, Schl09].

The C_n Bailey lemma

In this section we follow Milne and Lilly to derive a C_n analogue of the Bailey lemma.

If $\alpha = \{\alpha_N\}_{N\in\mathbb{N}^n}$ and $\beta = \{\beta_N\}_{N\in\mathbb{N}^n}$ are sequences such that

$$\beta_N = \sum_{0\subseteq r\subseteq N} \alpha_r \prod_{i,j=1}^{n} \frac{1}{(qx_i/x_j)_{N_i-r_j}(qx_ix_j)_{N_i+r_j}}, \tag{4.5}$$

then the pair (α,β) is a C_n *Bailey pair* relative to x. This definition is a slight rescaling of the original as given in [MiLil92, Equation 2.5]. For ease of comparison, we note that Milne and Lilly define their C_n Bailey pair as

$$\begin{aligned}\beta_N &= \sum_{0\subseteq r\subseteq N} \alpha_r \prod_{i,j=1}^{n} \frac{(qx_i/x_j)_{r_i-r_j}(qx_ix_j)_{r_i+r_j}}{(qx_i/x_j)_{N_i-r_j}(qx_ix_j)_{N_i+r_j}}\\ &= \sum_{0\subseteq r\subseteq N} \alpha_r \prod_{i,j=1}^{n} \frac{1}{(q^{r_i-r_j+1}x_i/x_j)_{N_i-r_i}(q^{r_i+r_j+1}x_ix_j)_{N_i-r_i}}.\end{aligned}$$

We now prove our corrected version of the C_n Bailey lemma [MiLil92, Equation 2.5].

Lemma 4.1 (C_n Bailey lemma)**.** *If (α,β) is a C_n Bailey pair relative to x, then (α',β') is a C_n Bailey pair relative to x where*

$$\alpha'_N = \alpha_N \prod_{i=1}^{n} \frac{(bx_i, cx_i)_{N_i}}{(qx_i/b, qx_i/c)_{N_i}} \left(\frac{q}{bc}\right)^{N_i}, \tag{4.6a}$$

$$\begin{aligned}\beta'_N = \sum_{0\subseteq r\subseteq N} \beta_r\, (q/bc)_{|N|-|r|} \left(\frac{q}{bc}\right)^{|r|} \prod_{i=1}^{n} \frac{(bx_i, cx_i)_{r_i}}{(qx_i/b, qx_i/c)_{N_i}}\\ \times \prod_{1\le i<j\le n} \frac{(qx_ix_j)_{r_i+r_j}}{(qx_ix_j)_{N_i+N_j}} \prod_{i,j=1}^{n} \frac{(qx_i/x_j)_{r_i-r_j}}{(qx_i/x_j)_{N_i-r_j}}.\end{aligned} \tag{4.6b}$$

Proof. Let $A_{N,r}$ be the summand of (4.5) without α_r, let $B_{N,r}$ be the summand of (4.6b) without β_r, and let (α,β) be a C_n Bailey pair. We will verify that (α',β') is a

C_n Bailey pair. This proof proceeds in an analogous fashion to that of the classical Bailey lemma.

Once more bearing in mind that $r \mapsto r+s$ means $(r_1, \ldots, r_n) \mapsto (r_1+s_1, \ldots, r_n+s_n)$, we perform steps essentially the same as those of the classical proof, i.e., we interchange sums and normalise the interior sum to 1 when r is the zero vector. Using our shorthand notation $B_{N,r}$ and $A_{N,r}$, this leads to

$$\begin{aligned}\beta'_N &= \sum_{0\subseteq s\subseteq N} \alpha_s A_{s,s} B_{N,s} \sum_{0\subseteq r\subseteq N-s} \frac{A_{r+s,s}B_{N,r+s}}{A_{s,s}B_{N,s}} \\ &= \sum_{0\subseteq s\subseteq N} \alpha_s A_{s,s} B_{N,s} \sum_{0\subseteq r\subseteq M} \frac{q^{\binom{|r|}{2}}}{(bcq^{-|M|})_{|r|}} \prod_{1\le i<j\le n} (qy_iy_j)_{r_i+r_j} \\ &\qquad \times \prod_{i,j=1}^{n} \frac{(qy_i/y_j)_{r_i-r_j}(q^{-M_j}y_i/y_j)_{r_i}}{(qy_i/y_j, qy_iy_j)_{r_i}} \Big(\frac{y_i}{y_j}\Big)^{-r_i} q^{-\binom{r_i}{2}} \prod_{i=1}^{n} (by_i, cy_i)_{r_i},\end{aligned}$$

where $y_i = y_i(s) = x_iq^{s_i}$ and $M = N - s$. We can carry out the sum over r using the C_n q-Pfaff–Saalschütz sum (4.4) and the elementary identity

$$\prod_{i,j=1}^{n} (qy_i/y_j)_{s_i-s_j} = \prod_{1\le i<j\le n} \frac{y_iq^{s_i} - y_jq^{s_j}}{y_i - y_j} \Big(-\frac{y_i}{y_j}\Big)^{s_i-s_j} q^{\binom{s_i-s_j}{2}-s_j}. \tag{4.7}$$

The result of completing the sum over r is then

$$\begin{aligned}\beta'_N = \sum_{0\subseteq s\subseteq N} \frac{B_{N,s}A_{s,s}}{(q/bc)_{|N-s|}} \alpha_s \prod_{1\le i<j\le n} (qy_iy_j)_{N_i+N_j-(s_i+s_j)} \\ \times \prod_{i,j=1}^{n} \frac{1}{(qy_iy_j)_{N_i-s_i}} \prod_{i=1}^{n} (qy_i/b, qy_i/c)_{N_i-s_i}.\end{aligned}$$

After eliminating the y_i in favour of the $x_iq^{s_i}$ this simplifies to

$$\begin{aligned}\beta'_N &= \sum_{0\subseteq s\subseteq N} A_{N,s}\alpha_s \prod_{i=1}^{n} \frac{(bx_i, cx_i)_{s_i}}{(qx_i/b, qx_i/c)_{s_i}} \Big(\frac{q}{bc}\Big)^{s_i} \\ &= \sum_{0\subseteq s\subseteq N} A_{N,s}\alpha'_s,\end{aligned}$$

as required. □

The C_n Andrews Transformation

In this section we employ the machinery of the C_n Bailey lemma to generate the C_n Andrews transformation. As in the proof of the C_n Bailey lemma, the key steps in the following procedure are precisely analogous to those of the classical setting.

We use a specialisation of the C_n Jackson sum to invert the C_n Bailey pair relation, which in turn yields a C_n analogue of the unit Bailey pair (3.16). From this C_n unit Bailey pair, the C_n Andrews transformation then follows by m-fold application of the C_n Bailey lemma.

We now carry out the inversion of the C_n Bailey pair relation.

Lemma 4.2 (C_n Bailey pair inversion [MiLil92])**.** *If (α, β) is a C_n Bailey pair, then*

$$\alpha_N = \frac{\Delta_{\mathrm{C}}(xq^N)}{\Delta_{\mathrm{C}}(x)} \sum_{0 \subseteq r \subseteq N} \beta_r \prod_{1 \le i < j \le n} \frac{x_i q^{r_i} - x_j q^{r_j}}{x_i - x_j} \frac{1 - x_i x_j q^{r_i + r_j}}{1 - x_i x_j} \\ \times q^{-(n-1)|r|} \prod_{i,j=1}^{n} \Big(-\frac{x_i}{x_j}\Big)^{N_i - r_j} q^{\binom{N_i - r_j}{2}} \frac{(x_i x_j)_{N_i + r_j}}{(qx_i/x_j)_{N_i - r_j}}. \tag{4.8}$$

Proof. We verify (4.8) by direct substitution into (4.5). For convenience of notation, let $A_{N,r}(x)$ be the summand of (4.5) without α_r and let $A'_{N,r}(x)$ be the summand of (4.8) without β_r, including the prefactor. As seen in the classical case, observe that both $A_{N,r}(x)$ and $A'_{N,r}(x)$ may be interpreted as the entries of two invertible, infinite-dimensional lower-triangular matrices, A and A'. The following calculations essentially show that $A' = A^{-1}$. This proof proceeds in a fashion analogous to the proof of the classical Bailey inversion. With the proviso that $r \mapsto r + s$ now means $(r_1, \dots, r_n) \mapsto (r_1 + s_1, \dots, r_n + s_n)$, the steps leading from (3.19) to (3.20) remain identically true. Now using the definitions of $A_{N,r}(x)$ and $A'_{N,r}(x)$, we arrive at a multiple analogue of (3.21)

$$\beta_N = \sum_{0 \subseteq s \subseteq N} \beta_s \prod_{i,j=1}^{n} \frac{1}{(qy_i/y_j)_{N_i - s_i} (qy_i y_j)_{N_i - s_i}} \\ \times \sum_{0 \subseteq r \subseteq N - s} \frac{\Delta_{\mathrm{C}}(yq^r)}{\Delta_{\mathrm{C}}(y)} q^{|N-s||r|} \prod_{i,j=1}^{n} \frac{(q^{-(N_j - s_j)} y_i/y_j, y_i y_j)_{r_i}}{(qy_i/y_j, q^{N_j - s_j + 1} y_i y_j)_{r_i}}, \tag{4.9}$$

where we have introduced the shorthand notation $y_i = y_i(s) = x_i q^{s_i}$ as before, and where we have used $A_{s,s}(x) A'_{s,s}(x) = 1$, which follows from (4.7).

Finally, applying (4.3) to (4.9) leads to

$$\beta_N = \sum_{0 \subseteq s \subseteq N} \delta_{N,s} \beta_s = \beta_N. \qquad \square$$

By choosing $\beta_N = \delta_{N,0}$, the C_n Bailey pair inversion (Lemma 4.2) yields the C_n unit Bailey pair

$$\alpha_N = \frac{\Delta_{\mathrm{C}}(xq^N)}{\Delta_{\mathrm{C}}(x)} \prod_{i,j=1}^{n} \frac{(x_i x_j)_{N_i}}{(qx_i/x_j)_{N_i}} \Big(-\frac{x_i}{x_j}\Big)^{N_i} q^{\binom{N_i}{2}}$$

$$\beta_N = \delta_{N,0}.$$

Naturally, this corresponds to the classical unit pair (3.16) when $n = 1$. Using Lemma 4.1, we may now generate the C_n Bailey chain corresponding to the C_n Andrews transformation. For m a nonnegative integer we obtain,

$$\alpha_N^{(m)} = \frac{\Delta_{\mathrm{C}}(xq^N)}{\Delta_{\mathrm{C}}(x)} \prod_{i=1}^{n} \Bigg[\prod_{\ell=1}^{m+1} \frac{(b_\ell x_i, c_\ell x_i)_{N_i}}{(qx_i/b_\ell, qx_i/c_\ell)_{N_i}} \Big(\frac{q}{b_\ell c_\ell}\Big)^{N_i} \times \prod_{j=1}^{n} \frac{(x_i x_j)_{N_i}}{(qx_i/x_j)_{N_i}} \Big(-\frac{x_i}{x_j}\Big)^{N_i} q^{\binom{N_i}{2}} \Bigg], \tag{4.10a}$$

$$\begin{aligned} \beta_N^{(m)} = \prod_{1\le i<j\le n} \frac{1}{(qx_i x_j)_{N_i+N_j}} &\times \sum_{r^{(1)},\dots,r^{(m)}\in\mathbb{Z}_+^n} \prod_{i,j=1}^{n} \frac{1}{(qx_i/x_j)_{N_i-r_j^{(1)}}} \prod_{\ell=1}^{m} f^{(0)}_{r^{(\ell)},r^{(\ell+1)}}(x;q) \\ &\times \prod_{\ell=1}^{m+1} \Bigg[(q/b_\ell c_\ell)_{|r^{(\ell-1)}|-|r^{(\ell)}|} \Big(\frac{q}{b_\ell c_\ell}\Big)^{|r^{(\ell)}|} \times \prod_{i=1}^{n} \frac{(b_\ell x_i, c_\ell x_i)_{r_i^{(\ell)}}}{(qx_i/b_\ell, qx_i/c_\ell)_{r_i^{(\ell-1)}}} \Bigg], \end{aligned} \tag{4.10b}$$

where in (4.10b), $r^{(0)} := N$ and $r^{(m+1)} := 0$ and we have introduced the notation

$$f^{(\tau)}_{r,s}(x;q) := \prod_{i=1}^{n} \big(x_i^{r_i} q^{\binom{r_i}{2}}\big)^{\tau} \prod_{i,j=1}^{n} \frac{(qx_i/x_j)_{r_i-r_j}}{(qx_i/x_j)_{r_i-s_j}}. \tag{4.11}$$

Note that since $1/(q)_n$ vanishes when $n < 0$, the function $f^{(\tau)}_{r,s}(x;q)$ is zero unless $s \subseteq r$. Equation (4.11) will appear frequently throughout the rest of this thesis.

By rescaling (4.10) according to $(b_\ell, c_\ell) \mapsto (b_\ell/a^{1/2}, c_\ell/a^{1/2})$, the classical Bailey chain described by (3.17a) and (3.17b) appears when $n = 1$ and $x_1^2 = a$. By substitution of (4.10) into the definition of a C_n Bailey pair, the C_n Andrews transformation then follows.

Theorem 4.3 (C_n Andrews transformation)**.** *For m a nonnegative integer and $N \in \mathbb{N}^n$,*

$$\sum_{0\subseteq r\subseteq N} \frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)} \prod_{i=1}^{n} \left[\prod_{\ell=1}^{m+1} \frac{(b_\ell x_i, c_\ell x_i)_{r_i}}{(qx_i/b_\ell, qx_i/c_\ell)_{r_i}} \Big(\frac{q}{b_\ell c_\ell}\Big)^{r_i} \prod_{j=1}^{n} \frac{(q^{-N_j} x_i/x_j, x_i x_j)_{r_i}}{(qx_i/x_j, q^{N_j+1} x_i x_j)_{r_i}} \, q^{N_j r_i} \right]$$
$$= \prod_{i,j=1}^{n} (qx_i x_j)_{N_i} \prod_{1\le i<j\le n} \frac{1}{(qx_i x_j)_{N_i+N_j}}$$
$$\times \sum_{r^{(1)},\dots,r^{(m)}\in\mathbb{N}^n} \prod_{i,j=1}^{n} \frac{(qx_i/x_j)_{N_i}}{(qx_i/x_j)_{N_i-r_j^{(1)}}} \prod_{\ell=1}^{m} f^{(0)}_{r^{(\ell)},r^{(\ell+1)}}(x;q)$$
$$\times \prod_{\ell=1}^{m+1} \left[(q/b_\ell c_\ell)_{|r^{(\ell-1)}|-|r^{(\ell)}|} \Big(\frac{q}{b_\ell c_\ell}\Big)^{|r^{(\ell)}|} \times \prod_{i=1}^{n} \frac{(b_\ell x_i, c_\ell x_i)_{r_i^{(\ell)}}}{(qx_i/b_\ell, qx_i/c_\ell)_{r_i^{(\ell-1)}}} \right], \tag{4.12}$$

where $r^{(0)} := N$ and $r^{(m+1)} := 0$.

A few remarks are in order. First we note that for $m = 0$ we recover the C_n Jackson sum (4.2). For $m = 1$ we obtain Milne and Lilly's C_n analogue of Watson's transformation (3.25) between a very-well poised ${}_8W_7$ and a balanced ${}_4\phi_3$ [Mi94, Theorem A3], [MiLil95, Theorem 6.6]. We point out that (4.12) is symmetric under simultaneous permutation of x and N. Hence, as all $N_1, \dots, N_n$ tend to infinity, both sides are symmetric in x.

Chapter 5

The modified Hall–Littlewood polynomials

In this chapter we obtain a new basic hypergeometric representation of the modified Hall–Littlewood polynomials. This representation stems from Jing's construction of these polynomials in terms of his q-Bernstein operators B_p [Ji91] and Garsia's subsequent explicit formulation of these latter objects as q-difference operators [Gar92]. Using this new formulation of the modified Hall–Littlewood polynomials we then prove a particular case of a conjectural identity for a Littlewood-type sum involving Q'_λ. One side of this conjectural identity is a q-hypergeometric sum. Later we will see that this side may be identified with the right-hand side of a specialisation of the C_n Andrews transformation.

An explicit formulation

We begin with a brief revision of Jing and Garsia's relevant theorems. For more details on this preliminary material, see also [Za00].

We will use the following notation. Let $f \in \Lambda_n$. Define the operator $f^\perp$ as the adjoint of f under multiplication with respect to the Hall inner product. That is, for $u, v \in \Lambda_n$,

$$\langle f^\perp(u), v\rangle = \langle u, f(v)\rangle.$$

This operator is sometimes called the *Foulkes derivative*; called derivative because its action on the order of a symmetric function shares some properties with that of an ordinary derivative. For example, if $f \in \Lambda_r$ and $g \in \Lambda_s$, then the homogenous symmetric function $f^\perp(g)$ has order $s-r$ (0 if $r > s$). The Foulkes derivatives of the elementary and complete symmetric functions appear prominently in the definition of the *q-Bernstein operators* $B_p := B_p(x;q)$,

$$B_p = \sum_{r,s=0}^{\infty} (-1)^r q^s h_{p+r+s}(x) e_r^\perp h_s^\perp = \sum_{r=0}^{\infty} h_{p+r}(x) h^\perp\big(x(q-1)\big),$$

where p is an integer and the second expression uses λ-ring notation. Alternatively, if $B(z) =: B(z;x;q)$ is the vertex operator $B(z) = \sum_m z^m B_m$, then

$$B(z)\big(f(x)\big) = f\Big(x - \frac{1-q}{x}\Big) \prod_{i\geq 1} \frac{1}{1-zx_i},$$

where we have again used λ-ring notation. In [Ji91] Jing showed that the modified Hall–Littlewood polynomials $Q'_\lambda = Q'_{(\lambda_1,\dots,\lambda_d)}$ may be expressed as

$$Q'_\lambda = B_{\lambda_1} \cdots B_{\lambda_d}(1).$$

In other words, the Q'_λ are uniquely determined by the recursion

$$Q'_\lambda = B_{\lambda_1}(Q'_\mu),$$

subject to the condition $Q'_0 = 1$, where $\mu = (\lambda_2, \dots, \lambda_d)$. This recursion was given an explicit form by Garsia [Gar92, Thm. 2.1] in terms of q-difference operators as

$$Q'_\lambda = \sum_{m=1}^{n} x_m^{\lambda_1} \Big(\prod_{i\neq m} \frac{x_m}{x_m - x_i} \Big) T_{q;x_m} Q'_\mu, \tag{5.1}$$

where the operator $T_{q;x_m}$ is defined by

$$T_{q;x_m} f(x_1, \dots, x_m, \dots, x_n) = f(x_1, \dots, qx_m, \dots, x_n).$$

Equation (5.1) may be used to quickly rederive the recursive forms for h_λ (1.9) and s_λ (1.15). When $q = 1$, $Q'_\lambda(x;1) = h_\lambda(x)$ and $T_{1;x_m}$ is the identity operator. Hence Q'_μ may be brought before the sum and (5.1) becomes (1.9). Recalling the notation

$$x^{(m)} = (x_1, \dots, x_{m-1}, x_{m+1}, \dots, x_n),$$

the second recursion arises for $q = 0$, where $Q'_\lambda(x;0) = s_\lambda(x)$. The stability property of s_λ ensures $T_{0;x_m} s_\lambda(x) = s_\lambda(x^{(m)})$ so that (5.1) becomes (1.15).

We will now briefly introduce a fourth family of Hall–Littlewood polynomials, the modified Hall–Littlewood polynomials $P'_\lambda(x;q) := P'_\lambda$, which appear as the dual basis of Q_λ in the Hall inner product,

$$\langle P'_\lambda, Q_\mu \rangle = \delta_{\lambda\mu}. \tag{5.2}$$

In many papers concerning the modified Hall–Littlewood polynomials [DeLeTh94, Gar92, GarPro92, Ki00, La05, Mi92], the P'_λ receive none of the attention given to P_λ, Q_λ and Q'_λ. The lack of interesting results involving P'_λ is likely due to the fact that these polynomials have coefficients in $\mathbb{Q}[q]$, are not Schur positive and do not interpolate between classical symmetric functions under specialisation of q. It should be understood that we will employ P'_λ simply because of the convenience provided by the scaling relation

$$Q'_\lambda = b_\lambda P'_\lambda,$$

where $b_\lambda := b_\lambda(q)$ is defined in (1.40). As such, we will freely switch between Q'_λ and P'_λ without further comment.

Recall the rational function $f^{(\tau)}_{r,s}(x;q)$ (4.11). For simplicity in the following statements, we define the convention

$$f_{r,s}(x;q) := f^{(1)}_{r,s}(x;q).$$

We now present two original representations of Q'_λ that we develop specifically for use in later sections. These formulas are probably too complicated to be of broader interest, but prove essential for the results that are the main focus of this thesis. This next theorem constitutes a closed-form solution to Garsia's recursion (5.1) for Q'_λ.

Theorem 5.1. *For all partitions λ with $x = (x_1, \ldots, x_n)$,*

$$P'_\lambda(x;q) = \sum \prod_{\ell \geq 1} f_{r^{(\ell)},r^{(\ell+1)}}(x;q) \tag{5.3}$$

where the sum is over all $r^{(1)} \supseteq r^{(2)} \supseteq r^{(3)} \cdots$ with each $r^{(\ell)} \in \mathbb{Z}^n_+$ such that $|r^{(\ell)}| := r^{(\ell)}_1 + \cdots + r^{(\ell)}_n = \lambda'_\ell$.

We remark that the product in (5.3) may be written with a finite upper bound. Since $|r^{(\ell)}| = 0$ when ℓ exceeds the length of λ', (i.e., when $\ell > \lambda_1$) and $f_{r^{(\ell)},r^{(\ell+1)}}(x;q) = 1$ for $r^{(\ell)} = 0$, we need only consider the product up to $\ell = \lambda_1$.

This discovery of this theorem is a critical step towards the combinatorial character identities of Chapter 6.

It is a highly non-trivial fact that the sum of rational functions on the right-hand side of (5.3) is a positive polynomial, and so we give an example in addition to a proof. Switching to Q'_λ, for $\lambda = (1^k)$ we have

$$Q'_{(1^k)}(x;q) = \sum_{|r|=k} f_{r,0}(x;q).$$

We now choose $\lambda = (1,1)$ and sum over $r = (r^{(1)})$ with $r^{(1)}$ running over $(2,0)$, $(0,2)$ and $(1,1)$, so that

$$\begin{aligned} Q'_{(1,1)}(x_1,x_2;q) &= \frac{q^2x_1^4}{(x_1-x_2)(qx_1-x_2)} + \frac{q^2x_2^4}{(x_2-x_1)(qx_2-x_1)} \\ &\quad + \frac{(1-q^2)x_1^2x_2^2}{(1-q)(qx_1-x_2)(qx_2-x_1)} \\ &= x_1x_2 + q(x_1^2+x_2^2+x_1x_2). \end{aligned}$$

Before we proceed with a proof of Theorem 5.1, we will give a useful lemma from [Mi88, Eq. (7.13)].

Lemma 5.2. *For all positive integers n,*

$$\sum_{m=1}^{n}(1-y_m)\Bigg(\prod_{\substack{i=1\\ i\neq m}}^{n}\frac{x_m-x_iy_i}{x_m-x_i}\Bigg)=1-y_1\cdots y_n, \tag{5.4}$$

In Milne's paper Equation (5.4) comes as part of a more general statement whose proof, while not complicated, is more than we need here. We will instead employ an elementary technique communicated by [Ros04, pp. 421].

Proof of Lemma 5.2. Given fixed formal parameters a and b, consider the following product and corresponding partial fraction expansion:

$$\prod_{i=1}^{n}\frac{b_i-z}{a_i-z}=1+\sum_{k=1}^{n}\frac{c_k}{a_k-z}, \tag{5.5}$$

where $c_k := c_k(a,b)$ is to be determined. To find this coefficient, we multiply both sides by a_m-z and set $z=a_m$. This yields

$$c_m=(b_m-a_m)\prod_{\substack{i=1\\ i\neq m}}^{n}\frac{b_i-a_m}{a_i-a_m}.$$

Now, substituting this into (5.5), we obtain

$$1-\prod_{i=1}^{n}\frac{b_i-z}{a_i-z}=\sum_{k=1}^{n}\frac{a_k-b_k}{a_k-z}\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{a_k-b_j}{a_k-a_j},$$

where, by setting $a_i=x_i$, $b_i=x_iy_i$ and $z=0$, we arrive at the result. □

Bearing Lemma 5.2 in mind, we proceed to the proof of Theorem 5.1. For any sequence $r=(r_1,\ldots,r_n)$, when we write $r\pm\epsilon_m$, we mean $(r_1,\ldots,r_m\pm 1,\ldots,r_n)$.

Proof of Theorem 5.1. Let $\mu=(\lambda_2,\ldots,\lambda_n)$ and let $\lambda_1=d$, so that

$$\lambda=(1^{m_1}\cdots d^{m_d}),$$

with $m_d=\lambda'_d>0$. Garsia's q-Bernstein recursion for Q'_λ (5.1) may be trivially restated for P'_λ:

$$P'_\lambda=\frac{1}{(1-q^{m_d})}\sum_{m=1}^{n}x_m^{\lambda_1}\Big(\prod_{i\neq m}\frac{x_m}{x_m-x_i}\Big)T_{q;x_m}P'_\mu, \tag{5.6}$$

where we have used $b_\mu(q)/b_\lambda(q) = 1/(1-q^{m_d})$. Equation (5.3) clearly satisfies the initial condition $P'_0(x) = 1$, so all we need to do is show that it satisfies (5.6). Now, using the remark after Theorem 5.1, we may write (5.3) as

$$P'_\lambda = \sum_{\substack{|r^{(k)}|=\lambda'_k \\ 1\le k\le d}} \prod_{\ell=1}^{d} f_{r^{(\ell)},r^{(\ell+1)}}, \tag{5.7}$$

where $r_i^{(d+1)} := 0$ for $1 \le i \le n$. Using $|\mu| = |\lambda| - d$ and

$$\sum_{i\ge1} \binom{\mu'_i}{2} = d - |\lambda| + \sum_{i\ge1} \binom{\lambda'_i}{2},$$

together with some simple but lengthy manipulations, it follows that

$$T_{q;x_m} P'_\mu(x;q) = x_m^{-d} \sum_{\substack{|r^{(k)}|=\lambda'_k-1 \\ 1\le k\le d}} \prod_{\ell=1}^{d} f_{r^{(\ell)}+\epsilon_m,r^{(\ell+1)}+\epsilon_m}(x;q).$$

After shifting $r_m^{(\ell)} \mapsto r_m^{(\ell)} - 1$ for $\ell = 1,\dots,d$ (while recalling that $r^{(d+1)} := 0$), this implies

$$T_{q;x_m} P'_\mu(x;q) = x_m^{-d} \sum_{\substack{|r^{(k)}|=\lambda'_k \\ 1\le k\le d}} \prod_{i=1}^{n} (1 - q^{r_i^{(d)}-1} x_i/x_j) \prod_{\ell=1}^{d} f_{r^{(\ell)},r^{(\ell+1)}}.$$

Therefore

$$\begin{aligned}\sum_{m=1}^{n} x_m^d \biggl(\prod_{i\ne m} \frac{x_m}{x_m - x_i}\biggr) T_{q;x_m} P'_\mu(x;q) \\ = \sum_{\substack{|r^{(k)}|=\lambda'_k \\ 1\le k\le d}} \sum_{m=1}^{n} \bigl(1 - q^{r_m^{(d)}}\bigr) \Biggl(\prod_{\substack{i=1 \\ i\ne m}}^{n} \frac{x_m - x_i q^{r_i^{(d)}}}{x_m - x_i}\Biggr) \prod_{\ell=1}^{d} f_{r^{(\ell)},r^{(\ell+1)}}.\end{aligned} \tag{5.8}$$

By Lemma 5.2 the sum over m in the summand simplifies to

$$1 - q^{r_1^{(d)}+\cdots+r_n^{(d)}} = 1 - q^{|r^{(d)}|} = 1 - q^{\lambda'_d} = 1 - q^{m_d}.$$

The right-hand side of (5.8) thus simplifies to $P'_\lambda(x;q)$, completing the proof. □

We also provide a more computationally efficient alternative representation of P'_λ.

Corollary 5.3. *Let* $\lambda' = (M_m^{\tau_m} \cdots M_2^{\tau_2} M_1^{\tau_1})$, *for* $M_1 \geq \cdots \geq M_m$ *and* $\tau_1, \ldots, \tau_m > 0$. *Then*

$$P'_\lambda(x;q) = \sum_{\substack{|r^{(l)}|=M_l \\ 1\leq l\leq k}} \prod_{l=1}^{m} f^{(\tau_\ell)}_{r^{(\ell)},r^{(\ell+1)}},$$

where $r_i^{(m+1)} := 0$.

Note that for $m = 1$ (5.3) simplifies to Milne's expression for P'_λ indexed by a rectangular partition of the form $\lambda = (m^n)$, which is implied by equating (2.7) and (2.17) in [Mi94].

In spite of the fact we have used the label *corollary*, at first glance the above formula may appear to be more general than Theorem 5.1. Indeed, Corollary 5.3 "reduces" to Theorem 5.1 when $\tau_1 = \tau_2 = \cdots = 1$, upon the identification of a_i with m_i and M_i with λ'_i. Corollary 5.3 arises from the observation that the sum in (5.3) is written with more summation indices than we need in most cases. In the following proof we work to get rid of these redundant indices.

Proof of Corollary 5.3. We use the following notation for ease of representation. Let the partition λ be given in multiplicity notation as

$$\lambda = \left(\tau_1^{a_1}(\tau_1+\tau_2)^{a_2}\cdots(\tau_1+\tau_2+\cdots+\tau_m)^{a_m}\right),$$

where $\tau_1, \ldots, \tau_m > 0$ and $a_1, \ldots, a_m \geq 0$ are integers. Note that this representation maintains full generality over the set of all partitions.

Now, observe that $m_i(\lambda) = 0$ for $\tau_1 + \cdots + \tau_\ell < i < \tau_1 + \cdots + \tau_{\ell+1}$ and $1 \leq \ell \leq m$. Then using $m_i(\lambda) = \lambda'_i - \lambda'_{i+1}$ we have

$$\lambda'_{\tau_1+\cdots+\tau_{\ell-1}+1} = \cdots = \lambda'_{\tau_1+\cdots+\tau_\ell} =: M_\ell \quad \text{for } 1 \leq \ell \leq m.$$

We apply this notation to Theorem 5.1. In the sum of (5.3), we then have

$$|r^{(\tau_1+\cdots+\tau_{\ell-1}+1)}| = \cdots = |r^{(\tau_1+\cdots+\tau_\ell)}| = M_\ell \quad \text{for } 1 \leq \ell \leq m. \tag{5.9}$$

Thanks to the occurrence of the product

$$\prod_{i,j=1}^{n} \frac{1}{(qx_i/x_j)_{r_i^{(\ell)}-r_j^{(\ell+1)}}}$$

in $f_{r^{(\ell)},r^{(\ell+1)}}$ and the fact that $1/(q)_{-n} = 0$ for $n > 0$, the summand of (5.3) vanishes unless $r_i^{(\ell)} \geq r_i^{(\ell+1)}$. Hence, if $|r^{(\ell)}| = |r^{(\ell+1)}|$, then we must have $r^{(\ell)} = r^{(\ell+1)}$. We may thus replace (5.9) by

$$r^{(\tau_1+\cdots+\tau_{\ell-1}+1)} = \cdots = r^{(\tau_1+\cdots+\tau_\ell)} =: s^{(\ell)} \quad \text{for } 1 \le \ell \le m \tag{5.10}$$

where $|s^{(\ell)}| = M_\ell$.

From (4.11) it is clear that

$$f_{r,r}^{(\tau)} = \prod_{i=1}^{n} \Big(x_i^{r_i} q^{\binom{r_i}{2}} \Big)^{\tau},$$

and we find that, subject to (5.10) and the remark immediately after Theorem 5.1,

$$\begin{aligned}\prod_{\ell=1}^{\tau_1+\cdots+\tau_m} f_{r^{(\ell)},r^{(\ell+1)}} &= \prod_{\ell=1}^{m} \big(f_{s^{(\ell)},s^{(\ell)}} \big)^{\tau_\ell - 1} f_{s^{(\ell)},s^{(\ell+1)}}, \\ &= \prod_{\ell=1}^{m} f^{(\tau_\ell)}_{s^{(\ell)},s^{(\ell+1)}}.\end{aligned}$$

It thus follows that

$$\sum_{\substack{|r^{(\ell)}|=\lambda'_\ell \\ \ell\ge 1}} \prod_{\ell\ge 1} f_{r^{(\ell)},r^{(\ell+1)}} = \sum_{\substack{|s^{(\ell)}|=M_\ell \\ 1\le\ell\le m}} \prod_{\ell=1}^{m} f^{(\tau_\ell)}_{s^{(\ell)},s^{(\ell+1)}}. \tag{5.11}$$

□

The Rogers–Szegő polynomials

In this short section we briefly review some of the elementary properties of the Rogers–Szegő polynomials [Rog1892, Sz26], which will appear alongside the Hall–Littlewood polynomials in our soon-to-come conjecture for a Littlewood-type sum.

The Rogers–Szegő polynomials $H_m(z;q)$ are defined by the generating function [An76, pp. 49]

$$\sum_{m=0}^{\infty} \frac{H_m(z;q)x^m}{(q)_m} = \frac{1}{(x,xz)_\infty}, \tag{5.12}$$

for $|x| < 1$. By two applications of the q-binomial theorem (3.2), one finds the explicit form

$$H_m(z;q) = \sum_{i=0}^{m} \begin{bmatrix} m \\ i \end{bmatrix} z^i, \tag{5.13}$$

where we have used the q-binomial coefficient $\begin{bmatrix} m \\ i \end{bmatrix}$, defined by

$$\begin{bmatrix} m \\ i \end{bmatrix}_q =: \begin{bmatrix} m \\ i \end{bmatrix} = \frac{(q)_m}{(q)_{m-i}(q)_i}.$$

It will be useful to follow the convention that $\begin{bmatrix}\infty\\ i\end{bmatrix} = 1/(q)_i$. Subject to the initial conditions $H_0(z;q) = 1$ and $H_1(z;q) = 1 - q$, the Rogers–Szegő polynomials satisfy the recursion

$$H_{m+1}(z;q) = (1+z)H_m(z;q) - (1-q^m)H_{m-1}(z;q). \tag{5.14}$$

In [Rog1892], the Rogers–Szegő polynomials are defined only implicitly by means of (what are now known as) the *continuous q-Hermite polynomials* $H_m(z|q)$, which are instrumental to Rogers' first proof of the Rogers–Ramanujan identities (3.22). These polynomials may be identified with the Rogers–Szegő polynomials by the assignment

$$H_m(z|q) = \mathrm{e}^{\mathrm{i}\, m\theta}\, H_m(\mathrm{e}^{-2\,\mathrm{i}\,\theta}; q),$$

where $z = \cos\theta$.

We remark that the continuous q-Hermite polynomials appear in the so-called *q-Askey scheme* for q-orthogonal polynomials, compiled by Koekoek and Swarttouw in two very detailed resource manuals [KoSw98, KoLesSw10]. In q-hypergeometric notation, the continuous q-Hermite polynomials may be written as

$$H_m(z|q) = \mathrm{e}^{\mathrm{i}\, m\theta}\, {}_2\phi_0\big(q^{-m}, 0\,; -\,; q, q^m\, \mathrm{e}^{-2\,\mathrm{i}\,\theta}\big).$$

Following the classical program, the q-Askey scheme organises many q-orthogonal polynomials into a tiered hierarchy, where each polynomial in the ith tier is a limiting specialisation of a polynomial in the $(i+1)$th tier, or may be specialised to obtain a polynomial in $(i-1)$th tier.

The Rogers–Szegő polynomials do not factorise in general, but there are a number of factorisations into cyclotomic polynomials for special values of z. Up to the symmetry

$$H_m(z;q) = z^m H_m(z^{-1};q), \tag{5.15}$$

these specialisations are listed exhaustively as follows [An76, Ch. 3, Examples 3–9]:

$$H_m(0;q) = 1, \tag{5.16a}$$

$$H_m(-1;q) = \begin{cases} (q;q^2)_{m/2}, & \text{for } m \text{ even,}\\ 0, & \text{for } m \text{ odd,}\end{cases} \tag{5.16b}$$

$$H_m(-q;q) = (q;q^2)_{\lceil m/2\rceil} \tag{5.16c}$$

$$H_m(\pm q^{1/2};q) = (\mp q^{1/2}; \pm q^{1/2})_m. \tag{5.16d}$$

It is easy to verify these expressions through the use of the following identity, due to Berkovich and Warnaar [BerWar05, Theorem 8.1]

$$H_m(z;q) = \sum_{r=0}^{\lfloor m/2 \rfloor} z^{2r}(-q/z;q^2)_r(-z;q^2)_{\lceil m/2 \rceil - r} \begin{bmatrix} \lfloor m/2 \rfloor \\ r \end{bmatrix}_{q^2},$$

or by using the standard identities for ${}_r\phi_s$ series that may be found in [GasRa90].

The occurrence of the Rogers–Szegő polynomials in the context of symmetric function theory is perhaps not as surprising as it may first seem. In particular we note that

$$H_m(z;q) = (q)_m h_m\Big(\frac{1+z}{1-q}\Big) \tag{5.17a}$$

$$= \sum_\lambda K_{\lambda'(1^m)} s_\lambda(1,z). \tag{5.17b}$$

Equation (5.17a), which employs λ-ring notation, immediately follows from (1.45b) and (5.12). Equation (5.17b) requires slightly more effort. Recalling the definition of hook length from page 5, the *hook length polynomial* is defined as

$$H_\lambda(q) := \prod_{s \in \lambda} \big(1 - q^{h(s)}\big).$$

Now, once more using (5.12), we have

$$\begin{aligned}\sum_{m=0}^{\infty} \frac{H_m(z)x^m}{(q)_m} &= \frac{1}{(x,xz)_\infty} \\ &= \sum_\lambda \frac{q^{n(\lambda)}x^{|\lambda|}}{H_\lambda(q)} s_\lambda(1,z) \\ &= \sum_{m=0}^{\infty} x^m \sum_{|\lambda|=m} \frac{q^{n(\lambda)}}{H_\lambda(q)} s_\lambda(1,z) \\ &= \sum_{m=0}^{\infty} \frac{x^m}{(q)_m} \sum_\lambda K_{\lambda'(1^m)}(q) s_\lambda(1,z),\end{aligned}$$

where the second equality follows from [Macd95, p. 66] and the final equality from [Macd95, p. 243]. Equating coefficients of x^m in the extremes of this equation settles (5.17b).

A conjectural q-hypergeometric identity for a Littlewood-type sum

Littlewood-type sums involving the Hall–Littlewood polynomials have appeared in the literature quite extensively, see e.g., [IsJoZe06, JoZe05, Kaw99, Macd95, Stem90, Ve12,War06]. In this section we present a novel type of Littlewood sum for the Hall–Littlewood polynomials, expressible as a q-hypergeometric multisum. We require only a little more notation before we proceed to state this conjecture.

Using the conventions of Warnaar [War06], $H_m(z;q)$ may be extended to partitions as follows:

$$h_\lambda(z;q) := \prod_{i\geq 1} H_{m_i(\lambda)}.$$

For example,

$$h_{(4,4,2,2,2,1,1)}(z;q) = H_2^2(z;q)H_3(z;q).$$

For a partition λ, let λ_{o} be the partition formed from the odd parts of λ. For example, if $\lambda = (5,4,3,3,2,1,1)$, then $\lambda_{\mathrm{o}} = (5,3,3,1,1)$.

With this notation, we present the following general conjecture for a Littlewood-type sum, which is supported by extensive computer-assisted checking.

Conjecture 5.4. *For $M = (M_1,\dots,M_m) \in \mathbb{N}^m$ and $m_0(\lambda) := \infty$*

$$\sum_{\substack{\lambda \\ \lambda_1 \leq 2m}} z^{\ell(\lambda_{\mathrm{o}})} P'_\lambda(x;q) h_{\lambda_{\mathrm{o}}}(w/z;q) \prod_{\ell=1}^{m} (wz)^{M_\ell - \lambda'_{2\ell-1}} \begin{bmatrix} m_{2\ell-2}(\lambda) \\ M_\ell - \lambda'_{2\ell-1} \end{bmatrix}$$
$$= \sum \left(\prod_{i=1}^{n} \left(-q^{1-r_i^{(1)}} w/x_i, -q^{1-r_i^{(1)}} z/x_i\right)_{r_i^{(1)}} \right) \prod_{\ell=1}^{m} f^{(2)}_{r^{(\ell)},r^{(\ell+1)}}(x;q), \quad (5.18)$$

where the sum on the right is over $r^{(1)},\dots,r^{(m)} \in \mathbb{N}^n$ such that $|r^{(\ell)}| = M_\ell$, and $r^{(m+1)} := 0$.

By summing both sides of (5.18) over all sequences $M \in \mathbb{Z}_+^m$, we obtain an expression that is more useful for later applications, where the right-hand side is simply a sum over $r^{(1)},\dots,r^{(m)} \in \mathbb{Z}_+^n$ For ease of representation we introduce the polynomial

$$h_\lambda^{(m)}(w,z;q) = \prod_{\substack{i=1 \\ i \text{ odd}}}^{2m-1} z^{m_i(\lambda)} H_{m_i(\lambda)}(w/z;q) \prod_{\substack{i=1 \\ i \text{ even}}}^{2m-1} H_{m_i(\lambda)}(wz;q).$$

For example, given $\lambda = (8,8,7,5,5,2,1,1)$ and $m = 4$:

$$h_\lambda^{(4)}(w,z;q) = z^5 H_1(w/z;q) H_2^2(w/z;q) H_1(wz;q) H_2(wz;q).$$

We remark that this representation hides the fact that $h_\lambda^{(m)}$ is symmetric in z and w, which becomes easy to see using (5.15).

We now use this polynomial to rewrite (5.18). On the left-hand side of (5.18) we interchange the λ and M sums and shift $M_\ell \mapsto M_\ell + \lambda'_{2\ell-1}$, to obtain

$$\sum_{\substack{\lambda \\ \lambda_1 \le 2m}} z^{\ell(\lambda_{\mathrm{o}})} P'_\lambda(x;q) h_{\lambda_{\mathrm{o}}}(w/z;q) \prod_{\ell=1}^{m} \sum_{M_\ell \ge 0} (wz)^{M_\ell} \begin{bmatrix} m_{2\ell-2}(\lambda) \\ M_\ell \end{bmatrix}. \tag{5.19}$$

Since $\left[\begin{smallmatrix} m_{2\ell-2}(\lambda) \\ M_\ell \end{smallmatrix}\right] = 1$ when $M_\ell > m_{2\ell-2}$, the sums inside the product terminate at $m_{2\ell-2}$. We may then apply (5.13) so that all but the $\ell = 1$ case of the product in (5.19) may be written together as

$$\prod_{\substack{i=1 \\ i \text{ even}}}^{2m-1} H_{m_i(\lambda)}(wz;q).$$

Since we define $m_0(\lambda) = \infty$, by application of the q-binomial theorem (3.2) for $a = 0$, the remaining M_1-sum may be completed to obtain the factor $1/(wz)_\infty$. We then arrive at the next corollary.

Corollary 5.5. *Let* $|wz| < 1$. *Assuming* (5.18), *we have*

$$\sum_{\substack{\lambda \\ \lambda_1 \le 2m}} h_\lambda^{(m)}(w,z;q) P'_\lambda(x;q)$$

$$= (wz)_\infty \sum \Bigg(\prod_{i=1}^{n} \big(-q^{1-r_i^{(1)}} w/x_i, -q^{1-r_i^{(1)}} z/x_i\big)_{r_i^{(1)}} \Bigg) \prod_{\ell=1}^{m} f^{(2)}_{r^{(\ell)},r^{(\ell+1)}}(x;q), \tag{5.20}$$

where the sum on the right is over $r^{(1)}, \dots, r^{(m)} \in \mathbb{N}^n$, *and* $r^{(m+1)} := 0$.

Later we will consider (5.18) in the case $m = 0$. For this purpose we set $h_0^{(0)} := (wz)_\infty$, so that here the left and right-hand sides are equal.

In the context of (5.20), consider the specialisations of (w,z) where $h_\lambda^{(m)}(w,z;q)$ is factorisable. Now, $h_\lambda^{(m)}(w,z;q)$ contains both $H_m(w/z;q)$ and $H_m(wz;q)$, and so by (5.16) the only choices of w and z that will yield a factorisable result are

$$(w,z) \mapsto (0,z), (1,q^{1/2}), (-1,-q^{1/2}), (q^{1/2},-q^{1/2}).$$

Note that for $w = 0$, the variable z is not restricted to the specialisations in (5.16). Note furthermore that we have dismissed the case $(w,z) \mapsto (1,-1)$ due to an issue of convergence; on the right-hand side of (5.20), the factor $(wz)_\infty$ tends to infinity. It is for a certain subset of these possible specialisations, applied to a later result, that we obtain characters of affine Lie algebras.

In the case of the specialisation $w = 0$ we can prove Conjecture 5.4 which, by application of (5.16a) to (5.18), leads us to the following proposition.

Proposition 5.6. *For* $M = (M_1, \dots, M_m) \in \mathbb{N}^n$

$$\sum z^{\ell(\lambda_{\mathrm{o}})} P'_\lambda(x;q) = \sum \left(\prod_{i=1}^{n} \left(-q^{1-r_i^{(1)}} z/x_i\right)_{r_i^{(1)}} \right) \prod_{\ell=1}^{m} f^{(2)}_{r^{(\ell)},r^{(\ell+1)}}(x;q), \tag{5.21}$$

where the sum on the left is over partitions λ *such that* $\lambda_1 \le 2m$ *and* $\lambda'_{2\ell-1} = M_\ell$*, and where the sum on the right is over* $r^{(1)}, \dots, r^{(m)} \in \mathbb{N}^n$ *such that* $|r^{(\ell)}| = M_\ell$*.*

The proof of Proposition 5.6 is simplified by the observation that z is essentially an overall scaling factor in (5.21) and therefore may be eliminated without loss of generality. This fact is demonstrated in the following remarks.

By the substitution $x \mapsto xz$ we have

$$\sum z^{\ell(\lambda_{\mathrm{o}})} P'_\lambda(xz;q) = \sum \left(\prod_{i=1}^{n} \left(-q^{1-r_i^{(1)}} /x_i\right)_{r_i^{(1)}} \right) \prod_{\ell=1}^{m} f^{(2)}_{r^{(\ell)},r^{(\ell+1)}}(xz;q). \tag{5.22}$$

Now, it follows trivially from the definition of $f^{(\tau)}_{r,s}(x;q)$ (4.11) that

$$f^{(\tau)}_{r,s}(xz;q) = z^{|r|\tau} f^{(\tau)}_{r,s}(x;q).$$

On the right-hand side of (5.22), we then have an overall factor of $z^{2M_1+\cdots+2M_m}$ arising from

$$\prod_{\ell=1}^{m} f^{(2)}_{r^{(\ell)},r^{(\ell+1)}}(xz;q) = z^{2M_1+\cdots+2M_m} \prod_{\ell=1}^{m} f^{(2)}_{r^{(\ell)},r^{(\ell+1)}}(x;q),$$

where we have used $|r^{(\ell)}| = M_\ell$.

Observe that since P'_λ is homogeneous of total degree $|\lambda|$, $P'_\lambda(xz;q) = z^{|\lambda|} P'_\lambda(x;q)$. Using $m_i(\lambda) = \lambda'_i - \lambda'_{i+1}$, we have $\ell(\lambda_{\mathrm{o}}) = |\lambda_{\mathrm{o}}| - |\lambda_{\mathrm{e}}|$ and so on the left-hand side of (5.22),

$$\sum z^{|\lambda_{\mathrm{o}}|-|\lambda_{\mathrm{e}}|+|\lambda|} P'_\lambda(x;q) = z^{2M_1+\cdots+2M_m} \sum P'_\lambda(x;q).$$

We may then rewrite (5.22) as

$$\sum P'_\lambda(x;q) = \sum \left(\prod_{i=1}^{n} \left(-q^{1-r_i^{(1)}} /x_i\right)_{r_i^{(1)}} \right) \prod_{\ell=1}^{m} f^{(2)}_{r^{(\ell)},r^{(\ell+1)}}(x;q),$$

which is precisely (5.21) for $z = 1$. It then suffices to prove Proposition 5.6 for $z = 1$.

In the following proof we will use the notation ${}_1\Phi_0$ to represent the A_{n-1} basic hypergeometric series

$${}_1\Phi_0(a;-;q,x) = \sum_{r\in\mathbb{N}^n} \prod_{i=1}^{n} \left((-1)^{r_i} x_i^{r_i} q^{\binom{r_i}{2}}\right)^{1-n} \prod_{i,j=1}^{n} \frac{x_i^{r_j} (a_j x_i/x_j)_{r_i} (q x_i/x_j)_{r_i - r_j}}{(q x_i/x_j)_{r_i}},$$

where $a = (a_1, \dots, a_n)$.

Proof of Proposition 5.6. Recall our earlier convention of writing $f_{r,s}$ for $f^{(1)}_{r,s}(x;q)$.

We apply Theorem 5.1 with λ a partition such that $\lambda_1 \le 2m$, so that in light of the remark after Theorem 5.1 we have

$$P'_\lambda = \sum \prod_{\ell=1}^{2m} f_{r^{(\ell)},r^{(\ell+1)}}.$$

summed over $r^{(1)} \supseteq \cdots \supseteq r^{(2m)} \in \mathbb{N}^n$ such that $|r^{(\ell)}| = \lambda'_\ell$, where $r^{(2m+1)} := 0$. We now replace $(r_{2\ell-1}, r_{2\ell}) \mapsto (r_\ell, s_\ell)$ for all $\ell = 1, \dots, m$. Hence

$$P'_\lambda = \sum \prod_{\ell=1}^{m} f_{r^{(\ell)},s^{(\ell)}} f_{s^{(\ell)},r^{(\ell+1)}}, \tag{5.23}$$

where the summation indices are now written $r^{(1)} \supseteq s^{(1)} \supseteq \cdots \supseteq r^{(m)} \supseteq s^{(m)} \in \mathbb{N}^n$, such that $|r^{(\ell)}| = \lambda'_{2\ell-1}$ and $|s^{(\ell)}| = \lambda'_{2\ell}$.

We now sum both sides of (5.23) over all partitions λ such that $\lambda_1 \le 2m$ and $\lambda'_{2\ell-1} = M_\ell$ for $1 \le \ell \le m$ to obtain

$$\sum P'_\lambda = \sum \prod_{\ell=1}^{m} f_{r^{(\ell)},s^{(\ell)}} f_{s^{(\ell)},r^{(\ell+1)}}, \tag{5.24}$$

so that the left-hand side of (5.24) is identical to the left-hand side of (5.21). On the right we have combined the summation conditions just mentioned with those of (5.23) so that the sum is over $r^{(1)}, \dots, r^{(m)} \in \mathbb{N}^n$ and $s^{(1)}, \dots, s^{(m)} \in \mathbb{N}^n$ such that $|r^{(\ell)}| = M_\ell$. What remains to be shown is that this is identical to the right-hand side of (5.21).

Now, the summand on the right-hand side vanishes unless $r^{(\ell+1)} \subseteq s^{(\ell)} \subseteq r^{(\ell)}$. We are then prompted to shift $s^{(\ell)} \mapsto s^{(\ell)} + r^{(\ell+1)}$ for the new bounds $0 \subseteq s^{(\ell)} \subseteq r^{(\ell)} - r^{(\ell+1)}$. Using the explicit form for $f_{r,s}$ given by (4.11) and manipulating some of the q-shifted factorials, the right-hand side of (5.24) is then equal to

$$\sum \prod_{\ell=1}^{m} f^{(\tau_\ell)}_{r^{(\ell)},r^{(\ell+1)}} \, {}_1\Phi_0\Big(q^{-(r^{(\ell)}-r^{(\ell+1)})}; -; q, -q^{r^{(\ell+1)}+|r^{(\ell)}|-|r^{(\ell+1)}|}\, x\Big), \tag{5.25}$$

where each $s^{(\ell)}$ sum now forms a ${}_1\Phi_0$ series and the remaining summation is over

$$r^{(1)}, \dots, r^{(m)} \in \mathbb{N}^n,$$

which is still subject to $|r^{(\ell)}| = M_\ell$, and where $(\tau_1, \dots, \tau_m) = (1, 2, \dots, 2)$. By Milne's A_{n-1} terminating q-binomial theorem [Mi97, Theorem 5.46]

$${}_1\Phi_0\big(q^{-N}; -; q, x\big) = \prod_{i=1}^{n} \big(q^{-|N|} x_i\big)_{N_i}.$$

By application of this theorem to each of the ${}_1\Phi_0$ series, (5.25) is then

$$\sum \left(\prod_{i=1}^{n} \prod_{\ell=1}^{m} (-q^{r_i^{(\ell+1)}} x_i)_{r_i^{(\ell)} - r_i^{(\ell+1)}} \right) \prod_{\ell=1}^{m} f^{(\tau_\ell)}_{r^{(\ell)}, r^{(\ell+1)}} = \sum \left(\prod_{i=1}^{n} (-x_i)_{r_i^{(1)}} \right) \prod_{\ell=1}^{m} f^{(\tau_\ell)}_{r^{(\ell)}, r^{(\ell+1)}}, \quad (5.26)$$

where the right-hand side follows by elementary manipulation, recalling that $r^{(m+1)} := 0$. We then apply $(-b)_r = (-q^{1-r}/b)_r \, b^r q^{\binom{r}{2}}$, to obtain the right-hand side of (5.21), as required. □

Chapter 6

Combinatorial character formulas

Introduction

In this penultimate chapter we bring together all the foundational material of previous chapters to derive combinatorial formulas for characters of affine Kac–Moody algebras of types $\mathrm{A}_{2n}^{(2)}$, $\mathrm{C}_n^{(1)}$ and $\mathrm{D}_{n+1}^{(2)}$ for a 1-parameter family of weights. These are the main results of this thesis.

Our mains results are achieved by carrying out, in any order, a three-step procedure on both sides of the C_n Andrews transformation (4.12), which in particular requires that

I. all of $b_2, c_2, \dots, b_{m+1}, c_{m+1}$ tend to infinity,

II. the bounds of summation $N_1, \dots, N_n$ all tend to infinity, and

III. the alphabet x is replaced with $x^{\pm} := (x_1, x_1^{-1}, \dots, x_n, x_n^{-1})$.

Note that step III effectively doubles the rank n. After steps I and II, and by using the conjectural (5.20), the right-hand side of the C_n Andrews transformation may be identified with a Littlewood-type sum over the modified Hall–Littlewood polynomials. Subsequent to all three steps, the left-hand side becomes a formula that unifies the Weyl–Kac forms of the characters $\mathrm{A}_{2n}^{(2)}$ (2.25), $\mathrm{C}_n^{(1)}$ (2.23) and $\mathrm{D}_{n+1}^{(2)}$ (2.28), where each individual character is obtained by certain specialisations of the parameters b_1 and c_1 that remain from step I.

We can motivate steps II and III by comparing the features of the left-hand side of the C_n Andrews transformation with the features of the character formulas (2.25), (2.23) and (2.28). Recall that the left-hand sum of the C_n Andrews transformation is a terminating series that is unilateral in the positive quadrant and symmetric under simultaneous permutation of x and N. In contrast, the expressions appearing on the right-hand side of the character formulas are bilateral sums over the full n-dimensional integer lattice and have signed-permutation symmetry in x. Steps

II and III serve to introduce the desired symmetry and range of support to the expression on the left-hand side of the C_n Andrews transformation.

In our results we also consider the same three-step procedure for the alphabet $x = (x_1, \dots, x_{n-1}, 1)$ where we abuse notation slightly to define

$$x^{\pm} := (x_1, x_1^{-1}, \dots, x_{n-1}, x_{n-1}^{-1}, 1),$$

and not $(x_1, x_1^{-1}, \dots, x_{n-1}, x_{n-1}^{-1}, 1, 1)$ as one might expect. Our result for this alphabet does not yield combinatorial character formulas, but will be important in the next chapter. The details for the execution of the three-step procedure are quite long and involved and so we leave them until after the discussion of our results.

Main results

We now present the general theorem that yields our main results under specialisation. Recall the polynomial

$$h_\lambda^{(m)}(w, z; q) = \prod_{\substack{i=1 \\ i \text{ odd}}}^{2m-1} z^{m_i(\lambda)} H_{m_i(\lambda)}(w/z; q) \prod_{\substack{i=1 \\ i \text{ even}}}^{2m-1} H_{m_i(\lambda)}(wz; q).$$

from page 69.

Theorem 6.1. *Let m be a nonnegative integer and $|q/bc| \le 1$. Then the following two identities are true for $b \to \infty$ and true for general b if the conjectural* (5.20) *holds. First,*

$$\begin{aligned} \frac{1}{D\big(x^{\pm}; b, c; q\big)} \sum_{r \in \mathbb{Z}^n} \frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)} \prod_{i=1}^{n} \frac{(bx_i, cx_i)_{r_i}}{(qx_i/b, qx_i/c)_{r_i}} \bigg(\frac{q^{1-n}}{bc}\bigg)^{r_i} \big(x_i^2 q^{r_i}\big)^{(m+n)r_i} \\ = \sum_{\substack{\lambda \\ \lambda_1 \le 2m}} q^{|\lambda|/2} h_\lambda^{(m)}\big(-q^{1/2}/b, -q^{1/2}/c; q\big) P'_\lambda\big(x^{\pm}; q\big), \end{aligned} \tag{6.1a}$$

where $x = (x_1, \dots, x_n)$ and

$$D\big(x^{\pm}; b, c; q\big) = (q)_\infty^n \prod_{i=1}^{n} \frac{(qx_i^{\pm 2})_\infty}{(qx_i^{\pm}/b, qx_i^{\pm}/c)_\infty} \prod_{1 \le i < j \le n} (qx_i^{\pm} x_j^{\pm})_\infty. \tag{6.1b}$$

Now using the alphabet $x = (x_1, \dots, x_{n-1}, 1)$, where

$$D\big(x^{\pm}; b, c; q\big) = \frac{(q)_\infty^n}{(q/b, q/c)_\infty} \prod_{i=1}^{n-1} \frac{(qx_i^{\pm 2}, qx_i^{\pm})_\infty}{(qx_i^{\pm}/b, qx_i^{\pm}/c)_\infty} \prod_{1 \le i < j \le n-1} (qx_i^{\pm} x_j^{\pm})_\infty, \tag{6.1c}$$

the second identity is given by

$$\frac{1}{D\big(x^{\pm};b,c;q\big)}\sum_{r\in\mathbb{Z}^n}\frac{\Delta_{\mathrm{B}}(-xq^r)}{\Delta_{\mathrm{B}}(-x)}\prod_{i=1}^{n}\frac{(bx_i,cx_i)_{r_i}}{(qx_i/b,qx_i/c)_{r_i}}\Big(-\frac{q^{3/2-n}}{bc}\Big)^{r_i}\big(x_i^2q^{r_i}\big)^{(m+n-1/2)r_i}$$
$$=\sum_{\substack{\lambda\\ \lambda_1\le 2m}} q^{|\lambda|/2}h_{\lambda}^{(m)}\big(-q^{1/2}/b,-q^{1/2}/c;q\big)P'_{\lambda}\big(x^{\pm};q\big). \quad (6.1\mathrm{d})$$

Recalling the definition $h_0^{(0)}(w,z;q)=(wz)_\infty$, we point out that for $m=0$, the identities (6.1a) and (6.1d) appear as specialisations of Gustafson's $\mathrm{C}_n^{(1)}$-analogue of Bailey's sum of a very-well-poised ${}_6\psi_6$ series [Gu87, Theorem 5.1]:

$$\sum_{r\in\mathbb{Z}^n}\frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)}\prod_{i=1}^{n}q^{-(n-i+1)r_i}\prod_{\ell=1}^{n+1}\frac{(b_\ell x_i,c_\ell x_i)_{r_i}}{(qx_i/b_\ell,qx_i/c_\ell)_{r_i}}\Big(\frac{q}{b_\ell c_\ell}\Big)^{r_i}$$
$$=\frac{(q)_\infty^n}{(q/b_1c_1\cdots b_{n+1}c_{n+1})_\infty}\prod_{k,\ell=1}^{n+1}(q/b_kc_\ell)_\infty\prod_{1\le k<\ell\le n+1}(q/b_kb_\ell,q/c_kc_\ell)_\infty\prod_{1\le i<j\le n}(qx_i^{\pm}x_j^{\pm})_\infty$$
$$\times\prod_{i=1}^{n}(qx_i^{\pm 2})_\infty\prod_{\ell=1}^{n+1}\frac{1}{(qx_i^{\pm}/b_\ell,qx_i^{\pm}/c_\ell)_\infty}. \quad (6.2)$$

In particular, (6.1a) is found from (6.2) in the limit $b_2,c_2,\dots,b_{n+1},c_{n+1}\to\infty$ and (6.1d) is found when $b_2,c_3,b_3,\dots,b_{n+1},c_{n+1}\to\infty$ and $c_2=1$, where

$$(x_1,\dots,x_n)\to(-x_1,\dots,-x_{n-1},-1).$$

We will now specialise (6.1a) to obtain manifestly positive representations of the characters $\mathrm{A}_{2n}^{(2)}$, $\mathrm{C}_n^{(1)}$ and $\mathrm{D}_{n+1}^{(2)}$. These specialisations are guided by the factor

$$h_{\lambda}^{(m)}\big(-q^{1/2}/b,-q^{1/2}/c;q\big)$$

on the right-hand side. Recall from page 70 that the arguments for which this function is factorisable (adjusted for $(w,z)\mapsto(-q^{1/2}b,-q^{1/2}c)$) are:

$$(b,c)\to(\infty,c),(-q^{1/2},-1),(q^{1/2},1),(-1,1).$$

We emphasise that it is only for specialisations where $b\to\infty$ that our results are fully supported, since the other specialisations are conjectural cases of Theorem 6.1. For $b\to\infty$ the right-hand side becomes

$$\sum_{\substack{\lambda\\ \lambda_1\le 2m}} q^{|\lambda|/2}(-q^{1/2}/c)^{\ell(\lambda_{\mathrm{o}})}P'_{\lambda}\big(x^{\pm};q\big) \quad (6.3)$$

There are three specialisations of c that yield characters of affine Lie algebras on the left-hand side of (6.1a). For $c \to \infty$ we have

$$\frac{1}{(q)_\infty^n \prod_{i=1}^n (qx_i^{\pm 2})_\infty \prod_{1\le i<j\le n}(qx_i^{\pm}x_j^{\pm})_\infty} \sum_{r\in\mathbb{Z}^n} \frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)} \prod_{i=1}^n q^{(n+m+1)r_i^2 - nr_i} x_i^{2(m+n+1)r_i},$$

which, by Lemma 2.1, is the character of the integrable highest-weight module $V(\Lambda)$ of type $\mathrm{C}_n^{(1)}$, where $\Lambda = m\Lambda_0$. For this same specialisation, the sum in (6.3) vanishes unless $\ell(\lambda_{\mathrm{o}}) = 0$, i.e., λ is a partition with only even parts. The corresponding identity for this specialisation is given in Theorem 6.2.

For the specialisation $c \to -q^{1/2}$ on the left-hand side of (6.1a), using

$$\frac{(aq)_\infty}{(-aq^{1/2})_\infty} = (aq^{1/2})_\infty (aq^2;q^2)_\infty$$

to compute $D(x^{\pm};\infty,-q^{1/2};q)$, we have

$$\frac{1}{(q)_\infty^n \prod_{i=1}^n (q^{1/2}x_i^{\pm})_\infty (q^2 x_i^{\pm 2};q^2)_\infty \prod_{1\le i<j\le n}(qx_i^{\pm}x_j^{\pm})_\infty} \\ \times \sum_{r\in\mathbb{Z}^n} \frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)} \prod_{i=1}^n q^{(n+m+1/2)r_i^2 - nr_i} x_i^{2(n+m+1/2)r_i},$$

which, by Lemma 2.2, is the character $V(\Lambda)$ of type $\mathrm{A}_{2n}^{(2)}$ for $\Lambda = m\Lambda_0$. The corresponding identity for this specialisation is given in Theorem 6.3.

Finally, for the specialisation $c \to -1$ on the left-hand side of (6.1a), using

$$\frac{(a^2q)_\infty}{(-aq)_\infty} = (aq)_\infty (a^2q;q^2)_\infty$$

to compute $D(x^{\pm};\infty,-1;q)$, we have

$$\frac{1}{(q)_\infty^n \prod_{i=1}^n (qx_i^{\pm})_\infty (qx_i^{\pm 2};q^2)_\infty \prod_{1\le i<j\le n}(qx_i^{\pm}x_j^{\pm})_\infty} \\ \times \sum_{r\in\mathbb{Z}^n} \frac{\Delta_{\mathrm{B}}(xq^r)}{\Delta_{\mathrm{B}}(x)} \prod_{i=1}^n q^{(m+n+1/2)r_i^2 - (n-1/2)r_i} x_i^{2(m+n+1/2)r_i},$$

which, by Lemma 2.3, is the mirrored form of the character of $V(\Lambda)$ of type $\mathrm{A}_{2n}^{(2)}$ for $\Lambda = m\Lambda_n$. The corresponding identity for this specialisation is given in Theorem 6.4.

With these specialisations we announce the following three theorems, which are the main results of this thesis.

Theorem 6.2. *Fix a nonnegative integer m and let*

$$q = \mathrm{e}^{-\delta} \quad \text{and } x_i = \mathrm{e}^{-\alpha_i - \cdots - \alpha_{n-1} - \alpha_n/2}.$$

Then, for $\mathfrak{g} = \mathrm{C}_n^{(1)}$ and $\Lambda = m\Lambda_0$,

$$\mathrm{e}^{-\Lambda} \operatorname{ch} V(\Lambda) = \sum_{\substack{\lambda \text{ even} \\ \lambda_1 \le 2m}} q^{|\lambda|/2} P'_\lambda\big(x^\pm; q\big). \tag{6.4}$$

Theorem 6.3. *Fix a nonnegative integer m and let*

$$q = \mathrm{e}^{-\delta} \quad \text{and } x_i = \mathrm{e}^{-\alpha_i - \cdots - \alpha_{n-1} - \alpha_n/2}.$$

Then, for $\mathfrak{g} = \mathrm{A}_{2n}^{(2)}$ and $\Lambda = 2m\Lambda_0$,

$$\mathrm{e}^{-\Lambda} \operatorname{ch} V(\Lambda) = \sum_{\substack{\lambda \\ \lambda_1 \le 2m}} q^{|\lambda|/2} P'_\lambda\big(x^\pm; q\big). \tag{6.5}$$

Theorem 6.4. *Fix a nonnegative integer m and let*

$$q = \mathrm{e}^{-\delta} \quad \text{and } x_i = \mathrm{e}^{-\alpha_0 - \cdots - \alpha_{n-i}}.$$

Then for $\mathfrak{g} = \mathrm{A}_{2n}^{(2)}$ and $\Lambda = m\Lambda_n$,

$$\mathrm{e}^{-\Lambda} \operatorname{ch} V(\Lambda) = \sum_{\substack{\lambda \\ \lambda_1 \le 2m}} q^{(|\lambda| + l(\lambda_{\mathrm{o}}))/2} P'_\lambda\big(x^\pm; q\big). \tag{6.6}$$

There is some evidence that suggests the combinatorial character formulas in Theorems 6.2, 6.3 and 6.4 hold more generally. After substantial computer-assisted investigation, we are convinced that (6.5) and (6.6) also hold for half-integer m. Furthermore, for the $\mathrm{C}_n^{(1)}$ character, we believe that

$$\mathrm{e}^{-\Lambda_1} \operatorname{ch} V(\Lambda_1) = x_1 \sum_{k=0}^{\infty} \frac{q^k}{(q)_k} Q'_{(2^k 1)}\big(x^\pm; q\big).$$

One more character identity may be obtained from (6.1a) under the specialisation $(b, c) \to (-1, -q^{1/2})$, which is one of the conjectural cases. We note that here the specialisations of each side are not completely independent of one another as was the case previously. The factor $h_\lambda^{(m)}(q^{1/2}, 1; q)$ on the right of (6.1a) simplifies by the straightforward use of the list of factorisations of $H_m(x; q)$ (5.16)

$$h_\lambda^{(m)}(q^{1/2}, 1; q) = \prod_{i=1}^{2m-1} (-q^{1/2}; q^{1/2})_{m_i(\lambda)}. \tag{6.7a}$$

To obtain the correct form for the following conjecture, it is important to introduce the factor $(-q^{1/2};q^{1/2})_\infty$ into the product above. For uniformity, we use the convention that $m_0(\lambda):=\infty$. The expression above then becomes

$$h_\lambda^{(m)}(q^{1/2},1;q)=\frac{1}{(-q^{1/2};q^{1/2})_\infty}\prod_{i=0}^{2m-1}(-q^{1/2};q^{1/2})_{m_i(\lambda)}, \tag{6.7b}$$

so the right-hand side of (6.1a) is given by

$$\frac{1}{(-q^{1/2};q^{1/2})_\infty}\sum_{\substack{\lambda\\ \lambda_1\le 2m}} q^{|\lambda|/2}\bigg(\prod_{i=0}^{2m-1}\big(-q^{1/2};q^{1/2}\big)_{m_i(\lambda)}\bigg)P'_\lambda\big(x^\pm;q\big). \tag{6.8}$$

On the left-hand side we have

$$\begin{aligned}\frac{1}{(q)_\infty^n}\prod_{i=1}^n\frac{(-qx_i^\pm,-q^{1/2}x_i^\pm)_\infty}{(qx_i^{\pm2})_\infty}&\prod_{1\le i<j\le n}\frac{1}{(qx_i^\pm x_j^\pm)_\infty}\\ &\times\sum_{r\in\mathbb{Z}^n}\frac{\Delta_{\mathrm{B}}(xq^r)}{\Delta_{\mathrm{B}}(x)}\prod_{i=1}^n q^{(m+n)r_i^2-(n-1/2)r_i}x_i^{2(m+n)r_i}.\end{aligned} \tag{6.9}$$

After shifting the introduced factor $1/(-q^{1/2};q^{1/2})$ from the right-hand side to the left and using the identity $(a^2q)_\infty/(-aq^{1/2},-aq)_\infty=(aq^{1/2};q^{1/2})_\infty$, we then rescale $q\to q^2$ and apply Lemma 2.4 to obtain the following conjecture. Again, we believe that this identity also holds for half-integer m, and so this condition has been incorporated.

Conjecture 6.5. *Let $\mathfrak{g}=\mathrm{D}_{n+1}^{(2)}$ and, for m a nonnegative integer or half-integer, let $\Lambda=2m\Lambda_0$, $q=\mathrm{e}^{-\delta}$ and $x_i=\mathrm{e}^{-\alpha_i-\cdots-\alpha_n}$. Then*

$$\mathrm{e}^{-\Lambda}\,\mathrm{ch}V(\Lambda)=\sum_{\substack{\lambda\\ \lambda_1\le 2m}} q^{|\lambda|}\bigg(\prod_{i=0}^{2m-1}\big(-q\big)_{m_i(\lambda)}\bigg)P'_\lambda\big(x^\pm;q^2\big). \tag{6.10}$$

It is natural to ask why it is only for the types $\mathrm{A}_{2n}^{(2)}$, $\mathrm{C}_n^{(1)}$ and $\mathrm{D}_{n+1}^{(2)}$ that we have theorems or conjectures. Now, no other specialisation of b and c in (6.1a) will yield further combinatorial character formulas but it is possible to obtain equations that are tantalisingly close. Under certain specialisations of b and c, the left-hand side of (6.1a) resembles the Kac–Peterson form of the other BC_n type characters; i.e., $\mathrm{B}_n^{(1)}$, $\mathrm{D}_n^{(1)}$ and $\mathrm{A}_{2n-1}^{(2)}$. The specialisations obtained are (up to a prefactor) of the form

$$\mathrm{LHS}=\sum_{r\in\mathbb{Z}^n} f_{\mathfrak{g},r}(x),$$

where $f_{\mathfrak{g},r}(x)$ is the summand of the Kac–Peterson representation of the affine character of type $\mathfrak{g}$. Where these near-character formulas differ importantly from the desired Kac–Peterson form of the other BC_n type characters is in the summation conditions: missing here is the condition $|r| \equiv 0 \pmod 2$. However, combinatorial character formulas for the missing types may be obtained from another C_n hypergeometric series, similar to the C_n Andrews Transformation, which arises from a C_n Bailey pair (α, β) where α_r vanishes whenever $r \equiv 1 \pmod 2$ [War]. Moreover, Milne and Lilly's A_n Bailey lemma [MiLil92] may be applied to generate an A_n Andrews Transformation, but perhaps surprisingly this does not seem to yield character formulas upon specialisation.

Proof of Theorem 6.1

In this section we carry out the three-step procedure described at the start of this chapter, which constitutes a proof of Theorem 6.1. We complete this proof in two parts, where the left and right-hand sides of the C_n Andrews transformation are treated separately. The details of the procedure for the left-hand side are dramatically more complicated than those of the right-hand side.

We will now define some useful notation. Recall the C_n Andrews transformation: for m a nonnegative integer and $N \in \mathbb{N}^n$,

$$\sum_{0 \subseteq r \subseteq N} \frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)} \prod_{i=1}^{n} \left[\prod_{\ell=1}^{m+1} \frac{(b_\ell x_i, c_\ell x_i)_{r_i}}{(qx_i/b_\ell, qx_i/c_\ell)_{r_i}} \Big(\frac{q}{b_\ell c_\ell}\Big)^{r_i} \prod_{j=1}^{n} \frac{(q^{-N_j} x_i/x_j, x_i x_j)_{r_i}}{(qx_i/x_j, q^{N_j+1} x_i x_j)_{r_i}} q^{N_j r_i} \right]$$
$$= \prod_{i,j=1}^{n} (qx_i x_j)_{N_i} \prod_{1 \le i < j \le n} \frac{1}{(qx_i x_j)_{N_i+N_j}}$$
$$\times \sum_{r^{(1)}, \dots, r^{(m)} \in \mathbb{N}^n} \prod_{i,j=1}^{n} \frac{(qx_i/x_j)_{N_i}}{(qx_i/x_j)_{N_i - r_j^{(1)}}} \prod_{\ell=1}^{m} f^{(0)}_{r^{(\ell)}, r^{(\ell+1)}}(x;q)$$
$$\times \prod_{\ell=1}^{m+1} \left[(q/b_\ell c_\ell)_{|r^{(\ell-1)}| - |r^{(\ell)}|} \Big(\frac{q}{b_\ell c_\ell}\Big)^{|r^{(\ell)}|} \right.$$
$$\left. \times \prod_{i=1}^{n} \frac{(b_\ell x_i, c_\ell x_i)_{r_i^{(\ell)}}}{(qx_i/b_\ell, qx_i/c_\ell)_{r_i^{(\ell-1)}}} \right], \tag{6.11}$$

where $r^{(0)} := N$ and $r^{(m+1)} := 0$. As shorthand, we will express this transformation as

$$L_N(x; b, c; b_2, c_2, \dots, b_{m+1}, c_{m+1}; q) = R_N(x; b, c; b_2, c_2, \dots, b_{m+1}, c_{m+1}; q) \tag{6.12}$$

where we have relabelled the variables b_1, c_1 as b, c.

The right-hand side

We will demonstrate that after application of steps I, II and III, the right-hand side of the C_n Andrews transformation becomes the right-hand side of (6.1a) when $x = (x_1, \dots, x_n)$, or (6.1d) when $x = (x_1, \dots, x_{n-1}, 1)$.

The right-hand side of C_n Andrews is a rational function and hence we may let all $b_2, c_2, \dots, b_{m+1}, c_{m+1}$ tend to infinity for fixed $N \in \mathbb{N}^n$. We then wish to let all $N_1, \dots, N_n$ tend to infinity, which will complete steps I and II. For this we must assume that $|q/bc| < 1$ so that we may apply dominated convergence to obtain

$$R_{(\infty^n)}(x; b, c; \underbrace{\infty, \dots, \infty}_{2m \text{ times}}; q) = (q/bc)_\infty D(x; b, c; q) \times \sum_{r^{(1)}, \dots, r^{(m)} \in \mathbb{N}^n} \prod_{i=1}^n \big(q^{1-r_i^{(1)}}/bx_i, q^{1-r_i^{(1)}}/cx_i\big)_{r_i^{(1)}} \prod_{\ell=1}^m q^{|r^{(\ell)}|} f^{(2)}_{r^{(\ell)}, r^{(\ell+1)}}(x; q), \quad (6.13)$$

where $r^{(m+1)} := 0$, $\infty^n = (\underbrace{\infty, \dots, \infty}_{n \text{ times}})$ and $D(x; b, c; q)$ is defined by

$$D(x; b, c; q) := \prod_{i=1}^n \frac{(qx_i^2)_\infty}{(qx_i/b, qx_i/c)_\infty} \prod_{1 \le i < j \le n} (qx_ix_j)_\infty. \quad (6.14)$$

By the replacement

$$(x, w, z) \mapsto (q^{1/2}x, -q^{1/2}/b, -q^{1/2}/c),$$

in the conjectural q-hypergeometric identity for a Littlewood-type sum (5.20) and the use of $f^{(2)}_{r,s}(q^{1/2}x; q) = q^{|r|} f^{(2)}_{r,s}(x; q)$, the right-hand side of (5.20) becomes an expression that matches the right-hand side of (6.13), up to the prefactor $D(x; b, c; q)$. Finally, step III is completed by replacing the alphabet x with $x^{\pm}$. For $|q/bc| < 1$ we then have

$$R_{(\infty^{2n})}(x^{\pm}; b, c; \infty, \dots, \infty; q) = D(x^{\pm}; b, c; q) \sum_{\substack{\lambda \\ \lambda_1 \le 2m}} q^{|\lambda|/2} h_\lambda^{(m)}\big(-q^{1/2}/b, -q^{1/2}/c; q\big) P'_\lambda(x^{\pm}; q). \quad (6.15)$$

where earlier we proved the conjectural (5.20) for $w = 0$ and so (6.15) holds for $b \to \infty$.

The case for $x = (x_1, \dots, x_{n-1}, 1)$ follows by exactly the same argument until (6.15) where ∞^{2n} is replaced by ∞^{2n-1}.

The left-hand side

We will now demonstrate that after the three-step procedure at the start of this chapter, the expression on the left-hand side of the C_n Andrews transformation becomes the left-hand side of (6.1a) when $x = (x_1, \ldots, x_n)$, or (6.1d) when $x = (x_1, \ldots, x_{n-1}, 1)$. In the execution of this procedure we are able to maintain finite bounds and full generality of the parameters $b_2, c_2, \ldots, b_{m+1}, c_{m+1}$ until the very end of our calculations and so, for this reason, we will carry out steps I, II and III in the order III, I, II.

Bilateralisation of the left-hand side of the C_n Andrews transformation

Recall that steps II and III amount to the restoration of signed-permutation symmetry and the bilaterialisation of the unilateral sum on the left-hand side of the C_n Andrews transformation. It is the carrying out of step III that demands the most effort in the following considerations.

Let the left-hand side of C_n Andrews be denoted more simply as

$$L_N(x; b, c; b_2, c_2, \ldots, b_{m+1}, c_{m+1}; q) =: L_N(x).$$

The notation $L_N(x)$ suppresses all but N and x as these are the only parameters that will have an active role until we later come to steps I and II.

We now outline our approach to step III. The main obstacle in the change of alphabet $x \to x^{\pm}$ arises from the denominator factor $\Delta_{C}(x)$, which vanishes whenever the product of two variables in x is 1. This obstacle may be overcome through a procedure in which we first double the alphabet $(x_1, \ldots, x_n) \to (x_1, y_1, \ldots, x_n, y_n)$ and then iteratively perform the limit $y_i \to x_i^{-1}$ for $1 \le i \le n$. This doubled alphabet is of course accompanied by a doubling of the number of summation indices and the number of summation bounds. We will denote the result of this procedure by

$$\lim_{y_n \to x_n^{-1}, \ldots, y_1 \to x_1^{-1}} L_{(N_1, M_1, \ldots, N_n, M_n)}\big(x_1, y_1, \ldots, x_n, y_n\big) =: L_{M,N}(x), \tag{6.16}$$

and, in Figure 6.1, we give a more precise summary of the features of the intermediate expressions at each stage. The application of step III in the $x = (x_1, \ldots, x_{n-1}, 1)$ case is very similar, and is achieved by carrying out the limit

$$\lim_{x_n \to 1, y_{n-1} \to x_{n-1}^{-1}, \ldots, y_1 \to x_1^{-1}} L_{(N_1, M_1, \ldots, N_n, M_n)}\big(x_1, y_1, \ldots, x_{n-1}, y_{n-1}, x_n\big) =: \hat{L}_{M,N}(\hat{x}), \tag{6.17}$$

where $\hat{x} = (x_1, \ldots, x_{n-1})$.

We now give the result of completing step III and provide the details of the computation shortly thereafter.

Step	III			II
	Initial	Doubled alphabet	$\lim y \to x^{-1}$	$N, M \to \infty$
Variables	x	$x_1, y_1, \ldots, x_n, y_n$	$x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}$	x
Bounds	$0 \subseteq r \subseteq N$	$0 \subseteq r, s \subseteq N, M$	$-M \subseteq r \subseteq N$	$-\infty \subseteq r \subseteq \infty$
Symmetry	S_n	S_{2n}	$(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$	$(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$

Figure 6.1: The features of left-to-right-sequential intermediate expressions during steps III and II. Note that the the desired bilateralisation of the sum and the introduction of signed-permutation symmetry are coupled together in the limit $y \to x^{-1}$.

Proposition 6.6. *For $x = (x_1, \ldots, x_n)$ and $M, N \in \mathbb{N}^n$,*

$$L_{M,N}(x) = \sum_{r \in \mathbb{Z}^n} \frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)} \prod_{i=1}^{n} \Bigg[\prod_{\ell=1}^{m+1} \frac{(b_\ell x_i, c_\ell x_i)_{r_i}}{(qx_i/b_\ell, qx_i/c_\ell)_{r_i}} \Big(\frac{q}{b_\ell c_\ell}\Big)^{r_i} \times \prod_{j=1}^{n} \frac{(q^{-M_j} x_i x_j, q^{-N_j} x_i/x_j)_{r_i}}{(q^{M_j+1} x_i/x_j, q^{N_j+1} x_i x_j)_{r_i}} q^{(M_j+N_j)r_i} \Bigg], \tag{6.18a}$$

and for $\hat{x} = (x_1, \ldots, x_{n-1})$, $x = (x_1, \ldots, x_{n-1}, 1)$ (i.e. $x_n = 1$), $M \in \mathbb{N}^{n-1}$ and $N \in \mathbb{N}^n$,

$$\hat{L}_{M,N}(\hat{x}) = \sum_{r \in \mathbb{Z}^n} \frac{\Delta_{\mathrm{B}}(-xq^r)}{\Delta_{\mathrm{B}}(-x)} \prod_{i=1}^{n} \Bigg[\prod_{\ell=1}^{m+1} \frac{(b_\ell x_i, c_\ell x_i)_{r_i}}{(qx_i/b_\ell, qx_i/c_\ell)_{r_i}} \Big(\frac{q}{b_\ell c_\ell}\Big)^{r_i} \times \prod_{j=1}^{n-1} \frac{(q^{-M_j} x_i x_j)_{r_i}}{(q^{M_j+1} x_i/x_j)_{r_i}} q^{M_j r_i} \prod_{j=1}^{n} \frac{(q^{-N_j} x_i/x_j)_{r_i}}{(q^{N_j+1} x_i x_j)_{r_i}} q^{N_j r_i} \Bigg]. \tag{6.18b}$$

We remark that the sums in (6.18a) and (6.18b) have natural bounds arising from the factors $(q^{M_j+1}x_i/x_j)_{r_i}$ and $(q^{-N_j}x_i/x_j)_{r_i}$, i.e., (6.18a) vanishes unless $-M_i \leq r_i \leq N_i$ for $1 \leq i \leq n$ and similarly, (6.18b) vanishes unless $-M_i \leq r_i \leq N_i$ and $r_n \leq N_n$, for $1 \leq i \leq n-1$.

Note that (6.18a) has the full symmetries of the group of signed permutations up to simultaneous permutation of the bounds of summation, as intended. For example, for $n = 2$,

$$\begin{aligned} L_{(M_1,M_2),(N_1,N_2)}(x_1, x_2) &= L_{(M_2,M_1),(N_2,N_1)}(x_2, x_1) = \\ L_{(M_1,N_2),(N_1,M_2)}(x_1, x_2^{-1}) &= L_{(N_2,M_1),(M_2,N_1)}(x_2^{-1}, x_1) = \\ L_{(N_1,M_2),(M_1,N_2)}(x_1^{-1}, x_2) &= L_{(M_2,N_1),(N_2,M_1)}(x_2, x_1^{-1}) = \\ L_{(N_1,N_2),(M_1,M_2)}(x_1^{-1}, x_2^{-1}) &= L_{(N_2,N_1),(M_2,M_1)}(x_2^{-1}, x_1^{-1}). \end{aligned}$$

Before we can give a proof of Proposition 6.6, we must develop an instrumental technical lemma. For this purpose we introduce the following function. For $r \in \mathbb{Z}^n$, $0 \le p \le n$ and $M := (M_1, \dots, M_p)$

$$L^{(p)}_{M,N;r}(x) := \frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)} \prod_{i=1}^{n} \Biggl[\prod_{\ell=1}^{m+1} \frac{(b_\ell x_i, c_\ell x_i)_{r_i}}{(qx_i/b_\ell, qx_i/c_\ell)_{r_i}} \Bigl(\frac{q}{b_\ell c_\ell}\Bigr)^{r_i} \times \prod_{j=1}^{n} \frac{(q^{-N_j} x_i/x_j, q^{-M_j} x_i x_j)_{r_i}}{(q^{M_j+1} x_i/x_j, q^{N_j+1} x_i x_j)_{r_i}} \, q^{(M_j+N_j)r_i} \Biggr], \tag{6.19}$$

where $M_{p+1} = \cdots = M_n := 0$. This function has identical vanishing conditions to those of (6.18a).

We use $L^{(p)}_{M,N}(x)$ to denote the sum of $L^{(p)}_{M,N;r}(x)$ over all $r \in \mathbb{Z}$. Observe that

$$L_N(x) = L^{(0)}_{-,N}(x) \tag{6.20}$$

and

$$L_{M,N}(x) = L^{(n)}_{M,N}(x). \tag{6.21}$$

Restricted to the following lemma and the proof of Proposition 6.6, we wish to introduce some important notation. Let $x^{(i)}$ denote the new alphabet derived from the alphabet x in which the variable in the ith *position* of x has been dropped. For example, if $x = (a, b, c)$, then $x^{(1)} = (b, c)$, $x^{(2)} = (a, c)$ and $x^{(3)} = (a, b)$.

Lemma 6.7. *Let $M = (M_1, \dots, M_p)$ and $M' = (M_1, \dots, M_p, N_{p+2})$. For $0 \le p \le n-2$,*

$$\lim_{x_{p+2} \to 1/x_{p+1}} L^{(p)}_{M,N}(x) = L^{(p+1)}_{M',N^{(p+2)}}(x^{(p+2)}). \tag{6.22}$$

Proof of Lemma 6.7. Let us first focus on those numerator and denominator terms in $L^{(p)}_{M,N;r}(x)$ that vanish when $x_{p+2} \to 1/x_{p+1}$. From the numerator, in the product

$$\prod_{i=1}^{n} \prod_{j=p+1}^{n} (x_i x_j)_{r_i},$$

we find the factors

$$(x_{p+1}x_{p+2})_{r_{p+1}} (x_{p+1}x_{p+2})_{r_{p+2}}, \tag{6.23}$$

where r_{p+1} and r_{p+2} are both nonnegative since $M_{p+1} = M_{p+2} = 0$. These factors in turn have the component $(1 - x_{p+1}x_{p+2})^2$ if r_{p+1} and r_{p+2} are both positive, $1 - x_{p+1}x_{p+2}$ if only one of these is positive and 1 if both are zero. From $\Delta_{\mathrm{C}}(xq^r)/\Delta_{\mathrm{C}}(x)$ comes the contribution

$$\frac{1 - x_{p+1}x_{p+2}q^{r_{p+1}+r_{p+2}}}{1 - x_{p+1}x_{p+2}}, \tag{6.24}$$

which is 1 if both r_p and r_{p+1} are zero, but leads to a factor $(1 - x_p x_{p+1})$ in the denominator if (at least) one of r_p, r_{p+1} is positive.

Combining contributions (6.24) and (6.23), it follows that the left-hand side of (6.22) vanishes unless one of r_{p+1} or r_{p+2} is zero. These two nonvanishing remainders are computed individually and then summed to obtain the final result.

It is now a long but elementary exercise to show that

$$\lim_{x_{p+2}\to 1/x_{p+1}} \Big(L^{(p)}_{M,N;r}(x)\big|_{r_{p+2}=0} \Big) = L^{(p+1)}_{M',N^{(p+2)};r^{(p+2)}}\big(x^{(p+2)}\big).$$

It takes only slightly more work show that

$$\lim_{x_{p+2}\to 1/x_{p+1}} \Big(L^{(p)}_{M,N;r}(x)\big|_{r_{p+1}=0} \Big) = L^{(p+1)}_{M',N^{(p+2)};\hat{r}^{(p+1)}}\big(x^{(p+2)}\big),$$

where $\hat{r}^{(i)} := (r_1, \dots, r_{i-1}, -r_{i+1}, r_{i+2}, \dots, r_n)$. In this last calculation we make use of

$$\frac{(a)_{-n}}{(b)_{-n}} = \frac{(q/b)_n}{(q/a)_n} \Big(\frac{b}{a}\Big)^n. \tag{6.25}$$

Recalling the natural bounds $-M_i \le r_i \le N_i$ and $M_{p+1} = \cdots = M_n := 0$, we have

$$\begin{aligned}
\lim_{x_{p+2}\to x_{p+1}^{-1}} L^{(p)}_{M,N}(x) &= \sum_{\substack{r_i \\ i=1,\dots,n \\ i\neq p+1,p+2}} \Bigg(\sum_{\substack{r_{p+1}=0 \\ r_{p+2}=0}}^{N_{p+1}} + \sum_{\substack{r_{p+2}=1 \\ r_{p+1}=0}}^{N_{p+2}} \Bigg) \lim_{x_{p+2}\to x_{p+1}^{-1}} L^{(p)}_{M,N;r}(x) \\
&= \sum_{\substack{r_i \\ i=1,\dots,n \\ i\neq p+1,p+2}} \Bigg(\sum_{r_{p+1}=0}^{N_{p+1}} L^{(p+1)}_{M',N^{(p+2)};r^{(p+2)}}\big(x^{(p+2)}\big) \\
&\qquad + \sum_{r_{p+2}=1}^{N_{p+2}} L^{(p+1)}_{M',N^{(p+2)};\hat{r}^{(p+1)}}\big(x^{(p+2)}\big) \Bigg).
\end{aligned}$$

Note that the case where $r_{p+1} = r_{p+2} = 0$ is captured in the first of the two internal sums. Renaming the summation index r_{p+2} as $-r_{p+1}$, this yields

$$\lim_{x_{p+2}\to x_{p+1}^{-1}} L^{(p)}_{M,N}(x) = \sum_{\substack{-M'_i \le r_i \le N_i \\ i=1,\dots,n \\ i\neq p+2}} L^{(p+1)}_{M',N^{(p+2)};r^{(p+2)}}\big(x^{(p+2)}\big) = L^{(p+1)}_{M',N^{(p+2)}}\big(x^{(p+2)}\big).$$

where $M'_{p+2} = \cdots = M'_n := 0$. □

The proof of Proposition 6.6 may now be completed. We treat separately the derivations of (6.18a) and (6.18b).

Proof of Proposition 6.6. We show that the limit in (6.16) may be carried out merely by repeated application of Lemma 6.7. Observe that by Lemma 6.7, we have

$$\lim_{y_1\to x_1^{-1}} L^{(0)}_{-,(N_1,M_1,\dots,N_n,M_n)}\big(x_1,y_1,\dots,x_n,y_n\big) = L^{(1)}_{(M_1),(N_1,N_2,M_2,\dots,N_n,M_n)}\big(x_1,x_2,y_2,\dots,x_n,y_n\big).$$

Recalling (6.20), it is then easy to see at the level of the following notation that by p applications of Lemma 6.7, we have

$$\begin{aligned}&\lim_{y_n\to x_n^{-1},\dots,y_1\to x_1^{-1}} L^{(0)}_{-,(N_1,M_1,\dots,N_n,M_n)}\big(x_1,y_1,\dots,x_n,y_n\big)\\ &= \lim_{y_n\to x_n^{-1},\dots,y_{p+1}\to x_{p+1}^{-1}} L^{(p)}_{(M_1,\dots,M_p),(N_1,\dots,N_{p+1},M_{p+1},\dots,N_n,M_n)}\big(x_1,\dots,x_{p+1},y_{p+1},\dots,x_n,y_n\big)\end{aligned}$$

We now set p to n and recall (6.21), which completes the derivation of (6.18a).

Now treating (6.18b), we apply Lemma 6.7 $n-1$ times to (6.17), which yields the expression

$$\lim_{x_n\to 1}\sum_{-M\subseteq r\subseteq N}\frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)}\prod_{i=1}^{n}\Bigg[\prod_{\ell=1}^{m+1}\frac{(b_\ell x_i,c_\ell x_i)_{r_i}}{(qx_i/b_\ell,qx_i/c_\ell)_{r_i}}\Big(\frac{q}{b_\ell c_\ell}\Big)^{r_i} \times\prod_{j=1}^{n}\frac{(q^{-N_j}x_i/x_j,q^{-M_j}x_ix_j)_{r_i}}{(q^{M_j+1}x_i/x_j,q^{N_j+1}x_ix_j)_{r_i}}\,q^{(M_j+N_j)r_i}\Bigg],$$

where $M_n:=0$. Letting x_n tend to 1, treating the $r_n=0$ and $r_n>0$ cases of the summand separately, results in

$$\begin{aligned}\hat{L}_{M,N}(\hat{x}) = \sum_{-M\subseteq r\subseteq N} u_{r_n}\frac{\Delta_{\mathrm{B}}(-xq^r)}{\Delta_{\mathrm{B}}(-x)}\prod_{i=1}^{n}\Bigg[&\prod_{\ell=1}^{m+1}\frac{(b_\ell x_i,c_\ell x_i)_{r_i}}{(qx_i/b_\ell,qx_i/c_\ell)_{r_i}}\Big(\frac{q}{b_\ell c_\ell}\Big)^{r_i}\\ &\times\prod_{j=1}^{n-1}\frac{(q^{-M_j}x_ix_j)_{r_i}}{(q^{M_j+1}x_i/x_j)_{r_i}}\,q^{M_jr_i}\prod_{j=1}^{n}\frac{(q^{-N_j}x_i/x_j)_{r_i}}{(q^{N_j+1}x_ix_j)_{r_i}}\,q^{N_jr_i}\Bigg],\end{aligned}$$

where $x=(x_1,\dots,x_{n-1},1)$ (so that $x_n:=1$), $M_n:=0$, $u_0=1$ and $u_i=2$ for $1\le i\le N_n$. Using (6.25) and the fact that for $x_n=1$

$$\frac{\Delta_{\mathrm{B}}(-xq^r)}{\Delta_{\mathrm{B}}(-x)}\bigg|_{r_n\mapsto -r_n} = q^{-(2n-1)r_n}\frac{\Delta_{\mathrm{B}}(-xq^r)}{\Delta_{\mathrm{B}}(-x)},$$

this can be rewritten in exactly the same functional form as the above but now with $M_n:=N_n$ and $u_i=1$ for all $-M_n\le i\le N_n$. □

Recall that we have already applied step III for (6.18a) and (6.18b), and so steps I and II remain. To carry out these steps we let all $b_2, c_2, \ldots, b_{m+1}, c_{m+1}$ tend to infinity, followed by letting all the entries of N and M also tend to infinity, which needs only

$$\lim_{a\to\infty} \frac{(ax)_k}{a^k} = (-x)^k q^{\binom{k}{2}}.$$

This completes the three step procedure on the left-hand side of the C_n transformation and the proof of Theorem 6.1.

Chapter 7

Generalised Macdonald eta-function identities

Prior to the discovery of Kac–Moody Lie algebras, in [Macd72a] Macdonald generalised the Weyl denominator formula (2.7) to the setting of affine root systems:

$$\sum_{w\in W} \operatorname{sgn}(w)\, \mathrm{e}^{w(\rho)-\rho} = \prod_{\alpha\in\Phi^+} (1-\mathrm{e}^{-\alpha})^{\operatorname{mult}(\alpha)}, \tag{7.1}$$

where W is an affine Weyl group, Φ^+ is the set of positive roots and ρ is a Weyl vector. Note that (7.1) is an exact match for the denominator formula that may be obtained from the Weyl–Kac character formula; i.e., (2.14) for $\Lambda = 0$. Macdonald's formula yields an identity for each affine root system, which together are known as the *Macdonald identities*. These results generalise several significant classical identities. For example, the Jacobi triple product (3.24) and the quintuple product identity [GasRa90, Exercise 5.6] correspond to (7.1) for the cases $\mathrm{A}_1^{(1)}$ and $\mathrm{A}_2^{(2)}$, respectively. Macdonald also considered certain specialisations of his denominator formula identities which yield expansions for powers of the famous weight $\frac{1}{2}$ modular form, the *Dedekind eta-function* $\eta(\tau)$:

$$\eta(\tau) = q^{1/24}\prod_{j=1}^{\infty}(1-q^j) = q^{1/24}(q)_\infty,$$

where $q = \exp(2\pi\,\mathrm{i}\,\tau)$ for $\operatorname{Im}(\tau) > 0$. The simplest of these *Macdonald eta-function identities* are for the non-twisted types $\mathrm{X}_r^{(1)}$, which yield summation formulas for $\eta(\tau)^{\dim(\mathrm{X}_r)}$. For example, in [Macd72a, p. 136, (6)] the Macdonald eta-function for $\mathrm{C}_n^{(1)}$ is given by

$$\eta(\tau)^{2n^2+n} = c_0 \sum q^{\frac{\|v\|^2}{4(n+1)}} \prod_{i=1}^{n} v_i \prod_{1\le i<j\le n} \big(v_i^2 - v_j^2\big), \tag{7.2}$$

where the sum is over $v \in \mathbb{Z}^n$ such that $v_i \equiv n-i+1 \pmod{2n+2}$ and we use the notation $c_0 = 1/(1!3!\cdots(2n-1)!)$ and $\|v\|^2 = v_1^2 + \cdots + v_n^2$.

In this chapter we provide combinatorial generalisations for almost all of the Macdonald eta-function identities. These generalisations arise from various choices for the parameters b and c in (6.1a) and (6.1d), subject always to the specialisation $(x_1,\dots,x_n)\to(1,\dots,1)$. Furthermore, we will see that particular results for types $\mathrm{A}_{2n}^{(2)}$, $\mathrm{B}_n^{(1)}$ and $\mathrm{D}_{n+1}^{(2)}$ may also be interpreted as generalisations of the $i=k$ instances of the Andrews–Gordon identities, Bressoud's even-modulus identities (for k odd) and Andrews' generalisation of the Göllnitz–Gordon identities, respectively.

Generalised eta-function identities

To properly state our results, we must first give a small part of the details of the specialisation procedure, the bulk of which appears later. Since Macdonald gave no indication of the methods employed in his specialisations, we develop our own approach using differential operators, which differs from Kac's [Kac90] algebraic technique.

Our efforts are highly concentrated upon the specialisation of $x\to(1,\dots,1)$ inside the sum on the left-hand sides of both (6.1a) and (6.1d); this specialisation is persued only after both b and c have been specialised. The functions under consideration may be uniformly represented by

$$\sum_{r\in\mathbb{Z}^n}\frac{\Delta_{\mathfrak{g}}(xq^r)}{\Delta_{\mathfrak{g}}(x)}\prod_{i=1}^{n}(-1)^{ar_i}q^{Kr_i^2-(n-\gamma/2)r_i}x^{2Kr_i}, \tag{7.3}$$

where K is an integer or half-integer; $a=0$ or 1; $\gamma=0,1,2$; $\mathfrak{g}=\mathrm{B},\mathrm{C},\mathrm{D}$ and $x=(x_1,\dots,x_n)$ or $(-x_1,\dots,-x_{n-1},-1)$. Figure 7.1 tabulates the values of $K,a,\gamma,\mathfrak{g}$ and x for the relevant specialisations of b and c.

In (7.3), it is the denominator factors Δ_{B}, Δ_{C} and Δ_{D} that present an obstacle: every term in these products vanish when any two of the variables in x are equal or reciprocal, or more specifically for our considerations, when $x\to(1,\dots,1)$ or $(-1,\dots,-1)$. This obstacle is overcome by application of Lemma 7.2 which prepares our specialisation technique employing differential operators.

Before the statement of our results there remains a couple of important remarks, and some notation to be introduced.

Some of the following identities depend on conjectural cases of Theorem 6.1. We will use the notation $\stackrel{?}{=}$ instead of $=$ to distinguish conjectural results from those that have been proven. Many of the following identities are also conjectured to hold more generally for m a half-integer. The relevant equations are: (7.7), (7.12), (7.13), (7.15), (7.17), (7.18) and (7.19).

The specialisations of parts of (6.1a) and (6.1d) not already discussed (i.e., $D(x^{\pm};b,c;q)$ and the right-hand side) is almost completely trivial. We point out that in each of the following results, the specialisation of $D(x^{\pm};b,c;q)$ results in

	Spec.		Values in (7.3)				
	b	c	K	a	γ	$\mathfrak{g}$	x
(6.1a)							
	-1	$-q^{\frac{1}{2}}$	$m+n$	0	1	B	$(x_1,\ldots,x_n)$
	∞	-1	$m+n+\frac{1}{2}$	0	1	B	
	∞	∞	$m+n+1$	0	0	C	
	∞	$-q^{\frac{1}{2}}$	$m+n+\frac{1}{2}$	0	0	C	
	1	-1	$m+n$	1	0	D	
(6.1d)							
	∞	∞	$m+n+\frac{1}{2}$	0	1	B	$(-x_1,\ldots,-x_{n-1},-1)$
	-1	$-q^{\frac{1}{2}}$	$m+n-\frac{1}{2}$	0	2	D	
	∞	-1	$m+n$	0	2	D	
	∞	$-q^{\frac{1}{2}}$	$m+n$	0	1	B	

Figure 7.1

the prefactors next to the left-hand sums, up to an additional factor of $q^{\|\rho\|^2/4(K-m)}$ which arises from introduction of the eta-function notation and, in some cases, up to a single q-factorial term absorbed from the right-hand side.

Though the next equations (7.4a) and (7.4b) are merely reformulations of $\Delta(v)$ (2.8a), we follow Macdonald [Macd72a] and introduce the characters $\chi_{\mathrm{B}}(v)$ and $\chi_{\mathrm{D}}(v)$ as

$$\chi_{\mathrm{B}}(v) := \prod_{i=1}^{n} v_i \prod_{1\le i<j\le n} (v_i^2 - v_j^2), \tag{7.4a}$$

$$\chi_{\mathrm{D}}(v) := \prod_{1\le i<j\le n} (v_i^2 - v_j^2), \tag{7.4b}$$

where $v = (v_1,\ldots,v_n)$ and for convenience we set $\chi_{\mathfrak{g}}(v/w) = \chi_{\mathfrak{g}}(v)/\chi_{\mathfrak{g}}(w)$. For example, using this notation (7.2) may be expressed as

$$\eta(\tau)^{2n^2+n} = \sum_{v} q^{\frac{\|v\|^2}{4(n+1)}} \chi_{\mathrm{B}}(v/\rho_{\mathrm{C}}),$$

where we have used the fact that $c_0 = 1/\chi_{\mathrm{B}}(\rho_{\mathrm{C}})$ (a number of facts of this kind are listed later in Lemma 7.1).

Finally, we remark that in many cases Macdonald refers to affine root systems by labels that differ from those which we have employed. Under Macdonald's labelling, the untwisted types $\mathrm{X}_r^{(1)}$ are referred to as X_r. Macdonald's labels for other types differ more radically and are indicated inside brackets adjacent to our labelling.

Type $\mathrm{B}_n^{(1)}$

The following is Macdonald's third eta-function identity for $\mathrm{B}_n^{(1)}$ [Macd72a, p. 135, (6c)]:

$$\eta(\tau/2)^{2n}\eta(\tau)^{2n^2-3n} = \sum_v (-1)^{|v|-|\rho|}\chi_{\mathrm{D}}(v/\rho)\, q^{\frac{\|v\|^2-\|\rho\|^2}{4n-2}+\frac{\|\rho\|^2}{4n-2}}, \tag{7.5}$$

where $\rho = \rho_{\mathrm{D}}$, $v \in \mathbb{Z}^n$ such that $v \equiv \rho \pmod{2n-1}$. We now provide a generalisation of this identity. If we set $b = -1$, $c = -q^{1/2}$ in (6.1d)—note that this choice of (b, c) is one of the conditional cases— and then specialise $x = (x_1, \dots, x_{n-1}, 1)$ to $(1, \dots, 1)$, on the right-hand side we obtain (6.8), except with $x = (1, \dots, 1)$; i.e.,

$$\frac{1}{(-q^{1/2};q^{1/2})_\infty} \sum_{\substack{\lambda \\ \lambda_1 \le 2m}} q^{|\lambda|/2} \bigg(\prod_{i=0}^{2m-1} (-q^{1/2};q^{1/2})_{m_i(\lambda)} \bigg) P'_\lambda(\underbrace{1,\dots,1}_{2n-1 \text{ times}};q). \tag{7.6}$$

Note that again we have introduced the factor $(-q^{1/2};q^{1/2})_\infty$ to match the power of the eta-function in our result with the classical case (7.5). For uniformity of expression on the right-hand side we again employ the convention that $m_0(\lambda) := \infty$.

On the left-hand side of (6.1d), we compute $D(1, \dots, 1; -1, -q^{1/2}; q)$ from (6.1c) and apply the operator $\mathcal{D}_{\mathrm{D},-}$ from Lemma 7.2 to the sum, which leads to

$$\frac{1}{(-q^{1/2};q^{1/2})_\infty (q^{1/2};q^{1/2})_\infty^{2n} (q)_\infty^{2n^2-3n}} \sum_v (-1)^{|v|-|\rho|}\chi_{\mathrm{D}}(v/\rho)\, q^{\frac{\|v\|^2-\|\rho\|^2}{2(2m+2n-1)}},$$

where $\rho = \rho_{\mathrm{D}}$, $v \in \mathbb{Z}^n$ such that $v \equiv \rho \pmod{2m+2n-1}$ and $m \ge 0$. Introducing the Dedekind eta-function notation and putting the two sides yields a generalisation of (7.5):

$$\begin{aligned}\frac{1}{\eta(\tau/2)^{2n}\eta(\tau)^{2n^2-3n}} &\sum_v (-1)^{|v|-|\rho|}\chi_{\mathrm{D}}(v/\rho)\, q^{\frac{\|v\|^2-\|\rho\|^2}{2(2m+2n-1)}+\frac{\|\rho\|^2}{2(2n-1)}} \\ &\stackrel{?}{=} \sum_{\substack{\lambda \\ \lambda_1 \le 2m}} q^{|\lambda|/2} \bigg(\prod_{i=0}^{2m-1} (-q^{1/2};q^{1/2})_{m_i(\lambda)} \bigg) P'_\lambda(\underbrace{1,\dots,1}_{2n-1 \text{ times}};q), \end{aligned} \tag{7.7}$$

where $\rho = \rho_{\mathrm{D}}$, $v \in \mathbb{Z}^n$ such that $v \equiv \rho \pmod{2m+2n-1}$. Note that (7.5) is recovered when $m = 0$. We believe that (7.7) also holds for half-integer m.

For $n = 1$, (7.7) may be interpreted as a generalisation of the $i = k$ case (i.e., Andrews' contribution) of Bressoud's even-modulus counterpart to the Andrews–Gordon identities (3.29) under the further restriction that k is odd. We demonstrate this as follows.

Using the congruence $v = (2m+1)r$, together with the factor $(-q^{1/2}; q^{1/2})$ from the right-hand side, the left-hand side of (7.7) becomes the expression

$$\frac{1}{(q^{1/2};q^{1/2})_\infty}\sum_{r\in\mathbb{Z}}(-q^{m+1/2})^r q^{(2m+1)\binom{r}{2}} = \frac{(q^{m+1/2},q^{m+1/2},q^{2m+1};q^{2m+1})_\infty}{(q)_\infty}, \tag{7.8}$$

where the second expression follows by application of the Jacobi triple product identity (3.24). Recall the identity $n(\lambda) = \sum_{i\geq 1}\binom{\lambda_i'}{2}$ (1.2) and the notation b_λ (1.40). By application of $P'_\lambda(1;q) = q^{n(\lambda)}/b_\lambda(q)$ (which is easy to see from (5.3)) and by recalling the facts that $m_i(\lambda) = \lambda_i' - \lambda_{i+1}'$ and $\ell(\lambda') = \lambda_1 \leq 2m$, the right-hand side of (7.7) becomes

$$\begin{aligned}\sum_{\substack{\lambda\\ \lambda_1\leq 2m}} &\frac{q^{|\lambda|/2+n(\lambda)}}{(q^{1/2};q^{1/2})_{\lambda_1'-\lambda_2'}\cdots(q^{1/2};q^{1/2})_{\lambda_{2m-1}'-\lambda_{2m}'}(q)_{\lambda_{2m}'}}\\ &= \sum_{n_1,\dots,n_{2m}\geq 0}\frac{q^{\frac{1}{2}(N_1^2+\cdots+N_{2m}^2)}}{(q^{1/2};q^{1/2})_{n_1}\cdots(q^{1/2};q^{1/2})_{n_{2m-1}}(q)_{n_{2m}}},\end{aligned} \tag{7.9}$$

where in the second expression we have introduced the integers $n_1,\dots,n_{2m}\geq 0$ such that $\lambda_i' := N_i = n_i + \cdots + n_{2m}$ for $1 \leq i \leq 2m$. Now equating (7.8) and (7.9) and rescaling $q \mapsto q^2$, we obtain Andrews' contribution to Bressoud's identity (3.29) for $k = 2m+1$.

Type $\mathrm{C}_n^{(1)}$

By application of the operator $\mathcal{D}_{\mathrm{C},+}$ from Lemma 7.2, the specialisation $b, c \to \infty$ and $x = (x_1,\dots,x_n)$ to $(1,\dots,1)$ in (6.1a) (or equivalently, x to $(1,\dots,1)$ in (6.4)) yields a generalisation of (7.2) (i.e., [Macd72a, p. 136, (6)]):

$$\frac{1}{\eta(\tau)^{2n^2+n}}\sum_v \chi_{\mathrm{B}}(v/\rho)\,q^{\frac{\|v\|^2-\|\rho\|^2}{4(m+n+1)}+\frac{\|\rho\|^2}{4(n+1)}} = \sum_{\substack{\lambda \text{ even}\\ \lambda_1\leq 2m}} q^{|\lambda|/2}P'_\lambda(\underbrace{1,\dots,1}_{2n \text{ times}};q) \tag{7.10}$$

where $\rho = \rho_{\mathrm{C}}$, $v \in \mathbb{Z}^n$ such that $v \equiv \rho \pmod{2m+2n+2}$ and $m \geq 0$. Note that the right-hand side may be equivalently expressed as

$$\sum_{\substack{\lambda\\ \lambda_1\leq m}} q^{|\lambda|}P'_{2\lambda}(\underbrace{1,\dots,1}_{2n \text{ times}};q).$$

There is another generalisation of (7.2) involving the Cartan matrix of type $\mathrm{A}_{2n} =: [C_{ab}]_{1\leq a,b\leq 2n}$, due to Feigin and Stoyanovsky ($n = 1$) [FeStoy94] and Stoyanovsky ($n > 1$) [Stoy98]:

$$\mathrm{LHS}(7.10) = \sum \frac{q^{\frac{1}{2}\sum_{a,b=1}^{2n}\sum_{i=1}^m C_{ab}R_i^{(a)}R_i^{(b)}}}{\prod_{a=1}^{2n}\prod_{i=1}^m (q)_{r_i^{(a)}}}, \tag{7.11}$$

where the sum is over $r_i^{(a)} \in \mathbb{N}$ for all $1 \le a \le 2n$ and $1 \le i \le m$, and we define the integers $R_i^{(a)} := r_i^{(a)} + \cdots + r_m^{(a)}$ for $1 \le i \le m$.

Type $\mathbf{A}_{2n}^{(2)}$ (or affine $\mathbf{BC}_n$)

By application of the operator $\mathcal{D}_{\mathrm{B},+}$ from Lemma 7.2, if we specialise $b \to \infty$, $c = -1$ and $x = (x_1, \dots, x_n)$ to $(1, \dots, 1)$ in (6.1a) (or equivalently, x to $(1, \dots, 1)$ in (6.6)) we obtain a generalisation of [Macd72a, page 138, (6a)]:

$$\frac{\eta(2\tau)^{2n}}{\eta(\tau)^{2n^2+3n}} \sum_v \chi_{\mathrm{B}}(v/\rho)\, q^{\frac{\|v\|^2-\|\rho\|^2}{2(2m+2n+1)} + \frac{\|\rho\|^2}{2(2n+1)}} = \sum_{\substack{\lambda \\ \lambda_1 \le 2m}} q^{(|\lambda|+l(\lambda_{\mathrm{o}}))/2} P'_\lambda(\underbrace{1, \dots, 1}_{2n \text{ times}}; q) \tag{7.12}$$

where $\rho = \rho_{\mathrm{B}}$ and $v \in (\mathbb{Z}/2)^n$ such that $v \equiv \rho \pmod{2m+2n+1}$. We believe that (7.12) also holds for half-integer m.

Using $\mathcal{D}_{\mathrm{C},+}$, if we specialise $b \to \infty$, $c = -q^{1/2}$ and $x = (x_1, \dots, x_n)$ to $(1, \dots, 1)$ in (6.1a) (or equivalently, x to $(1, \dots, 1)$ in (6.5)) we obtain a generalisation of [Macd72a, p. 138, (6b)]:

$$\frac{1}{\eta(\tau/2)^{2n}\eta(2\tau)^{2n}\eta(\tau)^{2n^2-3n}} \sum_v \chi_{\mathrm{B}}(v/\rho)\, q^{\frac{\|v\|^2-\|\rho\|^2}{2(2m+2n+1)} + \frac{\|\rho\|^2}{2(2n+1)}} = \sum_{\substack{\lambda \\ \lambda_1 \le 2m}} q^{|\lambda|/2} P'_\lambda(\underbrace{1, \dots, 1}_{2n \text{ times}}; q) \tag{7.13}$$

where $\rho = \rho_{\mathrm{C}}$ and $v \in \mathbb{Z}^n$ such that $v \equiv \rho \pmod{2m+2n+1}$. We believe that (7.13) also holds for half-integer m.

Using $\mathcal{D}_{\mathrm{B},-}$, If we let $b, c \to \infty$ in (6.1d) and then specialise $x = (x_1, \dots, x_{n-1}, 1)$ to $(1, \dots, 1)$ we obtain a generalisation of [Macd72a, page 138, (6c)]:

$$\frac{1}{\eta(\tau)^{2n^2-n}} \sum_v (-1)^{|v|-|\rho|} \chi_{\mathrm{D}}(v/\rho)\, q^{\frac{\|v\|^2-\|\rho\|^2}{2(2m+2n+1)} + \frac{\|\rho\|^2}{2(2n+1)}} = \sum_{\substack{\lambda \text{ even} \\ \lambda_1 \le 2m}} q^{|\lambda|/2} P'_\lambda(\underbrace{1, \dots, 1}_{2n-1 \text{ times}}; q) \tag{7.14}$$

where $\rho = \rho_{\mathrm{B}}$ and v is summed over $(\mathbb{Z}/2)^n$ such that $v \equiv \rho \pmod{2m+2n+1}$.

We should interpret (7.14) as a higher-rank generalisation of Andrews–Gordon identities (3.26), for $i = k$. We will see that for $n = 1$ and $m = k - 1$, (7.14) is identically equal to (3.26).

Again using $n(\lambda) = \sum_{i \ge 1} \binom{\lambda'_i}{2}$ (1.2) and now using the congruence $v - \rho = (2k+1)r$, the left-hand side of (7.14) becomes the right-hand side of (3.26):

$$\frac{1}{(q)_\infty} \sum_{r \in \mathbb{Z}} (-q^{k+1})^r q^{(2k+1)\binom{r}{2}} = \frac{(q^k, q^{k+1}, q^{2k+1}; q^{2k+1})_\infty}{(q)_\infty},$$

where the second expression follows by application of the Jacobi triple product identity (3.24), as before. Likewise, using $P'_\lambda(1;q) = q^{n(\lambda)}/b_\lambda(q)$ once again, the right-hand side of (7.14) becomes

$$\sum_{\substack{\lambda \\ \lambda_1 \le k-1}} \frac{q^{\lambda_1'^2+\cdots+\lambda_{k-1}'^2}}{(q)_{\lambda_1'-\lambda_2'}\cdots(q)_{\lambda_{k-1}'}} = \sum_{n_1,\dots,n_{k-1}\ge 0} \frac{q^{N_1^2+\cdots+N_{k-1}^2}}{(q)_{n_1}\cdots(q)_{n_{k-1}}},$$

where in the second expression we have introduced the integers $n_1,\dots,n_{k-1} \ge 0$ such that $\lambda_i' := N_i = n_i + \cdots + n_{k-1}$ for $1 \le i \le k-1$. This last expression may be immediately identified with the left-hand side of (3.26) by the relabelling $(N,n) = (M,n)$.

We point out that Warnaar and Zudilin recently discovered a conjectural formula (proven for $m = 1$) for the left-hand side of (7.14) in terms of Cartan matrix of A_{2n-1} [WarZu12, Theorem 4.1]:

$$\mathrm{LHS}(7.14) = \sum \frac{q^{\frac{1}{2}\sum_{a,b=1}^{2n-1}\sum_{i=1}^{m} C_{ab} R_i^{(a)} R_i^{(b)}}}{\prod_{a=1}^{2n-1}\prod_{i=1}^{m}(q)_{r_i^{(a)}}},$$

where the sum is over $r_i^{(a)} \in \mathbb{N}$ for all $1 \le a \le 2n-1$ and $1 \le i \le m$, and we define the integers $R_i^{(a)} := r_i^{(a)} + \cdots + r_m^{(a)}$ for $1 \le i \le m$. We note that this expression has exactly the same functional form as Feigen and Stoyanovsky's generalisation of Macdonald's $\mathrm{C}_n^{(1)}$ eta-function identity (7.11).

Type $\mathrm{A}_{2n-1}^{(2)}$ (or $\mathrm{B}_n^\vee$)

By application of the operator $\mathcal{D}_{\mathrm{D},-}$ from Lemma 7.2, if we let $b \to \infty$, $c \to -1$ in (6.1d) and then specialise $x = (x_1,\dots,x_{n-1},1)$ to $(1,\dots,1)$ we obtain a generalisation of [Macd72a, page 136 (6b)]

$$\frac{\eta(2\tau)^{2n-1}}{\eta(\tau)^{2n^2+n-1}} \sum (-1)^{\frac{|v|-|\rho|}{2(m+n)}} \chi_{\mathrm{D}}(v/\rho)\, q^{\frac{\|v\|^2-\|\rho\|^2}{4(m+n)}+\frac{\|\rho\|^2}{4n}} = \sum_{\substack{\lambda \\ \lambda_1 \le 2m}} q^{(|\lambda|+l(\lambda_{\mathrm{o}}))/2} P'_\lambda(\underbrace{1,\dots,1}_{2n-1 \text{ times}};q), \quad (7.15)$$

where $\rho = \rho_{\mathrm{D}}$, $v \in \mathbb{Z}^n$ such that $v \equiv \rho \pmod{2m+2n}$. Using $\mathcal{D}_{\mathrm{D},+}$, a somewhat different generalisation of the same eta-function identity arises if we take $b = -c = 1$ in (6.1a)—which again is one of the conditional cases—then use (5.16)

$$h_\lambda^{(m)}(-q^{1/2},q^{1/2};q) = \begin{cases} q^{l(\lambda_{\mathrm{o}})/2} \prod_{i=1}^{2m-1} (q;q^2)_{\lceil m_i(\lambda)/2 \rceil} & \text{for } m_{2i-1}(\lambda) \text{ even} \\ 0 & \text{otherwise,} \end{cases} \quad (7.16)$$

and $h_0^{(0)}(-q^{1/2}, q^{1/2}; q) = (-q)_\infty = (q^2; q^2)_\infty/(q)_\infty$, and finally specialise $x = (x_1, \dots, x_n)$ to $(1, \dots, 1)$. Then

$$\text{LHS(7.15)} \stackrel{?}{=} \sum_{\substack{\lambda \\ \lambda_1 \le 2m \\ (\lambda_o)' \text{ is even}}} q^{(|\lambda|+l(\lambda_o))/2} \left(\prod_{i=0}^{2m-1} (q; q^2)_{\lceil m_i(\lambda)/2 \rceil} \right) P'_\lambda(\underbrace{1, \dots, 1}_{2n \text{ times}}; q), \tag{7.17}$$

where again $m_0(\lambda) := \infty$. We believe that both (7.15) and (7.17) also hold for half-integer m.

Type $\mathrm{D}_{n+1}^{(2)}$ (or $\mathrm{C}_n^\vee$)

By application of the operator $\mathcal{D}_{\mathrm{B},+}$ from Lemma 7.2, if we specialise $b \to -1, c \to -q^{1/2}$ and $x = (x_1, \dots, x_n)$ to $(1, \dots, 1)$ in (6.1a) (or equivalently, x to $(1, \dots, 1)$ in (6.10)) we obtain a generalisation of [Macd72a, page 137, (6a)]:

$$\frac{1}{\eta(\tau)^{2n+1}\eta(2\tau)^{2n^2-n-1}} \sum_v \chi_{\mathrm{B}}(v/\rho)\, q^{\frac{\|v\|^2 - \|\rho\|^2}{2(m+n)} + \frac{\|\rho\|^2}{2n}} \stackrel{?}{=} \sum_{\substack{\lambda \\ \lambda_1 \le 2m}} q^{|\lambda|} \left(\prod_{i=0}^{2m-1} (-q)_{m_i(\lambda)} \right) P'_\lambda(\underbrace{1, \dots, 1}_{2n \text{ times}}; q^2), \tag{7.18}$$

where $\rho = \rho_{\mathrm{B}}$, $v \in (\mathbb{Z}/2)^n$ such that $v \equiv \rho \pmod{2m + 2n}$. We believe that (7.18) also holds for half-integer m.

Using $\mathcal{D}_{\mathrm{B},-}$, if we let $b \to \infty$ and $c = -q^{1/2}$ in (6.1d), then specialise $(x_1, \dots, x_{n-1}, 1) = (1, \dots, 1)$, and finally replace $q \mapsto -q$ we obtain a generalisation of [Macd72a, page 137, (6b)]:

$$\frac{1}{\eta(\tau)^{2n-1}\eta(4\tau)^{2n-1}\eta(2\tau)^{2n^2-5n+2}} \sum_v (-1)^{\frac{|v|-|\rho|}{2(m+n)}} \chi_{\mathrm{D}}(v/\rho)\, q^{\frac{\|v\|^2 - \|\rho\|^2}{2(m+n)} + \frac{\|\rho\|^2}{2n}} = \sum_{\substack{\lambda \\ \lambda_1 \le 2m}} q^{|\lambda|} P'_\lambda(\underbrace{1, \dots, 1}_{2n-1 \text{ times}}; q^2), \tag{7.19}$$

again with v as in (7.18). We believe that (7.19) also holds for half-integer m.

This last eta-function identity should be viewed as a higher-rank generalisation of Andrews' generalised Göllnitz–Gordon q-series (3.31); i.e., for $n = 1$ and $m = k - 1$, (7.19) is identically equal to (3.31). This fact is not quite as easy to see as the earlier generalisations of the Andrews–Gordon and Bressoud identities. To obtain the desired form, after specialisation the right-hand side of (7.19) must be transformed using an identity due to Bressoud, Ismail and Stanton.

By the congruence $v-\rho = 2kr$, the left-hand side of (7.19) becomes the right-hand side of (3.31):

$$\frac{(q^2;q^4)_\infty}{(q)_\infty}\sum_{r\in\mathbb{Z}}(-q^{2k+1})^r q^{4k\binom{r}{2}} = \frac{(q^{2k-1},q^{2k+1},q^{4k};q^{4k})_\infty (q^2;q^4)_\infty}{(q)_\infty}$$

where the second expression follows by application of the Jacobi triple product identity (3.24) once again. Using the now familiar identity $P'_\lambda(1;q) = q^{n(\lambda)}/b_\lambda(q)$, the right-hand side of (7.19) may be rewritten as

$$\sum_{\substack{\lambda \\ \lambda_1\le 2k-2}} \frac{q^{|\lambda|}q^{2n(\lambda)}}{(q^2;q^2)_{\lambda'_1-\lambda'_2}\cdots(q^2;q^2)_{\lambda'_{2k-2}}} = \sum_{n_1,\dots,n_{2k-2}\ge 0}\frac{q^{N_1^2+\cdots+N_{2k-2}^2}}{(q^2;q^2)_{n_1}\cdots(q^2;q^2)_{n_{2k-2}}}$$

where in the second expression we have introduced the integers $n_1,\dots,n_{2k-2}\ge 0$ such that $\lambda'_i := N_i = n_i+\cdots+n_{2k-2}$ for $1\le i\le 2k-2$. By application of [BrIsStan00, Theorem 5.1] for $i=k$, $a=1$ and $q\mapsto q^2$ we obtain

$$\sum_{n_1,\dots,n_{k-1}\ge 0}\frac{q^{2(N_1^2+\cdots+N_{k-1}^2)}(-q^{1-2N_1};q^2)_{N_1}}{(q^2;q^2)_{n_1}\cdots(q^2;q^2)_{n_{k-1}}},$$

which under the identification $(N,n)=(M,m)$ is the left-hand side of (3.31).

Details of specialisation procedure

In this section we prepare our specialisation lemma and demonstrate its use with an example.

Preliminaries

Using the Vandermonde determinant (2.8a), the characters (7.20) also have the following determinant forms:

$$\chi_{\mathrm{B}}(v) = (-1)^{\binom{n}{2}}\det_{1\le i,j\le n}\Big(v_i^{2j-1}\Big), \tag{7.20a}$$

$$\chi_{\mathrm{D}}(v) = (-1)^{\binom{n}{2}}\det_{1\le i,j\le n}\Big(v_i^{2j-2}\Big). \tag{7.20b}$$

To aid the calculations in our specialisation lemma we first provide a simple lemma containing a number of evaluations of these characters for various Weyl vectors (2.9).

Lemma 7.1. *We have*

$$\chi_{\mathrm{B}}(\rho_{\mathrm{B}}) = \frac{1}{2^n}\prod_{i=0}^{n-1}(2i+1)! \qquad \chi_{\mathrm{B}}(\rho_{\mathrm{C}}) = \prod_{i=0}^{n-1}(2i+1)!$$

$$\chi_{\mathrm{D}}(\rho_{\mathrm{B}}) = \prod_{i=0}^{n-1}(2i)! \qquad \chi_{\mathrm{D}}(\rho_{\mathrm{D}}) = \frac{1}{2^{n-1}}\prod_{i=0}^{n-1}(2i)!$$

Proof. These four simple facts require only elementary methods. We only give a proof of a typical example, in which most of the work is done by relabelling indices $j \mapsto j+i$ and then $i \mapsto n-i+1$.

$$\begin{aligned}\chi_{\mathrm{B}}(\rho_{\mathrm{C}}) &= \prod_{i=1}^{n}(n-i+1)\prod_{i=1}^{n}\prod_{j=i+1}^{n}\left((n-i+1)^2-(n-j+1)^2\right)\\ &= \prod_{i=1}^{n} i \prod_{j=1}^{i-1} i^2-(i-j)^2 = \prod_{i=1}^{n} i\prod_{j=1}^{i-1} j(2i-j)\\ &= \prod_{i=0}^{n-1}(2i+1)! \end{aligned}$$ □

We now proceed to our specialisation lemma.

Lemma 7.2 (Specialisation lemma). *Let $D_{x_i} := \partial/\partial x_i$, $m_{\mathrm{B}} = 1$, $m_{\mathrm{C}} = 0$ and $m_{\mathrm{D}} = 2$. Define the differential operators $\mathcal{D}_{\mathrm{C},\pm}$ and $\mathcal{D}_{\mathrm{B},+}$ (acting on formal Laurent series g in $x_1, \ldots, x_n$) by*

$$\mathcal{D}_{\mathfrak{g},\pm}\, g = D_{x_n}^{2n-1}\cdots D_{x_2}^{3} D_{x_1}^{1}\, \mathrm{e}^{(m_{\mathfrak{g}}/2-1)\sum_{i=1}^{n} x_i}\, g(\pm\,\mathrm{e}^{x})\Big|_{x_1=\cdots=x_n=0}, \tag{7.21}$$

and the operators $\mathcal{D}_{\mathrm{D},\pm}$ and $\mathcal{D}_{\mathrm{B},-}$ by

$$\mathcal{D}_{\mathfrak{g},\pm}\, g = D_{x_n}^{2n-2}\cdots D_{x_2}^{2} D_{x_1}^{0}\, \mathrm{e}^{(m_{\mathfrak{g}}/2-1)\sum_{i=1}^{n} x_i}\, g(\pm\,\mathrm{e}^{x})\Big|_{x_1=\cdots=x_n=0}. \tag{7.22}$$

If g is given by

$$g(x) = f(x)\Delta_{\mathfrak{g}}(x)\prod_{i=1}^{n} x_i^{1-n} \tag{7.23}$$

with f generic (i.e., g is free of zeros in the denominator at $x_i = x_j^{\pm}$ for $1 \le i < j \le n$ and $x_i = \pm 1$ for $1 \le i < j \le n$), then

$$\mathcal{D}_{\mathrm{C},\pm}\, g = 2^n(-1)^n(\mp 1)^{\binom{n}{2}} f(\pm 1, \ldots, \pm 1)\, \chi_{\mathrm{B}}(\rho_{\mathrm{C}}). \tag{7.24a}$$

$$\mathcal{D}_{\mathrm{B},+}\, g = 2^n(-1)^{n+\binom{n}{2}} f(1, \ldots, 1)\, \chi_{\mathrm{B}}(\rho_{\mathrm{B}}) \tag{7.24b}$$

$$\mathcal{D}_{\mathrm{D},\pm}\, g = 2^n(\mp 1)^{\binom{n}{2}} f(\pm 1, \ldots, \pm 1)\, \chi_{\mathrm{D}}(\rho_{\mathrm{D}}) \tag{7.24c}$$

$$\mathcal{D}_{\mathrm{B},-}\, g = 2^n f(-1, \ldots, -1)\, \chi_{\mathrm{D}}(\rho_{\mathrm{B}}) \tag{7.24d}$$

We give a proof for the $\mathcal{D}_{\mathrm{C},\pm}$ case only. Note that in this calculation (and the completely analogous working for the other cases), the factors $\mathrm{e}^{m_{\mathfrak{g}}/2-\sum_{i=1}^{n} x_i}$ and $\prod_{i=1}^{n} x_i^{1-n}$ play a purely passive role. In later calculations, their presence will be convenient.

Proof of Lemma 7.2. For $1 \le k \le n$, define $\big(\mathcal{D}^{(k)}_{\mathrm{C},\pm}\, g\big)(x)$ recursively as follows

$$\big(\mathcal{D}^{(k)}_{\mathrm{C},\pm}\, g\big)(x) = \big(D^{2k-1}_{x_k}\, \mathrm{e}^{-x_i}\, \mathcal{D}^{(k-1)}_{\mathrm{C},\pm}\, g\big)(\pm\, \mathrm{e}^{x})\Big|_{x_k=0},$$

where

$$\big(\mathcal{D}^{(0)}_{\mathrm{C},\pm}\, g\big)(x) = g(\pm\, \mathrm{e}^{x}).$$

Clearly,

$$\big(\mathcal{D}^{(n)}_{\mathrm{C},\pm}\, g\big)(x) = \big(\mathcal{D}_{\mathrm{C},\pm}\, g\big)(x).$$

We will now find a closed form for this recursion. At the kth iteration we have the following expression

$$\begin{aligned}\big(\mathcal{D}^{(k)}_{\mathrm{C},\pm}\, g\big)(x) =&(-2)^{k-1}(\mp 1)^{\binom{k}{2}+k(n-1)} \prod_{i=1}^{k-1}(2i-1)! \\ &\times \Delta_{\mathrm{C}}(\pm\, \mathrm{e}^{x_{k+1}},\dots,\pm\, \mathrm{e}^{x_n}) \prod_{i=k+1}^{n} (1-\mathrm{e}^{x_i})^{2(k-1)}\, \mathrm{e}^{(1-n)x_i} \\ &\times D^{2k-1}_{x_k}\Bigg[(1-\mathrm{e}^{x_k})^{2k-1}(1+\mathrm{e}^{x_k})\, \mathrm{e}^{-nx_k} \\ &\qquad \times f(\pm 1,\dots,\pm 1,\pm\, \mathrm{e}^{x_k},\dots,\pm\, \mathrm{e}^{x_n}) \\ &\qquad\qquad \times \prod_{i=k+1}^{n} (\mathrm{e}^{x_k}-\mathrm{e}^{x_i})(1-\mathrm{e}^{x_k+x_i})\Bigg]\Bigg|_{x_k=0}. \qquad (7.25)\end{aligned}$$

The action of the differential operator $D^{2k-1}_{x_k}$ on the square-bracketed factor is quite easy to describe. We repeatedly apply the product rule, and branch to a depth of $2k-1$. By looking ahead to the specialisation $x_k = 0$, and observing that under this specialisation $(1-\mathrm{e}^{x_k})$ vanishes, we need only write down the unique term containing the $(2k-1)$th derivative of $(1-\mathrm{e}^{x_k})^{2k-1}$. It then follows that

$$\begin{aligned}\big(\mathcal{D}^{(k)}_{\mathrm{C},\pm}\, g\big)(x) =&(-2)^{k}(\mp 1)^{\binom{k}{2}+k(n-1)} \prod_{i=1}^{k}(2i-1)! \\ &\times \Delta_{\mathrm{C}}(\pm\, \mathrm{e}^{x_{k+1}},\dots,\pm\, \mathrm{e}^{x_n}) \prod_{i=k+1}^{n} (1-\mathrm{e}^{x_i})^{2k}(\mathrm{e}^{x_i})^{1-n} \\ &\qquad \times f(\pm 1,\dots,\pm 1,\pm\, \mathrm{e}^{x_{k+1}},\dots,\pm\, \mathrm{e}^{x_n}). \qquad (7.26)\end{aligned}$$

By setting $k = n$ and using Lemma 7.1, we obtain the result. □

Example: specialisation for type $\mathbf{C}^{(1)}_n$

We now provide the complete details of the specialisation $(x_1,\dots,x_n) \to (1,\dots,1)$ in the left-hand sum of (6.1a) for the case $b, c \to \infty$. By (7.3) and Figure 7.1, we will specialise the function

$$f(x) := \sum_{r\in\mathbb{Z}^n} \frac{\Delta_{\mathrm{C}}(xq^r)}{\Delta_{\mathrm{C}}(x)} \prod_{i=1}^{n} q^{Kr_i^2-nr_i} x_i^{2Kr_i}, \tag{7.27}$$

where $K = m+n+1$. This corresponds to the generalised Macdonald eta-function identity of type $\mathrm{C}_n^{(1)}$ (7.10). The proofs for the other affine types may be completed by a completely analogous procedure, up to the changes specified in the statement of each case.

Specialisation of (7.27). By substitution of $f(x)$ into (7.23) we have

$$g(x) = \sum_{r\in\mathbb{Z}^n} \Delta_{\mathrm{C}}(xq^r) \prod_{i=1}^{n} q^{Kr_i^2-nr_i} x_i^{2Kr_i+1-n}. \tag{7.28}$$

In the next few steps we prepare (7.28) for the application of the differential operator $\mathcal{D}_{\mathrm{C},+}$, so that we may use Lemma 7.2. We expand the factor $\Delta_{\mathrm{C}}(xq^r)$ using (2.8c) and, by appeal to multilinearity, this yields

$$g(x) = \det_{1\le i,j\le n}\bigg(\sum_{r\in\mathbb{Z}} q^{Kr^2-(n-j+1)r} x_i^{2Kr-(n-j)} - \sum_{r\in\mathbb{Z}} q^{Kr^2+(n-j+1)r} x_i^{2Kr+n-j+2}\bigg).$$

Introducing $\rho = \rho_{\mathrm{C}}$ (2.9c) and sending $r \mapsto -r$ in the second sum we obtain

$$g(x) = \det_{1\le i,j\le n}\bigg(\sum_{r\in\mathbb{Z}} q^{Kr^2-\rho_j r} x_i \big(x_i^{2Kr-\rho_j} - x_i^{-2Kr+\rho_j}\big)\bigg).$$

We then further introduce the variables $v_i = \rho_i - 2Kr$, and note that $Kr^2 - \rho_i r = \frac{1}{4K}(v_i^2 - \rho_i^2)$, which leads to

$$g(x) = x_1\cdots x_n \sum q^{\frac{\|v\|^2-\|\rho\|^2}{4K}} \det_{1\le i,j\le n}\Big(x_j^{-v_i} - x_j^{v_i}\Big),$$

where the sum is over all $v\in\mathbb{Z}^n$ such that $v_i \equiv \rho_i \pmod{2K}$. By definition of the differential operator $\mathcal{D}_{\mathrm{C},+}$ (7.21) we have

$$\mathcal{D}_{\mathrm{C},+}\, g(x) = \sum q^{\frac{\|v\|^2-\|\rho\|^2}{4K}} \det_{1\le i,j\le n}\Big(D_{x_j}^{2j-1}\big(\mathrm{e}^{-v_i x_j} - \mathrm{e}^{v_i x_j}\big)\Big)\bigg|_{x_1=\cdots=x_n=0}.$$

After computing the derivative and setting $x_1 = \cdots = x_n = 0$, this yields

$$\mathcal{D}_{\mathrm{C},+}\, g(x) = (-2)^n \sum q^{\frac{\|v\|^2-\|\rho\|^2}{4K}} \det_{1\le i,j\le n}\Big(v_i^{2j-1}\Big).$$

Finally, we use (7.20a) and (7.24a) to obtain the desired expression. □

Bibliography

[AgAnBr87] A. K. Agarwal, G. E. Andrews and D. M. Bressoud, *The Bailey lattice*, J. Indian Math. Soc. (N.S.) **51** (1987), 57–73.

[An67] G. E. Andrews, *A generalization of the Göllnitz–Gordon partition theorems*, Proc. Amer. Math. Soc. **18** (1967), 945–952.

[An74] G. E Andrews, *An analytic generalization of the Rogers–Ramanujan identities for odd moduli*, Proc. Nat. Acad. Sci. U.S.A. **71** (1974),4082–4085.

[An75] G. E. Andrews, *Problems and prospects for basic hypergeometric functions*, in *Theory and Application of Special Functions*, pp. 191–224, Math. Res. Center, Univ. Wisconsin, 35, Academic Press, New York, 1975.

[An76] G. E. Andrews, *The Theory of Partitions*, Encyclopedia Math. Appl., Vol. 2, Addison–Wesley, Reading, MA, 1976.

[An79] G. E. Andrews, *Connection coefficient problems and partitions*, Relations between combinatorics and other parts of mathematics, Proc. Sympos. Pure Math. **34** (1979), 1–24.

[An84] G. E. Andrews, *Multiple series Rogers–Ramanujan type identities*, Pacific J. Math. **114** (1984), 267–283.

[An86] G. E. Andrews, *q-series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra.*, CBMS Regional Conf. Series in Mathematics **66**, Amer. Math. Soc., Providence, 1986.

[An00] G. E. Andrews, *Bailey's transform, lemma, chains and tree*, in *Special Functions 2000*, J. Bustoz *et al* eds., Kluwer, 2000.

[AnAsRoy99] G. E. Andrews, R. Askey and R. Roy, *Special Functions*, Encyclopedia of Mathematics and its Applications **71**, Cambridge University Press, Cambridge, 1999.

[AnScWar99] G. E. Andrews, A. Schilling and S. O. Warnaar, *An* A_2 *Bailey lemma and Rogers–Ramanujan type identities*, J. Amer. Math. Soc. **12** (1999), 677–702.

[Ba35] W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge University Press, Cambridge, 1935, reprinted by Stechert-Hafner, New York, 1964.

[Ba48] W. N. Bailey, *Identities of the Rogers–Ramanujan type*, Proc. London Math. Soc. (2) **50** (1948), 1–10.

[BerWar05] A. Berkovich and S. O. Warnaar, *Positivity preserving transformations for q-binomial coefficients*, Trans. Amer. Math. Soc. **357** (2005), 2291–351.

[Bh99] G. Bhatnagar, D_n *basic hypergeometric series*, Ramanujan J. **3** (1999), 175–203.

[BhSchl98] G. Bhatnagar and M. Schlosser, C_n *and* D_n *very-well-poised* ${}_{10}\phi_9$ *transformations*, Constr. Approx. **14** (1998), 531–567.

[Bo98] N. Bourbaki, *Lie groups and Lie algebras*, Chapters 1–3, Elements of Mathematics, Springer–Verlag, Berlin, 1998.

[Bo02] N. Bourbaki, *Lie groups and Lie algebras*, Chapters 4–6, Elements of Mathematics, Springer–Verlag, Berlin, 2002.

[Bo05] N. Bourbaki, *Lie groups and Lie algebras*, Chapters 7–9, Elements of Mathematics, Springer–Verlag, Berlin, 2005.

[Br80] D. M. Bressoud, *An analytic generalization of the Rogers–Ramanujan identities with interpretation*, Quart. J. Math. Oxford Ser. (2) **31** (1980), 385–399.

[Br80b] D. M. Bressoud, *Analytic and combinatorial generalizations of the Rogers–Ramanujan identities*, Mem. Amer. Math. Soc. **24** (1980).

[Br99] D. M. Bressoud, *Proofs and Confirmations, The Story of the Alternating Sign Matrix Conjecture*, Cambridge University Press, Cambridge, 1999.

[BrIsStan00] D. M. Bressoud, M. E. H. Ismail and D. Stanton, *Change of base in Bailey pairs*, Ramanujan J. **4** (2000), 435–453.

[Bu94] L. Butler, *Subgroup Lattices and Symmetric Functions*, Mem. Amer. Math. Soc. **112** (1994).

[Cao09] J. Cao, *New proofs of generating functions for Rogers–Szegő polynomials*, Appl. Math. Comput. **207** (2009), 486–492.

[Cap96] S. Capparelli, *A construction of the level 3 modules for the affine Lie algebra $A_2^{(2)}$ and a new combinatorial identity of the Rogers–Ramanujan type*, Trans. Amer. Math. Soc. **348** (1996), 481–501.

[Ch55] C. Chevalley, *Sur certains groupes simples*, Tôhoku Math. J. (2) **7** (1955), 14–66.

[DeLeTh94] J. Désarménien, B. Leclerc and J.-Y. Thibon, *Hall–Littlewood functions and Kostka–Foulkes polynomials in representation theory*, Sém. Lothar. Combin. **32** (1994), Art. B32c.

[De86] J. Désarménien, *La démonstration des identités de Gordon et MacMahon et de deux identités nouvelles*, Sém Lothar. Combin. **B15a** (1986), 11pp.

[FeStoy94] B. Feigin and A. V. Stoyanovsky, *Quasi-particles models for the representations of Lie algebras and geometry of flag manifold*, Funct. Anal. Appl. **28** (1994), 68–90.

[Fu97] W. Fulton, *Young Tableaux*, London Mathematical Society Student Texts, 35, Cambridge University Press, Cambridge, 1997.

[FuHa91] W. Fulton and J. Harris, *Representation Theory. A First Course.* Graduate Texts in Mathematics, 129. Readings in Mathematics. Springer–Verlag, New York, 1991.

[Fo74] H. O. Foulkes, *A survey of some combinatorial aspects of symmetric functions*, In *Permutations* (Actes Colloq., Univ. René-Descartes, Paris, 1972), Gauthier–Villars, Paris, 1974, pp. 79–92.

[GasRa90] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics and its Applications, Vol. 35, Cambridge University Press, Cambridge, 1990.

[Gar92] A. Garsia, *Orthogonality of Milne's polynomials and raising operators*, Discrete Mathematics **99** (1992),247–264.

[GarPro92] A. M. Garsia and C. Procesi, *On certain graded S_n-modules and the q-Kostka polynomials*, Adv. in Math. **94** (1992), 82–138.

[Gö60] H. Göllnitz, *Einfache Partitionen*, Diplomarbeit W. S. 1960, Gottingen.

[Go61] B. Gordon, *A combinatorial generalization of the Rogers–Ramanujan identities*, Amer. J. Math. **83** (1961), 393–399.

[Go65] B. Gordon, *Some continued fractions of the Rogers–Ramanujan type*, Duke Math. J. **31** (1965), 741–748.

[Gu87] R. A. Gustafson, *The Macdonald identities for affine root systems of classical type and hypergeometric series very-well-poised on semisimple Lie algebras*, in *Ramanujan International Symposium on Analysis*, pp. 187–224, Macmillan of India, New Delhi, 1989.

[Ha08] J. Haglund, *The q,t-Catalan Numbers and the Space of Diagonal Harmonics*, University lecture series, Vol. 41, American Mathematical Society, Providence, RI, 2008.

[Hu72] J. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in mathematics, Vol. 9. Springer-Verlag, New York–Berlin, 1972.

[IsJoZe06] M. Ishikawa, F. Jouhet and J. Zeng, *A generalization of Kawanaka's identity for Hall–Littlewood polynomials and applications*, J. Algebraic Combin. **23** (2006), 395–412.

[J1829] C. G. J. Jacobi, *Fundamenta nova theoriae functionum ellipticarum*, Sumtibis fratrum Borntraeger, Königsberg, 1829.

[Ja62] N. Jacobson, *Lie algebras*, Interscience Publishers, New York–London, 1962.

[Ji91] N. Jing, *Vertex operators and Hall–Littlewood symmetric functions*, Adv. in Math. **87** (1991), 226–248.

[JoZe05] F. Jouhet and J. Zeng, *New identities of Hall–Littlewood polynomials and applications*, Ramanujan J. **10** (2005), 89–112.

[Kac74] V. Kac, *Infinite-dimensional Lie algebras, and the Dedekind η-function*, Func. Anal. Appl. **8** (1974), 68–70.

[Kac90] V. G. Kac, *Infinite-dimensional Lie algebras, Third edition*, Cambridge University Press, Cambridge, 1990.

[KacPe84] V. G. Kac and D. H. Peterson, *Infinite-dimensional Lie algebras, theta functions and modular forms*, Adv. in Math. **53** (1984), 125–264.

[Kaw99] N. Kawanaka, *A q-series identity involving Schur functions and related topics*, Osaka J. Math. **36** (1999), 157–176.

[Ki00] A. N. Kirillov, *New combinatorial formula for modified Hall–Littlewood polynomials*, in *q-Series from a Contemporary Perspective*, 283–333, Contemp. Math. **254**, AMS, Providence, RI, 2000.

[KoSw98] R. Koekoek and R. F. Swarttouw, *The Askey-Scheme of Hypergeometric Orthogonal Polynomials and its q-Analogue*, Delft University of Technology, Faculty of Information Technology and Systems, Department of Technical Mathematics and Informatics, Report no. 98-17, 1998.

[KoLesSw10] R. Koekoek, P. A. Lesky and R. F. Swarttouw, *Hypergeometric Orthogonal Polynomials and their q-Analogues*, Springer Monographs in Mathematics, Springer–Verlag, Berlin, 2010.

[KoTe87] K. Koike and I. Terada, *Young-diagrammatic methods for the representation theory of the classical groups of type B, C and D*, J. Algebra **107** (1987), 466–511.

[Kr98] C. Krattenthaler, *Identities for classical group characters of nearly rectangular shape*, J. Algebra **209** (1998), 1–64.

[LaSchü78] A. Lascoux and M.-P. Schützenberger, *Sur une conjecture de H. O. Foulkes*, C. R. Acad. Sci. Paris Sér. A–B **286** (1978), A323–A324.

[La01] A. Lascoux, *Symmetric functions and combinatorial operators on polynomials*, CBMS regional conference series in mathematics, no. 99, American Mathematical Society with support from the National Science Foundation, 2001.

[La05] A. Lascoux, *Adding* ± 1 *to the argument of a Hall–Littlewood polynomial*, Sém. Lothar. Combin. **54** (2005/07), Art. B54n, 17 pp.

[Lep79] J. Lepowsky, *Generalized Verma modules, loop space cohomology and MacDonald-type identities*, Ann. Sci. Ecole Norm. Sup. (4) **12** (1979), 169–234.

[LepMi78a] J. Lepowsky and S. Milne, *Lie algebras and classical partition identities*, Proc. Nat. Acad. Sci. U.S.A. **75** (1978), 578–579.

[LepMi78b] J. Lepowsky and S. Milne, *Lie algebraic approaches to classical partition identities*, Adv. in Math. **29** (1978), 15–59.

[LepWi78] J. Lepowsky and R. L. Wilson, *A new family of algebras underlying the Rogers–Ramanujan identities and generalizations*, Proc. Nat. Acad. Sci. USA **78** (1981), 7254–7258.

[LepWi82] J. Lepowsky and R. L. Wilson, *A Lie theoretic interpretation and proof of the Rogers–Ramanujan identities*, Adv. in Math. **45** (1982), 21–72.

[LepWi84] J. Lepowsky and R. L. Wilson, *The structure of standard modules. I. Universal algebras and the Rogers–Ramanujan identities*, Inv. Math. **77** (1984), 199–290.

[Li36] D. E. Littlewood, *Polynomial concomitants and invariant matrices*, J. London Math. Soc. S1–11 (1936), 49–55.

[Li50] D. E. Littlewood, *The Theory of Group Characters and Matrix Representations of Groups*, Oxford University Press, 1950.

[LilMi93] G. Lilly and S. Milne, *The C_l Bailey transform and Bailey lemma.* Constr. Approx. **9** (1993), 473–500.

[Macd72a] I. G. Macdonald, *Affine root systems and Dedekind's η-function*, Inv. Math. **15** (1972), 91–143.

[Macd72b] I. G. Macdonald, *The Poincaré series of a Coxeter group*, Math. Ann. **199** (1972), 161–174.

[Macd95] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd edition, Oxford Univ. Press, New York, 1995.

[Macm16] P. A. MacMahon, *Combinatory Analysis*, Vol. 2, Cambridge University Press 1916.

[LauSiZi08] J. Mc Laughlin, A. V. Sills and P. Zimmer, *Dynamic Survey DS15: Rogers-Ramanujan-Slater Type Identities*, Electronic J. Combinatorics (2008) 1–59.

[MePri87] A. Meurman and M. Primc, *Annihilating ideals of standard modules of* $\widetilde{sl(2,\mathbb{C})}$ *and combinatorial identities*, Adv. in Math. **64** (1987), 177–240.

[MePri99] A. Meurman and M. Primc, *Annihilating fields of standard modules of* $\widetilde{sl(2,\mathbb{C})}$ *and combinatorial identities*, Mem. Amer. Math. Soc. **137** (1999), no. 652, 89 pp.

[Mi87] S. Milne, *Basic hypergeometric series very well-poised in $U(n)$*, J. Math. Anal. Appl. **122** (1987), 223–256.

[Mi88] S. Milne, *A q-analog of the Gauss summation theorem for hypergeometric series in* $\mathrm{U}(n)$, Adv. in Math. **72** (1988), 59–131.

[Mi92] S. Milne, *Classical partition functions and the $U(n+1)$ Rogers–Selberg identity*, Discrete Math. **99** (1992), 199–246.

[Mi94] S. Milne, *The C_l Rogers-Selberg identity*, SIAM J. Math. Anal. **25** (1994), 571–595.

[Mi97] S. Milne, *Balanced ${}_3\phi_2$ summation theorems for $U(n)$ basic hypergeometric series*, Adv. in Math. **131** (1997), 93–187.

[MiLil92] S. Milne and G. Lilly, *The A_l and C_l Bailey transform and lemma*, Bull. Amer. Math. Soc. (N.S.) **26** (1992), 258–263.

[MiLil95] S. Milne and G. Lilly, *Consequences of the A_l and C_l Bailey transform and Bailey lemma*, Discrete Math. **139** (1995), 319–346.

[Mo67] R. V. Moody, *Lie algebras associated with generalized Cartan matrices*, Bull. Amer. Math. Soc. **73** (1967), 217–221.

[Ok98] S. Okada, *Applications of minor summation formulas to rectangular-shaped representations of classical groups*, J. Algebra **205** (1998), 337–367.

[Pa82] P. Paule, *Zwei neue Transformationen als elementare Anwendungen der q-Vandermonde Formel*, Ph.D. Thesis, University of Vienna, 1982.

[Pa87] P. Paule, *A note on Bailey's lemma*, J. Combin. Theory Ser. A **44** (1987), 164–167.

[Pa88] P. Paule, *The concept of Bailey chains*, Publ. I.R.M.A. Strasbourg 358/S-18, (1988), 53–76.

[Pro90] R. A. Proctor, *New symmetric plane partition identities from invariant theory work of De Concini and Procesi*, European J. Combin. **11** (1990), 289–300.

[Rog1892] L. J. Rogers, *On the expansion of some infinite products*, Proc. London Math. Soc. **24** (1892), 337–352.

[Rog1894] L. J. Rogers, *Second memoir on the expansion of certain infinite products*, Proc. London Math. Soc. **25** (1894), 318–343.

[RogRa19] L. J. Rogers and S. Ramanujan, *Proof of certain identities in combinatory analysis*, Cambr. Phil. Soc. Proc. **19** (1919), 211–216.

[Ros04] H. Rosengren, *Elliptic hypergeometric series on root systems*, Adv. in Math. **181** (2004), 417–447.

[Schl09] M. J. Schlosser, *Multilateral inversion of A_r, C_r, and D_r basic hypergeometric series*, Ann. Comb. **13** (2009), 341–363.

[Schu17] I. J. Schur, *Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbrüche*, S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl. (1917), 302–321.

[Se36] A. Selberg, *Über einige arithmetische Identitäten*, Avhl. Norske Vid. **8** (1936).

[Si03] A. Sills, *Finite Rogers-Ramanujan type identities*, Electron. J. Combin. **10** (2003), Research Paper 13.

[Sl51] L. J. Slater, *A new proof of Roger's transformation of infinite series*, Proc. London Math. Soc. (2) **53** (1951), 460–475.

[Sl52] L. J. Slater, *Further identities of the Rogers–Ramanujan type*, Proc. London Math. Soc. (2) **54** (1952), 147–167.

[Sl66] L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, 1966.

[So66] L. Solomon, *The orders of the finite Chevalley groups*, J. Algebra **3** (1966), 376–393.

[Stan99] R. P. Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge Studies in Advanced Mathematics, 62, Cambridge University Press, Cambridge, 1999.

[Stem90] J. R. Stembridge, *Hall–Littlewood functions, plane partitions, and the Rogers–Ramanujan identities*, Trans. Amer. Math. Soc. **319** (1990), 469–498.

[Stem90b] J. R. Stembridge, *Nonintersecting paths, Pfaffians, and plane partitions*, Adv. in Math. **83** (1990), 96–131.

[Stoy98] A. V. Stoyanovsky, *Lie algebra deformations and character formulas*, Funct. Anal. Appl. **32** (1998), 66–68.

[Sz26] G. Szegő, *Ein Beitrag zur Theorie der Thetafunktionen*, Sitz. Preuss. Akad., Wiss. Phys. Math. Klasse **19** (1926), 242–252.

[Ve12] V. Venkateswaran, *Vanishing integrals for Hall–Littlewood polynomials*, Transform. Groups **17** (2012), 259–302.

[Wak01] M. Wakimoto, *Lectures on infinite-dimensional Lie algebra*, World Scientific Publishing Co., Inc., River Edge, NJ, 2001.

[War01] S. O. Warnaar, *50 years of Bailey's Lemma*, in *Algebraic Combinatorics and Applications*, Springer, Berlin, 2001, pp. 333–347.

[War06] S. O. Warnaar, *Rogers–Szegő polynomials and Hall–Littlewood symmetric functions*, J. Algebra **303** (2006), 810–830.

[War] S. O. Warnaar, Unpublished.

[WarZu12] S. O. Warnaar and W. Zudilin, *Dedekind's η-function and Rogers–Ramanujan identities*, Bull. Lond. Math. Soc. **44** (2012), 1–11.

[Wat29] G. N. Watson, *A New Proof of the Rogers-Ramanujan Identities* J. London Math. Soc. S1–4 **4** (1929), 4–9.

[We25] H. Weyl, *Theorie der Darstellung kontinuierlicher halb-einfacher Gruppen durch lineare Transformationen I*, Math. Z. **23** (1925), 271–309.

[We26] H. Weyl, *Theorie der Darstellung kontinuierlicher halb-einfacher Gruppen durch lineare Transformationen II & III*, Math. Z. **24** (1926), 328–395.

[Za00] M. Zabrocki, *Ribbon operators and Hall–Littlewood symmetric functions*, Adv. Math. **156** (2000), 33–43.