이학박사학위논문

# Exact Calculations of Indices and Partition Functions for 1D and 2D Supersymmetric Theories and Their String Theory Applications

1 차원과 2 차원 초대칭 이론에서의
지표와 분배 함수의 정확한 계산과 끈이론에의 응용

2014 년 8 월


서울대학교 대학원

물리천문학부

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Abstract

# Exact Calculations of Indices and Partition Functions for 1D and 2D Supersymmetric Theories and Their String Theory Applications

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In this thesis, we introduce two ideas of string theory which can examine geometrical structure of the spacetime compactified on a Calabi-Yau manifold. The first half of the thesis focuses on the Witten index and their applications in string theory. As one of the most interesting example, we review the counting problem of BPS states in four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory obtained from the Calabi-Yau compactification of type II string theory. Especially, we concentrate on how the Witten index can be used to prove the wall-crossing phenomena therein. At the second half, we outline recently revealed relation between two-dimensional partition functions and geometry of the Kahler moduli space of the Calabi-Yau manifolds. We show that the partition function of $\mathcal{N} = (2, 2)$ gauged linear sigma model on $S^2$, $D^2$ and $RP^2$ calculates the Kahler potential, central charge of D-brane and Orientifold of A-model respectively.

keywords: index, partition function, string theory, supersymmety, Calabi-Yau
Student number: 2009-20408

# Contents

# Chapter 1

# Introduction

String theory is a unique quantum theory in which the gravity consistently arises. As we have learned from the Einstein's theory of general relativity, the theory of gravity is closely related to the geometry of the spacetime which has sophisticated mathematical structures. Then what kind of geometry can string theory probe and which mathematics is associated to this new theory of gravity?

In order for the supersymmetric string theory to be well-defined, the spacetime should be ten-dimensional. In addition, if we want to obtain a four-dimensional field theory which preserves certain amount of supersymmetries, the six remaining directions should be compactified on a so called *Calabi-Yau* manifold. This space is defined by a complex manifold $X$ whose first Chern class $c_1(X)$ vanishes.

Understanding the mathematical structure of the Calabi-Yau manifold is extremly important since the field theories in four-dimensional spacetime are determined by the geometry and topology of the compactified manifold $X$. However, since the Calabi-Yau manifold is very complicated space that a single non-trivial metric is not known, the standard geometrical approach cannot be easily applied. Actually, these efforts of string theorists to describe the dynamics of two-dimensional worldsheet theory on Calabi-Yau manifolds offered a lot of new insights to modern mathematics.

The most prominent example is the *mirror symmetry*. This implies the equivalence between two topologically different Calabi-Yau manifold, which leads to a duality between symplectic geometry and complex geometery. It was first discovered by string theoriest [1], and later motivated the beginning of new branch of pure mathematics. Mirror symmetry is very non-trivial duality in mathematical point of view, but can be naturally understood in terms of the string theory. This can be easily seen from the mass spectrum of the string exitation with one direction compactified on a circle of radius $R$,

$$M^2 = \frac{n^2}{R^2} + \frac{m^2 R^2}{\alpha'} + \frac{2}{\alpha}(N + \tilde{N} - 2) \ . \tag{1.1}$$

Here $n$ is a momentum mode and $m$ is the winding mode of the string around the compactified direction. One can easily see that the spectrum is invariant under the exchange $(R, n, m) \leftrightarrow (\sqrt{\alpha'}/R, m, n)$. This is what we call a *T-duality* which is the prototype of the mirror symmetry.

Interestingly, the mirror symmetry relates the classical theory in one side (B-model) and the fully quantum corrected theory (A-model) on the other side. Hence, in order to calculate physical quantities for the A-model, one can use the mirror symmetry and readily obtain the results by looking at their B-model counterparts. However, until very recently, this duality has been proven only for very limited case, when the Calabi-Yau is obtained from the $U(1)$ quotient of the Kahler manifold. [2]

Having understood the structure of the backgroud spacetime, we consider objects which are embedded in the lower-dimensional submanifolds of the Calabi-Yau ambient space. This brings us to the study of D-branes in open string theory. Since the existence of D-branes breaks the translational symmetry of the spacetime, it also breaks the spacetime supersymmetry. In order to preserve the half of the supersymmetry, it should satisfy the BPS (Bogomol'nyi-Prasad-Sommerfield) bound, which imply

$$M = |Z| \ , \tag{1.2}$$

where $Z$ is the central charge of the supersymmetry algebra. This condition is translated into the volume minimizing cycles in the Calabi-Yau ambient space, whose solution is given by solving non-linear differential equations which are extremly difficult to solve in general. But it can be rather easily dealt with in view point of the supersymmetric theory, due to the *linear* killing spinor equations. Since the D-branes wrapping supersymmetric cycles inherit many properties of the ambient space, these objects are also crucial in studying the Calabi-Yau manifolds and mirror symmetry thereof.

Apart from its mathematical importance, studying BPS objects has strong motivations in string theory. Because of its topological nature, these states remain invarient along the continuous change of parameters of the theory. This property enables us to study the non-perturbative aspects of the supersymmetric theories. Since they can probe the strongly coupled regime of the theory as well, it plays a significant role in the proof of various dualities in string theory. Furthermore, BPS states are strongly believed to be candidates of the blackhole microstates, whose origin is one of the central questions that true quantum gravity should be able to answer.

In this thesis, we introduce recent developements of string theory to understand the structure of the Calabi-Yau manifold and D-branes wrapping supersymemtric cycles in it. The key framework is the *sigma model.* Superstring theory can be most easily described by a two-dimensional supersymmetric non-linear sigma model (NLSM) whose target space is the Calabi-Yau manifold $X$. The worldsheet scalar fields $\phi : \Sigma \to X$ provide coordinates of the target space, and the fermions are valued in pull-back of the tangent bundle $TX$. It provides very useful tool to understand the topology of the spacetime, as we shall see in the following chapters.

However, for a complicated manifold such as Calabi-Yau, since we do not have a metric it is difficult to explicitly write down the NLSM Lagrangian and calculate some useful quantities. In order to deal with this situation, Witten [3] introduced the concept of the gauged linear sigma model (GLSM). As the name indicates, it has a linear space such as $\mathbb{C}^N$ as a target space. Interestingly, when we properly choose the field contents and potentials of the theory, and do the renormalization group flow, it reduces to the NLSM whose target space is a Calabi-Yau manifolds at the infrared. This machinary turns out to be very powerful to investigate the special properties of the two-dimensional worldsheet theory such as Calabi-Yau/Landau-Ginzburg correspondence and the mirror symmetry.

Given these frameworks, we introduce two main tools which encode the topological and geometrical informations of the supersymmetric theories. These are the *Witten index* and *partition function*. Supersymmetric theories have exceptional property that theses quantities are *exactly* calculable. First of all, the Witten index is defined by the expression

$$\mathrm{Tr}(-1)^F \ , \tag{1.3}$$

where $F$ is the fermion number operator. Originally, the concept of the index was first introduced by mathematicians Atiyah and Singer [4] at the beginning of 60's, in order to characterize the topological properties of the solution space of differential equations. Witten later found that there are similar mathematical structures in the supersymmetric quantum mechanics, where (1.3) can be used as a measure of spontaneous breaking of the supersymmetry. [5] Furthermore, this quantity turns out to be topological, i.e., invariant under the continuous deformation of the theory. This enables us to deform the theory to the particular limit where we can exactly calculate this quantity.

The Witten index has tons of applications in various supersymmetric theories. The most prominent example is the gauge/gravitational anomalies in quantum field theory. Alvarez-Gaume and Witten [6] translated the anomaly calculation problem into the index theorem of the supersymmetric NLSM. From this work, one can easily extract one-loop anomalies of each field contents in a quantum field theory just by considering the Atiyah-Singer index theorem.

Secondly, one can use the Witten index to count the number of BPS states, so that it can be used to probe the non-perturbative properties of the field theories. Interesting observation is that, for theories with small number of supersymmetries, the number of BPS states are not constant, but only a piecewise constant on the moduli space. There exists a co-dimension one wall in the moduli space, and certain BPS states abruptly disappears across the wall. This is the *wall-crossing* of the BPS states. Understanding this phenomenon is of particular importance both in physics and mathematics, and we will see that again the Witten index and its variation play central roles in here.

More recently, the partition function arose as another powerful tool for probing the geometry of the Calabi-Yau space. From the pioneering work of Pestun [7], there has been much progress on calculating the partition funtions on spheres in various dimensions. For theories with superconformal symmetry, one can consistently map the flat Lagrangian on a sphere with proper curvature corrections. Via the localization procedure, it is possible to exactly calculate the partition functions for these theories.

Along the line of these developement, partition function of $\mathcal{N} = (2,2)$ GLSM on two-sphere has been calculated. [8, 9] Surprisingly, it turns out that this quantity calculates the exact Kahler potential of the A-model moduli space, which has been extremly difficult to probe due to the worldsheet instanton contributions.

This relation can be further investigated to the worldsheet with a boundary or a crosscap, which corresponds to the D-branes or Orientifolds wrapping the subsycles of the Calabi-Yau manifold. In this case, the corresponding GLSM is written on a hemisphere or a real projective plane respectively. We will see that, for these cases as well, the partition functions provide very useful information on the D-branes or Orientifolds coupled to the spacetime curvatures.

This thesis is organized as follow.

Chapter 2 summarizes the prerequisites which are required for discussions in the following sections. First of all, we present the mathematical and physical definitions of the index and relations between them. We provide various $0+1$ dimensional NLSM Lagrangian, and see how each of them can be used to derive the index theorems for various operators. Most importantly, we give the supersymmetric proof of the Atiyah-Singer index theorem, which will be prequently used in the various physical situations. After that, as a direct application of index theorem, we review the pioneering work of Alvarez-Gaume and Witten calculating the gauge/gravitational anomalies in quantum field theories. At the last section of this chapter, we include the brief summary of the technique of supersymmetric localization developed by Pestun, which will be mainly utilized for the calculations of various partition functions.

As an interesting application of the index theorem, Chapter 3 discusses the BPS states of $4d$ $\mathcal{N}=2$ theory and wall-crossing phenomena therein. We first ouline the $4d$ $\mathcal{N}=2$ supersymmetric gauge theories obtained from the Calabi-Yau compactification of type II string theory, and present the work [10] where the Coulomb branch wall-crossing formula was derived with the first principle.

Chapter 4 studies the relation between the two-dimensional partition function of $\mathcal{N}=(2,2)$ GLSM and various amplitudes in A-model string theory. First of all, we reviewed the basic properties of the two-dimensional $\mathcal{N}=(2,2)$ GLSM and topological string theories. [3, 11] The second section we summarized recent results on two sphere partition function that exactly calculates the Kahler potential of the A-model moduli space. [12, 13] Next, we turn to the discussion of the D-brane and

Orientifold. In this regard, at the third section, we review how we have determined the topological coupling of these obejects traditionally, via the anomaly inflow mechanism. This gives the central charges of such objects at the leading order of $\alpha'$. Finally, at section 4 and 5, we review the exact calculation of hemisphere and $\mathbb{RP}^2$ partition function, which turn out to give the $\alpha'$-exact central charge in the presence of D-branes and Orientifolds respectively. [14, 15] Especially, at section 5, we present the work [15].

# Chapter 2

# Exact Calculation of Supersymmetric Indices and Partition Functions

Supersymmetric index and partition function are the most important concepts in studying supersymmetric field theories. These quantities are exactly calculable for many cases, which enables us to study the non-perturbative aspects of the theory. First of all, the Witten index, defined by $\mathrm{Tr}(-1)^F$, was first taken from mathematics to physics by Witten [5], as a measure of spontaneous breaking of the supersymmetry. As a preliminary for the following sections, we review physical and mathematical definitions of the index, and present a physical proof of the Atiyah-Singer index theorem [4] which states that the analytical index is given by particular topological invariants of the theory. From this, we can see that the index calculates a topological invariant of the supersymmetric quantum field theory. After that, as an interesting application, we study the relation between gauge/gravitational anomalies and the index theorem, following the reference [6]. Finally, we briefly review the recently developed methods of calculating exact partition functions for supersymmetric gauge theories, on which the section 4 is based.

## 2.1 Mathematical and Physical Definition of Tr$(-1)^F$

Consider 0+1 dimensional supersymmetric quantum mechanics with finite volume which has complex supercarges $Q^i$, $Q^{*i}$. The algebra of these operators are

$$\{Q^i, Q^{*j}\} = 2\delta^{ij} H \ , \quad \{Q^i, Q^j\} = \{Q^{*i}, Q^{*j}\} = 0 \ . \tag{2.1}$$

The Hilbert space can be divided into the fermionic and bosonic states by introducing an fermion number operator $(-1)^F$, which satisfies $\{Q, (-1)^F\} = 0$. Then, we can see that every energy eigenstates with non-zero eigenvalue $E$ are paired: If we define a real supercharge $\mathbf{Q} = \frac{1}{\sqrt{2}}(Q + Q^*)$, we have

$$\mathbf{Q}|E_B\rangle = \sqrt{E}|E_F\rangle, \quad \mathbf{Q}|E_F\rangle = \sqrt{E}|E_B\rangle \ , \tag{2.2}$$

since $\mathbf{Q}^2 = H$. This equation further implies that supersymmetric states ($\mathbf{Q}|E\rangle = 0$) are always ground states ($H|E\rangle = 0$), and inverse is also true since $\mathbf{Q}$ is a Hermitian. On the other hand, the ground states are not necessarily paired, and their number $n_B(E = 0)$ and $n_F(E = 0)$ can be in general mismatch. Note that their difference $n_B(E = 0) - n_F(E = 0)$ does not change as we tune the parameters of the theory, since only paired states can be excited to the non-zero energy states and the theory is gapped by finite volume. From this simple argument, we can say that

$$\lim_{\beta \to \infty} \text{Tr}(-1)^F e^{-\beta H} = n_B(E = 0) - n_F(E = 0) \tag{2.3}$$

can be thought of as a topological quantity. If follows that this quantity is independent of the value of $\beta$. If we write our states as ${}^t(|E_B\rangle, |E_F\rangle)$, then we can express the supercharge as

$$\mathbf{Q} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & Q^* \\ Q & 0 \end{pmatrix} . \tag{2.4}$$

Then, fermionic and bosonic ground state are defined by $Q|E_B\rangle = 0$ and $Q^*|E_F\rangle = 0$, which leads

$$\lim_{\beta\to\infty} \mathrm{Tr}(-1)^F e^{-\beta H} = \ker Q - \ker Q^* . \tag{2.5}$$

This expression reminds us a parallel story in mathematics. Consider an elliptic operator[1] $D : \Gamma(M,E) \to \Gamma(M,F)$ defined on bundles $E, F$ over $M$. When we try to solve a differential equation $DX = Y$ with $X \in \Gamma(M,E)$, $Y \in \Gamma(M,F)$, elements of $\ker D$ and coker $D$ contains useful information about spectrum of the solutions: $\ker D$ is a space of solutions of homogeneous equation $DX = 0$, while coker $D \equiv Y/\mathrm{Im}D$ can be thought of as a space of constraints which $Y$ should satisfy to ensure the existence of a solution. If both of the space is of finite dimensional, we call $D$ a Fredholm operator. For a Fredholm $D$, we can define analytical index of $D$ as follow

$$\mathrm{Ind}\ D = \dim\ker D - \dim \mathrm{coker}\ D . \tag{2.7}$$

This equation can be further managed into

$$\begin{aligned} \mathrm{Ind}\ D &= \dim\ker D - \dim\mathrm{coker}\ D \\ &= \dim\ker D - (\dim Y - \dim \mathrm{Im}\ D) \\ &= \dim X - \dim Y \end{aligned}$$

[1] The operator $D : \Gamma(M,E) \to \Gamma(M,F)$ is said to be elliptic when the symbol of $s_\xi(D)$ is invertible. The symbol is defined by dim$E\times$dim$F$ matrix $s_\xi(D) \equiv \sum_{|M|=N} A^{M\alpha}{}_a(x)\xi_M$ where $N$ is order of $D$. Here, the matrix $A$ defines the operator $D$ by the relation

$$[Dy(x)]^\alpha = \sum_{|M|<N} A^{M\alpha}{}_a(x) D_M y^a(x) , \tag{2.6}$$

where $y(x)$ is a section of the bundle $E$. [16]

where we used the fact that these spaces are finite dimensional. As we can see from the last equation, this quantity does not depend on the details of the operator $D$, which can be regarded as a rigid quantity. Indeed, the Atiyah-Singer index theorem states that this is a topological invariant of the theory: For an elliptic operator $D$ over a compact complex manifold $M$ without boundary, the index is given by following quantity [17]

$$\text{Ind } D = (-1)^{n(n+1)/2} \int_M ch(\oplus_p (-1)^p E_p) \frac{Td(M)}{e(M)} \ , \tag{2.8}$$

where $n$ is complex dimension of $M$. Here the $ch(E)$, $Td(M)$, $e(M)$ are characteristic classes which quantify the non-triviality of bundles, being topological invariants of the theory. See Appendix A for definitions and properties of various characteristic classes. For a Fredholom operator, we can show that coker $D$ = ker $D^*$, where $D^*$ is an adjoint of $D$. This and eq. (2.5) allows us to identify $D$ with $Q$ in supersymmetric theories. Furthermore, we can say that there exists following correspondence between mathematical and physical definition of the index.

| Differential Eq. | SUSY QM |
|---|---|
| $D$ | $Q$ |
| ker $D$ | bosonic ground state |
| ker $D^*$ | fermionic ground state |
| $DD^* + D^*D$ | $H$ |
| $D$ is Fredholm | theory is gapped |
| Ind$(D)$ | Tr $(-1)^F$ |

Relying on this correspondence, we are going to prove the Atiyah-Singer index theorem with the supersymmetric quantum mechanics using the path integral representation of the regularized index $\text{Tr}(-1)^F e^{-\beta H}$. The standard argument of time slicing, we get the path integral of Euclidean action,

$$\text{Tr}(-1)^F e^{-\beta H} = \int [d\phi]\, [d\psi]_{\text{periodic}}\ e^{-\int_0^\beta d\tau\ \mathcal{L}_E(\phi(\tau),\psi(\tau))} \ . \tag{2.9}$$

Although the indices are defined in the limit $\beta \to \infty$, we are going to use topological property of this quantity which enables us to work in the limit $\beta \to 0$ instead, provided that the theory is gapped. In this limit, the higher order interaction terms become irrelevant, and the quadratic determinant gives an exact answer. Finally, the insertion of $(-1)^F$ acts as changing boundary condition of fermions from anti-periodic to periodic one. The following section reviews the original works by Friedan,Windy and Alvarez-Gaume. [18, 19]

## 2.2 Supersymmetric quantum mechanics and index theorem

### 2.2.1 Euler number

As the first example of (2.9), let us look at the following sigma model Lagrangian with $n$ real scalar fields $\phi^i$ and two-component real fermions $\psi^i_\alpha$ $(\alpha = 1, 2)$, on the even dimensional target space $M$ with metric $g_{ij}$.

$$L = \frac{1}{2} g_{ij}(\phi)\dot{\phi}^i\dot{\phi}^j + \frac{i}{2} g_{ij}\psi^i_\alpha D_t \psi^j_\alpha - \frac{1}{4} R_{ijkl}\psi^i_1\psi^j_1\psi^k_2\psi^l_2 \ . \tag{2.10}$$

In the Euclidean signature, it becomes

$$L_E = \frac{1}{2} g_{ij}(\phi)\dot{\phi}^i\dot{\phi}^j + \frac{1}{2} g_{ij}\psi^i_\alpha D_\tau \psi^j_\alpha + \frac{1}{4} R_{ijkl}\psi^i_1\psi^j_1\psi^k_2\psi^l_2 \ , \tag{2.11}$$

where $D_t\psi^j_\alpha = \frac{\partial}{\partial t}\psi^j_\alpha + \Gamma^j_{kl}\dot{\phi}^k\psi^l_\alpha$. Bosonic fields are quantized as

$$\left[p_i, \phi^j\right] = -i\delta^j_i \ , \tag{2.12}$$

while the fermionic fields should be quantized in terms of flat indices such that $\psi^a_\alpha = e_i{}^a \psi^i_\alpha$. If we complexify with $\chi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2)$,

$$\left\{\chi^a, \bar{\chi}^b\right\} = \delta^{ab}, \quad \left\{\chi^a, \chi^b\right\} = \left\{\bar{\chi}^a, \bar{\chi}^b\right\} = 0. \tag{2.13}$$

We can check that the Lagrangian (2.10) is invariant under the following supersymmetry transformation.

$$\begin{aligned} \delta\phi^i &= \epsilon\chi^{*i} - \epsilon^*\chi^i \ , \\ \delta\chi^i &= i\epsilon\dot{\phi}^i - \Gamma^i_{jk}\epsilon\bar{\chi}^j\chi^k \ , \end{aligned} \tag{2.14}$$

where $\epsilon$ is a one-component complex supersymmetry parameter. We will denote it as $\mathcal{N} = 2$ supersymmetry. Via the Noether procedure, we can find a complex supercharge,

$$\begin{aligned} Q &= \chi^i \left(p_i + i w_{iab}\bar{\chi}^a\chi^b\right) \ , \\ Q^* &= \bar{\chi}^i \left(p_i - i w_{iab}\bar{\chi}^a\chi^b\right) \ . \end{aligned} \tag{2.15}$$

In this simple non-linear sigma model, the bosonic fields maps worldsheet coordinate to the target space $M$, while fermions are spinors valued in pull-back of the tangent bundle, $\phi^*(TM)$. This and the quantization condition (2.13) imply that acting a $\bar{\chi}^i$ corresponds to generating a one-form $dx^i$ on $M$. The Hilbert space forms a exterior algebra $\Lambda^*(M)$, where the wave functions can be written as

$$\Omega_{i_1,\cdots,i_k}(x) e^{i_1}{}_{a_1} \cdots e^{i_k}{}_{a_k} \bar{\chi}^{a_1} \cdots \bar{\chi}^{a_k} |0\rangle \ . \tag{2.16}$$

The action of $Q^*$ on this state is

$$\begin{aligned}
& \bar{\chi}^b e^j{}_b(-i\partial_j \Omega_{i_1\cdots i_k}(x)) e^{i_1}{}_{a_1}\cdots e^{i_k}{}_{a_k}\bar{\chi}^{a_1}\cdots\bar{\chi}^{a_k}|0\rangle \\
+ & \sum_{i=1}^{k}\bar{\chi}^b e^j{}_b \Omega_{i_1\cdots i_k}(x)(-i\partial_j e^{i_i}{}_{a_i}) e^{i_1}{}_{a_1}\cdots \widehat{e^{i_i}{}_{a_k}}\cdots e^{i_k}{}_{a_k}\bar{\chi}^{a_1}\cdots\bar{\chi}^{a_k}|0\rangle \\
+ & \bar{\chi}^b e^j{}_b \Omega_{i_a\cdots i_k}(x)(i w_{icd}\bar{\chi}^c\chi^d) e^{i_1}{}_{a_1}\cdots e^{i_k}{}_{a_k}\bar{\chi}^{a_1}\cdots\bar{\chi}^{a_k}|0\rangle \ . 
\end{aligned} \tag{2.17}$$

Note that the last line can be rewritten as

$$\sum_{i=1}^{k}\bar{\chi}^b e^j{}_b \Omega_{i_a\cdots i_k}(x)(i w_{j a_i d} e^{i_i d}) e^{i_1}{}_{a_1}\cdots \widehat{e^{i_i}{}_{a_k}}\cdots e^{i_k}{}_{a_k}\bar{\chi}^{a_1}\cdots\bar{\chi}^{a_k}|0\rangle \ . \tag{2.18}$$

Then, using the torsion free condition

$$de + w\wedge e = 0 \ , \tag{2.19}$$

we can see that the second and the third line of eq. (2.17) cancels each other. It follows that the supercharges and Hamiltonian corresponds to

$$\begin{aligned}
Q &\longleftrightarrow d \ , \\
Q^* &\longleftrightarrow \delta \ (\equiv - * d *) \ , \\
H = \frac{1}{2}\{Q, Q^*\} &\longleftrightarrow d\delta + \delta d \ .
\end{aligned} \tag{2.20}$$

This relation implies that the ground states of this supersymmetric theory correspond to the harmonic forms on $M$. It leads to

$$\mathrm{Tr}(-1)^F e^{-\beta H} = \sum_{k=0}^{\dim\ M}(-1)^k b_k = \chi(M) \ , \tag{2.21}$$

where $b_k$ are $k$ th Betti number and $\chi$ is the Euler number of $M$.

Explicit computation of the index can be done by path integral of the Euclidean

Lagrangian (2.11). Since the saddle point of the Largrangian is given by the constant configuration of each fields, we can expand it as $\phi = \phi_0 + \delta\phi$ and $\psi = \psi_0 + \delta\psi$. Using the Riemann normal coordinate (defined by $g_{ij}(\phi_0) = \delta_{ij}$, $\partial_k g_{ij}(\phi_0) = 0$), quadratic expansion gives

$$L_E = \frac{1}{2}\delta_{ij}\delta\dot{\phi}^i\delta\dot{\phi}^j + \frac{1}{2}\delta_{ij}\delta\psi^i_\alpha\delta\dot{\psi}^j_\alpha + \frac{1}{4}R_{ijkl}(\phi_0)\psi^i_{01}\psi^j_{01}\psi^k_{02}\psi^l_{02} \ . \tag{2.22}$$

The path integral can be easily done by

$$\mathrm{Tr}(-1)^F e^{-\beta H} = \frac{(-1)^{d/2}}{\beta^{d/2}} \int \prod_{i=1}^{d} \frac{d\phi^i_0}{\sqrt{2\pi}} \prod_{i=1}^{d} d\psi^i_{01} d\psi^i_{02} \ e^{\frac{\beta}{4}R_{ijkl}(\phi_0)\psi^i_{01}\psi^j_{01}\psi^k_{02}\psi^l_{02}} \left[\frac{\det'(\partial_\tau)^2}{\det'(\partial_\tau^2)}\right]^{1/2} ,$$

where $d = \dim\, M$. The numerical pre-factor comes from the normlization of the bosonic and fermionic zero mode, (If $\hat{\phi}^i_0$ and $\hat{\psi}^i_0$ are unit normalized quantities, $d\hat{\phi}^i_0 = \frac{d\phi^i_0}{\sqrt{\beta}}$, $d\hat{\psi}^i_0 = \frac{d\psi^i_0}{\sqrt{\beta}} = \sqrt{\beta}d\psi^i_0$.) and the sign $(-i)^{d/2}$ comes from the fermionice zero mode measure,

$$d\bar{\chi}^1_0 d\chi^1_0 \cdots d\bar{\chi}^d_0 d\chi^d_0 = (-1)^{d/2} d\psi^1_{01} d\psi^1_{02} \cdots d\psi^d_{01} d\psi^d_{02} \ . \tag{2.23}$$

Note that, in order to saturate the fermion zero mode integral, only $d/2$ power of the exponential contributes, and it exactly cancels the $\beta$-dependence in front of the integral. Hence,

$$\begin{aligned} \mathrm{Tr}(-1)^F e^{-\beta H} &= \frac{(-1)^{d/2}}{(2\pi)^{d/2} 2^{d/2} (d/2)!} \int \prod_{i=1}^{d} d\phi^i_0 \ \epsilon^{i_1 j_1 \cdots i_{d/2} j_{d/2}} \epsilon^{k_1 l_1 \cdots k_{d/2} l_{d/2}} \\ &\qquad \times R_{i_1 j_1 k_1 l_1} \cdots R_{i_{d/2} j_{d/2} k_{d/2} l_{d/2}} \\ &= \frac{1}{(2\pi)^{d/2}} \int \mathrm{Pf}(\mathcal{R}_{ij}) \ , \end{aligned} \tag{2.24}$$

where $\mathcal{R}_{ij} = \frac{1}{2}R_{ijkl}dx^k \wedge dx^l$. This is nothing but the Gauss-Bonnet formula.

### 2.2.2 Hirzebruch signature

Note that the Lagrangian (2.10) has a following symmetry:

$$\psi^i_\alpha \to (\sigma_3 \psi^i)_\alpha \ , \tag{2.25}$$

which in terms of $\chi$

$$\chi^i \to \chi^{*i} \ , \quad \chi^{*i} \to -\chi^i \ . \tag{2.26}$$

This symmetry of exchanging $\chi$ and $\chi^*$ corresponds to the Hodge dual operation $*$ : $\Lambda^k(M) \to \Lambda^{d-k}(M)$ in the Hilbert space. Since $*$ commutes with the differential operator, the following is well-defined.

$$\mathrm{Tr} * e^{-\beta H} \ . \tag{2.27}$$

If we write the standard inner product on $\Lambda(M)$ as $\langle \alpha, \beta \rangle = \int \alpha \wedge *\beta$, the index defined by (2.27) can be thought of as the number of positive eigenvalue minus negative eigenvalue of "topological" inner product $\langle \alpha, *\beta \rangle$, for middle-dimensional forms $\alpha$ and $\beta$. Hence we can say that

$$\mathrm{Tr} * e^{-\beta H} = (\text{signature on } M) \ . \tag{2.28}$$

Note that this is non-zero only for $4n$ dimensional $M$. In the path integral representation, the inserted operator $*$ plays a role of flipping boundary condition of the negative chirality fermion. Hence we impose the periodic boundary condition to bosonic fields and negative chirality fermions $\psi_2$, and anti-periodic boundary condition to $\psi_1$. It follows that only $\psi_2$ and $\phi$ have zero modes, and quadratic expansion can be written as

$$L_E = \frac{1}{2}\delta_{ij}\delta\dot{\phi}^i\delta\dot{\phi}^j + \frac{1}{2}\delta_{ij}\psi^i_\alpha \frac{d}{d\tau}\psi^j_\alpha + \frac{1}{4}R_{ijkl}\delta\phi^i\delta\dot{\psi}^j\psi^k_{20}\psi^l_{20} + \frac{1}{4}R_{ijkl}\delta\psi^i_1\delta\psi^j_1\psi^k_{20}\psi^l_{20} \ . \tag{2.29}$$

Then the determinant reads

$$\begin{aligned}
\mathrm{Tr} * e^{-\beta H} &= \frac{(-i)^{d/2}}{\beta^{d/2}} \int \prod_{i=1}^{d} \frac{d\phi_0^i}{\sqrt{2\pi}} \prod_{i=1}^{d} d\psi_{20}^i \, \frac{\det\left[\delta_{ij}\partial_\tau + \frac{\beta}{2} R_{ijkl}\psi_{20}^k\psi_{20}^l\right]_{\mathrm{AP}}^{1/2} \det'\left[\partial_\tau\right]_P^{1/2}}{\det'\left[\delta_{ij}\partial_\tau^2 + \frac{\beta}{2} R_{ijkl}\psi_{20}^k\psi_{20}^l\partial_\tau\right]_{\mathrm{P}}^{1/2}} \\
&= (-i)^{d/2} \int \prod_{i=1}^{d} \frac{d\phi_0^i}{\sqrt{2\pi}} \prod_{i=1}^{d} d\psi_{20}^i \prod_{a=1}^{d/2} \frac{x_a \cosh x_a}{\sinh x_a} \ , \qquad (2.30)
\end{aligned}$$

where $x_a$'s are the skew eigenvalues of $\frac{1}{4\pi} R_{ijkl}\psi_{20}^k\psi_{20}^l$, valued in $SO(d)$. Note that from the first to second line, only $n/2$ power of $\frac{\beta}{2} R_{ijkl}\psi_{20}^k\psi_{20}^l$ contribute to saturate the fermionic zero mode integral, and this, combined with the definition of $x_a$ cancels the $2\pi\beta$ dependent prefactor exactly. The $'$ indicates the omission of the zero mode, which yields additional $x_a$ in the numerator. The sign $(-i)^{d/2}$ comes from the fermionic zero mode measure,

$$d\bar{\chi}_0^1 d\chi_0^1 \cdots d\bar{\chi}_0^{d/2} d\chi_0^{d/2} = (-i)^{d/2} d\psi_{20}^1 \cdots d\psi_{20}^d \ , \qquad (2.31)$$

where $\chi^i = \frac{1}{\sqrt{2}}(\psi_2^{2i-1} + i\psi_2^{2i})$, for $i = 1, \cdots, n/2$. By noting that we can express fermionic zero mode integral into the spacetime integral with the identity

$$\int dx^1 \cdots dx^d \int d\psi^1 \cdots d\psi^d \, C_{\mu_1 \cdots \mu_d} \psi^{\mu_1} \cdots \psi^{\mu_d} = (-1)^{d/2} \int_{M_d} C_{\mu_1 \cdots \mu_d} dx^{\mu_1} \cdots dx^{\mu_d} \ . \qquad (2.32)$$

This formula can be neatly summarized in terms of the $\mathcal{L}$-class,

$$\mathrm{Tr} * e^{-\beta H} = \frac{i^{d/2}}{(2\pi)^{d/2}} \int_{M_d} \mathcal{L}(TM) \ , \qquad (2.33)$$

where

$$\mathcal{L}(TM) = \prod_{a=1}^{d/2} \frac{x_a}{\tanh x_a} \ . \qquad (2.34)$$

For definitions and properties of various characteristic classes, see appendix A.

### 2.2.3 Dirac operator

One of the most important example of the index theorem is the Dirac $\mathcal{A}$-genus. For a $d$-dimensional target manifold which is a spin, we consider the following Lagrangian with $d$ real fermions.

$$L = \frac{1}{2} g_{ij}(\phi)\dot{\phi}^i\dot{\phi}^j + \frac{i}{2} g_{ij}(\phi)\psi^i D_t\psi^j \tag{2.35}$$

Note that this Lagrangian can be obtained from (2.10) by setting $\psi_1^i = \psi_2^i = \frac{1}{\sqrt{2}}\psi^i$. Fields are quantized as

$$\left[p_i, \phi^j\right] = -i\delta_i^j \ , \qquad \left\{\psi^a, \psi^b\right\} = \delta^{ab} \ . \tag{2.36}$$

Since the fermions satisfy the Clifford algebra, we find a correspondence

$$\psi^a \ \leftrightarrow \ \frac{1}{\sqrt{2}}\gamma^a \ . \tag{2.37}$$

This Lagrangian has one supercharge ($\mathcal{N} = 1$)

$$Q = \psi^\mu \left( p_\mu - \frac{i}{2} w_{\mu ab}\psi^a\psi^b \right) \ , \tag{2.38}$$

which plays a role of the Dirac operator

$$\not{D} = \frac{1}{\sqrt{2}}\gamma^\mu(\partial_\mu + \frac{1}{4} w_{\mu ab}\gamma^{ab}) \ , \tag{2.39}$$

and $H = Q^2 = \gamma^\mu\gamma^\nu D_\mu D_\nu$. Hence $\mathrm{Tr}(-1)^F$ in this case calculates the index of the Dirac operator. The quadratic expansion of the Euclidean Larangian, again in the Riemann normal coordinate reads

$$L_E^{(2)} = \frac{1}{2}\delta_{ij}\delta\dot{\phi}^i\delta\dot{\phi}^j + \frac{1}{2}\delta_{ij}\delta\psi^i\partial_\tau\psi^j + \frac{1}{4}R_{ijkl}\delta\phi^i\delta\dot{\phi}^j\psi_0^k\psi_0^l \ . \tag{2.40}$$

Then the index can be calculated as

$$\begin{aligned}\mathrm{Tr}(-1)^F e^{-\beta H} &= (-i)^{d/2}\int \prod_{i=1}^{d}\frac{d\phi_0^i}{\sqrt{2\pi}}\prod_{i=1}^{d}d\psi_0^i\,\frac{(\det{}'\,\partial_\tau)^{1/2}}{\det{}'\left[\partial_\tau^2+\frac{1}{2}R_{ijkl}\psi_0^k\psi_0^l\partial_\tau\right]^{1/2}}\\ &= (-i)^{d/2}\int \prod_{i=1}^{d}\frac{d\phi_0^i}{\sqrt{2\pi}}\prod_{i=1}^{d}d\psi_0^i\prod_{x_i}\left[\frac{\prod_{n>0}\left(\frac{2n\pi}{\beta}\right)}{\prod_{n>0}\left(\frac{2n\pi}{\beta}-ix_i\right)}\frac{1}{\prod_{n>0}\left(\frac{2n\pi}{\beta}\right)}\right]\end{aligned} \tag{2.41}$$

where $x_i$ are skew eigenvalue of $\frac{1}{2}R_{ijkl}\psi_0^k\psi_0^l$. The last factor can be regularized as

$$\begin{aligned}\prod_{n>0}\frac{2n\pi}{\beta} &= \lim_{s\to 0}\exp\sum_{n>0}\ln\left(\frac{2n\pi}{\beta}\right)n^{-s}\\ &= \lim_{s\to 0}\exp\sum_{n>0}\left[\ln\left(\frac{2\pi}{\beta}\right)n^{-s}+(\ln n)n^{-s}\right]\\ &= \lim_{s\to 0}\exp\left[\ln\left(\frac{2\pi}{\beta}\right)\zeta(0)-\zeta'(0)\right]\\ &= \sqrt{\beta}\end{aligned} \tag{2.42}$$

where we used $\zeta(0)=-\frac{1}{2}$ and $\zeta'(0)=-\ln\sqrt{2\pi}$. The first factor gives

$$\frac{\prod_{n>0}\left(\frac{2n\pi}{\beta}\right)}{\prod_{n>0}\left(\frac{2n\pi}{\beta}-ix_i\right)} = \frac{\beta x_i/2}{\sinh\beta x_i/2}\ , \tag{2.43}$$

which comes from the facts that the LHS has a pole at $x=-\frac{2n\pi i}{\beta}$, and the limit $x_i\to 0$ gives 1. To summarize,

$$\mathrm{Tr}(-1)^F e^{-\beta H} = \frac{(-i)^{d/2}}{\beta^{d/2}}\int\prod_{i=1}^{d}\frac{d\phi_0^i}{\sqrt{2\pi}}\prod_{i=1}^{d}d\psi_0^i\prod_{x_a}\frac{\beta x_a/2}{\sinh\beta x_a/2} \tag{2.44}$$

$$= \frac{i^{d/2}}{(2\pi)^{d/2}}\int\mathcal{A}(TM)\ . \tag{2.45}$$

Note that from the first to second line, $\beta$ pre-factor cancels when we saturate the fermionic zero modes, and the sign factor again comes from (2.31) and (2.32). For the last line, we defined $\mathcal{A}$ genus by

$$\mathcal{A}(TM) = \prod_a \frac{x_a/2}{\sinh x_a/2} \ . \tag{2.46}$$

### 2.2.4 Dirac operator coupled to external gauge fields

Now, let us consider the previous $\mathcal{N} = 1$ Lagrangian, now coupled to the non-abelian external gauge field. We introduce another complex ghost fermion $\eta^\alpha, \bar{\eta}_\alpha$ to incorporate the gauge symmetry labeled by $\alpha$. Lagrangian can be written as

$$L = \frac{1}{2} g_{ij}(\phi)\dot{\phi}^i\dot{\phi}^j + \frac{i}{2} g_{ij}(\phi)\psi^i D_t^g \psi^j + i\bar{\eta}_\alpha (D_t^A \eta)^\alpha + \frac{i}{2} F_{ij\alpha\beta}\psi^i\psi^j\bar{\eta}^\alpha\eta^\beta \ , \tag{2.47}$$

where $D_t^A \eta^\alpha = \partial_t \eta^\alpha + i\dot{\phi}^i A_i{}^\alpha{}_\beta \eta^\beta$ is a gauge covariant derivative. Quantization of fermionic fields are given by

$$\left\{\psi^\alpha, \psi^\beta\right\} = \delta^{\alpha\beta}, \quad \left\{\bar{\eta}^\alpha, \eta^\beta\right\} = \delta^{ab} \ . \tag{2.48}$$

We will restricts our trace to the one particle state of $\eta$ given by

$$\bar{\eta}^\alpha|0\rangle, \quad \text{where } \eta^\alpha|0\rangle = 0 \ , \tag{2.49}$$

which restricts states to be in a particular representation, not their tensor product generated by the multiparticle states of $\eta$. Supercharges are shifted by the gauge field:

$$Q = \psi^\mu \left( p_\mu - \frac{i}{2} w_{\mu ab}\psi^a\psi^b + \bar{\eta}^a A_{\mu,ab}\eta^b \right) \ . \tag{2.50}$$

This corresponds to a Dirac operator couple to external gravitational and gauge fields,

$$\not{D} = \frac{1}{\sqrt{2}}\gamma^\mu(\partial_\mu + \frac{1}{4} w_{\mu ab}\gamma^{ab} - iA_\mu^a T^a) \ . \tag{2.51}$$

Before we do the path integral, we first perfom the quantization of the ghost fields $\eta, \bar{\eta}$. As usual, $(-1)^F$ imposes the periodic boundary condition to the $\phi^i$ and $\psi^i$, and anti-periodic boundary condition to the $\eta$. The quadratic fluctuation near the saddle point $\phi = \phi_0$ and $\psi = \psi_0$ gives

$$\begin{aligned} L_E - \bar{\eta}^\alpha \dot{\eta}_\alpha &= \frac{1}{2}\delta_{ij}\delta\dot{\phi}^i\delta\dot{\phi}^j + \frac{1}{2}\delta_{ij}\delta\psi^i\delta\dot{\psi}^j + \frac{1}{2}F_{ij\alpha\beta}\delta\dot{\phi}^i\delta\phi^j\bar{\eta}^\alpha\eta^\beta + \frac{1}{4}R_{ijkl}\delta\phi^i\delta\dot{\phi}^j\psi_0^k\psi_0^l \\ &+\frac{1}{2}F_{ij\alpha\beta}\delta\psi^i\delta\psi^j\bar{\eta}^\alpha\eta^\beta + \frac{1}{2}F_{ij\alpha\beta}\psi_0^i\psi_0^j\bar{\eta}^\alpha\eta^\beta \ , \end{aligned} \tag{2.52}$$

where $F_{ij} = \partial_i A_j - \partial_i A_i + [A_i, A_j]$, written in Hermitian basis. One-loop determinant can be calculated as

$$\begin{aligned} &\lim_{\beta\to 0}\ (-i)^{d/2}\int\prod_{i=1}^{d}\frac{d\phi_0^i}{\sqrt{2\pi}}d\psi_0^i\prod_\alpha \mathrm{Tr}_\eta \exp\left(\frac{1}{2}F_{ij\alpha\beta}\delta\dot{\phi}^i\delta\phi^j\bar{\eta}^\alpha\eta^\beta\right) \\ &\qquad\times\frac{\det{}'(\partial_\tau + F_{ij\alpha\beta}\psi_0^i\psi_0^j\bar{\eta}^\alpha\eta^\beta)^{1/2}}{(\det{}'\,\partial_\tau)^{1/2}\det{}'(\partial_\tau + F_{ij\alpha\beta}\psi_0^i\psi_0^j\bar{\eta}^\alpha\eta^\beta + \frac{1}{2}R_{ijkl}\psi_0^k\psi_0^l)^{1/2}}\ , \end{aligned} \tag{2.53}$$

where $\mathrm{Tr}_\eta$ denotes for the trace in the $\eta$ space restricted to the one particle sector. The determinant factor in the second line are evaluated as

$$\det\prod_{n>0}\frac{\left(\frac{2n\pi}{\beta} - iF\right)}{\left(\frac{2n\pi}{\beta}\right)\left(\frac{2n\pi}{\beta} - iF - \frac{i}{2}R_{ijkl}\psi_0^k\psi_0^l\right)}\ , \tag{2.54}$$

where $F = F_{ij\alpha\beta}\psi_0^i\psi_0^j\bar{\eta}^\alpha\eta^\beta$. As before, $1/\prod_{n>0}(2n\pi/\beta)$ factors are regularized to yield overall factor $\frac{1}{\beta^{d/2}}$. We redefine the fermionic zero mode which absorbs this factor:

$$\psi_0 \to \frac{1}{\sqrt{\beta}}\psi_0\ , \quad d\psi_0 \to \sqrt{\beta}d\psi_0\ . \tag{2.55}$$

Then the integral becomes

$$\lim_{\beta\to 0}\ (-i)^{d/2}\int\prod_{i=1}^{d}\frac{d\phi_0^i}{\sqrt{2\pi}}d\psi_0^i\ \mathrm{Tr}_\eta\ e^{\frac{1}{2}F}\det\prod_{n>0}\frac{(2n\pi-i\beta F)}{2n\pi-i\beta F-\frac{i}{2}R_{ijkl}\psi_0^k\psi_0^l} \tag{2.56}$$

$$= \lim_{\beta\to 0}\ (-i)^{d/2}\int\prod_{i=1}^{d}\frac{d\phi_0^i}{\sqrt{2\pi}}d\psi_0^i\ \mathrm{Tr}_\eta\ e^{\frac{1}{2}F}\prod_i\frac{(x_i+\beta F)/2}{\sinh((x_i+\beta F)/2)} \tag{2.57}$$

$$= (-i)^{d/2}\int\prod_{i=1}^{d}\frac{d\phi_0^i}{\sqrt{2\pi}}d\psi_0^i\ \mathrm{Tr}_\eta\ e^{\frac{1}{2}F}\prod_i\frac{x_i/2}{\sinh(x_i/2)}\ . \tag{2.58}$$

The $\eta$ integral should be evaluated in the one-particle sector. Note that

$$\langle 0|\eta^d\exp\left(\frac{1}{2}\bar\eta^\alpha\bar\eta^\beta F_{\alpha\beta ij}\psi_0^i\psi_0^j\right)\bar\eta^d|0\rangle = \mathrm{Tr}\exp(\frac{1}{2}F_{ij}\psi_0^i\psi_0^j)\ , \tag{2.59}$$

we have

$$\mathrm{Tr}(-1)^F = \frac{i^{d/2}}{(2\pi)^{d/2}}\int_M ch(F)\wedge\mathcal{A}(TM)\ . \tag{2.60}$$

where $ch(F)$ is the Chern character of the gauge bundle on $M$.

### 2.2.5 Dolbeault complex

The last example of the index theorem is the Dolbeault index of Kahler manifold. Kahler manifold is defined as a Hermitian manifold which admit a metric

$$ds^2 = 2g_{i\bar j}d\phi^i d\bar\phi^{\bar j}\ , \tag{2.61}$$

where

$$\partial_\lambda g_{\mu\bar\nu} = \partial_\mu g_{\lambda\nu},\quad \bar\partial_{\bar\lambda}g_{\mu\bar\nu} = \bar\partial_{\bar\nu}g_{\mu\bar\lambda}\ , \tag{2.62}$$

or equivalently, equipped with Kahler form

$$\Omega = g_{i\bar j}dz^i\wedge d\bar z^j\ , \tag{2.63}$$

which is closed ($d\Omega = 0$). Note that, this condition ensures that there exist a function $K(\phi, \bar{\phi})$ such that $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K(\phi, \bar{\phi})$. A Hermitian manifold is Kahler if and only if there exist a almost complex structure $J_p : T_p M \to T_p M$ with $\nabla_\mu J = 0$. This can be locally written as

$$J_p = i dz^\mu \otimes \frac{\partial}{\partial z^\mu} - i d\bar{z}^\mu \otimes \frac{\partial}{\partial \bar{z}^\mu}. \tag{2.64}$$

From the definition, it is straightforward to see that only non-zero Chirstoffel symbols are

$$\Gamma^\mu_{\nu\rho}, \quad \Gamma^{\bar{\mu}}_{\bar{\nu}\bar{\rho}} \ , \tag{2.65}$$

and other components with mixed indices all vanish. It follows that, if we define $\mathcal{R}_{ij} = \frac{1}{2} R_{ijkl} dx^k \wedge dx^l$, only $\mathcal{R}_i{}^j$ and $\mathcal{R}_k{}^l$ are non-zero, which means that holonomy group is $U(n)$ subgroup of $O(2n)$. If there is additional constraint that the Ricci curvature, the trace of $\mathcal{R}_i{}^j$, vanishes, the holonomy group becomes $SU(n)$ subgroup. We call such a manifold as Calabi-Yau manifold.

The non-linear sigma model whose target space is Kahler manifold coupled with abelian gauge field can be written as

$$\begin{aligned} L &= g_{i\bar{j}} \dot{\phi}^i \dot{\bar{\phi}}^{\bar{j}} + \frac{i}{2} g_{i\bar{j}} \psi^i (\partial_t \bar{\psi}^{\bar{j}} + \Gamma^{\bar{j}}_{\bar{k}\bar{l}} \dot{\bar{\phi}}^{\bar{k}} \bar{\psi}^{\bar{l}}) + \frac{i}{2} g_{\bar{i}j} \bar{\psi}^{\bar{i}} (\partial_t \psi^j + \Gamma^j_{kl} \dot{\phi}^k \psi^l) \\ &\quad + i A_i \dot{\phi}^i - i A_{\bar{i}} \dot{\bar{\phi}}^{\bar{j}} + F_{i\bar{j}} \psi^i \bar{\psi}^{\bar{j}} \ . \end{aligned} \tag{2.66}$$

The quantization of fields are given by

$$[\pi_i, \phi^j] = -i\delta_i^j, \quad [\bar{\pi}_{\bar{i}}, \bar{\phi}^j] = -i\delta_{\bar{i}}^{\bar{j}}, \quad \{\psi^a, \bar{\psi}^{\bar{b}}\} = \delta^{a\bar{b}} \ , \tag{2.67}$$

where $\pi_i = p_i + A_i$, $\bar{\pi}_i = \bar{p}_{\bar{i}} + A_{\bar{i}}$. One can show that there are two supersymmetries preserved. By the Noether theorem, the supercharges are

$$\begin{aligned} Q &= \psi^i (p_i + iA_i + i w_{i\bar{a}b} \bar{\psi}^{\bar{a}} \psi^b) \ , \\ \bar{Q} &= \bar{\psi}^{\bar{i}} (\bar{p}_{\bar{i}} - iA_{\bar{i}} + i w_{\bar{i}a\bar{b}} \psi^a \bar{\psi}^{\bar{b}}) \ . \end{aligned} \tag{2.68}$$

Let us change the order of the fermions in the last term so that

$$\begin{aligned} Q &= \psi^i(p_i + iA_i + iw_i{}^a{}_a + iw_{ib\bar{a}}\psi^b\bar{\psi}^{\bar{a}}) \ , \\ \bar{Q} &= \bar{\psi}^{\bar{i}}(\bar{p}_{\bar{i}} - iA_{\bar{i}} + iw_{\bar{i}}{}^{\bar{a}}{}_{\bar{a}} + iw_{\bar{i}\bar{b}a}\bar{\psi}^{\bar{b}}\psi^a) \ . \end{aligned} \tag{2.69}$$

If we tune the external gauge field by $A_i = -w_i{}^a{}_a$, the trace part of the $U(n)$ holonomy, the supercharge $Q$ becomes the holomorphic dolbeault operator $\partial$, acting on

$$\Omega_{\mu_1\cdots\mu_k}(z,\bar{z})e^{\mu_1}{}_{a_1}\cdots e^{\mu_k}{}_{a_k}\psi^{a_1}\cdots\psi^{a_k}|0\rangle \ . \tag{2.70}$$

This is the element of $\Lambda^{k,0}(M)$. On the other hand, if we choose $A_{\bar{i}} = w_{\bar{i}}{}^{\bar{a}}{}_{\bar{a}}$, the $\bar{Q}$ becomes a anti-holomorphic dolbeault operator $\bar{\partial}$, which acts on elements of $\Lambda^{0,k}(M)$,

$$\Omega_{\bar{\mu}_1\cdots\bar{\mu}_k}(z,\bar{z})e^{\bar{\mu}_1}{}_{\bar{a}_1}\cdots e^{\bar{\mu}_k}{}_{\bar{a}_k}\bar{\psi}^{\bar{a}_1}\cdots\bar{\psi}^{\bar{a}_k}|\bar{0}\rangle \ . \tag{2.71}$$

This mechanism, locking the curvature with external symmetry is called *twisting*. In the higher dimensional supersymmetric field theory, it plays an important role in leaving some of the supersymmetries unbroken on the curved space.

The path integral can be easily evaluated with the procedure of the previous section. For the anti-holomorphic operator $\bar{\partial}$, we substitute $U(1)$ part of the curvature in the place of the gauge bundle,

$$\frac{i^n}{(2\pi)^n}\int_{M_{2n}}\prod_{i=1}^{n} e^{x_i/2}\frac{x_i/2}{\sinh(x_i/2)} = \frac{i^n}{(2\pi)^n}\int_{M_{2n}}\prod_{i=1}^{n}\frac{x_i}{1-e^{-x_i}} \ , \tag{2.72}$$

where $x_i$'s are eigenvalues of the matrix $R_{i\bar{j}k\bar{l}}d\phi^k\wedge d\bar{\phi}^{\bar{l}}$. This is defined as the Todd genus,

$$Td(M) = \int_M \prod_i \frac{x_i}{1-e^{-x_i}} \ , \tag{2.73}$$

which is defined only for the complex manifold.

## 2.3 Index and gravitational anomalies in string theory

### 2.3.1 Gauge/Gravitational anomalies in string theory

Anomalies refers to a breakdown of classical symmetry at the quantum level. In general, existence of global anomaly offers useful information of the theory, for example, in understanding pion decay into two photons in the standard model. On the other hand, gauge/gravitational symmetries cannot be violated at any quantum level since they govern unitarity and Lorentz invariance of the theory. Existence of such anomalies implies the breakdown of the theory, and it is crucial to check weather gauge/gravitational anomaly cancels out.

One of the most important application of the index theorems derived in the previous section is calculation of anomalies. Among the various ways of obtaining expression for anomalies, we are going to focus on the Fujikawa's method [20], which explicitly reveals the relation between definition of anomalies and their topological nature. The first example is the $\gamma_5$ anomalies of a dirac fermion. Consider a Dirac fermion coupled to the external spacetime curvature in even-dimensional spacetime:

$$L = \int d^d x \quad \bar{\psi}\gamma^\mu D_\mu \psi \ . \tag{2.74}$$

The action has a symmetry under

$$\psi \to e^{i\alpha(x)\gamma_{d+1}\psi}, \quad \bar{\psi} \to \bar{\psi}e^{i\alpha(x)\gamma_{d+1}} \ , \tag{2.75}$$

with a conserved current

$$j_5^\mu = \bar{\psi}\gamma^\mu\gamma_{d+1}\psi \ . \tag{2.76}$$

In order to check the full quantum invariance, we should ensure the invariance of the measure of the path integral. If $\psi_n$ are eigenvectors of the Dirac operator, we can expand the fields as

$$\psi(x) = \sum_n a_n \psi_n(x), \quad \bar{\psi}(x) = \sum_n \bar{b}_n \psi_n^\dagger(x) \ . \tag{2.77}$$

Then $[d\psi][d\bar{\psi}] = \prod_n da_n d\bar{b}_n$ transforms under (2.75) as

$$\prod_n da_n d\bar{b}_n \to \prod_n da_n d\bar{b}_n \exp\left[-2i \int dx \sum_n \psi_n(x)^\dagger \alpha(x) \gamma_{d+1} \psi_n(x)\right] \ , \tag{2.78}$$

which shows that the quantum theory is not invariant under this transformation. This is the source of the anomaly. We can say that

$$(\text{Anomaly}) = \int dx \sum_n \psi_n(x)^\dagger \gamma_{d+1} \psi_n(x) \ . \tag{2.79}$$

However this quantity is ill-defined since two fields are evaluated at the same point. We introduce a regulator as

$$\int dx \sum_n \psi_n(x)^\dagger \gamma_{d+1} \psi_n(x) = \lim_{\beta \to 0} \text{Tr } \gamma_{d+1} \ e^{-\beta (i \not{D})^2} \ . \tag{2.80}$$

Note that the quantity at RHS is exactly what is calculated in the section 2.2.3, the index of the Dirac operator. Quantum mechanics we need in order to evaluate this quantity is $\mathcal{N} = 1$ non-linear sigma model, which yields

$$(\text{Anomaly}) = \frac{i^{d/2}}{(2\pi)^{d/2}} \int_{M_d} \prod_{i=1}^{d/2} \frac{x_i/2}{\sinh x_i/2} \ . \tag{2.81}$$

As can be seen in this example, anomalies are closely related to the various index theorems, which will be further investigated in the next subsection. This method was widely extended in Alvarez-Gaume and Witten's work [6] where they calculated anomalies of string theory and their miraculous cancelation. Following this pioneering work, we will review the procedure of calculating gauge and gravitational anomalies of various basic fields contents in quantum field theory.

**Gravitaional anomalies of spin $1/2$ fields**

Gravitational anomaly implies quantum non-invariance of the theory under general coordinate transformation and local Lorentz transformation, which act as

$$\begin{aligned} \delta_c e_\mu{}^a &= \eta^\nu \partial_\nu e_\mu{}^a + (\partial_\mu \eta^\nu) e_\nu{}^a \\ \delta_l e_\mu{}^a &= \Lambda_a{}^b e_\mu{}^b \ . \end{aligned} \tag{2.82}$$

In the original work [6], they considered a particular combination of the two transformations, which acts covariantly on the chiral spinor as [2]

$$\delta_\eta = -\eta^\mu D_\mu \psi \ . \tag{2.84}$$

Since we are considering chiral fermions, the relevant operator is $i\not{D}_L = \frac{i}{2}\not{D}(1 - \gamma^{d+1})$. Since this operator is not self-adjoint, the determinant is not well-defined. Hence we expand as

$$\psi = \sum_n a_n \psi_n, \quad \bar{\psi} = \sum_n \bar{b}_n \chi_n \ , \tag{2.85}$$

where for this case $\psi_n$ and $\chi_n$ are eigenfunctions of $(i\not{D}_L)^\dagger(i\not{D}_L)$ and $(i\not{D}_L)(i\not{D}_L)^\dagger$ respectively. In Euclidean signature, we should integrate over both of them since they are independent degrees of freedom. As in the previous example, we examine the variation of the measure under the transformation (2.84). The Jacobian yields

$$\int dx\ \psi_n^\dagger(\eta^\mu D_\mu)\psi_n - \int dx\ \chi_n^\dagger(\eta^\mu D_\mu)\chi_n \ . \tag{2.86}$$

[2]Actually it can be shown that the two transformations essentially gives the same anomaly. [22] In particular, the anomaly only from the local Lorentz transformation can be written as [24]

$$(\text{covariant anomaly (2.84)})(\eta^\mu) = -2(\text{Lorentz anomaly})(D_{[\mu}\eta_{\nu]}) \tag{2.83}$$

This quantity can be calculated by proper regularization, which can be written as

$$\lim_{\beta\to 0} \text{Tr}\ \gamma^{d+1}(\eta^\mu D_\mu)\ e^{-\beta(i\not{D})^2}\ . \tag{2.87}$$

The operator insertion $(\eta^\mu D_\mu)$ can be exponentiated to the action, and we can recover the original quantity by taking terms linear in $\eta$ at the end. Exponentiated term amounts to $D_\mu\eta_\nu(x_0)x^\mu\dot{x}^\nu$ in the quadratic Lagrangian. The path integral reduces to the calculation of the Dirac index, which was done in section 2.2.3, where this operator insertion effectively shift the curvature as

$$R_{\mu\nu\rho\sigma}\psi_0^\rho\psi_0^\sigma \to R_{\mu\nu\rho\sigma}\psi_0^\rho\psi_0^\sigma + 4D_\mu\eta_\nu\ . \tag{2.88}$$

If we denote the shifted skew eigenvalue as $x_i'$, the result can be written as

$$I_{\text{gravity},1/2} = \frac{i^{d/2}}{(2\pi)^{d/2}}\int_{M_d}\prod_i \frac{x_i'/2}{\sinh x_i'/2}\bigg|_{D\eta\text{-linear}}\ . \tag{2.89}$$

Since the polynomial is even in $x_i'$, the integrand consists of the formal sum of $4n$-forms. Since we should pick only $D\eta$-linear terms, the anomaly can be obtained from the $2d+2$ form of the Dirac genus $\mathcal{A}(TM/2\pi)$. We clearly see that only in $4n+2$ dimension, spin-half fields have pure gravitational anomaly. Furthermore, they are closely related to the $\gamma_5$ anomaly of $4n$ dimensions. We will further investigate this relation in the next subsection.

#### Gauge and gravitational anomalies of spin 1/2 fields

Next, we consider spin-1/2 complex chiral fermion coupled to the external gauge field additionally. They transform as

$$\begin{aligned}\delta\psi &= i\bar{\eta}_\alpha T^\alpha\psi\\ \delta\bar{\psi} &= -i\bar{\psi}\eta_\alpha T^\alpha\ .\end{aligned} \tag{2.90}$$

The anomaly again comes from the transformation of the Jacobian which can be written as

$$\lim_{\beta\to 0} \text{Tr}\ \gamma_5\ i\eta_\alpha T^\alpha e^{-\beta(\not{D})^2}\ , \tag{2.91}$$

where $D_\mu$ is gauge covariant derivative. Similarly, this can be calculated by exponentiating $c^*\ i\eta_\alpha T^\alpha c^*$, and taking only terms linear in $\eta$. As noted in section 2.2.4, we are only interested in the one-particle state of $c$-fermions, to avoid generating tensor product of a given representation. The Lagrangian we need is exactly what was studied in section 2.2.4, whose path integral reads

$$I_{\text{gauge},1/2} = \frac{i^{d/2}}{(2\pi)^{d/2}} \int_{M_d} \text{Tr}\ e^{(\frac{1}{2}F^\alpha+\eta^\alpha)T^\alpha}\Big|_{\eta\text{-linear}}\ , \tag{2.92}$$

where $F^\alpha \equiv F^\alpha_{\mu\nu}dx^\mu dx^\nu$. When fermions couple to both of the gauge and gravitational field, we can combine the anomaly as

$$\begin{aligned} I_{\text{mixed},1/2} &= \frac{i^{d/2}}{(2\pi)^{d/2}} \int_{M_d} \text{Tr}\ e^{(\frac{1}{2}F^\alpha+\eta^\alpha)T^\alpha} \prod_i \frac{x_i'/2}{\sinh x_i'/2}\Big|_{\eta\text{-linear}} \\ &= \frac{i^{d/2}}{(2\pi)^{d/2}} \int_{M_d} ch(F')\wedge\mathcal{A}(TM')\Big|_{\eta\text{-linear}}\ , \end{aligned} \tag{2.93}$$

where $F'$ and $TM'$ denote that gauge field and curvature are shifted by $\eta$ and $D\eta$ respectively.

#### Gravitational anomalies of Rarita-Schwinger fields

The next chiral field which can carry the gravitational anomaly is the spin-3/2 Rarita-Schwinger field. This field $\psi_{\mu\alpha}$ can be thought of as the tensor product of spinor representation and the vector representation, $[1]\otimes[\frac{1}{2}] = [\frac{3}{2}]\oplus[\frac{1}{2}]$, where the latter spin-1/2 should be factored out at the end. The Dirac operator is in the $SO(2n)$ representation, where the ghost fields $c^*_a, c_a$ transform under $SO(2n)$ with $(T^{ab})_{cd} = \delta^a{}_c\delta^b{}_d - \delta^a{}_d\delta^b{}_c$. The same procedure as gauge anomaly of spin-1/2 fields can be applied to this, which results

$$I_{3/2} = \frac{i^{d/2}}{(2\pi)^{d/2}} \int_{M_n} \left(\text{Tr } e^{R'} - 1\right) \prod_i \frac{x_i'/2}{\sinh x_i'/2}\Bigg|_{\eta-\text{linear}} , \tag{2.94}$$

where $R$ is a two-form valued matrix $\frac{1}{2}R_{ijkl}dx^k dx^l$ and the trace is taken over the remaining indices. Prime denote for the fact that the eigenvalues are shifted by $\eta$ and $D\eta$ as in the previous example. $-1$ in the parenthesis is due to the factoring out spin-1/2 degrees of freedom. Note that the gauge field does not couple to the spin-3/2 field.

#### Gravitational anomalies of self-dual antisymmetric fields

In Euclidean, bosonic fields are always in real representations, which means that the complex conjugate should be integrated out together in the path integral. Hence they do not carry anomaly in general. The problem occurs for the fields which does not have a Lagrangian description. Anti-symmetric self-dual fields (ASD) are such examples. In Minkowskian, ASD exists in $4k+2$ dimensions, which satisfies

$$F_{\mu_1\cdots\mu_{2k+1}} = \frac{\sqrt{-g}}{(2k+1)!}\epsilon_{\mu_1\cdots\mu_{2k+1}\nu_1\cdots n_{2k+1}}F^{\nu_1\cdots\nu_{2k+1}} . \tag{2.95}$$

Since, in the Lagrangian representation of the index, we are working with Euclidean signature where such relation does not hold, we will complexify the fields and calculate the anomaly. Due to this, at the end we should divide the result by factor of 2. In order to calculate the anomaly, in [6], they introduced other bosonic antisymmetric tensor fields $F_{\mu_1\cdots\mu_i}$ with $i = 0, \cdots, 4k+2$. All of these can be constructed from the bi-spinor field $\phi_{\alpha\beta}$ defined by

$$\phi_{\alpha\beta} = \frac{1}{2^{k+1/2}} \sum_{i=1}^{4k+2} \gamma_{\alpha\beta}^{\mu_1\cdots\mu_i} F_{\mu_1\cdots\mu_i} . \tag{2.96}$$

It is enough to calculate the anomaly of $\phi_{\alpha\beta}$ instead of $F_{\mu_1\cdots\mu_i}$'s. Since other anti-symmetric tensor fields with $i \neq 2k+1$ does not carry anomaly, this is equivalent to calculate anomaly coming from self-dual fields only. We can treat the indices $\alpha$ and $\beta$ separately, then the procedure parallels that of spin-3/2 fields. The index $\alpha$ can be thought of as usual spinor index on which Dirac operator acts, while the index $\beta$ can be thought of as additional spinor representation. From the first factor, we get $\mathcal{A}(TM/2\pi)$, and from the second factor, we get $\mathrm{Tr}\exp\left(\frac{1}{4}R_{abcd}\psi_0^a\psi_0^b\gamma^{cd}\right)$. Hence,

$$\begin{aligned} I_{ASD} &= \frac{1}{4}\frac{i^{2k+1}}{(2\pi)^{2k+1}}\int_{M_{4k+2}} \mathrm{Tr}\exp\left(\frac{1}{4}R'_{cd}\gamma^{cd}\right)\prod_i \frac{x'_i/2}{\sinh x'_i/2} \\ &= \frac{1}{4}\frac{2^{2k+1}i^{2k+1}}{(2\pi)^{2k+1}}\int_{M_{4k+2}} \prod_i \cosh x'_i/2 \cdot \frac{x'_i/2}{\sinh x'_i/2} \\ &= \frac{1}{8}\frac{i^{2k+1}}{(2\pi)^{2k+1}}\int_{M_{4k+2}} \prod_i \frac{x'_i}{\tanh x'_i} \\ &= \frac{1}{8}\frac{i^{2k+1}}{(2\pi)^{2k+1}}\int_{M_{4k+2}} \mathcal{L}(TM)\ . \end{aligned} \tag{2.97}$$

In the first line, 1/4 factor comes from the reality of fields and chiral projection. $x'_i$ denotes for the eigenvalue of the matrix $\frac{1}{2}R_{ijkl}dx^k dx^l + D_i\eta_j - D_j\eta_i$. From the second to third line, $x'_i$'s absorbed factor $2^{2k+2}$. (Note that, in the integrand, we need linear term in $\eta$ and $2k+1$ power of $x_i$'s.) We can see that the answer is given by the $\mathcal{L}$-class.[3]

### 2.3.2 Relation between anomalies and index theorem

As can be seen in the last subsection, gauge/gravitational anomalies of $2n$ dimensions are closely related to the index theorems in $2n+2$ dimensions. In this

[3] One can directly calculate the anomaly for the ASD fields by noting that the Jacobian of diffeomorphism is given by $\mathrm{Tr} * \delta\eta$ [24]. Since the equation of motion of ASD fields is $\Box F_{i_1\cdots i_n} = 0$, the regulator can be chosen to be $\lim_{\beta\to 0}\mathrm{Tr} * \delta\eta\ e^{-\beta\Box}$. In this viewpoint, two fermion indices are equally treated and the supersymmetric Lagrangian is reduced to that of section 2.2.2, which has two supersymmetries.

subsection, we present the reason why the two quantities are related in general, following [21, 23]. Furthermore, we will see the universal shift of the curvature $R_{ij} \to R_{ij} + D_{[i}\eta_{j]}$ are related by so called descent procedure of anomaly polynomials. [22]

In order to see the effect of chiral mismatch, we consider the following Dirac operator,

$$\tilde{D} = \gamma^i \left( \partial_i + A_i \frac{1+\gamma_5}{2} \right) = \begin{pmatrix} 0 & \not{D}_+ \\ \not{\partial}_- & 0 \end{pmatrix} . \tag{2.98}$$

Determinant of $\tilde{D}$ can be calculated from the square root of

$$\mathrm{Det}(i\not{\partial}_+ i\not{\partial}_-)\mathrm{Det}(i\not{D}_+ i\not{D}_-) = C \cdot \mathrm{Det} \begin{pmatrix} 0 & \not{D}_+ \\ \not{D}_- & 0 \end{pmatrix} , \tag{2.99}$$

where $C$ is a constant independent of the gauge fields. Hence,

$$\mathrm{Det}(i\tilde{D}) = \sqrt{\mathrm{Det}(i\not{D})}\; e^{i\Phi[A]} . \tag{2.100}$$

Since the absolute value $\sqrt{\mathrm{Det}(i\not{D})}$ is a well-defined quantity, the source of the anomaly comes from the imaginary part of the Euclidean action $\Phi[A]$, which is a topological quantity in general. Consider $S^{2n}$ as $2n$-dimensional Eucildean spacetime. Let $g(\theta, x)$ is an element of the gauge group $G$, where $g(0, x) = g(2\pi, x) = 1$ and $x \in S^{2n}$. Domain of $g$ is therefore $S^{2n} \times [0, 2\pi]$ with two end point identified, which is $S^{2n+1}$. Now, let $A^\theta$ be a transformed gauge field $A^\theta \equiv A(\theta, x) = g(\theta, x)^{-1}(d + A(x))g(\theta, x)$. If we require well-definedness of $\mathrm{Det}(i\tilde{D}[A^\theta])$, from (2.100), it follows that $\Phi[A(0, x)] = 2\pi m + \Phi[A(2\pi, x)]$, where $m \in \mathbb{Z}$. Therefore, we can say that

$$\int_0^{2\pi} d\theta \frac{d\Phi[A, \theta]}{d\theta} = 2m\pi . \tag{2.101}$$

Now, consider the gauge variation of the Euclidean effective action $W_{\text{eff}}[A]$:

$$\begin{aligned}\frac{\partial W_{\text{eff}}[A(\theta,x)]}{\partial\theta} &= \int d^{2n}x\ \frac{\delta W_{\text{eff}}[A(\theta,x)]}{\delta A_j^\theta}\cdot\frac{\partial A_j^\theta}{\partial\theta}\\ &= -\int d^{2n}x\ \Lambda(\theta,x)\cdot D_j^\theta\left(\frac{\delta W_{\text{eff}}}{\delta A_j^\theta}\right)\ ,\end{aligned}\tag{2.102}$$

where $\Lambda = g^{-1}\partial_\theta g$, and we used the fact that

$$\frac{dA_j^\theta}{d\theta} = \partial_j\Lambda + [A_j^\theta,\Lambda] = D_j^\theta\Lambda\ .\tag{2.103}$$

The equation (2.102) is the definition of the consistent anomaly, which refers to the expression of anomaly obtained from the variation of effective action. We note that this is in fact related to the winding number $m$ defined above:

$$2\pi m i = i\frac{\partial\Phi[A,\theta]}{\partial\theta} = -\frac{\partial W_{\text{eff}}}{\partial\theta}.\tag{2.104}$$

This relation implies that the consistent anomaly is given by the winding number,

$$m = \frac{1}{2\pi i}\int_0^{2\pi}d\theta\int d^{2n}x\ \Lambda(\theta,x)\cdot D_j^\theta\left(\frac{\delta W_{\text{eff}}[A(\theta,x)]}{\delta A_j^\theta}\right)\ .\tag{2.105}$$

How is this quantity related to the index of $2n+2$ dimensional Dirac operator? Let us introduce yet another parameter $r$ such that $(r,\theta)$ parametrizes a two-dimensional disk. Furthermore, we define $A(r,\theta,x)$ so that $A(r=1,\theta,x) = A(\theta,x)$. Then, the following two statements can be shown to be true. [22] 1) The winding number around the boundary of the disk equals the sum of local winding number around each zeros of $\text{Det}(i\tilde{D}[A(\rho,\theta,x)])$. 2) Zeros of $\text{Det}(i\tilde{D}[A(\rho,\theta,x)])$ are in 1-1 correspondence with the zero modes of $i\not{D}_{2n+2}$. Especially, according to the chirality of each zero modes, it gives weight $\pm1$ to the winding number. Combining these facts together, from the definition of the index, we find that

$$\mathrm{ind}(i\not{D}_{2n+2}) = \frac{1}{2\pi i}\int d\theta \int d^{2n}x\ \Lambda(\theta,x)\cdot D_i^\theta\left(\frac{\delta W_{\mathrm{eff}}[A(\theta,x)]}{\delta A_j(\theta,x)}\right) . \quad (2.106)$$

As can be repeatedly seen in the previous subsections, actual anomaly in $2n$ dimensions can be obtained from $2n+2$-form of certain gauge invariant polynomials. This procedure of extracting anomaly can be summarized into so called descent procedure of the characteristic classes. In order to explicitly see this, let us consider the two dimensional auxiliary space parametrized by $(r,\theta)$. If we glue two such disks $D_+$ and $D_-$ at the boundaries where the gauge fields are defined through a gauge transformation,

$$A_+(r,\theta,x) = g^{-1}(\theta,x)(A(x)+d+d\theta\partial_\theta)g(\theta,x) , \quad \text{at } D_+ \quad (2.107)$$

$$A_-(r,\theta,x) = A(x) , \quad \text{at } D_- \quad (2.108)$$

where $d$ is a differential in the space of $x$. (Note that, since $g$ is independent of $r$, it can be thought of as a differential in the $x,r$ space.) Then, this additional two-dimensional space can be regarded as $S^{2n}$. Now, consider an index of $2n+2$ dimensional operator given by a characteristic class $P(F)$,

$$\mathrm{ind}(i\not{D}_{2n+2}(A)) = \frac{i^{n+1}}{(2\pi)^{(n+1)}}\int_{S^2\times S^{2n}} P_{2n+2}(F) . \quad (2.109)$$

Since the gauge field $F$ is closed, $dP(F)=0$, it follows that, locally there exists $2n+1$ form such that

$$P_{2n+2}(F) = dP^{(0)}_{2n+1}(F,A) . \quad (2.110)$$

Then we can rewrite the equation as

$$\begin{aligned}
\text{ind}(i\not{D}_{2n+2}(A)) &= \frac{i^{n+1}}{(2\pi)^{n+1}} \int_{D^+\times S^{2n}} dP^{(0)}_{2n+1}(A^+, F) + \int_{D^-\times S^{2n}} dP^{(0)}_{2n+1}(A^-, F) \\
&= \frac{i^{n+1}}{(2\pi)^{n+1}} \int_{S^1\times S^{2n}} P^{(0)}_{2n+1}(A^+, F) \\
&= \frac{i^{n+1}}{(2\pi)^{n+1}} \int_{S^1\times S^{2n}} P^{(0)}_{2n+1}(A^\theta(x) + d\theta\Lambda(\theta,x), F^\theta(x)) - P^{(0)}_{2n+1}(A^\theta(x), F^\theta(x)) \\
&= \frac{i^{n+1}}{(2\pi)^{n+1}} \int_{S^1} d\theta \int_{S^{2n}} P^{(1)}_{2n}(\Lambda(\theta,x), A^\theta(x), F(\theta,x)) \ . \qquad (2.111)
\end{aligned}$$

From the first to second line, the second term vanishes simply because $P^{(0)}_{2n+1}(A^-, F)$ does not contain any differential in $\theta$ direction. And we substracted the second term in third line for the same reason. For the last line, we further defined a $2n$ form by

$$\delta_\Lambda P^{(0)}_{2n+1}(A, F) = dP^{(1)}_{2n}(\Lambda, A, F) \ . \qquad (2.112)$$

Comparing (2.111) with (4.110) and (2.106), and fix $\theta$ at certain value, we obtain the expression for anomaly in $2n$ dimension,

$$\delta_\Lambda W_{\text{eff}} = \frac{i^n}{(2\pi)^n} \int_{S^{2n}} P^{(1)}_{2n}(\Lambda, A, F) \ . \qquad (2.113)$$

Although the actual anomalies are given by the descent $P^{(1)}_{2n}$ of the gauge invariant characteristic classes, when we check the cancelation of anomalies for given theory, it is enough to check the cancelation of the $2n+2$ form expression $P_{2n+2}$ only. It does not seem to be enough, since for given $P_{2n+2}$ form, its descent $P^{(1)}_{2n}$ is not unique. However, this fact precisely corresponds to the ambiguity in the definition of the 1-loop anomaly. The first ambiguity comes from the fact that we can always add a local counterterm $\Gamma = \int \alpha_{2n}$ to the action, which does not alter the equation of motion. Since $\Gamma$ does not need to be gauge-invariant, it can be source of the anomaly, which shift $P^{(1)}_{2n} + \delta\alpha_{2n}$. Furthermore, from the expression (2.113), we can see that addition of an exact term $d\beta_{2n-1}$ to $P_{2n^{(1)}}$ does not change the result, if the manifold does not have a boundary. From these facts, there are equivalence relation at the level of anomaly,

$$P_{2n}^{(1)} \sim P_{2n}^{(1)} + \delta\alpha_{2n} + d\beta_{2n-1} \ . \tag{2.114}$$

Using the fact that $d$ and $\delta$ commute, it is easy to see that this equivalence class defines unique gauge invariant form $P_{2n+2}(F)$.

## 2.4 Supersymmetric localization and exact partition functions

In this section, we turn to the study of partition functions in supersymmetric gauge theories, which will be substantially used for Chapter 4. Compared to the index studied in the previous sections which only contains information about ground states, partition function contains information of all the excited states as well, and in general very difficult to exactly calculate. However, recently, for theories equipped with certain amount of supersymmetries, there has been extensive developements of so called *supersymmetric localization technique*, which enables us to exactly calculate indices and partition functions of superconformal theories on spheres in various dimensions. The most prominent example is exact calculation of partition functions for $\mathcal{N} = 2$ supersymmetric Yang-Mills (SYM) theories and the Wilson loop expectation values thereof, which was done by Pestun 2007. [7] First of all, one should construct the Lagrangian on the four-sphere which preserves supersymmetries. In general, there are two different ways to put the supersymmetric theories on curved spaces. The first is so called *twisting*, which modifies the theory by turning on the current of global symmetry so that they cancel the curvature of the manifold. Then, part of the supercharge becomes scalar under new Lorentz symmetry, and they are preserved for curved spacetime. This procedure makes the theory topological. The second method is so called *rigid supersymmetry* on the curved manifolds, which is main concern of this section. Given a conformal field theory on a flat space, one can write the theory on a sphere by a stereographic projection. The Lagrangian can be written as flat Lagrangian with additional correction terms which depends on the curvature. For example, the action of $\mathcal{N} = 4$ SYM on $S^4$ can be written as [7]

$$S = \frac{1}{g_{YM}^2} \int_{S^4} \sqrt{g}\; d^4x\; \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \partial_\mu \Phi^a \partial^\mu \Phi^a + \frac{2}{r^2} \Phi^a \Phi^a - \Psi \Gamma^M D_M \Psi \;, \quad (2.115)$$

where fermions are combined and written in terms of a ten-dimensional Majorana-Weyl fermion $\Psi$. Here $r$ is a radius of four sphere, and the third term corresponds to the curvature correction to the flat Lagrangian. This action enjoys $\mathcal{N} = 4$ superconformal symmetry;

$$\begin{aligned} \delta_\epsilon A_M &= \epsilon \Gamma_M \Psi \\ \delta_\epsilon \Psi &= \frac{1}{2} F_{MN} \Gamma^{MN} \epsilon + \frac{1}{2} \Gamma_{\mu a} \Phi^a \nabla^\mu \epsilon \;. \end{aligned} \quad (2.116)$$

Here $\epsilon$ is a supersymmetric transformation parameter which satisfies the *conformal Killing spinor equation*,

$$\begin{aligned} \nabla_\mu \epsilon &= \tilde{\Gamma}_\mu \tilde{\epsilon} \\ \nabla_\mu \tilde{\epsilon} &= -\frac{1}{4r^2} \Gamma_\mu \epsilon \;, \end{aligned} \quad (2.117)$$

where $\Gamma_\mu : S^+ \to S^-$ is an off-diagonal component of the ten-dimensional gamma matrix and $\tilde{\Gamma}_\mu$ is its conjugate. Furthermore, one can show that square of the (2.116) gives

$$\delta_\epsilon^2 = -\mathcal{L}_{\epsilon \gamma^M \epsilon} - R - D_{\epsilon \tilde{\epsilon}} \;, \quad (2.118)$$

where the right hand side are bosonic symmetry generators of the theory which include the R-symmetry and dilatation. In order to calculate the partition function defined as

$$Z = \int [d\phi]\; e^{-S_E(\phi)} \;, \quad (2.119)$$

where $\phi$ denotes for all the dynamical fields in the theory, we consider following deformation,

$$Z(t) = \int [d\phi] \; e^{-S_E(\phi) - tQV} \;, \tag{2.120}$$

where $t$ is an arbitrary number, $Q$ is a supersymmetric variation

$$Q = \delta_\epsilon \phi \cdot \frac{\partial}{\partial \phi} \;, \tag{2.121}$$

and $V$ is some fermionic fields. When we choose $V$ such that $Q^2 V = 0$, in other words, when $V$ is invariant under the bosonic symmetries of the theory (see eq. (2.118)), one can show that the integral is independent of the parameter $t$.

$$\begin{aligned} \frac{\partial Z}{\partial t} &= -\int [d\phi] \; (QV) \; e^{-S_E(\phi) - tQV} \\ &= -\int [d\phi] \; Q\left(V \; e^{-S_E(\phi) - tQV}\right) = 0 \;, \end{aligned}$$

if the integrand behaves nicely at the infinity of field space $\phi$. From the first to second line, we used the fact that the classical action is closed under supersymmetric transformation. For the last equailty, we note that $Q$ becomes total derivative and the measure is invariant under the supersymmetric variation. This observation tells us that the result does not change if we take the limit $Z(t \to \infty)$. In this limit, the field field configurations are localized to the locus $\phi = \phi_0$ which satisfies $QV = 0$, and other contributions are exponentially suppressed. In this limit, all the fields can be expanded as

$$\phi = \phi_0 + \frac{1}{\sqrt{t}} \delta\phi \;, \tag{2.122}$$

and one-loop determinant of the fluctuation gives exact answer to the integral. Most convinient choice of the fermionic term $V$ is

$$V = \sum_\psi \langle \psi, \overline{Q\psi} \rangle \tag{2.123}$$

where the summation is over all the fermions in the theory, and $\langle\ ,\ \rangle$ is a inner product defined in the field space. By construction, $QV$ is positive definite, and the localization locus is given by $Q\psi = 0$. For $\mathcal{N} = 4$ SYM on $S^4$, the result is given by [7]

$$Z_{S^4} = \int [da]\ e^{\frac{-8\pi^2 r^2 a^2}{g_{YM}^2}} Z_{\text{1-loop}}(ia)|Z_{\text{inst}}(ia, r^{-1}, r^{-1}, q)|^2\ , \tag{2.124}$$

where the result is given in terms of the integral over real parameter $a$, which is saddle point value of the one component of vector multiplet scalar $\Phi_0$. The first factor is contribution from the classical action, and the second factor comes from the 1-loop determinant near the saddle configuration. The last factor is instanton contribution localized at north and south poles of the sphere respectively. Interestingly, $Z_{\text{inst}}(ia, r^{-1}, r^{-1}, q)$ is the Nekrasov's instanton partition function [25–27] with $\epsilon_1 = \epsilon_2 = r^{-1}$. This behavior of the factorization of partition function are observed to be general for sphere partition functions in other dimensions. [14, 28, 29]

After this seminal work, the partition functions on spheres in various dimensions are calculated. In three dimension, Kapustin, Willet and Yaakov obtained partition functions for the $\mathcal{N} = 2$ theories on $S^3$. [30] With this result, various conjectured three dimensional dualities are proved, and more recently, it was shown that this quantity calculates entanglement entropy across $S^1$ in $\mathbb{R}^{3,1}$. [31] In five dimensions, partition functions for SYM theory on $S^5$ are calculated by [32]. This result is of particular importance since this theory is known to probe the six dimensional $(2, 0)$ theories on M5-branes, which does not admit Lagrangian description. Most recently, along the lines of these developements, partition functions on $S^2$ are studied. [8, 9] Soon after that, it was realized through a series of works [12–15, 29] that these results calculates very useful quantity which is related to the geometry of Calabi-Yau manifolds where string theory is based on. This is the main subject of the Chapter 4 of this thesis.

# Chapter 3

# Applications of Index Theorems in String Theory

In this chapter, as another important application of the index theorem in string theory, we study the problem of counting BPS states of supersymmetric theories. We will mainly focus on theories with eight real supercharge ($\mathcal{N} = 2$) in four-dimensions, which can be obtained from compactifying Type II string theory on six dimensional Calabi-Yau manifold. In the first section, we briefly review basics of four-dimensional $\mathcal{N} = 2$ theories obtained in this way, and present the physical explanation of the wall-crossing phenomena which is the prominent feature of $4d$ $\mathcal{N} = 2$ theories. Contents after the section 3.2 are based on the work [10].

## 3.1 BPS States and Wall-Crossing in $4d$ $N = 2$ theories

Four-dimensional $N = 2$ theory has two supercharges which will be denoted as $Q^I_\alpha$, $\bar{Q}^I_{\dot{\alpha}}$, where $I = 1, 2$. They satisfy the following supersymmetry algebra,

$$\begin{aligned} \{Q^I_\alpha, \bar{Q}_{\dot{\beta}J}\} &= 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^I_J \\ \{Q^I_\alpha, Q^J_\beta\} &= 2\epsilon_{\alpha\beta}\epsilon^{IJ} Z \\ \{\bar{Q}_{\dot{\alpha}I}, \bar{Q}_{\dot{\beta}J}\} &= -2\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{IJ}\bar{Z} \ , \end{aligned} \tag{3.1}$$

where the conjugate supercharge is defined by $\bar{Q}_{\dot{\alpha}I} = (Q^I_\alpha)^\dagger$. Furthermore, the index $I$ can be lowered and raised by $\epsilon_{IJ}Q^J_\alpha = Q_{I\alpha}$ and $\epsilon^{12} = \epsilon_{21} = 1$. $Z$ is complex central charge of $N = 2$ algebra. In addition to these, we have bosonic R-symmetry which is $SU(2)_R \times U(1)_R$. Here $SU(2)_R$ rotates $I$ indices, and under $U(1)_R$, $Q^I$ and $\bar{Q}_I$ has charge 1 and $-1$ respectively. Let us assume $Z \neq 0$, and consider massive representations of Lorentz group, $SO(3)$. In order to see the particle spectrum of this algebra, define

$$\begin{aligned} a^I_\alpha &= e^{-i\delta/2}Q^I_\alpha + e^{i\delta/2}\bar{Q}^{\dot{\beta}I}\sigma^0_{\dot{\beta}\alpha} \\ b^I_\alpha &= e^{-i\delta/2}Q^I_\alpha - e^{i\delta/2}\bar{Q}^{\dot{\beta}I}\sigma^0_{\dot{\beta}\alpha} \ . \end{aligned} \tag{3.2}$$

Here the factor $e^{\pm i\delta/2}$ denotes for a possible $U(1)_R$ rotation of the supercharges. They satisfy

$$\begin{aligned} \{a^I_\alpha, (a^J_\beta)^\dagger\} &= 4(M - \mathrm{Re}(Ze^{-i\delta}))\delta_{\alpha\beta}\delta^{IJ} \ , \\ \{b^I_\alpha, (b^J_\beta)^\dagger\} &= 4(M + \mathrm{Re}(Ze^{-i\delta}))\delta_{\alpha\beta}\delta^{IJ} \ , \\ \{a^I_\alpha, (b^J_\beta)^\dagger\} &= \{a^I_\beta, (b^J_\alpha)^\dagger\} = 0 \ . \end{aligned} \tag{3.3}$$

Then, positivity of the Hilbert space requires

$$M \geq \mathrm{Re}(Ze^{-i\delta}) \ . \tag{3.4}$$

The most strict condition comes from the case $\delta = \arg Z$. If we require this, we have

$$M \geq |Z| \ , \tag{3.5}$$

which is called the BPS (Bogomol'nyi-Prasad-Sommerfield) bound. If we restrict to the strict inequality $M > |Z|$, the states generated by the above algebra are

$$\left\{ |0\rangle, \ \ a_2^1|0\rangle, \ \ a_2^2|0\rangle, \ \ a_2^1 a_2^2|0\rangle \right\} \otimes \left\{ |\tilde{0}\rangle, \ \ b_2^1|\tilde{0}\rangle, \ \ b_2^2|\tilde{0}\rangle, \ \ b_2^1 b_2^2|\tilde{0}\rangle \right\} , \tag{3.6}$$

where $a_1^I|0\rangle = b_1^I|\tilde{0}\rangle = 0$. These multiplets are sometimes called *long multiplets*. The spacetime spin contents are

$$\left(2\,[0] + \left[\frac{1}{2}\right]\right) \otimes \left(2\,[0] + \left[\frac{1}{2}\right]\right) \otimes [j] \tag{3.7}$$

where $[j]$ is a possible representation of the vacuum. When the equality of (3.45) is met, all the states in the second half of (3.6) vanish. We call these multiplets as *short multiplets*, or *BPS multipets*. The simplest BPS multiplet is

$$\left(2\,[0] + \left[\frac{1}{2}\right]\right) \ , \tag{3.8}$$

which is the half-hyper multiplet. In particular, two real scalars transform as spin $[\frac{1}{2}]$ in $SU(2)_R$ global symmetry. The next simplest BPS multiplet is obtained by assigning charge $[\frac{1}{2}]$ to the vacuum. This is the BPS vector multiplet,

$$2\left[\frac{1}{2}\right] + [1] + [0] \ . \tag{3.9}$$

Note that it consists of a massive vector field, a real scalar, and a Dirac spinor. These BPS multiplets are of particular interests when we study supersymmetric theories. First of all, they are very useful in investigating non-perturbative aspects of the supersymmetric theories, since they are rigid under continuous deformation of the theories. Secondly, these states are believed to be responsible for microscopic entropy of the extremal black-holes. Finally, they can be used to study properties of Calabi-Yau varieties and their cycles.

The four-dimensional $N = 2$ supersymmetric theories can be realized in string theory by compactifying Type II string theory on certain Calabi-Yau threefold (CY3). For simplicity, let us consider Type IIB string theory on $X$ which is CY3. For this case, BPS particles in non-compact four-dimensions can be obtained from wrapping D3-branes on special Lagrangian three-cycles in $X$. If there are $N$ coincident such D3-branes, dynamics the BPS particles obtained from them are subject to the gauged supersymmetric quantum mechanics with gauge group $U(N)$, which preserves four real supersymmetry. The resulting four-dimensional low-energy theory is $N = 2$ supergravity coupled to $h^{2,1}$ abelian massless gauge fields. In particular, the latter can be obtained from the self-dual five form Ramond-Ramond field strength $G_5$ coupled to D3-branes. We can write it as

$$G_5 \in \Omega^2(M_4) \otimes H^3(X, \mathbb{Z}) \ , \tag{3.10}$$

where $H^3(X, \mathbb{Z})$ is a symplectic lattice structure on $X$, whose basis $\alpha_I$ and $\beta^I$ $(I = 1, \cdots h^{2,1})$ satisfy

$$\begin{aligned} \langle \alpha_I, \alpha_J \rangle &= 0 \\ \langle \beta^I, \beta^J \rangle &= 0 \\ \langle \alpha_I, \beta^J \rangle &= \delta_I^J \ . \end{aligned} \tag{3.11}$$

As a result, in $M_4$, we have two-form abelian gauge field strength $F^I$ and their dual $\tilde{F}_I \equiv *_4 F^I$, by

$$G_5 = \alpha_I F^I + \beta^I \tilde{F}_I \ . \tag{3.12}$$

On the other hand, central charge of the BPS particle obtained from wrapping $D3$-brane on a three-cycle $\Gamma$ is naturally defined as

$$Z(\Gamma) = \int_X \Gamma \wedge \Omega, \tag{3.13}$$

where $\Omega$ is a holomorphic three-form of $X$. (We used same notation $\Gamma$ for its Poincare dual.) We can also define (topological) intersection product between cycles,

$$\langle \Gamma_1, \Gamma_2 \rangle = \int_X \Gamma_1 \wedge \Gamma_2 \ . \tag{3.14}$$

Now, let us restrict our attention to the four-dimensional supersymmetric gauge theory obtained as above. If the gauge group is $G$ of rank $r$, due to the potential of the scalar component of the vector multiplet,

$$\mathrm{Tr}[\phi^\dagger, \phi]^2 \ , \tag{3.15}$$

the moduli space of the coulomb branch is parametrized by abelian $U(1)^r$ Maxwell theory. The parameter of coulomb branch is $u^{i=1,\cdots r}$, which is made of VEV of the vector multiplet scalars. For example, if $G = SU(N)$, $u^i$ are chosen to be $\langle \mathrm{Tr}\ \phi^i \rangle$ for $i = 2 \cdots N$. Since the low-energy theory is rank $r$ $U(1)$ Maxwell theory, we can naturally define electric and magnetic charge associated to them. Then the theory is equipped with symplectic structure of lattice dimension $2r$, which is given by the Dirac-Schwinger-Zwanziger product, $\langle \gamma_a, \gamma_b \rangle \in \mathbb{Z}$. This quantization relation exactly corresponds to (3.14) of internel Calabi-Yau space.

From the seminal work of Seiberg and Witten [35, 36], it was shown that the low-energy effective theories are severely constrained by $N = 2$ supersymmetry. The effective action is determined by the prepotential $F$ which is a holomorphic function in $\langle \phi \rangle$. If we let $\langle \phi \rangle = \mathrm{diag}(a^1, \cdots, a^r)$, the bosonic part of the effective action can be written as

$$S_{\text{eff}} = \frac{1}{4\pi} \int_{M_4} \left( \text{Im} \frac{\partial^2 F}{\partial a^I \partial a^J} \right) \left( da^I \wedge *da^J + F^I \wedge *F^J \right) + \left( \text{Re} \frac{\partial^2 F}{\partial a^I \partial a^J} \right) \; F^I \wedge F^J \; . \tag{3.16}$$

Exact form of the prepotential $F$ can be determined from the information of BPS spectrum of the theory. The central charge of a BPS state with charge $\gamma$, which is a key quantity to study the BPS spectrum, is a function of moduli $u^i$ and $\gamma$. In particular, it is linear in $\gamma$, i.e.,

$$Z(u, \gamma_1) + Z(u, \gamma_2) = Z(\gamma_1 + \gamma_2, u) \; . \tag{3.17}$$

We require that $\alpha^I$, a basis of the symplectic lattice structure, satisfies

$$a_I = Z(u, \alpha_I) \; . \tag{3.18}$$

Then, the dual coordinate on the moduli space $a^{D,I}$ can be defined by

$$a^{D,I} \equiv \frac{\partial F(a)}{\partial a^I} = Z(u, \beta^I) \; . \tag{3.19}$$

Hence, for a given supersymmetric cycle $\gamma = p^I \alpha_I + q_I \beta^I$, in $X$, the central charge of corresponding particle is

$$Z(u, \gamma) = p^I a_I + q_I a^{D,I} \; . \tag{3.20}$$

The BPS states are known to be invariant under the continuous deformation of the parameters, but this is not true at every point of the moduli space. In the moduli space, we can find a co-dimension one *wall of marginal stability* such that as we cross that, certain BPS states suddenly dissappear. This is what is called the *wall-crossing* phenomenon, and this is one of the reason that makes $4d$ $N = 2$ theory much more interesting. This phenomenon can be easily seen from the examaple of pure $SU(2)$ Seiberg-Witten theory, as in the figure below.


FIGURE 3.1: Moduli space and BPS spectra of $4d$ $N = 2$ pure SU(2) theory. The line denotes for the wall of marginal stability defined as $\arg(a) = \arg(a_D)$, and the two points on that indicate where monopole of charge $(0, 1)$ and the dyon $(2, -1)$ become massless respectively. Being massless, they can be BPS states at both side of the wall. At the weak coupling regime, there are additional infinite tower of BPS states which can be understood as the bound states of the two states.

From the work of Denef and Moore, [45–47], the problem of identifying the BPS states that disappear across the wall has been translated in term of the bound state formation problem of the BPS states. To illustrate this, let us look at the BPS equation given by the supersymmetric variation of the fermions in the vector multiplet, which reads

$$\begin{aligned} F_{0i} - \frac{i}{2}\epsilon_{ijk}F_{jk} - iD_i(e^{-i\delta}\phi) = 0 \\ D_0(e^{-i\delta}\phi) - \frac{1}{2}[\phi^\dagger, \phi] = 0 \ , \end{aligned} \tag{3.21}$$

where $\delta$ is defined in equation (3.2), the phase of the central charge of this BPS configuration. For the abelian gauge group $U(1)^r$, they imply

$$F^{I+}_{0i} = \partial_i(e^{-i\delta}a^I) \ , \quad \partial_0 a^I = 0 \ , \tag{3.22}$$

where $F^{I+}$ is self-dual part of the gauge field strength. These BPS equations can be solved by assuming spherically symmetric field strength. It can be shown that they lead to

$$2\text{Im}[e^{-i\delta}Z(\gamma_p, u(r))] = -\frac{\langle\gamma_p, \gamma\rangle}{r} + 2\text{Im}[e^{-i\delta}Z(\gamma_p, u(\infty))] \ , \tag{3.23}$$

where $\gamma$ is a charge of the configuration (3.21), and $\gamma_p$ is that of the probe particle. If there are two such BPS particle, it becomes

$$2\text{Im}[e^{-i\delta_1}Z(\gamma_2, u(r))] = -\frac{\langle\gamma_1, \gamma_2\rangle}{r} + 2\text{Im}[e^{-i\delta_1}Z(\gamma_2, u(\infty))] \ . \tag{3.24}$$

Note that when $\arg Z(\gamma_1) = \arg Z(\gamma_2)$, i.e., when they enjoy the same supersymmetry, the following relation holds,

$$|Z(\gamma_1)| + |Z(\gamma_2)| = |Z(\gamma_1 + \gamma_2)| \ , \tag{3.25}$$

and these two states form a bound state which is BPS. One can easily extract the radius of the bound state,

$$R = \frac{1}{2}\frac{\langle\gamma_1, \gamma_2\rangle}{\text{Im}[e^{-i\delta}Z(\gamma_2, u(\infty))]} \ . \tag{3.26}$$

From this relation, normalizable bound state exists only when $\sin(\delta_2 - \delta_1) > 0$. When $\delta_2$ approaches $\delta_1$ so that eventually $\sin(\delta_2 - \delta_1) \leq 0$, the radius diverges and the corresponding bound states disappear. This is how we physically understand the wall-crossing phenomena. In order to explicitly calculate the number of BPS states at both side of the wall, we should define an index which is non-vanishing only for the BPS states. For this, we introduce the *the second helicity trace*, defined by

$$\Omega(u, \gamma) = -\frac{1}{2}\text{Tr}(-1)^{2J_3}(2J_3)^2 \ . \tag{3.27}$$

The angular momentum operator $J$ used to define the fermion number is $SU(2)_L$ generator which corresponds to the spatial rotation,

$$J = \frac{1}{2} \sum_{i<j} \langle \gamma_i, \gamma_j \rangle \, \frac{\vec{x}_i - \vec{x}_j}{|x_i - x_j|} \ . \tag{3.28}$$

Here $\vec{x}_i$ is a three-vector which defines the position of a BPS particle of charge $\gamma_i$. One can straightforwadly show that $\Omega(u, \gamma)$ vanishes for any long multiplet which include the factor of (3.7). On the other hand, for the half-hyper multiplet, $\Omega(u, \gamma)$ gives 1 and for the BPS vector multiplet, it gives $-2$. Moreover, for a BPS states in the representation

$$\mathcal{R} = \left( 2\,[0] + \left[\frac{1}{2}\right] \right) \otimes [j] \ , \tag{3.29}$$

where the first hypermultiplet factor denotes for the center of mass degrees of freedom, the index can be simplifed as

$$\Omega(u, \gamma) = -\frac{1}{2} \mathrm{Tr}_{\mathcal{R}} (-1)^{2J_3} (2J_3)^2 = \mathrm{Tr}_j (-1)^{2J_3} \ , \tag{3.30}$$

the usual Witten idex. For example, when a state $\gamma_1 + \gamma_2$ disappears across a marginal stability wall, and dissociate into $\gamma_1$ and $\gamma_2$ on the other side, the indices of these three kind of BPS particles are known to obey a universal formula

$$\Omega^-(\gamma_1 + \gamma_2) = (-1)^{|\langle \gamma_1, \gamma_2 \rangle| - 1} |\langle \gamma_1, \gamma_2 \rangle| \Omega^+(\gamma_1) \Omega^+(\gamma_2) \ , \tag{3.31}$$

where $\pm$ denote the two sides of the wall. This simplest wall-crossing formula has been studied in many examples, generalized to the so-called semi-primitive cases for $\gamma_1 + k\gamma_2$ states [47] and most recently embedded into an algebraic reformulation by Konsevitch and Soibelman [48], which in turn was explained in more physical basis [49–52].

In Ref. [53], a new approach to the low energy dynamics of dyons in generic $N = 2$ Seiberg-Witten theory was proposed. Assuming that bound states of interest are large, which is always true whenever the theory is near a wall of marginal stability, the author showed how a $\mathcal{N}=2$ supersymmetric dynamics can be explicitly written from the special Kahler data of the vacuum moduli space only. When applied in the limit of a simgle dynamical probe dyon in the presence of another (very massive) BPS state, the bound states can be constructed explicitly and counted, again confirming the above primitive wall-crossing formula. It is abundantly clear that his method can be used for an arbitrary number and varieties of dyons, as well, as long as the proximity to a marginal stability wall is satisfied. In the following sections, we substitute the reference [10], where they set up dynamics of arbitrary number of dyons near such a wall, with $\mathcal{N}=4$ supersymmetry, and generate wall-crossing formula via index theorem.

The first improvement concerns the question of what is the relevant index theorem. In the Denef's Coulomb phase approach, the most comprehensive studies to date involve a truncation of dynamics where one ends up with a geometric quantization problem on the classical moduli space of charge centers, which are typically compact. In this paper, we denote such moduli spaces for $n$ centers as $\mathcal{M}_n$. For two-center case, this manifold is always $S^2$. The Lagrangian has no kinetic term, but a minimal coupling to certain magnetic field induces a symplectic structure on the moduli space, making it a phase space. In turns out, however, the naive low energy dynamics on this classical moduli space on $\mathcal{M}_n$ end up with too many fermionic degrees of freedom. The anticipated and empirically correct answer, which is a Dirac index [57], results only if one can somehow remove half of the fermions. This deficiency has remained unresolved until now.

In the sections below, we will explain why the naive truncation to $\mathcal{M}_n$ was ill-motivated. It turns out that there is no separation of scales, and all $3n$ bosons and $4n$ fermions are of equal massgap. Instead, one can choose to reduce the index problem to $\mathcal{M}_n$ by deforming the theory with supersymmetry partially broken. As long as there is one supersymmetry left unbroken and since the quantum mechanics has a gap, the index is left invariant under the deformation. At the end of the day, we will thus have provided an ab initio derivation of the anticipated Dirac index on $\mathcal{M}_n$, for the first time.

The second concerns the physical interpretation of certain rational invariants, defined and extensively used by Manschot et.al. [54], of the form

$$\bar{\Omega}(\gamma) = \sum_{p|\gamma} \frac{\Omega(\gamma/p)}{p^2} , \tag{3.32}$$

where the sum is over divisors of $\gamma$. The expression naturally appears in other formulation of the wall-crossing, most notably in Konsevitch-Soibelmann. In the course of enumerating the bound states of Bosonic or Fermionic statistics, we will encounter $\Omega(\beta)/p^2$ as a universal effective degeneracy of $p$ identical particles of charge $\beta$. It appears as the multiplicative factor from the normal bundle as one computes contributions from a submanifold fixed by the permutation group of order $p$.[1]

Along the way, our work also clarifies relation between the field theory indices, namely the second helicity trace and the protected spin character, and the quantum mechanical ones. Quantum mechanical index usually suffers from ambiguity over the definition of $(-1)^F$. Usual index formulae relies on certain (mathematically) canonical choice of $(-1)^F$. Retaining three bosonic coordinates per dyons allow us to inherit both the spatial rotation group, denoted by $SU(2)_L$, and the R-symmetry of $N = 2$ field theory, $SU(2)_R$. The supersymmetries belong to $(2, 2)$ representation, so both $(-1)^{2J_3}$ of $SU(2)_L$ and $(-1)^{2I_3}$ of $SU(2)_R$ are chirality operators. The second helicity trace is then computed unambiguously by $\mathrm{Tr}(-1)^{2J_3}$. We in turn relate the latter to $\mathrm{Tr}(-1)^{2I_3}$ which turns out to be equivalent to the canonical choice leading to the usual Dirac index formula. This derives, for the first time, the well-known sign pre-factors in the wall-crossing formulae universally. In addition, we also explain why the protected spin character of the field theory is

[1] This same numerical factor $1/p^2$ had appeared before in the context of the D-brane bound state problems of 1990's [58, 59], where identical nature of the D-branes were also of some importance.

actually computed by equivariant index, by showing that the quantum mechanical "angular momentum" operator that appears in the latter is actually a diagonal sum, $J_3 + I_3$, from the spacetime viewpoint.

The chapter is organized as follows. Section 2 reviews Ref. [53] and generalize the low energy dynamics to the case of arbitrary number of dynamical charge centers, and note the universal nature of the potential terms. Section 3 defines the index as a method of BPS bound state counting, and in particular make contact with the field theory indices, commonly known as the 2nd helicity trace and its generalization known as the protected spin character. It turns out that the quantum mechanics found have $SU(2)_L \times SU(2)_R$ R-symmetry, each of which defines chirality operators $(-1)^{2J_3}$ and $(-1)^{2I_3}$. The field theory index corresponds to the former, while mathematical index formulae are more directly related to the latter. We discuss a universal relationship between the two, and conjecture that all BPS bound states in our quantum mechanics are all $SU(2)_R$ singlets.

Section 4 sets up index theorem for this dynamics and show how reduction to the classical moduli manifold may be achieved. Here we show why the naive derivative truncation leading to the geometric quantization is unjustified by demonstrating that there is no natural separation of scales between classically massive directions and classically massless directions. The main point is that $\mathcal{M}_n$ is of finite size, and the quantum gaps due to this are always equal to those along the classically massive directions. We show, nevertheless, how one can deform the theory while preserving the index, such that classically massive modes are decoupled from the evaluation of the index, at the cost of partially broken supersymmetry. We also observe that the reduction process keeps a diagonal subgroup $SU(2)_{\mathcal{J}}$, and identify the generator $\mathcal{J}_3 = J_3 + I_3$ as the operator usually used for equivariant index computations. This way, we show that the equivariant index of quantum mechanics on $\mathcal{M}_n$ actually computes the protected spin character of $N = 2$ field theory.

After the derivation of Dirac index in section 4, we go on to evaluate in section 5 the wall-crossing formula by taking into account the Bosonic or the Fermionic statistics. Projection operators are introduced for the purpose, and the index formula is decomposed into additive contributions from various fixed submanifolds associated with coincident identical particles. The reduced index problems on the fixed submanifolds appears in the full index with a universal degeneracy factor $\sim 1/p^2$, which arises from orbifolding action of the $p$-th order permutation group $S(p)$. Summing up all relevant contributions, we find an expression identical to Manschot et.al.'s wall-crossing formula. We close with summary and comments in section 6.

## 3.2 $\mathcal{N} = 4$ Moduli Mechanics for $n$ BPS Objects

In Ref. [53], a general framework for deriving moduli dynamics of dyons of Seiberg-Witten theory was given under the assumption that one works in the field theory vacuum where the central charge are almost aligned in terms of the phases of the respective central charges; in other words, very near the marginal stability wall. This program was then carried out explicitly when one can treat only one dyon as dynamical, with other dyons as external objects. In this note, we wish to generalize this to arbitrary number of charge centers, be they field theory dyons or charged black holes. For this, all dyons should be treated as dynamical, and we will denote their charges as $\gamma_A$'s. For the above derivation of one dynamical center, the proximity to a marginal stability wall played an essential role, allowing the nonrelativistic approximation and thus the moduli space approximation possible, so we need to retain this assumption.

While the moduli dynamics should have $\mathcal{N} = 4$ supersymmetry, as demanded by the BPS nature of the dyons, simple off-shell $\mathcal{N} = 4$ descriptions fail to accommodate key interaction terms. Furthermore, as we will see in section 4 where we compute the supersymmetric index, it is more convenient to take one of the four supersymmetries, say $Q_4$, and give up others. For these reasons, we employ the $\mathcal{N}=1$ superspace [60] where this supersymmetry, $Q_4$, is manifest. We package $3n$ bosonic coordinates, $x^{Aa}$, and $4n$ fermionic superpartners, $\psi^{Aa}$ and $\lambda^A$, as

$$\Phi^{Aa} = x^{Aa} - i\theta\psi^{Aa}, \quad \Lambda^A = i\lambda^A + i\theta b^A \ , \tag{3.33}$$

with $n$ auxiliary field $b^A$'s. The supertranslation generator and the supercovariant derivatives are then,

$$Q = \partial_\theta + i\theta\partial_t, \quad D = \partial_\theta - i\theta\partial_t \ . \tag{3.34}$$

### 3.2.1 Two Centers

The general structure of two dyon dynamics can be inferred from the results in Ref. [53]. The latter actually derived the effective action of a single dynamical dyon in the background of an infinitely heavy core BPS state. When the core state consists of a single dyon, the effective action derived there can also be regarded as the interacting "relative" part $\mathcal{L}^{rel}$ of a two-dyon effective action, upon the usual decomposition,

$$\mathcal{L} = \mathcal{L}^{c.m.} + \mathcal{L}^{rel} \ , \tag{3.35}$$

where the trivial center of mass part was understood to be

$$\mathcal{L}^{c.m.} = \int d\theta \ \frac{i}{2} M_{total} D\Phi^a_{c.m.}\partial_t\Phi^a_{c.m.} - \frac{1}{2} M_{total}\Lambda_{c.m.}D\Lambda_{c.m.} \ , \tag{3.36}$$

with $M_{total}\to\infty$ understood. Here, let us recall basic structures of $\mathcal{L}^{rel}$ as dictated by the supersymmetry.

$\mathcal{L}^{rel}$ involves only three bosonic coordinates and four fermionic ones and can be further decomposed as

$$\mathcal{L}^{rel} = \mathcal{L}_0^{rel} + \mathcal{L}_1^{rel} \ , \tag{3.37}$$

where

$$\mathcal{L}_0^{rel} = \int d\theta \left( \frac{i}{2} f(\Phi) D\Phi^a \partial_t \Phi^a - \frac{1}{2} f(\Phi) \Lambda D\Lambda + \frac{1}{4} \epsilon_{abc} \partial_a f(\Phi) D\Phi^b D\Phi^c \Lambda \right) \tag{3.38}$$

with $a = 1, 2, 3$, and

$$\mathcal{L}_1^{rel} = \int d\theta \ \left( i\mathcal{K}(\Phi)\Lambda - i\mathcal{W}(\Phi)_a D\Phi^a \right) \ , \tag{3.39}$$

with the condition

$$\partial_a \mathcal{K} = \epsilon_{abc} \, \partial_b \mathcal{W}_c \tag{3.40}$$

imposed. Note that this also implies $\partial_a \partial_a \mathcal{K} = 0$, which is solved by

$$\mathcal{K} = \mathcal{K}(\infty) - \frac{q}{|\vec{x}|} \ . \tag{3.41}$$

We will see shortly how $\mathcal{K}(\infty)$ and $q$ can be read off from the underlying Seiberg-Witten theory.

As was claimed, this Lagrangian is invariant under four supersymmetries,

$$\begin{aligned} \delta_\epsilon x^a &= i\eta^a_{mn} \epsilon^m \psi^n \ , \\ \delta_\epsilon \psi_m &= \eta^a_{mn} \epsilon^n \dot{x}^a + \epsilon_m b \ , \\ \delta_\epsilon b &= -i\epsilon_m \dot{\psi}^m \ , \end{aligned} \tag{3.42}$$

with four Grassman parameters $\epsilon^m$ and with $\psi^4 \equiv \lambda$. The $\mathcal{N} = 1$ superspace we employed is related to $\epsilon_4$, so $\mathcal{L}_0$ and $\mathcal{L}_1$ are manifestly and individually invariant under these supersymmetry transformation rules. A less obvious fact, which is nevertheless true, is that the two are also individually invariant under all four

$$\delta_\epsilon \int dt\, \mathcal{L}_0^{rel} = 0 = \delta_\epsilon \int dt\, \mathcal{L}_1^{rel}, \tag{3.43}$$

if the auxiliary field $b$ us kept off-shell. This is the feature that allows an easy generalization to $n$ dynamical centers. The auxiliary field $b$ takes the on-shell value,

$$b = b_{\text{onshell}} \equiv \frac{1}{f}\left(\mathcal{K} + \frac{i}{4}\eta^a_{pq}\partial_a f \psi^p \psi^q\right) , \tag{3.44}$$

which generates bosonic potential terms of type $\mathcal{K}^2/2f$ and mixes up terms in $\mathcal{L}_{0,1}$. Nevertheless, $\mathcal{N} = 4$ supersymmetry of $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ still holds, now in far more complicated on-shell form.

### 3.2.2 Seiberg-Witten

Before we extend this to $n$ dynamical dyons, we need to understand the role of the core-probe approximation and how it computes $f$, $\mathcal{K}$ and $\mathcal{W}$ [53] in terms of the quantities that appear in the Seiberg-Witten theory.

Let us consider a collection of charges $\gamma_A$, and represent it as a semiclassical state. The basic information about the semiclassical dyon state comes from the BPS equations of the Seiberg-Witten theory [61–64]

$$\vec{F}_i - i\zeta^{-1}\vec{\nabla}\phi_i = 0 \; , \quad \vec{F}_D^i - i\zeta^{-1}\vec{\nabla}\phi_D^i = 0 \; , \tag{3.45}$$

where $F = B + iE$ with magnetic field $B$'s and electric field $E$'s, $\phi$'s are unbroken part of the complex adjoint scalars, each of which are labeled by the Cartan index $i = 1, 2, \ldots, r$. $F_D$'s are defined through the low energy $U(1)$ coupling matrix as

$$\vec{F}_D^i \equiv \tau^{ij}\vec{F}_j \; , \qquad \tau^{ji} = \frac{\partial \phi_D^j}{\partial \phi^i} \; . \tag{3.46}$$

The pure phase factor $\zeta$ is determined by the supersymmetry left unbroken by the charge $\gamma$ in a given vacuum, and equals the phase factor of the central charge $Z_\gamma$ of the configuration.

In a core-probe approximation, we split $\gamma_T = \gamma_h + \sum_{A'} \gamma_{A'}$ and treat the latter $n-1$ as a fixed background of total charge $\gamma_c = \sum_{A'} \gamma_{A'}$. As we saw in the previous section, the Lagrangian for the dynamical dyon (of charge $\gamma_h$) is characterized by three objects.

The first is the mass function $f = |\mathcal{Z}_{\gamma_h}|$ as in

$$\mathcal{L} = \frac{1}{2} f \left( \frac{d\vec{x}}{dt} \right)^2 + \cdots \ , \tag{3.47}$$

where

$$\mathcal{Z}_\gamma = \gamma_h^e \cdot \phi + \gamma_h^m \cdot \phi_D \ , \tag{3.48}$$

with the electric part $\gamma_h^e$ and the magnetic part $\gamma_h^m$ of the charge vector $\gamma_h$. The scalar fields here solve the above BPS equation with the other $n-1$ charges $\gamma_{A'}$'s as the background point-like sources. The fact we treat such dyons as point-like objects is justified by going very near a marginal stability wall, since this tends to separate charge centers far apart from one another. As we will see shortly, this proximity to marginal stability wall plays a central role in allowing us to construct nonrelativistic low energy dynamics of dyons.

Clearly $|\mathcal{Z}_h|$ acts as the inertia of the probe dyon, which is position-dependent because of the background: this sort of identification is in accordance with general spirit of how one describe well-separated charged objects [65], which has been tested and used successfully for many soliton systems and even lead to exact moduli space metric in some cases [44, 66]. We also use the notation $Z_\gamma$ for the central charge of the charge $\gamma$ so that $Z_\gamma = \mathcal{Z}_\gamma(\infty)$.

The other two, more important for the discussion of BPS bound states, are the potential $\mathcal{K}^2/2f$ and the vector potential $\mathcal{W}$, so that

$$\mathcal{L} = \frac{1}{2} f \left(\frac{d\vec{x}}{dt}\right)^2 - \frac{\mathcal{K}^2}{2f} - \frac{d\vec{x}}{dt} \cdot \vec{\mathcal{W}} + \cdots \ , \tag{3.49}$$

where these two are determined entirely by the charge distribution of $\gamma_{A'}$'s as [53]

$$d\mathcal{W} = *d\mathcal{K} \ , \quad \mathcal{K} = \mathrm{Im}[\zeta^{-1}\mathcal{Z}_{\gamma_h}] = \mathrm{Im}[\zeta^{-1}Z_{\gamma_h}] - \sum_{A'} \frac{q_{hA'}}{|\,\vec{x} - \vec{x}_{A'}\,|} \ , \tag{3.50}$$

with[2]

$$q_{hA'} = \langle \gamma_h, \gamma_{A'} \rangle / 2$$

for the Schwinger product.

These are direct consequences of the equations (3.45), combined with the extra assumption of being near the marginal stability wall. Generically, the bosonic potential would have been

$$|\mathcal{Z}_h| - \mathrm{Re}[\zeta^{-1}\mathcal{Z}_h] \ , \tag{3.51}$$

but this reduces to

$$\mathcal{K}^2/2|\mathcal{Z}_h| = (\mathrm{Im}[\zeta^{-1}\mathcal{Z}_h])^2/|\mathcal{Z}_h| \ , \tag{3.52}$$

as we move near the marginal stability wall defined by alignment of $Z_h$ and $Z_c$ [53]. The reason why we need this proximity to the marginal stability wall is clearly not because of inherent properties of the system, but rather because of the non-relativistic quantum mechanics approximation we employed. Far away from the wall, the potential energy would be not small compared to rest mass of the

[2] This convention for the Schwinger product here follows the one used by Denef in Ref. [46, 47]. The original derivation of dyon dynamics from Seiberg-Witten theory in Ref. [53] used a different convention, such that

$$\langle \gamma, \gamma' \rangle = \langle \gamma, \gamma' \rangle_{\rm Denef} = 2\langle \tilde{\gamma}', \tilde{\gamma} \rangle_{\rm Lee-Yi}.$$

The tilde emphasizes the fact that the latter also used half-integral electric charges as opposed to integral ones, which is natural when we compute Coulomb energy. Magnetic charges are integral in either convention.

particles involved, which will bring dynamics to a relativistic one. However, we do not know how to handle interacting and relativistic particles at mechanical level.[3] Nevertheless, this approximation is good enough since we already know that BPS states are stable far away from marginal stability walls.

An important subtlety we wish to point out here is the choice of $\zeta$. In the core-probe limit, it appears that $\zeta = \zeta_c = Z_{\gamma_c}/|Z_{\gamma_c}|$ is the right choice, since we are treating $\gamma_h$ as an external particle in the background given by $\gamma_c = \sum_{A'} \gamma_{A'}$. However, $\zeta$ is tied to the supersymmetry left unbroken by the configuration and further more we are interested in the supersymmetric bound states of $\gamma_c$ and $\gamma_h$. Around such a state, the low energy dynamics should have supersymmetries associated with $\gamma_T = \gamma_c + \gamma_h$ rather than those associated with $\gamma_c$.

One can understand this as capturing the backreaction of the background due to the probe. Failing to do so clearly will give us nonsensical answers since, otherwise, the supersymmetry of the bound state in question would not be aligned with the supersymmetry of the moduli dynamics. In the core-probe approximation, the two happen to be the same, $\zeta_T = Z_{\gamma_T}/|Z_{\gamma_T}| = \zeta_c$, simply because the total central charge is dominated by that of the infinitely heavy core state. As we give up the core-probe dichotomy, this accidental identity will no longer hold, and the preceding discussion tells us that one must always use $\zeta_T$.

As we give up the core-probe approximation and treat all charge centers on equal footing, the moduli dynamics will become quite complicated. The part of the above action that remains least affected by this extension is the Lorentz force, coming from $-\dot{\vec{x}}\cdot\vec{\mathcal{W}}$ type couplings. The coefficient $q$ in $\mathcal{W}$ keeps track of how one particle's quantized electric (magnetic) charges see the other particle's quantized magnetic (electric one) charges. $\mathcal{W}$ is Dirac-quantized and topological, and furthermore can arise only from sum of two-body interactions. Therefore, this part of the interaction

[3] Importance of the wall in the derivation of low energy dynamics of dyons was also recognized by others [67].

can be reliably computed by adding up all pair-wise Lorentz forces, giving us

$$-\frac{d\vec{x}}{dt}\cdot\vec{\mathcal{W}} \quad \rightarrow \quad -\frac{d\vec{x}_A}{dt}\cdot\vec{\mathcal{W}}_A \tag{3.53}$$

with

$$\mathcal{W}_{Aa} = \sum_{B\neq A} q_{AB}\mathcal{W}_a^{Dirac}(\vec{x}_A - \vec{x}_B)\ , \tag{3.54}$$

where $q_{AB} = \langle\gamma_A, \gamma_B\rangle/2$ and $\mathcal{W}^{Dirac}$ is the Wu-Yang vector potential [68] of a $4\pi$ flux Dirac monopole. Note that the $4\pi$ flux of $\mathcal{W}^{Dirac}$ dovetails nicely with half-integer-quantized $q_{AB}$, as demanded by the Dirac quantization.

For general $n$ also, $\mathcal{N}=4$ supersymmetry constrains the Lagrangian greatly and, as we will see shortly, the potential energy is tied to such minimal couplings. Knowing the latter will allow us to fix, almost completely, the analog of $\mathcal{K}^2/2f$ as well. We will presently see how this works in $n$ center case. A more difficult question is how the kinetic terms would generalize, to which we will only give a general statement rather than precise solution. In this note, our primary interest is in the supersymmetric index for non-threshold bound states, which is independent of details of kinetic term.

### 3.2.3 Many Centers

For many centers, it is more convenient *not* to separate out the center of mass coordinate. Let us label the centers by $A = 1, 2, \ldots, n$ and denote their $R^3$ position as $x^{Aa}$ and the charge $\gamma_A$. The $\mathcal{N}=1$ superfield content is

$$\Phi^{Aa} = x^{Aa} - i\theta\psi^{Aa}\ , \quad \Lambda^A = i\lambda^A + i\theta b^A\ , \tag{3.55}$$

with $A = 1, 2, \ldots, n$ and $a = 1, 2, 3$. $\mathcal{N} = 4$ transformation rules are,

$$\begin{aligned} \delta_\epsilon x^A &= i\eta^a_{mn}\epsilon^m \psi^{An} , \\ \delta_\epsilon \psi^A_m &= \eta^a_{mn}\epsilon^n \dot{x}^{Aa} + \epsilon_m b^A , \\ \delta_\epsilon b^A &= -i\epsilon_m \dot{\psi}^{Am} , \end{aligned} \tag{3.56}$$

where as before $\psi^{A4} \equiv \lambda^A$. We again split the Lagrangian into the kinetic part and the potential part,

$$\mathcal{L} \;= \mathcal{L}_0 + \mathcal{L}_1 , \tag{3.57}$$

and look for $\mathcal{L}_{0,1}$ separately, with off-shell $b^A$'s.

The $n$-center version of $\mathcal{L}_1$ is, given (3.54), quite obvious,

$$\mathcal{L}_1 \;=\; \int d\theta \, \left( i\mathcal{K}_A(\Phi)\Lambda^A - i\mathcal{W}_{Aa}(\Phi) D\Phi^{Aa} \right) , \tag{3.58}$$

since the second term gives precisely the Lorentz force among dyons and while the first is induced from the second by $\mathcal{N} = 4$ supersymmetry; One can check easily that

$$\delta_\epsilon \int dt \, \mathcal{L}_1 = 0 \tag{3.59}$$

under all four supersymmtries, provided that

$$\partial_{Aa}\mathcal{K}_B \;=\; \frac{1}{2}\,\epsilon_{abc}\left(\partial_{Ab}\mathcal{W}_{Bc} - \partial_{Bc}\mathcal{W}_{Ab}\right) , \tag{3.60}$$

and

$$\begin{aligned} \epsilon_{abc}\partial_{Ab}\partial_{Bc}\mathcal{K}_C &= 0 , \\ \partial_{Aa}\partial_{Ba}\mathcal{K}_C &= 0 , \end{aligned} \tag{3.61}$$

for any $A, B, C$. We already learned that

$$\mathcal{W}_{Aa} = \sum_B \frac{\langle \gamma_A, \gamma_B \rangle}{2} \mathcal{W}_a^{Dirac}(\vec{x}^A - \vec{x}^B) \ ,$$

so $\mathcal{K}$'s also follow immediately via the $\mathcal{N} = 4$ constraints as

$$\mathcal{K}_A = \mathcal{K}_A(\infty) - \frac{1}{2} \sum_B \frac{\langle \gamma_A, \gamma_B \rangle}{|\,\vec{x}^A - \vec{x}^B|} \ , \tag{3.62}$$

Note that this obeys the constraints except at the submanifold, say $\Delta \equiv \{x^{Aa} : \vec{x}_A = \vec{x}_B,\, \langle \gamma_A, \gamma_B \rangle \neq 0\}$. The quantum mechanics can be very singular at such places also, meaning that we should excise $\Delta$ from $R^{3n}$ and impose the regular boundary condition instead.

It remains for us to determine $\mathcal{K}_A(\infty)$'s. These $\mathcal{K}$'s and $\mathcal{W}$'s can be traced back to the original BPS equations (3.45), and found by keeping track of how motion of each center is affected by the presence of the other $n-1$ centers. After solving the BPS equations, similarly as in the core-probe limit, we learn that

$$\mathcal{K}_A = \mathrm{Im}\left[\zeta^{-1} \mathcal{Z}_A\right] = \mathrm{Im}\left[\zeta^{-1} Z_A\right] - \frac{1}{2} \sum_{B \neq A} \frac{\langle \gamma_A, \gamma_B \rangle}{|\,\vec{x}_A - \vec{x}_B|} \ , \tag{3.63}$$

where $\mathcal{Z}_A$ is computed from the solution to (3.45) with the other $n-1$ charge centers taken as the background but, nevertheless, with the phase of the total charge, $\zeta = \sum_A Z_A / |\sum_A Z_A|$, used in the equations. As we noted above, this is because we must make sure to use the supersymmetries that are preserved by the bound state of all centers. This can be also seen from $\mathcal{K}_A(\infty) = \mathrm{Im}[\zeta^{-1} Z_A]$, which allows $\sum_A \mathcal{K}_A(\infty) = 0$ as demanded by the antisymmetric Schwinger product. Note that this consistency condition would have been violated if we had used different $\zeta$'s for different $\mathcal{K}_A$'s.

The other piece $\mathcal{L}_0$, containing kinetic terms, is a little more involved. The simplest way to find the most general $\mathcal{L}_0$ is via an $\mathcal{N} = 4$ superspace. For this, note that the collection $\{\Phi^a, \Lambda\}$ can be thought of as dimensional reduction of a $D = 4$ $N = 1$ vector superfield [69, 70].[4] In this map, $x^a$'s come from the spatial part of the vector field, the fermions from the gaugino, and the auxiliary field $b$ from that of the $D = 4$ $N = 1$ vector superfield. Here, we are mainly interested in $\mathcal{N} = 1$ form of such a general $\mathcal{L}_0$, which is available in Maloney et.al. [60],

$$\mathcal{L}_0 \;=\; \int d\theta\, \frac{i}{2}\, g_{AaBb} D\Phi^{Aa}\partial_t\Phi^{Bb} - \frac{1}{2}\, h_{AB}\Lambda^A D\Lambda^B - ik_{AaB}\dot{\Phi}^{Aa}\Lambda^B + \cdots \tag{3.64}$$

where the ellipsis denotes four cubic terms that we omit here for the sake of simplicity. This $\mathcal{L}_0$ is also invariant under the four supersymmetries we listed above,

$$\delta_\epsilon \int dt\, \mathcal{L}_0 = 0 \tag{3.65}$$

on its own with $b^A$'s off-shell, provided that various coefficient functions derive from a single real function $L(x)$ of $3n$ variables as

$$\begin{aligned} g_{AaBb}(\Phi) &= \left(\delta^e_a\delta^f_b + \epsilon_c{}^e{}_a\epsilon^{cf}{}_b\right)\partial_{Ae}\partial_{Bf}L(\Phi)\ , \\ h_{AB}(\Phi) &= \delta^{ab}\partial_{Aa}\partial_{Bb}L(\Phi)\ , \\ k_{AaB}(\Phi) &= \epsilon^{ef}{}_a\partial_{Ae}\partial_{Bf}L(\Phi)\ , \\ &\ \vdots \end{aligned} \tag{3.66}$$

Figuring out the precise form of $L$ for $n$ charge centers requires further work. For a single dynamical dyon in the core-probe limit, we know that it is related to the central charge function as $\partial^2 L = |\mathcal{Z}|$. We expect that there exists a similarly intuitive generalization for $n$ particles case as well. In this note, we are primarily interested in counting nonthreshold bound state, for which details of $L$ does not

[4]In this version of $\mathcal{N} = 4$ superspace, $\mathcal{L}_1$ is not obvious. On the other hand, a more extended harmonic superspace form has been found to accommodate both kinetic terms and potential terms [71].

enter. Determination of $L$ can become an important issue, when we begin to consider non-primitive charge states. See next subsection for related comments.

Again, the main point here is that $\mathcal{L}_0$ and $\mathcal{L}_1$ are invariant under the four supersymmetries separately when we keep the auxiliary fields $b^A$'s off-shell. Combining the two, it follows that the full Lagrangian

$$\mathcal{L}_0 + \mathcal{L}_1 \tag{3.67}$$

is also invariant under all four supersymmetries. Integrating out $b^A$'s generates potentials of type $\sim \mathcal{K}^2$ and mixes up terms in $\mathcal{L}_0$ and $\mathcal{L}_1$, but $\mathcal{N} = 4$ supersymmetries of the entire Lagrangian remain intact.

### 3.2.4 Kinetic Function $L$ : BPS Dyons vs BPS Black Holes

Note that the potential part $\mathcal{L}_1$ of the Lagrangian looks identical to the similar expression previously found by Denef [46], which has been later used extensively for counting BPS black holes bound states [54, 57]. The latter relied on $\mathcal{N} = 4$ quantum mechanical supersymmetry. Although we started with Seiberg-Witten theory for the derivation of $\mathcal{L}_1$, this part of Lagrangian is entirely determined by $\mathcal{N} = 4$ supersymmetry combined with long-distance Lorentz forces among charge centers. Thus appearance of the same $\mathcal{L}_1$ is hardly surprising. In fact, when we apply $\mathcal{L}_1$ to BPS black holes, it is even more trustworthy, since the Abelian approximation that would underlie such an interaction form is valid all the way to horizon. One cannot say the same for field theory dyons, since at short distance non-Abelian nature must be taken into account. Nevertheless, as long as we are near a marginal stability wall and only long-distance physics matters, it is clear that $\mathcal{L}_1$ is capable of describing both dyons and black holes.

This does not mean that the moduli dynamics of BPS dyons and those of BPS black holes are identical. The difference resides in the kinetic part $\mathcal{L}_0$ of the Lagrangian.

As demanded by $\mathcal{N}=4$ supersymmetry, $\mathcal{L}_0$ is determined by a single scalar function $L$ of the $n$ position vectors $\vec{x}_A$. For instance, $L$ for many BPS black holes of an identical charge was found by Maloney et. al. [60]

$$L(\vec{x}_1, \vec{x}_2, \dots) = -\frac{1}{16\pi} \int dx^3 \psi^4 \tag{3.68}$$

where $\psi = 1 + \sum_A (m/|\,\vec{x}_A - \vec{x}|)$ with the mass $m$. On the other hand, for two-center dyon case, we expect smooth behavior near $\vec{r} = 0$ [53] since, when the mutual distance is small, non-Abelian cores cannot be ignored and will smooth out Coulombic singularities. Even if we use the naive Abelian results, $\partial^2 L \sim 1/r$ at most. Comparing this to the two-body case of the supergravity result shows a substantial difference when the two objects begin to overlap.

Indeed, there are situations when the two theories are expected to give different answers. No example of $N=2$ field theory dyon which is a bound state of two or more identical dyons. For black holes, however, no such restriction seems to exist. If a BPS black hole of charge $\gamma$ exist, we expect BPS black holes of charge $N\gamma$ also to exist, in fact with large entropy. In the present context of moduli quantum mechanics, the latter corresponds to a collection of many charge centers with many flat directions extending to spatial infinities and may be realized as threshold bound states thereof. In such cases, the kinetic term of the effective action at both short distances and long distances could be important. This problem is an important outstanding issue in wall-crossing phenomena in general, for it provided much-needed input data on what dyons or black holes are available, to begin with, to form bound states.

Explicit forms of $L$ for $n$ BPS dyons and for $n$ BPS black holes, respectively, will be studied in a separate work.

## 3.3 R-Symmetry, Chirality Operators, and Indices

We wish to compute index of the preceding quantum mechanics

$$\mathrm{Tr}\left((-1)^F e^{-sH}\right) \ . \tag{3.69}$$

Since the quantum mechanics is gapped, of which much discussion will follow in next section, this quantity is truly independent of the parameter $s$. Thus, following the standard arguments, we will compute this in small $s$ limit. Before proceeding, however, it is important to clarify what we mean by the operator $(-1)^F$. In order for the index to make sense, this operator needs to anticommute with supercharge(s),

$$\{(-1)^F, Q\} = 0 \ ,$$

which is the condition needed for 1-1 matching and thus cancelation between bosonic and fermionic states for nonzero energy eigenvalues. Clearly this is not enough to fix the overall sign of $(-1)^F$ on the Hilbert space, and an index is also plagued by this ambiguity. When we compute an index of standard Dirac operator or de Rham operators, there is usually a canonical choice that is used widely. We will come back to this, later in next section, but the choice is a matter of convenience only and, a priori, has no physical significance.

At field theory level, however, we have an unambiguous and useful definition of such an index, say, the second helicity trace,

$$\Omega = -\frac{1}{2}\,\mathbf{Tr}\left((-1)^{2J_3}(2J_3)^2\right) \ , \tag{3.70}$$

where the trace **Tr** is over a single particle sector of a given charge. We wish to fix the sign of the quantum mechanical index, in accordance with this. Irreducible BPS multiplets, tensor products of half-hyper-multiplet and a spin $j$ multiplet, have the index

$$\Omega\left([j]\otimes([1/2]\oplus 2[0])\right) = (-1)^{2j}(2j+1)\ , \tag{3.71}$$

so often we also write,

$$\Omega = \mathbf{Tr}\left((-1)^{2J_3}\right)\ , \tag{3.72}$$

with the factored-out half-hypermultiplet understood. This naturally reduces to the low energy dynamics of dyons, which then must correspond to an index defined with a chirality operator that acts exactly like $(-1)^{2J_3}$

$$\Omega \quad \leftrightarrow \quad \mathrm{Tr}\left((-1)^{2J_3}\right)\ , \tag{3.73}$$

but of course we need to ask here how such an operator is realized in the quantum mechanics.

As can be inferred from discussions in Ref. [53], the quantum mechanics of previous section are equipped with $SO(4) = SU(2)_L \times SU(2)_R$ R-symmetry. This is easiest to see in how the fermion bilinear couplings to $d\mathcal{K}$ and $d\mathcal{W}$ combine to give,

$$-\frac{i}{2}\,\eta^a_{mn}\,\partial_{Aa}\mathcal{K}_B\,\psi^{Am}\psi^{Bn} \tag{3.74}$$

in the component form, where, as before, $\psi^{Am=1,2,3} = \psi^{Aa=1,2,3}$, $\psi^{A4} \equiv \lambda^A$, and $\eta$ is the 't Hooft self-dual symbol. The above form is precise when the metric is flat, but appropriately modified preserving $SO(4)$ symmetry when it is not. For each particle indexed by $A$, bosonic coordinates are in $(3,1)$ representations while the fermions are in $(2,2)$. Since spatial rotations rotate $\vec{x}^A$ as 3-vectors, $SU(2)_L$ should be interpreted as the rotation group, while $SU(2)_R$ must be descendant of $SU(2)_R$ R-symmetry of the underlying Seiberg-Witten theory. The latter rotates only fermions and leaves the position coordinate intact.[5]

[5] In the core-probe approximation of Ref. [53], only $SU(2)_R$ were generically there, but this was an artefact of treating some of dyon centers as fixed background.

In particular, the four supersymmetries are labeled by the $SO(4)$ vector index, and thus are in $(2,2)$ representations. Denoting generators of these two $SU(2)$'s by $J$ and $I$, respectively, we thus find

$$\{(-1)^{2J_3}, Q\} \;=\; 0 \;=\; \{(-1)^{2I_3}, Q\} \;. \tag{3.75}$$

The quantum mechanics have two unambiguous and physically meaningful chirality operators that can be used for index computation. The desired $(-1)^{2J_3}$ is one of them, therefore, we have an unambiguous way of computing the field theory index from the low energy quantum mechanics.

On the other hand, there is an interesting and universal relationship between these pair of chiral operators in the quantum mechanics. Restricting our attention to the relative part of the low energy dynamics again, we have

$$(-1)^{2J_3} = (-1)^{\sum_{A<B}\langle\gamma_A,\gamma_B\rangle+n-1}(-1)^{2I_3} \;. \tag{3.76}$$

This is easy to see by considering how the two $SU(2)$ generators are constructed in the quantum mechanics. For $SU(2)_R$, which rotate only fermions, we have

$$I_a = \sum_A \left(-\frac{i}{8}\,\epsilon_{abc}\,[\hat\psi^{Ab},\hat\psi^{Ac}] + \frac{i}{4}\,[\hat\psi^{Aa},\hat\lambda^{A}]\right) \;, \tag{3.77}$$

where the hat signifies the unit normalized fermion. The spatial rotation generators

$$J_a = L_a + \sum_A \left(-\frac{i}{8}\,\epsilon_{abc}\,[\hat\psi^{Ab},\hat\psi^{Ac}] - \frac{i}{4}\,[\hat\psi^{Aa},\hat\lambda^{A}]\right) \;, \tag{3.78}$$

are similar but differ in two aspects: first, since $SU(2)_R$ rotates $\vec{x}_A$'s, the generators include the orbital angular momentum $L$; secondly the fermions rotate differently, as reflected in the sign of the last term. This latter difference generates a relative sign between the two chiral operators for each $(2,2)$ representation of fermions, thus explaining $(-1)^{n-1}$. The other sign is equally simple, and come from well-known piece of charge-monopole physics, where the orbital angular momenta is schematically something like

$$\vec{L} \sim \sum_A (\vec{x}_A \times \vec{\pi}_A) + \sum_{A>B} \frac{\langle \gamma_A, \gamma_B \rangle}{2} \frac{\vec{x}_A - \vec{x}_B}{|\vec{x}_A - \vec{x}_B|} \tag{3.79}$$

with the covariantized momenta $\pi_A$. The orbital angular momentum is constructed from tensor product of spin $\langle \gamma_A, \gamma_B \rangle/2$ representations times usual integral angular momentum. Then regardless of which particular $SU(2)_L$ multiplet the state is, integrality vs half-integrality of the orbital angular momentum is unambiguously determined as

$$(-1)^{2L_3} = (-1)^{\sum_{A>B} \langle \gamma_A, \gamma_B \rangle} \ . \tag{3.80}$$

Note that this does not require $\vec{L}$ being symmetry operators.

Thus, we have the second helicity trace of $N = 2$ dyons which can be computed via the low energy quantum mechanics as

$$\Omega = \text{Tr}\left( (-1)^{2J_3} e^{-sH} \right) = (-1)^{\sum_{A<B} \langle \gamma_A, \gamma_B \rangle + n - 1} \times \text{Tr}\left( (-1)^{2I_3} e^{-sH} \right) \ . \tag{3.81}$$

In the subsequent computation, with this relation in mind, we will eventually identify $(-1)^{2I_3}$ as the canonical chirality operator $(-1)^F$. For this, there is another sign issue to settle, later when we begin to quote index formula from literature, since the latter come with a canonical choice of $(-1)^F$, which may or may not equal to our choice, $(-1)^{2I_3}$, but we postpone this to end of next section.

Another reason why $(-1)^{2I_3}$ is useful, even though we ultimately want $(-1)^{2J_3}$, can be found in the observation [73] that all explicitly constructed field theory BPS states, to date, are in $SU(2)_R$ singlets times the universal half-hypermultiplet (from the center of mass part in quantum mechanics viewpoint). If this is generally true, we can see that the index with $(-1)^{2I_3}$ is always positive and truly counts the degeneracy. An interesting question, therefore, is whether in the low energy quantum mechanics we derived all supersymmetric bound states are $SU(2)_R$ singlets.

An interesting variant of the second helicity trace is the protected spin character [73],[6]

$$\mathbf{Tr}\left((-1)^{2J_3}y^{2J_3+2I_3}\right) \ , \tag{3.82}$$

where again we took out the universal half-hypermultiplet from the trace for simplicity. This clearly reduces to, in quantum mechanics,

$$\mathrm{Tr}\left((-1)^{2J_3}y^{2J_3+2I_3}\right) \ . \tag{3.83}$$

Later we will also see how this quantity is naturally computed, after we reduce the index problem to the more familiar one that relies only the classical moduli space $\mathcal{K}=0$, by the equivariant index that counts "angular momentum" representations. As we will see, this reduction process cannot carry the entire $\mathcal{N}=4$ supersymmetry, and, of $SO(4)$ R-symmetry, only a diagonal $SU(2)$ subgroup generated by $J+I$ survives as global symmetry. The equivariant index on $\mathcal{K}=0$ space does not count representations under spatial rotations but under simultaneous rotation of spatial $SU(2)_L$ and $N=2$ R-symmetry $SU(2)_R$.[7] See section 4.4. for more detail.

## 3.4 Index Theorem for Distinguishable Centers

Now we turn to the problem of counting ground states of the above quantum mechanics, or equivalently counting BPS bound states of $n$ dyons. Since the quantum

[6] We are indebted to Boris Pioline and Jan Manschot for bringing the question of the protected spin character to our attention.

[7] Of course, if the $SU(2)_R$ singlet hypothesis actually holds for the ground state sector, the end result would not know about $I_3$, anyway. In fact, on the basis of this hypothesis, this equivalence was anticipated previously [54]. Our argument in section 4.4 will prove the identity without such an assumption.

mechanics has a potential, $\sim \mathcal{K}^2$, one may expect that the problem can be reduced naturally to another problem on the classical moduli space of $2(n-1)$ dimensions, say,

$$\mathcal{M}_n = \{x^{Aa} \mid \mathcal{K}_A = 0,\ A = 1, 2, \ldots, n\}/R^3 \ , \tag{3.84}$$

where the division by $R^3$ is to remove the flat center of mass part. This classical moduli space is generically a little more complicated since some of the centers could be associated with identical particles, which we will deal with in the next section.

This reduction is not as straightforward as one might think, however. Ref. [57], for example, suggested that one can ignore the (then unknown) kinetic part of the Lagrangian. Effectively, in our notation, this would involve a geometric quantization of $\mathcal{L}_1$,

$$\mathcal{L}_{geometric} = \mathcal{L}_1 = -b^A\mathcal{K}_A - \mathcal{W}_{Aa}\dot{x}^{Aa} + \frac{i}{2}\partial_{Aa}\mathcal{K}_B\eta^a_{mn}\psi^{Am}\psi^{Bn} \ , \tag{3.85}$$

which is obtained by truncating higher-derivative parts in $\mathcal{L}_0$. The auxiliary fields, $b^A$'s, are now Lagrange multipliers, imposing $\mathcal{K}_A = 0$ as constraints and leaving a lowest Landau level problem on $\mathcal{M}_n$ with the magnetic fields $\sum_A d\mathcal{W}_A$. However, computation of the resulting index, if we take $\mathcal{L}_{geometric}$ verbatim, generates wrong results relative to other known spectrum; The geometric quantization of $\mathcal{L}_{geometric}$ would lead to index formula that is known to generate empirically incorrect answers.

For two body case, for example, the degeneracy $2|q|$ has been known to be the correct answer for many explicit constructions. See, for example, Ref. [41] and Ref. [53] for explicit two-dyon bound state construction in the weakly coupled and in the strongly coupled regions of Seiberg-Witten theory, respectively. On the other hand, the naive lowest level Landau problem (or equivalently the geometric quantization problem) gives $2|q| + 1$. One would hope that the effect of fermions in $\mathcal{L}_{geometric}$ will fix this, but this apparently does not happen.

The truncation of the kinetic terms in the presence of fermions is quite subtle, since while bosons acquire a symplectic structure thanks to the magnetic field, there is no such analog for fermions. Setting the kinetic term of fermions to zero will cause the canonical commutator ill-defined, making the whole reduction process ambiguous. One can try to reinstate kinetic terms on $\mathcal{M}_n$ as a regulator, but then, the main issue is that the number of fermions in $\mathcal{L}_1$ is $4(n-1)$ real while the number of bosons is $2(n-1)$ real, and these lead to de Rham cohomology problem on $\mathcal{M}_n$. For not too small $q$ and when $\mathcal{M}_n$ is Käehler, for example, the index of such a quantum mechanics coincides precisely with the state counting of the bosonic geometric quantization problem,[8] again giving us wrong result for the index.

Really at the heart of the problem is, however, the fact that the classical massive directions are in fact no more massive than the classically massless directions. Because the classical moduli space $\mathcal{M}_n$ is of finite size[9], it comes with various gaps at quantum level, and it so happens that these quantum gaps are one-to-one matched and identical to the gaps associated with the classically massive directions: the dynamics cannot be really split into two distinct sectors of heavy and light modes, at all, and contrary to initial expectation, the reduction to $\mathcal{M}_n$ cannot be justified. In fact, this lack of separation of scales is easiest to see in how fermions enter the Hamiltonian. Half of fermions get mass from $d\mathcal{K}$ while the other hand get mass from $d\mathcal{W}$. However, $\mathcal{N}=4$ supersymmetry of the quantum mechanics tells us that the two are one and the same object, and fermions coupling to $d\mathcal{K}$ are no more heavier than those coupling to $d\mathcal{W}$.

Fortunately, we can still decouple these classically massive directions in the computation of the index problem. This involves a deformation that breaks all but one supersymmetry, yet because the quantum mechanics is gapped and the surviving

[8] See for example Ref. [72], where in effect a regularized version of these problems were considered with kinetic terms on $\mathcal{M}_n$ and for its fermionic partners present.

[9] There are also some exotic cases corresponding to the scaling solutions. In these cases, the moduli space is non-compact, from short distance side, but its volume in the naive flat metric is still finite.

supercharge is effectively a Fredholm operator, it can be done while preserving the index. Later in the section and in Appendix B, we explicitly show that, as far as computation of the index goes, we may reduce the moduli dynamics to an effective $\mathcal{N}=1$ supersymmetric quantum mechanics with target $\mathcal{M}_n$,

$$\mathcal{L}^{\mathcal{N}=1}_{\text{for index only}}(\mathcal{M}_n) = \frac{1}{2}\,G_{\mu\nu}\dot{z}^\mu\dot{z}^\nu + \frac{i}{2}\,G_{\mu\nu}\psi^\mu\dot{\psi}^\nu + \cdots - \mathcal{A}_\mu\dot{z}^\mu + \cdots \ , \tag{3.86}$$

where $\mathcal{A}$ is a gauge field on $\mathcal{M}_n$ such that

$$d\mathcal{A} = \mathcal{F} \equiv d\left(\sum_A \mathcal{W}_{Aa}dx^{Aa}\right)\Bigg|_{\mathcal{M}_n} . \tag{3.87}$$

and $G$ is the induced metric on $\mathcal{M}_n$. This Lagrangian must be used only for the purpose of computing index.

The key point here is that the number of fermions is exactly half of that in $\mathcal{L}_{geometric}$. Since these fermions live on the tangent bundle of $\mathcal{M}_n$, we have a nonlinear sigma model with real fermions. The relevant wavefunctions are spinors on $\mathcal{M}_n$ and the index in question becomes a Dirac index,

$$\mathcal{I}_n(\{\gamma_A\}) = \int_{\mathcal{M}_n} Ch(\mathcal{F})\hat{A}(\mathcal{M}_n) = \int_{\mathcal{M}_n} Ch(\mathcal{F}) \tag{3.88}$$

with the Chern character $Ch$ of $\mathcal{F}$. $\hat{A}$ is the A-roof genus of the tangent bundle, which will be shown to be trivial for all $\mathcal{M}_n$'s. This formula counts the index when we view individual charge centers as distinguishable; in section 5, we will extend the formula appropriately when identical particles are involved and along the way see why the rational invariants of the form $\sim \Omega/p^2$ with integer $p>1$ appears in various wall-crossing formulae.

The Dirac index found here is consistent with de Boer et. al.'s observation [57] that empirically correct answers emerge for $n=2$ and $n=3$ if one assumes that the relevant quantum mechanics admit spinors on $\mathcal{M}_n$ as the wavefunction. This can be then generalized to the refined index (or equivariant index) and make contact with a series of recent works by Manschot et.al [54, 55].

### 3.4.1 Two Centers: Reduction to $S^2$

Supersymmetric ground states were found and counted for $n = 2$ case in Ref. [53], which gave the correct answer of $2q$ at the end of the day. expected, the wavefunctions are all maximized near the classical "true" moduli space $\mathcal{K} = 0$, which was nothing but a two-sphere threaded by a flux of $4\pi q$. However, the wavefunctions can also be seen very diffuse, too much so to let us call it "localized" there.

Here, we will illustrate why a naive reduction to $\mathcal{M}_2 = S^2$ by throwing away entire kinetic term is wrong. After the latter procedure, one ends up with $\mathcal{L}_{geometric}$ for which we need to either geometrically quantize over $S^2$ or regularize the dynamics by reinserting kinetic term on $S^2$ and concentrate on the lowest Landau level. If we follow the second viewpoint, we end up with a two dimensional nonlinear sigma model with four real fermions, so effectively we will have thrown away only the bosonic radial coordinate from the original moduli dynamics.

Let us consider the zero point energy of the relative part of the two-center mechanics. Three bosons can be split into "radial" directions, on which $\mathcal{K}$ and the mass function $f$ depend, and flat "angular" directions. With $\mathcal{K} = a - q/r$ and positive $a$ and $q$, the ground state is at $r = r_0 = q/a$, and the radial direction becomes a harmonic oscillator of frequency $w = a^2/f(r_0)q$, so

$$E_{radial} \simeq \left( m_b^{radial} + \frac{1}{2} \right) w \geq \frac{a^2}{2f(r_0)q} \ . \tag{3.89}$$

The angular part, although classical flat, also comes with a gap due to the finite volume, and the energy quantization there goes as

$$E_{sphere} \simeq \frac{\vec{L}^2 - q^2}{2f(r_0)r_0^2} \geq \frac{a^2}{2f(r_0)q} \ , \tag{3.90}$$

since the angular momentum is bounded below, in the presence of the flux, by $q$. The four real fermions are paired up into two fermionic oscillators of the same frequency $w$ as above, so we get contribution from the fermion sector as

$$E_{fermion} \simeq \left(m_f + m_f' - 1\right) w \geq -\frac{a^2}{f(r_0)q} \ , \tag{3.91}$$

where we again see that there is only one scale in the fermion sector also. Of course, the behavior of fermionic degrees of freedom must be the same as the bosonic ones, since we have supersymmetry.

This shows that, without further deformation, the gap of the classically massive radial direction is exactly the same as the rest of the degrees of freedom. If we wish to localize the problem to $\mathcal{M}_2 = S^2$ by removing the radial mode, we must do something else so that the gap along the radial direction and the gap along $\mathcal{M}_2$ are different, but this seems impossible under the $\mathcal{N} = 4$ supersymmetry of the quantum mechanics.

Let us remember here that, for the evaluation of index, one needs only two things: a Dirac operator of some kind and a chirality operator that anticommutes with it. One would like to compute the index

$$\mathrm{Tr}(-1)^F e^{-sH} \ , \tag{3.92}$$

for interacting part of the theory. Let us, for the sake of definiteness, take $H = Q_4^2$, and evaluate

$$\mathrm{Tr}(-1)^F e^{-sQ_4^2} \ . \tag{3.93}$$

$\mathcal{N} = 4$ supersymmetry is useful since it constrains dynamics but all of them are not really necessary to define an index. It is clear that, as long as we preserve this quantity, we can even break $\mathcal{N} = 4$ supersymmetry.

Of course, $\mathcal{N} = 4$ supersymmetry is important when it comes to generating correct supermultiplet structure to the bound state, but that only concerns the free center of mass part. The index must be computed from relative interacting part of the dynamics, only for which we will break $\mathcal{N} = 4$ supersymmetry.

Thus we are motivated to give up $d\mathcal{K} = *d\mathcal{W}$ condition, thereby keeping only $Q = Q_4$ unbroken. Let us replace

$$\mathcal{K} \to \xi\mathcal{K} \tag{3.94}$$

with some arbitrarily large number $\xi$ while keeping $\mathcal{W}$ as it is. The ground state energy counting is now

$$E_{radial} + E_{sphere} + E_{fermion} \geq \frac{\xi w}{2} + \frac{w}{2} - \frac{\xi w + w}{2} \ , \tag{3.95}$$

since the half of the fermions ($\lambda$ and $\psi^r$) get the mass from $d(\xi\mathcal{K})$ and the other half from $d\mathcal{W}$. The angular momentum sector mass-gap, $w/2 = q/2f(r_0)r_0^2$, is unchanged since the classical vacuum, $\mathcal{K} = 0$ and thus the radius $r_0$, and $\mathcal{W}$ are intact under this rescaling.

It is not difficult to see that the reduced dynamics, after integrating out heavy modes, is a $\mathcal{N} = 1$ nonlinear sigma model onto $\mathcal{M}_2 = S^2$ coupled to an external vector field $\mathcal{W}$. See Appendix B for complete detail of the reduction process. We note that since $\mathcal{M}_2 = S^2$ happens to be Käehler, the unbroken supersymmetry gets accidentally extended to $\mathcal{N} = 2$, although this is not important for our purpose.

### 3.4.2 Many Centers: Reduction to $\mathcal{M}_n$

Similarly, we wish to deform the theory by rescaling $\mathcal{K}_A \to \xi\mathcal{K}_A$, when we have many dynamical charge centers, as well

$$\begin{aligned} \mathcal{L}'_{\text{deformed}} \ &= \int d\theta \left( \frac{i}{2} g_{AaBb} D\Phi^{Aa}\partial_t\Phi^{Bb} - \frac{1}{2} h_{AB}\Lambda^A D\Lambda^B - ik_{AaB}\dot{\Phi}^{Aa}\Lambda^B + \cdots \right. \\ &\qquad \left. + i\xi\mathcal{K}_A(\Phi)\Lambda^A - i\mathcal{W}_{Aa}(\Phi)D\Phi^{Aa} \right) , \end{aligned} \tag{3.96}$$

where $\xi$ is an arbitrarily large number. As in the two-center case, the bosonic potentials are quadratic in $\mathcal{K}_A$'s and there are $n-1$ "radial" directions that are of mass $\sim \xi$. There are also $2(n-1)$ fermions that couple to $d(\xi\mathcal{K}_A)$'s, so they are also of mass $\sim \xi$. The two sets can be decoupled together, thereby reducing the index problem to $\mathcal{M}_n$ with real fermions. It leaves behind a $\mathcal{N}=1$ supersymmetric quantum mechanics onto $\mathcal{M}_n$ with $2(n-1)$ bosons and $2(n-1)$ real fermions. The process does not affect the free center of mass part, so the latter still comes with 3 bosonic coordinates and 4 fermionic ones.

We may further deform the kinetic part, $\mathcal{L}_0$, by taking the simplest form of the kinetic function,

$$L = \frac{1}{2}\sum_A m_A \vec{x}^A \cdot \vec{x}^A \ , \tag{3.97}$$

which amounts to

$$g_{AaBb} = \delta_{AB}\delta_{ab}m_A, \qquad h_{AB} = \delta_{AB}m_A, \qquad k_{AaB} = 0, \tag{3.98}$$

and setting cubic terms to zero as well. The simplest way to justify this deformation is that the kinetic function approaches this flat metric when distances between charge centers approach infinity. This asymptotic form is more than good enough since we can always tune the field theory vacuum, so that we stay arbitrarily near the marginal stability wall. There, $\mathrm{Im}[\zeta^{-1}Z_A]$ approaches zero, and the submanifold $\mathcal{M}_n$ is arbitrarily large. Since the index cannot change under the continuous and sign-preserving deformation of $\mathrm{Im}[\zeta^{-1}Z_A]$, and since the ambient metric is effectively flat for large $\mathcal{M}_n$, the index will be unaffected by this choice of metric.

This leaves us with a very simple $\mathcal{N}=1$ quantum mechanics

$$\begin{aligned}\mathcal{L}_{\text{deformed}} &= \int d\theta \left(\frac{i}{2}m_A D\Phi^{Aa}\partial_t\Phi^{Aa} - \frac{1}{2}m_A\Lambda^A D\Lambda^A \right.\\ &\qquad \left. + i\xi\mathcal{K}_A(\Phi)\Lambda^A - i\mathcal{W}_{Aa}(\Phi)D\Phi^{Aa}\right) ,\end{aligned} \tag{3.99}$$

with target $R^{3n}$ modulo submanifolds given by $\mathcal{K}_A = \pm\infty$. Of this, the free center of mass positions $R^3$ and the accompanying four real fermions decouples, leaving behind the interacting part of the moduli dynamics onto $R^{3(n-1)}$. This free part is also essential since it generates the basic BPS multiplet structure (whose content equals half of a hypermultiplet) to the bound state. Then, by taking $\xi \to \infty$, we decouple $n-1$ "radial" directions and $2(n-1)$ accompanying heavy fermions, and end up with a nonlinear sigma model onto $\mathcal{M}_n$ with real $2(n-1)$ fermionic partners. See appendix for detailed derivation of this fact.

Thus, we arrive at the effective Lagrangian, which can be used for the purpose of computing the index of the original $n$ center problem,

$$\mathcal{L}^{\mathcal{N}=1}_{\text{for index only}} = \frac{1}{2}\,G_{\mu\nu}\dot{z}^\mu\dot{z}^\nu - \mathcal{A}_\mu\dot{z}^\mu + \frac{i}{2}\,G_{\mu\nu}\psi^\mu\dot{\psi}^\nu + \frac{i}{2}\,G_{\mu\nu}\psi^\mu\dot{z}^\lambda\Gamma^\nu_{\lambda\beta}\psi^\beta + \frac{i}{2}\,\mathcal{F}_{\mu\nu}\psi^\mu\psi^\nu \tag{3.100}$$

again with the induced metric $G$ on $\mathcal{M}_n$ and, as we already noted,

$$d\mathcal{A} = \mathcal{F} \equiv d\left(\sum_A \mathcal{W}_{Aa}dx^{Aa}\right)\Bigg|_{\mathcal{M}_n} . \tag{3.101}$$

Since each $\mathcal{W}_A$ is a sum of Dirac monopoles at $\vec{x} = \vec{x}_B$'s, we find

$$\begin{aligned}
\mathcal{F} &= d\left(\sum_A\sum_{B\neq A}\frac{\langle\gamma_A,\gamma_B\rangle}{2}\,\mathcal{W}^{Dirac}_a(\vec{x}_A-\vec{x}_B)\,dx^{Aa}\right)\Bigg|_{\mathcal{M}_n}\\
&= d\left(\sum_{A>B}\frac{\langle\gamma_A,\gamma_B\rangle}{2}\,\mathcal{W}^{Dirac}_a(\vec{x}_A-\vec{x}_B)\,d(x^{Aa}-x^{Ba})\right)\Bigg|_{\mathcal{M}_n}\\
&= \sum_{A>B}\frac{\langle\gamma_A,\gamma_B\rangle}{2}\,\mathcal{F}^{Dirac}(\vec{x}_A-\vec{x}_B)\ ,
\end{aligned} \tag{3.102}$$

where $\mathcal{F}^{Dirac}$ is the Dirac monopole of flux $4\pi$. Of four supercharges, $Q_4$ survives the deformation process above, which is then further reduced to $Q_{\mathcal{M}_n}$ as heavy modes are integrated out.

### 3.4.3 Index for $n$ Distinguishable Centers

Since this is the plain old nonlinear sigma model twisted by the minimal coupling to $\mathcal{A}$, the reduced supercharge is represented geometrically as the Dirac operator with a $U(1)$ gauge field

$$Q_4 \quad \rightarrow \quad Q_{\mathcal{M}_n} = \gamma^\mu \left( i\nabla_\mu + \mathcal{A}_\mu \right) \ , \tag{3.103}$$

whose index, according to Atiyah-Singer index theorem, is given by

$$\mathcal{I}_n(\{\gamma_A\}) = \mathrm{Tr}\left( (-1)^{F_{\mathcal{M}_n}} e^{-sQ^2} \right) = \int_{\mathcal{M}_n} Ch(\mathcal{F}) \hat{A}(\mathcal{M}_n) \ ,$$

as promised, where we must assume a canonical choice of the chirality operator. This is,

$$(-1)^{F_{\mathcal{M}_n}} = (2i)^{n-1} \hat{\psi}^1 \cdots \hat{\psi}^{2(n-1)} \ ,$$

in terms of properly normalized and ordered fermions. See next subsection for how this choice squares off with physically motivated chirality operators $(-1)^{2J_3}$ and $(-1)^{2I_3}$ of section 3 and how the latter chirality operators reduce on $\mathcal{M}_n$.

Curiously enough, the A-roof genus $\hat{A}$ does not contribute to the index, thanks to the simple topology of $\mathcal{M}_n$. To see this, let us first note that the ambient space, in which $\mathcal{M}_n$ is embedded is essentially $R^{3n}$. For instance, take $\vec{x}_1 = 0$ to remove the translation invariance and make the ambient space $R^{3(n-1)}$, and then impose $\mathcal{K}_A = 0$, of which $n-1$ are linearly independent. Therefore, $\mathcal{M}_n$ is a complete intersection in $R^{3(n-1)}$. Since A-roof genus is a multiplicative class, we have an identity,

$$\hat{A}(T\mathcal{M}_n)\hat{A}(N\mathcal{M}_n) = \hat{A}\left( TR^{3(n-1)}|_{\mathcal{M}_n} \right) = 1 \tag{3.104}$$

among the tangent and the normal bundles. However, $d\mathcal{K}_A$'s are nowhere vanishing normal vectors on $\mathcal{M}_n$, and thus the normal bundle $N\mathcal{M}_n$ is also topologically trivial[10], and

$$\hat{A}(T\mathcal{M}_n) = 1 \ . \tag{3.105}$$

It is important to note that this decoupling depends only on the topology of the ambient space, namely the original $3n$ dimensional moduli space, near the surface $\mathcal{K}_A = 0$.[11]

Note that similar argument will not lead to triviality of other multiplicative class since typically they require complex bundles in order to be defined. For instance, $Td(\mathcal{M}_{2k})$ or $c(\mathcal{M}_{2k})$ cannot be argued to be trivial in this manner since the normal bundle of $\mathcal{M}_{2k}$ inside the relative position space $R^{3(2k-1)}$ is of odd dimension and, if irreducible, cannot be complex. In particular $\mathcal{M}_2 = S^2$, which has a clearly nontrivial $c_1$, shows this clearly; its normal bundle has a real line as the fibre.

### 3.4.4 Reduced Symmetry, Index, and Internal Degeneracy

Since we arrived at the nonlinear sigma-model on $\mathcal{M}_n$ only after the deformation of the dynamics, which in particular removes the extended supersymmetry, we must first ask whether various operators survive this procedure of deformation and the subsequent reduction process $\xi \to \infty$. Of the four original supersymmetries, $Q_4$ survives the deformation. It's on-shell form will be smoothly deformed as well, which goes like

$$Q_4 = \cdots + \lambda^A \mathcal{K}^A \quad \to \quad Q_4 = \cdots + \xi \lambda^A \mathcal{K}^A \ . \tag{3.106}$$

The ellipsis denotes parts unaffected by the deformation. We emphasize again that this supersymmetry is explicitly preserved since the deformed Lagrangian (3.96) is written in the superspace associated with $Q_4$. Then, given that $Q_4$ is a gapped elliptic operator, at $\xi = 1$, this deformation preserves the index as we increase $\xi$ [43].

[10] We are indebted to Bumsig Kim for pointing this out to us.

[11] However, it turns out that $\hat{A}(\mathcal{M}_n)$ factor does make a difference when one evaluate the equivariant index [55].

This $Q_4$ reduces to $Q_{\mathcal{M}_n}$ of the nonlinear sigma model on $\mathcal{M}_n$, and obviously the Hamiltonian, $Q_4^2/2$, gets similarly deformed and eventually reduced to the natural one on $\mathcal{M}_n$.

This leaves the global symmetry operators and the chirality operators. With the $\mathcal{N}=4$ supersymmetry partially broken, the $SO(4)$ R-symmetry can be easily seen to be broken. On the other hand, the deformation commutes with rotation of $\vec{x}_A$'s, so we expect to see some $SU(2)$ symmetry does survive the process. The question is which $SU(2)$ in $SO(4) = SU(2)_L \times SU(2)_R$ remains unbroken. The answer is the diagonal subgroup, $SU(2)_{\mathcal{J}}$, generated by

$$\mathcal{J}_a = J_a + I_a \ . \tag{3.107}$$

One can see this in several different ways.

Firstly, both $J$ ($SU(2)_L$) and $I$ ($SU(2)_L$) are broken by themselves, since they both act nontrivially on heavy fermion sector. The diagonal generators $\mathcal{J}$'s, on the other hand does not involve $\lambda$ fermions and leave the heavy sector ground state untouched. Secondly, after deformation and reduction to $\mathcal{M}_n$, the dynamics is a nonlinear sigma model, where fermion transform identically to bosons. Recall that bosons and fermions used to belong to $(3,1)$ and $(2,2)$ of $SU(2)_L \times SU(2)_R$. In the reduced dynamics, symmetry properties of the bosons and fermions cannot be different, and indeed under the diagonal subgroup, bosons and fermion transforms identically. Finally, after the deformation, the dynamics has only one real supersymmetry $Q_4$ so no R-symmetry is expected. However this supercharge originate from a $(2,2)$ multiplet under $SU(2)_L \times SU(2)_R$, so has to transform nontrivially under either of the two individually. On the other hand, becaues $\mathcal{J}$ does not rotate $\lambda$'s, $\mathcal{J}$ commutes with $Q_4$ and also with its reduced version $Q_{\mathcal{M}_n}$. At the level of reduced dynamics on $\mathcal{M}_n$, this $SU(2)_{\mathcal{J}}$ is not an R-symmetry but a global symmetry that arises from the universal isometry of $\mathcal{M}_n$.

While we are on the question of symmetry, let us digress a little and consider the equivariant index or refined index one encounters in literature on wall-crossing, of the generic form

$$\mathrm{Tr}\left((-1)^F y^{2j_3}\right) \tag{3.108}$$

with a "rotation" generator $j_3$ along $z$-axis. Most such computations are based on some version of low energy quantum mechanics on the classical moduli spaces, our $\mathcal{M}_n$'s, but as we saw above, the "rotational symmetry" of $\mathcal{M}_n$ is in fact not the purely spatial rotation but a diagonal subgroup of spatial rotation $SU(2)_L$ and the field theory R-symmetry $SU(2)_R$. Therefore, the refined indices that have been computed are in fact

$$\mathrm{Tr}\left((-1)^F y^{2\mathcal{J}_3}\right) = \mathrm{Tr}\left((-1)^F y^{2J_3+2I_3}\right) \tag{3.109}$$

so actually would equal the protected spin character

$$\mathbf{Tr}\left((-1)^{2J_3} y^{2J_3+2I_3}\right) \tag{3.110}$$

of the field theory, if we are allowed to choose the chirality operator $(-1)^F$ of the quantum mechanics to be $(-1)^{2J_3}$.

So this brings the question of what happens to the two natural chirality operators, $(-1)^{2J_3}$ and $(-1)^{2I_3}$, when we deform and reduce the dynamics in favor of a $\mathcal{M}_n$ nonlinear sigma-model. As we saw, the two $SU(2)$ symmetries are lost individually, so operators like $J_3$ and $I_3$ can no longer be used to classify eigenstates. Nevertheless, $(-1)^{2J_3}$ and $(-1)^{2I_3}$ are still sensible chirality operators. Even after the deformation, one can show directly $(-1)^{2I_3}$ as a product of all fermions while $(-1)^{2J_3}$ is again the same product of all fermions times $(-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\rangle+n-1}$. Both anticommute with the surviving supercharge $Q_4$, so still defines chirality operators. This is not much of surprise since they simply measure the most rudimentary information about the states, i.e., whether, before deformation, the state was in a integral or in a half-integral representations.

When we reduced the dynamics to $\mathcal{M}_n$, however, we must properly redefine these chirality operators by evaluating them on vacuum of the heavy oscillators. For instance, consider $(-1)^{2I_3}$ for the simplest $n = 2$ case. The canonical chirality operator on $\mathcal{M}_n = S^2$ is, as noted before,

$$(-1)^{F_{S^2}} = 2i\hat{\psi}^1\hat{\psi}^2 \ ,$$

with the natural orientation arising from embedding of $S^2$ to $R^3$. To relate this to $(-1)^{2I_3}$, we remember to set the heavy fermions, $\psi^3$ and $\lambda$, to their ground state, which gives precisely

$$\langle 0|(-1)^{2I_3}|0\rangle_{heavy} = (-1)^{F_{S^2}} \ ,$$

it turns out.[12] Clearly, we may repeat this for each sector of 4 fermions labeled by $A$, and find

$$\langle 0|(-1)^{2I_3}|0\rangle_{heavy} = \prod_A 2i\hat{\psi}^1_A\hat{\psi}^2_A = (-1)^{F_{\mathcal{M}_n}} \ . \tag{3.111}$$

Therefore, the chirality operator $(-1)^{2I_3}$ prior to the deformation, smoothly descend to the canonical chirality operator on $(-1)^{F_{\mathcal{M}_n}}$, upon deformation and subsequent reduction of dynamics, and leads to the standard Dirac index $\mathcal{I}_n$.

Since the desired index $\Omega$ needs $(-1)^{2J_3}$ as the chirality operator, we then use (3.81) to relate $(-1)^{2J_3}$ to $(-1)^{2I_3}$, and find an unambiguous answer,

$$\Omega^{distinct} = (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\rangle+n-1} \times \mathcal{I}_n(\{\gamma_A\}) \ . \tag{3.112}$$

On the left hand side, we emphasized the fact we are yet to incorporate the statistics issue. We will see in next section how this generalizes when we impose statistics to the index computation. Before asking the question of statistics, however, there is still one more ambiguity to the expression above, since so far we did not take into account of the internal degeneracy and quantum numbers of the individual

[12] This can be seen most easily when we choose the ordering of $\gamma_A$'s such that $\langle\gamma_A,\gamma_B\rangle$ are all nonnegative for $A > B$, which is also the convention chosen in Ref. [54] for non-scaling cases.

charge centers. The left hand side is still defined with respect to $(-1)^{2J_3}$, so adding internal degeneracy factor can be accomodated by writing

$$\Omega^{distinct} = (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\rangle+n-1} \times \mathcal{I}_n(\{\gamma_A\}) \times \prod_{A=1}^{n} \Omega_A \tag{3.113}$$

where individual $\Omega_A$'s are also computed as the trace of $(-1)^{2J_3}$ (as usual modulo the universal half-hypermultiplet part). As usual, we assume that there is no significant coupling of these internal degeneracy to the quantum mechanical degrees of freedom. Sometimes, we will also write this as

$$(-\Omega^{distinct}) \times (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\rangle} = \mathcal{I}_n(\{\gamma_A\}) \times \prod_{A}(-\Omega_A) \tag{3.114}$$

which is more convenient when keeping track of statistics, since, for $SU(2)_R$ singlets, the Bose/Fermi statistics are naturally correlated with the sign of $-\Omega_A$'s.

## 3.5 Index with Bose/Fermi Statistics and Rational Invariants $\bar{\Omega}$

So far, we pretended that dyons involved are all distinct, and studied the supersymmetric bound state thereof. In reality, this is not quite good enough since we often need to understand bound states of many identical dyons, obeying either fermionic or bosonic statistics. The effective true moduli space, for example, has to be an orbifold of type

$$\mathcal{M}_n/\Gamma \tag{3.115}$$

where $\Gamma$ is a union of permutation groups that mix up labels for identical particles, with proper action on wavefunctions. Equivalently, the index should be computed with appropriate projection operator inserted,

$$\Omega = \mathrm{Tr}\left((-1)^{2J_3} e^{-sH} \mathcal{P}_\Gamma\right) \tag{3.116}$$

where $\mathcal{P}_\Gamma$ projects to wavefunctions obeying either Bose or Fermi statistics under the exchange of identical particles.

The orbifolding reduces the volume of the moduli space, so given the index formula which is an integral over the manifold, we should expect to see factors like $1/d!$ as a result of having $d$ identical centers. However, action of $\Gamma$ is not everywhere free, since when identical particles are on top of one another, the action is trivial. There are complicated fixed submanifolds under $\Gamma$, making the problem very involved, and in particular there should be additional contributions from the fixed manifolds under the orbifolding action.

### 3.5.1 The MPS Formula

Before we carry out such a computation directly, it is instructive to recall a recent result by Manschot, Pioline, Sen (MPS), who evaded this complication algebraically, and replaced it by a sum of many index problems with distinct charge centers [54]. They argued that one can recover the correct index, by adding indices for a series of artificial problems with a smaller number of charge centers. In this set of effective index problems, the trick requires the following rules: When the reduced problem has $d$ particles of the same kind, MPS divides the index by $1/d!$. When one has a particle of nonprimitive charge as a part of such a reduced problem, one must also use, in place of the true intrinsic degeneracies $\Omega$ of the particles, a mathematical one $\bar{\Omega}$,

$$\bar{\Omega}(\gamma) \equiv \sum_{p|\gamma} \frac{\Omega^+(\gamma/p)}{p^2} \ , \tag{3.117}$$

where the sum is over the positive integer $p$ such that $\gamma/p$ belongs to the quantized charge lattice of the theory. Note that $\Omega(\gamma) = \bar{\Omega}(\gamma)$ whenever $\gamma$ is primitive and $\bar{\Omega}(p\gamma) = \Omega(\gamma)/p^2$ if no non-primitive charge state exists.

For illustration, let us take two primitive charges $\beta_1$ and $\beta_2$. Suppose that, among all possible linear combinations of the two, only these two states exist on one side of the marginal stability wall. Labeling the degeneracy by $\pm$ depending on which side of the wall we are considering, we thus assume that

$$\Omega^+(m\beta_1 + k\beta_2) = 0, \quad \text{unless } (m,k) = \pm(1,0) \text{ or } (m,k) = \pm(0,1) \; . \tag{3.118}$$

The sign of $\Omega_{1,2} \equiv \Omega^+(\beta_{1,2})$ are correlated with the statistics assignment of the particle; a hypermultiplet has $\Omega = 1$ and must be treated as Fermions while a vector multiplet has $\Omega = -2$ and must be treated as Bosons. Under this assumption, we have $\bar{\Omega}(p\beta_{1,2}) = \Omega(\beta_{1,2})/p^2$. Manschot et.al.'s formula then simplifies to,

$$\begin{aligned}
& -\Omega^-(m\beta_1 + k\beta_2) \times (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\rangle} \\
= \; & \frac{1}{m!k!}\,\mathcal{I}_{m+k}(\beta_1,\beta_1,\ldots,\beta_2,\beta_2,\cdots)\,(-\Omega_1)^m(-\Omega_2)^k \\
+ \; & \frac{1}{(m-2)!k!}\,\mathcal{I}_{m-1+k}(2\beta_1,\beta_1,\beta_1,\ldots,\beta_2,\beta_2,\ldots)\,\frac{-\Omega_1}{2^2}\,(-\Omega_1)^{m-2}(-\Omega_2)^k \\
+ \; & \frac{1}{(m-3)!k!}\,\mathcal{I}_{m-2+k}(3\beta_1,\beta_1,\beta_1,\ldots,\beta_2,\beta_2,\ldots)\,\frac{-\Omega_1}{3^2}\,(-\Omega_1)^{m-3}(-\Omega_2)^k \\
+ \; & \frac{1}{2!(m-4)!k!}\,\mathcal{I}_{m-2+k}(2\beta_1,2\beta_1,\beta_1,\beta_1\ldots,\beta_2,\beta_2,\ldots)\left(\frac{-\Omega_1}{2^2}\right)^2(-\Omega_1)^{m-4}(-\Omega_2)^k \\
+ \; & \cdots \; ,
\end{aligned} \tag{3.119}$$

where the sum is over all unordered partitions of $m\beta_1 + k\beta_2$ respectively, although we listed above only part of the partitions of $m$. For the overall sign, we re-labeled the individual charges $\beta_1,\beta_1,\ldots,\beta_2,\beta_2,\ldots$ and called them $\gamma_A$'s. This sum and each term in it can be characterized by the following set of rules:

(i) The sum is over all unordered partition of $m\beta_1 + k\beta_2 = \sum_s d_s\beta_s$ where $\beta_s = (p_{s1}\beta_1 + p_{s2}\beta_2)$. For each $\beta_s$, we will have a factor $\bar{\Omega}(\beta_s)$, so we can, with the current assumption on $\Omega^+$'s, consider only a subset where only one of $p_{s1}$ and $p_{s2}$ is nonzero for each $s$.

(ii) The index $\mathcal{I}_{n'}$ with $n' \equiv \sum_s d_s$ effective charge centers. For $\mathcal{I}_{n'}$, we treat all charge centers as distinguishable, so it is computed by the index theorem of the previous section with $n' \leq n$ distinguishable centers.

(iii) The combinatoric factor of $1/d_s!$ for each $s$. This takes into account of the reduced volume of the moduli space due to the orbifolding by the permutation subgroup $S(d_s)$ acting on the reduced $n'$-center quantum mechanics, but does not address the contribution from the submanifolds fixed by $S(d_s)$.

(iv) For each effective particle of charge $p\beta$, with primitive $\beta$ and $p > 1$, that shows up in computation of $\mathcal{I}_{n'}$, one further assigns an effective internal degeneracy factor $-\bar{\Omega}(p\beta) = -\Omega(\beta)/p^2$, in addition to $(-\Omega_1)^{m'}(-\Omega_2)^{k'}$, which reflects the fact that $m'$ number of $\beta_1$ centers and $k'$ number of $\beta_2$ centers are left as individual.

The last $-\bar{\Omega}(p\beta) = -\Omega(\beta)/p^2$ should be compared to the naive $(-\Omega(\beta))^p$ degeneracy factor that would be the correct factor if we were considering $p$ separable particles of charge $\beta$ instead of one particle of charge $p\beta$. Finally, the appearance of $-\Omega$'s instead of $\Omega$'s is natural, since for example a half-hypermultiplet with $\Omega = 1$ acts like Fermions, while a vector multiplet with $\Omega = -2$ acts like Bosons.

From the quantum mechanics viewpoint, the decomposition (i) clearly has something do with the orbifolding action $\Gamma$. Each term in (3.119) arises from a submanifold which is fixed by the product of permutation groups of order $p_s$, $\prod_s S(p_s) \subset \Gamma$. For each sector, origins of (ii) and (iii) are also evident as coming from a reduced problem of $n'$ charge centers and the subsequent volume-reducing action of $\prod_s S(d_s) = \Gamma/\prod_s S(p_s)$. The only part of this formula which is not evident, so far, from the moduli quantum mechanics viewpoint is the rational degeneracy factor of (iv). Here, we would like to isolate where this comes from, and later derive it directly from the moduli dynamics.

After some careful thinking, it becomes evident that this rational degeneracy factor should come from quantum mechanical degrees of freedom normal to the submanifold fixed by $S(p)$'s. Let us consider only $p > 1$ cases and label them $p_{s'}$, since otherwise the internal degeneracy factor is $\Omega(\beta)$ as expected. Subgroup $S(p_{s'})$'s permuting these $p_{s'} > 1$ charges fixe a submanifold $\mathcal{M}_{n-\sum(p_{s'}-1)}$ inside $\mathcal{M}_n$. This fixed submanifold has a codimension $2\sum(p_{s'}-1)$ in $\mathcal{M}_n$, since it is spanned by coincidence of $p_{s'}$ centers, each of which span two directions in $\mathcal{M}_n$.

Consider the reduced dynamics on the intersection, $\mathcal{M}_{n'=n-\sum(p_{s'}-1)}$, for computation of $\mathcal{I}_{n'}$ with all such $p_{s'}\beta_{s'}$ center treated as single particle, respectively. If we start with this reduced index problem, impose the statistics, and pretend that the centers associated with $p_{s'}\beta_{s'}$ comes with a unit degeneracy we will find a contribution of type

$$\frac{1}{\prod_s d_s!}\,\mathcal{I}_{n'}\times(-\Omega_1)^{m'}(-\Omega_2)^{k'}\;, \tag{3.120}$$

where $m'\beta_1 + k'\beta_2 = m\beta_1 + k\beta_2 - \sum_{s'} p_{s'}\beta_{s'}$ and $n' = m' + k' + \sum_{s'} d_{s'}$. Note that we took care to include the volume-reducing effect of $\prod_s S(d_s) = \Gamma/\prod S(p_{s'})$ via the denominator $\prod d_s! = m'!k'!(\prod d_{s'}!)$.

This expression is obtained after ignoring the quantum degrees of freedom that are normal to the fixed manifold $\mathcal{M}_{n-\sum(p_{s'}-1)}$'s, and does not agree with MPS formula. The latter is

$$\frac{1}{\prod_s d_s!}\,\mathcal{I}_{n'}\times(-\Omega_1)^{m'}(-\Omega_2)^{k'}\times\prod_{s'}\frac{-\Omega(\beta_{s'})}{p_{s'}^2} \tag{3.121}$$

so the difference is precisely the rational degeneracy factor of (iv). Clearly it comes from quantizing the normal bundles of $\mathcal{M}_{n-\sum(p_{s'}-1)}$'s inside $\mathcal{M}_n$. On the other hand, for all intent and purpose, this part of quantum mechanics is free, since they have something to do with many identical particles and has no interaction of type $\mathcal{L}_1$, except for the statistics issue.[13]

This leads us to conclude that the factor, $\Omega(\beta)/p^2$, should arise from an index of $p$ noninteracting identical particles of charge $\beta$, modulo the center of mass part which already contributed to $\mathcal{I}_{n'=n-p+1}$. The relative dynamics of such identical particles carry $2(p-1)$ bosonic degrees of freedoms, $2(p-1)$ fermionic degrees of freedom, and additionally internal degeneracy of $|\Omega(\beta)|$ for all $p$ particles. In next subsection, we will show that precisely such a factor arises from the dynamics of non-interacting and identical $p$ particles with the internal degeneracy $\Omega(\beta)$.

The full MPS formula follows the same set of rules, except that one must in general consider an arbitrary set of charges on the + side, and all the partitions of the total charge $\gamma_T$ in terms of charges of states that exist on + side of the wall. Since the + side of spectrum may then include states of charges $h\beta_1 + j\beta_2$ with $h + j > 1$, more diverse charge centers will appear for the individual index problems on the right hand side. As we will discuss later, this can be incorporated by treating all such particles on the + side as independent. The only subtlety is when non-primitively charged states exist on the + side; this can be remedied by employing the fully general form, $\bar{\Omega}(\gamma) \equiv \sum_{p|\gamma} \Omega^+(\gamma/p)/p^2$ as the effective degeneracy factor. We will also see this most general $\bar{\Omega}$ emerging from our index computations.

[13] There is a subtlety, again related to whether threshold bound state of identical charges can form. Since we started with the assumption that such nonprimitive state do not exist, it is safe to assume this issue does not complicate our problem. Whether or not we can extend this to theories with threshold bound states, i.e. supergravity, is an open problem.

### 3.5.2 Physical Origin of $\Omega(\beta)/p^2$ from $p$ Non-Interacting Identical Particles

Let us first restrict ourselves to bound states involving several identical dyons of charge $\beta$ with $-\Omega^+(\beta) = \pm 1$.[14] As in the previous discussion, let us consider the bound state of $n$ charge centers, $\gamma_T = \sum_A \gamma_A$, $m$ of which are $\beta$'s. The identical nature of the $\beta$ dyons means that the orbifolding group includes the $m$-th order permutation group $S(m)$. We start with the assumption, for simplicity, that $k\beta$ state exists only for $|k| = 1$ on the $+$ side, and then come back for the fully general case in next subsection.

Consider the index reduced on the true moduli space $\mathcal{M}_n$ as described in the previous section, with proper account taken of the Bosonic or the Fermionic statistics,

$$\Omega^-\left(\sum_A \gamma_A\right) - \Omega^+\left(\sum_A \gamma_A\right) = \int_{\mathcal{M}_n} \text{tr}\left(\langle X|(-1)^{2J_3} e^{-sQ^2}\mathcal{P}_\Gamma|X\rangle\right) dX \ , \quad (3.122)$$

via the orbifolding projection operator $\mathcal{P}_\Gamma$. Here tr means the trace over the fermionic variables as well as other internal discrete degrees of freedom, and we integrate over the bosonic variables $X$ with an appropriate measure. Matching the sign of $-\Omega$ with $(-1)^F$ value of the component dyon states, as we noted in the case of bound state counting in distinguishable centers, this index naturally computes the degeneracy $\Omega^-$'s as

$$\text{Ind}(\{\gamma_A\};\Gamma) = \int_{\mathcal{M}_n} \text{tr}\left(\langle X|(-1)^{2J_3} e^{-sQ^2}\mathcal{P}_\Gamma|X\rangle\right) dX \ , \ , \quad (3.123)$$

so we would like to ask whether this reproduces (3.119) and rediscover the rational invariant $\bar{\Omega}$. For this, let us concentrate on the permutation group $S(m)$ part of $\Gamma$ and see how it generates a series of terms, similar to MPS's wall-crossing formula.

[14] Because of spin-statistics theorem, there is no irreducible BPS multiplet in $D = 4$ $N = 2$ theories with $\Omega = -1$. The half-hypermultiplet has 1 and vector multiplet has -2. The assumption here is strictly for the illustrative purpose only.

Inside $\mathcal{M}_n$ there are various fixed submanifolds, $\mathcal{M}_{n'}$, of dimension $2(n'-1)$. The simplest are $\mathcal{M}_{n-p+1}$, fixed under $S(p)$ subgroup of $S(m)$. Note that we use the same notation $\mathcal{M}$ for the fixed submanifolds as the full classical moduli manifold $\mathcal{M}$. This is because all of them are of exactly the same type. For example, the manifold $\mathcal{M}_{n-m+1}$ would also emerge if we started with a different low energy dynamics involving a single center of charge $m\beta$ in place of $m$ centers of charge $\beta$. Ignoring contributions from these fixed manifolds would simply give

$$(-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\rangle+n-1}\left(\frac{\mathcal{I}_n}{|\Gamma|}\right)\times\prod_A \Omega_A \tag{3.124}$$

due to the volume-reducing action of $\Gamma$ when it is acting freely. This is the very first term in the MPS formula. Since there are many fixed submanifolds, however, each of them will contribute additively on top of this.

Without loss of generality, let us consider the fixed manifold $\mathcal{M}_{n-p+1}$ associated with the partition $m\beta = p\beta+\beta+\beta+\cdots+\beta$, and label the coordinates along the fixed manifold $\mathcal{M}_{n-p+1}$ by $X'$ and those normal to it by $Y$. Note that among $X'$ are the two coincident (or center of mass) coordinates for the $p\beta$ charge center, so we can think of $Y$'s as the relative position coordinates among these $p$ charge centers; therefore there are $2(p-1)$ $Y$'s. We then formally write the additive contribution from the fixed submanifold $\mathcal{M}_{n-p+1}$ as

$$\begin{aligned}&\Delta_p\times \mathrm{Ind}(\{\gamma_{A'}\}=\{p\beta,\beta,\dots\};\Gamma')\\ =\;&\Delta_p\times\int_{\mathcal{M}_{n-p+1}}\mathrm{tr}'\left(\langle Y=0,X'|(-1)^{2J_3'}e^{-sH'}\mathcal{P}_{\Gamma'}|Y=0,X'\rangle\right)dX'\end{aligned} \tag{3.125}$$

where $\Gamma'=\Gamma/S(p)$ is the remaining orbifolding group that acts nontrivially on $\mathcal{M}_{n-p+1}$. Here $\mathrm{tr}'$ denotes trace over fermionic and other internal degrees of freedom, except those associated with the $p$ identical $\beta$'s that are held together at $\mathcal{M}_{n-p+1}$.

We factored out the contribution $\Delta_p$ from the normal directions, $Y$, and the superpartners thereof. On the other hand, the second factor is the index of a reduced $n-p+1$ center problem, modulo the internal degeneracy factor of $p\beta$ charge center. Other than this, the computation of this latter factor proceeds on equal footing as (3.122),

$$\int_{\mathcal{M}_{n-p+1}} \mathrm{tr}' \left( \langle Y=0, X'|(-1)^{2J_3'} e^{-sH'} \mathcal{P}_{\Gamma'} |Y=0, X'\rangle \right) dX'$$
$$\simeq (-1)^{\sum_{A'>B'}\langle\gamma_{A'},\gamma_{B'}\rangle+n-p} \left( \frac{\mathcal{I}_{n-p+1}(\{\gamma_{A'}\})}{|\Gamma'|} \right) \times \prod_{A'=2}^{n-p+1} \Omega_{A'} + \cdots \quad (3.126)$$

so we may compute the full index recursively. $\Delta_p$ plays the role of the missing internal degeneracy factor $\Omega_{1'}$ here, as it computes the effective contribution from these $p$ coincident $\beta$'s. The ellipsis denotes terms from other fixed submanifold inside $\mathcal{M}_{n-p+1}$ etc.

We will show that $\Delta_p = \pm 1/p^2$, regardless of precise nature of the $\beta$ particles, which also reproduces MPS formula for $\Omega(\beta) = \pm 1$ entirely from the dynamics. Schematically, this factor can be written as

$$\sim \int \mathrm{tr} \left( \langle Y|(-1)^{2J_3^\perp} e^{-sH^\perp} \mathcal{P}|Y\rangle \right) dY \ , \quad (3.127)$$

with $F_\perp$ and $H_\perp$ defined on $Y$'s and the superpartners, again with suitable measure for the bosonic integral. The projection operator

$$\mathcal{P} = \frac{1}{p!} \sum_{\pi\in S(p)} (\mp 1)^{\sigma(\pi)} M_\pi \quad (3.128)$$

ensures that we isolate wavefunctions of correct Bose/Fermi statistics. Note that the sign in the projection operator is the same as that of $-\Omega(\beta)$. $M_\pi$ is the $(p-1)$ dimensional representation of $\pi \in S(p)$, common for $(p-1)$ coordinate doublets $Y's$ and for their fermionic partners, $\psi$'s. Naturally $\sigma(\pi)$ is odd or even when $\pi$ is odd or even.

Since the embedding of $\mathcal{M}_{n-p+1}$'s in $\mathcal{M}_n$ could be very complicated, the exact nature of the decomposition is not entirely clear. On the other hand, the initial index problem is gapped and allows us to take $s \to 0$. At least in this limit, the decomposition makes sense intuitively, and, as we will see shortly it suffices to consider an arbitrarily small tubular neighborhood around the fixed manifold $\mathcal{M}_{n-p+1}$. We take the $Y$ directions as a ball $B_{2(p-1)}$ insides a flat $R^{2(p-1)}$. Therefore, we have

$$\Delta_p = \lim_{s\to 0} \int_{B_{2(p-1)}} \mathrm{tr}\left( \langle Y|\, (-1)^{J_3^\perp} e^{(s/2)\nabla^2} \mathcal{P}|Y\rangle \right) dY \ . \tag{3.129}$$

Precisely how we cut-off this neighborhood will not matter, as we will see shortly that a Gaussian integral of squared width $s$ emerges along $Y$ directions.

Interestingly, exactly the same kind of object was studied when solving for the famous D0 bound state problems in the 1990's. The first such computation appeared in Ref. [58] on two-body problem and was later expanded to many body case in Ref. [59]. Here we will adapt and expand the computations in these works. Since there are $2(p-1)$ real fermions, we will choose a polarization of type $\{\psi_i, \psi_i^\dagger\}$, so that a general wave function $|\,\Psi\rangle$ can be expanded as

$$|\Psi\rangle = \left( \Psi(Y) + \Psi^{\{i\}}(Y)\psi^\dagger_{(i)} + \frac{1}{2}\Psi^{\{i_1 i_2\}}(Y)\psi^\dagger_{(i_1)}\psi^\dagger_{(i_2)} + \dots \right) |0\rangle \ , \tag{3.130}$$

where $\Psi^{\{i_1\dots i_m\}}(Y) = \sum_k \lambda_k^{\{i_1\dots i_m\}} \Psi_k(Y)$ and $\{\Psi_k\}$ are complete basis of $Y$-space wave functions. Since the Hamiltonian is free and do not mix sectors of different fermion numbers, we may evaluate the bosonic and fermionic trace independently. The fermionic trace, for each of $M_\pi$, is given by

$$(\pm 1)^p\, \mathrm{tr}_\psi \left( (-1)^{p-1} (-1)^{N_{\psi^\dagger}} M_\pi \right) \ , \tag{3.131}$$

where $(\pm 1)^p$ arises from the value of $(-1)^{2J_3}$ on $p$ individual $\beta$ states. Also the orbital part of $J_3^\perp$ is always integral in the absence of the minimal coupling contribution, so $(-1)^{2J_3^\perp}$ acting on the quantum mechanical degrees of freedom becomes purely fermionic expression $(-1)^{p-1}(-1)^{N_{\psi^\dagger}}$. The sign in front of the latter comes from converting the chirality operator to a form involving the fermion number operator that counts the creation operators $\psi^\dagger$.

Then the contribution from $Y$ direction reads

$$\Delta_p = \frac{(\pm 1)^p}{p!} \sum_{\pi \in S(p)} (\mp 1)^{\sigma_\pi} (-1)^{p-1} \mathrm{tr}_\psi \left( (-1)^{N_{\psi^\dagger}} M_\pi \right) \times \int_{B_{2(p-1)}} \langle Y | \, e^{s\nabla^2/2} M_\pi | Y \rangle dY \ , \tag{3.132}$$

in the limit of $s \to 0$. A crucial observation that allows us to proceed systematically is

$$\begin{aligned} & \mathrm{tr}_\psi \left( (-1)^{F^\perp} M_\pi \right) \\ = \ & \langle 0|0\rangle - \langle 0| \psi^{a'} M_{\pi a'}{}^a \psi_a^\dagger |0\rangle + \frac{1}{2} \langle 0| \psi^{b'} \psi^{a'} M_{\pi a'}{}^a M_{\pi b'}{}^b \psi_a^\dagger \psi_b^\dagger |0\rangle + \cdots \\ = \ & \det(1 - M_\pi) \ , \end{aligned} \tag{3.133}$$

and furthermore

$$\det(1 - M_\pi) = \begin{cases} p \ , & \pi \text{ is a cyclic permutation of order } p \\ 0 \ , & \text{otherwise} \end{cases} \tag{3.134}$$

for which it is important to remember that $M_\pi$ is a $p-1$ (rather than $p$) dimensional representation of $S(p)$. Since $(-1)^{\sigma_\pi} = (-1)^{p-1}$ for any cyclic permutation of order $p$, we find

$$\begin{aligned} \Delta_p &= \frac{(\pm 1)^p}{p!} \sum_{\pi'} (\mp 1)^{p-1} (-1)^{p-1} \det(1 - M_{\pi'}) \int \langle Y | \, e^{s\nabla^2/2} M_{\pi'} | Y \rangle dY \\ &= \frac{\pm 1}{p!} \sum_{\pi'} \det(1 - M_{\pi'}) \times \frac{1}{(2\pi s)^{p-1}} \int e^{-(Y - M_{\pi'} Y)^2/2s} dY \\ &= \frac{\pm 1}{p!} \sum_{\pi'} \frac{1}{\det(1 - M_{\pi'})} \ , \end{aligned} \tag{3.135}$$

where the sum is now only over the cyclic permutations of order $p$. There are precisely $(p-1)!$ such permutations and they each contribute $1/p$, via the determinant, so the result is

$$\Delta_p = \frac{\pm 1}{p^2} \ , \tag{3.136}$$

as promised. Clearly, we can repeat this when there are several such factors simultaneously, to give,

$$\Delta_{\{p_{s'}\}} = \prod_{s'} \frac{\pm 1}{p_{s'}^2} \ , \tag{3.137}$$

reproducing $\Omega(\beta)/p_s^2$ of the MPS formula with $\Omega(\beta) = \pm 1$.

The more general case of $\Omega(\beta) = \pm d$ can be derived similarly. Let us write one particle state as

$$|\hat{\Psi}\rangle = \hat{\Psi}^{\eta}(x,\psi)|0;\eta\rangle \tag{3.138}$$

so that $\eta = 1, 2, \ldots, d$ labels the internal degeneracy, and $p$-particles wave function (without center of mass degree of freedom) can be written as a sum of terms like

$$\Psi^{\{ij\cdots k\}\{\eta_s\}}(Y)\psi^\dagger_{(i)}\psi^\dagger_{(j)}\cdots\psi^\dagger_{(k)}|0;\eta_1,\eta_2,\ldots,\eta_p\rangle \ , \tag{3.139}$$

none of which mixes under the free Hamiltonian. Thanks to this, Just as the fermionic part and the bosonic part separately contributed, this internal part also factorizes under each $\pi$. Expressing $\Delta_p$ as a sum over the elements of permutation group again, we now have an extra factor

$$\langle \eta_p, \ldots, \eta_1 | \eta_{\pi(1)}, \ldots, \eta_{\pi(p)} \rangle \ ,$$

for each permutation $\pi$, and the trace over these internal indices. Thus, we arrive at a similarly simple form,

$$\Delta_p = \frac{(\pm 1)}{p\,!} \sum_{\eta_1,\cdots,\eta_p} \sum_{\pi'} \frac{\langle \eta_p, \ldots, \eta_1 | \eta_{\pi(1)}, \ldots, \eta_{\pi(p)} \rangle}{\det(1 - M_{\pi'})} \tag{3.140}$$

As before, the sum is over the permutations of cyclic order $p$. For such $\pi'$, the inner product vanishes identically unless all $\eta_i$'s are equal to one another, and gives unit if all are equal. The sum over $\eta$'s thus collapses to a single sum over $\eta = \eta_1 = \eta_2 = \cdots = \eta_p$, and

$$\Delta_p = (\pm 1) \sum_{\eta} 1 \times \frac{1}{p\,!} \sum_{\pi'} \frac{1}{\det(1 - M_{\pi'})} = \frac{\pm d}{p^2} = \frac{\Omega(\beta)}{p^2}\ . \tag{3.141}$$

This gives us the only essential ingredient in confirming (3.122), and in fact the fully general version thereof.

For a complete derivation, a recursive argument is needed and we need to consider possibility of low energy dynamics with charge centers both primitive and non-primitive charges simultaneously. This naturally brings us to the most general wall-crossing formula, next.

### 3.5.3 General Wall-Crossing Formula

Most of what we derived generalizes to cases with arbitrary spectrum on + side, without much modification, but here we need to point out one subtlety. Suppose the + side of spectrum contains not only a pair of primitively charge states $\gamma$ and $\gamma'$ but also states like $h\gamma + j\gamma'$ a little more involved, and includes states of composite charges such as $m\gamma$ or linear combinations with other charges. (One can also have states with charges completely unrelated to these but those will not participate in the wall-crossing, and therefore irrelevant.) Such a state cannot be considered as a bound state of $h$ $\gamma$'s and $j$ $\gamma'$'s, since the two are mutually repulsive on the + side. Rather, it should be regarded as a completely independent particle of different origin. In fact, for $SU(2)$ theory with a single flavor, a monopole $\gamma$, a quark $\gamma'$ and a dyon $\gamma + \gamma'$ are known to coexist in the central part of the moduli space.

Let us denote charges of these independent particles as $\beta_v$. Since one can form bound states of a given total charge $\gamma_T$ on the "−" side from different combinations of these + side state,. We will label each of these physically distinct combination by the upper index inside a parenthesis such that

$$\gamma_T = \sum_A m_v^{(1)}\beta_v = \sum_v m_v^{(2)}\beta_v = \sum_v m_v^{(3)}\beta_v = \cdots \ . \tag{3.142}$$

In such circumstances, the total degeneracy for $\gamma_T$ has to be computed for each of such bound state problems and summed over,

$$\Omega^-(\gamma_T) - \Omega^+(\gamma_T) = \sum_q \Omega\left(\{m_v^{(q)}\}\right) \ , \tag{3.143}$$

where each term on the right hand side is computed from the $n^{(q)} = \sum m_A^{(q)}$ center quantum mechanics. For each of $\Omega(\{m_v^{(q)}\})$'s, computation of the previous subsection goes through without modification, and will be computed as

$$\Omega\left(\{m_v^{(q)}\}\right) \;\; = \;\; \Omega\left(\sum_A \gamma_A = \sum_v m_v^{(q)}\beta_v\right) \ . \tag{3.144}$$

One important detail to remember is that, even if some of charge $\beta_v$'s might be a linear combination of other $\beta_v$'s, each of them are physically unrelated independent particles: The permutation group is simply $\Gamma = \prod_v S(m_v^{(q)})$ for each of these index problems.

Combining this with the results of previous subsection, we reproduce MPS formula in its most general form. Note that, when we reorganize this formula in terms of the index of distinct particles of unit individual degeneracy,

$$\mathcal{I}_n(\{\ldots,\gamma,\ldots\}) = \int_{\mathcal{M}_n} Ch(\mathcal{F})\hat{A}(\mathcal{M}_n) \tag{3.145}$$

the rational invariants $\bar{\Omega}(\gamma)$, multiplying them, will accumulate additive contributions of the form, including $p = 1$ case,

$$\frac{\Omega^+(\gamma/p)}{p^2} \tag{3.146}$$

for each $\gamma/p = \beta_v$ that appears in one of the expansion $\gamma_T = \sum_v m_v^{(q)} \beta_v$. Since we are summing over all possible such expansions, it implies that

$$\bar{\Omega}(\gamma) = \sum_{p|\gamma} \frac{\Omega^+(\gamma/p)}{p^2} \tag{3.147}$$

will appear as the effective degeneracy factors that multiply $\mathcal{I}_n$'s. $p = 1$ terms arises only when $\gamma$ is one of the $\beta_v$'s, while $p > 1$ terms arise from the orbifold fixed sector as in the previous section when $\gamma/p$ is one of $\beta_v$'s. The final expression is

$$\begin{aligned}
&\Omega^-(\gamma_T) - \Omega^+(\gamma_T) = \\
&\cdots \\
&+(-1)^{-n'+1+\sum_{A'>B'}\langle\gamma'_{A'},\gamma'_{B'}\rangle} \times \frac{\mathcal{I}_{n'}(\{\gamma'_{1'}, \gamma'_{2'}, \dots\})}{|\Gamma'|} \times \prod_{A'} \bar{\Omega}(\gamma'_{A'}) \\
&+\cdots
\end{aligned} \tag{3.148}$$

where we wrote a representative form for the partition $\gamma_T = \sum_{A'=1}^{n'} \gamma'_{A'}$ into $n'$ centers and the associated orbifolding group $\Gamma'$ permuting among identical elements in $\{\gamma'_{A'}\}$.

## 3.6 Summary and Comments

In this section, we showed how $n$ generic BPS dyons of Seiberg-Witten theory interact with one another, and how the relevant low energy dynamics with $\mathcal{N} = 4$ supersymmetry can be derived in the vicinity of a wall of marginal stability. The resulting quantum mechanics is specified by three classes of quantities: kinetic term, potentials, and minimal couplings. The latter two turn out to be constrained to each other by supersymmetry and can be derived exactly, and are universal, in that the general structure is applicable to BPS black holes as well. The kinetic term may differ, but for counting non-threshold bound states via index theorem, we only need the asymptotic form of the kinetic terms, which fixes effectively the entire Lagrangian. Thanks to the universal form, this Lagrangian can also be used to compute non-threshold bound states of BPS black holes as well as those of Seiberg-Witten dyons.

We showed how the usual truncation (in the previous BPS black hole studies) down to zero locus, $\mathcal{M}_n$, of potentials is misleading because the massgaps along the classically massive direction are always the same as the quantum massgaps along $\mathcal{M}_n$, due to the latter's finite size. Instead, one must sacrifice $\mathcal{N} = 4$ supersymmetry, in favor of an index-preserving $\mathcal{N} = 1$ deformation, in order to reduce the problem to a nonlinear sigma model on $\mathcal{M}_n$.

This gives a definite prescription, hitherto unknown, on how to handle the fermionic superpartners, and the final form of the index is that of a Dirac operator on $\mathcal{M}_n$ with an Abelian gauge field $\mathcal{F}$ determined unambiguously by the minimal couplings among dyons/black holes. Along with $n - 1$ radial, classical massive directions, $2(n - 1)$ fermionic partners become decoupled from the problem, leaving behind a supersymmetric quantum mechanics on $\mathcal{M}_n$ with real supersymmetry. (Three bosonic and four fermionic variables decouple also, playing the role of the center of mass degrees of freedom.) This shows rigorously why the Dirac index is the relevant one, as was anticipated by de Boer et.al. [57].

Since typical wall-crossing problem involves only two linearly independent charge vectors, and thus bound states of many identical BPS states, statistics is of major importance. We address this directly for the index problem by inserting the relevant projection operator $\mathcal{P}_\Gamma$, and expanding the index to a series involving various fixed submanifolds. Each such contribution consists of two multiplicative factors: one is usual Dirac index on the fixed submanifold and the other is contribution from the normal direction. The latter turns out to be universal and generates a numerical factor $\sim 1/p^2$ for each $p$ coincident and identical particles, times the intrinsic degeneracy of the particle in question. This eventually lead to the rational invariants, $\bar{\Omega}(\gamma) = \sum_{p|\gamma} \Omega(\gamma/p)/p^2$, as the effective degeneracy factor, as was also noted by Manschot et.al. [54] In the end, we have derived the general wall-crossing formula, from the viewpoint of spatially loose BPS bound states by starting from Seiberg-Witten theory, ab initio.

After this work, Sen [74] showed that this wall-crossing formula agrees with that of Higgs branch and Kontsevich-Soibelman (KS) formula, in the limit where there is no scaling solution. The latter solution appears when the quiver description has closed loops, so when the theory allows the superpotential. Although the Coulomb description described in this section offers clear physical picture in understanding the wall-crossing phenomena, when there exist such solutions, this description becomes ill-defined and fails to capture all the BPS states. These states which are accessible only in the Higgs branch description are called the Higgs invariants, and they are known to be insensitive to the wall crossing phenomena. Furthermore, theses states are singlet under the spatial rotation, and believed to be responsible for the exponentially large microstates of the single centered blackhole of the supergravity. Recently, there has been several developments which try to clarify the properties of these states. [75–80] They started from the original quiver quantum mechanics, and investigated the Higgs branch solution of the theory in various ways. In this description, the index calculation in question is translated into the cohomology counting of the moduli space of the given quiver theory. Especially, we would like to emphasize the recent work [81], where they started from the gauged quiver quantum mechanics Lagrangian, and carried out the path integral honestly to obtain the index of the theory. This work generalizes the relation between supersymmetric quantum mechanics and the index theorem which was reviewed in section 2.2 of this thesis, to the level of gauged linear sigma models.

# Chapter 4

# D-branes and Orientifolds From 2D Partition Functions

Two-dimensional gauged linear sigma model (GLSM) with $\mathcal{N} = (2,2)$ supersymmetry provides a very useful tool to investigate the Calabi-Yau spaces which the string theory is based on. At the first section, we study the basic features of this model, which include brief summary of two-dimensional mirror symmetry and Calabi-Yau (CY)/Landau-Ginzburg (LG) correspondences, based on reviews of [3] and [11]. From the second to the last section, we intoduce the new framework of studying the mirror symmetry and the properties of CY, recently developed in [12–15]. We will see that the exact partition functions on various two-dimensional manifold calculated via the method reviewed in section 2.4 plays a crucial role. Especially, at the second section, we will see that the two-sphere partition function exactly calculates Kahler potential of the A-model conformal manifold. From the third section, we focus on the D-branes/Orientifolds which wrap subcycles of ambient Calabi-Yau manifolds. We first present the review on how we traditionally have determined the topological coupling of the D-brane/Orientifolds, which gives the central charges at the tree-level of $\alpha'$. Finally, we will see that the exact calculations of partition functions on hemisphere/$\mathbb{RP}^2$ gives the $\alpha'$-exact central charges in the presence of D-branes and Orientifolds. All these series of works provide a way to exactly calculate the fully quantum corrected A-model quantities. We will see that these works clarify the several subtle issues regarding RR-charges as well.

## 4.1 Basics of $2d$ $\mathcal{N}=(2,2)$ Gauged Linear Sigma Model

### 4.1.1 $2d$ $\mathcal{N}=(2,2)$ algebra

Let us denote the two-dimensional wordsheet coordinate as $x^{\pm}=x^0\pm x^1$. In order to deal with the $\mathcal{N}=(2,2)$ supersymmetry, we include fermionic coordinates $\theta^{\pm}$, $\bar{\theta}^{\pm}$. It is particularly useful to study the theory in terms of various superfields. Define supersymmtry transformation and derivatives acting on superfields as

$$\begin{aligned} Q_{\pm} &= \frac{\partial}{\partial\theta^{\pm}} + i\bar{\theta}^{\pm}\partial_{\pm} \\ \bar{Q}_{\pm} &= -\frac{\partial}{\partial\bar{\theta}^{\pm}} - i\theta^{\pm}\partial_{\pm} \\ D_{\pm} &= \frac{\partial}{\partial\theta^{\pm}} - i\bar{\theta}^{\pm}\partial_{\pm} \\ \bar{D}_{\pm} &= -\frac{\partial}{\partial\bar{\theta}^{\pm}} + i\theta^{\pm}\partial_{\pm} \ . \end{aligned} \tag{4.1}$$

The *chiral multiplet* $\Phi(x^{\pm},\theta^{\pm},\bar{\theta}^{\pm})$ is a superfield defined by

$$\bar{D}_{\pm}\Phi = 0 \ . \tag{4.2}$$

The solution of this equation can be written in terms of

$$\Phi = \phi(y^{\pm}) + \theta^{\alpha}\psi_{\alpha}(y^{\pm}) + \theta^{+}\theta^{-}F(y^{\pm}) \ , \tag{4.3}$$

where $y^{\pm} = x^{\pm} - i\theta^{\pm}\bar{\theta}^{\pm}$, $\phi$ is a complex scalar field and $\phi_\alpha$ is a Dirac fermion. The *anti chiral multiplet* is defined as

$$D_{\pm}\bar{\Phi} = 0 \; , \tag{4.4}$$

which is solved by

$$\bar{\Phi} = \bar{\phi}(y^{\pm}) + \bar{\theta}^{\alpha}\bar{\psi}_{\alpha} + \bar{\theta}^{+}\bar{\theta}^{-}\bar{F}(y^{\pm}) \; . \tag{4.5}$$

For two-dimensional $\mathcal{N} = (2,2)$ theory, we can additionally define *twisted chiral multiplet*, which satisfies

$$\bar{D}_{+}\Psi = D_{-}\Psi = 0 \; . \tag{4.6}$$

This equations is solved by

$$\Psi = v(\tilde{y}^{\pm}) + \theta^{+}\bar{\chi}_{+}(\tilde{y}^{\pm}) + \bar{\theta}^{-}\chi_{-}(\tilde{y}^{\pm}) + \theta^{+}\bar{\theta}^{-}G(\tilde{y}^{\pm}) \; , \tag{4.7}$$

where $\tilde{y}^{\pm} = x^{\pm} \mp i\theta^{\pm}\bar{\theta}^{\pm}$. Similary, *the twisted anti-chiral multiplet* can be defined by

$$D_{+}\bar{\Psi} = \bar{D}_{-}\Psi = 0 \; , \tag{4.8}$$

and this is solved by

$$\bar{\Psi} = \bar{v}(\tilde{y}^{\pm}) + \bar{\theta}^{+}\chi_{+}(\tilde{y}^{\pm}) + \theta^{-}\bar{\chi}_{-}(\tilde{y}^{\pm}) + \bar{\theta}^{+}\theta^{-}\bar{G}(\tilde{y}^{\pm}) \; . \tag{4.9}$$

For the theories with gauge symmetry, we define the real multiplet $V$ which contains a gauge fields. If the chiral field transforms under the gauge transformation as $\Phi \to e^{i\Lambda}\Phi$, where $\Lambda$ is another chiral fields, the gauge invariant kinetic action can be written as

$$\int d^4\theta \; \bar{\Phi} e^{V} \Phi \; , \tag{4.10}$$

and the gauge field transform as $V \to V + i(\bar{\Lambda} - \Lambda)$ at the same time. In the Wess-Zumino gauge, it can be expended as

$$\begin{aligned} V &= \theta^-\bar{\theta}A_- + \theta^+\bar{\theta}^+A_+ - \theta^-\bar{\theta}^+\sigma - \theta^+\bar{\theta}^-\bar{\sigma} \\ &\quad + i\theta^-\theta^+(\bar{\theta}^-\bar{\lambda}_- + \bar{\theta}^+\bar{\lambda}_+) + i\bar{\theta}^+\bar{\theta}^-(\theta^-\lambda_- + \theta^+\lambda_+) + \theta^-\theta^+\bar{\theta}^+\bar{\theta}^-D \end{aligned} \tag{4.11}$$

One thing to note is that the gauge invariant vector multiplet can be written in the twisted chiral multiplet and its conjugate, which is

$$\Sigma = \bar{D}_+ D_- V \ . \tag{4.12}$$

This is automatically invariant under $V \to V + i(\bar{\Lambda} - \Lambda)$. In the component form,

$$\Sigma = \sigma(\tilde{y}^\pm) + \bar{\theta}^+\lambda_+(\tilde{y}^\pm) + \theta^-\bar{\chi}_-(\tilde{y}^\pm) + \bar{\theta}^+\theta^-(D + iF_{12})(\tilde{y}^\pm) \ . \tag{4.13}$$

Given these ingredients, the supersymmetric action can be easily written with following three possibilities. 1) *D-term* for chiral and twisted chiral fields,

$$\int d^2x d^4\theta \ K(\Phi, \bar{\Phi}) \ . \tag{4.14}$$

2) *F-term* for chiral fields,

$$\int d^2x d\theta^- d\theta^+ \ W(\Phi) \ + \ \int d^2x d\bar{\theta}^- d\bar{\theta}^+ \ \bar{W}(\bar{\phi}) \ , \tag{4.15}$$

3) *Twisted F-term* for anti chiral fields,

$$\int d^2x d\bar{\theta}^- d\theta^+ \tilde{W}(\Psi) \ + \ \int d^2x \theta^- d\bar{\theta}^+ \bar{\tilde{W}}(\bar{\Psi}) \ . \tag{4.16}$$

As an example, we present the action for the gauged linear sigma model (GLSM) which is the main subject of this chapter. The D-term is given by the chiral multiplet charged under the gauge group $G$ and the vector multiplet kinetic action,

$$S_{kinetic} = \int d^2x d^4\theta \ \bar{\Phi}^i e^{QV} \Phi^i \ + \ \frac{1}{2e^2}\bar{\Sigma}\Sigma \ . \tag{4.17}$$

We allow the F-term which is given by

$$S_W = \int d^2x d\theta^- d\theta^+ \ W(\Phi) \ + \ \int d^2x d\bar{\theta}^- d\bar{\theta}^+ \ \bar{W}(\bar{\phi}) \ . \tag{4.18}$$

Finally we have the twisted F-term which is linear in $\Sigma$,

$$S_{FI} = t\int d^2x d^2\tilde{\theta} \ \Sigma \ + \ \bar{t}\int d^2x d^2\bar{\tilde{\theta}} \ \bar{\Sigma} \ , \tag{4.19}$$

where $t = \xi - i\frac{\theta}{2\pi}$, is a combination of the Fayet-Iliopoulos parameter and the topological theta term parameter. When $G = U(1)$, all of these action can be written as, in the component form,

$$\begin{aligned} L &= L_{kinetic} + L_W + L_{FI} \\ &= -D^\mu\bar{\phi}^i D_\mu\phi^i + i\bar{\psi}^i_- D_+\psi^i_- + i\bar{\psi}^i_+ D_-\psi^i_+ + D|\phi^i|^2 + |F^i|^2 - |\sigma|^2|\phi^i|^2 \\ &\quad -\bar{\psi}^i_-\sigma\psi^i_+ - \bar{\psi}^i_+\bar{\sigma}\psi^i_- - i\bar{\phi}^i\lambda_-\psi^i_+ + i\bar{\phi}^i\lambda_+\psi^i_- + i\bar{\psi}^i_+\bar{\lambda}_-\phi^i - i\bar{\psi}^i_-\bar{\lambda}_+\phi^i \\ &\quad +\frac{1}{2e^2}\left[-\partial^\mu\bar{\sigma}\partial_\mu\sigma + i\bar{\lambda}_-\partial_+\lambda_- + i\bar{\lambda}_+\partial_-\lambda_+ + F_{01}^2 + D^2\right] \\ &\quad +\frac{\partial W}{\partial\phi^i}F^i - \frac{\partial^2 W}{\partial\phi^i\partial\phi^j}\psi^i_+\psi^j_- + \frac{\partial\bar{W}}{\partial\bar{\phi}^i}\bar{F}^i + \frac{\partial^2\bar{W}}{\partial\bar{\phi}^i\partial\bar{\phi}^j}\bar{\psi}^i_+\bar{\psi}^j_- \\ &\quad -\xi D + \frac{\theta}{2\pi}F_{01} \ . \end{aligned} \tag{4.20}$$

Note that this action can be obtained from the dimensional reduction of the $4d$ $\mathcal{N} = 1$ theory. One can show that the action is invariant under the supersymmetry transformation given by

$$
\begin{aligned}
\delta A_\pm &= i\bar{\epsilon}_\pm\lambda_\pm + i\epsilon_\pm\bar{\lambda} \\
\delta\lambda_+ &= i\epsilon_+(D + iF_{01}) + 2\epsilon_-\partial_+\bar{\sigma} \\
\delta\lambda_- &= i\epsilon_-(D - iF_{01}) + 2\epsilon_+\partial_-\sigma \\
\delta\sigma &= -i\epsilon_+\lambda_- - i\epsilon_-\bar{\lambda}_+ \\
\delta D &= -\bar{\epsilon}_+\partial_-\lambda_+ - \bar{\epsilon}_-\partial_+\lambda_- + \epsilon_+\partial_-\bar{\lambda}_+ + \epsilon_-\partial_+\bar{\lambda}_- \\
\delta\phi &= \epsilon_+\psi_- - \epsilon_-\psi_+ \\
\delta\psi_+ &= i\bar{\epsilon}_-D_+\phi + \epsilon_+F - \bar{\epsilon}_+\bar{\sigma}\phi \\
\delta\psi_- &= -i\bar{\epsilon}_+D_-\phi + \epsilon_-F + \bar{\epsilon}_-\sigma\phi \\
\delta F &= -i\bar{\epsilon}_+D_-\psi_+ - i\bar{\epsilon}_-D_+\psi_- i\bar{\epsilon}_-\bar{\lambda}_+\phi - i\bar{\epsilon}_+\bar{\lambda}_-\phi + \bar{\epsilon}\sigma\psi_- + \bar{\epsilon}_-\sigma\psi_+
\end{aligned} \quad (4.21)
$$

Two-dimensional $\mathcal{N} = (2,2)$ theories have important global symmetries, which are $U(1)_V \times U(1)_A$. The vector-like R-symmetry $U(1)_V$ is inherited from the $U(1)_R$ R-symmetry of the $4d$ $N = 1$ algebra. It acts on the fields as

$$\Phi(x^\mu, \theta^\pm, \bar{\theta}^\pm) \;\to\; e^{iq_V\alpha}\Phi(x^\mu, e^{-i\alpha}\theta^\pm, e^{i\alpha}\bar{\theta}^\pm)\ . \quad (4.22)$$

On the other hand, the axial R-symmetry, $U(1)_A$ comes from the rotation of dimensionally reduced two internal coordinates. It acts as

$$\Phi(x^\mu, \theta^\pm, \bar{\theta}^\pm) \;\to\; e^{iq_A\alpha}\Phi(x^\mu, e^{\mp i\alpha}\theta, e^{\pm i\alpha}\bar{\theta}^\pm)\ . \quad (4.23)$$

These imply that they acts on the supersymmetry generator as

$$
\begin{aligned}
[U(1)_V, Q_\pm] &= -Q_\pm \\
[U(1)_V, \bar{Q}_\pm] &= \bar{Q}_\pm \\
[U(1)_A, Q_\pm] &= \mp Q_\pm \\
[U(1)_A, \bar{Q}_\pm] &= \pm\bar{Q}_\pm
\end{aligned}
$$

As can be clearly seen from (4.23), $U(1)_A$ couples asymmetrically to the left and the right moving fermion. This implies that there can be an anomaly associated to this symmetry. Anomaly comes from the 2-gon 1-loop Feynman diagram, with currents for the gauge field and $U(1)_A$ symmetry attatched at both ends respectively. This diagram evaluates to

$$\frac{1}{2\pi}\int_\Sigma \mathrm{Tr}F \;=\; \sum_i Q_i \;, \tag{4.24}$$

where $Q_i$ is a charges of the chiral fields under the $U(1)$ factor of the gauge group. This result can be reproduced from the index theorem studied in the Chapter 2, which gives

$$\frac{1}{2\pi}\int_\Sigma ch(F)\wedge A(T) = \frac{1}{2\pi}\int_\Sigma c_1(F) = \frac{1}{2\pi}\int_\Sigma \mathrm{Tr}F \;. \tag{4.25}$$

If this is equal to $k$, it means that the number of $\psi_+, \bar{\psi}_-$ zero modes are $k$ larger than that of $\psi_-, \bar{\psi}_+$. This mismatch implies thet the measure of the path integral becomes

$$\int d\psi_0^{1+} d\bar{\psi}_0^{1-} \cdots d\psi_0^{k+} d\bar{\psi}_0^{k-} [d\psi^+][d\psi^-][d\bar{\psi}^+][d\bar{\psi}^-] \;, \tag{4.26}$$

where $[d\psi]$'s are non-zero modes as well as remaining zero modes of the fermion whose numbers match. From this expression, the measure is definitely non-invariant under the $U(1)_A$ action, which rotates the partition function as

$$Z \;\to\; e^{2ik\alpha} Z \;. \tag{4.27}$$

Note that there remains unbroken subgroup $Z_{2k}$, which corresponds to $\alpha = \frac{2\pi i p}{2k}$ with $p = 1, \cdots 2k$.

One of the most important remarks on the global symmetries of the $\mathcal{N} = (2,2)$ theory is that, one can find a duality that exchanges

$$U(1)_A \;\leftrightarrow\; U(1)_V \;, \tag{4.28}$$

which can be done by exchanging

$$\theta^- \;\leftrightarrow\; \bar{\theta}^-, \quad \text{i.e.,} \quad Q_- \;\leftrightarrow\; \bar{Q}_- . \tag{4.29}$$

This is what is called *mirror symmetry*. Equivalently, the mirror symmetry can be rephrased as exchanging the chrial and twisted chiral multiplet,

$$\Phi \;\leftrightarrow\; \Psi \; . \tag{4.30}$$

### 4.1.2 Phases of $2d$ $\mathcal{N} = (2,2)$ GLSM and the mirror symmetry

The theory of two-dimensional string worldsheets in ten-dimensional spacetime can be most easily described by the non-linear sigma model (NLSM) whose target space is $\mathbb{R}^{1,3} \times M_6$, where $M_6$ is a compact six-dimensional Calabi-Yau manifolds. However, since the geometry of the CY is very complicated and no explicit metric is known, this NLSM is very hard to deal with. The reason why we have been interested in the GLSM of $2d$ $\mathcal{N} = (2,2)$ theory is that, it can serve a very useful tool to investigate the properties of CY and the string theory based on it.

The $\mathcal{N} = (2,2)$ GLSM, whose action is given in (4.20), flows under renormalization group (RG) action to the NLSM whose target space is the Kahler manifold. In order to explicitly show this, take the simplest example of GLSM with $G = U(1)$ gauge group and $N$ fundamental chiral fields coupled to $G$ with charge 1. From the action given in (4.20), after integrating out the auxiliary field D, we can read off the bosonic potential, which is

$$V = \sum_i |\sigma|^2 |\phi^i|^2 + \frac{e^2}{2}(\sum_i |\phi|^2 - \xi)^2 \; . \tag{4.31}$$

The energy is minimized when

$$\sum_i |\phi^i|^2 = \xi, \quad \sigma = 0 \ . \tag{4.32}$$

Hence the moduli space is given by

$$\mathcal{M} = \{\phi^i | \sum_i \phi^i \bar{\phi}^i = \xi\} // U(1) = \mathbb{CP}^{N-1} \ . \tag{4.33}$$

As can be seen from this example, the moduli space is discribed by the zero of the D-term potential quotiented by the gauge group, and this naturally produces the toric manifold when gauge group is abelian. Under the RG flow, we can see that this GLSM flows to the NLSM whose target space is $\mathbb{CP}^{N-1}$. First, let us look into the RG flow of FI parameter $\xi$, which determines the volume of the moduli space. It comes from the one-loop diagram as follows.

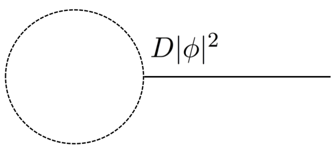


FIGURE 4.1: One-loop diagram for $D|\phi|^2$

This diagram evaluates to

$$\int_{\mu}^{\Lambda_{UV}} \frac{d^2k}{(2\pi)^2} \frac{1}{k^2} \ = \ \log\left(\frac{\Lambda_{UV}}{\mu}\right) \ . \tag{4.34}$$

In general, when gauge group has $U(1)^r$ factor, and if there are chiral multiplet with charge $Q_{ia}$ under the $a$-th $U(1)$ factor, the FI-parameter runs as

$$\xi^a = \xi^a(\mu) + \sum_i Q_{ia} \log(\Lambda_{UV}/\mu) \ . \tag{4.35}$$

When moduli space is $\mathbb{CP}^{N-1}$, along the radial direction, there are one bosonic and fermionic massive modes for the chiral multiplet whose mass is equal to $e\sqrt{2\xi}$. Hence the effective theory of massless modes only can be obtained by taking $e \to \infty$ limit. In this limit, the kinetic terms for the vector multiplet vanish and $A_\mu$ and $\sigma$ become auxiliary. The equations of motion of these give

$$A_\mu = \frac{i}{2}\frac{1}{\sum_{i=1}^N |\phi^i|^2}\left(\sum_{i=1}^N \bar{\phi}^i \partial_\mu \phi^i - \partial_\mu \bar{\phi}^i \phi^i + \bar{\psi}^i \rho_\mu \psi^i\right) , \tag{4.36}$$

$$\sigma = -\frac{\sum_{i=1}^N \bar{\psi}^i_+ \psi^i_-}{\sum_{i=1}^N |\phi^i|^2} . \tag{4.37}$$

If we put (4.36) back to the $\sum_{i=1}^N |D_\mu \phi^i|^2$, it becomes $g_{ij}^{FS}\partial_\mu \bar{\phi}^i \partial^\mu \phi^j$, where $g_{ij}^{FS}$ is homogeneous Fubini-Study metric of $\mathbb{CP}^{N-1}$,

$$ds^2 = \frac{\sum_{i,j=1}^N |\phi^i|^2 |d\phi^j|^2 - \bar{\phi}^i d\phi_i \phi_j d\bar{\phi}^j}{\sum_{i=1}^N |\phi^i|^2} . \tag{4.38}$$

Secondly, the topological theta term $\frac{\theta}{2\pi}\int F_{01}$ reduces to the anti-symmetric B-field in the NLSM,

$$\int_\Sigma B_{ij} d\bar{\phi}^i \wedge d\phi^j \tag{4.39}$$

where $B = \frac{\theta}{2\pi} w_{FS}$, and $w_{FS}$ is the Kahler two form of the Fubini-Study metric. Finally, subsitution of the fermion part of (4.36) and the equation (4.37) to $|\sigma|^2|\phi^i|^2 + \bar{\psi}^i_- \sigma \psi^i_+ + \bar{\psi}^i_+ \bar{\sigma} \psi^i_-$ give a four fermion interaction term in the NLSM.

**Calabi-Yau/Landau-Ginzburg correspondence**

The Calabi-Yau manifold can be obtained if we restrict to special cases that the charges of each $U(1)$ factor satisfy

$$\sum_i Q_i = 0 \tag{4.40}$$

For example, consider $U(1)$ theory with N chiral multiplet $\Phi$ with charge 1, and a chiral multiplet $P$ with charge $-N$. Then we can add a superpotential

$$W = PG(\Phi) \ , \tag{4.41}$$

where $G(\Phi)$ is a homogeneous polynimial of degree $N$. In order to preserve the $U(1)_V$ symmetry of the action, the vector R-charge of this superpotential should add up to 2. Furthermore, for later convenience, we assume that solution for $\frac{\partial G}{\partial \phi^i} = 0$ is given only by $\phi^i = 0$. The bosonic potential for this GLSM reads

$$|\sigma|^2 \left( \sum_i |\phi^i|^2 + N^2|p|^2 \right) + \left( \sum_i |\phi^i|^2 - N|p|^2 - \xi \right)^2 + p^2 \left| \frac{\partial G}{\partial \phi} \right|^2 + |G(\phi)|^2 \ . \tag{4.42}$$

The moduli space, zero loci of the above potential crucially depends on the sign of the FI parameter $\xi$. On the other hand, due to the condition (4.40) and equation (4.35), $\xi$ does not vary under RG flow. Hence we can define a notion of *phase*, distinguished by the sign of $\xi$. First of all, consider the case $\xi > 0$. The solution that minimizes the potential is

$$\sum_i |\phi^i|^2 = \xi, \ p = 0, \ \sigma = 0, \ G(\phi) = 0 \ . \tag{4.43}$$

Due to the last condition, the moduli space is a degree $d$ hypersurface in $\mathbb{CP}^{N-1}$, where $d$ is a degree of the polynomial $G$. For this case, $d = N$, it is well-known that the moduli space is CY hypersurface in $\mathbb{CP}^{N-1}$. Note that, the CY condition is equivalent to the non-anomalous axial R-symmetry condition. The moduli of the CY target space obtained like this is particulary easy to deal with, by tuning the parameters of the GLSM. Volume of the CY is controlled by the FI parameter $\xi$, and the complex structure is determined by the form of the superpotential $W(\Phi)$. Next, let us consider the case $\xi < 0$. For this case, the moduli space is given by

$$\phi^i = 0, \ |p|^2 = \xi/N, \ \frac{\partial G}{\partial \phi^i} = 0, \ \sigma = 0 \ . \tag{4.44}$$

The low-energy theory given by the above equation is nothing but the Landau-Ginzburg (LG) theory with potential

$$W_{eff} = \sqrt{\frac{\xi}{N}} G(\phi) \ . \tag{4.45}$$

Note that we fix the VEV of $p$ to be real using the $U(1)$ gauge symmetry. Since the charge of $P$ is $-N$, after this gauge fixing, there still remains $Z_N$ subgroup which we should mod out at the end. Finally, we get the LG $Z_N$ orbifold theory.

The singularity which divides two phases of low-energy theory lies at $\xi = 0$. In order to determine the singularity structure, we should note an important subtlety in the presence of the topological $\theta$ term. For 1+1 dimension, when $\theta \neq 0$, it is very well-known [83] that the topological $\theta$ term induces additional electric field and the vacuum energy is shifted. For example, consider a pure gauge theory with the action

$$\int d^2x \ \frac{1}{2e^2} F_{01}^2 + \frac{\theta}{2\pi} F_{01} \ . \tag{4.46}$$

We can integrate out $F_{01}$, which yields the vacuum energy

$$E_{vac} = \frac{e^2 \tilde{\theta}^2}{8\pi^2} \ , \tag{4.47}$$

where $\tilde{\theta}$ satisfies $|\tilde{\theta}| \leq \pi$ and $\tilde{\theta} = \theta \bmod 2\pi\mathbb{Z}$. This implies that the real singularity where the phase transition can occur exists only at the point $\xi = \theta = 0$. This is a real codimension two singularity which always can be by-passed. The argument allows us to conclude that the CY and LG phases can be thought of as a pair of equivalent theory. This is a crucial difference from the one-dimensional theory we have seen in the previous chapter.

### 4.1.3 Twisting and Topological Field Theories

On a curved worldsheet, it is not possible to preserve all the supersymmetries in general. In order to deal with the super-string worldsheet with arbitrary curvature, we introduce a notion of twisting. This is done by mixing two-dimensional Lorentz symmetry which is $SO(2) = U(1)_E$ with one of the global symmetry of the theory, so that the Killing spinor equation becomes

$$(\partial_\mu + w_\mu + A_\mu^{\text{global}})\epsilon = 0 \ . \tag{4.48}$$

If we tune $A_\mu^{\text{global}} = -w_\mu$, the equation admits a constant Killing spinor as a solution, and the corresponding supersymmetry is preserved. Since $\mathcal{N} = (2,2)$ theory has two global symmetries, we have two possibilities as follow.

$$\begin{aligned} U(1)'_E &= U(1)_E + U(1)_V \quad : \text{A-twist} \\ U(1)'_E &= U(1)_E + U(1)_A \quad : \text{B-twist} \end{aligned} \tag{4.49}$$

In order to see which supersymmetries are left preserved under these two choices, let us look at the following table of charges of supersymmetries under the global and original Lorentz symmetries.

| | $Q_+$ | $Q_-$ | $\bar{Q}_+$ | $\bar{Q}_-$ |
|---|---|---|---|---|
| $U(1)_V$ | $-1$ | $-1$ | $1$ | $1$ |
| $U(1)_A$ | $-1$ | $1$ | $1$ | $-1$ |
| $U(1)_E$ | $-1$ | $1$ | $-1$ | $1$ |

(4.50)

This ensures that for the A-twisted theory, $Q_-$ and $\bar{Q}_+$ becomes a scalar, hence a preserved supersymmetries. On the other hand, for the B-twisted theory, $\bar{Q}_+$ and $\bar{Q}_-$ are preserved supersymmetries. We are going to use the particular combination of these suparcharges by defining $Q_A = \bar{Q}_+ + Q_-$ and $Q_B = \bar{Q}_+ + \bar{Q}_-$. Then we have

$$Q_{A,B}^2 = 0 \ , \tag{4.51}$$

meaning that $Q_{A,B}$ respectively defines cohomology which are one to one correspondence to the supersymmetric ground state of the theory. In order to explicitly construct the states, we define chiral operator $\phi$ which satisfies

$$[Q_B, \phi] = 0 \ , \tag{4.52}$$

and twisted chiral operator $y$ by

$$[Q_A, y] = 0 \ . \tag{4.53}$$

We can see that the lowest component of chiral and twisted chiral multiplet serves as chiral and twisted chiral operators respectively, if we look at their supersymmetry transformation rules. Furthermore, when $\phi_1$ and $\phi_2$ are chiral operator, it is obvious that $\phi_1\phi_2$ also is a chiral operator. It means $\{\phi^i\}$ form a chiral ring, with a relation

$$\phi_i\phi_j = C_{ij}^k\phi_k + (Q\text{-exact}) \ , \tag{4.54}$$

where $C_{ij}^k$ is a structure constant of the ring. One of the most important aspects of this chiral ring theory with A- or B-twist is that the twisted energy momentum tensor

$$\tilde{T}_{\mu\nu} = T_{\mu\nu} + \frac{1}{4}(\epsilon_\mu^\nu\partial_l J_\nu^R + \epsilon_\nu^\lambda\partial_\lambda J_\mu^R) \tag{4.55}$$

can be shown to be $Q_A$ or $Q_B$-exact. This implies that the metric variation of an arbitrary correlation function of chiral operator becomes

$$\delta_g\langle\phi_1\cdots\phi_k\rangle = \langle\int_\sigma d^2x\{Q_B, A_{\mu\nu}\}g^{\mu\nu}\phi_1\cdots\phi_k\rangle \tag{4.56}$$

which vanishes in $Q_B$ cohomology. The same happens for the $Q_A$ cohomology with twisted chiral operators. This analysis implies that the correlation functions are independent of the metric, which means the theory is topological.

Then, on which parameters do the twisted theories depend? let us consider the A-twisted theory with a twisted chiral multiplet $Y$ and chiral multiplet $\Phi$. First of all, the D-term for $Y$ and $\Phi$ can be shown to be $Q_A$ exact:

$$\begin{aligned}\int d^4\theta\ K(Y,\bar{Y}) &= \left\{\bar{Q}_+, \left[Q_-, \int d\bar{\theta}^- d\theta^+ K(Y,\bar{Y})|_{\theta^-=\bar{\theta}^+=0}\right]\right\} \\ &= \left\{\bar{Q}_+ + Q_-, \left[Q_-, \int d\bar{\theta}^- d\theta^+ K(Y,\bar{Y})|_{\theta^-=\bar{\theta}^+=0}\right]\right\} \end{aligned} \tag{4.57}$$

Secondly, the chiral and anti-chiral superpotentials are also $Q_A$ exact:

$$\begin{aligned}\int d^2x \int d\theta^+ d\theta^-\ W(\Phi) &= \{Q_+ [Q_-, W(\phi)]\} \\ &= \int d^2x \left\{Q_+ \left[Q_- + \bar{Q}_+, W(\phi)\right]\right\} \\ &= -\int d^2x \left\{Q_- + \bar{Q}_+, [Q_+, W(\phi)]\right\}\ , \end{aligned} \tag{4.58}$$

where for the second line, we used the definition of the chiral field, and for the third line, we applied the Jacobi identity and ignored a total derivative term. In a similar manner, one can show that $\int d\bar{\theta}^+ d\bar{\theta}^- \bar{W}(\bar{\Phi})$ can be also written as a $Q_A$ exact term. Finally, the anti-twisted superpotential is also $Q_A$ exact:

$$\begin{aligned}\int d\bar{\theta}^+ d\theta^- W(Y) &= \{\bar{Q}_+, [Q_-, W(y)]\} \\ &= \{\bar{Q}_+ + Q_-, [Q_-, W(y)]\}\ . \end{aligned} \tag{4.59}$$

Hence, the A-twisted theory only holomorphically depends on the twisted chiral parameters. On the other hand, the B-twisted theory only depends holomorphically on the chiral superpotential parameters. Recall the example of GLSM which reduces to the NLSM with compact CY target space. The A-twist of this theory only depends on the twisted chiral parameters of the theory, which is nothing but the FI parameter. On the other hand, the B-twist of this theory only depends on the form of the superpotential. As can be seen from this example, The A-twisted theory carries information about the volume of given CY space, while the B-twisted theory carries infomation about the complex structure of the CY space.

The statement of the mirror symmetry, (4.29), relates A-twisted theory of given CY and and B-twisted theory of mirror CY, obtained via the T-duality. One of the reason this duality is useful is that, while the B-twisted theories are classical (due to the F-term non-renormalization theorem), the A-twisted theories get quantum corrections in general, including non-perturbative ones. However, since the mirror symmetry is proven only for the abelian gauge theories [2], there has been no general prescription to obtain exact correlation functions for A-twisted theory until very recently.

On the other hand, after the pioneering work of Pestun [7], there has been much progress on calculating exact partition functions on supersymmetric gauge theories on spheres in various dimensions. [7–9, 30, 32] Along the line of these works, exact partition function of $\mathcal{N} = (2,2)$ GLSM on $S^2$ was recently calculated by [8] and [9]. Suprisingly, it was claimed [12, 13] that the partition function on $S^2$ exactly calculates the Kahler potential of the A-model moduli space, which provides a direct method of computing wordsheet instanton contributions to various correlation functions.

These works has been further exptended to the partition function on a hemisphere [14, 29, 127], and on a real projective plane [15], which was argued to compute the central charge of D-brane and Orientifolds respectively, in the A-twisted theory. In the following sections, we are going to review the series of these works in details, focusing on the Orientifold case.

Before going into the detailed calculations, let us briefly summarize the vacuum structure of the topologically twisted $\mathcal{N} = (2,2)$ theories developed in [126], and explain the properties of quantities we are going to study in the following sections.

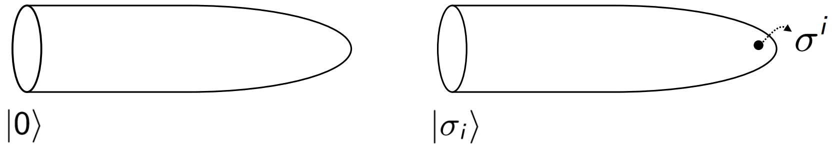


FIGURE 4.2: Ground states $\sigma^i|0\rangle \equiv |\sigma_i\rangle$ of A-twisted theory can be realized as infinite hemisphere with a twisted chiral field insertion at the tip of the hemisphere. We associate the field configuration at the boundary of the hemisphere with a ground state $|\sigma_i\rangle$. This is illustrated in the right figure. Among the ground states constructed as such procedure, we can define a distinguished (canonical) ground state $|0\rangle$ with identity operator insertion (no insertion) as depicted in the left.

First, let us consider an A or B twisted theory with supercharge $Q = Q_A$ or $Q_B$. Then, the ground states $|0\rangle$ of the theory are defined as

$$Q|0\rangle = Q^\dagger|0\rangle = 0 \ . \tag{4.60}$$

Due to the relation (4.52) and (4.53), for a given ground state $|0\rangle$, we can construct the other ground states by acting

$$\phi^i|0\rangle \ , \tag{4.61}$$

where $\phi^i$ are twisted chiral and chiral operator for A- and B-twisted theory respectively. Due to the property of chiral operators, this set of the ground states $\phi^i|0\rangle \equiv |i\rangle$ also forms a ring. In order to realize these set of ground states, we consider the infinite hemisphere with (twisted-) chial field inserted at the tip of it, as depicted in the above figures. Since the wavefunction propagates along the infinite time direction along the neck of the hemisphere, which is equivalent to acting the operator $\lim_{\beta\to\infty} e^{-\beta H}$, the states at the end circle of the hemisphere are projected to the ground state. We associate the field configuration at the equator of the hemisphere with a state $|i\rangle$. Then, there exists a canonical choice of the distinguished ground state $|0\rangle$ which corresponds to the identity operator insertion. All the other ground states are then related to this state by the relation $\phi^i|0\rangle \equiv |i\rangle$. Note that, thanks to the topological nature of the theory (4.56), this states are invariant under the metric variation of the hemisphere or the position of the operator insertion.

Due to the $\mathcal{N} = (2,2)$ supersymmetry, the moduli space of these theory has a complex structure. One of the most important quantity in the study of this moduli space is a Hermitian metric of the moduli space which is given by

$$g_{\bar{i}j} = \langle \bar{i}|j\rangle \ , \tag{4.62}$$

where $\langle \bar{i}|$ denotes the states obtained from the anti-topological ($\bar{A}$- or $\bar{B}$-) twist. Furthermore, one can show that the connection on these vacuum bundle over the parameter space is holomorphic, i.e.,

$$A_{\bar{i}}{}^{k}{}_{j} \equiv \langle k|\bar{\partial}_i|j\rangle = 0 \ , \tag{4.63}$$

and satisfies the so called $tt^*$ equations [126], which guarantee the existence of the Gauss-Manin flat connection. Using these relations, one can show that the Zamolodchikov metric on the moduli space can be obtained from the relation

$$G_{i\bar{j}} \equiv -\frac{g_{i\bar{j}}}{g_{0\bar{0}}} = \bar{\partial}_{\bar{j}}\partial_i \ln g_{0\bar{0}} \ , \tag{4.64}$$

which means that $\ln\langle 0|\bar{0}\rangle$ equals to the Kahler potential of the Zamolodchikov metric for the family of the Calabi-Yau space. As can be inferred from the discussion above, $\langle 0|\bar{0}\rangle$ can be obtained from the partition function on the sphere with topological twisting in the northern hemisphere and anti-topological twisting in the southern hemisphere. For B-twisted theory, this quantity can be easily calculated since there are no quantum correction to the F-term potential. This is given by

$$\langle 0|\bar{0}\rangle = \int_X \Omega \wedge \bar{\Omega} = e^{-K} \ , \tag{4.65}$$

where $K$ is a Kahler potential for the complex structure moduli space, and $\Omega$ is a holomorphic volume form of the Calabi-Yau space. On the other hand, for

A-twsted theory, the corresponding quantity is not easy to calculate because of the non-trivial worldsheet instanton correction. The following section presents the recently developed ways to calculate this quantity exactly for A-twisted theories, via the method of the supersymmetric localization reviewed in the section 2.4.

Since the worldsheet topology is a sphere in the above case, the string theory in question is naturally the closed string theory. Next obvious step is to consider the case when the worldsheet has a boundary, which is the open string theory. Since these strings can end on the subspace of the Calabi-Yau ambient, which has been dubbed D-brane, this theory naturally captures information of such D-brane which wraps the sub-cycles of the Calabi-Yau ambient space. For this case, we should impose certain boundary condition at the boundary of the hemisphere which is properly twisted. Note that this boundary condition should compatible with the supersymmetry of the worldsheet. Then we can consider the following overlap between A- (or B-) twisted canonical ground state and the boundary state $|B(\gamma_i)\rangle$ defined by the sub-cycle $\gamma_i$ that the string ends,

$$\Pi_i = \langle 0|B(\gamma_i)\rangle \ . \tag{4.66}$$

This quantity is called the period integral. For a B-twisted theory, it is known that [141] the boundary cycle $\gamma_i$ should be a middle dimensional cycle called the *Lagrangian subcycle*. Hence the period integral can be naturally mapped to the integral

$$\Pi_i^B = \int_{\gamma_i} \Omega \ , \tag{4.67}$$

which again is not quantum corrected. This quantity is nothing but the central charge of the $\mathcal{N} = 2$ string theory, which appears on the right hand side of the commutation relation of the two supersymmetries. The A-twisted counterpart is

$$\Pi_i^A = \int_{\tilde{\gamma}_i} e^{-B-iJ} + \mathcal{O}(\alpha') \ , \tag{4.68}$$

which depends holomorphically on the complexified Kahler parameter $B+iJ$. This quantity, on the other hand, can not be easily calculated in a direct manner because of the worldsheet instanton correction. In section 4.3, we are going to review what have been known about this quantity before recent developement of the exact calculations, by means of the anomaly inflow. In section 4.4, we introduce a new method to directly calculate the central charge of given D-brane data including all the $\alpha'$ corrections, and will see how these works corrected and improved the previously known results reviewed in section 4.3.

Finally, there are one more class of defect localized at a subcycle of the Calabi-Yau ambient space, which is the Orientifold planes. For this case, the relevant quantity is

$$\widetilde{\Pi}_i = \langle 0 | C(\gamma_i) \rangle \ , \tag{4.69}$$

where $|C(\gamma_i)\rangle$ is a crosscap state which is a fixed point of a certain $Z_2$ projection of fields accompanied by exchange of both ends of the string. This quantity is known to calculate the central charge of the type IIA string theory in the presence of such $Z_2$ action. As in the D-brane case, for the A-twisted theory, this quantity has been calculated only for the tree-level until very recently. In the last section of this chapter, we are going to present the work [15] where the exact calculation of the central charges of the Orientifold planes are given. In this work, several subtleties regarding the RR-charges of D-brane/Orientifolds and issue with *Spin*$^c$ manifolds are also addressed and solved partially.

Studying the vacuum structure of these topological theories has an important applications in Calabi-Yau compactfication of the string theory. For example, when we consider Type II stirng theory compactified on six-dimensional Calabi-Yau, the vacuum-to-vacuum amplitude $\langle \bar{0} | 0 \rangle$ we mentioned above, exactly determines the gauge coupling of the four-dimensional $\mathcal{N} = 2$ effective theory

$$\partial_i \partial_j F_0 = \tau_{ij} \ , \tag{4.70}$$

hence the Seiberg-Witten prepotential $F_0$. The D-branes/Orientifolds to the vacuum amplitudes $\langle 0|B(\gamma_i)\rangle$ and $\langle 0|C(\gamma_i)\rangle$ also play crucial role in the Calabi-Yau compactification, which calculates the charge and mass of the BPS states in four-dimensional spacetime obtained from compactifying D-branes wraping supersymmetric cycles.

## 4.2 Kahler Potential and the Two-Sphere Partition Function

Recently, there has been drastic improvement in understanding two-dimensional $\mathcal{N} = (2,2)$ theories along with the development of the localization technique which enables us to exactly calculate the quantities for supersymmetric theories. As advertised in the last section, we are going to review such development for the 2d GLSM, and study how these series of works can be used to exactly calculate the fully quantum corrected quantities in the A-twisted theory. In this section, we start with a short review of [8, 9], where the partition function of 2d $\mathcal{N} = (2,2)$ GLSM on a two-sphere was calculated.

First of all, we should write down the action which preserves $\mathcal{N} = (2,2)$ supersymmetry on the sphere. As was reviewed in section 2.4, this can be done by adding proper curvature correction terms to the flat Lagrangian. For 2d GLSM, it turns out to be

$$\mathcal{L} = \mathcal{L}_{vector} + \mathcal{L}_{chiral} + \mathcal{L}_W + \mathcal{L}_{FI} \ , \tag{4.71}$$

where the kinetic terms for the vector and the charged chiral multiplets are, respectively,

$$\begin{aligned}\mathcal{L}_{vector} = \frac{1}{2g^2}\mathrm{Tr}\Bigg[&\left(F_{12} + \frac{\sigma_1}{r}\right)^2 + (D_\mu\sigma_1)^2 + (D_\mu\sigma_2)^2 - [\sigma_1,\sigma_2]^2 + D^2 \\ &+ i\bar{\lambda}\gamma^\mu D_\mu\lambda + i\bar{\lambda}[\sigma_1,\lambda] + i\bar{\lambda}\gamma^3[\sigma_2,\lambda]\Bigg] ,\end{aligned} \tag{4.72}$$

$$\begin{aligned}\mathcal{L}_{chiral} = \bar{\phi}&\left(-D^\mu D_\mu + \sigma_1^2 + \sigma_2^2 + iD + i\frac{q-1}{r}\sigma_2 + \frac{q(2-q)}{4r^2}\right)\phi + \bar{F}F \\ &- i\bar{\psi}\left(\gamma^\mu D_\mu - \sigma_1 - i\gamma^3\sigma_2 + \frac{q}{2r}\gamma^3\right)\psi + i\bar{\psi}\lambda\phi - i\bar{\phi}\bar{\lambda}\psi ,\end{aligned} \tag{4.73}$$

and the potential terms take the following form,

$$\mathcal{L}_{\mathcal{W}} = \sum_i \frac{\partial\mathcal{W}}{\partial\phi^i}F^i - \frac{1}{2}\sum_{i,j}\frac{\partial^2\mathcal{W}}{\partial\phi^i\partial\phi^j}\psi^i\psi^j + \text{c.c.} . \tag{4.74}$$

Finally the Fayet-Illiopoulos (FI) coupling and the two-dimensional topological term are

$$\mathcal{L}_{FI} = -\frac{\tau}{2}\mathrm{Tr}\Big[D - \frac{\sigma_2}{r} + iF_{12}\Big] + \frac{\bar{\tau}}{2}\mathrm{Tr}\Big[D - \frac{\sigma_2}{r} - iF_{12}\Big] , \tag{4.75}$$

where $\tau = i\xi + \frac{\theta}{2\pi}$, ($\xi \in \mathbb{R}$, $\theta \in [0, 2\pi]$). Note that the superpotential $\mathcal{W}(\phi)$ should carry $R$-charge two to preserve the supersymmetry. This Lagrangian is invariant under the following supersymmetry transformation rules,

$$\begin{aligned}
\delta\lambda =&(iV_1\gamma^1 + iV_2\gamma^2 + iV_3\gamma^3 - \mathrm{D})\epsilon \ , \\
\delta\bar{\lambda} =&(i\bar{V}_1\gamma^1 + i\bar{V}_2\gamma^2 + i\bar{V}_3\gamma^3 + \mathrm{D})\bar{\epsilon} \ , \\
\delta A_i =& -\frac{i}{2}\Big(\bar{\epsilon}\gamma_i\lambda - \bar{\lambda}\gamma_i\epsilon\Big) \ , \\
\delta\sigma_1 =&\frac{1}{2}\Big(\bar{\epsilon}\lambda - \bar{\lambda}\epsilon\Big) \ , \\
\delta\sigma_2 =& -\frac{i}{2}\Big(\bar{\epsilon}\gamma_3\lambda - \bar{\lambda}\gamma_3\epsilon\Big) \ , \\
\delta D =& -\frac{i}{2}\bar{\epsilon}\gamma^\mu D_\mu\lambda - \frac{i}{2}\big[\sigma_1, \bar{\epsilon}\lambda\big] - \frac{1}{2}\big[\sigma_2, \bar{\epsilon}\gamma^3\lambda\big] \ , \\
&+ \frac{i}{2}\epsilon\gamma^\mu D_\mu\bar{\lambda} - \frac{i}{2}\big[\sigma_1, \bar{\lambda}\epsilon\big] - \frac{1}{2}\big[\sigma_2, \bar{\lambda}\gamma^3\epsilon\big] \ ,
\end{aligned} \tag{4.76}$$

with

$$\begin{aligned}
\vec{V} \equiv&\Big( + D_1\sigma_1 + D_2\sigma_2, \ +D_2\sigma_1 - D_1\sigma_2, \ F_{12} + i[\sigma_1, \sigma_2] + \frac{1}{r}\sigma_1\Big) \ , \\
\vec{\bar{V}} \equiv&\Big( - D_1\sigma_1 + D_2\sigma_2, \ -D_2\sigma_1 - D_1\sigma_2, \ F_{12} - i[\sigma_1, \sigma_2] + \frac{1}{r}\sigma_1\Big) \ ,
\end{aligned} \tag{4.77}$$

and

$$\begin{aligned}
\delta\phi =&\bar{\epsilon}\psi \ \ , \\
\delta\bar{\phi} =&\epsilon\bar{\psi} \ , \\
\delta\psi =&i\gamma^\mu\epsilon D_\mu\phi + i\epsilon\sigma_1\phi + \gamma^3\epsilon\sigma_2\phi + i\frac{q}{2r}\gamma_3\epsilon\phi + \bar{\epsilon}F \ \ , \\
\delta\bar{\psi} =&i\gamma^\mu\bar{\epsilon}D_\mu\bar{\phi} + i\bar{\epsilon}\bar{\phi}\sigma_1 - \gamma^3\bar{\epsilon}\bar{\phi}\sigma_2 - i\frac{q}{2r}\gamma_3\bar{\epsilon}\bar{\phi} + \epsilon\bar{F} \ \ , \\
\delta F =&\epsilon\Big(i\gamma^i D_i\psi - i\sigma_1\psi + \gamma^3\sigma_2\psi - i\lambda\phi\Big) - i\frac{q}{2}\psi\gamma^i D_i\epsilon \ \ , \\
\delta\bar{F} =&\bar{\epsilon}\Big(i\gamma^i D_i\bar{\psi} - i\bar{\psi}\sigma_1 - \gamma^3\bar{\psi}\sigma_2 + i\bar{\phi}\lambda\Big) - i\frac{q}{2}\bar{\psi}\gamma^i D_i\bar{\epsilon} \ .
\end{aligned} \tag{4.78}$$

Here the spinors $\epsilon$ and $\bar{\epsilon}$ can be chosen to be

$$\epsilon = e^{i\varphi/2} \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix} , \qquad \bar{\epsilon} = e^{-i\varphi/2} \begin{pmatrix} \sin\theta/2 \\ \cos\theta/2 \end{pmatrix} , \tag{4.79}$$

which are solutions to the Killing spinor equations

$$\nabla_\mu \epsilon = \frac{1}{2r}\gamma_\mu \gamma^3 \epsilon \ , \qquad \nabla_\mu \bar{\epsilon} = -\frac{1}{2r}\gamma_\mu \gamma^3 \bar{\epsilon} \ . \tag{4.80}$$

There are two different localization method which can be applied in general. The first one is the Coulomb branch localization where we use the fact that the above Lagrangian $\mathcal{L}_{vector}$, $\mathcal{L}_{chiral}$ and $\mathcal{L}_W$ can be expressed as a Q-exact terms. For example,

$$\mathcal{L}_{vector} = \frac{1}{g^2}\delta_\epsilon \delta_{\bar{\epsilon}} \mathrm{Tr}\Big[\frac{1}{2}\bar{\lambda}\gamma^3\lambda - 2iD\sigma_2 + \frac{i}{r}\sigma_2^2\Big] \ , \tag{4.81}$$

and

$$\mathcal{L}_{chiral} = -\delta_\epsilon \delta_{\bar{\epsilon}}\Big[\bar{\psi}\gamma^3\psi - 2\bar{\phi}(\sigma_2 + i\frac{q}{2r})\phi + \frac{i}{r}\bar{\phi}\phi\Big] \ . \tag{4.82}$$

Since these terms are Q-exact themselves, the one-loop determinant calculated from these quadratic action yields exact partition function. The saddle configuration is

$$\begin{aligned} &A = \frac{B}{2}(\kappa - \cos\theta)d\phi \ , \quad \sigma_1 = -\frac{B}{2r} \ , \quad \sigma_2 = \sigma \ , \quad D = 0 \ , \\ &\phi = 0 \ , \quad F = 0 \ , \end{aligned} \tag{4.83}$$

where $\sigma$ is arbitrary constant and $B$ is quantized magnetic flux. Since the fields are localized to the constant value of the lowest component of the vector multiplet, this solution is referred to as a Coulomb branch localization. The result can be written as

$$
\begin{aligned}
Z_{S^2} &= \sum_B \frac{1}{|W|} \int d\sigma e^{-4\pi i\xi \mathrm{Tr}\sigma + i\theta \mathrm{Tr} B} \\
&\times \prod_{\alpha\in\Delta^+} \left[ \left( \frac{\alpha\cdot B}{2r} \right)^2 + (\alpha\cdot\sigma)^2 \right] \prod_{w\in\mathbf{R}} \frac{\Gamma\left(\frac{q}{2} - irw\cdot\sigma + \frac{|w\cdot B|}{2}\right)}{\Gamma\left(1-\frac{q}{2} + irw\cdot\sigma + \frac{|w\cdot B|}{2}\right)}
\end{aligned}
\tag{4.84}
$$

In the integrand, the first line correponds to the classical contribution of the action. At the second line, the first factor comes from the gauge multiplet with roots $\alpha$, and the rest of them which is expressed in terms of the ratio of the gamma function are chiral multiplet contribution, charged with weight $w$.

The other choice of the Q-exact term is also possible and presented in [8, 9]. This is the Higgs branch localization. For this choice, one can clearly see that the solutions are localized at the north and south pole of the sphere. Furthermore, these solutions are the vortex and anti-vortex configuration on $\mathbb{R}^2$ with $\Omega$ backgroud parameter $\epsilon = 1/r$. Since the choice of Q-exact term should not affect the results, the result must agree with that of the Coulomb branch calculation, and it can be shown to be true.

As can be clearly seen in (4.84), the result only depends on the parameter of A-model moduli space $\tau = i\xi + \frac{\theta}{2\pi}$ and $\bar{\tau}$. Then what does this quantity calculate? A few month later, there appeared a conjecture [12] that (4.84) exactly gives the Kahler potential of the A-model moduli space. I.e.,

$$
Z_{S^2}(\tau,\bar{\tau}) = e^{-K_{Kahler}(\tau,\bar{\tau})} = \langle\bar{0}|0\rangle \ . \tag{4.85}
$$

This conjecture has been checked against various known examples of the mirror symmetry. Especially, it was shown that (4.84) captures the well-known term in the perturbative correction of $e^{-K}$ which is proportional to $\zeta(3)$.

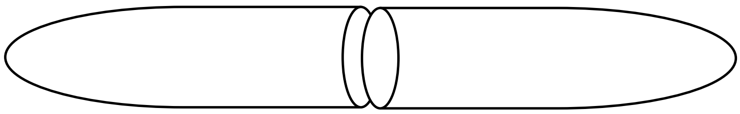


Figure 4.3: Geometry of the squashed two-sphere and the Kahler potential. At the tip of the hemisphere, the preserved supersymmetry is $A$- and $\bar{A}$-type.

Later, the physical argument on why this conjecture should hold was offered by [13]. They observed two crucial facts. 1) The $Z_{S^2}(\tau, \bar{\tau})$ is invariant under the squashing of the sphere which preserves $U(1)$ isometry. 2) At the two poles of the sphere, the preserved supersymmetry (4.79) is exactly $A$- and $\bar{A}$-type respectively. Combining these two facts, the geometry of the partition function calculation are very similar to the $tt^*$ picture. The state at the tip of the hemisphere, which corresponds to the the identity operator of the $A$-twisted theory, propagates through the infinite time direction and projected to the canonical ground state $|0\rangle$. On the other half of the hemisphere, anti-topological counterpart happens and the partition function over the whole squashed sphere yields a overlap amplitute between these two states. Although this is not exactly the same as the original $tt^*$ situation since the supersymmetry is continuously interpolating between $A$- and $\bar{A}$-type in this case, it gives very plausible argument why the conjecture (4.85) holds in general. More recently, there appeared another proof of this conjecture without using the localization argument. [128] In this work, it was also claimed that the similar relation between the sphere partition function and the Kahler potential of the conformal manifolds holds for four-dimensional superconformal field theories as well.

Compared to the fact that the original mirror symmetry was proven only for the GLSM with $U(1)$ gauge symmetry, this results can be easily generalized to the non-abelian gauge groups. We can say that these results offer a new way to suggest and prove the mirror symmetry for general Calabi-Yau manifold obtained from non-abelian GLSMs.

## 4.3 Ramond-Ramond Charges from the Anomaly Inflow

Having understood that the two-sphere partition function of $\mathcal{N} = (2, 2)$ GLSM calculates the Kahler potential of the A-model moduli space, the next natural step is to consider the worldsheet with boundary, which would be applicable for the open string theory. For this case, the worldsheet topology is a disk (a hemisphere). As can be inferred from the discussion of section 4.1.3, we can expect that the exact partition function of the same GLSM on a hemisphere would calculate the overlap amplitude

$$\langle 0|B(\gamma_i)\rangle \ , \tag{4.86}$$

where the boundary state $|B(\gamma_i)\rangle$ is determined by the cycle $\gamma_i$ which the D-brane attached at the end wraps. This boundary data will be translated to GLSM language properly. For the exact calculation of this quantity, we should check whether the saddle configuration and fluctuations satisfies the boundary conditions. More importantly, if the Killing spinors (4.79) satisfy the boundary condition, we expect that, for the same reason in the two-sphere case, (4.86) obtained via the localization would give the overlap amplitute between the A-twisted canonical ground state and the boundary state which corresponds to the even-dimensional holomorphic subcycle embedded in the ambient space. As studied before, this quantity maps to the period integral at the level of NLSM, and the period integral gives the central charge of the $\mathcal{N} = 2$ supersymmetry algebra. On the other hand, central charge are closely related to the BPS states of the supersymmetric theory, which are realized as D-branes coupled to the Ramond-Ramond (RR) fields. As a consequence, if we denote this coupling $Y(\mathcal{F}, \mathcal{R})$, where $\mathcal{F}, \mathcal{R}$ are gauge and gravitational field strength respectively, the general form of the central charge can be written as

$$(\text{D-brane central charge}) = \int_X e^{-B-iJ} \wedge Y(\mathcal{F}, \mathcal{R}) + \mathcal{O}(\alpha') \ . \tag{4.87}$$

This formula is analogous to the $Z(p,q) = pa + qa_D$ in the $4d$ $\mathcal{N} = 2$ Seiberg-Witten theory obtained from a Calabi-Yau compactification. Here $Y(\mathcal{F}, \mathcal{R})$ is the topological RR coupling to the spacetime curvature, which has been renowned to be exactly determined by the anomaly cancelation mechanism of the D-brane worldvolume theories. [97, 98, 140] In the presence of the Orientifold planes, the analogous quantity is

$$\langle 0|C(\gamma_i)\rangle \ , \tag{4.88}$$

where $|C\rangle$ denotes for the crosscap states whose relavant Orientifold planes are even-dimensional holomorphic manifold. The precise definition will be given in the section 4.5.2. The worldsheet topology now becomes $\mathbb{RP}^2$, which is non-orientable. RR coupling to the spacetime curvature also has been calculated in a similar manner, which yields

$$(\text{O-plane central charge}) = \pm \int_X e^{-iJ} \wedge Z(\mathcal{R}) + \mathcal{O}(\alpha') \ . \tag{4.89}$$

Note that, for this case, the NSNS two-form field $e^{-B}$ takes discrete value denoted as factor $\pm$, and the polynomial $Z$ does not depend on the gauge fields since the strings cannot ends on the Orientifold plane. Before going into detailed discussion of the exact form of the equation (4.87), in this section, we review the well-established mechanism of the anomaly inflow which determines polynomial $Y(\mathcal{F}, \mathcal{R})$ and $Z(\mathcal{R})$, hence the tree-level central charge. We follow the recent work [144], which reviewed and improved the original study done by [97, 98, 140].

### 4.3.1 Anomaly Inflow for Intersecting Branes

The axial and gravitational anomaly are quite prevalent and in fact most supersymmetric Yang-Mills theories with $d \geq 4$ have such anomalies. Many of these theories are realizable as world-volume theories from D-branes and Orientifold planes. From the pioneering work of Alvarez-Gaume and Witten which we reviewed in section

2.3.1, one can straightforwardly calculate the one-loop anomalies of such theories. For a $d$-dimensional worldvolume theory of intersecting branes, anomaly polynomial can be collectively written as

$$I^{1-loop}_{d=p+1} = (-1)^{(p+1)/2}\pi \cdot \left([ch(\mathcal{F}) \wedge \mathcal{A}(\mathcal{T}) \wedge [ch_{S^+}(\mathcal{N}) - ch_{S^-}(\mathcal{N})]\right)\Big|_{(d+2)-form}, \tag{4.90}$$

where $\mathcal{T}$ and $\mathcal{N}$ are tangent and normal bundle of the worldvolume respectively. These 1-loop anomaly can be canceled via so called *anomaly inflow* mechanism, which was developed by [97, 98, 140]. The main claim is that the topological couplings between Ramond-Ramond (RR) tensor fields and the spacetime curvature should be properly chosen so that they exactly cancel (4.90). This mechanism has been widely discussed in the context M5-brane worldvolume theory as well [96], which is well known to produce the $N^3$ behaviour of the six dimensional $\mathcal{N} = (2,0)$.

We will shortly review how this anomaly inflow mechanism can determine the RR couplings of the D-branes and Orientifolds. The original anomaly inflow mechanism of [97, 98, 140] contains essentially all the necessary ingredients. However, there has been several unsatisfiactory issues in these arguments until very recently. One is that, for self-dual brane configurations such as D3 brane or D1-D5 intersecting brane system, they fail to generate any anomaly inflow although they apparently suffer from the 1-loop anomalies. In the following, we present the work [144] which settled this problem by modifying the inflow mechanism properly, and showed that the RR couplings are allowed to be written in a more natural form. Note that, for simplicity, we restrict to the case where D-brane wrap a Spin manifold.

Consider the Chern-Simons coupling in a form[1]

$$S_{CS} = \frac{\mu_p}{2} \int_{Dp} \sum_{r \le p} s^*(C_{r+1}) \wedge Y_{p-r} \ , \tag{4.92}$$

where $Y = ch(\mathcal{F})\mathcal{A}(\mathcal{T})^{1/2}\mathcal{A}(\mathcal{N})^{-1/2}$ and $s^*$ is the pull-back to the world-volume.

The equation of motion that follows from this coupling is

$$d\left(*(H_{r+2})\right) = -(-1)^r \sum_B 2\kappa_{10}^2 \mu_q \, Y_{q-r}^B \wedge \Delta_{9-q}^B \ , \tag{4.93}$$

with some "delta function" $(9-q)$-form, $\Delta^B_{9-q}$, representing the D-brane position. Because this is not a scalar object, however, the expression becomes ill-defined unless we carefully regularize and covariantize it. This smearing of the magnetic source is a recurring and necessary step when we discuss the anomaly inflow, especially when the anomaly associated with normal bundle needs to be discussed. Thus, we write instead,

$$d\left(*(H_{r+2})\right) = -(-1)^r \sum_B 2\kappa_{10}^2 \mu_q \, Y_{q-r}^B \wedge \tau_{9-q}^B \ , \tag{4.94}$$

where we smeared the sources due to the D$q$-branes by introducing a "delta-function" $(9-q)$-form $\tau^B_{9-q}$, well-identified in the mathematical literatures as the Thom class of the normal bundle $\mathcal{N}$ [110].

$$\tau_{9-q} = d(\rho \, \hat{e}_{8-q}) \ . \tag{4.95}$$

[1] Note that this not equivalent to the originally conjectured formula

$$S'_{CS} = \frac{\mu_p}{2} \int_{Dp} \left( s^*(C_{p+1}) \wedge Y_0 + (-1)^\epsilon \sum_{r<p} s^*(H_{r+2}) \wedge Y^{(0)}_{p-r-1} \right) \ , \tag{4.91}$$

of [97, 98].

The "radial" function $\rho$, whose support determines the smearing of the source, interpolates between $-1$ on the brane and 0 at infinity. The global angular form $\hat{e}_{8-q}$ is essentially a covariantized volume-form, normalized to unit volume, of a $(8-q)$-sphere surrounding the D$p$-brane. In particular $\delta\hat{e}_{8-q} = 0$, and $d\hat{e}_{8-q} = 0$ for even $q$ and $d\hat{e}_{8-q} = -\chi(\mathcal{N})_{9-q}$ with the Euler class $\chi$ for odd $q$. By choosing $\rho$ to have increasingly small support near the origin, we can localize the source with arbitrary precision, and with diffeomorphism invariance preserved. In addition we will also choose $\rho'(0) = 0$. With arbitrary small support of $\rho$, we can take $Y$'s to be uniform along the normal direction, which allows (4.94) to make sense.

Since this equation of motion exists for all $C_{q+1}$'s, it also implies, with $*H_n = (-1)^{(n-2+\epsilon)/2} H_{10-n}$, the modified Bianchi identities

$$dH_{8-r} = -\sum_B 2\kappa_{10}^2 \mu_q \, (-1)^{(-q+\epsilon)/2} \wedge \bar{Y}^B_{q-r} \wedge \tau^B_{9-q} \ , \tag{4.96}$$

with $\bar{Y}$'s being the complex conjugated $Y$'s,

$$\bar{Y}^A_n = \left[ ch(-\mathcal{F}_A) \wedge \sqrt{\frac{\mathcal{A}(\mathcal{T}_A)}{\mathcal{A}(\mathcal{N}_A)}} \, \right] \Bigg|_n \ . \tag{4.97}$$

Before solving this Bianchi identity, we need to clarify an important difference between the Thom classes of even and odd dimensional bundles. For odd fibre dimensions (applicable to even $q$ and thus type IIA branes), $\tau_{9-q} = d(\rho \cdot \hat{e}_{8-q})$ behaves in much the same way as $\tau_{M5}$ of the previous section. For even fibre dimensions (applicable to odd $q$ and thus type IIB branes), the global angular form decomposes into two pieces [110, 112]

$$\hat{e}_{8-q} = \upsilon_{8-q} + \Omega_{8-q}(\mathcal{N}) \ , \tag{4.98}$$

where the first term involves at least one normal vector field $\hat{y}$ and can be written locally as

$$v_{8-q} = d\psi_{7-q} \ , \tag{4.99}$$

while the last term is nothing but the Chern-Simons term of the Euler class with a sign flip, i.e.,

$$d\Omega_{8-q}(\mathcal{N}) = -\chi(\mathcal{N})_{9-q} \ . \tag{4.100}$$

Clearly, this behavior of the Thom class is responsible, with $\rho(0) = -1$, for the identity $s^*(\tau) = \chi$. Finally the gauge-invariance of $\hat{e}$ implies that

$$\delta\psi_{7-q} = -\Omega^{(1)}_{7-q} = \chi(\mathcal{N})^{(1)}_{7-q} \ . \tag{4.101}$$

$\Omega$ exists for even-dimensional normal bundles, and so this is relevant for all type IIB branes.

Note that $v_{8-q}$ (and its descent $\psi_{7-q}$) is singular at the origin, being a normalized volume form of $S^{8-q}$. In contrast, $\Omega(\mathcal{N})_{8-q}$ is composed only of the gauge fields of the normal bundle and is well-defined and smooth everywhere. For regular solutions of $H$, we must then choose the following descent for $\tau$,

$$\tau^{(0)}_{8-q} = -d\rho \wedge \psi_{7-q} + \rho \cdot \Omega_{8-q} \ , \tag{4.102}$$

which results in

$$\tau^{(1)}_{7-q} = -\rho \cdot \chi(\mathcal{N})^{(1)}_{7-q} \ . \tag{4.103}$$

Note that both expressions are regular at the origin, with $\rho'(0) = 0$. This gives

$$H_{s+2} = d(C_{s+1}) - \sum_B 2\kappa^2_{10}\mu_q \ (-1)^{(-q+\epsilon)/2} \left(\bar{Y}^B \wedge \tau^B\right)^{(0)}_{s+2} \ , \tag{4.104}$$

where, for type IIB theory,

$$\big(\bar{Y}^B \wedge \tau^B\big)^{(0)}_{s+2} = \beta(\bar{Y}^B)^{(0)}_{q+s-7} \wedge \tau^B_{9-q} + (1-\beta)(\bar{Y}^B)_{q+s-6} \wedge (-d\rho \wedge \psi_{7-q} + \rho \cdot \Omega_{8-q})^B \ . \tag{4.105}$$

Although $\beta$ is an arbitrary real number in general, we must take $\beta = 0$ when $\bar{Y}$ on the left hand side is a 0-form (here, $q+s=6$). Its gauge variation gives

$$\big(\bar{Y}^B \wedge \tau^B\big)^{(1)}_{s+1} = \beta(\bar{Y}^B)^{(1)}_{q+s-8} \wedge \tau^B_{9-q} + (1-\beta)\bar{Y}^B_{q+s-6} \wedge (-\rho \cdot \chi^{(1)}_{7-q})^B \ . \tag{4.106}$$

With this understood, the gauge transformation of $C$ is,

$$\delta C_{s+1} = \sum_B 2\kappa^2_{10}\mu_q \, (-1)^{(-q+\epsilon)/2} \big(\bar{Y}^B \wedge \tau^B\big)^{(1)}_{s+1} \ . \tag{4.107}$$

Let us concentrate on the case of a single stack of type IIB D$p$-branes. The gauge variation of $S_{CS}$ (4.92) is

$$\delta S_{CS} = (-1)^{(-p+1)/2}\pi \int_{Dp} \sum_r s^* \Big(\big(\bar{Y}_{p+r-6} \wedge \tau_{9-p}\big)^{(1)}\Big) \wedge Y_{p-r} \ . \tag{4.108}$$

Just as $s^*(\tau) = \chi$, it is easy to show that

$$s^*(\tau^{(1)}) = s^*(-\rho\chi^{(1)}) = \chi^{(1)} \ , \tag{4.109}$$

and that

$$\delta S_{CS} = (-1)^{(-p+1)/2}\pi \int_{Dp} \sum_r \big(\bar{Y}_{p+r-6} \wedge \chi_{9-p}(\mathcal{N})\big)^{(1)} \wedge Y_{p-r} \ , \tag{4.110}$$

which equals

$$-(-1)^{(p+1)/2}\pi \int_{\mathrm{Dp}} \left( ch(-\mathcal{F}) \wedge \sqrt{\frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})}} \wedge \chi(\mathcal{N}) \right)^{(1)} \wedge \left( ch(\mathcal{F}) \wedge \sqrt{\frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})}} \right) \ . \tag{4.111}$$

With $p<9$, $\chi(\mathcal{N})_{9-p}$ is never 0-form, allowing us to rewrite this as, up to local counter terms,[2]

$$\begin{aligned}\delta S_{CS} &= -(-1)^{(p+1)/2}\pi\int_{\mathrm{D}p}\left(ch(\mathcal{F})\wedge ch(-\mathcal{F})\wedge\frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})}\wedge\chi(\mathcal{N})\right)^{(1)} \\ &= -(-1)^{(p+1)/2}\pi\int_{\mathrm{D}p}\left([ch^{SU(n)}_{\mathrm{adj}}(\mathcal{F})+1]\wedge\mathcal{A}(\mathcal{T})\wedge[ch_{+}(\mathcal{N})-ch_{-}(\mathcal{N})]\right)^{(1)}.\end{aligned} \tag{4.112}$$

Of these, for $p=1$, the expression is null and no inflow is generated. For others, $p=3,5,7$, this is precisely the right inflow to cancel one-loop anomaly (4.90) for $d=4,6,8$.

We have re-analyzed the Bianchi identities of RR field strengths by requiring the regularity of physical variables. This is not by a choice but required, since the D-brane inflow analysis must have the magnetic sources regulated anyway. To summarize, the RR coupling should be written in a form

$$S_{CS}=\sum_{D_p}\frac{\mu_p}{2}\int_{D_p}\sum_{r\leq p}s^*(C_{r+1})\wedge ch(\mathcal{F})\wedge\sqrt{\frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})}}\,, \tag{4.113}$$

in order to cancel all the possible one-loop anomalies. In the following paragraph, we present an example of the D3-brane worldvolume theory, where precise definition of the global angular forms are explicitly given, emphasizing the fact that this mechanism can be safely applied to the self-dual systems.

**Axial Anomaly Inflow onto D3-Branes**

One-loop anomaly of the maximal $U(N_3)$ super Yang-Mills theory can be completely canceled by the anomaly inflow onto $N_3$ coincident D3-branes. Previous analysis [97, 98] produced a null inflow for this case, seemingly requiring another

[2] $p=9$ requires a separate discussion since this case involves Orientifold planes. See next section.

inflow mechanism. The crucial difference between the old and the revised inflow is whether one has a 6-form $s^*(\tau_6) = \chi_6$ as a blind overall factor (which kills off all terms) or one also has an exceptional term with 4-form $s^*(\tau_4^{(1)}) = \chi_4^{(1)}$ instead. Here we wish to retrace the case of D3-branes, with more care given to details of the Thom class, for a pedagogical reason.

Upon close inspection of the inflow, one can see easily that, for D$p$-branes, only those RR gauge fields from $C_{p+1}$ down to its dual $C_{7-p}$ contribute to the inflow. For an $N_3$ coincident D3, $C_4$ is self-dual, and the only relevant term for D3-brane inflow is the minimal coupling

$$S_{CS}^{D3} = \frac{\mu_3 N_3}{2} \int_{D3} s^*(C_4) \ , \tag{4.114}$$

with the constant 0-form $Y_0 = N_3 = \bar{Y}_0$. This is also related to the fact that $s^*(\tau^{(1)}) = \chi^{(1)}$ is already a 4-form, saturating all the world-volume dimensions. From this, combining with the self-duality constraint on $C_4$, we have the Bianchi identity of $H_5$

$$dH_5 = 2\kappa_{10}^2 \mu_3 N_3 \tau_6(D3) \ , \tag{4.115}$$

again with the regularized and covariantized $\tau_6(D3)$.

Recall that this Thom class is defined by

$$\tau_6(D3) = d(\rho \cdot \hat{e}_5) \ , \tag{4.116}$$

with the global angular five-form $\hat{e}_5$ of unit volume. More explicitly,

$$\begin{aligned} \hat{e}_5 \;=\; & -\frac{1}{15}\epsilon_{a_1\cdots a_6} D\hat{y}^{a_1} D\hat{y}^{a_2} D\hat{y}^{a_3} D\hat{y}^{a_4} D\hat{y}^{a_5} \hat{y}^{a_6} \\ & -\frac{1}{6}\epsilon_{a_1\cdots a_6} F_R^{a_1 a_2} D\hat{y}^{a_3} D\hat{y}^{a_4} D\hat{y}^{a_5} \hat{y}^{a_6} - \frac{1}{8}\epsilon_{a_1\cdots a_6} F_R^{a_1 a_2} F_R^{a_3 a_4} D\hat{y}^{a_5} \hat{y}^{a_6} \end{aligned} \tag{4.117}$$

which can be decomposed as

$$\hat{e}_5 = d\psi_4 + \Omega_5 \ , \tag{4.118}$$

with

$$d\psi_4 = -\frac{1}{120\pi^3}\ \epsilon_{a_1\cdots a_6} d\hat{y}^{a_1} d\hat{y}^{a_2} \cdots d\hat{y}^{a_5} \hat{y}^{a_6} + \cdots \ , \tag{4.119}$$

and

$$\Omega_5 = \frac{1}{384\pi^3}\,\epsilon_{a_1\cdots a_6}\left[F_R^{a_1a_2} F_R^{a_3a_4} A_R^{a_5a_6} + \cdots\right] \ , \qquad d\Omega_5 = -\chi_6(F_R)\ . \tag{4.120}$$

Of course the six-form $\chi_6$ and the five-form $\Omega_5$ vanish identically when evaluated on the four dimensional world-volume of D3, but what matters at the end is the appearance of the 4-form $\chi_4^{(1)}$ from the variation of $\psi_4$. In what follows, we obtain the same final answer if we remove $\chi_6$ and $\Omega_5$ from all the formulae but remember that $\delta\psi_4$ is trivially closed on the D3 world-volume.

As before, from the regularity requirement of $H_5$ and $C_4$, we must choose among many naive choices of $[\tau_6(D3)]^{(0)}$,

$$H_5 = dC_4 + 2\kappa_{10}^2\mu_3 N_3\,(\tau(D3))_5^{(0)} = dC_4 + 2\kappa_{10}^2\mu_3 N_3\,[\rho\wedge\hat{e}_5 - d(\rho\wedge\psi_4)]\ . \tag{4.121}$$

On the other hand, since

$$\delta\hat{e}_5 = 0\ , \qquad \delta\psi_4 = \chi_4^{(1)}\ , \tag{4.122}$$

the gauge invariance of $H_5$ yields

$$s^*(\delta C_4) = -2\kappa_{10}^2\mu_3 N_3 \times s^*\left(\tau_6(D3)^{(1)}\right) = -2\kappa_{10}^2\mu_3 N_3 \times \chi_4^{(1)}\ . \tag{4.123}$$

If we substitute this to $\delta S_{CS}^{D3}$, we finally have

$$\delta S_{CS}^{D3} = -\kappa_{10}^2\mu_3^2 N_3^2 \int_{D3}\chi_4^{(1)} = N_3^2 \times \left(-\pi\int_{D3}\chi_4^{(1)}\right)\ , \tag{4.124}$$

with $\kappa_{10}^2\mu_3^2 = [(2\pi)^7(\alpha')^4/2] \times [1/(2\pi)^3(\alpha')^2]^2 = \pi$. This cancels exactly the one-loop anomaly on the D3-branes.

As we saw in the introduction, the $\mathrm{SO}(6)_R$ axial anomaly polynomial at one-loop of the $U(N_3)$ theory is

$$\begin{aligned} I_6 &= \frac{N_3^2}{24\pi^2}\,\mathrm{tr}_{S^+} F_R^3 = N_3^2 \cdot 2\pi \cdot ch_{S^+}(F_R)\Big|_{6-form} \\ &= N_3^2 \cdot \pi \cdot [ch_{S^+}(F_R) - ch_{S^-}(F_R)]\Big|_{6-form} , \end{aligned} \tag{4.125}$$

where $F_R$ is the curvature tensor of an external $SO(6)_R$ in the Weyl representation. The bracket in the last line equals the Euler class divided by the A-roof genus, and the Euler class is already 6-form, so the one-loop anomaly polynomial equals

$$I_6 = N_3^2 \times \pi\chi(F_R) , \tag{4.126}$$

which is precisely canceled by the inflow (4.124).

The case of D3 is special in that the minimal coupling to $C_4$ alone generates the anomaly inflow and there is no need to invoke lower-rank RR gauge fields. This happens due to the self-dual nature of D3. A toy model of such self-dual objects, namely dyonic string in six dimensions, was studied previously in Refs. [118–120]. Our inflow argument is related most directly to that of Ref. [119]. There is also some relation to Ref. [118] in that $v_5 = d\psi_4$ is the generalization of the Wess-Zumino-Witten term of the latter, but the inflow here is a direct consequence of the standard topological coupling, rather than with additional modifications. In particular, the smearing function $\rho$ plays a crucial role here.

### 4.3.2 Chern-Simons Couplings on Orientifold Planes

Extending all of these to the presence of Orientifold planes should be straightforward. The main extra ingredient is how the various Orientifold planes couple to the space-time curvature. For O$p^-$ plane, the relevant Chern-Simons coupling is known to be,

$$S_{Op^-} = \frac{1}{2} \times \left( -2^{p-4}\frac{\mu_p}{2} \int_{Op^-} \sum_r s^*(C_{r+1}) \wedge \sqrt{\frac{\mathcal{L}(\mathcal{T}/4)}{\mathcal{L}(\mathcal{N}/4)}} \right) , \tag{4.127}$$

where $\mathcal{L}$ is the Hirzebruch class [99–102]. There are various studies in the past that worked out analog of this for other three classes of Orientifold planes, but the answers seem to disagree partially with one another [103–107].

In this section, we will show that the one-loop anomaly from the gauge sector cancels away by the anomaly inflow, if we assume the most obvious choices of the Orientifold Chern-Simons couplings, which in addition to the above O$^-$,

$$S_{\widetilde{Op^-}} = \frac{1}{2} \times \left( -\frac{\mu_p}{2} \int_{\widetilde{Op^-}} \sum_r s^*(C_{r+1}) \wedge \left[ \sqrt{\frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})}} - 2^{p-4}\sqrt{\frac{\mathcal{L}(\mathcal{T}/4)}{\mathcal{L}(\mathcal{N}/4)}} \right] \right) , \tag{4.128}$$

reflecting the usual statement that this case has a single, unpaired D-brane stuck at the Orientifold plane. For O$p^+$,

$$S_{Op^+} = \frac{1}{2} \times \left( 2^{p-4}\frac{\mu_p}{2} \int_{Op^+} \sum_r s^*(C_{r+1}) \wedge \sqrt{\frac{\mathcal{L}(\mathcal{T}/4)}{\mathcal{L}(\mathcal{N}/4)}} \right) , \tag{4.129}$$

and the same expression for $S_{\widetilde{Op^+}}$. This last one associated with symplectic type orbifolding agrees with Refs. [103, 104].

As before, the overall factor $1/2$ exists only when we write the kinetic terms of RR tensors in the duality symmetric form, and does not enter the equation of motion. The other $1/2$ factor accompanying $\mu_p$ is due to the Orientifolding projection.

**D$p$-O$p$ Inflow**

We work in the covering space of the Orientifold and take care to divide by two at the end of everything. For example, equation of motion and the Bianchi identity are unaffected by this, but the action written in the covering space must be either divided by two (e.g., world-volume part) or restricted to the half space (e.g., spacetime part). Similarly, the D-brane Chern-Simons couplings are

$$S_{Dp} = \frac{1}{2} \times \left( \frac{1}{2} \sum_A \mu_p \int_A \sum_{r \le p} s^*(C_{r+1}) \wedge ch_{2k}(\mathcal{F}_\mathcal{A}) \wedge \sqrt{\frac{\mathcal{A}(\mathcal{T}_A)}{\mathcal{A}(\mathcal{N}_A)}} \right) . \qquad (4.130)$$

Note that here we assumed these $2k$ D$p$ branes are on the top of the O$p^-$ plane, so they share the Thom class $\tau$, the tangent bundle $\mathcal{T}$, and the normal bundle $\mathcal{N}$.

Note that, upon the Orientifold projection, some of the RR tensor fields are absent. With O$p$ planes, $C_{p-1\pm 4n}$ maps to its negative and thus are projected out, while $C_{p+1\pm 4n}$ remains intact. This can potentially modify inflow argument. However, we do not really lose any term since $ch_{2k}(\mathcal{F})$ is a sum of $4n$-forms for $SO(2k)$ and $Sp(k)$ gauge groups, and since the Euler character $\chi_{9-p}$ is a $(9-p)$-form monomial. An exception to this is $p = 9$, for which one of the relevant RR gauge field, $C_{10=9+1}$, does not exist, and $Y_{10}^{(1)}$ type of inflow cannot be generated. This is precisely what leads to the tadpole condition $2k = 32$ for type I string theory.

With this, we may proceed as before except that $Y = ch_{2k}(\mathcal{F})\mathcal{A}(\mathcal{T})^{1/2}\mathcal{A}(\mathcal{N})^{-1/2}$ is shifted by $-2^{p-4}$ times

$$Z \equiv \sqrt{\frac{\mathcal{L}(\mathcal{T}/4)}{\mathcal{L}(\mathcal{N}/4)}} , \qquad (4.131)$$

and the Bianchi identity reads

$$d(H_{s+2}) = -\sum_B 2\kappa_{10}^2 \mu_q (-1)^{(-q+\epsilon)/2} (\bar{Y}^B_{q+s-6} - 2^{p-4} \bar{Z}^B_{q-r}) \wedge \tau^B_{9-p} , \qquad (4.132)$$

from which we repeat the procedure of the D-brane cases and arrive at the world-volume expressions,

$$\begin{aligned}\delta(S_{Op^-}+S_{Dp}) &= -(-1)^{(p+1)/2}\cdot\frac{\pi}{2}\int\left(\bar{Y}\wedge Y\wedge\chi(\mathcal{N})\right)^{(1)}\\ &\quad +(-1)^{(p+1)/2}\cdot\frac{\pi}{2}\cdot 2^{p-4}\int\left((\bar{Y}\wedge Z+\bar{Z}\wedge Y)\wedge\chi(\mathcal{N})\right)^{(1)}\\ &\quad -(-1)^{(p+1)/2}\cdot\frac{\pi}{2}\cdot 2^{2(p-4)}\int\left(\bar{Z}\wedge Z\wedge\chi(\mathcal{N})\right)^{(1)}\\ &\equiv (-1)^{(p+1)/2}\int\left(\Delta_{BB}+\Delta_{BO^-+O^-B}+\Delta_{O^-O^-}\right)\ ,\end{aligned} \tag{4.133}$$

where in the last line we classified the contribution to brane-brane($BB$), brane-plane($BO$), and plane-plane($OO$) type.

Again we denote by $ch_\rho$ the trace over $\rho$ representation of $SO(2k)$. In particular, $ch_{2k}=ch_{\overline{2k}}$ and $ch_{2k\otimes\overline{2k}}=ch_{2k\otimes 2k}=[ch_{2k}]^2$, thanks to the reality of the vector representation of $SO$ groups. Then, we find contributions with gauge group factors

$$\Delta_{BB}=-\frac{\pi}{2}\left(ch_{2k\otimes\overline{2k}}(\mathcal{F})\wedge\frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})}\wedge\chi(\mathcal{N})\right)^{(1)}_{p+1}\ , \tag{4.134}$$

and[3]

$$\begin{aligned}\Delta_{BO^-+O^-B} &= \frac{\pi}{2}\cdot 2^{p-4}\left(\left[ch_{2k}(\mathcal{F})+ch_{\overline{2k}}(\mathcal{F})\right]\wedge\sqrt{\frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})}}\wedge\sqrt{\frac{\mathcal{L}(\mathcal{T}/4)}{\mathcal{L}(\mathcal{N}/4)}}\wedge\chi(\mathcal{N})\right)^{(1)}_{p+1}\\ &= \frac{\pi}{2}\cdot 2^{p-4}\left(\left[ch_{2k}(\mathcal{F})+ch_{\overline{2k}}(\mathcal{F})\right]\wedge\frac{\mathcal{A}(\mathcal{T}/2)}{\mathcal{A}(\mathcal{N}/2)}\wedge\chi(\mathcal{N})\right)^{(1)}_{p+1}\\ &= \frac{\pi}{2}\left(ch_{2k}(2\mathcal{F})\wedge\frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})}\wedge\chi(\mathcal{N})\right)^{(1)}_{p+1}\ ,\end{aligned}$$

[3] A useful identity throughout here is

$$\sqrt{\mathcal{A}(\mathcal{T})}\sqrt{\mathcal{L}(\mathcal{T}/4)}=\mathcal{A}(\mathcal{T}/2)$$

which combine to

$$\begin{aligned}
& (-1)^{(p+1)/2}\left(\Delta_{BB}+\Delta_{BO^-+O^-B}\right) \\
= \quad & -(-1)^{(p+1)/2}\left(\frac{\pi}{2}\left[ch_{2k\otimes 2k}(\mathcal{F})-ch_{2k}(2\mathcal{F})\right]\wedge\frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})}\wedge\chi(\mathcal{N})\right)^{(1)}_{p+1} \qquad (4.135)
\end{aligned}$$

Purely Orientifold contribution is

$$\begin{aligned}
(-1)^{(p+1)/2}\Delta_{O^-O^-} \quad = \quad & -(-1)^{(p+1)/2}\frac{\pi}{2}\cdot 2^{2(p-4)}\left(\frac{\mathcal{L}(\mathcal{T}/4)}{\mathcal{L}(\mathcal{N}/4)}\wedge\chi(\mathcal{N})\right)^{(1)}_{p+1} \\
= \quad & -(-1)^{(p+1)/2}\frac{\pi}{8}\left(\frac{\mathcal{L}(\mathcal{T})}{\mathcal{L}(\mathcal{N})}\wedge\chi(\mathcal{N})\right)^{(1)}_{p+1} . \qquad (4.136)
\end{aligned}$$

We will see later how these cancel various one-loop contributions.

Extending this to O$p^+$ plane is immediate with

$$S_{Op^+}=-S_{Op^-} \ , \qquad (4.137)$$

as motivated by the fact that the two planes differ by a sign of the charge. Again writing

$$\delta(S_{Op^+}+S_{Dp}) \quad = \quad (-1)^{(p+1)/2}\int\left(\Delta_{BB}+\Delta_{BO^++O^+B}+\Delta_{O^+O^+}\right) \ ,$$

the only change from O$^-$ case is the sign flip of $\Delta_{BO^++O^+B}=-\Delta_{BO^-+O^-B}$. As such, we have

$$\begin{aligned}
& (-1)^{(p+1)/2}\left(\Delta_{BB}+\Delta_{BO^++O^+B}\right) \\
= \quad & -(-1)^{(p+1)/2}\left(\frac{\pi}{2}\left[ch_{2k\otimes 2k}(\mathcal{F})+ch_{2k}(2\mathcal{F})\right]\wedge\frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})}\wedge\chi(\mathcal{N})\right)^{(1)}_{p+1} \qquad (4.138)
\end{aligned}$$

where the trace in $ch_\rho$ should be understood as taken in $\rho$ representations of $Sp(k)$ gauge group. The defining representation $2k$ is pseudo-real, so the algebra goes the same as $SO(2k)$ cases. The Orientifold contribution

$$\Delta_{O^+O^+} = \Delta_{O^-O^-} \tag{4.139}$$

remains the same, begins quadratic in the $p$-brane charge.

Inflow in the presence of $\widetilde{\mathrm{O}p^-}$'s can be similarly obtained. Since the charge of $\widetilde{\mathrm{O}p^-}$ equals to that of an O$p^-$ plus an half D-brane, the obvious candidate for the CS coupling of $\widetilde{\mathrm{O}p^-}$ is

$$S_{\widetilde{Op^-}} = \frac{1}{2} \times \left( -\frac{\mu_p}{2} \int \sum_r s^*(C_{r+1}) \wedge \left[ \sqrt{\frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})}} - 2^{p-4} \sqrt{\frac{\mathcal{L}(\mathcal{T}/4)}{\mathcal{L}(\mathcal{N}/4)}} \right] \right) . \tag{4.140}$$

$\Delta_{BB}$ is unaffected as before, while $\Delta_{\widetilde{O^-}B+B\widetilde{O^-}}$ is modified as

$$\Delta_{B\widetilde{O^-}+\widetilde{O^-}B} = \frac{\pi}{2} \left( [ch_{2k}(2\mathcal{F}) - 2ch_{2k}(\mathcal{F})] \wedge \frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})} \wedge \chi(\mathcal{N}) \right)^{(1)}_{p+1} . \tag{4.141}$$

Thus, the analog of (4.135) and (4.138) here is

$$-(-1)^{(p+1)/2} \left( \frac{\pi}{2} [ch_{2k\otimes 2k}(\mathcal{F}) - ch_{2k}(2\mathcal{F}) + 2ch_{2k}(\mathcal{F})] \wedge \frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})} \wedge \chi(\mathcal{N}) \right)^{(1)}_{p+1} \tag{4.142}$$

Finally, the purely Orientifold contribution may look more involved than before, but turns out to be the same:

$$\begin{aligned}
\Delta_{\widetilde{O^-}\widetilde{O^-}} &= -\frac{\pi}{2}\left(\left(\sqrt{\frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})}} - 2^{p-4}\sqrt{\frac{\mathcal{L}(\mathcal{T}/4)}{\mathcal{L}(\mathcal{N}/4)}}\right)^2_{2p-6}\right)^{(1)} \wedge \chi(\mathcal{N})_{9-p} \\
&= -\frac{\pi}{2}\left(\left(\frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})} - 2^{p-3}\frac{\mathcal{A}(\mathcal{T}/2)}{\mathcal{A}(\mathcal{N}/2)} + 2^{2(p-4)}\frac{\mathcal{L}(\mathcal{T}/4)}{\mathcal{L}(\mathcal{N}/4)}\right)_{2p-6}\right)^{(1)} \wedge \chi(\mathcal{N})_{9-p} \\
&\simeq -\frac{\pi}{2}\cdot 2^{2(p-4)}\left(\left(\frac{\mathcal{L}(\mathcal{T}/4)}{\mathcal{L}(\mathcal{N}/4)} \wedge \chi(\mathcal{N})\right)_{p+3}\right)^{(1)} \\
&= -\frac{\pi}{8}\left(\frac{\mathcal{L}(\mathcal{T})}{\mathcal{L}(\mathcal{N})} \wedge \chi(\mathcal{N})\right)^{(1)}_{p+1} \;=\; \Delta_{O^-O^-}\,. \qquad (4.143)
\end{aligned}$$

where the equalities hold because we are supposed to extract $p+3$-form parts of the anomaly polynomial.

## One-Loop from Open String Sector

Consider the situation where $2k$ coincident D-branes are on the top of one of an $\mathrm{O}^-$, an $\mathrm{O}^+$, or an $\widetilde{\mathrm{O}^-}$ plane. There is one more type of Orientifold plane $\widetilde{\mathrm{O}^+}$, but this leads to the same gauge group as the $\mathrm{O}^+$ case and thus the same world-volume one-loop anomaly is induced.

First, in the presence of the $\mathrm{O}^-$ planes, the gauge group of the open strings ending on D$p$-branes is enhanced from $U(k)$ to $SO(2k)$. Hence a $SO(2k)$ adjoint fermion contributes to the world-volume anomaly polynomial of amount

$$2\pi \cdot ch_{\frac{1}{2}2k(2k-1)} \wedge \mathcal{A}(\mathcal{T}) \wedge ch_{S^+}(\mathcal{N}) \qquad (4.144)$$

for $4n$-dimensions, and

$$\pi \cdot ch_{\frac{1}{2}2k(2k-1)} \wedge \mathcal{A}(\mathcal{T}) \wedge [ch_{S^+}(\mathcal{N}) - ch_{S^-}(\mathcal{N})] \qquad (4.145)$$

for $4n+2$-dimensions. Thanks to the reality of $SO(2k)$, two of these can be written uniformly as

$$I^{SO(2k)}_{1-loop} = \pi \cdot ch_{\frac{1}{2}2k(2k-1)} \wedge \mathcal{A}(\mathcal{T}) \wedge [ch_{S^+}(\mathcal{N}) - ch_{S^-}(\mathcal{N})] \ . \tag{4.146}$$

By the way, we have an identity

$$ch_{\frac{1}{2}2k(2k\pm1)}(\mathcal{F}) = \frac{1}{2}ch_{2k\otimes 2k}(\mathcal{F}) \pm \frac{1}{2}ch_{2k}(2\mathcal{F}) \tag{4.147}$$

and it leads to

$$I^{SO(2k)}_{1-loop} = \frac{\pi}{2}[ch_{2k\otimes 2k} - ch_{2k}(2\mathcal{F})] \wedge \mathcal{A}(\mathcal{T}) \wedge [ch_{S^+}(\mathcal{N}) - ch_{S^-}(\mathcal{N})] \tag{4.148}$$

Again, with the identity

$$\frac{\chi(\mathcal{N})}{\mathcal{A}(\mathcal{N})} = ch_{S^+}(\mathcal{N}) - ch_{S^-}(\mathcal{N}) \ , \tag{4.149}$$

we see that they have the precise form and the factor that can cancel inflows (4.135) from $BB$ and $BO+OB$ intersection.

Similarly, the other cases follow. The symplectic case is

$$\begin{aligned} I^{Sp(k)}_{1-loop} &= \pi \cdot ch_{\frac{1}{2}2k(2k+1)} \wedge \mathcal{A}(\mathcal{T}) \wedge [ch_{S^+}(\mathcal{N}) - ch_{S^-}(\mathcal{N})] \\ &= \frac{\pi}{2}\ [ch_{2k\otimes 2k}(\mathcal{F}) + ch_{2k}(2\mathcal{F})] \wedge \mathcal{A}(\mathcal{T}) \wedge [ch_{S^+}(\mathcal{N}) - ch_{S^-}(\mathcal{N})] \ , \end{aligned}$$

which are again canceled by the anomaly inflow $\Delta_{BB} + \Delta_{BO^+ + O^+B}$ (4.138) in the presence of an $O^+$ plane. $SO(2k+1)$ type gauge theory can be also dealt with by expanding its adjoint representation in terms of the $SO(2k)$ representation as

$$ch^{SO(2k+1)}_{adj.} = ch_{\frac{1}{2}2k(2k-1)+2k} = \frac{1}{2}\,ch_{2k\otimes 2k}(\mathcal{F}) - \frac{1}{2}\,ch_{2k}(2\mathcal{F}) + ch_{2k}(\mathcal{F}) \ , \tag{4.150}$$

whereby the world-volume anomaly can be decomposed as

$$\begin{aligned} I^{SO(2k+1)}_{1-loop} &= \frac{\pi}{2}\left[ch_{2k\otimes 2k}(\mathcal{F}) - ch_{2k}(2\mathcal{F}) + 2ch_{2k}(\mathcal{F})\right] \\ &\quad \wedge\mathcal{A}(\mathcal{T})\wedge\left[ch_{S^+}(\mathcal{N}) - ch_{S^-}(\mathcal{N})\right] , \end{aligned} \tag{4.151}$$

which again is neatly canceled by $\Delta_{BB} + \Delta_{B\widetilde{O^-}+\widetilde{O^-}B}$ (4.142).

Hence, we conclude that the part of anomaly and inflow that depend on the gauge group exactly cancel regardless of the brane types, after the overall chirality (or the orientation issue) is properly taken into account.

**On Universal Inflow $\Delta_{OO}$**

As $\Delta_{BB} + \Delta_{BO+OB}$ are canceled by the open string sector one-loop, $\Delta_{OO}$ is left uncanceled so far. Clearly this part of inflow has nothing to do with the open string degrees of freedom; it exists even in the absence of any D-branes. As such, $\Delta_{OO}$ should be canceled by one-loop anomaly from the closed string spectrum. We wish to emphasize here that, even before checking cancelation against closed string one-loop, the proposed Chern-Simons couplings stand out because they lead to a universal inflow

$$\Delta_{O^-O^-} = \Delta_{O^+O^+} = \Delta_{\widetilde{O^-}\widetilde{O^-}} = -\frac{\pi}{8}\left(\frac{\mathcal{L}(\mathcal{T})}{\mathcal{L}(\mathcal{N})}\wedge\chi(\mathcal{N})\right)^{(1)}_{p+1} , \tag{4.152}$$

from all types of Orientifold planes. This has to be the case, as the closed string part of the low energy spectrum does not care what kind of projections are taken on the Chan-Paton factors. This obvious and basic requirement is met by our Chern-Simons couplings, which may be compared to those in Refs. [105–107].

Checking the cancelation of $\Delta_{OO}$ by closed string one-loop for $p < 9$ is a bit nontrivial, however. The simplest thing to try would be the compact version of the same problem of $T^{9-p}/Z_2$ with $2^{9-p}$ Orientifold planes distributed, one at each fixed point. The low energy spectra here would be identical to type I theory compactified on $T^{9-p}$, producing one gravity multiplet and $(9-p)$ vector multiplets, transforming as vector representation under $SO(9-p)_R$. For $p = 5, 7$, in particular, one can see that the one-loop of this spectra does not completely cancel $2^{9-p}\Delta_{OO}$. That is, unless we set the normal bundle $\mathcal{N}$ to be trivial. In the latter case, both the inflow and the one-loop vanish individually.

In retrospect, this mismatch is to be expected since the one-loop computation based on the massless spectra in $p+1$ dimensions only is really computing smeared version of the anomaly, over $T^{9-p}$, rather than the localized ones. As such, the normal bundle information, which measures nontrivial curvature effect along $T^{9-p}$ direction to begin with, is inevitably lost along the way [121]. One must rely on more complete information, where higher modes such as Kaluza-Klein modes are taken into account, along the line of Ref. [122]. This is not an easy task, since one must also keep track of nontrivial internal curvatures. Instead we will consider $p = 9$ case that sidesteps this complication.

## 4.4 D-brane Central Charges and Hemisphere Partition Function

In the last section, we have learned that the central charge of the D-brane wrapping a spin manifold can be written as

$$\text{(D-brane central charge)} = \int_X e^{-B-iJ} \wedge ch(\mathcal{F}) \wedge \sqrt{\frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})}} + \mathcal{O}(\alpha') \ , \qquad (4.153)$$

at the tree-level of $\alpha'$. Following the exact calculation of the Kahler potentials via the two-sphere partition function, we naturally expect that the GLSM partition function on the hemisphere with proper boundary condition will exactly calculate the central charge of the D-brane, as was briefly reviewed in section 4.1.3. The calculation has been carried out recently by [14, 29, 127], which found various interesting features for the complete expression of (4.153).

### 4.4.1 Partition Function of $2d$ $\mathcal{N}=(2,2)$ GLSM with a boundary

Constructing the supersymmetric Lagrangian for $\mathcal{N}=(2,2)$ GLSM on the hemisphere are exactly same as that of the two-sphere, except for the additioanl boundary terms at the equator. Especially, in order to preserve the supersymmetry chosen as in (4.79) on the northern hemisphere, one has to add the Chan-Paton factor

$$\mathrm{Tr}_{\mathcal{V}}\Big[\mathcal{P}\exp\Big(-i\int d\varphi\ \mathcal{A}_\varphi\Big)\Big] \tag{4.154}$$

with

$$\begin{aligned}\mathcal{A}_\varphi =&\rho_*\left(A_\varphi+i\sigma_2\right)-\frac{r_*}{2r}-i\Big\{\mathcal{Q},\bar{\mathcal{Q}}\Big\}\\&+\frac{1}{\sqrt{2}}\left(\psi^i_+-\psi^i_-\right)\partial_i\mathcal{Q}+\frac{1}{\sqrt{2}}\left(\bar\psi^i_+-\bar\psi^i_-\right)\partial_i\bar{\mathcal{Q}}\ .\end{aligned} \tag{4.155}$$

Here $\mathcal{V}$ denotes a $Z_2$ graded Chan-Paton vector space. The tachyon profile $\mathcal{Q}(\phi)$ is an operator acting on the vector space $\mathcal{V}$, anti-commuting with fermions, and obeys the following relation

$$Q^2=\mathcal{W}\cdot\mathbf{1}_{\mathcal{V}}\ , \tag{4.156}$$

where $\mathcal{W}(\phi)$ denotes a given superpotential. The $G\times U(1)_v$ representation of the Chan-Paton vector space $\mathcal{V}$ is specified by $\rho_*$ and $r_*$,

$$\begin{aligned}&\rho(g)Q(\phi)\rho(g)^{-1}=Q(g\phi)\ ,\\&\lambda\cdot\lambda^{r_*}Q(\phi)\lambda^{-r_*}=Q(\lambda^q\phi)\ ,\end{aligned} \tag{4.157}$$

where $g\in G$.

In order to study the field configuration on hemisphere, we need to choose the boundary condition of chiral multiplets, either Neumann or Dirichlet boundary condition. This choice would determine the dimension of the D-branes wrapping subcycles of the ambient space. At first sight, it seems to be natural to impose the Dirichlet boundary condition to the chiral multiplets which become a coordinates of normal direction and Neumann to the tangential direction. However, for some reason which will be explained in the next subsection, it turns out to be better to work with imposing the Neumann boundary condition to the all the chiral multiplet. After that, we can obtain the lower dimensional brane via well-established tachyon condensation mechanism. Among the saddle solution of the Coulomb branch localization for the sphere (4.83), surviving configuration after imposing the boundary condition is

$$A = 0, \quad \sigma_1 = 0, \quad \sigma_2 = \sigma, \quad D = 0, \quad \phi = 0, \quad F = 0 \; . \tag{4.158}$$

For the one-loop determinant, we again pick out modes which satisfies the boundary condition. As a result, the exact hemisphere partition function can be expressed by

$$Z_{D^2} \propto \frac{1}{|W(G)|} \int_{\mathfrak{t}} d\sigma \; e^{-2\pi i \xi_{\text{ren}} \text{tr}\sigma - \theta \text{tr}\sigma} \times \text{Tr}_{\mathcal{V}} \left[ e^{2\pi \rho_*(\sigma) + i\pi r_*} \right] \times Z_{\text{1-loop}} \tag{4.159}$$

with

$$Z_{\text{1-loop}} = \prod_{\alpha > 0} \left[ \alpha \cdot \sigma \sinh \alpha \cdot \sigma \right] \prod_{w_a \in \mathbf{R^a}} \Gamma\left( \frac{q_a}{2} - i w_a \cdot \sigma \right) \; , \tag{4.160}$$

where $\mathfrak{t}$ is the Cartan subalgebra of the gauge group $G$, and $W(G)$ is the Weyl group.

In what follows, we consider the $U(1)$ GLSM describing the degree $N$ hypersurface of $\mathbb{CP}^{N-1}$ studied in section 5 for simplicity.

$$\mathcal{W} = PG_N(X^i) \ , \tag{4.161}$$

where $G_N(X^i)$ denotes a homogeneous polynomial of degree $N$. This model describes the non-linear sigma model whose target space is a CY hypersurface $X$ in $\mathbb{CP}^{N-1}$.

Taking into account for the Knörrer map to relate the GLSM brane $\mathfrak{B}_{UV}$ to the NLSM brane $\mathfrak{B}_{IR}$ [141], one can show that

$$ch\big[\mathfrak{B}_{IR}\big] = \frac{1}{1-e^{-2\pi i N(q/2-i\sigma)}} \times \mathrm{Tr}_{\mathcal{V}}\Big[e^{2\pi\rho_*(\sigma)+i\pi r_*}\Big] \ , \tag{4.162}$$

where $\mathcal{V}$ denotes the Chan-Paton vector space of $\mathfrak{B}_{UV}$. Note that the Knörrer map also leads to the shift of the theta angle

$$\theta_{UV} = \theta_{IR} - \pi N \ . \tag{4.163}$$

The central charge of the NLSM brane $\mathfrak{B}_{IR}$ then takes the following form

$$Z\left(\mathfrak{B}_{IR}\right) = 2i\pi\beta \int_{q/2-i\infty}^{q/2+i\infty} \frac{d\epsilon}{2\pi i}\ e^{2\pi\xi\epsilon - i\theta_{IR}\epsilon} \times \frac{N}{\epsilon^{N-1}} \times \frac{\Gamma(1+\epsilon)^N}{\Gamma(1+N\epsilon)} \times ch\big[\mathfrak{B}_{IR}\big] \tag{4.164}$$

with $\beta = (r\Lambda)^{c/6}/(2\pi)^{(N-2)/2}$. Focusing on the perturbative part of the central charge, one can finally obtain the large-volume expression [14, 29]

$$\begin{aligned} Z_{\text{pert}}\left(\mathfrak{B}_{IR}\right) &= (2\pi i)^{N-2}\,\beta \int_X \ e^{-i\xi H - \frac{\theta_{IR}}{2\pi}H} \times \frac{\Gamma(1+\frac{H}{2\pi i})^N}{\Gamma(1+\frac{NH}{2\pi i})} \times ch\big[\mathfrak{B}_{IR}\big] \\ &= (2\pi i)^{N-2}\,\beta \int_X \ e^{-iJ-B} \wedge \hat{\Gamma}_c(X) \wedge ch\big[\mathfrak{B}_{IR}\big] \ , \end{aligned} \tag{4.165}$$

where $H$ is the hyperplane class of $\mathbb{CP}^{N-1}$. For the last line, we defined the new characteristic class $\hat{\Gamma}_c(X)$, which is defined by

$$\hat{\Gamma}_c(X) \equiv \prod_i \Gamma\left(1+\frac{x_i}{2\pi i}\right) \ . \tag{4.166}$$

In terms of the Chern characters, it can be expanded as

$$\hat{\Gamma}_c(\mathcal{F}) \ = \ \exp\left[\frac{i\gamma}{2\pi}ch_1(\mathcal{F})+\sum_{k\geq 2}\left(\frac{i}{2\pi}\right)^k (k-1)!\zeta(k)ch_k(\mathcal{F})\right] \ , \tag{4.167}$$

where $\gamma = 0.577...$ is the Euler-Mascheroni constant, and $\zeta(k)$ is the Riemann zeta function. Gamma class satisfies an identity

$$\hat{\Gamma}_c(\mathcal{F})\hat{\Gamma}_c(-\mathcal{F}) = \mathcal{A}(\mathcal{F}) \ , \tag{4.168}$$

which will turns out to be useful. In fact, the appearence of this new characteristic class has been very well-known in the mirror symmetry literature [129–131], where their physical derivation was understood recently by the localization technique. [14, 132]

Expression for lower-dimensional branes can be obtained as well, via the tachyon condensation procedure. To be more concrete let us consider a tachyon profile $\mathcal{Q}$

$$\mathcal{Q} = X^a\eta_a + P\tilde{\eta} + G_N\bar{\tilde{\eta}} \ , \tag{4.169}$$

where the fermionic oscillators satisfy the following anti-commutation relations

$$\{\tilde{\eta},\bar{\tilde{\eta}}\} = 1 \ , \qquad \{\eta_a,\bar{\eta}_b\} = \delta_{ab} \tag{4.170}$$

with $a = 1, 2, .., n$. Since the boundary potential becomes

$$\left\{\mathcal{Q}, \bar{\mathcal{Q}}\right\} = \sum_{a=1}^{n} |X^a|^2 + |P|^2 + |G_N(X^i)|^2 \ , \qquad (4.171)$$

the above tachyon profile describes a lower-dimensional brane wrapping a submanifold at $X^a = 0$ in the Calabi-Yau space $X$ in the geometric phase. One can easily show that the Chern character of the brane $\mathfrak{B}_{IR}$ is

$$ch\big[\mathfrak{B}_{IR}\big] = e^{-\pi i n\epsilon}\Big(2i\sin(\pi\epsilon)\Big)^n \ , \qquad (4.172)$$

where $\epsilon = q/2 - i\sigma$. Then, the central charge of the brane in the large volume limit (4.165) can be written as

$$\begin{aligned} Z_{\rm pert}\left(\mathfrak{B}_{IR}\right) &= (2i\pi)^{N-1}\,\beta\int_X e^{-i\xi H - \frac{\theta_{IR}}{2\pi}H} \times \frac{\Gamma(1+\frac{H}{2\pi i})^{N-n}}{\Gamma(1-\frac{H}{2\pi i})^n\Gamma(1+\frac{NH}{2\pi i})} \times H^n \times e^{-\frac{nH}{2}} \\ &= (2i\pi)^{N-1}\,\beta\int_X e^{-iJ-B}\wedge\frac{\hat{\Gamma}_c(\mathcal{T})}{\hat{\Gamma}_c(-\mathcal{N})}\wedge e(\mathcal{N})\wedge e^{-\frac{1}{2}c_1(\mathcal{N})} \ . \qquad (4.173) \end{aligned}$$

Note that one can see the very subtle factor $e^{-c_1(\mathcal{N})/2}$ emerges from the partition function computation again. As a byproduct, we also confirmed that the overall normalization factor $(2\pi i)^{N-1}\beta = (2\pi)^{N/2} i^{N-1}(r\Lambda)^{c/6}$ are the same for any dimensional D-branes, which turns out to be consistent with the tadpole cancelation condition in the presence of Orientifold planes.

### 4.4.2 Freed-Witten Global Anomaly and Spin$^c$ Structure

A well-known subtlety with D-branes occurs when they wrap a manifold $\mathcal{M}$ which is not Spin. This causes a global anomaly in 2D boundary CFT, whereby the world-sheet fermion determinant has an ill-defined sign. As pointed out by Freed and Witten [138] this ambiguity is cancelable by additional phase factor, provided that $\mathcal{M}$ is Spin$^c$,

$$\exp\left(i\int_{\partial\Sigma}\hat{A}\right)\ , \tag{4.174}$$

with some world-volume Abelian "gauge field" $\hat{A}$. The latter is equally ill-defined, precisely such that the sign flip due to the world-sheet global anomaly is canceled by the sign ambiguity of the latter.

A related observation is that spacetime spinor is ill-defined on a Spin$^c$ manifold, which is nevertheless correctable if we think of the spinor as a section of $\hat{L}^{1/2}\otimes\mathcal{S}(\mathcal{TM})$, where $\hat{A}$ is the "connection" on the ill-defined bundle $\hat{L}^{1/2}$. This implies that the Dirac index on $\mathcal{M}$ is equally ill-defined unless we twist the Dirac operator by $\hat{L}^{1/2}$ and once this is done we have an index theorem,

$$\int_{\mathcal{M}} e^{\hat{F}/2\pi}\wedge\mathcal{A}(\mathcal{TM})\wedge\cdots \tag{4.175}$$

with $\hat{F}=d\hat{A}$, where the ellipsis denotes contributions from the well-defined part of the gauge bundle. A little experiment with this index formula[4] suggests that a good de Rham cohomology representative for $\hat{F}/2\pi$ is $c_1(\mathcal{M})/2$. One can understand this from the fact that it is $c_1(\mathcal{M})$, or more precisely the 2nd Stiefel-Whitney class

$$w_2(\mathcal{M})\quad=\quad c_1(\mathcal{M})\ \text{mod}\ Z_2$$

that determines whether the manifold is Spin. With $w_3(\mathcal{M})=0$, therefore, $c_1(\mathcal{M})/2$ determines whether the manifold is Spin or Spin$^c$.

For $\mathcal{M}$ embedded in an Calabi-Yau ambient $\mathcal{X}$ so that $c_1(\mathcal{T})+c_1(\mathcal{N})=0$, this implies an additional factor

$$e^{\hat{F}/2\pi}=e^{-c_1(\mathcal{N})/2} \tag{4.176}$$

in the central charge (and in the RR-charge) of the D-brane, whose presence was argued by Minasian and Moore [140]: the correct central charge must have this

[4] With the aim at obtaining integer values of the index for completely smooth an compact examples like $\mathbb{CP}^{2k}$ or other toric Spin$^c$ manifold. See also [139].

extra factor,

$$Z_{D^2} \sim \int_{\mathcal{M}} e^{-B-iJ} \wedge ch(\mathcal{E}) \wedge \cdots \wedge e^{-c_1(\mathcal{N})/2} \; . \tag{4.177}$$

In view of its origin as the "half line bundle" $\hat{L}^{1/2}$, it makes more sense to think of it as part of the "gauge bundle" $\mathcal{E} \to \mathcal{E} \otimes \hat{L}^{1/2}$.

When $\mathcal{M}$ is Spin, however, this is a mere redefinition of $\mathcal{E}$ since $\hat{L}^{1/2}$ is a proper line bundle when $\hat{F}/2\pi = c_1(\mathcal{M})/2$ is integral. The D-brane spectra is, as expected, not affected by such factor when $\mathcal{M}$ is Spin. For this reason (and also because the Orientifold cannot admit gauge bundles), the right thing to do is to keep this factor explicitly only when $\mathcal{M}$ is Spin$^c$. With this in mind, we will write, instead

$$Z_{D^2} \sim \int_{\mathcal{M}} e^{-B-iJ} \wedge ch(\mathcal{E}) \wedge \cdots \wedge e^{d(\mathcal{M})/2} \; , \tag{4.178}$$

where

$$d(\mathcal{M}) = \left\{ \begin{array}{ll} 0 & \mathcal{M} \;\text{ is Spin} \\ c_1(\mathcal{T}\mathcal{M}) = -c_1(\mathcal{N}\mathcal{M}) & \mathcal{M} \;\text{ is Spin}^c \end{array} \right\} \; ,$$

again by redefinition of the gauge bundle $\mathcal{E}$.

For D-branes, section 4.4.1 outlines how one can compute the hemisphere partition functions, starting with the result in [14, 29], via the tachyon condensation. In this approach, one does find the factor $e^{-c_1(\mathcal{N})/2}$, where the key point lies with charge assignment for the Hilbert space vacua [14, 29, 141] associated with the boundary degrees of freedom. With "correct" choice of the charges, we find Eq. (4.320). In view of the global anomaly, this result is quite natural. Since the original Calabi-Yau manifold is always Spin and thus free of the global anomaly, the lower dimensional D-brane induced from it must be equipped with the necessary twist to countermand the potential anomaly on the induced D-brane, as it must flow to a well-defined boundary CFT again.

However, if one imposes the Dirichlet boundary condition from the outset, to obtain lower dimensional D-branes in the hemisphere partition function [14], the origin of such a factor is at best subtle. The naive computation from imposing the Dirichlet boundary condition, in contrast to the tachyon condensation above, does not seem to generate the factor in question. It remains an open question to clarify the GLSM origin of the hidden global anomaly when the Dirichlet boundary conditions are explicitly imposed.

## 4.5 Orientifold Central Charges and $RP^2$ Partition Functions

In this section, we initiate extending these works to the presence of Orientifold planes. The simplest quantity one can compute is the vacuum-to-crosscap amplitude,

$$ {}_R\langle 0|\mathcal{C}\rangle_R \ . \tag{4.179}$$

Pictorially, this is computed by a cigar-like geometry with the identity operator at the tip and a crosscap at the other end. There are two possible choices for the crosscap, say, A-type and B-type. The former corresponds to Orientifold planes wrapping Lagrange subcycles. In this note, we are led to consider B-type parity for GLSM, for much the same reason as in Ref. [133], which corresponds to Orientifold planes wrapping the holomorphic cycles. Topologically the world-sheet is that of $\mathbb{RP}^2$, and the same squashing deformation as in Ref. [13] is allowed, the partition function of GLSM on $S^2/Z_2 = \mathbb{RP}^2$ is expected to compute the vacuum-to-crosscap amplitude,

$$ {}_R\langle 0|\mathcal{C}\rangle_R = Z_{\mathbb{RP}^2}(O,\tau) \ . \tag{4.180}$$

In the convention of Brunner-Hori [133], the relevant parity action for our purpose here is of type B, which leads to, generally, holomorphically embedded Orientifold planes.

For Orientifold plane that wraps the Calabi-Yau $\mathcal{X}$ entirely, we also take the large volume limit of the central charge. As we have seen in section 3, Orientifold planes, $O^{\pm}$, have $\mathcal{L}^{1/2}$ class as the counterpart of D-branes' $\mathcal{A}^{1/2}$ class. Here we find that one must also replace

$$\sqrt{\mathcal{L}(\mathcal{T}/4)} \quad \rightarrow \quad \frac{\mathcal{A}(\mathcal{T}/2)}{\hat{\Gamma}_c(-\mathcal{T})} \,. \tag{4.181}$$

The parity action on $S^2$ can be augmented by additional $Z_2$ action on the chiral fields, which induces various combinations of $O_{2(d-s)}$ planes, say wrapping a submanifold $\mathcal{M}$. For these cases, we must also replace

$$\sqrt{\frac{\mathcal{L}(\mathcal{T}/4)}{\mathcal{L}(\mathcal{N}/4)}} \quad \rightarrow \quad \frac{\mathcal{A}(\mathcal{T}/2)}{\hat{\Gamma}_c(-\mathcal{T})} \wedge \frac{\hat{\Gamma}_c(\mathcal{N})}{\mathcal{A}(\mathcal{N}/2)} \,, \tag{4.182}$$

with the normal bundle $\mathcal{N}$ and the tangent bundle $\mathcal{T}$ of holomorphically embedded $\mathcal{M}$ in the Calabi-Yau $\mathcal{X}$. For more complete expression for the large volume limit, see section 5.

The results found here should be consistent with the hemisphere computation of the D-brane central charges. Among those issues discussed are anomaly inflow and a twist that is known to be present when the world-volume wraps a Spin$^c$ (rather than Spin, i.e.) submanifold. Also, one outfall from having both D-brane and Orientifold plane central charges available is the interpretation of exactly what the Gamma class corrects. The central charge does not by itself tells us whether the correction goes to the RR-charge or the vacuum expectation values of spacetime scalars, or equivalently the quantum volumes. Our conclusion is that the correction should be attributed entirely to the $\alpha'$ correction of volumes.

### 4.5.1 GLSM on $\mathbb{RP}^2$ and Squashing

In this section, we start with a brief review on general aspects of parity symmetries in 2d (2,2) theory on $\mathbb{R}^{1+1}$, which were thoroughly studied in Ref. [133]. To begin with, the parity action on the 2-dimensional superspace ($x^{\pm} = x^0 \pm x^1, \theta^{\pm}, \bar{\theta}^{\pm}$) is $x^1 \to -x^1$, accompanied by the proper action in the fermionic coordinates. Depending on the latter there are two distinct possibilities,

$$\begin{aligned} \Omega_A &: (x^{\pm}, \theta^{\pm}, \bar{\theta}^{\pm}) \to (x^{\mp}, -\bar{\theta}^{\mp}, -\theta^{\mp}) \ , \\ \Omega_B &: (x^{\pm}, \theta^{\pm}, \bar{\theta}^{\pm}) \to (x^{\mp}, \theta^{\mp}, \bar{\theta}^{\mp}) \ , \end{aligned} \tag{4.183}$$

which we will call A and B-parity respectively. Under this action, the four supercharges transform as

$$\begin{aligned} A &: Q_{\pm} \to \bar{Q}_{\mp}, \quad \bar{Q}_{\pm} \to Q_{\mp}, \\ &\quad D_{\pm} \to \bar{D}_{\mp}, \quad \bar{D}_{\pm} \to D_{\mp}, \\ B &: Q_{\pm} \to Q_{\mp}, \quad \bar{Q}_{\pm} \to \bar{Q}_{\mp}, \\ &\quad D_{\pm} \to D_{\mp}, \quad \bar{D}_{\pm} \to \bar{D}_{\mp} \ . \end{aligned} \tag{4.184}$$

Hence, under the A-parity action, half of the supersymmetry is broken, leaving $Q_A \equiv Q_+ + \bar{Q}_-$ and $Q_A^\dagger$ invariant. Under B-parity, and $Q_B \equiv \bar{Q}_+ + \bar{Q}_-$ and $Q_B^\dagger$ survive.

Furthermore, the simplest transformation rule for a chiral field $(\phi, \psi, F)$ is

$$\begin{aligned} A &: \phi(x) \to \bar{\phi}(x') \ , \\ &\quad \psi_{\pm}(x) \to \bar{\psi}_{\mp}(x') \ , \\ &\quad F(x) \to \bar{F}(x') \ , \end{aligned} \tag{4.185}$$

$$\begin{aligned} B &: \phi(x) \to \phi(x') \ , \\ &\quad \psi_{\pm}(x) \to \psi_{\mp}(x') \ , \\ &\quad F(x) \to -F(x') \ , \end{aligned} \tag{4.186}$$

and one can check that these leave the kinetic lagrangian of the chiral multiplet invariant. For a twisted chiral multiplet, transformation rules under A and B-parities are exchanged.

For each parity projection, we can associate a crosscap state denoted by $|\mathcal{C}_{A,B}\rangle$. Then we can think of the overlap between this state and a (twisted) chiral ring element, such as

$$\langle a|\mathcal{C}_B\rangle \ . \tag{4.187}$$

We naturally expect that this quantity calculates the Orientifold analogue of the D-brane central charge. Among these overlaps, there are distinguished element $\langle 0|\mathcal{C}_B\rangle$ that no chiral field is inserted at the tip of the hemisphere. The path integral can be done by doubling of the hemisphere by gluing its mirror image. Topology of the world-sheet is that of a two sphere with antipodal points identified, i.e., $\mathbb{RP}^2$.

**GLSM on $\mathbb{RP}^2$**

The supersymmetric Lagrangian we are considering is the same as that used in [8, 9];

$$\mathcal{L} = \mathcal{L}_{vector} + \mathcal{L}_{chiral} + \mathcal{L}_W + \mathcal{L}_{FI} \ , \tag{4.188}$$

where the kinetic terms for the vector and the charged chiral multiplets are, respectively,

$$\begin{aligned}\mathcal{L}_{vector} = \frac{1}{2g^2}\mathrm{Tr}\Bigg[\left(F_{12}+\frac{\sigma_1}{r}\right)^2 + (D_\mu\sigma_1)^2 + (D_\mu\sigma_2)^2 - [\sigma_1,\sigma_2]^2 + D^2 \\ + i\bar{\lambda}\gamma^\mu D_\mu\lambda + i\bar{\lambda}[\sigma_1,\lambda] + i\bar{\lambda}\gamma^3[\sigma_2,\lambda]\Bigg] \ ,\end{aligned} \tag{4.189}$$

$$\begin{aligned}\mathcal{L}_{chiral} = \bar{\phi}\left(-D^\mu D_\mu + \sigma_1^2 + \sigma_2^2 + iD + i\frac{q-1}{r}\sigma_2 + \frac{q(2-q)}{4r^2}\right)\phi + \bar{F}F \\ - i\bar{\psi}\left(\gamma^\mu D_\mu - \sigma_1 - i\gamma^3\sigma_2 + \frac{q}{2r}\gamma^3\right)\psi + i\bar{\psi}\lambda\phi - i\bar{\phi}\bar{\lambda}\psi \ ,\end{aligned} \tag{4.190}$$

and the potential terms take the following form,

$$\mathcal{L}_{\mathcal{W}} = \sum_i \frac{\partial \mathcal{W}}{\partial \phi^i} F^i - \frac{1}{2} \sum_{i,j} \frac{\partial^2 \mathcal{W}}{\partial \phi^i \partial \phi^j} \psi^i \psi^j + \text{c.c.} \ . \tag{4.191}$$

Finally the Fayet-Illiopoulos (FI) coupling and the two-dimensional topological term are

$$\mathcal{L}_{FI} = -\frac{\tau}{2} \text{Tr}\Big[D - \frac{\sigma_2}{r} + iF_{12}\Big] + \frac{\bar{\tau}}{2} \text{Tr}\Big[D - \frac{\sigma_2}{r} - iF_{12}\Big] \ , \tag{4.192}$$

where $\tau = i\xi + \frac{\theta}{2\pi}$, ($\xi \in \mathbb{R}$, $\theta \in [0, 2\pi]$). Note that the superpotential $\mathcal{W}(\phi)$ should carry $R$-charge two to preserve the supersymmetry on $\mathbb{RP}^2$.

The Lagrangian is invariant under the supersymmetry transformation rules,

$$\begin{aligned}
\delta\lambda =& (iV_1\gamma^1 + iV_2\gamma^2 + iV_3\gamma^3 - \text{D})\epsilon \ , \\
\delta\bar{\lambda} =& (i\bar{V}_1\gamma^1 + i\bar{V}_2\gamma^2 + i\bar{V}_3\gamma^3 + \text{D})\bar{\epsilon} \ , \\
\delta A_i =& -\frac{i}{2}\Big(\bar{\epsilon}\gamma_i\lambda - \bar{\lambda}\gamma_i\epsilon\Big) \ , \\
\delta\sigma_1 =& \frac{1}{2}\Big(\bar{\epsilon}\lambda - \bar{\lambda}\epsilon\Big) \ , \\
\delta\sigma_2 =& -\frac{i}{2}\Big(\bar{\epsilon}\gamma_3\lambda - \bar{\lambda}\gamma_3\epsilon\Big) \ , \\
\delta D =& -\frac{i}{2}\bar{\epsilon}\gamma^\mu D_\mu\lambda - \frac{i}{2}\big[\sigma_1, \bar{\epsilon}\lambda\big] - \frac{1}{2}\big[\sigma_2, \bar{\epsilon}\gamma^3\lambda\big] \ , \\
& + \frac{i}{2}\epsilon\gamma^\mu D_\mu\bar{\lambda} - \frac{i}{2}\big[\sigma_1, \bar{\lambda}\epsilon\big] - \frac{1}{2}\big[\sigma_2, \bar{\lambda}\gamma^3\epsilon\big] \ ,
\end{aligned} \tag{4.193}$$

with

$$\begin{aligned}
\vec{V} \equiv& \Big( + D_1\sigma_1 + D_2\sigma_2, \ +D_2\sigma_1 - D_1\sigma_2, \ F_{12} + i[\sigma_1, \sigma_2] + \frac{1}{r}\sigma_1\Big) \ , \\
\vec{\bar{V}} \equiv& \Big( - D_1\sigma_1 + D_2\sigma_2, \ -D_2\sigma_1 - D_1\sigma_2, \ F_{12} - i[\sigma_1, \sigma_2] + \frac{1}{r}\sigma_1\Big) \ ,
\end{aligned} \tag{4.194}$$

and

$$\begin{aligned}
\delta\phi =& \bar{\epsilon}\psi \ , \\
\delta\bar{\phi} =& \epsilon\bar{\psi} \ , \\
\delta\psi =& i\gamma^\mu\epsilon D_\mu\phi + i\epsilon\sigma_1\phi + \gamma^3\epsilon\sigma_2\phi + i\frac{q}{2r}\gamma_3\epsilon\phi + \bar{\epsilon}F \ , \\
\delta\bar{\psi} =& i\gamma^\mu\bar{\epsilon}D_\mu\bar{\phi} + i\bar{\epsilon}\bar{\phi}\sigma_1 - \gamma^3\bar{\epsilon}\bar{\phi}\sigma_2 - i\frac{q}{2r}\gamma_3\bar{\epsilon}\bar{\phi} + \epsilon\bar{F} \ , \\
\delta F =& \epsilon\Big(i\gamma^i D_i\psi - i\sigma_1\psi + \gamma^3\sigma_2\psi - i\lambda\phi\Big) - i\frac{q}{2}\psi\gamma^i D_i\epsilon \ , \\
\delta\bar{F} =& \bar{\epsilon}\Big(i\gamma^i D_i\bar{\psi} - i\bar{\psi}\sigma_1 - \gamma^3\bar{\psi}\sigma_2 + i\bar{\phi}\lambda\Big) - i\frac{q}{2}\bar{\psi}\gamma^i D_i\bar{\epsilon} \ .
\end{aligned} \tag{4.195}$$

Here the spinors $\epsilon$ and $\bar{\epsilon}$ are given by[5]

$$\epsilon = e^{i\varphi/2}\begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix} \ , \qquad \bar{\epsilon} = e^{-i\varphi/2}\begin{pmatrix} \sin\theta/2 \\ \cos\theta/2 \end{pmatrix} \ , \tag{4.196}$$

satisfying the Killing spinor equations

$$\nabla_\mu\epsilon = \frac{1}{2r}\gamma_\mu\gamma^3\epsilon \ , \qquad \nabla_\mu\bar{\epsilon} = -\frac{1}{2r}\gamma_\mu\gamma^3\bar{\epsilon} \ . \tag{4.197}$$

Note that the surviving supersymmetry (4.196) becomes A-type and B-type supersymmetry at the pole ($\theta = 0$) and the equator ($\theta = \pi/2$), respectively.

In order to define the theory on $\mathbb{RP}^2$, we further impose a suitable parity projection condition on the dynamical fields so that the Lagrangian is invariant under the parity. Particularly, one has to consider the type B-parity in the following discussion. This is because the Killing spinors (4.196) transform as

$$\epsilon_\pm \to i\epsilon_\mp \ , \qquad \bar{\epsilon}_\pm \to -i\bar{\epsilon}_\mp \ , \tag{4.198}$$

under the parity action $(\theta, \varphi) \to (\pi - \theta, \varphi + \pi)$. It implies that the B-type Orientifold plane can be naturally placed at the equator $\theta = \pi/2$.

[5]See Appendix A for our gauge choice.

We remark here that, as in the case of the $S^2$, the Lagrangian except $\mathcal{L}_{FI}$ can be made $Q$-exact with the supersymmetry chosen by (4.196). For instance,

$$\mathcal{L}_{vector} = \frac{1}{g^2}\delta_\epsilon\delta_{\bar{\epsilon}}\mathrm{Tr}\Big[\frac{1}{2}\bar{\lambda}\gamma^3\lambda - 2iD\sigma_2 + \frac{i}{r}\sigma_2^2\Big] \ , \tag{4.199}$$

and

$$\mathcal{L}_{chiral} = -\delta_\epsilon\delta_{\bar{\epsilon}}\Big[\bar{\psi}\gamma^3\psi - 2\bar{\phi}(\sigma_2 + i\frac{q}{2r})\phi + \frac{i}{r}\bar{\phi}\phi\Big] \ . \tag{4.200}$$

Consequently, the partition function on $\mathbb{RP}^2$ contains only the A-model data.

### 4.5.2 Squashed $\mathbb{RP}^2$ and Crosscap Amplitudes

We propose that the partition function of $\mathcal{N} = (2,2)$ GLSM on $\mathbb{RP}^2$ computes the overlap between the supersymmetric ground state and the type B-crosscap state in the Ramond sector

$$Z_{\mathbb{RP}^2} = \ {}_R\langle 0|\mathcal{C}_B\rangle_R \ , \tag{4.201}$$

which is the central charge of the Orientifold plane. To understand the above proposal, it is useful to consider a squashed $\mathbb{RP}^2$, denoted by $\mathbb{RP}^2_b$, where the Hilbert space interpretation of the results in section 3 becomes clear.

The squashed $\mathbb{RP}^2$ can be described by

$$\frac{x_1^2 + x_2^2}{l^2} + \frac{x_3^2}{\tilde{l}^2} = 1 \tag{4.202}$$

with $Z_2$ identification below

$$Z_2 \ : \ (x_1, x_2, x_3) \ \to \ (-x_1, -x_2, -x_3) \ . \tag{4.203}$$

The metric on this space is

$$ds^2 = f^2(\theta)d\theta^2 + l^2 \sin^2\theta d\varphi^2 \ , \tag{4.204}$$

where $f^2(\theta) = \tilde{l}^2 \sin^2\theta + l^2 \cos^2\theta$. The world-sheet parity $Z_2$ acts on the polar coordinates as follows

$$Z_2 \ : \ (\theta, \varphi) \ \to (\pi - \theta, \pi + \varphi) \ . \tag{4.205}$$

An Orientifold plane is placed at the equator $\theta = \pi/2$. By turning on a suitable background gauge field coupled to the $U(1)_V$ current,

$$V = \frac{1}{2}\left(1 - \frac{l}{f(\theta)}\right) d\varphi \ , \tag{4.206}$$

valid in the region $0 < \theta < \pi$, one can show the Killing spinors (4.196) on the squashed $\mathbb{RP}^2$ satisfying the generalized Killing spinor equations

$$D_m\epsilon = \frac{1}{2f}\gamma_m\gamma^3\epsilon \ , \qquad D_m\bar{\epsilon} = -\frac{1}{2f}\gamma_m\gamma^3\bar{\epsilon} \ , \tag{4.207}$$

where the covariant derivative denotes $D_m = \partial_m - iV_m$. Here we normalize the $R$-charge so that the Killing spinor $\epsilon$ ($\bar{\epsilon}$) carries $+1$ $(-1)$ $R$-charge.

As in Ref. [13], one can show that the partition function is invariant no matter how much we squash the space $\mathbb{RP}^2$, i.e., it is independent of the squashing parameter $b = l/\tilde{l}$. Appendix D shows detailed computations for this. In the limit $b \to 0$, we have an infinitely stretched cigar-like geometry where the type B-crosscap state $|\mathcal{C}_B\rangle$ is prepared at $\theta = \pi/2$. Near $\theta \simeq \pi/2$, all the fields can be made periodic along the circle $S^1$ due to the background gauge field $V \simeq \frac{1}{2}d\varphi$, which implies that the theory is in the Ramond sector near $\theta \simeq \pi/2$. Moreover, as mentioned earlier, the partition function on the squashed $\mathbb{RP}^2$ contains only the A-model data.

Combining all these facts, we can identify the partition function on $\mathbb{RP}^2_b$ as the overlap in the Ramond sector between A-model ground state corresponding to the identity operator at the tip and the B-type crosscap state defined by an appropriate projection condition we discuss soon,

$$Z_{\mathbb{RP}^2} = Z_{\mathbb{RP}^2_b} \overset{b\to 0}{=} {}_R\langle 0|\mathcal{C}_B\rangle_R \ . \tag{4.208}$$

### 4.5.3 Exact $\mathbb{RP}^2$ Partition Function

In this section, we compute the partition function of GLSM on $\mathbb{RP}^2$ exactly, via the localization technique. The analysis is parallel to the computation of the two-sphere partition function [8, 9].

As we will be working with the Coulomb phase saddle points, the gauge group is effectively reduced to the Cartan subgroup $U(1)^{r_G}$, whose scalar partners will be collectively denoted by $\sigma$. The relevant gauge charges are expressed via weights and roots. For chiral multiplets in the $G$-representation $\mathbf{R}$, these $U(1)^{r_G}$ gauge charges will be denoted collectively as $w$, so the 1-loop determinant of a chiral multiplet with weight $w$ is a function of $w\cdot\sigma$. When the gauge group is Abelian as in sections 4, 5, and 6, we also use the notation $Q$ for the gauge charges, so $w\cdot\sigma$ is written as $Q\cdot\sigma$. Similarly, contribution from each massive "off-diagonal" vector multiplet is determined entirely by its charge under the unbroken $U(1)^{r_G}$; the determinant is then written in terms of $\alpha\cdot\sigma$. In the end, we take a product over all the weights, $w$, and all the roots, $\alpha$.

**Saddle points**

To apply the localization technique, we choose the kinetic terms $\mathcal{L}_{vector}$ and $\mathcal{L}_{chiral}$ as the $Q$-exact deformation and scale them up to infinitely. The path-integral then localizes at the supersymmetric saddle points satisfying the equations

$$F_{12} = -\frac{\sigma_1}{r} = \frac{B}{2r^2}, \quad D_\mu \sigma_1 = D_\mu \sigma_2 = [\sigma_1, \sigma_2] = 0, \quad D + \frac{\sigma_2}{r} = 0 \ , \tag{4.209}$$

with all the other fields vanishing. Among these saddle configurations, the only one invariant under the B-type Orientifold projection is

$$F_{12} = 0, \quad \sigma_1 = 0, \quad D_\mu \sigma_2 = 0, \quad D + \frac{\sigma_2}{r} = 0 \ . \tag{4.210}$$

However, since $\mathbb{RP}^2$ has a non-contractible loop $C$ which connects two antipodal points in the equator, $F_{12} = 0$ is solved by a flat connection with a discrete $Z_2$ holonomy

$$\mathcal{P} \exp \left[ i \int_C A \right] \in Z_2 \ . \tag{4.211}$$

Hence there are two kinds of saddle points, which we call even and odd holonomy. Near the odd holonomy, fields effectively satisfy twisted boundary condition that picks up additional sign along the loop.

Finally, using $U(N)$ gauge transformation, we can make $A_\mu$ holonomy and constant mode of $\sigma_2$ both diagonal, as the two must commute with each other. Then the saddle point configurations all reduce to

$$\sigma_2 = \sigma, \quad D = -\frac{\sigma}{r} \ , \tag{4.212}$$

where $\sigma$ is arbitrary constant element in the Cartan subalgebra. The classical action at the saddle points is,

$$Z_{classical} = e^{-i2\pi r \xi \sigma}. \tag{4.213}$$

**Chiral multiplets**

In this section, we calculate one-loop determinants of chiral multiplets, say, in the representation $\mathbf{R}$ of the gauge group $G$. To compute the one-loop determinant, we truncate the regulator action up to quadratic order in small fluctuation, around each saddle point

$$S_{chiral} = S^b_{chiral} + S^f_{chiral}$$

with

$$S^b_{chiral} = \int d^2x\sqrt{g}\ \bar{\phi}\Big[-D^2_\mu + \sigma^2 + i\frac{q-1}{r}\sigma + \frac{q(2-q)}{4r^2}\Big]\phi\ , \tag{4.214}$$

and

$$S^f_{chiral} = \int d^2x\sqrt{g}\ \bar{\psi}\gamma^3\Big[-i\gamma^3\gamma^\mu D_\mu - \Big(\sigma + i\frac{q}{2r}\Big)\Big]\psi\ . \tag{4.215}$$

We refer readers to Appendix C for properties of the relevant spherical harmonics.

*Even Holonomy* First, we will calculate the contribution near the first saddle point, where the holonomy is trivial. For this, we impose the B-type Orientifold projection[6],

$$\begin{aligned} \phi(\pi-\theta,\pi+\varphi) &= +\,\phi(\theta,\varphi)\ , \\ \psi_\pm(\pi-\theta,\pi+\varphi) &= -\,i\psi_\mp(\theta,\varphi)\ , \\ \bar{\psi}_\pm(\pi-\theta,\pi+\varphi) &= +\,i\bar{\psi}_\mp(\theta,\varphi)\ , \\ F(\pi-\theta,\pi+\varphi) &= +\,F(\theta,\varphi)\ . \end{aligned} \tag{4.216}$$

[6]This choice of projection condition is consistent with the supersymmetry (4.195) and (4.196).

For simplicity, let us first consider a single chiral multiplet of charge +1 under a $U(1)$ gauge group. Thanks to the property, with our gauge choice,

$$Y_{\mathbf{q},jm}(\pi-\theta,\pi+\varphi) =(-1)^j e^{-i\pi|\mathbf{q}|}Y_{-\mathbf{q},jm}(\theta,\varphi) \ , \tag{4.217}$$

we can write scalar fluctuations that survive under the projection (4.216) as

$$\phi(\theta,\varphi) = \sum_{\substack{j=2k\\k\geq 0}}\sum_{m=-j}^{j}\phi_{jm}Y_{jm} \ . \tag{4.218}$$

The bosonic part of the quadratic action then becomes

$$S^b_{chiral} = \frac{1}{2}\sum_{\substack{j=2k\\k\geq 0}}\sum_{m=-j}^{j}\bar{\phi}_{jm}\Big[\Big(j+\frac{q}{2}-ir\sigma\Big)\Big(j+1-\frac{q}{2}+ir\sigma\Big)\Big]\phi_{jm} \ , \tag{4.219}$$

which leads to

$$\mathrm{Det}_\phi = \prod_{k\geq 0}\Big(2k+\frac{q}{2}-ir\sigma\Big)^{4k+1}\Big(2k+1-\frac{q}{2}+ir\sigma\Big)^{4k+1} \ . \tag{4.220}$$

Next, the mode expansion of the fermion fluctuation invariant under the projection (4.216) takes the form

$$\begin{aligned}\psi &= \sum_{\substack{j=2k+1/2\\k\geq 0}}\sum_{m=-j}^{j}\psi^+_{jm}\Psi^+_{jm} + \sum_{\substack{j=2k+3/2\\k\geq 0}}\sum_{m=-j}^{j}\psi^-_{jm}\Psi^-_{jm} \ ,\\ \bar{\psi} &= \sum_{\substack{j=2k+1/2\\k\geq 0}}\sum_{m=-j}^{j}\bar{\psi}^+_{jm}\bar{\Psi}^+_{jm} + \sum_{\substack{j=2k+3/2\\k\geq 0}}\sum_{m=-j}^{j}\bar{\psi}^-_{jm}\bar{\Psi}^-_{jm} \ ,\end{aligned} \tag{4.221}$$

where the spinor harmonics $\Psi^{\pm}_{j,m}$ are

$$\Psi^{\pm}_{jm} = \begin{pmatrix} Y_{-\frac{1}{2},jm} \\ \pm Y_{\frac{1}{2},jm} \end{pmatrix} \ , \qquad \bar{\Psi}^{\pm}_{jm} = \begin{pmatrix} Y^{*}_{\frac{1}{2},jm} \\ \pm Y^{*}_{-\frac{1}{2},jm} \end{pmatrix} \ . \tag{4.222}$$

In terms of the mode variables, the fermionic part of the quadratic action can be expressed as

$$\begin{aligned} S^{f}_{chiral} = & + i \sum_{\substack{j=2k+1/2 \\ k\geq 0}} \sum_{m=-j}^{j} \bar{\psi}^{+}_{jm} \Big[ j + \frac{1}{2} - \frac{q}{2} + ir\sigma \Big] \psi^{+}_{jm} \\ & - i \sum_{\substack{j=2k+3/2 \\ k\geq 0}} \sum_{m=-j}^{j} \bar{\psi}^{-}_{jm} \Big[ j + \frac{1}{2} + \frac{q}{2} - ir\sigma \Big] \bar{\psi}^{-}_{jm} \ . \end{aligned} \tag{4.223}$$

As a consequence, the determinant for the fermion modes equals to

$$\mathrm{Det}_{\psi} = \prod_{k\geq 0} \Big( 2k + 1 - \frac{q}{2} + ir\sigma \Big)^{4k+2} \Big( 2k + \frac{q}{2} - ir\sigma \Big)^{4k} \ . \tag{4.224}$$

One can easily generalize the above results for a chiral multiplet of weight $w$ under $G$ by the replacement $\sigma \to w \cdot \sigma$.

Combining these two expressions, we find that the one-loop contribution from a chiral multiplet in the representation $\mathbf{R}$ under the gauge group $G$ is

$$Z^{chiral}_{\text{1-loop}} = \frac{\mathrm{Det}_{\phi}}{\mathrm{Det}_{\psi}} = \prod_{w\in\mathbf{R}} \prod_{k\geq 0} \frac{2k + 1 - \frac{q}{2} + irw \cdot \sigma}{2k + \frac{q}{2} - irw \cdot \sigma} \ . \tag{4.225}$$

This can be regularized with Gamma function representation

$$\Gamma(a) = \lim_{n_{max}\to\infty} \frac{n_{max}!(n_{max})^{a}}{\prod_{n=0}^{n_{max}} (a+n)} \ , \tag{4.226}$$

where we should take care to introduce the UV cutoff $\Lambda$ via $r\Lambda \simeq 2k_{max}$ since $(2k + \cdots)/r$ are the physical eigenvalues. Then,

$$\begin{aligned} Z_{\text{1-loop}}^{chiral} &= \prod_{w\in\mathbf{R}} \lim_{k_{max}\to\infty} (k_{max})^{\frac{1}{2}-\frac{q}{2}+irw\cdot\sigma} \cdot \frac{\Gamma\left(\frac{q}{4} - irw\cdot\sigma/2\right)}{\Gamma\left(\frac{1}{2}-\frac{q}{4}+irw\cdot\sigma/2\right)} \\ &= \prod_{w\in\mathbf{R}} e^{\left[\frac{1-q}{2}+irw\cdot\sigma\right]\log(r\Lambda/2)} \cdot \frac{\Gamma\left(\frac{q}{4} - irw\cdot\sigma/2\right)}{\Gamma\left(\frac{1}{2}-\frac{q}{4}+irw\cdot\sigma/2\right)} \cdot \frac{\Gamma\left(-\frac{q}{4} + irw\cdot\sigma/2\right)}{\Gamma\left(-\frac{q}{4}+irw\cdot\sigma/2\right)} \\ &= \prod_{w\in\mathbf{R}} \frac{1}{2\sqrt{2\pi}} \cdot e^{\left[\frac{1-q}{2}+irw\cdot\sigma\right]\log(r\Lambda)} \frac{\Gamma\left(\frac{q}{4} - \frac{irw\cdot\sigma}{2}\right)\cdot\Gamma\left(-\frac{q}{4}+\frac{irw\cdot\sigma}{2}\right)}{\Gamma\left(-\frac{q}{2}+irw\cdot\sigma\right)} , \end{aligned} \tag{4.227}$$

where we used

$$\Gamma\left(\frac{1}{2}+x\right)\Gamma(x) = 2^{1-2x}\sqrt{\pi}\,\Gamma(2x) , \tag{4.228}$$

for the last equality. The exponential factor which diverges when $\Lambda \to \infty$ is understood to be one-loop running of the FI-parameter and appearance of central charge defined as $c \equiv 3(\sum_i(1-q_i) - d_G)$ when combined with vector multiplet contribution.

*Odd Holonomy* Let us now in turn consider the fluctuation near the second saddle point with nontrivial holonomy. At the odd holonomy fixed point, the boundary condition for charged field must be twisted by $e^{iw\cdot h} = \pm 1$, where $e^{ih\cdot H}$ is the $Z_2$ holonomy with unit-normalized Cartan generators $H$. The chiral fields can then be classified into two classes, with even charge $w_e$ and with odd charge $w_o$, respectively, depending on the above sign. For even ones, $w_e$, one-loop determinant is unchanged from the even holonomy case, so we focus on a chiral multiplet with odd charge $w_o$

$$\exp\left[i\int_C w_o\cdot A\right] = -1 . \tag{4.229}$$

Effectively, we impose the twisted projection condition on those carrying odd charges $w_o$ as

$$\begin{aligned}\phi(\pi-\theta,\pi+\varphi) &= -\,\phi(\theta,\varphi)\ ,\\ \psi_{\pm}(\pi-\theta,\pi+\varphi) &= +\,i\psi_{\mp}(\theta,\varphi)\ ,\\ \bar{\psi}_{\pm}(\pi-\theta,\pi+\varphi) &= -\,i\bar{\psi}_{\mp}(\theta,\varphi)\ ,\\ F(\pi-\theta,\pi+\varphi) &= -\,F(\theta,\varphi)\ ,\end{aligned} \tag{4.230}$$

without a background gauge field. Thus the spectral analysis is parallel to the previous one except the twisted projection picks exactly opposite eigenvalues, which were projected out under the original B-type parity action. Therefore, one obtains

$$\mathrm{Det}_\phi = \left[\prod_{k\geq 0}\left(2k+1+\frac{q}{2}-irw_o\cdot\sigma\right)^{4k+3}\prod_{k\geq 1}\left(2k-\frac{q}{2}+irw_o\cdot\sigma\right)^{4k-1}\right]\ , \tag{4.231}$$

for bosons, and

$$\mathrm{Det}_\psi = \prod_{k\geq 0}\left(2k-\frac{q}{2}+irw_o\cdot\sigma\right)^{4k}\left(2k+1+\frac{q}{2}-irw_o\cdot\sigma\right)^{4k+2}\ , \tag{4.232}$$

for fermions. Hence the one-loop determinant at this saddle point becomes

$$Z^{chiral}_{\text{1-loop}} = \prod_{w_o\in\mathbf{R}}\prod_{k\geq 0}\frac{2k+2-\frac{q}{2}+irw_o\cdot\sigma}{2k+1+\frac{q}{2}-irw_o\cdot\sigma}\ . \tag{4.233}$$

With the same procedure, we can further simplify this expression as

$$\begin{aligned} Z^{chiral}_{\text{1-loop}} &= \prod_{w_o\in\mathbf{R}}\lim_{k_{max}\to\infty}(k_{max})^{\frac{1}{2}-\frac{q}{2}+irw_o\cdot\sigma}\frac{\Gamma\left(\frac{1}{2}+\frac{q}{4}-\frac{irw_o\cdot\sigma}{2}\right)}{\Gamma\left(1-\frac{q}{4}+\frac{irw_o\cdot\sigma}{2}\right)} \\ &= \prod_{w_o\in\mathbf{R}} 2\sqrt{2\pi}\cdot e^{\left[\frac{1-q}{2}+irw_o\cdot\sigma\right]\log(r\Lambda)}\frac{\Gamma\left(\frac{q}{2}-irw_o\cdot\sigma\right)}{\Gamma\left(-\frac{q}{4}+\frac{irw_o\cdot\sigma}{2}\right)\Gamma\left(\frac{q}{4}-\frac{irw_o\cdot\sigma}{2}\right)}\cdot\frac{1}{-\frac{q}{2}+irw_0\cdot\sigma}\ . \end{aligned} \tag{4.234}$$

**Parity Accompanied by Flavor Rotations**

For theories with non-trivial flavor symmetry, we can enrich the $Z_2$ projection by combination with flavor rotations, i.e.,

$$\begin{aligned} \phi^i(x) &\rightarrow M^i{}_j \phi^j(x') \ , \\ \psi^i_\pm(x) &\rightarrow M^i{}_j \psi^j_\mp(x') \ , \end{aligned} \tag{4.235}$$

where $M^i{}_j$ is a flavor rotation which squares to the identity. Let us consider the simplest example where $M^i{}_j$ exchanges two chiral multiplets $\Phi^1(x) \leftrightarrow \Phi^2(x')$. The contribution of these modes to the 1-loop determinant is easily obtained, by noting that fluctuations of one of $\Phi^{1,2}$ is completely determined by that of the other in the opposite hemisphere. Hence, these two effectively contribute as one chiral multiplet without $Z_2$ projection, i.e., that of the full two-sphere partition function

$$\prod_{w\in\mathbf{R}} e^{[1-q+2irw\cdot\sigma]\log(r\Lambda)} \cdot \frac{\Gamma\left(\frac{q}{2} - irw\cdot\sigma\right)}{\Gamma\left(1-\frac{q}{2}+irw\cdot\sigma\right)} \ , \tag{4.236}$$

calculated in Ref. [8, 9].

All other $Z_2$ flavor transformations are generated by combination of the above rotation and a gauge transformation. For example, we can consider a projection of type $\Phi^1(x) \rightarrow -\Phi^1(x')$, when the superpotential respects such symmetry. The result of this sign flip is the same as in (4.233), so we find

$$\prod_{w\in\mathbf{R}} 2\sqrt{2\pi}\cdot e^{\left[\frac{1-q}{2}+irw\cdot\sigma\right]\log(r\Lambda)} \cdot \frac{\Gamma\left(\frac{q}{2}-irw\cdot\sigma\right)}{\Gamma\left(-\frac{q}{4}+\frac{irw\cdot\sigma}{2}\right)\Gamma\left(\frac{q}{4}-\frac{irw\cdot\sigma}{2}\right)} \cdot \frac{1}{-\frac{q}{2}+irw\cdot\sigma} \ . \tag{4.237}$$

These observations will be useful in the next section where we consider lower-dimensional Orientifold planes embedded as a hypersurface in the Calabi-Yau ambient space.

**Vector Multiplets**

Finally, we come to the vector multiplets. We follow the Fadeev-Popov method to deal with the gauge symmetry, and introduce ghost fields $c, \bar{c}$. Up to the quadratic order, the action around the saddle point is

$$S_{vector} = S^{b}_{vector} + S^{f}_{vector} + S^{FP}_{vector} \ , \tag{4.238}$$

where

$$\begin{aligned} S^{b}_{vec} &= \int \frac{1}{2}\mathrm{Tr}\Big[Da \wedge *Da - \big[\sigma, a\big] \wedge \big[\sigma, *a\big] + D\sigma_1 \wedge *D\sigma_1 - \big[\sigma, \sigma_1\big] \wedge \big[\sigma, *\sigma_1\big] \\ &\quad + \frac{1}{r^2}\sigma_1 \wedge *\sigma_1 + D\varphi \wedge *D\varphi + \frac{2}{r} Da \wedge \sigma_1 + iD\varphi \wedge \big[\sigma, *a\big] + i\big[\sigma, a\big] \wedge *D\varphi\Big] \ , \\ S^{f}_{vec} &= \int d^2x\sqrt{g}\ \frac{1}{2}\mathrm{Tr}\Big[\bar{\lambda}\gamma^3 \left(i\gamma^3\gamma^i D_i\lambda + \big[\sigma, \lambda\big]\right)\Big] \ , \\ S^{FP}_{vec} &= \int d^2x\sqrt{g}\ \mathrm{Tr}\Big[D_\mu\bar{c}D_\mu c + \frac{1}{2} f \wedge *f\Big] \ , \end{aligned} \tag{4.239}$$

with the gauge fixing functional

$$f = *D * a \ . \tag{4.240}$$

Here $a$ and $\varphi$ are the small fluctuation part of the gauge field and of the scalar field $\sigma_2$, respectively,

$$A = A_{\text{flat}} + a \ , \qquad \sigma_2 = \sigma + \varphi \ . \tag{4.241}$$

*Even Holonomy* When the holonomy is trivial, we impose the ordinary type B projection condition

$$\begin{aligned}
A(\pi-\theta,\pi+\varphi) &= +\,A(\theta,\varphi)\ ,\\
\sigma_1(\pi-\theta,\pi+\varphi) &= -\,\sigma_1(\theta,\varphi)\ ,\\
\sigma_2(\pi-\theta,\pi+\varphi) &= +\,\sigma_2(\theta,\varphi)\ ,\\
\lambda_\pm(\pi-\theta,\pi+\varphi) &= +\,i\lambda_\mp(\theta,\varphi)\ ,\\
\bar{\lambda}_\pm(\pi-\theta,\pi+\varphi) &= -\,i\bar{\lambda}_\mp(\theta,\varphi)\ ,\\
D(\pi-\theta,\pi+\varphi) &= +\,D(\theta,\varphi)\ .
\end{aligned} \tag{4.242}$$

First, decompose all the fluctuation fields into Cartan-Weyl basis, and then consider the off-diagonal modes carrying the charge $\alpha$, a root of $G$. In terms of the one-form and the scalar spherical harmonics $C^{\lambda}_{jm}$[7], $Y_{jm}$, one can expand the bosonic fluctuations $a^\alpha$, $\varphi^\alpha$, and $\sigma_1^\alpha$ as

$$\begin{aligned}
a &= \sum_{\substack{j=2k\\ k\geq 1}}\sum_{m=-j}^{j} a^1_{jm}C^1_{jm} + \sum_{\substack{j=2k+1\\ k\geq 0}}\sum_{m=-j}^{j} a^2_{jm}C^2_{jm}\ ,\\
\sigma_1 &= \sum_{\substack{j=2k+1\\ k\geq 0}}\sum_{m=-j}^{j} \sigma^1_{jm}Y_{jm}\ ,\\
\varphi &= \sum_{\substack{j=2k\\ k\geq 0}}\sum_{m=-j}^{j} \varphi_{jm}Y_{jm}\ ,
\end{aligned} \tag{4.243}$$

under the projection condition (4.242). From now on, the superscript $\alpha$ is suppressed unless it causes any confusion. The Laplacian operator $\mathcal{O}^{(1)}_b$ acting on

[7] Useful properties of $C^{\lambda}_{jm}$ are summarized in appendix C.

$(a^2_{jm}, \sigma^1_{jm})$ can be summarized into

$$\mathcal{O}_b^{(1)} \doteq \begin{pmatrix} j(j+1)+(\sigma\cdot\alpha)^2 & \sqrt{j(j+1)} \\ \sqrt{j(j+1)} & j(j+1)+(\sigma\cdot\alpha)^2+1 \end{pmatrix} , \tag{4.244}$$

with $j = 2k+1$ $(k \geq 0)$. The determinant of this operator is therefore,

$$\begin{aligned} \sqrt{\det \mathcal{O}_b^{(1)}} = &\prod_{k\geq 0} \Big[(2k+1)(2k+2)\Big]^{(4k+3)r_G} \\ &\times \prod_{\alpha\in\Delta_+}\prod_{k\geq 0}\Big[\big((2k+1)^2+(\alpha\cdot\sigma)^2\big)\big((2k+2)^2+(\alpha\cdot\sigma)^2\big)\Big]^{4k+3} , \end{aligned} \tag{4.245}$$

where $r_G$ is rank of the gauge group. The operator $\mathcal{O}_b^{(2)}$ acting on the modes $(a^1_{jm}, \varphi_{jm})$ with $j = 2k$ $(k \geq 1)$ can be read from (4.239),

$$\mathcal{O}_b^{(2)} \doteq \begin{pmatrix} j(j+1)+(\sigma\cdot\alpha)^2 & i\sqrt{j(j+1)}(\sigma\cdot\alpha) \\ -i\sqrt{j(j+1)}(\sigma\cdot\alpha) & j(j+1) \end{pmatrix} . \tag{4.246}$$

When $j = 0$, the operator has a vanishing eigenvalue that corresponds to the shift of the saddle point $\sigma_2 = \sigma$. The determinant of this operator is therefore

$$\sqrt{\det{}'\mathcal{O}_b^{(2)}} = \prod_{k=1}\Big[2k(2k+1)\Big]^{(4k+1)d_G} , \tag{4.247}$$

where $d_G$ is dimension of the gauge group $G$, and the prime in $\det'$ denotes the fact that the zero mode of $\sigma_2$ is removed. For the ghosts, we require the same projection condition as $\varphi, \bar{\varphi}$, and find

$$\det \mathcal{O}_{FP} = \prod_{k=1}\Big[2k(2k+1)\Big]^{d_G(4j+1)} , \tag{4.248}$$

which cancels with $\mathcal{O}_b^{(2)}$ determinant exactly. For fermions, the structure of determinants are essentially the same as that of the adjoint chiral multiplet with the twisted projection condition. Therefore, gaugino with root $\alpha$ contributes

$$\det \mathcal{O}_\lambda = \prod_{k\geq 0}\Big[(2k+1)(2k+2)\Big]^{r_G(4k+2)} \\ \times \prod_{\alpha\in\Delta_+}\prod_{k\geq 0}\Big[\left((2k+1)^2+(\alpha\cdot\sigma)^2\right)\left((2k+2)^2+(\alpha\cdot\sigma)^2\right)\Big]^{4k+2} . \quad (4.249)$$

Let us combine all these contributions from vector multiplets together. The Cartan part of the vector multiplets contributes,

$$\prod_{j=0}\left(\frac{2j+2}{2j+3}\right)^{r_G} = \left[\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(1)}\cdot e^{-\frac{1}{2}\log(r\Lambda/2)}\right]^{r_G} = \left(\frac{\pi}{2}\right)^{\frac{r_G}{2}} e^{-\frac{r_G}{2}\log(r\Lambda)} . \quad (4.250)$$

while the "off-diagonal part" regularize to

$$\begin{aligned} &\prod_{\alpha\in\Delta_+}\prod_{k=0}\frac{(2k)^2+(\alpha\cdot\sigma)^2}{(2k+1)^2+(\alpha\cdot\sigma)^2}\cdot\frac{1}{(\alpha\cdot\sigma)^2} \\ =\;& e^{-\frac{1}{2}(d_G-r_G)\log(r\Lambda/2)}\prod_{\alpha\in\Delta_+}\frac{\Gamma\left(\frac{1}{2}+\frac{i\alpha\cdot\sigma}{2}\right)\Gamma\left(\frac{1}{2}-\frac{i\alpha\cdot\sigma}{2}\right)}{4\cdot\Gamma\left(1+\frac{i\alpha\cdot\sigma}{2}\right)\Gamma\left(1-\frac{i\alpha\cdot\sigma}{2}\right)} \\ =\;& e^{-\frac{1}{2}(d_G-r_G)\log(r\Lambda/2)}\prod_{\alpha\in\Delta_+}\frac{2\pi\sin\left[\frac{\pi\alpha\cdot\sigma}{2}\right]}{\sin\pi\alpha\cdot\sigma}\cdot\frac{\sin\left[\frac{\pi\alpha\cdot\sigma}{2}\right]}{2\pi\alpha\cdot\sigma} \\ =\;& e^{-\frac{1}{2}(d_G-r_G)\log(r\Lambda)}\prod_{\alpha\in\Delta_+}\frac{1}{\alpha\cdot\sigma}\cdot\tan\left(\frac{\pi\alpha\cdot\sigma}{2}\right) . \end{aligned} \quad (4.251)$$

As the zero mode part contributes

$$\frac{1}{|W_G|}\int d^{r_G}\sigma\prod_{\alpha\cdot\sigma>0}(\alpha\cdot\sigma)^2 , \quad (4.252)$$

with the Vandermonde determinant and the Weyl factor, we obtain the even holonomy part of the partition function, where the vector multiplet contributions in the even holonomy sector can be displayed explicitly as

$$Z^{even} = \frac{1}{|W_G|}\int d^{r_G}\sigma \left(\frac{\pi}{2}\right)^{\frac{r_G}{2}} \cdot e^{-\frac{d_G}{2}\log(r\Lambda)} \tag{4.253}$$

$$\times \prod_{\alpha\in\Delta_+} \alpha\cdot\sigma \tan\left(\frac{\pi\alpha\cdot\sigma}{2}\right) \times \cdots , \tag{4.254}$$

where the ellipsis reminds us that for the GLSM partition function, we need to insert, multiplicatively, the 1-loop contributions from the chiral multiplets in the integrand.

*Odd Holonomy* At the odd holonomy fixed point, the boundary condition for the vector multiplet fluctuation must be twisted by $e^{i\alpha\cdot h} = \pm 1$, where, as before, $e^{ih\cdot H}$ is the $Z_2$ holonomy with the Cartan generators $H$. Thus, we only need to modify, in Eq. (4.253), as

$$\tan\left(\frac{\pi\alpha\cdot\sigma}{2}\right) \quad\rightarrow\quad \cot\left(\frac{\pi\alpha\cdot\sigma}{2}\right) , \tag{4.255}$$

for each and every root with $e^{i\alpha\cdot h} = -1$. So, splitting the positive root space $\Delta_+$ into the even part $\Delta_+^e$ and the odd part $\Delta_+^e$, relative to the holonomy $e^{ih\cdot H}$, we find that the odd holonomy sector contributes additively to the partition function

$$Z^{odd} = \frac{\eta}{|W_G|}\int d^{r_G}\sigma \left(\frac{\pi}{2}\right)^{\frac{r_G}{2}} \cdot e^{-\frac{d_G}{2}\log(r\Lambda)} \tag{4.256}$$

$$\times \prod_{\alpha_e\in\Delta_+^e} \alpha_e\cdot\sigma \tan\left(\frac{\pi\alpha_e\cdot\sigma}{2}\right) \prod_{\alpha_o\in\Delta_+^o} \alpha_o\cdot\sigma \cot\left(\frac{\pi\alpha_o\cdot\sigma}{2}\right) \times \cdots ,$$

where, again, the ellipsis in the integrand denotes multiplicative contributions from the chiral multiplet 1-loop determinants.

The numerical factor $\eta = \pm 1$ represents our ignorance regarding fermion determinants. As with any determinant computation involving fermions, the signs of various 1-loop factors are difficult to fix. Among such, $\eta$ which is the relative sign between the two additive contributions, from the even holonomy and the odd holonomy sectors, is an important physical quantity but is not accessible from the Coulomb-phase GLSM computation. One needs a different approach that the full partition function without worrying about the two holonomy sectors. For this reason, and also as a consistency check, we make a short excursion to the mirror LG computation for the Abelian GLSM, in next section, which will teach about how this sign distinguishes $O^-$ type and $O^+$ type Orientifold planes.

### 4.5.4 Landau-Ginzburg Model and Mirror Symmetry

Before we consider examples and the large volume limit, let us make a brief look at the mirror pair of the Abelian GLSM. In particular, we consider $U(1)$ theory with chiral multiplets $\Phi_a$ with gauge charges $Q_a$. As shown by Hori and Vafa [2], the mirror theory is a Landau-Ginzburg (LG) type with twisted chiral multiplet $Y_a$'s and the twisted superpotential $W(Y_a)$, generated by the vortex instantons. On $\mathbb{RP}^2$, the supersymmetric Lagrangian of a LG model with twisted chiral multiplets takes the following form

$$\mathcal{L} = \mathcal{L}_{twisted} + \mathcal{L}_W \ , \tag{4.257}$$

with

$$\mathcal{L}_{twisted} = D^\mu \bar{Y} D_\mu Y + i\bar{\chi}\gamma^m D_m \chi + \bar{G}G \ , \tag{4.258}$$

and the twisted superpotential terms,

$$\begin{aligned}\mathcal{L}_W = &+ \Big[ - iW'(Y)G - W''(Y)\bar{\chi}\gamma_- \chi + \frac{i}{r}W(Y)\Big] \\ &+ \Big[ - i\bar{W}'(\bar{Y})\bar{G} + \bar{W}''(\bar{Y})\bar{\chi}\gamma_+ \chi + \frac{i}{r}\bar{W}(\bar{Y})\Big] \ ,\end{aligned} \tag{4.259}$$

where $\gamma_\pm = \frac{1+\gamma^3}{2}$. One can show that the above Lagrangian is invariant under the supersymmetric variation rules given by

$$\begin{aligned}
\delta Y &= +\, i\bar{\epsilon}\gamma_-\chi - i\epsilon\gamma_+\bar{\chi} \ , \\
\delta \overline{Y} &= -\, i\bar{\epsilon}\gamma_+\chi + i\epsilon\gamma_-\bar{\chi} \ , \\
\delta \chi &= +\, \gamma^\mu\gamma_+\epsilon D_\mu Y - \gamma^\mu\gamma_-\epsilon D_\mu \overline{Y} - \gamma_+\epsilon\bar{G} - \gamma_-\epsilon G \ , \\
\delta \bar{\chi} &= +\, \gamma^\mu\gamma_+\bar{\epsilon} D_\mu \overline{Y} - \gamma^\mu\gamma_-\bar{\epsilon} D_\mu Y + \gamma_+\bar{\epsilon} G + \gamma_-\bar{\epsilon}\bar{G} \ , \\
\delta G &= -\, i\bar{\epsilon}\gamma^\mu\gamma_- D_\mu\chi + i\epsilon\gamma^\mu\gamma_+ D_\mu\bar{\chi} \ , \\
\delta \bar{G} &= -\, i\bar{\epsilon}\gamma^\mu\gamma_+ D_\mu\chi + i\epsilon\gamma^\mu\gamma_- D_\mu\bar{\chi} \ ,
\end{aligned} \tag{4.260}$$

where $\epsilon$ and $\bar{\epsilon}$ are the Killing spinors (4.196). The kinetic terms are again Q-exact [13, 134],

$$\mathcal{L}_{\text{twisted}} = \delta_\epsilon \delta_{\bar{\epsilon}} \left[ \frac{i}{r}\bar{Y}Y - i\bar{G}Y - i\bar{Y}G \right] \ . \tag{4.261}$$

Type B-parity action on the twisted chiral fields resembles the type A-parity on the chiral fields, naturally, which we first outline. One important fact, perhaps not too obvious immediately, is that the parity action which flips $Y$ to $\bar{Y}$ should be accompanied by a half-shift of the imaginary part, in order to preserve the action. Due to this, the fixed submanifolds are spanned by

$$Y = x + in\frac{\pi}{2} \ , \tag{4.262}$$

with $n = \pm 1$.

On this mirror side, the role of $\theta$ angle becomes more visible. $\theta = 0, \pi$ are the two allowed values, corresponding to the two types of Orientifolds, $O^\mp$. From the equation of motion for the vector multiplet, we learn allowed values of $n$'s have to be such that

$$\frac{1}{2}\sum_a Q_a n^a = \frac{\theta}{\pi} \mod Z_2 \ , \tag{4.263}$$

which restricts the sum over $n_a = \pm 1$ into two disjoint sets, depending on the value of $\theta$. Comparing the result to the GLSM computation, we learn that the difference between $O^{\mp}$ lies in the choice of the relative sign $\eta$ between the even holonomy contribution (4.253) and the odd holonomy contribution (4.256).

#### Parity on the Mirror

Under the type B-parity (4.198), one can show that the projection conditions are

$$Y(\pi-\theta,\pi+\varphi) = \bar{Y}(\theta,\varphi) + \text{constant} \ , \tag{4.264}$$

and

$$\begin{aligned} \chi_{\pm}(\pi-\theta,\pi+\varphi) &= +\, i\chi_{\mp}(\theta,\varphi) \ , \\ \bar{\chi}_{\pm}(\pi-\theta,\pi+\varphi) &= -\, i\bar{\chi}_{\mp}(\theta,\varphi) \ , \\ G(\pi-\theta,\pi+\varphi) &= +\, \bar{G}(\theta,\varphi) \end{aligned} \tag{4.265}$$

are consistent to the SUSY variation rules, for free theories. In order to fix the constant term in (4.264), we need to consider interactions such as twisted superpotential terms.

First, recall that the gauge multiplet can be written as a twisted chiral $\Sigma$, where

$$\begin{aligned} Y = \sigma_2 + i\sigma_1 \ , \qquad & G = D + i\left(F_{12} + \frac{\sigma_1}{r}\right) \ , \\ \chi = \lambda \ , \qquad & \bar{\chi} = \bar{\lambda} \ . \end{aligned} \tag{4.266}$$

As discussed above, we impose the projection conditions

$$\begin{aligned}\sigma_1(\pi-\theta,\pi+\varphi) &= -\,\sigma_1(\theta,\varphi)\ ,\\ \sigma_2(\pi-\theta,\pi+\varphi) &= +\,\sigma_2(\theta,\varphi)\ ,\end{aligned} \tag{4.267}$$

in order to introduce a minimal coupling of a charged chiral multiplet. It implies that

$$\Sigma(\pi-\theta,\pi+\varphi) = \bar{\Sigma}(\theta,\varphi)\ . \tag{4.268}$$

Note also that $\Sigma$ enters the tree-level twisted superpotential linearly as

$$W = -\frac{i}{2}\tau\Sigma\ , \tag{4.269}$$

with $\tau = i\xi + \frac{\theta}{2\pi}$, which leads to the FI coupling and 2d topological term

$$\mathcal{L}_W + \mathcal{L}_{\bar{W}} = -i\xi\left(D - \frac{\sigma_2}{r}\right) - i\frac{\theta}{2\pi}F_{12}\ . \tag{4.270}$$

Note that the complexified FI parameter is periodic $\tau \simeq \tau + n$ ($n \in \mathbb{Z}$). In order to make the interaction invariant under the type B Orientifold action, the parameter $\tau$ has to satisfy the following condition,

$$\tau + \bar{\tau} = n\ , \qquad n \in \mathbb{Z}\ . \tag{4.271}$$

In other words, the allowed value for the two-dimensional theta angle is either

$$\theta = 0 \ \text{ or } \ \pi\ . \tag{4.272}$$

Second, let us consider a simple example mirror to the $U(1)$ GLSM with $n$ chiral multiplets of gauge charge $Q_a$ where $a$ runs from 1 to $n$. The chiral multiplets also carry $U(1)_V$ $R$-charges $q^a$ so that the superpotential $\mathcal{W}$ carries the $R$-charge two.

The mirror Landau-Ginzburg model involves $n$ neutral twisted chiral multiplets $Y^a$ with period $2\pi i$. The dual description also comes with the following twisted superpotential

$$W = -\frac{1}{4\pi}\left[\Sigma\left(\sum_{a=1}^{n} Q_a Y^a + 2\pi i\tau\right) + \frac{i}{r}\sum_{a=1}^{n} e^{-Y^a}\right] . \tag{4.273}$$

At low-energy, the field-strength multiplet $\Sigma$ is effectively a Lagrange multiplier, leading to the constraint:

$$\sum_{a=1}^{n} Q_a Y^a = -2\pi i\tau \ . \tag{4.274}$$

To make these Toda-like interaction terms invariant under the type B-parity, one has to fix the constant piece in (4.264) by $i\pi$. That is,

$$Y(\pi-\theta,\pi+\varphi) = \bar{Y}(\theta,\varphi) + i\pi \ . \tag{4.275}$$

**Partition Function on $\mathbb{RP}^2$**

Choosing the kinetic terms $\mathcal{L}_{twisted}$ as Q-exact deformation terms, one can show that the path-integral localizes onto

$$Y = x + iy \ , \tag{4.276}$$

where $x$ and $y$ are real constants [13]. To obey the projection conditions (4.268) and (4.275), the supersymmetric saddle points are

$$\sigma_2 = \sigma \ , \qquad \sigma_1 = 0 \ , \qquad F_{12} = 0 \ , \tag{4.277}$$

and

$$Y^a = x^a + \frac{i\pi}{2} n^a \ , \tag{4.278}$$

where $x^a$ and $\sigma$ are real constants over $\mathbb{RP}^2$. Here $n^a = \pm 1$ obeying the constraint, for $\theta = 0$,

$$\frac{1}{2} \sum_a Q_a n^a = 2m \ , \qquad m \in \mathbb{Z} \ , \tag{4.279}$$

and for $\theta = \pi$,

$$\frac{1}{2} \sum_a Q_a n^a = 2m + 1 \ , \qquad m \in \mathbb{Z} \ , \tag{4.280}$$

obeying the constraint

$$\sum_a Q_a Y^a = -2\pi i \tau \ . \tag{4.281}$$

It is easy to show that one-loop determinants around the above supersymmetric saddle points are trivial in a sense that they are independent of $\sigma$ and $x^a$. One can show that the partition function of the mirror LG model with the twisted superpotential (4.273) on $\mathbb{RP}^2$ reduces to an ordinary contour integral,[8]

[8] We used for the last equality an integral formula

$$\int_{-\infty}^{\infty} dx \ e^{ipx} \cos\left[e^{-x} + z\right] = \cos\left[\frac{i\pi p}{2} - z\right] \Gamma[-ip] \ , \ \text{if} \ -1 < \mathfrak{Re}[ip] < 0 \ .$$

$$
\begin{aligned}
Z_{\rm LG} \simeq & \int_{-\infty}^{\infty} d\sigma \prod_a \left[\int_{-\infty}^{\infty} dx^a e^{-\frac{q_a}{2}x^a}\right] \sum_{n^a=\pm 1} \frac{1}{2}\left(1 \pm e^{i\pi Q_a n^a/2}\right) \cdot e^{ir\sigma(Q_a x^a - 2\pi\xi)} \cdot e^{ie^{-x^a}\sin(\pi n_a/2)} \\
= & \int_{-\infty}^{\infty} d\sigma e^{-2i\pi r\sigma\xi} \prod_a \left\{ \int_{-\infty}^{\infty} dx^a\, e^{-\frac{q_a}{2}x^a} e^{ir\sigma Q_a x^a} \left(\cos\left[e^{-x^a}\right] \pm \cos\left[\frac{\pi}{2}Q_a + e^{-x^a}\right]\right) \right\} \\
= & \int_{-\infty}^{\infty} d\sigma e^{-2i\pi r\sigma\xi} \left\{ \prod_a \cos\left[\frac{\pi}{2}\left(\frac{q_a}{2} - irQ_a\sigma\right)\right] \Gamma\left[\frac{q_a}{2} - irQ_a\sigma\right] \right. \\
& \left. \pm \prod_b \cos\left[\frac{\pi}{2}\left(\frac{q_b}{2} - irQ_b\sigma\right) - \frac{\pi}{2}Q_b\right] \Gamma\left[\frac{q_b}{2} - irQ_b\sigma\right] \right\} ,
\end{aligned}
\tag{4.282}
$$

where "$\simeq$" symbol in the first line reflects our ignorance of the overall numerical normalization of the integration measure. Here the factors $e^{-\frac{q_a}{2}x^a}$ reflect the important fact that the proper variables describing the mirror LG model are $X^a = e^{-\frac{q_a}{2}Y^a}$ rather than $Y^a$ [13]. Below, we compare to the GLSM side up to this normalization issue. The signs $\pm$ are for $\theta = 0$ and $\theta = \pi$ respectively.

The parity projection that leads to Eq. (4.282) assumes no specific flavor symmetry in the original GLSM, and thus must be the mirror of the spacetime-filling case of section 3.2. In the trivial holonomy sector, we start with the last line of Eq. (4.227) and use the identities

$$
\Gamma\left(\frac{1}{2}+x\right)\Gamma(x) = 2^{1-2x}\sqrt{\pi}\Gamma(2x) , \qquad \Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin \pi x} , \tag{4.283}
$$

to massage the one-loop determinant into

$$
Z^{\rm trivial}_{\text{1-loop}} = \Gamma\left[\frac{q}{2} - iQ\sigma\right] \cos\left[\frac{\pi}{2}\left(\frac{q}{2} - iQ\sigma\right)\right] \times \sqrt{\frac{2}{\pi}} e^{\left[\frac{1-q}{2}+iQ\sigma\right]\log(r\Lambda)} . \tag{4.284}
$$

In the nontrivial holonomy, a chiral multiplet carrying the even charge $Q_a = Q_e$, the same result holds,

$$Z^{\text{nontrivial}}_{\text{1-loop},Q_e} = \Gamma\left[\frac{q}{2} - iQ_e\sigma\right]\cos\left[\frac{\pi}{2}\left(\frac{q}{2} - iQ_e\sigma\right)\right] \times \sqrt{\frac{2}{\pi}} e^{\left[\frac{1-q}{2} + iQ\sigma\right]\log(r\Lambda)} , \quad (4.285)$$

while for the odd charge, $Q_a = Q_o$, the partition function becomes

$$\begin{aligned} Z^{\text{nontrivial}}_{\text{1-loop},Q_o} &= \prod_{k\geq 0} \frac{2k + 2 - \frac{q}{2} + iQ_o\sigma}{2k + 1 - \frac{q}{2} + iQ_o\sigma} \\ &= \frac{\Gamma\left(\frac{1}{2} + \frac{q}{4} - \frac{iQ_o\sigma}{2}\right)}{\Gamma\left(1 - \frac{q}{4} + \frac{iQ_o\sigma}{2}\right)} \times e^{\left[\frac{1-q}{2} + iQ_o\sigma\right]\log(r\Lambda/2)} \\ &= \Gamma\left[\frac{q}{2} - iQ_o\sigma\right]\sin\left[\frac{\pi}{2}\left(\frac{q}{2} - iQ_o\sigma\right)\right] \times \sqrt{\frac{2}{\pi}} e^{\left[\frac{1-q}{2} + iQ\sigma\right]\log(r\Lambda)} . \quad (4.286) \end{aligned}$$

Thus one can conclude that the first term in the final expression (4.282) of the LG partition function corresponds to the partition function of GLSM with the even holonomy, while the second term corresponds to the partition function with the odd holonomy.

After interpreting the exponentiated log piece as the renormalization of $\xi$, we learn two additional facts. First, the common overall normalization $\sqrt{2/\pi}$ should be incorporated into the measure on the mirror LG side. Second, an additional relative sign $\eta \equiv \pm \prod_a (-1)^{[Q_a/2]}$ (for $\theta = 0, \pi$, respectively) should sit between the trivial and the nontrivial holonomy contributions in the GLSM side, and distinguishes $O^-$ type and $O^+$ type Orientifold planes.[9]

### 4.5.5 Orientifolds in Calabi-Yau Hypersurface

In this section, we consider the Orientifolds for a prototype Calabi-Yau manifold $\mathcal{X}$, i.e., a degree $N$ hypersurface of $CP^{N-1}$. At the level of GLSM, the chiral field

[9] Recall that $\tilde{O}^{\pm}$ type Orientifolds involve turning on discrete RR-flux [135], and thus are not accessible from GLSM.

contents are

$$\begin{array}{c|cc} & U(1)_G & U(1)_V \\ \hline X_{i=1,\cdots N} & 1 & q \\ P & -N & 2-Nq \end{array} \tag{4.287}$$

where we displayed the gauge and the vector $R$-charges. As usual, the superpotential takes the form $P \cdot G_N(X)$ with degree $N$ homogeneous polynomial $G_N$. For simplicity, we will call $\epsilon = q/2 - ir\sigma$ below, and assume $N$ odd. For $N =$ even, the $P$ multiplet contributions from even and odd holonomy are exchanged. The number $q$ is in principle arbitrary as it can be shifted by mixing $U(1)_G$ and $U(1)_V$, but we restrict it to be in the range $0 < q < 2/N$ [13].

The main goal of this section is to extract the large volume expressions for the central charges of Orientifold planes. Traditionally, the latter were expressed in terms of the $\sqrt{\mathcal{L}}$ class, but just as with D-brane central charge, we will see that $\hat{\Gamma}_c$ class enters and corrects the expression. $\hat{\Gamma}_c$ is a multiplicative class associated with the function [14, 129–132]

$$\Gamma\left(1+\frac{x}{2\pi i}\right) \ , \tag{4.288}$$

so that, for any holomorphic bundle $\mathcal{F}$, an important identity

$$\hat{\Gamma}_c(\mathcal{F})\hat{\Gamma}_c(-\mathcal{F}) = \mathcal{A}(\mathcal{F}) \tag{4.289}$$

holds. In terms of the Chern characters, it can be expanded as

$$\hat{\Gamma}_c(\mathcal{F}) \;=\; \exp\left[\frac{i\gamma}{2\pi}ch_1(\mathcal{F}) + \sum_{k\geq 2}\left(\frac{i}{2\pi}\right)^k (k-1)!\zeta(k)ch_k(\mathcal{F})\right] \ , \tag{4.290}$$

where $\gamma = 0.577...$ is the Euler-Mascheroni constant, and $\zeta(k)$ is the Riemann zeta function.

The results from this hypersurface examples suggest that, for a general Orientifold plane that wraps a cycle $\mathcal{M}$ in the Calabi-Yau $\mathcal{X}$, with the tangent bundle $\mathcal{TM}$ and the normal bundle $\mathcal{NM}$ with respect to $\mathcal{X}$, we must correct the characteristic class that appear in the central charge as

$$\frac{\sqrt{\mathcal{L}(\mathcal{TM}/4)}}{\sqrt{\mathcal{L}(\mathcal{NM}/4)}} \quad \rightarrow \quad \frac{\mathcal{A}(\mathcal{TM}/2)}{\hat{\Gamma}_c(-\mathcal{TM})} \wedge \frac{\hat{\Gamma}_c(\mathcal{NM})}{\mathcal{A}(\mathcal{NM}/2)} \,. \tag{4.291}$$

We devote the rest of this section to derivation of this, by isolating the perturbative contributions for Orientifolds wrapping (partially) Calabi-Yau hypersurfaces in $\mathbb{CP}^{N-1}$.

**Spacetime-Filling Orientifolds**

First, let us consider the case where the Orientifold plane wraps $\mathcal{X}$ entirely, i.e., no flavor symmetry action is mixed with the B-parity projection. With the classical contribution

$$Z_{classical} = e^{-i2\pi r\xi\sigma} = e^{-2\pi\xi(q/2-\epsilon)}, \tag{4.292}$$

we find

$$\begin{aligned} Z_{\mathbb{RP}^2} \;=\; \int_{q/2-i\infty}^{q/2+i\infty} \frac{d\epsilon}{2\pi i} & \left(\beta_1 \cdot e^{2\pi\xi\epsilon}\left[\frac{\Gamma\left(\frac{\epsilon}{2}\right)\Gamma\left(-\frac{\epsilon}{2}\right)}{\Gamma(-\epsilon)}\right]^N \cdot \left[\frac{\Gamma\left(\frac{1-N\epsilon}{2}\right)\Gamma\left(-\frac{1-N\epsilon}{2}\right)}{\Gamma(-1+N\epsilon)}\right]\right. \\ & \left. +\eta\cdot\beta_2\cdot e^{2\pi\xi\epsilon}\left[\frac{\Gamma(\epsilon)/\epsilon}{\Gamma\left(\frac{\epsilon}{2}\right)\Gamma\left(-\frac{\epsilon}{2}\right)}\right]^N \cdot \left[\frac{\Gamma(1-N\epsilon)/(1-N\epsilon)}{\Gamma\left(\frac{1-N\epsilon}{2}\right)\Gamma\left(-\frac{1-N\epsilon}{2}\right)}\right]\right), \end{aligned} \tag{4.293}$$

where the constants $\beta_{1,2}$ are

$$\begin{aligned} \beta_1 &= e^{-\pi\xi q}\cdot(2\pi)^{-N/2+1}\cdot 2^{-(N+2)}\cdot e^{\frac{c}{6}\cdot\log(r\Lambda)}\,, \\ \beta_2 &= e^{-\pi\xi q}\cdot(2\pi)^{N/2+2}\cdot 2^{N}\cdot e^{\frac{c}{6}\log(r\Lambda)}\,, \end{aligned} \tag{4.294}$$

and the sign $\eta = \pm\prod_a(-1)^{[Q_a/2]}$ (for $\theta = 0, \pi$, respectively) chooses either $O^-$ type or $O^+$ type Orientifold. The two lines are, respectively, contributions from the even and the odd holonomy sector. Another common factor in $\beta_{1,2}$, $e^{\frac{c}{6}\log(r\Lambda)}$, renormalizes the partition function. Because $\mathcal{X}$ is Calabi-Yau, $\xi$ is not renormalized but the partition function itself is multiplicatively renormalized with the exponent $c/6 = (N-2)/2$ for this model.

The first factor in $\beta_{1,2}$, i.e., $e^{-\pi\xi q}$, with an explicit dependence on the ambiguous $R$-charge, should be in principle removable by shift of the $R$-charges by the gauge charges; $q \to q+\delta$ for any $\delta$ is such a shift for the present model. This is however easier said than done. For ratios of correlators, as in computation of the Zamolodchikov metric, the invariance is automatic. For the central charges which do depend on the overall normalization and thus on the normalization of the measure, it is not completely clear how this unphysical dependence is removed. Below, we choose to set $q \to 0^+$ to satisfy the charge integrality condition, following Ref. [14], and thus suppress this exponential factor. (Integral converges only when $q$ is positive real [13].)

When $\xi > 0$, the GLSM flows to the geometric phase in IR and we should close the contour to the left infinity. For the even holonomy sector, the relevant poles are those of $\Gamma(\epsilon/2)$ at $\epsilon = -2k$ $(k = 0,1,2,\cdots)$. For the odd holonomy sector, the relevant poles are those of $\Gamma(\epsilon)/\Gamma(\epsilon/2)$ at $\epsilon = -2k+1$ $(k = 0,1,2,\cdots)$. Poles of other factors either cancel out among themselves or are located outside of the contour. Of these, poles at $\epsilon < 0$ capture the world-sheet instanton contributions, which are suppressed exponentially in the large volume limit $\xi \gg 1$.

The perturbative part of the partition function, appropriate for the large volume limit, comes entirely from the pole at $\epsilon = 0$. With (4.293), therefore, only the even holonomy sector contributes, giving us

$$Z^{pert.}_{\mathbb{RP}^2} = \beta_1 \oint_{\epsilon=0} \frac{d\epsilon}{2\pi i} \; e^{2\pi\xi\epsilon} \left[\frac{\Gamma\left(\frac{\epsilon}{2}\right)\Gamma\left(-\frac{\epsilon}{2}\right)}{\Gamma(-\epsilon)}\right]^N \cdot \left[\frac{\Gamma\left(\frac{1-N\epsilon}{2}\right)\Gamma\left(-\frac{1-N\epsilon}{2}\right)}{\Gamma(-1+N\epsilon)}\right] \tag{4.295}$$

We first invoke the identity

$$\Gamma\left(\frac{1}{2}+x\right)\Gamma(x) = 2^{1-2x}\sqrt{\pi}\,\Gamma(2x) \ . \tag{4.296}$$

to rewrite this as

$$\begin{aligned} Z^{pert.}_{\mathbb{RP}^2} &= 8\pi\beta_1 \oint_{\epsilon=0} \frac{d\epsilon}{2\pi i}\, e^{2\pi\xi\epsilon} \left[\frac{\Gamma\left(\frac{\epsilon}{2}\right)\Gamma\left(-\frac{\epsilon}{2}\right)}{\Gamma(-\epsilon)}\right]^N \Bigg/ \left[\frac{\Gamma\left(\frac{N\epsilon}{2}\right)\Gamma\left(-\frac{N\epsilon}{2}\right)}{\Gamma(-N\epsilon)}\right] \qquad (4.297) \\ &= 8\pi\beta_1 \cdot 2^{2(N-1)} \oint_{\epsilon=0} \frac{d\epsilon}{2\pi i}\, e^{2\pi\xi\epsilon} \frac{N}{\epsilon^{N-1}} \\ &\qquad \times \left[\frac{\Gamma\left(1+\frac{\epsilon}{2}\right)\Gamma\left(1-\frac{\epsilon}{2}\right)}{\Gamma(1-\epsilon)}\right]^N \Bigg/ \left[\frac{\Gamma\left(1+\frac{N\epsilon}{2}\right)\Gamma\left(1-\frac{N\epsilon}{2}\right)}{\Gamma(1-N\epsilon)}\right] \ . \end{aligned}$$

This can be further rewritten as an integral over $\mathcal{X}$, with $H$ the hyperplane class of $\mathbb{CP}^{N-1}$,

$$Z^{pert.}_{\mathbb{RP}^2} = C_0 \int_{\mathcal{X}} e^{-i\xi H} \left[\frac{\Gamma\left(1+\frac{H}{4\pi i}\right)\Gamma\left(1-\frac{H}{4\pi i}\right)}{\Gamma(1-\frac{H}{2\pi i})}\right]^N \Bigg/ \left[\frac{\Gamma\left(1+\frac{NH}{4\pi i}\right)\Gamma\left(1-\frac{NH}{4\pi i}\right)}{\Gamma(1-\frac{NH}{2\pi i})}\right] \tag{4.298}$$

with $C_0 = i^{N-2}(2\pi)^{N/2}2^{N-2}(\Lambda r)^{c/6}$. We used $\int_{\mathcal{X}} H^{N-2} = N\int_{\mathbb{CP}^{N-1}} H^{N-1} = N$.

Since $\mathcal{X}$ is a Calabi-Yau hypersurface embedded in $\mathbb{CP}^{N-1}$, we may also write

$$\hat{\Gamma}_c(\mathcal{T}\mathcal{X}) = \frac{\hat{\Gamma}_c(\mathcal{T}\mathbb{CP}^{N-1})}{\hat{\Gamma}_c(\mathcal{N}\mathcal{X})} \ , \tag{4.299}$$

so that

$$\begin{aligned} Z^{pert.}_{\mathbb{RP}^2} &= C_0 \int_{\mathcal{X}} e^{-iJ}\, \frac{\hat{\Gamma}_c\left(\frac{\mathcal{T}\mathcal{X}}{2}\right)\hat{\Gamma}_c\left(-\frac{\mathcal{T}\mathcal{X}}{2}\right)}{\hat{\Gamma}_c(-\mathcal{T}\mathcal{X})} \\ &= C_0 \int_{\mathcal{X}} e^{-iJ}\, \frac{\mathcal{A}\left(\frac{\mathcal{T}\mathcal{X}}{2}\right)}{\hat{\Gamma}_c(-\mathcal{T}\mathcal{X})} \ , \qquad (4.300) \end{aligned}$$

where $\mathcal{A}$ is the $\hat{A}$ class. This shows that in the large volume limit, the conventional overlap amplitude between RR-ground state and a crosscap state (see e.g., [133]) are corrected by replacing

$$\sqrt{\mathcal{L}(\mathcal{T}\mathcal{X}/4)} \quad \rightarrow \quad \frac{\mathcal{A}(\mathcal{T}\mathcal{X}/2)}{\hat{\Gamma}_c(-\mathcal{T}\mathcal{X})}\,. \tag{4.301}$$

In section 6, we will come back to this expression and explore the consequences.

**Orientifolds with a Normal Bundle**

Lower dimensional Orientifold planes, from B-parity projection, may wrap a holomorphically embedded surface $\mathcal{M}$ in the ambient Calabi-Yau $\mathcal{X}$, if $\mathcal{X}$ admits $Z_2$ discrete symmetries. At the level of GLSM, this is achieved by combining the parity projection with such a flavor symmetry, as we considered in section 4.5.3.

For example, the simplest such Calabi-Yau has a superpotential $P \cdot G_N = P \cdot \sum_{a=1}^N X_i^N$ which is invariant under exchange of $X$'s among themselves. Exchanging a pair of chiral fields $X^1 \leftrightarrow X^2$ gives rise to a fixed locus defined by $X^1 + X^2 = 0$, a complex co-dimension one hypersurface as well as a complex co-dimension $(N-2)$ subspace, i.e., a point at $X^3 = \cdots = X^N = 0$. We can do the similar analysis for the symmetry exchanging $X^1 \leftrightarrow X^2$ and $X^3 \leftrightarrow X^4$ simultaneously. This action gives complex co-dimension 2 fixed locus defined as $(X^1, \cdots X^N) = (X, X, Y, Y, X^5, \cdots, X^N)$, and co-dimension $(N-3)$ fixed locus, $(X^1, \cdots X^N) = (X, -X, Y, -Y, 0, \cdots, 0)$. For the quintic, both of these correspond to $O5$ planes. These results are summarized in the following table [133].

| | |
|---|---|
| $(X_1, X_2, X_3, X_4 X_5) \rightarrow (X_1, X_2, X_3, X_4 X_5)$ | $O9$ (spacetime filling) |
| $(X_1, X_2, X_3, X_4 X_5) \rightarrow (X_2, X_1, X_3, X_4 X_5)$ | $O7$ at $(X, X, X_3, X_4, X_5)$<br>$O3$ at $(X, -X, 0, 0, 0)$ |
| $(X_1, X_2, X_3, X_4 X_5) \rightarrow (X_2, X_1, X_4, X_3 X_5)$ | $O5$ at $(X, X, Y, Y, X_5)$<br>$O5$ at $(X, -X, Y, -Y, 0)$ |

(4.302)

As this shows, we generically end up with more than one Orientifold planes, given a parity projection. The central charges must be all present in the $\mathbb{RP}^2$ partition function, so the latter must be in general composed of more than one additive terms. What allows this is the holonomy sectors we encountered in section 3. For a GLSM gauge group $U(n)$, for example, one has $n+1$ such distinct holonomy sectors, and can accommodate several Orientifold planes. For the current example of $U(1)$ GLSM, we have exactly two such holonomy sectors, and thus up to two Orientifolds planes.[10]

In the end, our examples below, combined with the spacetime-filling case above, will suggest a universal formula for the large volume central charge

$$\begin{aligned} Z^{pert.}_{\mathbb{RP}_2} &= C_{-s} \int_{\mathcal{X}} e^{-iJ}\, \frac{\mathcal{A}(\mathcal{TM}/2)}{\hat{\Gamma}_c(-\mathcal{TM})} \wedge \frac{\hat{\Gamma}_c(\mathcal{NM})}{\mathcal{A}(\mathcal{NM}/2)} \wedge e(\mathcal{NM}) \\ &= C_{-s} \int_{\mathcal{M}} e^{-iJ} \wedge \frac{\mathcal{A}(\mathcal{TM}/2)}{\hat{\Gamma}_c(-\mathcal{TM})} \wedge \frac{\hat{\Gamma}_c(\mathcal{NM})}{\mathcal{A}(\mathcal{NM}/2)}\ , \end{aligned} \tag{4.303}$$

for an Orientifold plane $\mathcal{M}$ of real co-dimension $2s$ in a Calabi-Yau $d$-fold $X$, with $C_{-s} = i^{d-s} 2^{d-2s}\ (2\pi)^{(d+2)/2}\ (r\Lambda)^{c/6}$.

*Orientifold Planes of Complex Co-Dimensions* 1 *&* $N-2$ Let us consider the projection involving $X^1 \leftrightarrow X^2$. As the table above shows, this produces two different fixed planes; An hyperplane with $X^1 = X^2$ and an isolated point at $X^3 = \cdots = X^N = 0$. Thus, we expect to recover additive contributions from these two planes, for which existence of the two holonomy sectors is crucial.

[10] For the spacetime-filling case of section 5.1, only even sector contributed to the large-volume limit, and there was only one type of Orientifold plane. However, the odd holonomy piece is still important in the following sense: Thanks to the $U(1)$ gauge symmetry of GLSM, one can alternatively project with $X \to -X$ and $P \to (-1)^N P$ without changing the theory. However, this flips the even and the odd holonomy sector precisely, which implies that the large-volume central charge of the spacetime-filling Orientifold planes resides in the odd holonomy sector instead.

As we are considering the ambient Calabi-Yau $\mathcal{X}$ as a hypersurface embedded in $\mathbb{CP}^{N-1}$, the results of section 4.5.3 reads

$$(2\pi)^{-N/2+2}\, 2^{-N}\, (r\Lambda)^{c/6}\, \mathrm{res}_{\epsilon=0} \frac{\Gamma(\epsilon)}{\Gamma\left(1-\epsilon\right)} \cdot \left[\frac{\Gamma\left(\frac{\epsilon}{2}\right)\Gamma\left(-\frac{\epsilon}{2}\right)}{\Gamma(-\epsilon)}\right]^{N-2} \cdot \left[\frac{\Gamma\left(\frac{1-N\epsilon}{2}\right)\Gamma\left(-\frac{1-N\epsilon}{2}\right)}{\Gamma(-1+N\epsilon)}\right], \tag{4.304}$$

from the even holonomy sector,

$$(2\pi)^{N/2+1}\, 2^{N-2}\, (r\Lambda)^{c/6}\, \mathrm{res}_{\epsilon=0} \frac{\Gamma(\epsilon)}{\Gamma\left(1-\epsilon\right)} \cdot \left[\frac{\Gamma(\epsilon)/\epsilon}{\Gamma\left(\frac{\epsilon}{2}\right)\Gamma\left(-\frac{\epsilon}{2}\right)}\right]^{N-2} \cdot \left[\frac{\Gamma(1-N\epsilon)/(1-N\epsilon)}{\Gamma\left(\frac{1-N\epsilon}{2}\right)\Gamma\left(-\frac{1-N\epsilon}{2}\right)}\right], \tag{4.305}$$

from the odd holonomy sector. Note that, for this case, both sectors contribute to the residue at $\epsilon = 0$.

First, let us consider the even holonomy sector contribution. With (4.296), we may write (4.304) as

$$\begin{aligned}
&-(2\pi)^{-N/2+3}\, 2^{-N+2}\, (r\Lambda)^{c/6}\, \mathrm{res}_{\epsilon=0}\, \frac{1}{\epsilon} \cdot \frac{\Gamma(\epsilon)}{\Gamma\left(\frac{\epsilon}{2}\right)\Gamma\left(-\frac{\epsilon}{2}\right)} \\
&\qquad \times \left[\frac{\Gamma\left(\frac{\epsilon}{2}\right)\Gamma\left(-\frac{\epsilon}{2}\right)}{\Gamma(-\epsilon)}\right]^{N-1} \frac{\Gamma(-N\epsilon)}{\Gamma\left(\frac{N\epsilon}{2}\right)\Gamma\left(-\frac{N\epsilon}{2}\right)} \\
=\;& (2\pi)^{-N/2+3}\, 2^{N-4}\, (r\Lambda)^{c/6}\, \mathrm{res}_{\epsilon=0}\, \frac{N}{\epsilon^{N-2}} \cdot \frac{\Gamma(1+\epsilon)}{\Gamma\left(1+\frac{\epsilon}{2}\right)\Gamma\left(1-\frac{\epsilon}{2}\right)} \\
&\qquad \times \left[\frac{\Gamma\left(1+\frac{\epsilon}{2}\right)\Gamma\left(1-\frac{\epsilon}{2}\right)}{\Gamma(1-\epsilon)}\right]^{N-1} \frac{\Gamma(1-N\epsilon)}{\Gamma\left(1+\frac{N\epsilon}{2}\right)\Gamma\left(1-\frac{N\epsilon}{2}\right)}\,.
\end{aligned} \tag{4.306}$$

Expressing the residue integral at $\epsilon = 0$ via an integral over $CX$ with the hyperplane class $H$, we find

$$\begin{aligned}
Z^{pert.,\, even}_{\mathbb{RP}_2} \;=\;& C_{-1} \int_{\mathcal{X}} e^{-i\xi H} \wedge H \wedge \left[\frac{\Gamma(1+\frac{H}{2\pi i})}{\Gamma\left(1+\frac{H}{4\pi i}\right)\Gamma\left(1-\frac{H}{4\pi i}\right)}\right] \\
&\wedge \left[\frac{\Gamma\left(1+\frac{H}{4\pi i}\right)\Gamma\left(1-\frac{H}{4\pi i}\right)}{\Gamma(1-\frac{H}{2\pi i})}\right]^{N-1} \Bigg/ \left[\frac{\Gamma\left(1+\frac{NH}{4\pi i}\right)\Gamma\left(1-\frac{NH}{4\pi i}\right)}{\Gamma(1-\frac{NH}{2\pi i})}\right],
\end{aligned} \tag{4.307}$$

with $C_{-1} = i^{N-3}(2\pi)^{N/2}2^{N-4}(r\Lambda)^{c/6}$. Note that, again in terms of the $\mathcal{A}$ and $\hat{\Gamma}_c$ classes, this formula can be organized as

$$Z^{pert.,\ even}_{\mathbb{RP}_2} = C_{-1}\int_{\mathcal{M}_{-1}} e^{-iJ} \wedge \frac{\mathcal{A}(\mathcal{TM}_{-1}/2)}{\hat{\Gamma}_c(-\mathcal{TM}_{-1})} \wedge \frac{\hat{\Gamma}_c(\mathcal{NM}_{-1})}{\mathcal{A}(\mathcal{NM}_{-1}/2)}\ , \tag{4.308}$$

where $\mathcal{M}_{-1}$ denotes for a complex co-dimension 1 fixed locus, parameterized by $(X^1,\cdots X^N) = (X, X, X^3,\cdots X^N)$.

Contribution from the odd holonomy sector can be similarly written as

$$\begin{aligned}(-1)^{N-1}2^{-N+2}\ (2\pi)^{N/2}\ (r\Lambda)^{c/6}\ \mathrm{res}_{\epsilon=0}\frac{1}{\epsilon}\cdot\frac{\Gamma\left(1+\frac{\epsilon}{2}\right)\Gamma\left(1-\frac{\epsilon}{2}\right)}{\Gamma(1-\epsilon)} \\ \times\left[\frac{\Gamma(1+\epsilon)}{\Gamma\left(1+\frac{\epsilon}{2}\right)\Gamma\left(1-\frac{\epsilon}{2}\right)}\right]^{N-1}\cdot\frac{\Gamma\left(1+\frac{N\epsilon}{2}\right)\Gamma\left(1-\frac{N\epsilon}{2}\right)}{\Gamma\left(1+N\epsilon\right)}\ ,\end{aligned} \tag{4.309}$$

which is equivalent to

$$\begin{aligned}Z^{pert.,\ odd}_{\mathbb{RP}_2} &= C_{-(N-2)}\int_{\mathcal{X}} e^{-iJ}\wedge\frac{H^{N-2}}{N}\wedge\frac{\Gamma\left(1+\frac{H}{4\pi i}\right)\Gamma\left(1-\frac{H}{4\pi i}\right)}{\Gamma(1-\frac{H}{2\pi i})} \\ &\wedge\ \left[\frac{\Gamma(1+\frac{H}{2\pi i})}{\Gamma\left(1+\frac{H}{4\pi i}\right)\Gamma\left(1-\frac{H}{4\pi i}\right)}\right]^{N-1}\Bigg/\left[\frac{\Gamma\left(1+\frac{NH}{2\pi i}\right)}{\Gamma\left(1+\frac{NH}{4\pi i}\right)\Gamma\left(1-\frac{NH}{4\pi i}\right)}\right]\ ,\end{aligned} \tag{4.310}$$

where $C_{-(N-2)} = (-1)^{N-1}2^{-N+2}\ (2\pi)^{N/2}\ (r\Lambda)^{c/6}$. Again we may rewrite this as an integral

$$Z^{pert.,\ odd}_{\mathbb{RP}_2} = C_{-(N-2)}\int_{\mathcal{M}_{-(N-2)}} e^{-iJ}\ \frac{\mathcal{A}(\mathcal{TM}_{-(N-2)}/2)}{\hat{\Gamma}_c(-\mathcal{TM}_{-(N-2)})} \wedge \frac{\hat{\Gamma}_c(\mathcal{NM}_{-(N-2)})}{\mathcal{A}(\mathcal{NM}_{-(N-2)}/2)}\ , \tag{4.311}$$

over $\mathcal{M}_{N-2}$ which, in this case, is actually evaluation at the fixed point at $(X^1,\cdots X^N) = (X, -X, 0, \cdots, 0)$.

*Orientifold Planes of Complex Co-Dimensions* 2 *&* $N-3$ Next, we consider the B-parity action that exchanges $X^1 \leftrightarrow X^2$ and $X^3 \leftrightarrow X^4$ simultaneously. Similarly, from the even holonomy sector, we have

$$(2\pi)^{-N/2+3}\, 2^{-N+2}\, (r\Lambda)^{c/6}\, \mathrm{res}_{\epsilon=0} \left[\frac{\Gamma(\epsilon)}{\Gamma\left(1-\epsilon\right)}\right]^2 \left[\frac{\Gamma\left(\frac{\epsilon}{2}\right)\Gamma\left(-\frac{\epsilon}{2}\right)}{\Gamma(-\epsilon)}\right]^{N-4} \left[\frac{\Gamma\left(\frac{1-N\epsilon}{2}\right)\Gamma\left(-\frac{1-N\epsilon}{2}\right)}{\Gamma(-1+N\epsilon)}\right], \tag{4.312}$$

and from the odd holonomy sector,

$$(2\pi)^{N/2}\, 2^{N-4}\, (r\Lambda)^{c/6}\, \mathrm{res}_{\epsilon=0} \left[\frac{\Gamma(\epsilon)}{\Gamma\left(1-\epsilon\right)}\right]^2 \left[\frac{\Gamma(\epsilon)/\epsilon}{\Gamma\left(\frac{\epsilon}{2}\right)\Gamma\left(-\frac{\epsilon}{2}\right)}\right]^{N-4} \left[\frac{\Gamma(1-N\epsilon)/(1-N\epsilon)}{\Gamma\left(\frac{1-N\epsilon}{2}\right)\Gamma\left(-\frac{1-N\epsilon}{2}\right)}\right]. \tag{4.313}$$

Again, both holonomy sectors contribute for the residue at $\epsilon = 0$.

For the even holonomy sector, a similar procedure gives

$$\begin{aligned} Z_{\mathbb{RP}_2}^{pert.,\ even} &= C_{-2}\int_{\mathcal{X}} e^{-i\xi H}\wedge H^2 \wedge \left[\frac{\Gamma(1+\frac{H}{2\pi i})}{\Gamma\left(1+\frac{H}{4\pi i}\right)\Gamma\left(1-\frac{H}{4\pi i}\right)}\right]^2 \\ &\wedge \left[\frac{\Gamma\left(1+\frac{H}{4\pi i}\right)\Gamma\left(1-\frac{H}{4\pi i}\right)}{\Gamma(1-\frac{H}{2\pi i})}\right]^{N-2} \Big/ \left[\frac{\Gamma\left(1+\frac{NH}{4\pi i}\right)\Gamma\left(1-\frac{NH}{4\pi i}\right)}{\Gamma(1-\frac{NH}{2\pi i})}\right]. \end{aligned} \tag{4.314}$$

where $C_{-2} = i^{N-4}(2\pi)^{N/2}2^{N-6}(r\Lambda)^{c/6}$. In terms of the characteristic classes, we rewrite this

$$Z_{\mathbb{RP}_2}^{pert.,\ even} = C_{-2}\int_{\mathcal{M}_{-2}} e^{-iJ}\wedge \frac{\mathcal{A}(\mathcal{TM}_{-2}/2)}{\hat{\Gamma}_c(-\mathcal{TM}_{-2})}\wedge\frac{\hat{\Gamma}_c(\mathcal{NM}_{-2})}{\mathcal{A}(\mathcal{NM}_{-2}/2)}, \tag{4.315}$$

with $\mathcal{M}_{-2}$ is complex co-dimension 2 fixed locus, $(X_1,\cdots X_N) = (X,X,Y,Y,X_5,\cdots X_N)$.

Finally, from the odd holonomy sector, we have

$$\begin{aligned} Z_{\mathbb{RP}_2}^{pert.,\ odd} &= C_{-(N-3)}\int_{\mathcal{X}} e^{-i\xi H}\wedge\frac{H^{N-3}}{N}\wedge\left[\frac{\Gamma\left(1+\frac{H}{4\pi i}\right)\Gamma\left(1-\frac{H}{4\pi i}\right)}{\Gamma(1-\frac{H}{2\pi i})}\right]^2 \\ &\wedge \left[\frac{\Gamma(1+\frac{H}{2\pi i})}{\Gamma\left(1+\frac{H}{4\pi i}\right)\Gamma\left(1-\frac{H}{4\pi i}\right)}\right]^{N-2} \Big/ \left[\frac{\Gamma\left(1+\frac{NH}{2\pi i}\right)}{\Gamma\left(1+\frac{NH}{4\pi i}\right)\Gamma\left(1-\frac{NH}{4\pi i}\right)}\right], \end{aligned} \tag{4.316}$$

where $C_{-(N-3)} = i(-1)^{N-1}(2\pi)^{N/2}\, 2^{-N+4}$. This again can be summarized as

$$Z_{\mathbb{RP}_2}^{pert.,\ odd} = C_{-(N-3)} \int_{\mathcal{M}_{-(N-3)}} e^{-iJ} \wedge \frac{\mathcal{A}(\mathcal{TM}_{-(N-3)}/2)}{\hat{\Gamma}_c(-\mathcal{TM}_{-(N-3)})} \wedge \frac{\hat{\Gamma}_c(\mathcal{NM}_{-(N-3)})}{\mathcal{A}(\mathcal{NM}_{-(N-3)}/2)}\ , \tag{4.317}$$

where $\mathcal{M}_{-(N-3)}$ is a co-dimension $N-3$ locus spanned by $(X, -X, Y, -Y, 0, \cdots, 0)$.

### 4.5.6 Consistency Checks and Subtleties

In this last section, we explore the disk amplitudes ${}_R\langle 0|\mathcal{B}\rangle_R$ and the crosscap amplitudes ${}_R\langle 0|\mathcal{C}\rangle_R$ further. The most immediate question is whether these two types of amplitudes, or equivalently the central charges, come out with the correct relative normalization, for which we kept the overall coefficients carefully in the above. We will then ask subtler questions of what should happen when $\mathcal{M}$ is not Spin but only Spin$^c$, for which we can only offer a guess for the final expression but not a derivation.

We then move on to the anomaly inflow and also how we should extract, from the computed central charge, the RR-tensor Chern-Simons coupling. Having both ${}_R\langle 0|\mathcal{B}\rangle_R$ and ${}_R\langle 0|\mathcal{C}\rangle_R$ explicitly is most telling in this regard, whereby we discover that the difference between the conventional central charges and the newly computed ones is universal; the extra multiplicative factor due to $\hat{\Gamma}_c$ class is common for both D-branes and Orientifold planes and the same again makes appearance in $S^2$ partition function as well. This strongly suggests that the change should be attributed to the quantum volume of the cycles in $\mathcal{X}$, rather than to the characteristic class that appears in the world-volume Chern-Simons coupling to the spacetime RR tensor fields.

**Tadpole**

The simplest consistency check comes from the tadpole cancelation condition of the RR ground states, which can be written as [136, 137]

$$ {}_R\langle 0|\mathcal{C}\rangle_R + {}_R\langle 0|\mathcal{B}\rangle_R = 0 \ , \tag{4.318} $$

and demand the boundary state be constrained to satisfy this equality. From the spacetime viewpoint, this is the Gauss constraint for the RR-tensor fields, integrated over the compact Calabi-Yau manifold. Recall that the RR-charge of a single D$p$-brane and that of an $Op^{\pm}$ Orientifold plane must have a relative weight of

$$ \pm\, 2^{p-4} \tag{4.319} $$

in the covering space. Obviously, the same numerical factor must appear in the central charges.

For this numerical factor, we start with Hori and Romo [14], and consider tachyon condensation to obtain the disk partition function for a D-brane wrapping $\mathcal{M}$ in $\mathcal{X}$

$$ Z_{D^2} = (r\Lambda)^{c/6}\ (2\pi)^{(d+2)/2} \int_{\mathcal{M}} e^{-B-iJ} \wedge ch(\mathcal{E}) \wedge \frac{\hat{\Gamma}_c(\mathcal{T})}{\hat{\Gamma}_c(-\mathcal{N})} \wedge e^{-c_1(\mathcal{N})/2}\ , \tag{4.320} $$

where $d$ is the complex dimension of the Calabi-Yau $\mathcal{X}$. See Appendix C for details of this procedure. On the other hand, the result of section 4.5.5 can be written as

$$ Z_{\mathbb{RP}^2} = \ 2^{d-2s}(r\Lambda)^{c/6}(2\pi)^{(d+2)/2} \int_{\mathcal{M}} e^{-iJ} \wedge \frac{\mathcal{A}(\mathcal{T}/2)}{\hat{\Gamma}_c(-\mathcal{T})} \wedge \frac{\hat{\Gamma}_c(\mathcal{N})}{\mathcal{A}(\mathcal{N}/2)}\ , \tag{4.321} $$

where the complex co-dimension of $\mathcal{M}$ is denoted by $s$. The last factor in (4.320) and its apparent absence in (4.321) is the subject of the next subsection; for tadpole issue, it suffices to know that the 0-form part of the two expressions differ by the numerical factor of rank($\mathcal{E}$), prior to the projection, and also by $2^{d-2s}$. For the familiar Ramond-Ramond tadpole cancelation condition to emerge correctly, therefore, $2^{d-2s}$ must equal $2^{p-4}$. For ten-dimensional spacetime, $d = 10/2 = 5$ and $p = 9 - 2s$, so $d - 2s = p - 4$, precisely as needed.

### Anomaly Inflow and Indices

Let $|a\rangle_{RR}$ denote one of the crosscap or boundary states in the Ramond-Ramond sector. Then one can naturally define the Witten index as

$$I(a, b) = \lim_{T\to\infty} {}_{RR}\langle a|e^{-TH}|b\rangle_{RR} \ , \tag{4.322}$$

which calculates the indices of open strings attached between D-branes and Orientifold planes. Following figures are three distinguished topologies which give rise to the indices for brane-brane, brane-plane, and plane-plane respectively.

Due to the Riemann bilinear identity, these indices can be expressed in terms of the partition functions as follows [133].

$$I(\mathcal{B}_E, \mathcal{B}_F) = \sum_{ij} \langle \mathcal{B}_E|i\rangle \eta^{ij} \langle j|\mathcal{B}_F\rangle \ , \tag{4.323}$$

$$I(\mathcal{B}_E, \mathcal{C}) = \sum_{ij} \langle \mathcal{B}_E|i\rangle \eta^{ij} \langle j|\mathcal{C}\rangle \ , \tag{4.324}$$

$$I(\mathcal{C}, \mathcal{C}) = \sum_{ij} \langle \mathcal{C}|i\rangle \eta^{ij} \langle j|\mathcal{C}\rangle \ , \tag{4.325}$$

where all the states are in the Ramond-Ramond sector, and $\eta^{ij}$ is the topological metric of the chiral ring elements. Since the overlap between the RR ground states and the boundary/crosscap states measures the coupling to the RR gauge fields, this formula can be thought of as inflow mechanism which cancels the one-loop anomaly from each open string sector. Since the expression for these indices in the geometric

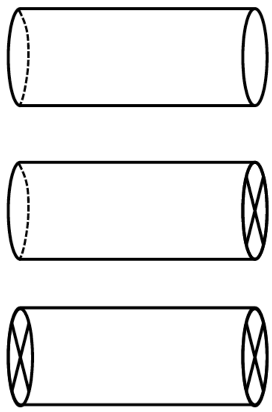

Figure 4.4: Two dimensional topologies where indices are defined. The first one denotes for a cylinder with two boundaries at the ends, and the second one corresponds to the Möbius strip with one boundary and one crosscap. The last one is the Klein bottle, with two crosscap states at the ends.

limit are well-known in the literature, we can check whether our results generate expected indices, and consistency with the original inflow mechanism [97, 98, 140].

Following the discussion of the previous subsection, here we assume that an extra factor $e^{d(\mathcal{M})/2}$ is present not only on the world-volumes of D-branes but also on the world-volumes of Orientifold planes. Otherwise, amplitudes involving boundary states only and amplitude involving a boundary state and a crosscap cannot be summed up; this would lead to net world-volume anomaly and make the spacetime theory inconsistent. Because we assume $\mathcal{X}$ itself to be Spin, $d(\mathcal{M})/2$ is always expressed as a sum over $-c_1/2$ of the normal bundles of the world-volumes.

*Cylinder* Index on the cylinder and relation to the disk partition function were studied in [14] and [29].

We start with Eq. (4.320) and use the relation (4.323) to calculate the open string index stretched between two branes with $(\mathcal{E}_1, \mathcal{M}_1)$ and $(\mathcal{E}_2, \mathcal{M}_2)$ as

$$\begin{aligned} & I(\mathcal{B}_{\mathcal{E}_1}, \mathcal{B}_{\mathcal{E}_2}) \\ \sim & \int_{\mathcal{M}_1 \cap \mathcal{M}_2} e^{-B-iJ} \wedge ch(\mathcal{E}_1) \wedge \frac{\hat{\Gamma}_c(\mathcal{T}_1)}{\hat{\Gamma}_c(-\mathcal{N}_1)} \wedge e^{d(\mathcal{M}_1)/2} \\ & \qquad \wedge e^{B+iJ} \wedge ch(-\mathcal{E}_2) \wedge \frac{\hat{\Gamma}_c(-\mathcal{T}_2)}{\hat{\Gamma}_c(\mathcal{N}_2)} \wedge e^{-d(\mathcal{M}_2)/2} \wedge e(\mathcal{N}_{12}) \\ = & \int_{\mathcal{M}_1 \cap \mathcal{M}_2} ch(\mathcal{E}_1) \wedge ch(-\mathcal{E}_2) \wedge \frac{\mathcal{A}(\mathcal{T}(\mathcal{M}_1 \cap \mathcal{M}_2))}{\mathcal{A}(\mathcal{N}(\mathcal{M}_1 \cap \mathcal{M}_2))} \wedge e^{(d(\mathcal{M}_1)-d(\mathcal{M}_2))/2} \wedge e(\mathcal{N}_{12}) \;, \end{aligned} \tag{4.326}$$

where $\mathcal{T}_i$ and $\mathcal{N}_i$ denote for tangent and normal bundles of $\mathcal{M}_i$ and $\mathcal{N}_{12} \equiv \mathcal{N}_1 \cap \mathcal{N}_2$. From the first to the second line, we used

$$\frac{\hat{\Gamma}_c(\mathcal{T}_1) \wedge \hat{\Gamma}_c(-\mathcal{T}_2)}{\hat{\Gamma}_c(-\mathcal{N}_1) \wedge \hat{\Gamma}_c(\mathcal{N}_2)} = \frac{\hat{\Gamma}_c(\mathcal{T}_1 \cap \mathcal{T}_2)\hat{\Gamma}_c(-\mathcal{T}_1 \cap \mathcal{T}_2)}{\hat{\Gamma}_c(-\mathcal{N}_1 \cap \mathcal{N}_2)\hat{\Gamma}_c(\mathcal{N}_1 \cap \mathcal{N}_2)} = \frac{\mathcal{A}(\mathcal{T}_1 \cap \mathcal{T}_2)}{\mathcal{A}(\mathcal{N}_1 \cap \mathcal{N}_2)} \;, \tag{4.327}$$

since

$$\mathcal{T}_1 \backslash (\mathcal{T}_1 \cap \mathcal{T}_2) = \mathcal{N}_2 \backslash (\mathcal{N}_1 \cap \mathcal{N}_2) \;. \tag{4.328}$$

Note that, for the first equality, complex conjugation of the normal bundle in the denominator of Eq. (4.320) is essential.

The factor $e^{(d(\mathcal{M}_1)-d(\mathcal{M}_2))/2}$ in (4.326) can be understood from the fact that the I-brane fermions on $\mathcal{M}_1 \cap \mathcal{M}_2$ are naturally sections of $\mathcal{S}(\mathcal{T}_1 \cap \mathcal{T}_2 \oplus \mathcal{N}_1 \cap \mathcal{N}_2)$. When the latter fails to be Spin, the 2nd Stiefel-Whitney class that measures this failure is

$$w_2(\mathcal{T}_1 \cap \mathcal{T}_2 \oplus \mathcal{N}_1 \cap \mathcal{N}_2) = w_2(\mathcal{T}_1) - w_2(\mathcal{T}_2) \;,$$

where the equality follows from the assumption that the ambient $\mathcal{X}$ is Spin. Since $w_2 = c_1 \bmod Z_2$, the relevant correcting factor for the Spin$^c$ case is $e^{(c_1(\mathcal{T}_1)-c_1(\mathcal{T}_2))/2}$. Note that this factor reduces to 1 when $\mathcal{M}_1$ and $\mathcal{M}_2$ are coincident, which is expected since $\mathcal{T} \oplus \mathcal{N} = \mathcal{T}\mathcal{X}$ is Spin. Next, we show how this extends to amplitudes involving Orientifold planes.

*Möbius strip* Similarly, the index on the Möbius strip can be obtained via the relation (4.324). If we let $\mathcal{M}_1$ and $\mathcal{M}_2$ are locus where D-branes and Orientifolds exist, we have[11]

$$\begin{aligned} I(\mathcal{B}_{\mathcal{E}},\mathcal{C}) &\sim 2^{p-4}\int_{\mathcal{M}_1\cap\mathcal{M}_2} ch_{2k}(\mathcal{F})\wedge\frac{\hat{\Gamma}(\mathcal{T}_1)}{\hat{\Gamma}(-\mathcal{N}_1)}\wedge\frac{\mathcal{A}(\mathcal{T}_2/2)\hat{\Gamma}(-\mathcal{N}_2)}{\mathcal{A}(\mathcal{N}_2/2)\hat{\Gamma}(\mathcal{T}_2)}\wedge e^{(d(\mathcal{M}_1)-d(\mathcal{M}_2))/2}\wedge e(\mathcal{N}_{12}) \\ &= 2^{p-4}\int_{\mathcal{M}_1\cap\mathcal{M}_2} ch_{2k}(\mathcal{F})\wedge\sqrt{\frac{\mathcal{A}(\mathcal{T}_1)\mathcal{L}(\mathcal{T}_2/4)}{\mathcal{A}(\mathcal{N}_1)\mathcal{L}(\mathcal{N}_2/4)}}\wedge e^{(d(\mathcal{M}_1)-d(\mathcal{M}_2))/2}\wedge e(\mathcal{N}_{12})\ , \end{aligned} \tag{4.329}$$

which exactly reproduce the index formula of the Möbius strip calculated at the level of non-linear sigma model [99, 133]. Here, $p+1$ is the dimension of the Orientifold plane.

When D$p$-branes are on the top of an O$p$-plane, in particular, we can read off $p+3$-form from $I(\mathcal{B}_E,\mathcal{C})+I(\mathcal{C},\mathcal{B}_E)$, which gives anomaly inflow on the $p+1$ dimensional world-volume as

$$\begin{aligned} & \pm\, 2^{p-4}\cdot[ch_{2k}(\mathcal{F})+ch_{\overline{2k}}(\mathcal{F})]\wedge\frac{\hat{\Gamma}(\mathcal{T})}{\hat{\Gamma}(-\mathcal{N})}\wedge\frac{\mathcal{A}(\mathcal{T}/2)\hat{\Gamma}(-\mathcal{N})}{\mathcal{A}(\mathcal{N}/2)\hat{\Gamma}(\mathcal{T})}\wedge e(\mathcal{N})\Bigg|_{p+3} \\ =\ & \pm\, 2^{p-4}\cdot\left[ch_{2k}(\mathcal{F})+ch_{\overline{2k}}(\mathcal{F})\right]\wedge\frac{\mathcal{A}(\mathcal{T}/2)}{\mathcal{A}(\mathcal{N}/2)}\wedge e(\mathcal{N})\Bigg|_{p+3} \\ =\ & \pm\, ch_{2k}(2\mathcal{F})\wedge\frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})}\wedge e(\mathcal{N})\Bigg|_{p+3}\ . \end{aligned} \tag{4.330}$$

Note that, since $U(k)$ gauge group is enhanced to $SO(2k)$ or $Sp(k)$ group, we used the relation $ch_{2k}(\mathcal{F})=ch_{\overline{2k}}(\mathcal{F})$. Adding two contributions from the cylinder and the Möbius indices, we recover the open string Witten index, i.e., anomaly inflow

[11] From the first to second line, we used the identity $\sqrt{\mathcal{A}(\mathcal{T})}\sqrt{\mathcal{L}(\mathcal{T}/4)}=\mathcal{A}(\mathcal{T}/2)$ .

for the $SO(2N)$ or $Sp(N)$ gauge group according to the sign of (4.330),

$$\begin{aligned} I_{SO(2k),Sp(k)} &= \left[ch_{2k\otimes 2k}(\mathcal{F}) \pm ch_{2k}(2\mathcal{F})\right] \wedge \frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})} \wedge e(\mathcal{N})\bigg|_{p+3} \\ &= 2\cdot ch_{\frac{1}{2}2k(2k\pm1)} \wedge \frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})} \wedge e(\mathcal{N})\bigg|_{p+3} . \end{aligned} \tag{4.331}$$

*Klein bottle* Finally, if there are two crosscap states as in the last diagram of the figure, we have topology of the Klein bottle whose index is given by the relation (4.325). Substituting our formula for the crosscap overlap into this identity, we have

$$\begin{aligned} I(\mathcal{C},\mathcal{C}) &\sim 2^{p_1+p_2-8} \int_{\mathcal{M}_1\cap\mathcal{M}_2} \frac{\mathcal{A}(\mathcal{T}_1/2)\hat{\Gamma}(\mathcal{N}_1)}{\mathcal{A}(\mathcal{N}_1/2)\hat{\Gamma}(-\mathcal{T}_1)} \wedge \frac{\mathcal{A}(\mathcal{T}_2/2)\hat{\Gamma}(-\mathcal{N}_2)}{\mathcal{A}(\mathcal{N}_2/2)\hat{\Gamma}(\mathcal{T}_2)} \wedge e^{(d(\mathcal{M}_1)-d(\mathcal{N}_2))/2} \wedge e(\mathcal{N}_{12}) \\ &= 2^{p_1+p_2-8} \int_{\mathcal{M}_1\cap\mathcal{M}_2} \left(\frac{\mathcal{A}(\mathcal{T}_1\cap\mathcal{T}_2/2)}{\mathcal{A}(\mathcal{N}_1\cap\mathcal{N}_2/2)}\right)^2 \wedge \frac{\mathcal{A}(\mathcal{N}_1\cap\mathcal{N}_2)}{\mathcal{A}(\mathcal{T}_1\cap\mathcal{T}_2)} \wedge e^{(d(\mathcal{M}_1)-d(\mathcal{M}_2))/2} \wedge e(\mathcal{N}_{12}) \\ &= 2^{p_1+p_2-8} \int_{\mathcal{M}_1\cap\mathcal{M}_2} \frac{\mathcal{L}(\mathcal{T}_1\cap\mathcal{T}_2/4)}{\mathcal{L}(\mathcal{N}_1\cap\mathcal{N}_2/4)} \wedge e^{(d(\mathcal{M}_1)-d(\mathcal{M}_2))/2} \wedge e(\mathcal{N}_{12}) . \end{aligned} \tag{4.332}$$

This again gives the well-known formula for the Klein bottle index calculated in non-linear sigma model. Since the B-type parity action corresponds to the Hodge star operation of the target space, it reproduces the Hirzebruch signature theorem [133]. Obviously, this index is independent of the open string degrees of freedom, or the types of planes [144]. For type-I string theory, this inflow precisely cancels the one-loop anomaly of supergravity multiplet.

### 4.5.7 RR-Charges and Quantum Volumes

This brings us, finally, to a natural question of what part of the central charge should be attributed to the RR-charges. Recall that the conventional RR-charges, or the Chern-Simons coupling to RR-tensors, was deduced indirectly via anomaly inflow. For instance, for the simplest case of the spacetime-filling D-brane, the relevant anomaly polynomial is $\mathcal{A}(\mathcal{T})$, the $\hat{A}$ class, which is then reconstructed via inflow as

$$\Omega(\mathcal{T}) \wedge \Omega(-\mathcal{T}) = \mathcal{A}(\mathcal{T}) \ , \tag{4.333}$$

where $\Omega$ is the characteristic class that appears in the Chern-Simons coupling. With an implicit assumption that $\log \Omega$ is "even," i.e., includes $4k$-forms only, this leads to $\Omega = \mathcal{A}^{1/2}$ [97, 98, 140]. Some of early literatures were casual about distinction between $\Omega(\mathcal{T})$ and $\Omega(-\mathcal{T})$, although more careful computations show the conjugation has to occur for one of the two factors [97, 144]. Thus, in hindsight, the anomaly cancelation argument fixes only "even" part of $\log \Omega$.

As was noted previously, $\Omega = \hat{\Gamma}_c$ is one multiplicative class that is consistent with the anomaly inflow $\mathcal{A}$ in the above sense. This happens precisely because "even" part of $\log \hat{\Gamma}_c$ coincides exactly with $\log \mathcal{A}^{1/2}$. Our discussion in the previous section demonstrated that replacements like

$$\mathcal{A}^{1/2}(\mathcal{T}) \to \hat{\Gamma}_c(\mathcal{T}), \qquad \mathcal{L}^{1/2}(\mathcal{T}/4) \to \mathcal{A}(\mathcal{T}/2)/\hat{\Gamma}_c(-\mathcal{T}) \ , \tag{4.334}$$

for D-branes and Orientifold planes, respectively, would be still consistent with anomaly inflow. However, since the central charge is made from RR-charges and quantum volumes of various cycles, it is hardly clear whether such a change in the central charge should be attributed to the RR-charge or not.

More generally, for a D-brane wrapping a cycle $\mathcal{M}$ in Calabi-Yau $\mathcal{X}$, the gravitational curvature contribution to the central charge is

$$\frac{\hat{\Gamma}_c(\mathcal{T})}{\hat{\Gamma}_c(-\mathcal{N})} = \sqrt{\frac{\mathcal{A}(\mathcal{T})}{\mathcal{A}(\mathcal{N})}} \wedge \exp\left( i \sum_{k\geq 1} \frac{(-1)^k (2k)! \zeta(2k+1)}{(2\pi)^{2k}} ch_{2k+1}(\mathcal{X}) \right) \ , \tag{4.335}$$

so the deviation depends only on $\mathcal{X}$. As shown in the present work, something quite similar happens for the Orientifold planes,

$$\frac{\mathcal{A}(\mathcal{T}/2)}{\hat{\Gamma}_c(-\mathcal{T})}\frac{\hat{\Gamma}_c(\mathcal{N})}{\mathcal{A}(\mathcal{N}/2)} = \sqrt{\frac{\mathcal{L}(\mathcal{T}/4)}{\mathcal{L}(\mathcal{N}/4)}} \wedge \exp\left( i\sum_{k\geq 1}\frac{(-1)^k(2k)!\zeta(2k+1)}{(2\pi)^{2k}} ch_{2k+1}(\mathcal{X})\right) , \tag{4.336}$$

where the deviation is identical to its D-brane counterpart. So the difference between the new central charges and the conventional ones can be expressed by a universal factor, determined by $\mathcal{X}$ only, is independent of the choice of the cycle $\mathcal{M}$, and its logarithm is purely imaginary.

These properties all suggest that this factor should be interpreted as a $\alpha'$ modification of volumes, in the sense,

$$\exp(-iJ) \quad \rightarrow \quad \exp\left(-iJ + i\sum_{k\geq 1}\frac{(-1)^k(2k)!\zeta(2k+1)}{(2\pi)^{2k}} ch_{2k+1}(\mathcal{X})\right) , \tag{4.337}$$

rather than as a shift of RR-charges, or the Chern-Simons couplings, themselves. In fact, this is precisely the same shift of $J$ that appears in $S^2$ partition function, or its large volume expression,

$$\begin{aligned} Z_{S^2} \quad &\sim \quad \int_{\mathcal{X}} e^{-2iJ} \wedge \frac{\hat{\Gamma}_c(\mathcal{T}\mathcal{X})}{\hat{\Gamma}_c(-\mathcal{T}\mathcal{X})} \\ &= \quad \int_{\mathcal{X}} \exp\left(-2iJ + 2i\sum_{k\geq 1}\frac{(-1)^k(2k)!\zeta(2k+1)}{(2\pi)^{2k}} ch_{2k+1}(\mathcal{X})\right) . \end{aligned} \tag{4.338}$$

Here, the "even" part of the two Gamma classes cancel out completely, suggesting that they, but not "odd" parts, carry RR-charge information. For Calabi-Yau 3-fold, the piece $\int_{\mathcal{X}} ch_3(\mathcal{X})$ is proportional to the Euler number and represents exactly the quantum shift of the volume that has been seen in the mirror map [12, 132]. This viewpoint also conforms with the fact that there is no modification for Calabi-Yau

2-fold (times remaining flat directions), for which the ten-dimensional spacetime theory has as many as 16 supercharges.

The ambiguity in determining RR-charge from the anomaly inflow remains, as the D-brane and the I-brane inflow mechanisms always conjugate one of the two factors as in (4.333).[12] However, once we accept (4.337) as the quantum version of the exponentiated Kähler class, this ambiguity is lifted, and we come back to the same old Chern-Simons coupling to spacetime RR-tensors for D-branes and Orientifold planes [97–99, 101, 102, 140, 144].

[12] Although, in principle, the Chern-Simons coupling may be computable by direct string world-sheet method along the line of Refs. [99, 100, 145, 146], which had confirmed the first few terms of Refs. [97, 98].

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# Appendix A

# Characteristic Classes

Throughout this thesis, the role of the characteristic classes are essential for studying the topological properties of the theories. For a base manifold $M$ and fiber $F$ over it, characteristic class measures non-triviality of twisting of such bundle. They are defined in terms of the polynomial of gauge invariant curvatures, which is referred to as the *invariant polynomials*. For a detailed discussion of properties of them, consult the section 2.3.2 and also the references [16, 147].

When $E \xrightarrow{\pi} M$ is a complex vector bundle whose fiber is isomorphic to $\mathbb{C}^n$, the most prequently used characteristic classes are the Chern class and the Todd class. First of all, the Chern class is defined as

$$ch_{\mathbf{R}}(\mathcal{F}) \equiv \mathrm{tr}_{\mathbf{R}} e^{\mathcal{F}/2\pi} = \sum_i e^{x_i} \ , \tag{A.1}$$

where $\mathbf{R}$ denotes the relevant representation, and $x_i$ are the two-form-valued eigenvalues of

$$\frac{\mathcal{F}}{2\pi} = \frac{F_{ij}}{2\pi} \ dx^i \wedge dx^j \ . \tag{A.2}$$

in the representation **R**. The Todd class, which appears in the discussion of the Dolbeaux complex of complex manifolds, is defined as

$$Td(\mathcal{F}) = \prod_i \frac{x_i}{1 - e^{-x_i}} \ . \tag{A.3}$$

Note that the $Td(\mathcal{F})$ is defined only for a complex manifold.

For real bundles, we have the $\mathcal{A}$-roof genus and the Hirzbruch $\mathcal{L}$-class. These can be expressed in terms of 2-form skew-eigenvalues $y_i$ of

$$\frac{R}{2\pi} = \frac{1}{4\pi} R_{ijkl} \ dx^k \wedge dx^l, \tag{A.4}$$

$$\mathcal{A}(R) \equiv \prod_i \frac{y_i/2}{\sinh(y_i/2)} \ , \qquad \mathcal{L}(R) \equiv \prod_i \frac{y_i}{\tanh(y_i)} \ . \tag{A.5}$$

These two can also be expanded in term of Pontryagin classes,

$$p_1(R) = \sum_i y_i^2 \ , \quad p_2(R) = \sum_{i<k} y_i^2 y_k^2 \ , \quad p_3(R) = \sum_{i<k<l} y_i^2 y_k^2 y_l^2 \ , \tag{A.6}$$

and so on. Note that $p_k(R)$ are $4k$-forms, which is consistent with the fact that for a real bundle $R$ and invariant polynomial $I$, $I(R) = I(R^T) = I(-R)$. Finally, the Euler class is

$$\chi(R) = \prod_i y_i \ , \tag{A.7}$$

which is given by a top form. We sometimes denote the Euler class with the notation $e(R)$. With these definition, we can prove various useful identities. For example, we see

$$\frac{\chi(R)}{\mathcal{A}(R)} = \prod_i \frac{\sinh(y_i/2)}{y_i/2} \prod_j y_j = \prod_i \left( e^{y_i/2} - e^{-y_i/2} \right) = ch_{S^+}(R) - ch_{S^-}(R) \ , \tag{A.8}$$

which is a crucial identity when we match 1-loop anomaly to the inflows. Furthermore, we also have

$$\mathcal{A}(R)\mathcal{L}(R/4) = \prod_i \frac{2(y_i/4)^2}{\sinh(y_i/2)\tanh(y_i/4)} = \prod_i \frac{(y_i/4)^2}{\sinh(y_i/4)^2} = \mathcal{A}(R/2)^2 \ . \quad \text{(A.9)}$$

Finally, we discuss the Stiefel-Whitney classes which is valued in $H^p(M, \mathbb{Z}_2)$. The first Stiefel-Whitney class $w_1(M)$ is zero if and only if the manifold is orientable. The second Stiefel-Whitney class $w_2(M)$ measures the obstrunction to define spinors on $M$, i.e., $w_2(M)$ is zero if and only if the bundle admits a Spin structure. However, even if $w_2(M)$ is odd, there is a way to define a spinors on $M$ by turning on additional half-line bundle which cancels the global anomaly. For the manifold where this procedure is possible, the third Stiefel-Whitney class $w_3(M)$ is zero and we say that the bundle admits a Spin$^c$ structure.

# Appendix B

# Reduction to Nonlinear Sigma Model on $\mathcal{M}_n$

With $n$ centers, one starts with $3(n-1)$ bosonic coordinates and $4(n-1)$ fermionic ones, after the free center of mass part is removed from the dynamics. It is convenient to work with a coordinate system where $n-1$ of them equal to independent linear combinations of $\mathcal{K}_A$'s. In a slight abuse of notation we will denote these again by $\mathcal{K}^A$, now with $A = 1, \dots, n-1$; although there are $n$ $\mathcal{K}$'s, only $n-1$ of them are linearly independent. Thus, we split the relative part of $r^{Aa}$ and $\psi^{Aa}$ as $Z^M = (\mathcal{K}^A, y^\mu)$ and $\psi^M = (\psi^A, \psi^\mu)$, with $M = 1, \dots, 3(n-1)$ and $\mu = n, \dots, 3(n-1)$. Along the same spirit, we also denote by $\lambda^A$, $n-1$ linearly combinations of $\lambda$'s that belong to the relative part of the low energy dynamics. What do we mean by $\psi^M$? We wish to preserve at least one supersymmetry, say $Q_4$, and naturally $\psi^M$ is the superpartner of $Z^M$,

$$\psi^M = \frac{\partial Z^M}{\partial r^{Aa}} \psi^{Aa} , \tag{B.1}$$

and the kinetic term of $\psi^M$ includes two factors of $\partial r^{Aa}/\partial Z^M$.

As argued in section 4, it suffices to consider the dynamics with flat metric, which after taking out the center of mass part becomes

$$g_{AaBb} = m_{AB}\,\delta_{ab}\ ,$$

where $m_{AB}$ is the $(n-1)\times(n-1)$ reduced mass matrix. Expressing this in the curved coordinate system, $Z^M$,

$$g_{MN} = \sum m_{AB}\frac{\partial r^{Aa}}{\partial Z^M}\frac{\partial r^{Ba}}{\partial Z^N}\ ,$$

we find that partial derivatives of metric coefficients $g_{MN}$ are nontrivial. In contrast, nothing much happens to $\lambda$'s, other than one of them being taken out as the center of mass part, so their metric is the same reduced mass matrix,

$$h_{AB} = m_{AB}\ ,$$

and is constant. Thus, no coordinate-dependent transformations are needed for $\lambda$'s. The deformed Lagrangian with flat kinetic term reads in this coordinate,

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\,g_{MN}(Z)\dot{Z}^M\dot{Z}^N - \frac{1}{2}\,\xi^2(m^{-1})^{AB}\mathcal{K}_A(Z)\mathcal{K}_B(Z) - \mathcal{W}(Z)_M\dot{Z}^M \\ &+ \frac{i}{2}\,g_{MN}(Z)\psi^M\dot{\psi}^N - \frac{i}{2}\,\partial_L g_{MN}(Z)\dot{Z}^N\psi^L\psi^M + \frac{i}{2}\,m_{AB}\lambda^A\dot{\lambda}^B \\ &+ i\xi\partial_B\mathcal{K}_A\psi^B\lambda^A + i\partial_M\mathcal{W}_N(Z)\psi^M\psi^N \end{aligned} \tag{B.2}$$

where the crucial middle term in the second line follows from Eq. (B.1) and the anticommuting nature of fermions. We also used $\partial_\mu\mathcal{K}_A = 0$.

Since we anticipate that $\mathcal{K}$ directions will decouple as $\xi\to\infty$, we split the metric as

$$[g_{MN}] = \begin{pmatrix} H_{AB} & C_{A\mu} \\ C^T_{\mu A} & G_{\mu\nu} \end{pmatrix}$$

and the likewise for its inverse

$$[g^{MN}] = \begin{pmatrix} (H - CG^{-1}C^T)^{-1} & -(H - CG^{-1}C^T)^{-1}CG^{-1} \\ -G^{-1}C^T(H - CG^{-1}C^T)^{-1} & G^{-1} + G^{-1}C^T(H - CG^{-1}C^T)^{-1}C^TG^{-1} \end{pmatrix}$$

Ignoring $\mathcal{W}$ and fermion contributions to the conjugate momentum for now for simplicity, the bosonic part of Hamiltonian will then looks something like

$$\begin{aligned} \mathcal{H} &\simeq \frac{1}{2}g^{\mu\nu}p_\mu p_\nu + g^{A\mu}p_A p_\mu + \frac{1}{2}g^{AB}p_A p_B + \cdots \\ &= \frac{1}{2}(G^{-1})^{\mu\nu}p_\mu p_\nu + \frac{1}{2}g^{AB}P_A P_B + \cdots \end{aligned} \tag{B.3}$$

where

$$P_A \equiv p_A + (H - CG^{-1}C^T)_{AC}\, g^{C\mu}p_\mu = p_A - (CG^{-1})_A^{\ \mu}p_\mu$$

$P_A$'s have the standard canonical commutator with $\mathcal{K}$'s, so it is clear that, together with $\sim \xi^2\mathcal{K}^2$ terms, they form very heavy harmonic oscillators of frequency $\sim \xi$, settle down to its ground state sector, and decouple from ground state counting. This leaves behind

$$\mathcal{H} \simeq \frac{1}{2}(G^{-1})^{\mu\nu}p_\mu p_\nu + \cdots \tag{B.4}$$

Denoting the canonical conjugate of $p_\mu$ in this reduced dynamics again by $y^\mu$,[1] the corresponding Lagrangian would be

$$\mathcal{L} \simeq \frac{1}{2}\, G_{\mu\nu}\Big|_{\mathcal{K}=0} \dot{y}^\mu \dot{y}^\nu + \cdots \tag{B.5}$$

This makes clear that we could have done the same more simply by imposing $\mathcal{K} = 0$ at the level of Lagrangian.

[1] Generally $\mathcal{K}$ will mix in the definition of this new $y^\mu$ coordinates, to reflect the shift of the conjugate momenta, but this becomes irrelevant because dynamics forces $\mathcal{K} = 0$. Therefore, the same old $y$ coordinates can be used here.

Procedure leading up to (B.4) can be repeated in the presence of $\mathcal{W}$'s, which simply shift the conjugate momenta in the Hamiltonian, and it is clear that only $\mathcal{W}_\mu$'s will survive. We should ask whether this is consistent, since after all $d\mathcal{W}$'s are Dirac quantized magnetic fields, and removing some part of the gauge connection could make the remainder ill-defined. However, we have

$$d\mathcal{W} = \partial_B \mathcal{W}_A d\mathcal{K}^B d\mathcal{K}^A + (\partial_\mu \mathcal{W}_A - \partial_A \mathcal{W}_\mu)\, dy^\mu d\mathcal{K}^A + \partial_\nu \mathcal{W}_\mu dy^\nu dy^\mu$$

and the pull-back onto $\mathcal{M}_n$ is simply

$$\mathcal{M}_n^*(d\mathcal{W}) = \partial_\nu \mathcal{W}_\mu dy^\nu dy^\mu \tag{B.6}$$

The pull-back of a well-defined bundle to a smoothly embedded submanifold is still a well-defined bundle, so the reduced gauge connection $\mathcal{W}_\mu(\mathcal{K}=0)$ is consistent. Thus, the bosonic part of the action reduces to

$$\mathcal{L} \simeq \frac{1}{2}\, G_{\mu\nu}\Big|_{\mathcal{K}=0} \dot{y}^\mu \dot{y}^\nu - \mathcal{W}_\mu\Big|_{\mathcal{K}=0} \dot{y}^\mu + \cdots \tag{B.7}$$

leaving us with the question of how to reduce fermion sector.

The fermions enter the Hamiltonian in two places. One is as bilinear connection term added to the conjugate momenta, and the other is an additive contribution of the form

$$-i\xi \partial_B \mathcal{K}_A \psi^B \lambda^A - i\partial_A \mathcal{W}_B \psi^A \psi^B - i(\partial_A \mathcal{W}_\mu - \partial_\mu \mathcal{W}_A)\psi^A \psi^\mu - i\partial_\mu \mathcal{W}_\nu \psi^\mu \psi^\nu$$

with the canonical anticommutator among $\psi^M$'s equal to $g^{MN}$. To disentangle heavy $\psi^A$ from light $\psi^\mu$, we shift the light fermions as

$$\tilde{\psi}^\mu \equiv \psi^\mu + \psi^A (H - CG^{-1}C^T)_{AC}\, g^{C\mu} = \psi^\mu - \psi^A (CG^{-1})_A^{\;\;\mu}\ ,$$

such that

$$\{\psi^A, \tilde{\psi}^\mu\} = 0 \ , \qquad \{\tilde{\psi}^\mu, \tilde{\psi}^\nu\} = (G^{-1})^{\mu\nu} \ .$$

Let us categorize these fermion bilinears into three difference pieces,

$$-i\partial_\mu \mathcal{W}_\nu \tilde{\psi}^\mu \tilde{\psi}^\nu - i\mathcal{E}_{A\mu}\psi^A\tilde{\psi}^\mu + \left[-i\xi\partial_B\mathcal{K}_A\psi^B\lambda^A + \cdots\right] \ .$$

Terms in the last bracket involve only $\psi^A$ and $\lambda^A$'s with eigenvalues $\sim \xi$, so these will decouple from the low energy spectrum. The potential mixing between heavy and light modes are in

$$\mathcal{E}_{A\mu} = \partial_A\mathcal{W}_\mu - \partial_\mu\mathcal{W}_A + (CG^{-1})_A^{\ \nu}(\partial_\nu\mathcal{W}_\mu - \partial_\mu\mathcal{W}_\nu) \ .$$

For heavy sector, this is of course a minor perturbation and ignorable as $\xi \to 0$. For light sector, things looks less innocent since the size of this operator is itself not negligible. However, the heavy fermion enters this operator linearly, and always will connect excited states and ground states of heavy fermion sector. This forces the energy eigenvalue differences $(E_n - E_k)$ in the denominator of the perturbation series to be of order $\sim \xi$, such that the perturbation is suppressed by powers of $\sim \mathcal{E}/\xi$. In the end, again, the net effect is to turn off the heavy modes $\psi^A$ and $\lambda^A$ completely, leaving behind

$$-i\partial_\mu\mathcal{W}_\nu\psi^\mu\psi^\nu$$

only, where well call this light fermion again as $\psi^\mu$'s. The simplest way to understand this is to recall that any operator linear in heavy fermions will vanish when sandwiched between heavy sector vacuum.

Combining the reduction processes of the bosonic and the fermionic sectors, it is clear that the connection term can be equally reduced to

$$\frac{i}{2}\partial_L g_{MN}\dot{Z}^N\psi^L\psi^M \quad \to \quad \frac{i}{2}\partial_\delta G_{\alpha\beta}\dot{y}^\beta\psi^\delta\psi^\alpha \ .$$

Now we can revert from the Hamiltonian to the Lagrangian, after putting all the heavy modes to their ground states, and arrive at the following reduced Lagrangian,

$$\begin{aligned}
&\mathcal{L}^{\mathcal{N}=1}_{\text{for index only}} \\
&= \frac{1}{2}G_{\mu\nu}(\dot{y}^\mu\dot{y}^\nu + i\psi^\mu\dot{\psi}^\nu) - \mathcal{A}_\mu\dot{y}^\mu + \frac{i}{2}\mathcal{F}_{\mu\nu}\psi^\mu\psi^\nu - \frac{i}{2}\partial_\delta G_{\alpha\beta}\dot{y}^\beta\psi^\delta\psi^\alpha \\
&= \frac{1}{2}G_{\mu\nu}\dot{y}^\mu\dot{y}^\nu + \frac{i}{2}\psi^\mu G_{\mu\nu}(\dot{\psi}^\nu + \Gamma^\mu_{\gamma\delta}\dot{y}^\gamma\psi^\delta) - \mathcal{A}_\mu\dot{y}^\mu + \frac{i}{2}\mathcal{F}_{\mu\nu}\psi^\mu\psi^\nu \ ,
\end{aligned} \tag{B.8}$$

where we introduced the notation, also used in the main text, $\mathcal{F} = \mathcal{M}_n^*(d\mathcal{W})$ and its gauge field $\mathcal{A}$. We already defined $G$ as the appropriate block of $g$, but now valued at $\mathcal{M}_n$. In other words, $G = \mathcal{M}_n^*(g)$. Remaining fermions live in the cotangent bundle of $\mathcal{M}_n$, so the resulting Lagrangian is $\mathcal{N} = 1$ non-linear sigma model on $2(n-1)$-dimensional manifold $\mathcal{M}_n$, coupled to an Abelian gauge field $\mathcal{W}$. Supercharge of this dynamics is a Dirac operator on $\mathcal{M}_n$ coupled to Abelian gauge field $\mathcal{A}$, and therefore the index is, under the canonical choice of the chirality operator,

$$\int_{\mathcal{M}_n} Ch(\mathcal{F})\hat{A}(\mathcal{M}_n) \ .$$

# Appendix C

# Spherical Harmonics

We summarize basic facts about the (monopole) spherical harmonics. In order to discuss the projection condition under the parity, it is convenient to choose a gauge where the monopole background vector field takes the following form

$$A = -\frac{B}{2} \cos\theta d\varphi \ , \tag{C.1}$$

valid in the region $0 < \theta < \pi$. In addition, we also need to choose a gauge for the spin connection, as it affects the harmonics for spinors and vectors. Our choice,

$$w^{\hat{\theta}}{}_{\hat{\phi}} = -\cos\theta d\varphi \ , \tag{C.2}$$

is such that spinor spherical harmonics are antiperiodic along $\phi \to \phi + 2\pi$.

The scalar monopole harmonics $Y_{\mathbf{q},jm}$ with $\mathbf{q} = \frac{B}{2}Q$ satisfy

$$-D_m^2 Y_{\mathbf{q},jm} = j(j+1) - \mathbf{q}^2 \ , \qquad j = l + |\mathbf{q}| \ (l = 0, 1, 2, ..) \ , \tag{C.3}$$

where the covariant derivative denotes

$$D = d - iQA \ . \tag{C.4}$$

For later convenience, we present an explicit expression of the scalar monopole harmonics below,

$$Y_{\mathbf{q},jm}(\theta,\varphi) = M_{\mathbf{q},jm}(1-x)^{\alpha/2}(1+x)^{\beta/2}P_n^{\alpha\beta}(x)e^{im\varphi} \ , \tag{C.5}$$

with

$$x = \cos\theta \ , \ \alpha = -\mathbf{q} - m \ , \ \beta = \mathbf{q} - m \ , \ n = j + m \ , \tag{C.6}$$

and

$$M_{\mathbf{q},jm} = 2^m \sqrt{\frac{2j+1}{4\pi}\frac{(j-m)!(j+m)!}{(j-\mathbf{q})!(j+\mathbf{q})!}} \ . \tag{C.7}$$

Here the Jacobi polynomial $P_n^{\alpha\beta}(x)$ is defined by

$$P_n^{\alpha\beta}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\Big[(1-x)^{\alpha+n}(1+x)^{\beta+n}\Big] \ . \tag{C.8}$$

Using the fact that

$$P_n^{\alpha\beta}(-x) = (-1)^n P_n^{\beta\alpha}(x) \ , \tag{C.9}$$

it is straightforward to show that, for $0 < \theta < \pi$,

$$\begin{aligned} Y_{\mathbf{q},jm}(\pi-\theta,\pi+\varphi) =& (-1)^n e^{i\pi m}\, Y_{-\mathbf{q},jm}(\theta,\varphi) \\ =& (-1)^l e^{-i\pi|\mathbf{q}|}\, Y_{-\mathbf{q},jm}(\theta,\varphi) \ . \end{aligned} \tag{C.10}$$

For instance,

$$Y_{\pm\frac{1}{2},jm}(\pi-\theta,\pi+\varphi) = (-i)(-1)^l Y_{\mp\frac{1}{2},jm}(\theta,\varphi) \text{ for } j = l+\frac{1}{2} \ . \tag{C.11}$$

The complex conjugate of the monopole harmonics satisfy the following two relations,

$$Y^*_{\mathbf{q},jm}(\theta,\varphi) = (-1)^{\mathbf{q}+m} Y_{-\mathbf{q},j(-m)}(\theta,\varphi) \ , \tag{C.12}$$

and

$$\int_{S^2} Y^*_{\mathbf{q},jm}(\theta,\varphi) Y_{\mathbf{q}',j'm'}(\theta,\varphi) = \delta_{qq'}\delta_{jj'}\delta_{mm'} \ . \tag{C.13}$$

We now move on to the spinor monopole harmonics. It is useful to consider the eigenmodes $\Psi^{\pm}_{\mathbf{q},jm}$ of a modified Dirac operator

$$-i\gamma^3\gamma^m D_m \Psi^{\pm}_{\mathbf{q},jm} = i\lambda_{\pm}\Psi^{\pm}_{\mathbf{q},jm} \ , \qquad \lambda_{\pm} = \pm\sqrt{\left(j+\frac{1}{2}\right)^2 - \mathbf{q}^2} \ , \tag{C.14}$$

where

$$\Psi^{\pm}_{\mathbf{q},jm} = \begin{pmatrix} Y_{\mathbf{q}-\frac{1}{2},jm} \\ \pm Y_{\mathbf{q}+\frac{1}{2},jm} \end{pmatrix} \ . \tag{C.15}$$

Here the covariant derivative is

$$D = d - iQA + \frac{1}{4}\omega_{ab}\gamma^{ab} \ . \tag{C.16}$$

Using the property of the monopole harmonics (C.11), one can show

$$\Psi^{\pm}_{\mathbf{q}=0,jm}(\pi-\theta,\pi+\varphi) = \mp i(-1)^l \begin{pmatrix} \pm Y_{\frac{1}{2},jm}(\theta,\varphi) \\ Y_{-\frac{1}{2},jm}(\theta,\varphi) \end{pmatrix} \ , \tag{C.17}$$

with $0 < \theta < \pi$.

Finally let us discuss about the one-form spherical harmonics defined by

$$\begin{aligned} C^1_{jm} &= + \frac{1}{\sqrt{j(j+1)}} dY_{lm} \ , \\ C^2_{jm} &= - \frac{1}{\sqrt{j(j+1)}} * dY_{jm} \ , \end{aligned} \tag{C.18}$$

where $j \geq 1$. Useful properties of the vector spherical harmonics can be summarized as follow,

$$*C^2_{jm} = C^1_{jm} \ , \qquad *dC^2_{jm} = \sqrt{j(j+1)} Y_{lm} \ , \qquad *dC^1_{jm} = 0 \ , \tag{C.19}$$

which lead to

$$\begin{aligned} *d * dC^2_{jm} &= - j(j+1) C^2_{jm} \ , \\ *d * dC^1_{jm} &= 0 \ . \end{aligned} \tag{C.20}$$

Under the parity action, they transform as

$$\begin{aligned} C^1_{jm}(\pi - \theta, \pi + \varphi) &= (-1)^j C^1_{jm}(\theta, \varphi) \ , \\ C^2_{jm}(\pi - \theta, \pi + \varphi) &= (-1)^{j+1} C^2_{jm}(\theta, \varphi) \ . \end{aligned} \tag{C.21}$$

# Appendix D

# One-Loop Determinant on $\mathbb{RP}_b^2$

We will show that the partition function on the squashed real projective space $\mathbb{RP}_b^2$ is independent of the squashing parameter $b$. This section largely relies on the discussion in [13]. For details, please refer to Appendix A of the reference.

To compute the one-loop determinant around the SUSY saddle points, it is not necessary to know all the eigenmodes of boson and fermion kinetic operators. This is because, as we see in section 3, the huge cancelation between boson and fermion eigenmodes occurs. It is therefore sufficient to understand how the boson and fermion eigenmodes are paired by the supersymmetry.

## Chiral multiplet

We start with a chiral multiplet of unit $U(1)$ gauge charge. To simplify the computation, we choose a $\mathcal{Q}$-exact regulator

$$\mathcal{L}_{\text{reg}} = -\delta_\epsilon \delta_{\bar{\epsilon}} \Big[ \bar{\psi}\gamma^3\psi - 2\bar{\phi}\sigma_2\phi \Big] \ , \tag{D.1}$$

different to the one used in the main context. The above choice leads to the kinetic operators around the saddle points (4.209), (4.210)

$$\begin{aligned}\Delta_b =& - D_m^2 + \sigma^2 + \frac{q}{4}\mathcal{R} + \frac{q-1}{f} v^m D_m + + \frac{q^2-2q}{4f^2} \ , \\ \Delta_f =& - i\gamma^m D_m - \sigma\gamma^3 - i\frac{1}{2f}\gamma^3 + i\frac{q-1}{2f} v_m \gamma^m + i\frac{q-1}{2f} w \ ,\end{aligned} \tag{D.2}$$

where the covariant derivative involves the background gauge field $V$ given in (4.206),

$$\begin{aligned}D_m\phi =& \left(\partial_m - iA_m + iqV_m\right)\phi \ , \\ D_m\psi =& \left(\partial_m - iA_m + \frac{1}{4}w_{ab}\gamma^{ab} + i(q-1)V_m\right)\psi \ ,\end{aligned} \tag{D.3}$$

and

$$v^m = \bar\epsilon\gamma^m\epsilon \ , \qquad w = \bar\epsilon\epsilon \ . \tag{D.4}$$

Here $\mathcal{R}$ denotes the scalar curvature of $\mathbb{RP}^2$. As in section 3, it is convenient to consider spinor eigenmodes for an operator $\gamma^3\Delta_f$ instead of $\Delta_f$.

One can show that there is a pair between a scalar eigenmode for $\Delta_b \doteq -M(M+2\sigma)$ and two spinor eigenmodes for $\gamma^3\Delta_f \doteq M, -(M+2\sigma)$, subject to either (4.216) or (4.230) projection conditions. The precise map which pairs the scalar and spinor eigenmodes is the following; Given a spinor eigenmode $\Psi$ for $\gamma^3\Delta_f \doteq M$, one can show that

$$\bar\epsilon\Psi \tag{D.5}$$

is a scalar eigenmode for $\Delta_b \doteq -M(M+2\sigma)$. On the other hand, one can define a pair of spinors

$$\Psi_1 = \gamma^3 \epsilon \Phi \ , \qquad \Psi_2 = i\gamma^m \epsilon D_m \Phi + \gamma^3 \epsilon \left( \sigma \Phi + i \frac{q}{2f} \right) \Phi \ , \tag{D.6}$$

where $\Phi$ is a scalar eigenmode for $\Delta_b \doteq -M(M+2\sigma)$. One can show that

$$M\Psi_1 + \Psi_2 \ , \qquad -(M+2\sigma)\Psi_1 + \Psi_2 \tag{D.7}$$

are the eigenmodes for $\gamma^3 \Delta_f \doteq M$ and $\gamma^3 \Delta_f \doteq -(M+2\sigma)$ respectively.

Any modes in such a pair can not contribute to the one-loop determinant due to the cancelation. As a consequence, the nontrivial contributions arise from the eigenmodes where either the map (D.5) or the map (D.7) becomes ill-defined.

*Unpaired spinor eigenmode* If a spinor eigenmode vanishes when contracted with $\bar{\epsilon}$, there is no scalar partner. Such an unpaired spinor eigenmode takes the following form

$$\Psi = e^{-iJ\varphi} h(\theta) \bar{\epsilon} \ , \tag{D.8}$$

where

$$iJ = \left( Ml + \sigma l + i \frac{q-2}{2} \right) \ , \tag{D.9}$$

and

$$\frac{1}{f} \partial_\theta h = \tan\theta \left( \frac{J}{l} - \frac{q-2}{2l} + i \frac{q-2}{2f} \right) h \ . \tag{D.10}$$

For the normalizability, one has to require $J$ to be non-negative. Note that the function $h(\theta)$ is even under the parity, i.e., $h(\theta) = h(\pi - \theta)$. One can show that $J$ should be further restricted to be even (odd) to satisfy the projection conditions in the even (odd) holonomy, i.e.,

$$\begin{aligned} Ml =& i\left(2k+1+i\sigma l - \frac{q}{2}\right) \text{ for even holonomy },\\ Ml =& i\left(2k+2+i\sigma l - \frac{q}{2}\right) \text{ for odd holonomy }, \end{aligned} \tag{D.11}$$

with $k \geq 0$.

*Missing spinor eigenmode* Suppose that a scalar eigenmode $\Phi$ for $\Delta_b \doteq -M(M+2\sigma)$ fails to provide two independent spinor eigenmodes via the map (D.7). It happens when

$$\Psi_2 = -M\Psi_1 , \tag{D.12}$$

which leads to a missing spinor eigenmode for $\gamma^3\Delta_f \doteq M$. One can verify that such a scalar eigenmode $\Phi$ missing a spinor eigenmode takes the following form

$$\Phi = e^{iJ\varphi}\chi(\theta) , \tag{D.13}$$

where

$$iJ = -\left(Ml + \sigma l + i\frac{q}{2}\right) , \tag{D.14}$$

with $J \geq 0$ for the normalizability, and

$$\frac{1}{f}\partial_\theta \chi = \tan\theta \left(\frac{J}{l} + \frac{q}{2l} - \frac{q}{2f}\right)\chi . \tag{D.15}$$

To satisfy the projection condition in the even (odd) holonomy, one can show that

$$\begin{aligned} Ml =& -i\left(2k - i\sigma l + \frac{q}{2}\right) \text{ for even holonomy },\\ Ml =& -i\left(2k+1 - i\sigma l + \frac{q}{2}\right) \text{ for odd holonomy }, \end{aligned} \tag{D.16}$$

with $k \geq 0$.

*One-loop determinant* Combining all the results (D.11) and (D.16), one can show

$$\frac{\det \Delta_f}{\det \Delta_b} \simeq \frac{\det \gamma^3 \Delta_f}{\det \Delta_b} \simeq \begin{cases} \prod_{k\geq 0} \frac{2k+1+i\sigma l - \frac{q}{2}}{2k - i\sigma l + \frac{q}{2}} & \text{for even holonomy} \\ \prod_{k\geq 0} \frac{2k+2+i\sigma l - \frac{q}{2}}{2k+1 - i\sigma l + \frac{q}{2}} & \text{for odd holonomy} \end{cases} , \tag{D.17}$$

where the symbol $\simeq$ represents the equality up to a sign independent of $\sigma$. From the comparison to the results in section 3, one can fix the sign factor by the unity. These results are in perfect agreement to those for $\mathbb{RP}^2$.

## Vector multiplet

We now in turn compute the one-loop determinant from the vector multiplet. Denoting the various fluctuation fields as follows

$$A = A_{\text{flat}} + a \ , \qquad \sigma_1 = \zeta \ , \qquad \sigma_2 = \sigma + \eta \ , \tag{D.18}$$

let us decompose all the adjoint fields $(a, \zeta, \eta)$ into Cartan-Weyl basis. From now on, we focus on the W-boson of charge $\alpha$, a root of $G$, and its super partners. The kinetic Lagrangian for the vector multiplet is chosen as a $\mathcal{Q}$-exact regulator.

As explained in [9] and [13] that the four bosonic modes contain two longitudinal modes with $a \sim D\eta$ that correspond to a gauge rotation and the volume of the gauge group $G$. Using the standard Fadeev-Popov method, one can argue that these longitudinal modes can not contribute to the one-loop determinant. Thus we need to find how two transverse modes with $*D*a = 0$ can be paired with spinor eigenmodes.

The kinetic operators of our interest are

$$\begin{aligned}\Delta_b =& \begin{pmatrix} - * d * d + (\alpha \cdot \sigma)^2 & - * d \frac{1}{f} \\ + \frac{1}{f} * d & - * d * d + \frac{1}{f^2} + (\alpha \cdot \sigma)^2 \end{pmatrix} \ , \\ \Delta_f =& i\gamma^m D_m + (\alpha \cdot \sigma)\gamma^3 \ , \end{aligned} \tag{D.19}$$

with the gauge choice $*d * a = 0$. The operator $\Delta_b$ acts on the fluctuation fields $(a, \zeta)$ subject to the projection conditions (4.242) for the even holonomy and the twisted projection conditions for the odd holonomy. Instead of $\Delta_b$, it is convenient to consider the following operator

$$\delta_b \equiv \begin{pmatrix} i\alpha \cdot \sigma & - * d \\ *d & \frac{1}{f} + i\alpha \cdot \sigma \end{pmatrix} \ . \tag{D.20}$$

One can show that the operator $\delta_b$ satisfies the relation $\delta_b^2 = \Delta_b + 2i(\alpha \cdot \sigma)\delta_b$, or equivalently,

$$\delta_b \doteq -iM \ , +i(M + 2\alpha \cdot \sigma) \ \leftrightarrow \ \Delta_b \doteq -M(M + 2\alpha \cdot \sigma) \ . \tag{D.21}$$

Let $(\mathcal{A}, \Sigma)$ and $\Lambda$ be bosonic eigenmodes for $\delta_b \doteq -iM$ and fermionic eigenmodes for $\gamma^3 \Delta_f \doteq -M$. They can be shown to be mapped to each other by

$$\mathcal{A} = -i\left(M + \alpha \cdot \sigma\right) \bar{\epsilon}\gamma_m \Lambda e^m - d\left(\bar{\epsilon}\gamma^3 \Lambda\right) \ , \qquad \Sigma = (M + \alpha \cdot \sigma)\bar{\epsilon}\Lambda \ , \tag{D.22}$$

and

$$\Lambda = \left(\gamma^3 \gamma^m \mathcal{A}_m + i\Sigma\gamma^3\right) \epsilon \ . \tag{D.23}$$

Again, one can have nontrivial contribution to the one-loop determinant from either unpaired or missing spinor eigenmodes.

*Unpaired spinor eigenmodes* An unpaired spinor eigenmode, annihilated by the map (D.22), takes the following form

$$\Lambda = e^{-iJ\varphi} h(\theta)\bar{\epsilon} \ , \tag{D.24}$$

where

$$i\,(J+1) = Ml + \alpha \cdot \sigma l \ , \tag{D.25}$$

with $J \geq 0$ due to the normalizability, and

$$\frac{1}{f}\partial_\theta h + \tan\theta \left( \frac{1}{f} - \frac{J+1}{l} \right) h = 0 \ . \tag{D.26}$$

Note that the function $h(\theta)$ is even under the parity, $h(\pi - \theta) = h(\theta)$. In order to satisfy the projection conditions in the even (odd) holonomy, the non-negative integer $J$ should be further constrained to be odd (even), i.e.,

$$\begin{aligned} Ml =& i\,(2k+2+i\alpha\cdot\sigma) \ \text{ for even holonomy} \ , \\ Ml =& i\,(2k+1+i\alpha\cdot\sigma) \ \text{ for odd holonomy} \ , \end{aligned} \tag{D.27}$$

with $k \geq 0$.

*Missing spinor eigenmodes* One can show from the map (D.23) that a bosonic eigenmode with missing spinor partner can take the following form

$$\mathcal{A} = e^{i(J+1)\varphi}\chi(\theta)\left(e^1 + i\cos\theta\epsilon^2\right) \ , \qquad \Sigma = ie^{i(J+1)\varphi}\chi(\theta)\sin\theta \ , \tag{D.28}$$

where $e^m$ denotes the vielbein of $\mathbb{RP}_b^2$, and

$$i(J+1) = -\left(Ml + \alpha\cdot\sigma l\right) \ , \tag{D.29}$$

and

$$\frac{1}{f}\partial_\theta \chi + \tan\theta \left(\frac{1}{f} - \frac{J+1}{l}\right)\chi = 0 \ . \tag{D.30}$$

The normalizability requires $J$ to be non-negative. The projection conditions in even (odd) holonomy are satisfied if $J$ are even (odd), i.e.,

$$\begin{aligned} Ml &= -\, i\,(2k+1-i\alpha\cdot\sigma) \ \text{ for even holonomy} \ , \\ Ml &= -\, i\,(2k+2-i\alpha\cdot\sigma) \ \text{ for odd holonomy} \ , \end{aligned} \tag{D.31}$$

with $k \geq 0$.

*One-loop determinant* Collecting all the results (D.27) and (D.31), the one-loop determinant from the vector multiplet becomes

$$\frac{\det \Delta_f}{\sqrt{\det \Delta_b}} \simeq \frac{\det \gamma^3 \Delta_f}{\det \delta_b} \simeq \begin{cases} \prod_{\alpha\in\Delta}\prod_{k\geq 0} \frac{2k+2+i\alpha\cdot\sigma}{2k+1-i\alpha\cdot\sigma} & \text{for even holonomy} \\ \prod_{\alpha\in\Delta}\prod_{k\geq 0} \frac{2k+1+i\alpha\cdot\sigma}{2k+2-i\alpha\cdot\sigma} & \text{for odd holonomy} \end{cases} . \tag{D.32}$$

By comparing the results to those in section 3, one can fix the sign factor by the unity. Again, these results perfectly agree with those for $\mathbb{RP}^2$.

# 초록

## 1 차원과 2 차원 초대칭 이론에서의 지표와 분배 함수의 정확한 계산과 끈이론에의 응용

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이 논문에서는, 칼라비-야우 다양체에 옹골화된 초끈 이론의 기하학적 구조를 조사할 수 있는 두 가지 방법을 소개한다. 첫째로, 위튼 지표를 소개하고, 이것이 초끈 이론에 어떻게 사용되는지 살펴 본다. 가장 흥미로운 예로, II 형 초끈 이론을 칼라비-야우 다양체에 옹골화하여 얻어지는 4 차원 $N = (2,2)$ 초대칭 게이지 이론의 BPS 상태의 개수를 세는 문제에 대하여 살펴 본다. 특히, 이 이론의 벽넘기 현상을 설명하는 데에 위튼 지표가 어떻게 사용되는 지에 집중한다. 둘째로, 최근 개발된 2 차원 분배 함수와 칼라비-야우 다양체의 케일러-모듈라이 공간의 기하학 사이의 관계에 대하여 살펴본다. 특히, 2 차원 구, 반구 그리고 실사영 평면 위에서 계산한 $N = (2,2)$ 게이지 선형 시그마 모형의 분배 함수가 케일러 퍼텐셜, D-브레인과 오리엔티폴드의 중심 전하를 계산한다는 사실을 보인다.

주요어: 지표, 분배 함수, 끈이론, 초대칭, 칼라비-야우 다양체

학번: 2009-20408