
DUALITY AND GEOMETRY OF STRING THEORY

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Declaration of Authorship

Unless otherwise referenced, the work presented in this thesis is my own - Nipol Chaemjumrus. The main results of the project are outlined in chapters 5, chapter 6, chapter 7 and chapter 8. These follow the work published in

[1] N. Chaemjumrus and C. M. Hull, “Finite Gauge Transformations and Geometry in Extended Field Theory,” *Phys. Rev. D* **93** (2016) no.8, 086007 [arXiv:1512.03837 [hep-th]],

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[4] N. Chaemjumrus and C. M. Hull, “The Doubled Geometry of Nilmanifold Reductions,” in preparation.

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Abstract

String theory possesses duality symmetries that relate different string backgrounds. One symmetry is known as T-duality symmetry. In general, when n dimensions are toroidally compactified, the T-duality group is $O(n, n, \mathbb{Z})$. String theory has another duality symmetry known as S-duality, which does not commute with T-duality. The full S-duality group is $SL(2, \mathbb{Z})$. The last duality symmetry is U-duality symmetry, which is $E_{n+1, (n+1)}(\mathbb{Z})$ for type II string theory on T^n . Duality symmetries tell us that strings experience geometry differently from particles. In order to understand string theory, a new way to understand string geometry is required.

In this thesis, first we introduce some basic ideas on duality symmetries in string theory, namely, T-duality, S-duality, and U-duality. Next, we review string field theory. We, then, provide the basic constructions of DFT and EFT. Next, we consider the finite gauge transformations of DFT and EFT. The expressions for finite gauge transformations in double field theory with duality group $O(n, n)$ are generalized to give expressions for finite gauge transformations for extended field theories with duality groups $SL(5, \mathbb{R})$, $SO(5, 5)$ and E_6 .

Another topic is the T-duality chain of special holonomy domain wall solutions. This example can arise in string theory in solutions in which these backgrounds appear as fibres over a line. The cases with 3-torus with H -flux over a line were obtained from identifications of suitable NS5-brane solutions, and are dual to D8-brane solutions. This T-duality chain implies that $K3$ should have a limit in which it degenerates to a long neck of the form $\mathcal{N} \times \mathbb{R}$ capped off by suitable smooth geometries. A similar result applies for the higher dimensional analogues of the nilfold. In each case, the space admits a multi-domain wall type metric that has special holonomy, so that taking the product of the domain wall solution with Minkowski space gives a supersymmetric solution.

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1. Introduction

In the standard model of particle physics, three forces in nature, namely, electromagnetic, weak interaction, and strong interaction can be explained by quantum field theory. On the other hand, gravity can be described in terms of the geometry of the spacetime in the theory of general relativity. At some energy scale, one believes that gravity could be unified with the other three forces. However, due to the non-renormalizable property of gravity, it is difficult to combine gravity into quantum theory. String theory [5]- [8] is an alternative theory that might show a way to quantum gravity because it contains the graviton in the spectrum.

In string theory, the concept of point particles is broken down and replaced by strings. The interactions between particles are also replaced by the string world-sheet interaction. The divergence from the non-renormalizable property of gravity is softened when the interaction is given by the exchange of strings. The extended objects like strings have an infinite number of degrees of freedom, which is given by the Fourier modes or the shapes of the objects. By replacing particles by extended objects, it smeared out the space-time divergence, however, it introduces new divergences from the internal degrees of freedom, which will become worse as the dimension of the extended objects increases. String theory is the only one that the divergences for both space-time and internal are under control. That makes string theory more interesting than the higher dimensional object theory.

Another interesting feature of string theory is that it possesses duality symmetries that relates different string backgrounds. One symmetry is known as T-duality symmetry [9]. When one of the dimensions where strings propagate is compactified into a circle, strings can wind along this compact direction. A number of times that strings curl along the circle gives rise to the winding mode w . The mass spectrum of the closed string state with one circular direction is given by

$$M^2 = (N + \bar{N} - 2) + p^2 \frac{l_s^2}{R^2} + w^2 \frac{R^2}{l_s^2}, \quad (1.0.1)$$

where N and \bar{N} are number operators for right and left-movers respectively, p is a momentum mode along the circle, w is the winding mode and l_s is a string length. This closed string state

also satisfies the level-matching condition

$$N - \bar{N} = pw. \quad (1.0.2)$$

If the momentum mode p is exchanged with the winding mode w as well as the quantity R/l_s becomes l_s/R , the mass spectrum (1.0.1) and the level-matching condition (1.0.2) are still invariant. It implies that in the string point of view, strings cannot distinguish between propagating along the circle with the radius R or $1/R$. In general, when n dimensions are toroidally compactified, T-duality is generalized into the T-duality group $O(n, n, \mathbb{Z})$.

T-duality symmetry tell us that strings experience the geometry differently from particles. In order to understand string theory, a new way to understand the string geometry is required. In [10], T-duality is realized as a symmetry of string field theory [11, 12]. In string field theory on the torus, the winding modes are treated on an equal footing as the momentum modes and this gives rise to coordinates that are dual to the winding modes. Although the full closed string field theory on a torus is so complicated and cannot be studied in more detail, the massless sector has been developed and it is known as double field theory (DFT) [13], with fields depending on a doubled spacetime in which the periodic coordinates x^i on the torus are supplemented by dual periodic coordinates \tilde{x}_i conjugate to the winding numbers.

Key features of double field theory are that T-duality is manifest and the fields depend on all the doubled coordinates. The double field theory corresponding to the metric, b -field and dilaton of the bosonic string was derived from string field theory in [13] to cubic order in the fields. The full theory has proved rather intractable, and much work has been done on a small subsector of the theory, obtained by imposing the ‘strong constraint’, which locally implies that locally all fields and parameters depend on only half the doubled coordinates.

The origin of the strong constraint arises from the level-matching condition $L_0 - \bar{L}_0 = 0$. In terms of field representation, it implies that fields A are annihilated by $\partial_i \tilde{\partial}^i(A) = 0$. This constraint is known as the weak constraint. Additionally, the generalized diffeomorphism is also considered in order to construct the invariant action. The gauge transformation in DFT [14] is given by the generalized Lie derivative generated by a vector field ξ^i and a one-form $\tilde{\xi}_i$. These gauge parameters can be rearranged into an $O(D, D)$ vector representation such that $\xi^M = (\xi^i, \tilde{\xi}_i)$. In the limit when the theory is independent of dual coordinates, the gauge transformation has reduced into the ordinary diffeomorphism and the two-form gauge transformation. For the closure of the generalized Lie derivative, the constraint that is stronger than the weak constraint is required and known as the strong constraint.

The strongly constrained theory has been found to all orders in the fields [15, 16], and is locally equivalent to the conventional field theory of metric, b -field and dilaton, and DFT reduces to the duality-covariant formulation of field theory proposed by Siegel [17], and can be thought of as a formulation in terms of generalized geometry [18] - [31].

In [15], the background independent action of double field theory has been constructed and taken the form

$$S = \int dx d\tilde{x} e^{-2d} \left(-\frac{1}{4} g^{ik} g^{jl} \mathcal{D}^p \mathcal{E}_{kl} \mathcal{D}_p \mathcal{E}_{ij} + \frac{1}{4} (\mathcal{D}^j \mathcal{E}_{ik} \mathcal{D}^i \mathcal{E}_{jl} + \bar{\mathcal{D}}^j \mathcal{E}_{ki} \bar{\mathcal{D}}^i \mathcal{E}_{lj}) \right. \\ \left. + (\mathcal{D}^i d \bar{\mathcal{D}}^j \mathcal{E}_{ij} + \bar{\mathcal{D}}^i d \mathcal{D}^j \mathcal{E}_{ji}) + 4 \mathcal{D}^i d \mathcal{D}_i d \right), \quad (1.0.3)$$

where derivatives \mathcal{D}_i and $\bar{\mathcal{D}}_i$ are defined by

$$\mathcal{D}_i \equiv \frac{\partial}{\partial x^i} - \mathcal{E}_{ik} \frac{\partial}{\partial \tilde{x}_k}, \quad \bar{\mathcal{D}}_i \equiv \frac{\partial}{\partial x^i} + \mathcal{E}_{ik} \frac{\partial}{\partial \tilde{x}_k}, \quad (1.0.4)$$

and the field \mathcal{E}_{ij} is defined as $\mathcal{E}_{ij} = g_{ij} + b_{ij}$, and the field d is related to the dilaton field via $e^{-2d} = \sqrt{g} e^{-2\phi}$. In this action, indices are raising and lowering with the metric g_{ij} and each terms is invariant under the $O(D, D)$ T-duality group. The gauge transformations of fields are given by

$$\delta_\xi \mathcal{E}_{ij} = \mathcal{D}_i \tilde{\xi}_j - \bar{\mathcal{D}}_j \tilde{\xi}_i + \xi^M \partial_M \mathcal{E}_{ij} + \mathcal{D}_i \xi^k \mathcal{E}_{kj} + \bar{\mathcal{D}}_j \xi^k \mathcal{E}_{ik}, \quad (1.0.5)$$

$$\delta_\xi d = -\frac{1}{2} \partial_M \xi^M + \xi^M \partial_M d, \quad (1.0.6)$$

where $\xi^M \partial_M = \xi^i \partial_i + \tilde{\xi}_i \tilde{\partial}^i$ and $\partial_M \xi^M = \partial_i \xi^i + \tilde{\partial}^i \tilde{\xi}_i$. However, proving the gauge invariance of this action is so difficult and requires a long calculation. Therefore, in [16], a new action that related to (1.0.3) has been created from the generalized metric \mathcal{H}_{MN} and field d as

$$S = \int dx d\tilde{x} e^{-2d} \left(4 \mathcal{H}^{MN} \partial_M \partial_N d - \partial_M \partial_N \mathcal{H}^{MN} - 4 \mathcal{H}^{MN} \partial_M d \partial_N d + 4 \partial_M \mathcal{H}^{MN} \partial_N d \right. \\ \left. + \frac{1}{8} \mathcal{H}^{MN} \partial_M \mathcal{H}^{KL} \partial_N \mathcal{H}_{KL} - \frac{1}{2} \mathcal{H}^{MN} \partial_M \mathcal{H}^{KL} \partial_K \mathcal{H}_{NL} \right). \quad (1.0.7)$$

The gauge transformations of \mathcal{H}_{MN} and d are given by the generalized Lie derivative

$$\delta_\xi \mathcal{H}_{MN} = \mathcal{L}_\xi \mathcal{H}_{MN} = \xi^P \partial_P \mathcal{H}_{MN} + (\partial_M \xi^P - \partial^P \xi_M) \mathcal{H}_{PN} + (\partial_N \xi^P - \partial^P \xi_N) \mathcal{H}_{MP}, \quad (1.0.8)$$

$$\delta_\xi (e^{-2d}) = \mathcal{L}_\xi e^{-2d} = \partial_M (\xi^M e^{-2d}). \quad (1.0.9)$$

From the action (1.0.7), the $O(D, D)$ structure of each terms is manifest and proving gauge invariant property is simpler than (1.0.3).

String theory also has a duality symmetry known as S-duality [32]. S-duality is a non-perturbative symmetry of type IIB superstring theory, which is not accessible from string perturbation theory. It relates a strong coupling theory of type IIB with a weak coupling theory of type IIB. It also exchanges the fields in the NS-NS sector with the R-R sector. For example, two-form the b -field is exchanged with the RR two-form. With the fields in the NS-NS

sector and the R-R sector are exchanged, the objects charged under these gauge fields are also exchanged. For example, the role of the F1-string is swapped with the solitonic string, D1. The full S-duality group is expected to be $SL(2, \mathbb{Z})$ [32]. This group relates the F1-string to a whole set of strings with quantum numbers (p, q) , where p is the number of the F1-string and q is the number of the D1-string for p, q are relatively prime.

There is another duality group in string theory. This duality group is known as U-duality group [32] gives rise the idea of M-theory, which is an eleven-dimensional theory. In M-theory, there exist extended objects known as M2-branes and M5-branes. When M-theory is compactified on an n -dimensional torus, M2-branes and M5-branes can be wrapped along cycles of the torus, which gives rise to wrapping modes. U-duality which relates momentum modes with M2-brane and M5-brane wrapping modes is given by the exceptional group E_n [32]. Upon toroidal compactification, M-theory can be related to type IIA superstring, which is also related to type IIB superstring via T-duality. That means U-duality symmetry is also a duality symmetry of type II superstring. To be precise, type II superstring on n -torus would have U-duality group E_{n+1} . The extension from T-duality group to U-duality can be explained by the extension of momentum modes and winding modes of strings to wrapping modes of Dp -branes in superstring theory. That is U-duality transformation in type II superstring will mix momentum and winding modes of strings with wrapping modes of Dp -branes.

U-duality symmetry in M-theory and superstring theory tells us that in order to obtain a duality invariant geometry one needs a formulation that includes momentum modes and winding modes of strings as well as wrapping modes of Dp -branes. In this formulation, the space-time coordinates are extended to include coordinates conjugated to wrapping modes. This theory goes by the name of ‘Extended Field Theory’ (EFT). Extended field theories [33] - [59] generalize strongly constrained DFT to a theory on an extended geometry that is covariant under E_n U-duality transformations. The total number of extended spaces in each case are given by the below table

d	$E_{d,(d)}$	R_1
3	$SL(3) \times SL(2)$	(3,2)
4	$SL(5)$	10
5	$Spin(5, 5)$	16
6	$E_{6,(6)}$	27
7	$E_{7,(7)}$	56
8	$E_{8,(8)}$	248

In EFT, the actions are constructed in terms of the generalized metric, which consists of the metric, the 3-form gauge field and possibly the 6-form gauge field on the internal space. The gauge symmetries of the theories are given by the generalized Lie derivative. The closure of the gauge algebra requires a constraint to be imposed on the fields and gauge parameters in

the theories. This constraint is known as the section condition. One way to solve the section constraint is to demand fields and gauge parameters to be independent of dual coordinates. This would lead to generalized geometry formulation, where the tangent space is extended to include winding modes of string and branes.

In DFT and EFT, the truncation to strong constraint or section constraint results in fields depending on the conventional coordinates, x^m . This leads to the field theory on the space parameterised by the coordinates, x^m . The formulation of strongly constraint DFT may be formulated on a general double manifold M [61] - [64]. The strong constraint results in the fields depend on an n -dimensional submanifold $N \subset M$. This is a conventional field theory on a space-time N , with a metric, $g_{ij}(x^m)$, a 2-form gauge field, $b_{ij}(x^m)$, and a dilaton, $\phi(x^m)$. The symmetry of this theory is the diffeomorphism of N and the 2-form gauge transformation, which leads to the symmetry group $\text{Diff}(N) \ltimes \Lambda_{\text{closed}}^2(N)$.

In general, this picture need only to be true locally. For example, let us consider the double space with a coordinate patch \mathcal{U} . The solution of the strong constraint can be chosen such that all the fields are independent of winding coordinates, \tilde{x}_m . By taking the quotient of \mathcal{U} by the action of $\partial/\partial\tilde{x}_m$, this give some patch U of \mathbb{R}^n . The strong constraint allows to take the quotient so that the fields, such as, the generalized metric, are defined on $U \subset \mathcal{U}$. The generalized Lie derivative of DFT reduces to the generalized Lie derivative for generalized geometry on U . In general case, these patch U will not necessary form the submanifold N . They can formulate a non-geometric space [65] - [72], such as T-fold [67], where the transition functions between the patches are given by the T-duality transformation. In the double space formalism, the transition functions are given by the $O(n, n, \mathbb{Z})$ transformation, which acts geometrically through large diffeomorphism on the double space.

In order to understand the geometry of DFT, there have been a number of attempts to explore the relationship between the gauge symmetries of DFT and the diffeomorphisms of the doubled space [73] - [77]. However, the gauge group and the diffeomorphism group are not isomorphic [78] because the former acts through the generalized Lie derivative while the latter acts through the Lie derivative. The DFT gauge transformations acts on fields at a point $X \in \mathcal{M}$. This transforms fields at X to fields at X , $A(X) \rightarrow A'(X)$. Recall that diffeomorphisms can be written in terms of an active or a passive form. It is natural to ask if the DFT gauge transformation can be written in the form that the coordinate X transforms.

To address such questions requires a better understanding of transition functions and a global structure, and for this one needs formulae for gauge transformations with finite parameters. The infinitesimal gauge transformation of DFT is given by the generalized Lie derivative. The expression for the gauge transformation with finite parameters in which fields transform at a point X , $A(X) \rightarrow A'(X)$, is obtained by exponentiating the generalized Lie derivative. For

example, for a generalized tensor A_M ,

$$A'_M(X) = e^{\mathcal{L}^\epsilon} A_M(X), \quad (1.0.10)$$

where all fields and parameters depend on X and satisfy the strong constraint. In [78], explicit forms were found for finite gauge transformations in DFT that have the correct gauge algebra. This finite transformation is in agreement with the finite transformations for the metric and b -field, and make explicit contact with generalized geometry. The same question can be asked for the EFT. In [1], the result of finite transformations of EFT was found. In that paper, the cases $E_4 = SL(5, \mathbb{R})$, $E_5 = SO(5, 5)$ and E_6 were considered. The result agrees with the finite transformation for the metric and the 3-form gauge field.

Another interesting question is the interpretation of the double space M . The double space of string theory on the toroidal background T^n is the double tori T^{2n} . The extra coordinates arise from the winding modes. However, in a non-toroidal background, there is no generic interpretation of winding coordinates. In [64], the double space M was constructed from the quotient space of a group manifold \mathcal{G} with its discrete cocompact subgroup Γ , such that $M = \mathcal{G}/\Gamma$ is a compact space. This group structure arises from the gauge algebra of the compactified theory [62, 64].

For example, first let us consider a theory on a $(d + n + 1)$ -dimensional space-time. By compactifying on T^n , this gives a theory with a gauge group $U(1)^{2n}$ and an $O(n, n, \mathbb{Z})$ symmetry. The generators of $U(1)^{2n}$ consist of Z_a , which generate the $U(1)^n$ action on T^n , and X^a , which generate the b -field gauge transformation with one leg on T^n and the other on the external space. Next, consider a Scherk-Schwarz reduction [79] on a circle with a periodic coordinate, $x \sim x + 1$, with an $O(n, n)$ duality twist around the circle. This results in the theory with a non-abelian gauge symmetry. One can construct the double geometry in terms of T^{2n} bundle over S^1 , which can be given by a quotient of $(2n + 1)$ dimensional group by a discrete subgroup. It is natural to consider doubling the coordinates on the base space. This results in a $(2n + 2)$ -dimensional space, which can be thought of as the quotient space of a $(2n + 2)$ -dimensional group manifold with a discrete subgroup. The Lie algebra of this group manifold is given by

$$[T_M, T_N] = t_{MN}{}^P T_P. \quad (1.0.11)$$

The group generators T_M can be decomposed into Z_m and X^m [64].

On the double group manifold \mathcal{G} , there are two sets of globally defined vector fields [64], namely, left-invariant vector fields, K_M , and right-invariant vector fields, \tilde{K}_M . The left-invariant vector fields K_M generate a right action of \mathcal{G} and the right-invariant vector fields \tilde{K}_M generate a left action of \mathcal{G} . On $M = \mathcal{G}/\Gamma$, where Γ acts on the left, only the left-invariant vector fields K_M are globally defined. The gauge symmetry acts through the right action generated by K_M .

At a given point on M , the basis of the tangent space is given by the right-invariant, \tilde{K}_M , which can be split as \tilde{Z}_m, \tilde{X}^m . The conventional space can be obtained by the quotient of M by the action of \tilde{X}^m [64].

In the case that \tilde{X}^m generate a subgroup \tilde{G}_L , if the subgroup \tilde{G}_L is preserved by Γ , that is for all $\gamma \in \Gamma$ and $k \in \tilde{G}_L$

$$\gamma k \gamma^{-1} = k', \quad (1.0.12)$$

for some $k' \in \tilde{G}_L$, the quotient of \mathcal{G}/\tilde{G}_L by Γ is well-defined and defines a subspace of $M = \mathcal{G}/\Gamma$. In general, Γ will not preserve the subgroup \tilde{G}_L , in this case taking quotient by Γ is inconsistent with taking the quotient by \tilde{G}_L . The conventional space is locally given in the local patches, which is locally \mathcal{G}/\tilde{G}_L . These patches will not fit together to form a submanifold of M .

The double space M is described as a universal background which includes many different string backgrounds. The well-known example is a T^3 with H -flux [80]. This background is T-dual to a nilmanifold, which is a T^2 bundle over S^1 . By performing T-duality along one of T^2 -direction, T-fold background is obtained. These backgrounds can be described in terms of 6 dimensional space, which is a quotient of a nilpotent Lie group by its discrete subgroup. The detail of this double space is given by [64].

These examples are instructive but have the drawback of not defining a CFT and so not giving a solution of string theory. However, these examples can arise in string theory in solutions in which these backgrounds appear as fibres over some base, related by a T-duality acting on the fibres. The simplest case is that in which these solutions are fibred over a line, defining a solution that is sometimes referred to as a domain wall background. The cases with 3-torus or nilfold fibred over a line were obtained in [81] from identifications of suitable NS5-brane or KK-monopole solutions, and are dual to D8-brane solutions.

The NS5-brane with the transverse space given by $\mathbb{R} \times T^3$ maps to a Kaluza-Klein monopole with a Gibbons-Hawking metric [82] which is a circle bundle over a base space $\mathbb{R} \times T^2$, giving the product of \mathbb{R} with a circle bundle over T^2 known as a nilfold or a nilmanifold. On the other hand, the same chain of dualities takes the type I string on T^4 to type IIA string theory on $K3$ [83] - [85]. Remarkably, a recent work on a limit of $K3$ [86] reconciles these two pictures, providing confirmation of our approach. There is a region near the boundary of $K3$ moduli space in which the $K3$ develops a long neck which is locally of the form of a the product of a nilfold with a line, with Kaluza-Klein monopoles inserted in that space. The ends of the long neck are capped with hyperkähler spaces asymptotic to the product of a nilfold with a line, known as Tian-Yau spaces [87]. These Tian-Yau caps can be viewed as the duals of the regions around the ON [88] - [90] or orientifold planes [91] - [93] and it is remarkable that these are realised as smooth geometries, similar to the realisation of certain other duals of orientifold planes as smooth Atiyah-Hitchin spaces [94]. Moreover, the singular domain walls

of the supergravity solution obtained by dualising D8-brane solutions are also smoothed out in the $K3$ geometry as Kaluza-Klein monopole geometries. The classification of Tian-Yau spaces leads to the maximum number of Kaluza-Klein monopoles being 18, which is precisely the maximum number of D8 branes possible in the type I' theory [95].

Taking the product of the 3-dimensional nilfold with the real line gave a space admitting a hyperkähler metric. Remarkably, a similar result applies for the nilmanifolds arising as higher dimensional analogues of the 3-dimensional nilfold [96]. Each of the spaces is a T^n bundle over T^m for some m, n . In each case, the space $\mathcal{M} \times \mathbb{R}$ admits a multi-domain wall type metric that has special holonomy [96], so that taking the product of the domain wall solution with Minkowski space gives a supersymmetric solution. Duality transformation of this supersymmetric solution gives an intersecting brane background [97], which preserves the same amount of supersymmetry. In each case, the multi-domain wall solution is dual to D4-D8 brane system [98, 99].

The objective of this thesis is to represent the relation between a geometry and duality symmetries in string theory. Particularly, we will consider the relation between a double space formalism and a T-duality transformation. In chapter 2, we begin by introducing a bosonic string theory. Next, we consider the bosonic string theory on a toroidal background. This leads to the T-duality symmetry on the string spectrum. After that we will provide a basic introduction to superstring theories and M-theory. The last section in this chapter will be the U-duality symmetry.

In chapter 3, we introduce the string field theory. We explain the basic features of the string field theory, such as, a closed string state, a closed string field action, and a gauge transformation of a closed string state. We also provide the example of a first massive closed string field state and a T-duality transformation of this state.

In chapter 4, we introduce Double Field theory and Extended Field theory. In this chapter, we will construct the Double Field Theory action from the massless closed string state on a toroidal background up to the cubic order. The action of DFT can be written in terms of the background metric \mathcal{E} and the dilaton d or in terms of the generalized metric \mathcal{H}_{MN} and the dilaton d . Next, we will discuss the gauge transformation of the DFT and the importance of strong constraint in order for the closure of gauge algebra. After that we will move to EFT and the strong constraint in EFT, which is known as the section constraint.

In chapter 5, we discuss the finite gauge transformations of DFT and EFT. In the first section, we consider the finite transformation of DFT. In the second section, the finite transformations of $E_4 = SL(5, \mathbb{R})$, $E_5 = SO(5, 5)$ and E_6 are considered. Next, we will discuss the finite transformation of the generalized tensor of DFT and EFT. The last section will devote to the generalized metric of DFT and EFT.

In chapter 6, we focus on the T-duality chain of special holonomy domain wall solutions.

This example can arise in string theory in solutions in which these backgrounds appear as fibres over a line. The cases with a 3-torus or a nilfold fibred over a line were obtained from identifications of suitable NS5-brane solutions, and are dual to D8-brane solutions. This T-duality chain implies that $K3$ should have a limit in which it degenerates to a long neck of the form of a nilfold fibred over a line capped off by suitable smooth geometries.

In chapter 7, we generalize the 3-dimensional nilfold to the higher dimensional analogues of the nilfold. In each case, the space admits a multi-domain wall type metric that has special holonomy, so that taking the product of the domain wall solution with Minkowski space gives a supersymmetric solution. This solution is dual to D4-D8-brane system.

In chapter 8, we will provide the double space formulation to the nilmanifold.

2. String Theories, M-theory, and Dualities

The interesting feature of string objects [5] - [8] is that strings can wrap along the compact dimension. As a result, it leads to the existence of winding modes that have not been seen in the particle theory. Along with the momentum mode, there exists a symmetry that exchanges momentum modes and winding modes known as “Target Space Duality” or T-Duality [9]. When supersymmetry is included, string theory becomes superstring theory. There are five different string theories in ten dimensions, namely, type IIA, type IIB, type I, Het $SO(32)$, and Het $E_8 \times E_8$. Upon a toroidal compactification, these theories are related to each other. In some limit, such as a strong coupling limit, type IIA theory becomes an eleven-dimensional theory, known as, M-theory. When M-theory is compactified on n -torus, the theory has another symmetry, known as U-duality [32]. In this chapter, the background on string theories is provided as well as dualities in string theories.

2.1. String theory

First, let us consider a string moving on a d -dimensional target space-time with coordinates $X^0 = t, X^1, \dots, X^{d-1}$. The two-dimensional string worldsheet can be parametrised by worldsheet coordinates (τ, σ) . The string evolution in the target space-time is given by the evolution of functions $X^\mu(\tau, \sigma)$, which is governed by the string action

$$\begin{aligned}
 S = & -\frac{1}{4\pi\alpha'} \int d^2\sigma \{ \sqrt{-\gamma} \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X^\rho) + \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X^\rho) \} \\
 & + \frac{1}{4\pi} \int d^2\sigma \sqrt{-h} \Phi(X^\rho) R^{(2)},
 \end{aligned} \tag{2.1.1}$$

where α' is a parameter relating to the string tension, $T = (2\pi\alpha')^{-1}$. The first term in the action (2.1.1) describes strings propagating on some general manifold \mathcal{M} with the metric $G_{\mu\nu}$. The second term introduces the effect of the Kalb-Ramond two-form $B_{\mu\nu}$ on the world-sheet. This term can be thought of as a pull-back of the two-form gauge field $B_{\mu\nu}$ on the target space-time \mathcal{M} on to the string world-sheet. Note that the second term changes by a total divergence

under the gauge transformation

$$\delta B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu, \quad (2.1.2)$$

where Λ is a one-form. The gauge-invariant field strength H , which is a three-form, can be defined as

$$H = dB, \quad (2.1.3)$$

or in terms of components

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \text{cyclic permutation}. \quad (2.1.4)$$

The last term in the action (2.1.1) describes the interaction between the dilaton field $\Phi(X^\mu)$ and the string world-sheet.

The conditions that the string action (2.1.1) is conformally invariance are given by [5]

$$\begin{aligned} 0 &= R_{\mu\nu} + \frac{1}{4} H_\mu{}^{\lambda\rho} H_{\nu\lambda\rho} - 2\nabla_\mu \nabla_\nu \Phi, \\ 0 &= \nabla_\lambda H^\lambda{}_{\mu\nu} - 2(\nabla_\lambda \Phi) H^\lambda{}_{\mu\nu} \\ 0 &= \frac{(D-26)}{3\alpha'} + 4(\nabla_\mu \Phi)^2 - 4\nabla_\mu \nabla^\mu \Phi + R + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho}, \end{aligned} \quad (2.1.5)$$

where $R_{\mu\nu}$ is a Ricci tensor, R is a Ricci scalar, and ∇_μ is a covariant derivative. The equations (2.1.5) has a physical interpretation. They are the equations of motion of the supergravity action

$$S = -\frac{1}{2\kappa^2} \int d^m x \sqrt{-G} e^{-2\Phi} \left(R - 4\nabla_\mu \Phi \nabla^\mu \Phi + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{(D-26)}{3\alpha'} \right). \quad (2.1.6)$$

This action describes the the long-wavelength limit of the interactions of the massless modes of the bosonic string.

2.2. Toroidal compactification

Following from [9], let us consider string theory on a D -dimensional space with n directions are toroidal compactified. The target space manifold can be expressed as a product between a d -dimensional Minkowski space-time and an n -torus, such that $\mathbb{R}^{d-1,1} \times T^n$ where $D = n + d$. In this case, the critical string theory, which has no Weyl anomalies, is considered. That means it can be either $D = 26$ for the bosonic string theory or $D = 10$ for superstring theory. The

string action is given by

$$S = -\frac{1}{4\pi\alpha'} \int_0^{2\pi} d\sigma \int d\tau \left(\sqrt{\gamma} \gamma^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j G_{ij} + \epsilon^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j B_{ij} \right), \quad (2.2.1)$$

where $\gamma_{\alpha\beta}$ is a world-sheet metric, $\epsilon^{\alpha\beta}$ is an antisymmetric tensor with $\epsilon^{01} = -1$, G_{ij} is a constant target space metric, and B_{ij} is a constant target space two-form.

In the action (2.2.1), the string coordinates X^i are split into non-compact directions represented by X^μ and compact directions represented by X^m ,

$$X^i = \{X^m, X^\mu\}, \quad (2.2.2)$$

where $\mu = 0, \dots, d-1$ and $m = 1, \dots, n$.

By using a notation and following from [9, 13], the constant background metric G_{ij} with an inverse metric G^{ij} satisfying $G^{ij}G_{jk} = \delta_k^i$ is written as

$$G_{ij} = \begin{pmatrix} \check{G}_{mn} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix}, \quad (2.2.3)$$

where \check{G}_{mn} is a flat metric on the n -torus T^n and $\eta_{\mu\nu}$ is a Minkowski metric on the $\mathbb{R}^{d-1,1}$. Similarly, the constant background two-form B_{ij} is written as

$$B_{ij} = \begin{pmatrix} \check{B}_{mn} & 0 \\ 0 & 0 \end{pmatrix}. \quad (2.2.4)$$

For later convenience, the background matrix E_{ij} [9] is defined by

$$E_{ij} \equiv G_{ij} + B_{ij} = \begin{pmatrix} \check{E}_{mn} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix}, \quad (2.2.5)$$

where $\check{E}_{mn} = \check{G}_{mn} + \check{B}_{mn}$.

In this case, it is restricted to the closed string theory, so that the string boundary conditions in compact directions and non-compact directions are given by

$$X^m(\sigma + 2\pi) = X^m(\sigma) + 2\pi w^m, \quad (2.2.6)$$

$$X^\mu(\sigma + 2\pi) = X^\mu(\sigma), \quad (2.2.7)$$

respectively, where w^m is a winding number and takes an integer value. It represents the number of times that string wraps along X^m coordinate.

Recall the action (2.2.1), since the critical string theory is considered, the string world-sheet metric can be chosen such that it is a Minkowski metric in 2-dimension,

$$\gamma_{\alpha\beta} = \eta_{\alpha\beta}. \quad (2.2.8)$$

By substituting this metric into (2.2.1), the action becomes

$$\begin{aligned} S &= -\frac{1}{4\pi\alpha'} \int_0^{2\pi} d\sigma \int d\tau \{ \eta^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j G_{ij} + \epsilon^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j B_{ij} \}, \\ &= -\frac{1}{4\pi\alpha'} \int_0^{2\pi} d\sigma \int d\tau \{ -\dot{X}^i \dot{X}^j G_{ij} + X'^i X'^j G_{ij} - 2\dot{X}^i X'^j B_{ij} \}, \end{aligned} \quad (2.2.9)$$

where $\dot{}$ and $'$ represent derivatives with respect to the world-sheet time-like coordinate τ and the space-like coordinate σ , respectively. The canonical momentum P_i conjugated to the coordinate X^i is defined as

$$P_i = \frac{\delta S}{\delta \dot{X}^i}. \quad (2.2.10)$$

Therefore, from the action (2.2.9), the canonical momentum is given by

$$2\pi P_i(\sigma, \tau) = G_{ij} \dot{X}^j(\sigma, \tau) + B_{ij} X'^j(\sigma, \tau). \quad (2.2.11)$$

A momentum excitation p_i from the canonical momentum is defined by

$$p_i = \int_0^{2\pi} d\sigma P_i. \quad (2.2.12)$$

Recall that from the Kaluza-Klein theory, the momentum excitation along the compact dimension p_m is quantised and normalised such that it takes an integer value. The reason for the Kaluza-Klein momentum must be quantised is because $\exp(ip_m Y^m)$ must be a single value function.

The expansion of modes for coordinate X^i is given by

$$X^i(\sigma, \tau) = x^i + w^i \sigma + \tau G^{ij} (p_j - B_{jk} w^k) + \frac{i}{\sqrt{2}} \sum_{n \neq 0} \frac{1}{n} (\bar{\alpha}_n^i e^{-in(\tau+\sigma)} + \alpha_n^i e^{-in(\tau-\sigma)}), \quad (2.2.13)$$

where x^i is the centre of mass of the string, α_n^i and $\bar{\alpha}_n^i$ are the n -mode oscillators for right-mover and left-mover, respectively. In this expression, there is no winding number in non-compact

directions,

$$w^i = \{w^m, 0\} \quad (2.2.14)$$

By substituting the coordinate expression (2.2.13) into the conjugate momentum expression (2.2.11), it becomes

$$2\pi P_i = p_i + \frac{1}{\sqrt{2}} \sum_{n \neq 0} (E_{ij} \bar{\alpha}_n^i e^{-in(\tau+\sigma)} + E_{ij}^T \alpha_n^i e^{-in(\tau-\sigma)}), \quad (2.2.15)$$

where E_{ij} is the background matrix defined in (2.2.5).

2.3. Hamiltonian and level-matching condition

In order to determine the spectrum of string theory, the Hamiltonian will be determined and its definition is given by

$$H = \int_0^{2\pi} d\sigma \underline{H}(\sigma, \tau), \quad (2.3.1)$$

where $\underline{H}(\sigma, \tau)$ is a world-sheet Hamiltonian density given by

$$\underline{H}(\sigma, \tau) = P_i \dot{X}^i + \frac{1}{4\pi} \left(-\dot{X}^i \dot{X}^j G_{ij} + X'^i X'^j G_{ij} - 2\dot{X}^i X'^j B_{ij} \right). \quad (2.3.2)$$

By substituting the coordinate expression (2.2.13) and the momentum expression (2.2.15) into the above equation, the Hamiltonian density becomes

$$4\pi \underline{H} = \begin{pmatrix} X' & 2\pi P \end{pmatrix} \mathcal{H}(E) \begin{pmatrix} X' \\ 2\pi P \end{pmatrix}, \quad (2.3.3)$$

where $\mathcal{H}(E)$ is a $2D \times 2D$ symmetric matrix and constructed from the metric G_{ij} and the two-form B_{ij} . It is known as the generalized metric and takes the form

$$\mathcal{H}(E) = \begin{pmatrix} G_{ij} - B_{ik} G^{kl} B_{lj} & B_{ik} G^{kj} \\ -G^{ik} B_{kj} & G^{ij} \end{pmatrix}. \quad (2.3.4)$$

Therefore, the Hamiltonian can be calculated by substituting the expression of the coordinate (2.2.13) and the canonical momentum (2.2.15) into the Hamiltonian density (2.3.3). The result

is

$$H = \frac{1}{2}Z^T \mathcal{H}(E)Z + \frac{1}{2} \sum_{n \neq 0} (\bar{\alpha}_{-n}^i G_{ij} \bar{\alpha}_n^i + \alpha_{-n}^i G_{ij} \alpha_n^j). \quad (2.3.5)$$

However the Hamiltonian (2.3.5) is not in the normal-ordering due to ambiguous order in the second term. By performing normal-ordering and discarding the constant from the normal-ordering, the Hamiltonian becomes

$$H = \frac{1}{2}Z^T \mathcal{H}(E)Z + N + \bar{N}, \quad (2.3.6)$$

where Z is a generalized momentum, that unifies the momentum excitations p_i with the winding modes w^i , and defined by

$$Z = \begin{pmatrix} w^i \\ p_i \end{pmatrix}, \quad (2.3.7)$$

and N, \bar{N} are number operators for right and left-moving modes, and written by

$$N = \sum_{n>0} (\alpha_{-n}^i G_{ij} \alpha_n^j), \quad (2.3.8)$$

$$\bar{N} = \sum_{n>0} (\bar{\alpha}_{-n}^i G_{ij} \bar{\alpha}_n^i). \quad (2.3.9)$$

In the string theory, the physical state $|\phi\rangle$ satisfies the Virasoro constraints

$$L_0 - a|\phi\rangle = 0, L_m|\phi\rangle = 0, \quad (2.3.10)$$

$$\bar{L}_0 - a|\phi\rangle = 0, \bar{L}_m|\phi\rangle = 0, \text{ with } m > 0. \quad (2.3.11)$$

These conditions give rise to the level-matching condition which takes the form,

$$L_0 - \bar{L}_0|\phi\rangle = 0. \quad (2.3.12)$$

After substitute the expression of L_0 and \bar{L}_0 , the level-matching condition becomes

$$L_0 - \bar{L}_0 = N - \bar{N} - p_i w^i = 0. \quad (2.3.13)$$

As a result, the level-matching condition gives

$$\begin{aligned} N - \bar{N} &= p_i w^i, \\ &= \frac{1}{2}Z^T \eta Z, \end{aligned} \quad (2.3.14)$$

where Z is the generalized momentum defined in (2.3.7) and η is a constant matrix which will

play a major role in the next section and defined as

$$\eta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad (2.3.15)$$

with $\mathbb{1}$ is an identity $D \times D$ matrix.

2.4. T-duality and $O(n, n, \mathbb{Z})$

From the previous section, the Hamiltonian (2.3.6) and the level-matching condition (2.3.14) are obtained. Now let us consider the transformation symmetry of the generalized momentum Z such that

$$Z \rightarrow Z' = h^T Z, \quad (2.4.1)$$

where h is a transformation matrix that mixes w^m and p_m after operating on the generalized momentum. The requirement of this transformation is that the level-matching condition and the Hamiltonian are preserved. Therefore, from the level-matching condition and (2.4.1), it gives

$$\begin{aligned} N - \bar{N} = \frac{1}{2} Z'^T \eta Z' &= \frac{1}{2} Z^T \eta Z \\ &= \frac{1}{2} Z'^T h \eta h^T Z'. \end{aligned} \quad (2.4.2)$$

From the above relation, the transformation matrix h must preserve η

$$h \eta h^T = \eta. \quad (2.4.3)$$

That means h is an element of $O(D, D, \mathbb{R})$ group and η is an $O(D, D, \mathbb{R})$ invariant metric. Since we must encounter this group several times in this thesis, let us introduce the basic feature of this group.

The element h belongs to the $O(D, D, \mathbb{R})$ group if it preserves the $O(D, D, \mathbb{R})$ invariant metric η

$$O(D, D, \mathbb{R}) = \{h \in GL(2D, \mathbb{R}) : h \eta h^T = \eta\}. \quad (2.4.4)$$

Let a , b , c , and d be $D \times D$ matrices, h can be represented in terms of these matrices such that

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad (2.4.5)$$

The condition in which h preserves η gives the conditions for a , b , c , and d , namely,

$$a^T c + c^T a = 0, b^T d + d^T b = 0, \text{ and } a^T d + c^T b = \mathbb{1}. \quad (2.4.6)$$

From (2.3.6), let us consider the first term which is

$$H_0 = \frac{1}{2} Z^T \mathcal{H}(E) Z. \quad (2.4.7)$$

This term which is invariant under the $O(D, D, \mathbb{R})$ transformation induces the transformation property for $\mathcal{H}(E)$

$$\begin{aligned} Z'^T \mathcal{H}(E') Z' &= Z^T \mathcal{H}(E) Z, \\ &= Z'^T h \mathcal{H}(E) h^T Z'. \end{aligned} \quad (2.4.8)$$

From the above equation, the generalized metric transforms as

$$\mathcal{H}(E') = h \mathcal{H}(E) h^T. \quad (2.4.9)$$

From (2.4.9), it leads to the transformation rule for E by the following method. First, the generalized metric is formulated in terms of a vielbein h_E which is an $O(D, D, \mathbb{R})$ element

$$\mathcal{H}(E) = h_E h_E^T, \quad (2.4.10)$$

and h_E is defined by

$$h_E = \begin{pmatrix} e & B(e^T)^{-1} \\ 0 & (e^T)^{-1} \end{pmatrix}, \quad (2.4.11)$$

where e is a vielbein of the metric $G = ee^T$. Next, the action of $O(D, D, \mathbb{R})$ group element h on $D \times D$ matrix F is defined by

$$h(F) = (aF + b)(cF + d)^{-1}. \quad (2.4.12)$$

From this group action, the background matrix E is obtained from

$$E = h_E(\mathbb{1}). \quad (2.4.13)$$

From (2.4.9), the transformed vielbein h'_E is obtained from the original h_E

$$h_{E'} = h h_E. \quad (2.4.14)$$

Therefore, the transformation rule for E is obtained by

$$E' = h_{E'}(\mathbb{1}) = h h_E(\mathbb{1}) = h(E) = (aE + b)(cE + d)^{-1}.. \quad (2.4.15)$$

In order that the full Hamiltonian is invariant under $O(D, D, \mathbb{R})$ transformation, N , and \bar{N} should be invariant under this transformation. From the transformation rule for E (2.4.15), the symmetric part of E' is corresponding to G' , then we get the relation between G and G' [9]

$$(d + cE)^T G' (d + cE) = G, \quad (2.4.16)$$

$$(d - cE^T)^T G' (d - cE^T) = G. \quad (2.4.17)$$

After the transformation of the metric is obtained, and using the commutation relations between the oscillator

$$[\alpha_m^i(E), \alpha_n^j(E)] = [\bar{\alpha}_m^i(E), \bar{\alpha}_n^j(E)] = m G^{ij} \delta_{m+n,0}. \quad (2.4.18)$$

The transformation rules for α_m^i and $\bar{\alpha}_m^i$ are obtained [9]

$$\alpha_n(E) \rightarrow (d - cE^T)^{-1} \alpha_n(E'), \quad (2.4.19)$$

$$\bar{\alpha}_n(E) \rightarrow (d + cE)^{-1} \bar{\alpha}_n(E'). \quad (2.4.20)$$

Therefore, the number operators are invariant. This means the full spectrum is invariant under the $O(D, D, \mathbb{R})$ transformation.

Moreover, there is another symmetry which is known as the world-sheet parity. The operation of the symmetry flips the sign of the two-form ($B \rightarrow -B$) and exchanges the right-moving and left-moving oscillators into each other as

$$\alpha_n \leftrightarrow \bar{\alpha}_n. \quad (2.4.21)$$

The full Hamiltonian is also invariant under this action.

As we mention before, from the restriction that w^m and p_m take the discrete values due to the boundary condition of the n -dimensional toroidal space, so that the symmetry group should be restricted to $O(n, n, \mathbb{Z})$ subgroup of $O(D, D, \mathbb{R})$. This $O(n, n, \mathbb{Z})$ is known as the T-duality group in string theory. However, it is useful to represent $h \in O(n, n, \mathbb{Z})$ in terms of $O(D, D, \mathbb{R})$ representation and represented as

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (2.4.22)$$

with

$$a = \begin{pmatrix} \check{a} & 0 \\ 0 & \mathbf{1} \end{pmatrix}, b = \begin{pmatrix} \check{b} & 0 \\ 0 & 0 \end{pmatrix}, c = \begin{pmatrix} \check{c} & 0 \\ 0 & 0 \end{pmatrix}, d = \begin{pmatrix} \check{d} & 0 \\ 0 & \mathbf{1} \end{pmatrix}, \quad (2.4.23)$$

where \check{a} , \check{b} , \check{c} , and \check{d} are $n \times n$ matrices and can be rearranged in terms of $O(n, n, \mathbb{Z})$ element \check{h} as

$$\check{h} = \begin{pmatrix} \check{a} & \check{b} \\ \check{c} & \check{d} \end{pmatrix}. \quad (2.4.24)$$

In this report, the representation of $O(D, D)$ and $O(n, n)$ are both used.

2.5. Example of $O(n, n, \mathbb{Z})$ transformation

In previous section, the string theory on the space with n dimensions are toroidal compactified background leads to the existence T-duality $O(n, n, \mathbb{Z})$ group. In this section, the examples of the $O(n, n, \mathbb{Z})$ element are provided. At the point, one wonders that every $O(n, n, \mathbb{Z})$ can be used to generate the transformation. However, the answer is no because there are some group elements that break the upper triangle of the vielbein (2.4.11) after transformation. These kinds of group elements do not give the metric and the two-form in the transformed theory, whereas they introduce the bivector β^{ij} . So that in this section, we will focus only on group elements that preserve the upper triangle of (2.4.11).

Integer theta-parameter shift Θ_{mn}

The first $O(n, n, \mathbb{Z})$ element that we would like to introduce is the theta-parameter shift Θ_{mn} . In the string-world sheet action, the term that correspond to the constant two-form in fact gives the total derivative. That means if the two-form is shifted by the constant integer, it will not contribute to the path integral because it gives only the topological contribution. On the other hand, this transformation can be thought of as a two-form gauge transformation such that

$$B_{mn} \rightarrow B_{mn} + \Theta_{mn}. \quad (2.5.1)$$

The group elements that correspond to the theta-parameter shift are

$$\check{h}_\Theta = \begin{pmatrix} \mathbf{1} & \Theta \\ 0 & \mathbf{1} \end{pmatrix}, \quad (2.5.2)$$

where $\Theta_{mn} \in \mathbb{Z}$ and $\Theta_{mn} = -\Theta_{mn}$.

Basis change A

The n -torus T^n is the quotient space of \mathbb{R}^n with the lattice Λ . The transformation of the lattice Λ by the $GL(n, \mathbb{Z})$ transformation does not change the torus. Thus, the spectrum is invariant under this transformation. The group element of this transformation can be represented as

$$\check{h}_A = \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix}, \quad (2.5.3)$$

where $A \in GL(n, \mathbb{Z})$.

Factorized duality T_k

The factorized duality T_k is corresponding to the exchange of the radius $R_k \rightarrow 1/R_k$ along the circle in X^k direction and leaves the other direction unchanged. This gives rise to the interchange between the winding mode and the momentum mode in this direction,

$$w^k \leftrightarrow p_k. \quad (2.5.4)$$

In the literature, this transformation is referred to the T-duality along X^k direction. The group elements that represent this transformation are

$$\check{h}_{T_k} = \begin{pmatrix} \mathbb{1} - e_k & e_k \\ e_k & \mathbb{1} - e_k \end{pmatrix}, \quad (2.5.5)$$

where e_k is a matrix that has zero component everywhere except kk component. Not only the winding mode and momentum excitation are exchanged, but also some component of the metric and the 2-form in the compact direction. This is known as the Buscher rules [100, 101].

Inversion

The transformation that interchange $R_k \rightarrow 1/R_k$ have been previously discussed. If one try to do n successive factorized T-dualities in all the n -dimensional compact space, it gives the inversion of the background matrix E ,

$$\check{E} = \check{G} + \check{B} \rightarrow \check{E}' = \check{G}' + \check{B}' = E^{-1}. \quad (2.5.6)$$

The group element is represented by the $O(n, n)$ invariant metric

$$\check{h}_I = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \quad (2.5.7)$$

2.6. Superstring theories and M-theory

In the previous section, the toroidal compactification of the bosonic string is discussed. In this section, we will consider superstring theories [5] - [8]. There are five different superstring theories in ten dimensions, namely, type IIA, type IIB, type I, heterotic $SO(32)$, and heterotic $E_8 \times E_8$. Upon compactification on a circle, these superstring theories are dual to each others. Moreover, the strong coupling limit of type IIA and heterotic $E_8 \times E_8$ can be described by an 11-dimensional theory known as M-theory.

2.6.1. Open superstring

Consider the generalization of the open bosonic string theory to include fermions, in the conformal gauge, the action is given by

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left(\partial X^\mu \bar{\partial} X_\mu - i\psi^\mu \bar{\partial} \psi_\mu - i\tilde{\psi}^\mu \partial \tilde{\psi}_\mu \right), \quad (2.6.1)$$

where the world-sheet of the open string is the strip $0 < \sigma < \pi, -\infty < \tau < \infty$. The world-sheet fermions admit two possible boundary conditions, namely Ramond sector (R), and Neveu-Schwarz sector (NS).

$$\begin{aligned} \text{R sector} \quad : \quad \psi^\mu(0, \tau) &= \tilde{\psi}^\mu(0, \tau) \\ \psi^\mu(\pi, \tau) &= \tilde{\psi}^\mu(\pi, \tau) \end{aligned} \quad (2.6.2)$$

$$\begin{aligned} \text{NS sector} \quad : \quad \psi^\mu(0, \tau) &= -\tilde{\psi}^\mu(0, \tau) \\ \psi^\mu(\pi, \tau) &= \tilde{\psi}^\mu(\pi, \tau). \end{aligned} \quad (2.6.3)$$

The mode expansion of the fermions is given by

$$\psi^\mu(z) = \sum_r \frac{\psi_r^\mu}{z^{r+\frac{1}{2}}}, \text{ where } r \in \mathbb{Z} \text{ (R) or } r \in \mathbb{Z} + \frac{1}{2} \text{ (NS)}. \quad (2.6.4)$$

Along with the mode expansion of the bosonic excitation α_m^μ , the mass formula of the open string is given by

$$M^2 = \frac{1}{\alpha'} \left(\sum_{r,n} \alpha_n \cdot \alpha_n + r\psi_{-r} \cdot \psi_r - a \right), \quad (2.6.5)$$

where a is related to the zero point energy

$$a = \frac{1}{2} \text{ (NS) or } 0 \text{ (R)}. \quad (2.6.6)$$

The ground state of NS sector is a tachyon and is a Lorentz singlet with odd fermion number, which is a state of eigenvalue -1 of the operator $(-1)^F$.

In order to get the spacetime supersymmetry in this theory, the tachyon state need to be projected out, while keeping the massless state. This projection is known as the GSO projection [102]. The GSO projection will project out the state with odd fermion number in the NS sector. The tachyon state has $(-1)^F = -1$ so it is removed from the spectrum. After the GSO projection, the ground state in the NS sector is tachyon-free. This massless state of the NS sector is a vector field, A^μ ,

$$\psi_{-\frac{1}{2}}^\mu |k\rangle, \quad M^2 = 0 \quad (2.6.7)$$

Since a massless particle state in ten dimensions is classified by $SO(8)$, which is little group of $SO(1,9)$, this massless state is a vector representation of $SO(8)$, $\mathbf{8}_v$.

In the R sector, the ground state is massless since the zero point energy vanishes. This state is a spinor state, which has 32 complex components in ten dimensions. However, by imposing Majorana and Weyl constraints, these will reduce the number of components to 16 real numbers. With the Dirac equation in ten dimensions, the degrees of freedom of Majorana-Weyl spinor will be 8 propagating modes. There are two types of massless Majorana-Weyl spinors in ten dimensions. Both of them have 8 degrees of freedom, labeled by $\mathbf{8}_s$, and $\mathbf{8}_c$. In the R sector, the GSO projection picks out the $\mathbf{8}_s$. The other choice, $\mathbf{8}_c$, can also be chosen. The difference between these two spinors is meaningful if only they are both present.

From the GSO projection of the NS sector and the R sector, the ground state spectrum of the open string theory is $\mathbf{8}_v \oplus \mathbf{8}_s$. This is a vector multiplet in ten dimensions with $\mathcal{N} = 1$ supersymmetry. When the Chan-Paton factors are included, this gives the $U(N)$ gauge theory in the oriented theory and the $SO(N)$ or the $USp(N)$ in the unoriented theory.

This theory on its own is inconsistent. One of the reason is this theory is a chiral theory, for example, the fermion $\mathbf{8}_s$ has a specific chirality. The chiral theory contains gauge and gravity anomalies in ten dimensions.

2.6.2. Closed string theory: type II

The closed string state is the product of two open string states with the level matching condition. The two choices of GSO projection in the open string theory are equivalent. However, in the closed string theory there are two inequivalent choices of the ground state, namely,

$$\begin{aligned} \text{Type IIA} & : (\mathbf{8}_v \oplus \mathbf{8}_s) \otimes (\mathbf{8}_v \oplus \mathbf{8}_c) \\ \text{Type IIB} & : (\mathbf{8}_v \oplus \mathbf{8}_s) \otimes (\mathbf{8}_v \oplus \mathbf{8}_s). \end{aligned} \quad (2.6.8)$$

The ground state of the closed string theory can be classified into 4 sectors, namely, NS-NS, NS-R, R-NS, and R-R sectors.

Type IIA and IIB have the same NS-NS sector, which is

$$\mathbf{8}_v \otimes \mathbf{8}_v = G_{\mu\nu} \oplus B_{\mu\nu} \oplus \Phi. \quad (2.6.9)$$

In the R-R sector, the states are the product of two spinors giving the space of n -form as

$$\text{Type IIA} : \mathbf{8}_s \otimes \mathbf{8}_c = [1] \oplus [3] \quad (2.6.10)$$

$$\text{Type IIB} : \mathbf{8}_s \otimes \mathbf{8}_s = [0] \oplus [2] \oplus [4]_+, \quad (2.6.11)$$

where $[n]$ represents n -form of $SO(8)$, and $[4]_+$ represents self-dual 4-form. In NS-R and R-NS, the products $\mathbf{8}_v \otimes \mathbf{8}_c$ and $\mathbf{8}_s \otimes \mathbf{8}_s$ are given by

$$\mathbf{8}_v \otimes \mathbf{8}_c = \mathbf{8}_s \oplus \mathbf{56}_c, \quad (2.6.12)$$

$$\mathbf{8}_v \otimes \mathbf{8}_s = \mathbf{8}_c \oplus \mathbf{56}_s, \quad (2.6.13)$$

where $\mathbf{56}_c$ and $\mathbf{56}_s$ represents gravitinos with different chiralities.

The massless spectrums of type IIA and type IIB are summary as

$$\text{IIA} : G_{\mu\nu} \oplus B_{\mu\nu} \oplus \Phi \oplus [1] \oplus [3] \oplus \mathbf{8}_s \oplus \mathbf{56}_c \oplus \mathbf{8}_c \oplus \mathbf{56}_s \quad (2.6.14)$$

$$\text{IIB} : G_{\mu\nu} \oplus B_{\mu\nu} \oplus \Phi \oplus [0] \oplus [2] \oplus [4]_+ \oplus \mathbf{8}_c \oplus \mathbf{56}_s \oplus \mathbf{8}_c \oplus \mathbf{56}_s \quad (2.6.15)$$

In the type IIA, the two gravitinos (and supercharges) have different chiralities, while they are the same in the type IIB. Type IIB is a chiral theory, therefore, should have a gravitational anomaly. However, the massless spectrum is anomaly free. This is because the cancellation between the anomalies of $\mathbf{8}_c$, $\mathbf{56}_s$ and $[4]_+$.

Now, let us consider the role of each fields in the massless spectrums of type IIA and IIB. The first one is the Kalb-Ramond two-form, $B_{\mu\nu}$. This field is electrically coupled to the fundamental string, F1, and magnetically coupled to the NS5-brane. This two-form are present in both type IIA and IIB, therefore, both theories have the F1-string and the NS5-brane.

In the type IIA, the R-R sector contains odd-forms, namely, 1-form and 3-form. This 1-form is electrically coupled to the D0-brane and is magnetically coupled to the D6-brane. The 3-form is electrically coupled to the D2-brane and is magnetically to the D4-brane. Therefore, apart from F1-string and NS5-brane, type IIA string also contains Dp -branes, where p is even. These objects are solutions of the supergravity theory of type IIA, which is a low-energy limit of string theory of type IIA. There is also an D8-brane, which is a solution of massive supergravity [103].

In the type IIB, the R-R sector contains fields of even-forms, namely, 0-form, 2-form, and self-

dual 4-form. the 0-form is electrically coupled to the $D(-1)$ -brane and is magnetically coupled to the D7-brane. The 2-form is electrically coupled to the D1-brane and is magnetically coupled to the D5-brane. the self-dual 4-form are both electrically and magnetically coupled to the D3-brane. These Dp -branes are solutions of supergravity theory of type IIB. There is also an D9-brane, which appears in type I string.

2.6.3. Type I

From the last section, the spectra of closed string theories have been shown. In the type IIB, the left-moving and the right moving sections have the same GSO projection, for example, they have the same chiralities in the R sector. Therefore, the spectrum of type IIB is invariant under the world-sheet parity Ω , which is a symmetry that reverses left- and right-moving oscillators. By gauging type IIB with Ω , the unoriented string theory is obtained. This theory is known as type I string theory.

The spectrum of type I consists of even states under Ω of type IIB. In the NS-NS sector, $G_{\mu\nu}$ and Φ are symmetric under Ω and survive the projection, while, $B_{\mu\nu}$ is projected out. In the R-R sector, $[2]$ state is an even state of Ω . In the NS-R sector and the R-NS sector, one linear combination of gravitinos survives the projection so as the supercharges. Therefore, the massless closed string states of type I consist of $G_{\mu\nu}$, Φ , an RR two-form, and a gravitino. These are an $\mathcal{N} = 1$ supergravity multiplet in ten dimensions. This theory in fact contains an anomaly because the state $\mathbf{8}_c$, $\mathbf{56}_s$ and $[4]_+$ has been projected out. This theory needs an additional sector to cancel the anomaly. The additional sector is an $\mathcal{N} = 1$ vector multiplet with the gauge group $SO(32)$, which is the massless sector of the unoriented open string theory.

Type I can be thought of another way as when projecting the type IIB with Ω , the spacetime of type IIB is filled with O9-brane making the unoriented string theory. The anomaly appears because of the O9-charge which has -16 unit of the D9-brane charge. Therefore, to get a consistent theory, 16 D9-branes need to be filled in the spacetime to cancel the O9-brane charge. This is equivalent to introduce open string states that end on D9-branes in the theory. When the 16 D9-branes are coincide with O9-brane, the gauge group $SO(32)$ is obtained. This theory contains the RR two-form, therefore, it includes D1-brane and D5-brane.

2.6.4. Heterotic string theory

In additions to the three superstring theories, namely, type IIA, IIB, I, there are two more superstring theories in ten dimensions. They have non-Abelian gauge symmetries, namely, $SO(32)$, or $E_8 \times E_8$. The word ‘heterotic’ means that these theories are a hybrid of two different constructions of the string theory on the left and the right sectors. On the right moving sector, this is a state of superstring theory, while the left moving state is the bosonic

string theory, which is the consistent theory in 26 dimensions.

The heterotic theory makes sense in ten dimensions if the left moving sector is compactified on a 16-torus, T^{16} . The generic gauge group $U(1)^{16}$ can be enhanced to one of two rank 16 gauge groups, namely, $SO(32)$, and $E_8 \times E_8$. This enhance gauge symmetry happens when the lattice Λ , which defines 16-torus as \mathbb{R}^{16}/Λ , are self-dual even lattice. There are two self-dual even lattices in 16 dimensions corresponding to Γ^{16} or $\Gamma^8 \times \Gamma^8$. The Γ^{16} gives the $SO(32)$ gauge group, while $\Gamma^8 \times \Gamma^8$ gives the $E_8 \times E_8$ gauge group. The massless spectra of these theories are the same of type I, which consist of $N = 1$ supergravity multiplet and $N = 1$ vector multiplet with gauge group $SO(32)$, or $E_8 \times E_8$.

2.6.5. M-theory

In the type IIA string theory, there are two free parameters, namely, a string coupling, g_s , and a string length, l_s . The tension of the D p -brane in type IIA is given by

$$T_{Dp} \sim \frac{1}{g_s l_s^{p+1}}. \quad (2.6.16)$$

Consider the tension of the D0-brane which is

$$T_{D0} \sim \frac{1}{g_s l_s}. \quad (2.6.17)$$

at the strong coupling limit, $g_s \rightarrow \infty$, this state becomes light. For any number n of D0-branes, there is a short multiplet of a bounded state with masses nT_{D0} . This mass is exact since D0-brane is a BPS state. This spectrum matches the spectrum of the momentum state of the higher dimensional theory compactified on a circle with a radius R given by

$$R = g_s l_s. \quad (2.6.18)$$

Therefore, as $g_s \rightarrow \infty$, the eleventh dimension appears. This strong coupling theory of type IIA string is known as M-theory. There are 2 parameters in M-theory on a circle, namely, the radius of eleventh dimension, R , and the Planck's constant, l_p . These two parameters of M-theory are relate to those of type IIA as

$$R = g_s l_s \quad (2.6.19)$$

$$l_p^3 = g_s l_s^3. \quad (2.6.20)$$

The low-energy limit of M-theory is the eleven-dimensional supergravity. The field contents of this supergravity are the metric, G_{MN} , the three-form, A_{MNP} , and the gravitino, Ψ_M . The three-form is electrically coupled to the M2-brane and is magnetically coupled to the M5-brane.

Tensions of the M2-brane and the M5-brane are proportional to the Planck constant, l_p , in the 11 dimensions as

$$T_{\text{M2}} \sim \frac{1}{l_p^3}, \quad T_{\text{M5}} \sim \frac{1}{l_p^6}. \quad (2.6.21)$$

Upon compactification on S^1 , the ten dimensional supergravity of type IIA can be obtained. For example, the eleven-dimensional metric gives the ten-dimensional metric and the 1-form,

$$G_{MN} \rightarrow G_{\mu\nu}, A_{(1)}. \quad (2.6.22)$$

The three-form gives the ten-dimensional three form and the two-form,

$$A_{MNP} \rightarrow B_{\mu\nu}, A_{(3)}. \quad (2.6.23)$$

The BPS states in type IIA, namely, the F1-string, the NS5-brane, and Dp -branes, where p is even, can be obtained from the M2 and the M5-brane. F1-strings are obtained from wrapped M2-branes along the circle. The tension of wrapped M2-branes is

$$T_{\text{wrapped M2}} \sim \frac{R}{l_p^3}. \quad (2.6.24)$$

From the relations (2.6.19) and (2.6.20), the tension of F1-string is obtained

$$T_{\text{F1}} \sim \frac{1}{l_s^2}. \quad (2.6.25)$$

The other BPS-states are summary in the below table [60].

M-theory	Mass/tension	type IIA
wrapped M2-brane	$\frac{R}{l_p^3} = \frac{1}{l_s^2}$	F1-string
M2-brane	$\frac{1}{l_p^3} = \frac{1}{g_s l_s^3}$	D2-brane
wrapped M5-brane	$\frac{R}{l_p^6} = \frac{1}{g_s l_s^5}$	D4-brane
M5-brane	$\frac{1}{l_p^6} = \frac{1}{g_s^2 l_s^6}$	NS5-brane
KK-mode	$\frac{1}{R} = \frac{1}{g_s l_s}$	D0-brane
KK-monopole	$\frac{R^2}{l_p^9} = \frac{1}{g_s l_s^7}$	D6-brane

The origin of D6-brane in M-theory is the Kaluza-Klein monopole solution. This solution is a product of the flat-metric and the Taub-NUT space,

$$\begin{aligned} ds_{11}^2 &= ds^2(\mathbb{R}^{1,6}) + ds_{TN}^2(y), \\ ds_{TN}^2 &= H dy^i dy^i + H^{-1} (d\psi + V_i(y) dy^i)^2, \end{aligned} \quad (2.6.26)$$

where $H = 1 + \frac{k}{|y|}$ and V_i satisfies $\nabla \times V = \nabla \cdot H$. This solution is localised in the four Taub-NUT directions. Upon compactification in ψ -direction, it becomes D6-brane in type IIA.

2.7. U-duality

In the previous section, the T-duality group of the bosonic string theory is considered. In this section, the T-duality of superstring theory is considered. Along with the S-duality of Type IIB, the large group of duality, which is known as U-duality [32] is constructed.

2.7.1. T-duality of type IIA and IIB

As we mentioned before, upon compactification on the n -torus, T^n , the T-duality group is $O(n, n, \mathbb{Z})$. The momentum state, p_m , and winding state, w^m , of strings is the fundamental representation of $O(n, n)$ group. There are also states coming from D p -brane wraps along the n -torus which will be the representation of $O(n, n)$ group.

For example, let us consider a simplest case, the compactification of type IIA on the S^1 with the radius R . The momentum mode with the mass $\frac{1}{R}$ is T-dual to winding mode with the mass $\frac{R}{l_s^2}$ as we mentioned this before. The interesting part is the D p -brane wrapping state. In type IIA, there are D0-branes, D2-branes, D4-branes, and D6-branes. Let us consider the D2-brane wrapping on the S^1 . The tension is

$$T \sim \frac{R}{g_s l_s^3}. \quad (2.7.1)$$

Using T-duality along S^1 ,

$$R \leftrightarrow \frac{l_s^2}{R}, g_s \leftrightarrow g_s \frac{l_s}{R}, \quad (2.7.2)$$

the D2-brane wrapping state becomes

$$T \sim \frac{R}{g_s l_s^3} \rightarrow \frac{1}{g_s l_s^2}. \quad (2.7.3)$$

This is a tension of the D1-brane. That means T-duality transformation transforms type IIA into IIB, and vice versa.

The wrapped states of the D p -brane in type IIA transform into the unwrapped state of D($p-1$)-brane. In ten dimensions, type IIA and type IIB is distinguishable. Upon compactifying on the n -torus, these two theories become the same theory. The states between two theories are related by $O(n, n, \mathbb{Z})$, T-duality group.

2.7.2. S-duality of type IIB

In type IIB, there is another duality that relates the strong coupling limit of type IIB with the weak coupling limit of type IIB. This duality is known as S-duality. In type IIB, there are two types of strings, namely, the F1-string, and the D1-string. The simplest S-duality exchange the role of the F1-string and the D1-string. The full S-duality group is $SL(2, \mathbb{Z})$ [32]. This $SL(2, \mathbb{Z})$ relates the F1-string not only to the D1-string but to the (p, q) -string which is a bound state of p F1-strings and q D1-strings for p, q are relatively prime.

2.7.3. U-duality

Upon toroidal compactification on the n -torus, type IIA and type IIB are related by $O(n, n, \mathbb{Z})$, T-duality group. This T-duality group does not commute with $SL(2, \mathbb{Z})$, S-duality group. They are the discrete subgroup of a larger group, which known as U-duality group. The U-duality for type II string compactifies on the n -torus or M-theory on the $(n+1)$ -torus is $E_{n+1, (n+1)}(\mathbb{Z})$ [32]. The notation $E_{d, (d)}$ denotes a non-compact form of the exceptional group E_d .

The scalar fields in these theories are given by the coset space [104, 105]

$$E_{n+1, (n+1)} / H_{n+1}. \quad (2.7.4)$$

The scalar manifolds in each dimension are given by

D	d	$E_{d, (d)}$	H_d	$E_{d, (d)} / H_d$
10	1	\mathbb{R}^+	1	\mathbb{R}^+
9	2	$SL(2, \mathbb{R}) \times \mathbb{R}^+$	$U(1)$	$SL(2, \mathbb{R}) / U(1) \times \mathbb{R}^+$
8	3	$SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$	$SO(3) \times U(1)$	$SL(3, \mathbb{R}) / SO(3) \times SL(2, \mathbb{R}) / U(1)$
7	4	$SL(5, \mathbb{R})$	$SO(5)$	$SL(5, \mathbb{R}) / SO(5)$
6	5	$SO(5, 5, \mathbb{R})$	$SO(5) \times SO(5)$	$SO(5, 5, \mathbb{R}) / (SO(5) \times SO(5))$
5	6	$E_{6, (6)}$	$USp(8)$	$E_{6, (6)} / USp(8)$
4	7	$E_{7, (7)}$	$SU(8)$	$E_{7, (7)} / SU(8)$
3	8	$E_{8, (8)}$	$SO(16)$	$E_{8, (8)} / SO(16)$

In each dimension, the representations of $E_{d, (d)}$ can be interpreted in terms of momentum states along T^d , and M2-and M5-brane wrapping states. U-duality will relate momentum states with M2- or M5-brane wrapping states. The detail in each dimension will be mentioned in chapter 4, when we consider extended field theory.

3. String Field Theory

The main objective of the string field theory is to explain string dynamics in terms of target space fields by introducing a string field $\Psi(X^i)$. Due to a gauge fixing of the world-sheet symmetry, there are ghost and anti-ghost modes, which also contribute to the string field $\Psi(X^i, c, b)$. Based on [12], the ways of constructing string fields in terms of a basis state of conformal field theory will be discussed in the next section.

3.1. Conformal field theory

In a two-dimensional conformal field theory with a complex coordinate $z = \exp(\tau + i\sigma)$, a holomorphic field $\phi(z)$ is a primary field of dimension d if

$$T(z)\phi(w) \sim \frac{d}{(z-w)^2}\phi(w) + \frac{1}{(z-w)}\partial_w\phi(w) + \dots, \quad (3.1.1)$$

where $T(z)$ is a stress tensor. A transformation law of the conformal transformation $z' = f(z)$ is

$$\phi'(z')(dz')^d = \phi(z)(dz)^d. \quad (3.1.2)$$

The primary field $\phi(z)$ of dimension d can be expanded in terms of Laurent series as

$$\phi(z) = \sum_n \frac{\phi_n}{z^{n+d}}, \quad (3.1.3)$$

where

$$\phi_n = \int \frac{dz}{2\pi i} z^{n+d-1} \phi(z). \quad (3.1.4)$$

A conformal vacuum $|0\rangle$, which is equivalent to an asymptotic past $z = 0$, is defined by

$$\phi_n|0\rangle = 0 \text{ for } n \geq -d + 1. \quad (3.1.5)$$

Its dual $\langle 0|$, which is equivalent to an asymptotic future $z = \infty$, is defined by

$$\langle 0|\phi_n = 0 \text{ for } n \leq d-1. \quad (3.1.6)$$

The bosonic string theory is a conformal field theory and the action is given by

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-\gamma} \gamma^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j \eta_{ij}, \quad (3.1.7)$$

where η_{ij} is a flat target space metric, $\gamma_{\alpha\beta}$ is a world-sheet metric and $X^i(\sigma)$ are target space coordinates. By fixing the gauge, $\gamma_{\alpha\beta} = e^\phi \eta_{\alpha\beta}$, the string action becomes

$$S = -\frac{1}{4\pi\alpha'} \int d^2z \partial X^i \bar{\partial} X^j \eta_{ij} + \frac{1}{\pi} \int d^2z (b\bar{\partial}c + \bar{b}\partial\bar{c}), \quad (3.1.8)$$

where c and \bar{c} are ghost fields, and b and \bar{b} are anti-ghost fields. String theory with the gauge fixing, therefore, is a conformal field theory with bosonic fields $i\partial X(z), i\bar{\partial} X(\bar{z})$ and fermionic fields $c(z), \bar{c}(\bar{z}), b(z),$ and $\bar{b}(\bar{z})$.

The bosonic fields $i\partial X(z), i\bar{\partial} X(\bar{z})$ are primary fields of conformal dimension $(1,0)$ and $(0,1)$, where (d, \bar{d}) means the field has holomorphic part of conformal dimension d and anti-holomorphic part of conformal dimension \bar{d} . The fields $i\partial X^i(z), i\bar{\partial} X^j(\bar{z})$ can be expanded in terms of Laurent series as

$$i\partial X^i(z) = \sum_n \frac{\alpha_n^i}{z^{n+1}}, \quad i\bar{\partial} X^j(\bar{z}) = \sum_n \frac{\bar{\alpha}_n^j}{\bar{z}^{n+1}}, \quad (3.1.9)$$

where the zero-modes $\alpha_0^i, \bar{\alpha}_0^j$ correspond to a centre of mass momentum p^i . The non-vanishing commutation relations of $\alpha_m^i, \bar{\alpha}_n^j$ are given by

$$[\alpha_m^i, \alpha_n^j] = [\bar{\alpha}_m^i, \bar{\alpha}_n^j] = m\eta^{ij} \delta_{m+n,0}. \quad (3.1.10)$$

The conformal vacuum $|0\rangle$ and its dual $\langle 0|$ of the bosonic string field theory are defined by

$$\alpha_n^i |0\rangle = \bar{\alpha}_n^i |0\rangle = 0 \text{ for } n \geq 0, \quad (3.1.11)$$

$$\langle 0| \alpha_n^i = \langle 0| \bar{\alpha}_n^i = 0 \text{ for } n \leq 0. \quad (3.1.12)$$

A string state is defined by acting the vacuum with creation operators,

$$\alpha_{-n_1}^{i_1} \dots \alpha_{-n_m}^{i_m} |0\rangle. \quad (3.1.13)$$

However, the state is not completely determined because one degree of freedom which is the

centre of mass momentum missing. Therefore, a new vacuum, $|0, p\rangle$, can be defined such that

$$\hat{p}|0, p\rangle = p|0, p\rangle, \quad (3.1.14)$$

where \hat{p} is the momentum operator. This conformal vacuum and its dual are annihilated by the following operators,

$$\alpha_n^i|0, p\rangle = \bar{\alpha}_n^i|0, p\rangle = 0 \text{ for } n > 0, \quad (3.1.15)$$

$$\langle 0, p|\alpha_n^i = \langle 0, p|\bar{\alpha}_n^i = 0 \text{ for } n < 0. \quad (3.1.16)$$

The ghost fields $c(z)$ and $\bar{c}(\bar{z})$ are primary fields of conformal dimension $(-1, 0)$ and $(0, -1)$. The anti-ghost fields $b(z)$ and $\bar{b}(\bar{z})$ are primary fields of conformal dimension $(2, 0)$ and $(0, 2)$. These fields can be expanded in terms of Laurent series as

$$c(z) = \sum_n \frac{c_n}{z^{n-1}}, \quad \bar{c}(\bar{z}) = \sum_n \frac{\bar{c}_n}{\bar{z}^{n-1}}, \quad (3.1.17)$$

$$b(z) = \sum_n \frac{b_n}{z^{n+2}}, \quad \bar{b}(\bar{z}) = \sum_n \frac{\bar{b}_n}{\bar{z}^{n+2}}. \quad (3.1.18)$$

The non-vanishing anticommutation relations of the mode expansions of these fields are

$$\{b_m, c_n\} = \{\bar{b}_m, \bar{c}_n\} = \delta_{m+n, 0}. \quad (3.1.19)$$

The following modes annihilate the vacuum and its dual as

$$c_n|0, p\rangle = \bar{c}_n|0, p\rangle = 0 \quad \text{for } n \geq -1, \quad (3.1.20)$$

$$b_n|0, p\rangle = \bar{b}_n|0, p\rangle = 0 \quad \text{for } n \geq 2, \quad (3.1.21)$$

$$\langle 0, p|c_n = \langle 0, p|\bar{c}_n = 0 \quad \text{for } n \leq 1, \quad (3.1.22)$$

$$\langle 0, p|b_n = \langle 0, p|\bar{b}_n = 0 \quad \text{for } n \leq -2, \quad (3.1.23)$$

By fixing the conformal gauge, the string action is no longer gauge invariant. However, it is invariant under the BRST symmetry with the BRST operator given by

$$Q = \int \frac{dz}{2\pi i} c(z) \left(T_b(z) + \frac{1}{2} T_g(z) \right) + \int \frac{d\bar{z}}{2\pi i} \bar{c}(\bar{z}) \left(\bar{T}_b(\bar{z}) + \frac{1}{2} \bar{T}_g(\bar{z}) \right), \quad (3.1.24)$$

where $T_b(z)$ and $T_g(z)$ are stress tensors of the bosonic sector and ghost sector, respectively,

and are defined as

$$T_b(z) = -\frac{1}{2} : (\partial X(z))^2 :, \quad (3.1.25)$$

$$T_g(z) = - : 2b(z) \cdot \partial c(z) : - : \partial b(z) \cdot c(z) : . \quad (3.1.26)$$

The total stress tensor is given by

$$T(z) = T_b(z) + T_g(z), \quad (3.1.27)$$

and can be expanded in terms of Laurent series as

$$T(z) = \sum_n \frac{L_n}{z^{n+2}} \quad , \quad \bar{T}(\bar{z}) = \sum_n \frac{\bar{L}_n}{\bar{z}^{n+2}}. \quad (3.1.28)$$

The vacuum $|0, p\rangle$ is not annihilated by L_0 and \bar{L}_0 . This means there is another vacuum state that has L_0 and \bar{L}_0 lower than $|0, p\rangle$. This state is known as a BRST vacuum and defined as

$$|\Downarrow, p\rangle = c_1 \bar{c}_1 |0, p\rangle. \quad (3.1.29)$$

3.2. BPZ conjugate

Given a conformal field theory, a linear inner product can be defined from a BPZ conjugate state [106]. For a state $|A\rangle = A(0)|0\rangle$, where $A(0)$ is a normal order operator which is constructed from creation operators: $\alpha_m^I, \bar{\alpha}_n^i, c_p, \bar{c}_q, b_r$, and \bar{b}_s . Its BPZ state is defined as

$$\langle A| \equiv \langle 0| I \cdot A(0), \quad (3.2.1)$$

where I is a conformal transformation $I(z) = 1/z$, which relates the state at $z = 0$ to the state at $z = \infty$. The BPZ conjugate of the state $c a_m b_n c_p \dots |0\rangle$ is

$$\langle 0| c (a_m)^T (b_n)^T (c_p)^T \dots, \quad (3.2.2)$$

where

$$\begin{aligned} (\phi)_n^T = I \cdot \phi_n &= \int \frac{dz'}{2\pi i} z'^{n+d+1} \phi'(z'), \\ &= \int \frac{dz'}{2\pi i} z'^{n+d+1} \frac{-\frac{1}{z'^2}}{\phi} \left(\frac{1}{z} \right) = (-1)^d \phi_n. \end{aligned} \quad (3.2.3)$$

In the BPZ conjugation, the order of operators does not change and the c -number is not complex conjugate. For the BRST vacuum, its BPZ conjugate is then given by

$$\langle \downarrow\downarrow, p | = \langle 0, -p | c_{-1} \bar{c}_{-1}. \quad (3.2.4)$$

Due to the fact that $c_{-1}, c_0, c_1, \bar{c}_1, \bar{c}_0$ and \bar{c}_1 do not annihilate the states $|0, p\rangle$ and $\langle 0, p|$, the ghost overlap can be defined as

$$\langle 0, p | c_{-1} \bar{c}_{-1} c_0^+ c_0^- c_1 \bar{c}_1 | 0, p' \rangle = (2\pi)^d \delta^d(p - p'), \quad (3.2.5)$$

where

$$c_0^\pm = \frac{1}{2}(c_0 \pm \bar{c}_0). \quad (3.2.6)$$

From the BPZ conjugation, the inner product can be defined as

$$\langle A, B \rangle = \langle A | c_0^- | B \rangle. \quad (3.2.7)$$

3.3. Closed string field action and gauge transformation

A closed string field $|\Psi\rangle$ is a vector in the Hilbert space of a conformal field theory,

$$|\Psi\rangle = \sum_s \psi_s |\Phi_s\rangle, \quad (3.3.1)$$

where ψ_s , which is a target space field and is a component of the closed string field, and $|\Phi_s\rangle$ is a string state constructed from string and ghost oscillations. The closed string field is not arbitrary. It must satisfy certain conditions. The first condition is known as the level-matching condition and is given by

$$(L_0 - \bar{L}_0)|\Psi\rangle = (b_0 - \bar{b}_0)|\Psi\rangle = 0. \quad (3.3.2)$$

Furthermore, the closed string field is restrict to the field with ghost number two and grassmannian even.

From the closed string field $|\Psi\rangle$, the free theory action can be constructed from the inner product

$$S = \frac{1}{2} \langle \Psi, Q\Psi \rangle = \frac{1}{2} \langle \Psi | c_0^- Q | \Psi \rangle. \quad (3.3.3)$$

This action has ghost number six, which are two from $c_0^- Q$ and four from the string field state.

By varying the action, the field equation is given by

$$Q|\Psi\rangle = 0. \quad (3.3.4)$$

The field equation (3.3.4) is invariant under a gauge transformation

$$\delta|\Psi\rangle = Q|\Lambda\rangle, \quad (3.3.5)$$

because of the nilpotent property of Q . The gauge parameter $|\Lambda\rangle$ satisfies the following conditions

$$(L_0 - \bar{L}_0)|\Lambda\rangle = (b_0 - \bar{b}_0)|\Lambda\rangle = 0. \quad (3.3.6)$$

The gauge parameter has ghost number one and is grassmanian odd.

3.4. 1st-massive open string state

In the last section, we provide the construction for the closed string field theory. The open string field theory can be constructed in the same way as the closed string field theory. Instead of having two ghost modes and two anti-ghost modes, the open string field theory will have only one ghost mode and one anti-ghost mode. The general open string field will be a state with ghost number one.

The first massive open bosonic string field in can be expanded as

$$|\Psi\rangle = \int [dp] \left(iV_i(p)\alpha_{-2}^i c_1 + \frac{1}{2}H_{ij}(p)\alpha_{-1}^i \alpha_{-1}^j c_1 + iB_i(p)\alpha_{-1}^i c_0 + D(p)b_{-2}c_0 c_1 + E(p)c_{-1} \right) |p\rangle. \quad (3.4.1)$$

For the free theory, this string field is invariant under the gauge transformation

$$\delta|\Psi\rangle = Q|\Lambda\rangle, \quad (3.4.2)$$

where Q is a BRST operator, and a gauge parameter $|\Lambda\rangle$ is given by

$$|\Lambda\rangle = \int [dp] \left(i\lambda(p)_i \alpha_{-1}^i + \chi b_{-2} c_1 \right) |p\rangle. \quad (3.4.3)$$

This gauge transformation leads to transformations of the component fields as

$$\delta V_i = \lambda_i - \alpha_{0i} \chi, \quad (3.4.4)$$

$$\delta H_{ij} = i\alpha_{0i} \lambda_j + i\alpha_{0j} \lambda_i + \chi \eta_{ij}, \quad (3.4.5)$$

$$\delta B_i = \left(\frac{1}{2} \alpha_0^2 + 1 \right) \lambda_i, \quad (3.4.6)$$

$$\delta D = - \left(\frac{1}{2} \alpha_0^2 + 1 \right) \chi, \quad (3.4.7)$$

$$\delta E = i\alpha_0^i \lambda_i + 3\chi. \quad (3.4.8)$$

The free theory equation of motion is

$$Q|\Psi\rangle = 0, \quad (3.4.9)$$

which leads to the field equations of the component fields as

$$i \left(\frac{1}{2} \alpha_0^2 + 1 \right) V_i - iB_i + \alpha_{0i} D = 0, \quad (3.4.10)$$

$$\left(\frac{1}{2} \alpha_0^2 + 1 \right) H_{ij} - i\alpha_{0i} B_j - i\alpha_{0j} B_i + D \eta_{ij} = 0, \quad (3.4.11)$$

$$\left(\frac{1}{2} \alpha_0^2 + 1 \right) E - i\alpha_0^i B_i + 3D = 0, \quad (3.4.12)$$

$$2iV_i + \alpha_0^k H_{ki} - 2iB_i - \alpha_{0i} E = 0, \quad (3.4.13)$$

$$2i\alpha_0^i V_i + \frac{1}{2} H + 4D - 3E = 0, \quad (3.4.14)$$

where H is a trace of H_{ij} .

By define a new field A ,

$$A = \frac{1}{20}(H - 2E), \quad (3.4.15)$$

it transforms as

$$\delta A = \chi. \quad (3.4.16)$$

From equations (3.4.11) and (3.4.12), the equation of motion of A is identified

$$\left(\frac{1}{2} \alpha_0^2 + 1 \right) A + D = 0. \quad (3.4.17)$$

By using the field A , a new vector field \tilde{V}_i can be defined such that it transforms only under

λ_i parameter

$$\tilde{V}_i = V_i + i\alpha_{0i}A, \quad (3.4.18)$$

$$\delta\tilde{V}_i = \lambda_i. \quad (3.4.19)$$

Its equation of motion is given by

$$\left(\frac{1}{2}\alpha_0^2 + 1\right) V_i - B_i = 0. \quad (3.4.20)$$

Similarly, a new tensor field \tilde{H}_{ij} can be defined as

$$\tilde{H}_{ij} = H_{ij} - A\eta_{ij}, \quad (3.4.21)$$

which transforms as

$$\delta\tilde{H}_{ij} = i\alpha_{0i}\lambda_j + i\alpha_{0j}\lambda_i. \quad (3.4.22)$$

Its field equation is

$$\left(\frac{1}{2}\alpha_0^2 + 1\right) \tilde{H}_{ij} - i\alpha_{0i}B_j - i\alpha_{0j}B_i = 0. \quad (3.4.23)$$

By using (3.4.13) and (3.4.14), the auxiliary field B_i is

$$B_i = \tilde{V}_i - i\frac{1}{2}\alpha_0^k\tilde{H}_{ki} - \frac{1}{2}\alpha_{0i}\alpha_0^k\tilde{V}_k, \quad (3.4.24)$$

$$= \tilde{V}_i - i\frac{1}{2}\alpha_0^k\tilde{H}_{ki} + \frac{1}{4}\alpha_{0i}\tilde{H}, \quad (3.4.25)$$

where \tilde{H} is a trace of \tilde{H}_{ij} .

From the tensor field \tilde{H}_{ij} and the vector field \tilde{V}_i , a new gauge invariant field h_{ij} can be defined as

$$h_{ij} = \tilde{H}_{ij} - i\alpha_{0i}V_j - i\alpha_{0j}V_i. \quad (3.4.26)$$

Its field equation is

$$\left(\frac{1}{2}\alpha_0^2 + 1\right) h_{ij} - \frac{1}{2}\alpha_{0i}\alpha_0^k h_{kj} - \frac{1}{2}\alpha_{0j}\alpha_0^k h_{ki} + \frac{1}{2}\alpha_{0i}\alpha_{0j}h = 0. \quad (3.4.27)$$

This is an equation of motion from Fierz-Pauli action describing the massive gravity theory. It

can also be shown that

$$i\alpha_0^k h_{kj} = h = 0, \quad (3.4.28)$$

which leads to

$$\left(\frac{1}{2}\alpha_0^2 + 1\right) h_{ij} = 0. \quad (3.4.29)$$

The free open string field theory at 1st massive level gives the theory of massive gravity described by the gauge invariant field h_{ij} which is transverse and traceless. h_{ij} is, therefore, a 2nd rank symmetric tensor of $SO(25)$.

3.5. The first massive closed string state

The first massive closed string field can be expanded in terms of 23 fields as

$$\begin{aligned} |\Psi\rangle = \int [dp] \quad & \left(H_{ij\bar{j}}(p) \alpha_{-2}^i \bar{\alpha}_{-2}^{\bar{j}} c_1 \bar{c}_1 + A_{ij\bar{k}}(p) \alpha_{-1}^i \alpha_{-1}^j \bar{\alpha}_{-2}^{\bar{k}} c_1 \bar{c}_1 + B_{ij\bar{k}}(p) \alpha_{-2}^i \bar{\alpha}_{-1}^{\bar{j}} \bar{\alpha}_{-1}^{\bar{k}} c_1 \bar{c}_1 \right. \\ & + C_{ij\bar{k}\bar{l}}(p) \alpha_{-1}^i \alpha_{-1}^j \bar{\alpha}_{-1}^{\bar{k}} \bar{\alpha}_{-1}^{\bar{l}} c_1 \bar{c}_1 + D_{ij\bar{k}}(p) \alpha_{-1}^i \alpha_{-1}^j \bar{\alpha}_{-1}^{\bar{k}} c_0^+ c_1 \\ & + \bar{D}_{ij\bar{k}}(p) \alpha_{-1}^i \alpha_{-1}^j \bar{\alpha}_{-1}^{\bar{k}} c_0^+ \bar{c}_1 + W_i(p) \alpha_{-1}^i c_0^+ \bar{c}_{-1} + \bar{W}_{\bar{i}}(p) \bar{\alpha}_{-1}^{\bar{i}} c_0^+ c_{-1} \\ & + V_{ij\bar{j}}(p) \alpha_{-2}^i \alpha_{-1}^j c_0^+ c_1 + \bar{V}_{ij\bar{j}}(p) \alpha_{-1}^i \bar{\alpha}_{-2}^{\bar{j}} c_0^+ \bar{c}_1 + X_{ij}(p) \alpha_{-1}^i \alpha_{-1}^j c_1 \bar{c}_{-1} \\ & + \bar{X}_{ij\bar{j}}(p) \bar{\alpha}_{-1}^{\bar{i}} \bar{\alpha}_{-1}^{\bar{j}} \bar{c}_1 c_{-1} + Y_i(p) \alpha_{-2}^i c_1 \bar{c}_{-1} + \bar{Y}_{\bar{i}} \bar{\alpha}_{-2}^{\bar{i}} \bar{c}_1 c_{-1} \\ & + G_{ij\bar{j}}(p) \alpha_{-1}^i \bar{\alpha}_{-1}^j c_1 c_{-1} + \bar{G}_{ij\bar{j}}(p) \alpha_{-1}^i \bar{\alpha}_{-1}^j \bar{c}_1 \bar{c}_{-1} + Z(p) c_{-1} \bar{c}_{-1} \\ & + E_i(p) \alpha_{-2}^i \bar{b}_{-2} c_0^+ c_1 \bar{c}_1 + \bar{E}_{\bar{i}}(p) \bar{\alpha}_{-2}^{\bar{i}} b_{-2} c_0^+ c_1 \bar{c}_{-1} + F_{ij}(p) \alpha_{-1}^i \alpha_{-1}^j \bar{b}_{-2} c_0^+ c_1 \bar{c}_1 \\ & + \bar{F}_{ij\bar{j}}(p) \bar{\alpha}_{-1}^{\bar{i}} \bar{\alpha}_{-1}^{\bar{j}} b_{-2} c_0^+ c_1 \bar{c}_1 + K_i(p) \alpha_{-1}^i \bar{b}_{-2} c_1 c_{-1} \bar{c}_1 \\ & \left. + \bar{K}_{\bar{i}} \bar{\alpha}_{-1}^{\bar{i}} b_{-2} c_1 \bar{c}_1 \bar{c}_{-1} \right) |p\rangle. \end{aligned} \quad (3.5.1)$$

In the case of free theory, it is invariant under gauge transformation

$$\delta|\Psi\rangle = Q|\Lambda\rangle, \quad (3.5.2)$$

where $|\Lambda\rangle$ is given by

$$\begin{aligned} |\Lambda\rangle = \int [dp] \quad & \left(\xi_{ij\bar{k}}(p) \alpha_{-1}^i \alpha_{-1}^j \bar{\alpha}_{-1}^{\bar{k}} c_1 + \bar{\xi}_{ij\bar{k}}(p) \alpha_{-1}^i \bar{\alpha}_{-1}^{\bar{j}} \bar{\alpha}_{-1}^{\bar{k}} \bar{c}_1 + \psi_{ij\bar{j}}(p) \alpha_{-2}^i \bar{\alpha}_{-1}^{\bar{j}} c_1 \right. \\ & + \bar{\psi}_{ij\bar{j}}(p) \alpha_{-1}^i \bar{\alpha}_{-2}^{\bar{j}} \bar{c}_1 + \theta_i(p) \alpha_{-1}^i \bar{c}_{-1} + \bar{\theta}_{\bar{i}}(p) \bar{\alpha}_{-1}^{\bar{i}} c_{-1} + \zeta_{ij\bar{j}}(p) \alpha_{-1}^i \bar{\alpha}_{-1}^{\bar{j}} c_0^+ \\ & + \lambda_i(p) \alpha_{-2}^i \bar{b}_{-2} c_1 \bar{c}_1 + \bar{\lambda}_{\bar{i}}(p) \bar{\alpha}_{-2}^{\bar{i}} b_{-2} c_1 \bar{c}_1 + \rho_{ij}(p) \alpha_{-1}^i \alpha_{-1}^j \bar{b}_{-2} c_1 \bar{c}_1 \\ & \left. + \bar{\rho}_{ij\bar{j}}(p) \bar{\alpha}_{-1}^{\bar{i}} \bar{\alpha}_{-1}^{\bar{j}} b_{-2} c_1 \bar{c}_1 \right) |p\rangle. \end{aligned} \quad (3.5.3)$$

It implies gauge transformations of component fields as

$$\delta H_{i\bar{j}} = -\psi_{i\bar{j}} + \bar{\psi}_{i\bar{j}} + \alpha_{0i}\bar{\lambda}_{\bar{j}} + \bar{\alpha}_{0\bar{j}}\lambda_i, \quad (3.5.4)$$

$$\delta A_{ij\bar{k}} = \xi_{ij\bar{k}} + \alpha_{0i}\bar{\psi}_{j\bar{k}} + \frac{1}{2}\delta_{ij}\bar{\lambda}_{\bar{k}} + \bar{\alpha}_{0\bar{k}}\rho_{ij} \quad (3.5.5)$$

$$\delta B_{i\bar{j}\bar{k}} = \bar{\xi}_{i\bar{j}\bar{k}} - \bar{\alpha}_{0\bar{k}}\psi_{i\bar{j}} + \frac{1}{2}\lambda_i\delta_{\bar{j}\bar{k}} + \alpha_{0i}\bar{\rho}_{\bar{j}\bar{k}} \quad (3.5.6)$$

$$\delta C_{ij\bar{k}\bar{l}} = \alpha_{0i}\bar{\xi}_{j\bar{k}\bar{l}} + \bar{\alpha}_{0\bar{k}}\xi_{ij\bar{k}} + \frac{1}{2}\delta_{ij}\bar{\rho}_{\bar{k}\bar{l}} + \frac{1}{2}\rho_{ij}\delta_{\bar{k}\bar{l}} \quad (3.5.7)$$

$$\delta D_{ij\bar{k}} = \left(-\frac{1}{2}\square + 2\right)\xi_{ij\bar{k}} - \alpha_{0i}\zeta_{j\bar{k}} \quad (3.5.8)$$

$$\delta \bar{D}_{i\bar{j}\bar{k}} = \left(-\frac{1}{2}\square + 2\right)\bar{\xi}_{i\bar{j}\bar{k}} - \bar{\alpha}_{0\bar{j}}\zeta_{i\bar{k}} \quad (3.5.9)$$

$$\delta W_i = \left(-\frac{1}{2}\square + 2\right)\theta_i - \bar{\alpha}_0^{\bar{j}}\zeta_{i\bar{j}} \quad (3.5.10)$$

$$\delta \bar{W}_{\bar{i}} = \left(-\frac{1}{2}\square + 2\right)\bar{\theta}_{\bar{i}} - \alpha_0^j\zeta_{j\bar{i}} \quad (3.5.11)$$

$$\delta V_{i\bar{j}} = \left(-\frac{1}{2}\square + 2\right)\psi_{i\bar{j}} - \zeta_{i\bar{j}} \quad (3.5.12)$$

$$\delta \bar{V}_{i\bar{j}} = \left(-\frac{1}{2}\square + 2\right)\bar{\psi}_{i\bar{j}} - \zeta_{i\bar{j}} \quad (3.5.13)$$

$$\delta X_{ij} = -\bar{\alpha}_0^{\bar{k}}\xi_{ij\bar{k}} + \alpha_{0i}\theta_j \quad (3.5.14)$$

$$\delta \bar{X}_{i\bar{j}} = -\alpha_0^k\xi_{k\bar{i}\bar{j}} + \bar{\alpha}_{0\bar{i}}\theta_{\bar{j}} \quad (3.5.15)$$

$$\delta G_{i\bar{j}} = -2\alpha_0^k\xi_{k\bar{i}\bar{j}} - 2\psi_{i\bar{j}} + \zeta_{i\bar{j}} + \alpha_{0i}\bar{\theta}_{\bar{j}} \quad (3.5.16)$$

$$\delta \bar{G}_{i\bar{j}} = -2\bar{\alpha}_0^{\bar{k}}\bar{\xi}_{i\bar{j}\bar{k}} - 2\bar{\psi}_{i\bar{j}} + \zeta_{i\bar{j}} + \bar{\alpha}_{0\bar{j}}\theta_i \quad (3.5.17)$$

$$\delta Y_i = -\bar{\alpha}^{\bar{j}}\psi_{i\bar{j}} + \theta_i \quad (3.5.18)$$

$$\delta \bar{Y}_{\bar{i}} = -\alpha^j\bar{\psi}_{j\bar{i}} + \bar{\theta}_{\bar{i}} \quad (3.5.19)$$

$$\delta Z = \alpha_0^i\theta_i - \bar{\alpha}^{\bar{i}}\bar{\theta}_{\bar{i}} \quad (3.5.20)$$

$$\delta E_i = -\left(-\frac{1}{2}\square + 2\right)\lambda_i \quad (3.5.21)$$

$$\delta \bar{E}_{\bar{i}} = -\left(-\frac{1}{2}\square + 2\right)\bar{\lambda}_{\bar{i}} \quad (3.5.22)$$

$$\delta F_{ij} = -\left(-\frac{1}{2}\square + 2\right)\rho_{ij} \quad (3.5.23)$$

$$\delta \bar{F}_{i\bar{j}} = -\left(-\frac{1}{2}\square + 2\right)\bar{\rho}_{i\bar{j}} \quad (3.5.24)$$

$$\delta K_i = 2\lambda_i + 2\alpha_0^j\rho_{ji} \quad (3.5.25)$$

$$\delta \bar{K}_{\bar{i}} = -2\bar{\lambda}_{\bar{i}} - 2\bar{\alpha}^{\bar{j}}\bar{\rho}_{i\bar{j}}. \quad (3.5.26)$$

3.5.1. T-duality transformation of the first massive level

Next, let us consider a T-duality transformation of the first massive level. Let the background field E_{ij} is given by

$$E_{ij} = G_{ij} + B_{ij}, \quad (3.5.27)$$

where G_{ij} is the metric on a torus and B_{ij} is a constant 2-form gauge field on the torus.

The T-duality of this background field is given by

$$E' = (aE + b)(cE + d)^{-1}, \quad (3.5.28)$$

where a, b, c , and d are the components of $O(n, n, \mathbb{Z})$ matrix h as

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad (3.5.29)$$

The oscillators α_n^i and $\bar{\alpha}_n^i$ transform under T-duality as [9]

$$\alpha_n(E) \rightarrow (d - cE^T)^{-1} \alpha_n(E'), \quad (3.5.30)$$

$$\bar{\alpha}_n(E) \rightarrow (d + cE)^{-1} \bar{\alpha}_n(E'). \quad (3.5.31)$$

Define a matrix $M_i{}^j$ and a matrix $\bar{M}_{\bar{i}}{}^{\bar{j}}$ as

$$M = d^T - Ec^T, \quad (3.5.32)$$

$$\bar{M} = d^T + E^T c^T. \quad (3.5.33)$$

That means under the T-duality transformation the oscillators transform as

$$\alpha'_n(E') = M^T \alpha_n(E), \quad (3.5.34)$$

$$\bar{\alpha}'_n(E') = \bar{M}^T \bar{\alpha}_n(E). \quad (3.5.35)$$

The fields will transform non-linearly under the $O(n, n, \mathbb{Z})$ transformation. For example, consider the first field in the the first massive level, which is

$$\int [dp] H_{i\bar{j}}(p) \alpha_{-2}^i \bar{\alpha}_{-2}^{\bar{j}} c_1 \bar{c}_1 |p\rangle. \quad (3.5.36)$$

Under the T-duality transformation, this state transform into

$$H'_{i'\bar{j}'}(p') \alpha_{-2}^{i'} \bar{\alpha}_{-2}^{\bar{j}'} c_1 \bar{c}_1 |p'\rangle. \quad (3.5.37)$$

The string field state is invariant under the T-duality transformation. This implies the transformation of field $H_{i\bar{j}(p)}$ as

$$H_{i\bar{j}}(p) = H'_{k\bar{l}}(p') M_i{}^k \bar{M}_{\bar{j}}{}^{\bar{l}}. \quad (3.5.38)$$

In general, a field with indices such as $A_{i_1 \dots i_n \bar{j}_1 \dots \bar{j}_m}(p)$ will transform under the T-duality as

$$A_{i_1 \dots i_n \bar{j}_1 \dots \bar{j}_m}(p) = A'_{k_1 \dots k_n \bar{l}_1 \dots \bar{l}_m}(p') M_{i_1}{}^{k_1} \dots M_{i_n}{}^{k_n} \bar{M}_{\bar{j}_1}{}^{\bar{l}_1} \dots \bar{M}_{\bar{j}_m}{}^{\bar{l}_m}. \quad (3.5.39)$$

4. Double Field Theory and Extended Field Theory

In this chapter, Double Field Theory (DFT) [13] will be constructed from the closed string field theory on the toroidal background base on references [10, 12, 13]. In [10], it is well known that T-duality is a symmetry of the closed string field theory. DFT, therefore, inherits the T-duality symmetry from the closed string field theory. We will also provide a brief introduction to Extended Field Theory (EFT) in this chapter.

4.1. Double field theory

4.1.1. Free Theory action

In a $\mathbb{R}^{d-1,1} \times T^n$ background, a general string field is given by [13]

$$|\Psi\rangle = \sum_I \int dk \sum_{p,w} \Phi^I(k, p, w) \mathcal{O}^I |k_\mu, p_a, w^a\rangle, \quad (4.1.1)$$

where I represents all possible states of string fields, $\Phi^I(k, p, w)$ is a target space field, and \mathcal{O}^I is an operator constructed from string and ghost oscillators. This string state is characterized by continuous momenta k_μ in the Minkowski directions, discrete momenta p_a in the n -torus directions, and discrete winding numbers along cycles of the n -torus. The target space field is a function of momenta and winding modes. By performing Fourier transform on the target space field, the field will depend on the usual coordinates x^i as well as another set of periodic coordinates, \tilde{x}_a conjugated to winding modes,

$$\Phi^I(k_\mu, p_a, w^a) \rightarrow \Phi^I(x^\mu, x^a, \tilde{x}_a). \quad (4.1.2)$$

This new set of coordinates is a coordinate of a dual n -torus, \tilde{T}^n . The target space field on the $\mathbb{R}^{d-1,1} \times T^n$ becomes a function of the $\mathbb{R}^{d-1,1} \times T^{2n}$.

By defining $\tilde{x}_i = (0, \tilde{x}_a)$, a commutation relation between winding modes and its winding

coordinates is given by

$$[\tilde{x}_i, w^j] = i\hat{\delta}_i{}^j \text{ with } \hat{\delta}_i{}^j = \text{diag}\{\delta_a{}^b, 0\}. \quad (4.1.3)$$

In this background, the zero-modes of string oscillations depend on the momentum modes and the winding modes,

$$\alpha_0^i = \frac{1}{\sqrt{2}}G^{ij}(p_j - E_{jk}w^k), \quad \bar{\alpha}_0^i = \frac{1}{\sqrt{2}}G^{ij}(p_j + E_{jk}^T w^k), \quad (4.1.4)$$

where $E_{ij} = G_{ij} + B_{ij}$. The lower indices of these operators can be used to define a new set of derivatives

$$\alpha_{0i} = -iD_i = \frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial x^i} - E_{ik}\frac{\partial}{\partial \tilde{x}_k}\right), \quad \bar{\alpha}_{0i} = -i\bar{D}_i = \frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial x^i} + E_{ik}^T\frac{\partial}{\partial \tilde{x}_k}\right). \quad (4.1.5)$$

By using these derivatives, operators which are quadratic in α_0 and $\bar{\alpha}_0$ can be defined as

$$\square = \frac{1}{2}(D^2 + \bar{D}^2), \quad \triangle = \frac{1}{2}(D^2 - \bar{D}^2), \quad (4.1.6)$$

where $D^2 = D^i D_i$ and $\bar{D}^2 = \bar{D}^i \bar{D}_i$. Note that these quadratic operators can be written in terms of ordinary and winding derivatives as

$$\square = \frac{1}{2}\mathcal{H}_{IJ}\partial^I\partial^J, \quad \triangle = \frac{1}{2}\eta_{IJ}\partial^I\partial^J, \quad (4.1.7)$$

where \mathcal{H} is the generalized metric, and $\partial^I = (\bar{\partial}^i, \partial_i)$ is the generalized derivative. The level-matching condition of the closed string state with $N = \bar{N}$ gives a condition on the target space field as,

$$(L_0 - \bar{L}_0)|\Psi\rangle = 0 \leftrightarrow \triangle\Phi^I = 0. \quad (4.1.8)$$

Double field theory is constructed from the closed string field theory on the massless level. The massless closed string field is given by [13]

$$|\Psi\rangle = \int [dp] \left(-\frac{1}{2}e_{ij}(p)\alpha_{-1}^i\bar{\alpha}_{-1}^j c_1\bar{c}_1 + e(p)c_1c_{-1} + \bar{e}(p)\bar{c}_1\bar{c}_{-1} \right. \\ \left. + i(f_i(p)c_0^+c_{-1}^i + \bar{f}_j(p)c_0^+\bar{c}_1\bar{\alpha}_{-1}^j) \right) |p\rangle, \quad (4.1.9)$$

where $\int [dp]$ represents an integration over continuous momenta in Minkowski directions, and summations over discrete momenta and winding numbers in compact directions, and $e_{ij}(p) \equiv e_{ij}(k_\mu, p_a, w^a)$. This closed string field has ghost number two because each terms contains two

ghost oscillators and the field satisfies the conditions [12]

$$(L_0 - \bar{L}_0)|\Psi\rangle = (b_0 - \bar{b}_0)|\Psi\rangle = 0. \quad (4.1.10)$$

The first condition leads to the fields e_{ij}, e, \bar{e}, f , and \bar{f} satisfying the conditions,

$$\Delta e_{ij} = \Delta e = \Delta \bar{e} = \Delta f = \Delta \bar{f} = 0. \quad (4.1.11)$$

The latter condition constrains the closed string field in order not to have the form of $c_0^-|p\rangle$.

From the closed string field theory action (3.3.3), the free theory action is given by

$$\begin{aligned} S &= \frac{1}{2} \langle \Psi | c_0^- Q | \Psi \rangle, \\ &= \int [dx d\tilde{x}] \left(\frac{1}{4} e_{ij} \square e^{ij} + 2\bar{e} \square e - f_i f^i - \bar{f}_i \bar{f}^i - f^i (\bar{D}^j e_{ij} - 2D_i \bar{e}) \right. \\ &\quad \left. + \bar{f}^j (D^i e_{ij} + 2\bar{D}_j e) \right), \end{aligned} \quad (4.1.12)$$

where $\int [dx d\tilde{x}]$ represents an integration over $\mathbb{R}^{d-1,1} \times T^{2n}$. This free theory action is invariant under the gauge transformation

$$\delta |\Psi\rangle = Q |\Lambda\rangle, \quad (4.1.13)$$

where $|\Lambda\rangle$ is a string field gauge parameter and given by

$$|\Lambda\rangle = \int [dp] \left(\frac{i}{2} \lambda_i(p) \alpha_{-1}^i c_1 - \frac{i}{2} \bar{\lambda}_i(p) \bar{\alpha}_{-1}^i \bar{c}_1 + \mu(p) c_0^+ \right) |p\rangle. \quad (4.1.14)$$

This gauge parameter has ghost number one because each term contains one ghost oscillator and it is grassmanian odd. This gauge parameter also satisfies the following conditions

$$(L_0 - \bar{L}_0)|\Lambda\rangle = (b_0 - \bar{b}_0)|\Lambda\rangle = 0, \quad (4.1.15)$$

which constrains $\lambda_i, \bar{\lambda}_i$, and μ such that

$$\Delta \lambda_i = \Delta \bar{\lambda}_i = \Delta \mu = 0. \quad (4.1.16)$$

The gauge transformation (4.1.13) gives the following field transformations

$$\delta e_{ij} = D_i \bar{\lambda}_j + \bar{D}_j \lambda_i, \quad (4.1.17)$$

$$\delta f_i = -\frac{1}{2} \square \lambda_i + D_i \mu, \quad (4.1.18)$$

$$\delta \bar{f}_i = \frac{1}{2} \square \bar{\lambda}_i + \bar{D}_i \mu, \quad (4.1.19)$$

$$\delta e = -\frac{1}{2} D^i \lambda_i + \mu, \quad (4.1.20)$$

$$\delta \bar{e} = \frac{1}{2} \bar{D}^i \bar{\lambda}_i + \mu. \quad (4.1.21)$$

By introducing new fields d and χ such that

$$d = \frac{1}{2}(e - \bar{e}), \text{ and } \chi = \frac{1}{2}(e + \bar{e}). \quad (4.1.22)$$

The gauge transformations of d and χ become

$$\delta d = -\frac{1}{4}(D^i \lambda_i + \bar{D}^i \bar{\lambda}_i), \quad (4.1.23)$$

$$\delta \chi = -\frac{1}{4}(D^i \lambda_i - \bar{D}^i \bar{\lambda}_i) + \mu. \quad (4.1.24)$$

In this case, μ can be used to fix a gauge condition $\chi = 0$. This gauge choice will affect the gauge transformations of f_i and \bar{f}_i . However, f_i and \bar{f}_i are auxiliary fields and will be eliminated by their equations of motion.

After choosing the gauge choice $\chi = 0$, which leads to the condition $d = e = -\bar{e}$, and eliminating fields f_i and \bar{f}_i , the free theory action becomes

$$S = \int [dx d\tilde{x}] \left(\frac{1}{4} e_{ij} \square e^{ij} + \frac{1}{4} (\bar{D}^j e_{ij})^2 + \frac{1}{4} (D^i e_{ij})^2 - 2d D^i \bar{D}^j e_{ij} - 4d \square d \right). \quad (4.1.25)$$

The remaining fields in this action are e_{ij} and d . The former is the fluctuation of the metric and the two-form gauge field,

$$e_{ij} = h_{ij} + b_{ij}, \quad (4.1.26)$$

while the latter is the dilaton field. This action has gauge symmetries generated by λ_i and $\bar{\lambda}_i$, which are

$$\delta_\lambda e_{ij} = \bar{D}_j \lambda_i, \quad \delta_\lambda d = -\frac{1}{4} D^i \lambda_i, \quad (4.1.27)$$

$$\delta_{\bar{\lambda}} e_{ij} = D_i \bar{\lambda}_j, \quad \delta_{\bar{\lambda}} d = -\frac{1}{4} \bar{D}^i \bar{\lambda}_i. \quad (4.1.28)$$

4.1.2. Cubic interaction action

A cubic interaction of the string action is the result of a following term [12],

$$\{\Psi, \Psi, \Psi\} = \langle \Psi | c_0^- [\Psi, \Psi] \rangle, \quad (4.1.29)$$

where $[\Psi, \Psi]$ is a closed string product. The gauge transformation is modified as

$$\delta_\Lambda \Psi = Q\Lambda + [\Lambda, \Psi] + \dots, \quad (4.1.30)$$

where \dots represents the higher-order terms of string fields. The string field action is constructed by keeping the terms with a number of derivative less than or equal to two. After fixing the gauge $\chi = 0$, setting $d = e = -\bar{e}$ and fields redefinitions, the complete action is

$$\begin{aligned} S = & \int [dx d\tilde{x}] \left(\frac{1}{4} e_{ij} \square e^{ij} + \frac{1}{4} (\bar{D}^j e_{ij})^2 + \frac{1}{4} (D^i e_{ij})^2 - 2d D^i \bar{D}^j e_{ij} - 4d \square d \right. \\ & + \frac{1}{4} e_{ij} \left((D^i e_{kl})(\bar{D}^j e^{kl}) - (D^i e_{kl})(\bar{D}^l e^{kj}) - (D^k e^{il})(\bar{D}^j e_{kl}) \right) \\ & + \frac{1}{2} d \left((D^i e_{ij})^2 + (\bar{D}^j e_{ij})^2 + \frac{1}{2} (D_k e_{ij})^2 + \frac{1}{2} (\bar{D}_k e_{ij})^2 \right) \\ & \left. + d e^{ij} (D_i D^k e_{kj} + \bar{D}_j \bar{D}^k e_{ik}) + 4e_{ij} d D^i \bar{D}^j d + 4d^2 \square d \right). \end{aligned} \quad (4.1.31)$$

This action is invariant under the following gauge transformations,

$$\delta_\lambda e_{ij} = \bar{D}_j \lambda_i + \frac{1}{2} [(D_i \lambda^k) e_{kj} - (D^k \lambda_i) e_{kj} + \lambda_k D^k e_{ij}], \quad (4.1.32)$$

$$\delta_\lambda d = -\frac{1}{4} D^i \lambda_i + \frac{1}{2} (\lambda^i D_i) d, \quad (4.1.33)$$

$$\delta_{\bar{\lambda}} e_{ij} = D_i \bar{\lambda}_j + \frac{1}{2} [(\bar{D}_i \bar{\lambda}^k) e_{kj} - (\bar{D}^k \bar{\lambda}_i) e_{kj} + \bar{\lambda}_k \bar{D}^k e_{ij}], \quad (4.1.34)$$

$$\delta_{\bar{\lambda}} d = -\frac{1}{4} \bar{D}^i \bar{\lambda}_i + \frac{1}{2} (\bar{\lambda}^i \bar{D}_i) d. \quad (4.1.35)$$

This action is also T-duality invariant.

4.1.3. Double field theory action

Double field theory is focused on a cubic interaction rather than higher order interaction terms. The cubic interaction hints a way to construct the action in terms of full fields instead of background fields and fluctuations. In [15], a background independent action is constructed

in terms of the field $\mathcal{E}_{ij} = g_{ij} + b_{ij}$ and the dilaton d , as

$$S = \int dx d\tilde{x} e^{-2d} \left(-\frac{1}{4} g^{ik} g^{jl} \mathcal{D}^p \mathcal{E}_{kl} \mathcal{D}_p \mathcal{E}_{ij} + \frac{1}{4} g^{kl} (\mathcal{D}^j \mathcal{E}_{ik} \mathcal{D}^i \mathcal{E}_{jl} + \bar{\mathcal{D}}^j \mathcal{E}_{ki} \bar{\mathcal{D}}^i \mathcal{E}_{lj}) \right. \\ \left. + (\mathcal{D}^i d \bar{\mathcal{D}}^j \mathcal{E}_{ij} + \bar{\mathcal{D}}^i d \mathcal{D}^j \mathcal{E}_{ji}) + 4 \mathcal{D}^i d \mathcal{D}_i d \right),$$

where the derivatives are defined as

$$\mathcal{D}_i \equiv \partial_i - \mathcal{E}_{ik} \tilde{\partial}^k, \quad \bar{\mathcal{D}}_i \equiv \partial_i + \mathcal{E}_{ki} \tilde{\partial}^k. \quad (4.1.36)$$

This action is invariant under the $O(D, D)$ transformation and the following gauge transformations,

$$\delta_\xi \mathcal{E}_{ij} = \mathcal{D}_i \tilde{\xi}_j - \bar{\mathcal{D}}_j \tilde{\xi}_i + \xi^M \partial_M \mathcal{E}_{ij} + \mathcal{D}_i \xi^k \mathcal{E}_{kj} + \bar{\mathcal{D}}_j \xi^k \mathcal{E}_{ik}, \quad (4.1.37)$$

$$\delta_\xi d = -\frac{1}{2} \partial_M \xi^M + \xi^M \partial_M d, \quad (4.1.38)$$

where new gauge parameters are defined by

$$\lambda_i = -\tilde{\xi} + E_{ij} \xi^j, \quad \bar{\lambda}_i = \tilde{\xi}_i + E_{ij} \xi^j. \quad (4.1.39)$$

These new gauge parameters ξ^i and $\tilde{\xi}_i$ form a vector representation of the $O(D, D)$ group,

$$\xi^M = \begin{pmatrix} \xi^i \\ \tilde{\xi}_i \end{pmatrix}. \quad (4.1.40)$$

This action require a strong constraint, $\partial_i \tilde{\partial}^i (\dots) = 0$, where \dots represents fields and gauge parameters, in order to make the gauge transformation closed. The strong constraint restricts the theory depending on half of coordinates. When the set of coordinates is in the supergravity frame, i.e. $\tilde{\partial}^i = 0$, this action reduces to the bosonic part of the supergravity action,

$$S = \int dx \sqrt{-g} e^{-2\phi} \left(R + 4(\partial\phi)^2 - \frac{1}{12} H^2 \right). \quad (4.1.41)$$

In order to verify the gauge invariant property, one needs to go through a long calculation. In [16], an alternative form of the action is provided in terms of the generalized metric \mathcal{H}_{MN} , which is defined as

$$\mathcal{H}_{MN} = \begin{pmatrix} g_{ij} - b_{ik} g^{kl} b_{lj} & b_{ik} g^{kj} \\ -g^{ik} b_{kj} & g^{ij} \end{pmatrix}, \quad (4.1.42)$$

and the dilaton d

$$S = \int dx d\tilde{x} e^{-2d} \left(\frac{1}{8} \mathcal{H}^{MN} \partial_M \mathcal{H}^{KL} \partial_N \mathcal{H}_{KL} - \frac{1}{2} \mathcal{H}^{MN} \partial_N \mathcal{H}^{KL} \partial_L \mathcal{H}_{MK} \right. \\ \left. - 2 \partial_M d \partial_N \mathcal{H}^{MN} + 4 \mathcal{H}^{MN} \partial_M d \partial_N d \right). \quad (4.1.43)$$

It is obvious that this action is $O(D, D)$ invariant because every field in this action is written in terms of $O(D, D)$ objects and their indices are totally contracted. For example, \mathcal{H}_{MN} is an $O(D, D)$ tensor, the dilaton d is an $O(D, D)$ singlet, and the gauge parameters are combined into an $O(D, D)$ vector. An $O(D, D)$ index, such as M, N , runs from 1 to $2D$ and can be raised and lowered by the $O(D, D)$ invariant metric, η_{MN} .

The gauge transformations of the generalized metric and the dilaton are given by

$$\delta_\xi \mathcal{H}^{MN} = \xi^P \partial_P \mathcal{H}^{MN} + (\partial^M \xi_P - \partial_P \xi^M) \mathcal{H}^{PN} + (\partial^N \xi_P - \partial_P \xi^N) \mathcal{H}^{MP} \quad (4.1.44)$$

$$\delta_\xi d = \xi^M \partial_M d - \frac{1}{2} \partial_M \xi^M. \quad (4.1.45)$$

When the strong constraint is imposed, i.e. $\tilde{\partial}^i = 0$, the gauge transformation of the generalized metric becomes

$$\delta_\xi g_{ij} = \mathcal{L}_\xi g_{ij}, \quad (4.1.46)$$

$$\delta_\xi b_{ij} = \mathcal{L}_\xi b_{ij} + \partial_i \tilde{\xi}_j - \partial_j \tilde{\xi}_i. \quad (4.1.47)$$

The $\xi^i(x)$ is the parameter of the ordinary Lie derivative, while $\tilde{\xi}_i(x)$ is the parameter of the two-form gauge transformation.

From the gauge transformation of the generalized metric, the generalized Lie derivative can be defined such that

$$\delta_\xi \mathcal{H}^{MN} = \hat{\mathcal{L}}_\xi \mathcal{H}^{MN}. \quad (4.1.48)$$

The generalized tensor $T^{M_1 \dots M_m}{}_{N_1 \dots N_n}$ can be defined such that under the gauge transformation, it transforms as

$$\delta_\xi T^{M_1 \dots M_m}{}_{N_1 \dots N_n} = \hat{\mathcal{L}}_\xi T^{M_1 \dots M_m}{}_{N_1 \dots N_n}. \quad (4.1.49)$$

For example, the generalized Lie derivative of the generalized vector with the lower index, A_M ,

and with the upper index, B^M , are given by

$$\hat{\mathcal{L}}_\xi A_M = \xi^P \partial_P A_M + (\partial_M \xi^P - \partial^P \xi_M) A_P, \quad (4.1.50)$$

$$\hat{\mathcal{L}}_\xi B^M = \xi^P \partial_P B^M + (\partial^M \xi_P - \partial_P \xi^M) B^P. \quad (4.1.51)$$

Furthermore, the invariant metric η_{MN} and the Kronecker delta δ^M_N are invariant under the gauge transformation, which are given by the vanishing of the generalized Lie derivative,

$$\hat{\mathcal{L}}_\xi \eta_{MN} = 0, \quad \hat{\mathcal{L}}_\xi \delta^M_N = 0. \quad (4.1.52)$$

When $\xi^M = \partial^M \chi$, where χ is a function, and the strong constraint holds, the generalized Lie derivative of any generalized tensor with this gauge parameter vanishes

$$\hat{\mathcal{L}}_{\partial\chi} T = 0. \quad (4.1.53)$$

This case is known as the redundant gauge transformation.

The commutation relation of any two generalized Lie derivative gives rise to a C-bracket,

$$\left[\hat{\mathcal{L}}_{\xi_1}, \hat{\mathcal{L}}_{\xi_2} \right] = \hat{\mathcal{L}}_{[\xi_1, \xi_2]_C}, \quad (4.1.54)$$

where $[\ , \]_C$ is the C-bracket and is defined by

$$[\xi_1, \xi_2]_C^M \equiv \xi_1^N \partial_N \xi_2^M - \frac{1}{2} \xi_{1N} \partial^M \xi_2^N - (1 \leftrightarrow 2). \quad (4.1.55)$$

This C-bracket has a non-vanishing Jacobiator $J(\xi_1, \xi_2, \xi_3)$,

$$[\xi_1, [\xi_2, \xi_3]_C]_C + \text{cyclic} = J(\xi_1, \xi_2, \xi_3). \quad (4.1.56)$$

However, this Jacobiator corresponds to the redundant transformation, $J^M = \partial^M \mathcal{N}$. The generalized Lie derivative with the strong constraint, therefore, satisfies the Jacobi identity.

4.2. Extended field theory

The idea of extended field theory is to make a U-duality manifest theory by enlarging the internal space to accommodate brane wrapping modes as

$$M^{11} \rightarrow M^{11-d} \times M^d \rightarrow M^{11-d} \times M^{\dim R_1}, \quad (4.2.1)$$

where $M^{\dim R_1}$ is an enlarged space with dimensions equal to $\dim R_1$, and R_1 is a representation of the generalized vector given in the below table.

d	$E_{d,(d)}$	R_1
3	$SL(3) \times SL(2)$	(3,2)
4	$SL(5)$	10
5	$Spin(5, 5)$	16
6	$E_{6,(6)}$	27
7	$E_{7,(7)}$	56
8	$E_{8,(8)}$	248

This generalized vector transforms under the generalized Lie derivative as [26]

$$\mathcal{L}_U V^M = L_U V^M + Y^{MN}{}_{PQ} \partial_N U^P V^Q, \quad (4.2.2)$$

where the Y -tensor, $Y^{MN}{}_{PQ}$, can be constructed for each symmetry group, $E_{d,(d)}$. It measures the deviation of the extended geometry from the Riemannian geometry. The algebra of the generalized Lie derivative is closed and the Jacobi identity is satisfied if a strong constraint is imposed. The strong constraint in extended field theory is known as the section constraint, which is given by [26]

$$Y^{MN}{}_{PQ} \partial_M(\dots) \partial_N(\dots) = 0. \quad (4.2.3)$$

This constrain will ensure that the fields are locally depend on an d -dimensional subspace of extended space.

The bosonic degrees of freedom of the theory are given by the metric and 3-form gauge field. These fields parameterized the coset space $E_{(d,(d)}/H_d$ and are known as the generalized metric.

5. Finite Transformation for Double Field Theory and Extended Field theory

In order to understand the geometry of DFT, there have been a number of attempts to explore the relationship between the gauge symmetries of DFT and the diffeomorphisms of the doubled space [73] - [77]. However, the gauge group and the diffeomorphism group are not isomorphic [78] because the former acts through the generalized Lie derivative while the latter acts through the Lie derivative. In this chapter, we generalize the result from [78], which is a finite transformation for DFT to a finite transformation for EFT. The cases $SL(5, \mathbb{R})$, $Spin(5, 5)$ and $E_{6,(6)}$ are considered.

5.1. Finite transformations for double field theory

In this section, we review the results of [78]. Double field theory has fields on a doubled spacetime M . Consider a patch \mathcal{U} of M with coordinates

$$X^M = \begin{pmatrix} x^m \\ \tilde{x}_m \end{pmatrix} \quad (5.1.1)$$

where $m = 1, \dots, D$. A generalized vector W^M transforms as a vector under $O(D, D)$ and decomposes as

$$W^M = \begin{pmatrix} w^m \\ \tilde{w}_m \end{pmatrix}, \quad (5.1.2)$$

under $GL(D, \mathbb{R})$. The strong constraint is solved by having all fields independent of \tilde{x}_m so that

$$\tilde{\partial}^m = 0 \quad (5.1.3)$$

on all fields and parameters. Then the fields depend on the coordinates x^m , parameterizing a D -dimensional patch U (which can be thought of as the quotient of \mathcal{U} by the isometries generated by $\partial/\partial\tilde{x}^m$).

The generalized Lie derivative

$$\hat{\mathcal{L}}_V W^M = V^P \partial_P W^M + W^P (\partial^M V_P - \partial_P V^M) \quad (5.1.4)$$

for $V^M(x), W^M(x)$ then gives

$$(\hat{\mathcal{L}}_V W)^m = v^p \partial_p w^m - w^p \partial_p v^m = \mathcal{L}_v w^m \quad (5.1.5)$$

and

$$(\hat{\mathcal{L}}_V W)_m = v^p \partial_p \tilde{w}_m + \tilde{w}_p \partial_m v^p + w^p (\partial_m \tilde{v}_p - \partial_p \tilde{v}_m) \quad (5.1.6)$$

$$= \mathcal{L}_v \tilde{w}_m + w^p (\partial_m \tilde{v}_p - \partial_p \tilde{v}_m) \quad (5.1.7)$$

where \mathcal{L}_v is the usual Lie derivative on U .

Under an infinitesimal transformation with a parameter V^M , W^M transforms as

$$\delta W^M = \hat{\mathcal{L}}_V W^M \quad (5.1.8)$$

giving

$$\delta w^m = \mathcal{L}_v w^m \quad (5.1.9)$$

$$\delta \tilde{w}_m = \mathcal{L}_v \tilde{w}_m + w^p (\partial_m \tilde{v}_p - \partial_p \tilde{v}_m) \quad (5.1.10)$$

It will be convenient to rewrite the components of the generalized vector W as

$$w = w^m e_m \quad \tilde{w}_{(1)} = \tilde{w}_m e^m, \quad (5.1.11)$$

where $e_m = \partial/\partial x^m$ and $e^m = dx^m$ are the coordinate bases for TU and T^*U , respectively. Then the generalized vector can be written as

$$W = w \oplus \tilde{w}_{(1)}. \quad (5.1.12)$$

Under an infinitesimal transformation with a parameter V , these transform as

$$\delta_V w = \mathcal{L}_v w, \quad (5.1.13)$$

$$\delta_V \tilde{w}_{(1)} = \mathcal{L}_v \tilde{w}_{(1)} - \iota_w d\tilde{v}_{(1)}, \quad (5.1.14)$$

where \mathcal{L}_v is a Lie derivative on patch U .

Next, we introduce a gerbe connection $B_{(2)}$ on U ,

$$B_{(2)} = \frac{1}{2} B_{mn} e^m \wedge e^n, \quad (5.1.15)$$

which transforms under the gauge transformation as

$$\delta_V B_{(2)} = \mathcal{L}_v B_{(2)} + d\tilde{v}_{(1)}. \quad (5.1.16)$$

Then

$$\hat{w}_{(1)} = \tilde{w}_{(1)} + \iota_w B_{(2)}, \quad (5.1.17)$$

transforms as

$$\delta_V \hat{w}_{(1)} = \mathcal{L}_v \hat{w}_{(1)}. \quad (5.1.18)$$

and so is a 1-form on U , and is invariant under the \tilde{v} transformations. Then

$$\hat{W} = w \oplus \hat{w}_{(1)} \quad (5.1.19)$$

is a section of $(T \oplus T^*)U$.

The finite transformation of \hat{W} is given by

$$w'(x') = w(x) \quad \hat{w}'_{(1)}(x') = \hat{w}_{(1)}(x), \quad (5.1.20)$$

where $x'(x) = e^{-v^m \partial_m} x$. Using the finite transformation of the coordinate bases

$$e'_m(x') = e_n(x) \frac{\partial x^n}{\partial x'^m} \quad e'^m(x') = e^n(x) \frac{\partial x'^m}{\partial x^n}, \quad (5.1.21)$$

the finite transformation of the components of \hat{W} are then

$$w'^m(x') = w^n(x) \frac{\partial x'^m}{\partial x^n} \quad \hat{w}'_m(x') = \hat{w}_n(x) \frac{\partial x^n}{\partial x'^m}. \quad (5.1.22)$$

The finite transformation of the gerbe connection can be taken to be

$$B'_{(2)}(x') = B_{(2)}(x) + d\tilde{v}_{(1)}(x), \quad (5.1.23)$$

so that

$$\hat{w}'_{(1)}(x') = \tilde{w}'_{(1)}(x') + \iota_{w'} B'_{(2)}(x'). \quad (5.1.24)$$

This then gives the finite transformations of $\tilde{w}_{(1)}$:

$$\begin{aligned}\tilde{w}'_{(1)}(x') &= \hat{w}'_{(1)}(x') - \iota_{w'} B'_{(2)}(x'), \\ &= \hat{w}_{(1)}(x) - \iota_w (B_{(2)}(x) + d\tilde{v}_{(1)}(x)), \\ &= \tilde{w}_{(1)}(x) - \iota_w d\tilde{v}_{(1)}(x).\end{aligned}\tag{5.1.25}$$

To summarize, the transformation of W is given by

$$w'(x') = w(x),\tag{5.1.26}$$

$$\tilde{w}'_{(1)}(x') = \tilde{w}_{(1)}(x) - \iota_w d\tilde{v}_{(1)}(x).\tag{5.1.27}$$

which implies the components (w^m, \tilde{w}_m) transform as

$$w'^m(x') = w^n(x) \frac{\partial x'^m}{\partial x^n}\tag{5.1.28}$$

$$\tilde{w}'_m(x') = [\tilde{w}_p(x) - w^n(x)(\partial_n \tilde{v}_p(x) - \partial_p \tilde{v}_n(x))] \frac{\partial x^p}{\partial x'^m}.\tag{5.1.29}$$

5.2. Finite Transformations for Extended Field Theory

5.2.1. $SL(5, \mathbb{R})$ Extended field theory

In $SL(5, \mathbb{R})$ extended field theory [35, 42], a generalized vector W^M transforms as a **10** of $SL(5, \mathbb{R})$ where the indices $M, N = 1, \dots, 10$ label the **10** representation of $SL(5, \mathbb{R})$. It decomposes under $GL(4, \mathbb{R}) \subset SL(5, \mathbb{R})$ into a vector and 2-form:

$$W^M = \begin{pmatrix} w^m \\ \tilde{w}_{mn} \end{pmatrix},\tag{5.2.1}$$

where $m, n = 1, \dots, 4$ and $\tilde{w}_{mn} = -\tilde{w}_{nm}$. The coordinates in a patch \mathcal{U} consist of 7 spacetime coordinates y^μ , $\mu = 0, \dots, 6$, together with 10 internal coordinates X^M transforming as a **10** of $SL(5, \mathbb{R})$. This decomposes under $GL(4, \mathbb{R}) \subset SL(5, \mathbb{R})$ as

$$X^M = \begin{pmatrix} x^m \\ \tilde{x}_{mn} \end{pmatrix},\tag{5.2.2}$$

where, in a suitable duality frame, x^m are the usual coordinates on T^4 and \tilde{x}_{mn} are periodic coordinates conjugate to M2-brane wrapping numbers on T^4 . Fields and gauge parameters depend on both y^μ and X^M , but we will suppress dependence on y^μ in what follows.

The strong constraint of $SL(5, \mathbb{R})$ EFT is given by

$$\epsilon^{iMN} \epsilon_{iPQ} \partial_M(\dots) \partial_N(\dots) = 0, \quad (5.2.3)$$

where (\dots) represents fields and gauge parameters, and the indices $i, j = 1, \dots, 5$ label the fundamental representation $\mathbf{5}$ of $SL(5, \mathbb{R})$. An index M can be regarded as an antisymmetric pair of indices $[m_1 m_2]$, so that $\epsilon^{iMN} = \epsilon^{im_1 m_2 n_1 n_2}$. The strong constraint can be solved such that the fields are independent of wrapping coordinates \tilde{x}_{mn} so that

$$\tilde{\partial}^{mn}(\dots) = 0, \quad (5.2.4)$$

where $\tilde{\partial}^{mn} = \frac{\partial}{\partial \tilde{x}_{mn}}$. The gauge transformations of $SL(5, \mathbb{R})$ extended field theory are given by the generalized Lie derivative, which is defined as

$$\hat{\mathcal{L}}_V W^M = V^N \partial_N W^M - W^N \partial_N V^M + \epsilon^{iMN} \epsilon_{iPQ} \partial_N V^P W^Q. \quad (5.2.5)$$

It is convenient to rewrite the components of the generalized vector as

$$w = w^m e_m, \quad (5.2.6)$$

$$\tilde{w}_{(2)} = \frac{1}{2!} \tilde{w}_{mn} e^m \wedge e^n. \quad (5.2.7)$$

Then the generalized vector W is

$$W = w \oplus \tilde{w}_{(2)}. \quad (5.2.8)$$

Under a gauge transformation with gauge parameter V , a generalized vector W transforms as

$$\delta_V W = \hat{\mathcal{L}}_V W. \quad (5.2.9)$$

This decomposes into

$$\delta_V w = \mathcal{L}_v w, \quad (5.2.10)$$

$$\delta_V \tilde{w}_{(2)} = \mathcal{L}_v \tilde{w}_{(2)} - \iota_w d\tilde{w}_{(2)}, \quad (5.2.11)$$

where \mathcal{L}_v is the ordinary Lie derivative with a parameter v .

Next we introduce a gerbe connection $C_{(3)}$,

$$C_{(3)} = \frac{1}{3!} C_{mnp} e^m \wedge e^n \wedge e^p, \quad (5.2.12)$$

which transforms under a gauge transformation as

$$\delta_V C_{(3)} = \mathcal{L}_v C_{(3)} + d\tilde{v}_{(2)}. \quad (5.2.13)$$

This allows us to define $\hat{w}_{(2)}$ by

$$\hat{w}_{(2)} = \tilde{w}_{(2)} + \iota_w C_{(3)}. \quad (5.2.14)$$

Under a gauge transformation, this transforms as a 2-form

$$\delta_V \hat{w}_{(2)} = \mathcal{L}_v \hat{w}_{(2)} \quad (5.2.15)$$

and is invariant under the \tilde{v} transformations. Therefore, $\hat{W} = w \oplus \hat{w}_{(2)}$ is a section of $(T \oplus \Lambda^2 T^*)U$. This allows us to immediately write down the finite transformation of \hat{W} , which is given by

$$w'(x') = w(x), \quad (5.2.16)$$

$$\hat{w}'_{(2)}(x') = \hat{w}_{(2)}(x), \quad (5.2.17)$$

where $x' = e^{-v^m \partial_m} x$.

Using the finite transformation of the coordinate bases given by

$$e'_m(x') = e_n(x) \frac{\partial x^n}{\partial x'^m} \quad e'^m(x') = e^n(x) \frac{\partial x'^m}{\partial x^n}, \quad (5.2.18)$$

the finite transformation of the components of \hat{W} are then

$$w'^m(x') = w^n(x) \frac{\partial x'^m}{\partial x^n}, \quad (5.2.19)$$

$$\hat{w}'_{mn}(x') = \hat{w}_{pq}(x) \frac{\partial x^p}{\partial x'^m} \frac{\partial x^q}{\partial x'^n}. \quad (5.2.20)$$

The finite transformation of the gerbe connection can be taken to be

$$C'_{(3)}(x') = C_{(3)}(x) + d\tilde{v}_{(2)}(x), \quad (5.2.21)$$

so that

$$\hat{w}'_{(2)}(x') = \tilde{w}'_{(2)}(x') + \iota_{w'} C'_{(3)}(x'). \quad (5.2.22)$$

This then gives the finite transformation of $\tilde{w}_{(2)}$:

$$\begin{aligned}
\tilde{w}'_{(2)}(x') &= \hat{w}'_{(2)}(x') - \iota_{w'} C'_{(3)}(x'), \\
&= \hat{w}_{(2)}(x) - \iota_w (C_{(3)}(x) + d\tilde{v}_{(2)}(x)), \\
&= \hat{w}_{(2)}(x) - \iota_w C_{(3)}(x) - \iota_w d\tilde{v}_{(2)}(x), \\
&= \tilde{w}_{(2)}(x) - \iota_w d\tilde{v}_{(2)}(x).
\end{aligned} \tag{5.2.23}$$

To summarize, the transformation of W is given by

$$w'(x') = w(x), \tag{5.2.24}$$

$$\tilde{w}'_{(2)}(x') = \tilde{w}_{(2)}(x) - \iota_w d\tilde{v}_{(2)}(x), \tag{5.2.25}$$

which implies the components (w^m, \tilde{w}_{mn}) transform as

$$w'^m(x') = w^n(x) \frac{\partial x'^m}{\partial x^n}, \tag{5.2.26}$$

$$\tilde{w}'_{mn}(x') = (\tilde{w}_{pq}(x) - 3w^r(x)(d\tilde{v}_{(2)})_{rpq}(x)) \frac{\partial x^p}{\partial x'^m} \frac{\partial x^q}{\partial x'^n}, \tag{5.2.27}$$

where $(d\tilde{v}_{(2)})_{rpq} = \partial_{[r}\tilde{v}_{pq]}$.

5.2.2. $SO(5, 5)$ Extended field theory

In $SO(5, 5)$ extended field theory [41], a generalized vector W^M transforms as a spinor of $SO(5, 5)$, where the indices $M, N = 1, \dots, 16$ label the positive chirality spinor representation. It decomposes under $GL(5, \mathbb{R}) \subset SO(5, 5)$ into

$$W^M = \begin{pmatrix} w^m \\ \tilde{w}_{mn} \\ \tilde{w}_{mnpqr} \end{pmatrix}, \tag{5.2.28}$$

where $m, n = 1, \dots, 5$, $\tilde{w}_{mn} = -\tilde{w}_{nm}$, and $\tilde{w}_{mnpqr} = \tilde{w}_{[mnpqr]}$.

The coordinates in a patch \mathcal{U} consist of 6 spacetime coordinates y^μ , $\mu = 0, \dots, 5$, together with 16 internal coordinates X^M transforming as a **16** of $SO(5, 5)$. This decomposes under $GL(5, \mathbb{R}) \subset SO(5, 5)$ into

$$X^M = \begin{pmatrix} x^m \\ \tilde{x}_{mn} \\ \tilde{x}_{mnpqr} \end{pmatrix}, \tag{5.2.29}$$

where, in a suitable duality frame, x^m are the usual coordinates on T^5 and \tilde{x}_{mn} are periodic coordinates conjugate to M2-brane wrapping numbers on T^5 and \tilde{x}_{mnpqr} are periodic coordinates

conjugate to M5-brane wrapping numbers on T^5 . Fields and gauge parameters depend on both y^μ and X^M , but we will suppress dependence on y^μ in what follows.

The strong constraint of $SO(5, 5)$ EFT is given by

$$\gamma_a^{MN} \gamma_{PQ}^a \partial_M(\dots) \partial_N(\dots) = 0, \quad (5.2.30)$$

where (\dots) represents fields and gauge parameters, γ_{MN}^a is a gamma matrix of $SO(5, 5)$ and $a = 1, \dots, 10$ labels the vector representation of $SO(5, 5)$. The strong constraint can be solved such that the fields are independent of wrapping coordinates \tilde{x}_{mn} and \tilde{x}_{mnpqr} so that

$$\tilde{\partial}^{mn}(\dots) = 0 \text{ and } \tilde{\partial}^{mnpqr}(\dots) = 0, \quad (5.2.31)$$

where $\tilde{\partial}^{mn} = \frac{\partial}{\partial \tilde{x}_{mn}}$ and $\tilde{\partial}^{mnpqr} = \frac{\partial}{\partial \tilde{x}_{mnpqr}}$. The gauge transformation of $SO(5, 5)$ extended field theory is given by the generalized Lie derivative, which is

$$\hat{\mathcal{L}}_V W^M = V^N \partial_N W^M - W^N \partial_N V^M + \frac{1}{2} \gamma_a^{MN} \gamma_{PQ}^a \partial_N V^P W^Q, \quad (5.2.32)$$

It is convenient to rewrite the components of the generalized vector as

$$w = w^m e_m, \quad (5.2.33)$$

$$\tilde{w}_{(2)} = \frac{1}{2!} \tilde{w}_{mn} e^m \wedge e^n, \quad (5.2.34)$$

$$\tilde{w}_{(5)} = \frac{1}{5!} \tilde{w}_{mnpqr} e^m \wedge e^n \wedge e^p \wedge e^q \wedge e^r. \quad (5.2.35)$$

Then the generalized vector W is

$$W = w \oplus \tilde{w}_{(2)} \oplus \tilde{w}_{(5)}. \quad (5.2.36)$$

Under a gauge transformation with gauge parameter V , the generalized vector W transforms as

$$\delta_V W = \hat{\mathcal{L}}_V W, \quad (5.2.37)$$

which decomposes into

$$\delta_V w = \mathcal{L}_v w, \quad (5.2.38)$$

$$\delta_V \tilde{w}_{(2)} = \mathcal{L}_v \tilde{w}_{(2)} - \iota_w d\tilde{w}_{(2)}, \quad (5.2.39)$$

$$\delta_V \tilde{w}_{(5)} = \mathcal{L}_v \tilde{w}_{(5)} - \tilde{w}_{(2)} \wedge d\tilde{w}_{(2)}. \quad (5.2.40)$$

where \mathcal{L}_v is an ordinary Lie derivative with a parameter v .

Next we define a gerbe connection $C_{(3)}$,

$$C_{(3)} = \frac{1}{3!} C_{mnp} e^m \wedge e^n \wedge e^p, \quad (5.2.41)$$

which transforms under a gauge transformation as

$$\delta_V C_{(3)} = \mathcal{L}_v C_{(3)} + d\tilde{v}_{(2)}. \quad (5.2.42)$$

This allows us to define $\hat{w}_{(2)}$ and $\hat{w}_{(5)}$ by

$$\hat{w}_{(2)} = \tilde{w}_{(2)} + \iota_w C_{(3)}, \quad (5.2.43)$$

$$\hat{w}_{(5)} = \tilde{w}_{(5)} + \tilde{w}_{(2)} \wedge C_{(3)} + \frac{1}{2} \iota_w C_{(3)} \wedge C_{(3)}. \quad (5.2.44)$$

Under a gauge transformation, these objects transform as a 2-form and a 5-form, respectively,

$$\delta_V \hat{w}_{(2)} = \mathcal{L}_v \hat{w}_{(2)}, \quad (5.2.45)$$

$$\delta_V \hat{w}_{(5)} = \mathcal{L}_v \hat{w}_{(5)}, \quad (5.2.46)$$

and are invariant under the \tilde{v} transformations. Therefore, $\hat{W} = w \oplus \hat{w}_{(2)} \oplus \hat{w}_{(5)}$ is a section of $(T \oplus \Lambda^2 T^* \oplus \Lambda^5 T^*)U$. The finite transformations are then

$$w'(x') = w(x), \quad (5.2.47)$$

$$\hat{w}'_{(2)}(x') = \hat{w}_{(2)}(x), \quad (5.2.48)$$

$$\hat{w}'_{(5)}(x') = \hat{w}_{(5)}(x), \quad (5.2.49)$$

where $x' = e^{-v^m \partial_m} x$.

Using the finite transformation of the coordinate bases given by

$$e'_m(x') = e_n(x) \frac{\partial x^n}{\partial x'^m} \quad e'^m(x') = e^n(x) \frac{\partial x'^m}{\partial x^n}, \quad (5.2.50)$$

the finite transformation of the components of \hat{W} are

$$w'^m(x') = w^n(x) \frac{\partial x'^m}{\partial x^n}, \quad (5.2.51)$$

$$\hat{w}'_{mn}(x') = \hat{w}_{mn}(x) \frac{\partial x^p}{\partial x'^m} \frac{\partial x^q}{\partial x'^n}, \quad (5.2.52)$$

$$\hat{w}'_{mnpqr}(x') = \hat{w}_{stuvw}(x) \frac{\partial x^s}{\partial x'^m} \frac{\partial x^t}{\partial x'^n} \frac{\partial x^u}{\partial x'^p} \frac{\partial x^v}{\partial x'^q} \frac{\partial x^w}{\partial x'^r} \quad (5.2.53)$$

The finite transformation of the gerbe connection can be taken to be

$$C'_{(3)}(x') = C_{(3)}(x) + d\tilde{v}_{(2)}(x), \quad (5.2.54)$$

so that

$$\hat{w}'_{(2)}(x') = \tilde{w}'_{(2)}(x') + \iota_{w'} C'_{(3)}(x'). \quad (5.2.55)$$

$$\hat{w}'_{(5)}(x') = \tilde{w}'_{(5)}(x') + \tilde{w}'_{(2)}(x') \wedge C'_{(3)}(x') + \frac{1}{2} \iota_{w'} C'_{(3)}(x') \wedge C'_{(3)}(x'). \quad (5.2.56)$$

This give the finite transformation of $\tilde{w}_{(2)}$:

$$\begin{aligned} \tilde{w}'_{(2)}(x') &= \hat{w}'_{(2)}(x') - \iota_{w'} C'_{(3)}(x'). \\ &= \hat{w}_{(2)}(x) - \iota_w (C_{(3)}(x) + d\tilde{v}_{(2)}(x)), \\ &= \hat{w}_{(2)}(x) - \iota_w C_{(3)}(x) - \iota_w d\tilde{v}_{(2)}(x), \\ &= \tilde{w}_{(2)}(x) - \iota_w d\tilde{v}_{(2)}(x). \end{aligned} \quad (5.2.57)$$

Furthermore, the finite transformations of $\tilde{w}_{(5)}$ are

$$\begin{aligned} \tilde{w}'_{(5)}(x') &= \hat{w}'_{(5)}(x') - \tilde{w}'_{(2)}(x') \wedge C'_{(3)}(x') - \frac{1}{2} \iota_{w'} C'_{(3)}(x') \wedge C'_{(3)}(x'), \\ &= (\tilde{w}_{(5)}(x) + \tilde{w}_{(2)}(x) \wedge C_{(3)}(x) + \frac{1}{2} \iota_w C_{(3)}(x) \wedge C_{(3)}(x)) \\ &\quad - (\tilde{w}_{(2)}(x) - \iota_w d\tilde{v}_{(2)}(x)) \wedge (C_{(3)}(x) + d\tilde{v}_{(2)}(x)) \\ &\quad - \frac{1}{2} \iota_w (C_{(3)}(x) + d\tilde{v}_{(2)}(x)) \wedge (C_{(3)}(x) + d\tilde{v}_{(2)}(x)), \\ &= \tilde{w}_{(5)}(x) - \tilde{w}_{(2)}(x) \wedge d\tilde{v}_{(2)}(x) + \frac{1}{2} \iota_w d\tilde{v}_{(2)}(x) \wedge d\tilde{v}_{(2)}(x). \end{aligned} \quad (5.2.58)$$

To summary, the transformation of W is given by

$$w'(x') = w(x), \quad (5.2.59)$$

$$\tilde{w}'_{(2)}(x') = \tilde{w}_{(2)}(x) - \iota_w d\tilde{v}_{(2)}(x), \quad (5.2.60)$$

$$\tilde{w}'_{(5)}(x') = \tilde{w}_{(5)}(x) - \tilde{w}_{(2)}(x) \wedge d\tilde{v}_{(2)}(x) + \frac{1}{2} \iota_w d\tilde{v}_{(2)}(x) \wedge d\tilde{v}_{(2)}(x), \quad (5.2.61)$$

which implies the components $(w^m, \tilde{w}_{mn}, \tilde{w}_{mnpqr})$ transform as

$$w'^m(x') = w^n(x) \frac{\partial x'^m}{\partial x^n}, \quad (5.2.62)$$

$$\tilde{w}'_{mn}(x') = (\tilde{w}_{pq}(x) - 3w^r(x)(d\tilde{v}_{(2)})_{rpq}(x)) \frac{\partial x^p}{\partial x'^m} \frac{\partial x^q}{\partial x'^n}, \quad (5.2.63)$$

$$\begin{aligned} \tilde{w}'_{mnpqr}(x') &= (\tilde{w}_{stuvw}(x) - 30\tilde{w}_{[st}(x)(d\tilde{v}_{(2)})_{uvw]}(x) + 15w^l(x)(d\tilde{v}_{(2)})_{l[st}(x)(d\tilde{v}_{(2)})_{uvw]}(x)) \\ &\quad \times \frac{\partial x^s}{\partial x'^m} \frac{\partial x^t}{\partial x'^n} \frac{\partial x^u}{\partial x'^p} \frac{\partial x^v}{\partial x'^q} \frac{\partial x^w}{\partial x'^r}, \end{aligned} \quad (5.2.64)$$

where $(d\tilde{v}_{(2)})_{rpq} = \partial_{[r}\tilde{v}_{pq]}$.

5.2.3. E_6 Extended field theory

In E_6 extended field theory [26, 27, 55], a generalized vector W^M transforms in the fundamental **(27)** representation of E_6 with $M, N = 1, \dots, 27$. It decomposes under $GL(6, \mathbb{R}) \subset E_6$ into

$$W^M = \begin{pmatrix} w^m \\ \tilde{w}_{mn} \\ \tilde{w}_{mnpqr} \end{pmatrix}, \quad (5.2.65)$$

where $m, n = 1, \dots, 6$, $\tilde{w}_{mn} = -\tilde{w}_{nm}$, and $\tilde{w}_{mnpqr} = \tilde{w}_{[mnpqr]}$.

The coordinates in a patch \mathcal{U} consist of 5 spacetime coordinates y^μ , $\mu = 0, \dots, 4$, together with 27 internal coordinates X^M transforming as a **27** of E_6 . This decomposes under $GL(6, \mathbb{R}) \subset E_6$ into

$$X^M = \begin{pmatrix} x^m \\ \tilde{x}_{mn} \\ \tilde{x}_{mnpqr} \end{pmatrix}, \quad (5.2.66)$$

where, in a suitable duality frame, x^m are the usual coordinates on T^6 and \tilde{x}_{mn} are periodic coordinates conjugate to M2-brane wrapping numbers on T^6 and \tilde{x}_{mnpqr} are periodic coordinates conjugate to M5-brane wrapping numbers on T^6 . Fields and gauge parameters depend on both y^μ and X^M , but we will suppress dependence on y^μ in what follows.

The strong constraint of E_6 EFT is given by

$$c^{MNR} c_{PQR} \partial_M(\dots) \partial_N(\dots) = 0, \quad (5.2.67)$$

where (\dots) represents fields and gauge parameters, and c^{MNP} and c_{MNP} are the E_6 invariant tensors. The strong constraint can be solved such that the fields are independent of wrapping coordinates \tilde{x}_{mn} and \tilde{x}_{mnpqr} so that

$$\tilde{\partial}^{mn}(\dots) = 0 \text{ and } \tilde{\partial}^{mnpqr}(\dots) = 0, \quad (5.2.68)$$

where $\tilde{\partial}^{mn} = \frac{\partial}{\partial \tilde{x}_{mn}}$ and $\tilde{\partial}^{mnpqr} = \frac{\partial}{\partial \tilde{x}_{mnpqr}}$. The gauge transformation of E_6 extended field theory is given by the generalized Lie derivative, which is defined as

$$\hat{\mathcal{L}}_V W^M = V^N \partial_N W^M - W^N \partial_N V^M + 10c^{MNR} c_{PQR} \partial_N V^Q W^R, \quad (5.2.69)$$

It is convenient to rewrite the components of the generalized vector as

$$w = w^m e_m, \quad (5.2.70)$$

$$\tilde{w}_{(2)} = \frac{1}{2!} \tilde{w}_{mn} e^m \wedge e^n, \quad (5.2.71)$$

$$\tilde{w}_{(5)} = \frac{1}{5!} \tilde{w}_{mnpqr} e^m \wedge e^n \wedge e^p \wedge e^q \wedge e^r. \quad (5.2.72)$$

Then the generalized vector W is

$$W = w \oplus \tilde{w}_{(2)} \oplus \tilde{w}_{(5)}. \quad (5.2.73)$$

Under a gauge transformation with a gauge parameter V , the generalized vector W transform as

$$\delta_V W = \hat{\mathcal{L}}_V W, \quad (5.2.74)$$

which decomposes into

$$\delta_V w = \mathcal{L}_v w, \quad (5.2.75)$$

$$\delta_V \tilde{w}_{(2)} = \mathcal{L}_v \tilde{w}_{(2)} - \iota_w d\tilde{v}_{(2)}, \quad (5.2.76)$$

$$\delta_V \tilde{w}_{(5)} = \mathcal{L}_v \tilde{w}_{(5)} - \tilde{w}_{(2)} \wedge d\tilde{v}_{(2)} - \iota_w d\tilde{v}_{(5)}. \quad (5.2.77)$$

where \mathcal{L}_v is an ordinary Lie derivative with a parameter v .

Next we introduce gerbe connections $C_{(3)}$ and $C_{(6)}$,

$$C_{(3)} = \frac{1}{3!} C_{mnp} e^m \wedge e^n \wedge e^p, \quad (5.2.78)$$

$$C_{(6)} = \frac{1}{6!} C_{mnpqrs} e^m \wedge e^n \wedge e^p \wedge e^q \wedge e^r \wedge e^s, \quad (5.2.79)$$

which transform under gauge transformation as

$$\delta_V C_{(3)} = \mathcal{L}_v C_{(3)} + d\tilde{v}_{(2)}, \quad (5.2.80)$$

$$\delta_V C_{(6)} = \mathcal{L}_v C_{(6)} + d\tilde{v}_{(5)} - \frac{1}{2} C_{(3)} \wedge d\tilde{v}_{(2)}. \quad (5.2.81)$$

This allows us to define $\hat{w}_{(2)}$ and $\hat{w}_{(5)}$ by

$$\hat{w}_{(2)} = \tilde{w}_{(2)} + \iota_w C_{(3)}, \quad (5.2.82)$$

$$\hat{w}_{(5)} = \tilde{w}_{(5)} + \tilde{w}_{(2)} \wedge C_{(3)} + \frac{1}{2} \iota_w C_{(3)} \wedge C_{(3)} + \iota_w C_{(6)}. \quad (5.2.83)$$

Under a gauge transformation, these objects transform as a 2-form and a 5-form, respectively,

$$\delta_V \hat{w}_{(2)} = \mathcal{L}_v \hat{w}_{(2)}, \quad (5.2.84)$$

$$\delta_V \hat{w}_{(5)} = \mathcal{L}_v \hat{w}_{(5)}, \quad (5.2.85)$$

and are invariant under the \tilde{v} transformation. Therefore, $\hat{W} = w \oplus \hat{w}_{(2)} \oplus \hat{w}_{(5)}$ is a section of $(T \oplus \Lambda^2 T^* \oplus \Lambda^5 T^*)U$. Their finite transformations are given by

$$w'(x') = w(x), \quad (5.2.86)$$

$$\hat{w}'_{(2)}(x') = \hat{w}_{(2)}(x), \quad (5.2.87)$$

$$\hat{w}'_{(5)}(x') = \hat{w}_{(5)}(x), \quad (5.2.88)$$

where $x' = e^{-v^m \partial_m} x$.

Using the finite transformation of the coordinate bases given by

$$e'_m(x') = e_n(x) \frac{\partial x^n}{\partial x'^m} \quad e'^m(x') = e^n(x) \frac{\partial x'^m}{\partial x^n}, \quad (5.2.89)$$

the finite transformations of the components of the \hat{W} can be written as

$$w'^m(x') = w^n(x) \frac{\partial x'^m}{\partial x^n}, \quad (5.2.90)$$

$$\hat{w}'_{mn}(x') = \hat{w}_{pq}(x) \frac{\partial x^p}{\partial x'^m} \frac{\partial x^q}{\partial x'^n}, \quad (5.2.91)$$

$$\hat{w}'_{mnpqr}(x') = \hat{w}_{stuvw}(x) \frac{\partial x^s}{\partial x'^m} \frac{\partial x^t}{\partial x'^n} \frac{\partial x^u}{\partial x'^p} \frac{\partial x^v}{\partial x'^q} \frac{\partial x^w}{\partial x'^r}. \quad (5.2.92)$$

The finite transformation of the gerbe connections can be taken to be

$$C'_{(3)}(x') = C_{(3)}(x) + d\tilde{v}_{(2)}(x), \quad (5.2.93)$$

$$C'_{(6)}(x') = C_{(6)}(x) + d\tilde{v}_{(5)}(x) - \frac{1}{2} C_{(3)}(x) \wedge d\tilde{v}_{(2)}(x), \quad (5.2.94)$$

so that

$$\hat{w}'_{(2)}(x') = \tilde{w}'_{(2)}(x') + \iota_{w'} C'_{(3)}(x') \quad (5.2.95)$$

$$\begin{aligned} \hat{w}'_{(5)}(x') &= \tilde{w}'_{(5)}(x') + \tilde{w}'_{(2)}(x') \wedge C'_{(3)}(x') + \frac{1}{2} \iota_{w'} C'_{(3)}(x') \wedge C'_{(3)}(x') + \iota_{w'} C'_{(6)}(x'). \end{aligned} \quad (5.2.96)$$

This then gives the finite transformation of $\tilde{w}_{(2)}$:

$$\begin{aligned} \tilde{w}'_{(2)}(x') &= \hat{w}'_{(2)}(x') - \iota_{w'} C'_{(3)}(x'). \\ &= \hat{w}_{(2)}(x) - \iota_w (C_{(3)}(x) + d\tilde{v}_{(2)}(x)), \\ &= \hat{w}_{(2)}(x) - \iota_w C_{(3)}(x) - \iota_w d\tilde{v}_{(2)}(x), \\ &= \tilde{w}_{(2)}(x) - \iota_w d\tilde{v}_{(2)}(x), \end{aligned} \quad (5.2.97)$$

and the finite transformation of $\tilde{w}_{(5)}$:

$$\begin{aligned} \tilde{w}'_{(5)}(x') &= \hat{w}'_{(5)}(x') - \tilde{w}'_{(2)}(x') \wedge C'_{(3)}(x') - \frac{1}{2} \iota_{w'} C'_{(3)}(x') \wedge C'_{(3)}(x') - \iota_{w'} C'_{(6)}(x'), \\ &= (\tilde{w}_{(5)}(x) + \tilde{w}_{(2)}(x) \wedge C_{(3)}(x) + \frac{1}{2} \iota_w C_{(3)}(x) \wedge C_{(3)}(x) + \iota_w \tilde{C}_{(3)}(x)) \\ &\quad - (\tilde{w}_{(2)}(x) - \iota_w d\tilde{v}_{(2)}(x)) \wedge (C_{(3)}(x) + d\tilde{v}_{(2)}(x)) \\ &\quad - \frac{1}{2} \iota_w (C_{(3)}(x) + d\tilde{v}_{(2)}(x)) \wedge (C_{(3)}(x) + d\tilde{v}_{(2)}(x)) \\ &\quad - \iota_w (C_{(6)}(x) + d\tilde{v}_{(5)}(x) - \frac{1}{2} C_{(3)}(x) \wedge d\tilde{v}_{(2)}(x)), \\ &= \tilde{w}_{(5)}(x) - \tilde{w}_{(2)}(x) \wedge d\tilde{v}_{(2)}(x) + \frac{1}{2} \iota_w d\tilde{v}_{(2)}(x) \wedge d\tilde{v}_{(2)}(x) - \iota_w d\tilde{v}_{(5)}(x). \end{aligned} \quad (5.2.98)$$

In summary, the transformation of W is given by

$$w'(x') = w(x), \quad (5.2.99)$$

$$\tilde{w}'_{(2)}(x') = \tilde{w}_{(2)}(x) - \iota_w d\tilde{v}_{(2)}(x), \quad (5.2.100)$$

$$\begin{aligned} \tilde{w}'_{(5)}(x') &= \tilde{w}_{(5)}(x) - \tilde{w}_{(2)}(x) \wedge d\tilde{v}_{(2)}(x) + \frac{1}{2} \iota_w d\tilde{v}_{(2)}(x) \wedge d\tilde{v}_{(2)}(x) - \iota_w d\tilde{v}_{(5)}(x), \end{aligned} \quad (5.2.101)$$

which implies the components $(w^m, \tilde{w}_{mn}, \tilde{w}_{mnpqr})$ transform as

$$w'^m(x') = w^n(x) \frac{\partial x'^m}{\partial x^n}, \quad (5.2.102)$$

$$\tilde{w}'_{mn}(x') = (\tilde{w}_{pq}(x) - 3w^r(x)(d\tilde{v}_{(2)})_{rpq}(x)) \frac{\partial x^p}{\partial x'^m} \frac{\partial x^q}{\partial x'^n}, \quad (5.2.103)$$

$$\begin{aligned} \tilde{w}'_{mnpqr}(x') = & (\tilde{w}_{stuvw}(x) - 30\tilde{w}_{[st}(x)(d\tilde{v}_{(2)})_{uvw]}(x) + 15w^l(x)(d\tilde{v}_{(2)})_{l[st}(x)(d\tilde{v}_{(2)})_{uvw]}(x) \\ & - 6w^l(x)(d\tilde{v}_{(5)})_{lstuvw}(x)) \frac{\partial x^s}{\partial x'^m} \frac{\partial x^t}{\partial x'^n} \frac{\partial x^u}{\partial x'^p} \frac{\partial x^v}{\partial x'^q} \frac{\partial x^w}{\partial x'^r}, \end{aligned} \quad (5.2.104)$$

where $(d\tilde{v}_{(2)})_{rpq} = \partial_{[r}\tilde{v}_{pq]}$ and $(d\tilde{v}_{(5)})_{mnpqrs} = \partial_{[m}\tilde{v}_{npqrs]}$.

5.3. Generalized tensors

In this section, we review the construction of tensors and twisted tensors in DFT of [78] and then generalize this to EFT .

5.3.1. Generalized tensors of double field theory

A generalized vector W^M transforms as a vector under $O(D, D)$, so that under $GL(D, \mathbb{R}) \subset O(D, D)$ it transforms reducibly as

$$W \rightarrow \hat{R}(\lambda)W \quad (5.3.1)$$

where $\lambda^m_n \in GL(D, \mathbb{R})$ and

$$\hat{R}(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & (\lambda^{-1})^t \end{pmatrix} \quad (5.3.2)$$

The untwisted version of W is \hat{W}^M , which can be written as

$$\hat{W} = LW \quad (5.3.3)$$

where

$$L = \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} \quad (5.3.4)$$

denotes the matrix with components

$$L^M_N = \begin{pmatrix} \delta^m_n & 0 \\ -B_{mn} & \delta_m^n \end{pmatrix}. \quad (5.3.5)$$

The transformation of the untwisted vector \hat{W} is then

$$\hat{W}'(X') = \hat{R}(\Lambda)\hat{W}(X) \quad (5.3.6)$$

where

$$\Lambda^m{}_n = \frac{\partial x'^m}{\partial x^n}. \quad (5.3.7)$$

The coordinate transformation acts only on the x :

$$X^M \rightarrow X'^M = \begin{pmatrix} x'^m \\ \tilde{x}'_m \end{pmatrix}, \quad (5.3.8)$$

with

$$x^m \rightarrow x'^m(x), \quad \tilde{x}_m \rightarrow \tilde{x}'_m = \tilde{x}_m \quad (5.3.9)$$

The transformation of the twisted vector W was found by twisting the untwisted transformation and is

$$W'(X') = RW(X) \quad (5.3.10)$$

where

$$R = L'(X')^{-1} \hat{R}(\Lambda) L(X) = \hat{R}(\Lambda) S \quad (5.3.11)$$

and

$$L'(X') = \begin{pmatrix} 1 & 0 \\ -B'(x') & 1 \end{pmatrix} \quad (5.3.12)$$

with $B'(x')$ given by (5.1.23), and

$$S = \begin{pmatrix} \delta^m{}_n & 0 \\ 2\partial_{[m}\tilde{v}_{n]} & \delta_m{}^n \end{pmatrix}. \quad (5.3.13)$$

The matrices R, \hat{R}, L, S are all in $O(D, D)$.

Lowering indices with η gives similar formulae for a generalized vector with lower index

$$U_M = \begin{pmatrix} \tilde{u}_m \\ u^m \end{pmatrix}. \quad (5.3.14)$$

The untwisted vector

$$\hat{U}_M = \begin{pmatrix} \hat{u}_m \\ u^m \end{pmatrix} = \begin{pmatrix} \tilde{u}_m - B_{mn}u^n \\ u^m \end{pmatrix} \quad (5.3.15)$$

transforms with

$$\delta \hat{U}_M = \mathcal{L}_v \hat{U}_M \quad (5.3.16)$$

and is invariant under \tilde{v} transformations. Then the untwisted vector is

$$\hat{U} = UL^{-1} \quad (5.3.17)$$

(i.e. $\hat{U}_M = U_N(L^{-1})^N_M$; recall $\eta L \eta^{-1} = (L^t)^{-1}$ as L is in $O(D, D)$) and transforms under a finite transformation as

$$\hat{U}'(X') = \hat{U}(X) \hat{R}^{-1} \quad (5.3.18)$$

where here and in what follows $\hat{R} = \hat{R}(\Lambda)$. For the twisted vectors

$$U'(X') = U(X) R^{-1} \quad (5.3.19)$$

This extends to arbitrary generalized tensors $T^{MN\dots}_{PQ\dots}$. We define the untwisted tensor

$$\hat{T}^{MN\dots}_{PQ\dots} = L^M{}_R L^N{}_S \dots T^{RS\dots}_{TU\dots} (L^{-1})^T{}_P (L^{-1})^U{}_Q \dots \quad (5.3.20)$$

which transforms as

$$\hat{T}'^{MN\dots}_{PQ\dots}(X') = \hat{R}^M{}_R \hat{R}^N{}_S \dots T^{RS\dots}_{TU\dots} (\hat{R}^{-1})^T{}_P (\hat{R}^{-1})^U{}_Q \dots \quad (5.3.21)$$

so that the original tensor transforms as

$$T'^{MN\dots}_{PQ\dots}(X') = R^M{}_R R^N{}_S \dots T^{RS\dots}_{TU\dots} (R^{-1})^T{}_P (R^{-1})^U{}_Q \dots \quad (5.3.22)$$

Raising all lower indices with η gives a generalized tensor $T^{M_1\dots M_p}$ of some rank p which is a section of E^p while $\hat{T}^{M_1\dots M_p}$ is a section of $(T \oplus T^*)^p$. In particular,

$$\hat{\eta}_{MN} = \eta_{MN} \quad (5.3.23)$$

as $L \in O(D, D)$, and is invariant, $\eta' = \eta$.

5.3.2. Generalized Tensors of Extended Field Theory

For each of the extended field theories, a similar structure applies, with matrices L, L', R, \hat{R}, S , all of which are in the duality group G which is $SL(5, \mathbb{R})$, $SO(5, 5)$ or E_6 . The untwisted form \hat{W}^M of a generalized vector W^M can be written as

$$\hat{W} = L W. \quad (5.3.24)$$

The generalized vector transforms as a representation of G and decompose into a reducible representation of $GL(d, \mathbb{R})$ (where d is the rank of G) under which $\lambda \in GL(d, \mathbb{R})$ acts on W to give $W \rightarrow \hat{R}(\lambda) W$. The transformation of the untwisted vector \hat{W} is then

$$\hat{W}'(X') = \hat{R}(\lambda) \hat{W}(X) \quad (5.3.25)$$

where

$$\Lambda^m_n = \frac{\partial x'^m}{\partial x^n}. \quad (5.3.26)$$

The transformation of the twisted vector W was found by twisting the untwisted transformation and can be written as

$$W'(X') = RW(X) \quad (5.3.27)$$

where

$$R = L'(X')^{-1} \hat{R}(\Lambda) L(X) = \hat{R}(\Lambda) S \quad (5.3.28)$$

where L' generates shifts of the antisymmetric tensor gauge fields and S is the corresponding gauge transformation; see below for explicit forms.

As for the DFT case, this extends to arbitrary generalized tensors $T^{MN\dots}_{PQ\dots}$. We define the untwisted tensor

$$\hat{T}^{MN\dots}_{PQ\dots} = L^M_R L^N_S \dots T^{RS\dots}_{TU\dots} (L^{-1})^T_P (L^{-1})^U_Q \dots \quad (5.3.29)$$

which transforms as

$$\hat{T}^{MN\dots}_{PQ\dots}(X') = \hat{R}^M_R \hat{R}^N_S \dots T^{RS\dots}_{TU\dots} (\hat{R}^{-1})^T_P (\hat{R}^{-1})^U_Q \dots \quad (5.3.30)$$

so that the original tensor transforms as

$$T^{MN\dots}_{PQ\dots}(X') = R^M_R R^N_S \dots T^{RS\dots}_{TU\dots} (R^{-1})^T_P (R^{-1})^U_Q \dots \quad (5.3.31)$$

We now give the explicit forms of the matrices appearing above. For the $SL(5, \mathbb{R})$ case, the untwisted vector is

$$\hat{W}^M = L^M_N W^N = \begin{pmatrix} \delta^m_l & 0 \\ C_{lmn} & \delta_{mn}{}^{pq} \end{pmatrix} \begin{pmatrix} w^l \\ \tilde{w}_{pq} \end{pmatrix}, \quad (5.3.32)$$

where $\delta_{mn}{}^{pq} = \frac{1}{2}(\delta_m^p \delta_n^q - \delta_m^q \delta_n^p)$. For the $SO(5, 5)$ case, the untwisted vector is

$$\hat{W}^M = L^M_N W^N = \begin{pmatrix} \delta^m_l & 0 & 0 \\ C_{lmn} & \delta_{mn}{}^{pq} & 0 \\ 5C_{l[mn} C_{pqr]} & 10\delta_{[mn}{}^{pq} C_{rst]} & \delta_{mnpqr}{}^{stuvw} \end{pmatrix} \begin{pmatrix} w^l \\ \tilde{w}_{pq} \\ \tilde{w}_{stuvw} \end{pmatrix}, \quad (5.3.33)$$

where $\delta_{mnpqr}{}^{stuvw} = \delta_{[m}^s \dots \delta_{r]}^w$. For E_6 case, the untwisted vector is

$$\hat{W}^M = L^M_N W^N = \begin{pmatrix} \delta^m_l & 0 & 0 \\ C_{lmn} & \delta_{mn}{}^{pq} & 0 \\ C_{lmnpqr} + 5C_{l[mn} C_{pqr]} & 10\delta_{[mn}{}^{pq} C_{rst]} & \delta_{mnpqr}{}^{stuvw} \end{pmatrix} \begin{pmatrix} w^l \\ \tilde{w}_{pq} \\ \tilde{w}_{stuvw} \end{pmatrix} \quad (5.3.34)$$

Then the L matrix for the $SL(5, \mathbb{R})$ theory is

$$L^M{}_N = \begin{pmatrix} \delta^m{}_l & 0 \\ C_{lmn} & \delta_{mn}{}^{pq} \end{pmatrix}, \quad (5.3.35)$$

while for the $SO(5, 5)$ theory it is

$$L^M{}_N = \begin{pmatrix} \delta^m{}_l & 0 & 0 \\ C_{lmn} & \delta_{mn}{}^{pq} & 0 \\ 5C_{l[mn}C_{pqr]} & 10\delta_{[mn}{}^{pq}C_{rst]} & \delta_{mnpqr}{}^{stuvw} \end{pmatrix} \quad (5.3.36)$$

and for the E_6 theory it is

$$L^M{}_N = \begin{pmatrix} \delta^m{}_l & 0 & 0 \\ C_{lmn} & \delta_{mn}{}^{pq} & 0 \\ C_{lmnpqr} + 5C_{l[mn}C_{pqr]} & 10\delta_{[mn}{}^{pq}C_{rst]} & \delta_{mnpqr}{}^{stuvw} \end{pmatrix}. \quad (5.3.37)$$

The matrices \hat{R} for the $SL(5, \mathbb{R})$, $SO(5, 5)$ and E_6 theories are given by

$$\hat{R}^M{}_N = \begin{pmatrix} \Lambda^m{}_l & 0 \\ 0 & (\Lambda^{-1})_m{}^p(\Lambda^{-1})_n{}^q \end{pmatrix}, \quad (5.3.38)$$

$$\hat{R}^M{}_N = \begin{pmatrix} \Lambda^m{}_l & 0 & 0 \\ 0 & (\Lambda^{-1})_m{}^p(\Lambda^{-1})_n{}^q & 0 \\ 0 & 0 & (\Lambda^{-1})_m{}^s(\Lambda^{-1})_n{}^t(\Lambda^{-1})_p{}^u(\Lambda^{-1})_q{}^v(\Lambda^{-1})_r{}^w \end{pmatrix}, \quad (5.3.39)$$

$$\hat{R}^M{}_N = \begin{pmatrix} \Lambda^m{}_l & 0 & 0 \\ 0 & (\Lambda^{-1})_m{}^p(\Lambda^{-1})_n{}^q & 0 \\ 0 & 0 & (\Lambda^{-1})_m{}^s(\Lambda^{-1})_n{}^t(\Lambda^{-1})_p{}^u(\Lambda^{-1})_q{}^v(\Lambda^{-1})_r{}^w \end{pmatrix}, \quad (5.3.40)$$

respectively.

The L' matrix for the $SL(5, \mathbb{R})$ theory is

$$L'^M{}_N(X') = \begin{pmatrix} \delta^m{}_l & 0 \\ C'_{lmn}(X') & \delta_{mn}{}^{pq} \end{pmatrix}, \quad (5.3.41)$$

while for the $SO(5, 5)$ theory it is

$$L'^M{}_N(X') = \begin{pmatrix} \delta^m{}_l & 0 & 0 \\ C'_{lmn}(X') & \delta_{mn}{}^{pq} & 0 \\ 5C'_{l[mn}(X')C'_{pqr]}(X') & 10\delta_{[mn}{}^{pq}C'_{rst]}(X') & \delta_{mnpqr}{}^{stuvw} \end{pmatrix} \quad (5.3.42)$$

and for the E_6 theory it is

$$L'^M{}_N(X') = \begin{pmatrix} \delta^m{}_l & 0 & 0 \\ C'_{lmn}(X') & \delta_{mn}{}^{pq} & 0 \\ C'_{lmnpqr}(X') + 5C'_{l[mn}(X')C'_{pqr]}(X') & 10\delta_{[mn}{}^{pq}C'_{rst]}(X') & \delta_{mnpqr}{}^{stuvw} \end{pmatrix}. \quad (5.3.43)$$

Finally, the matrix S for the $SL(5, \mathbb{R})$ theory is

$$S^M{}_N = \begin{pmatrix} \delta^m{}_l & 0 \\ -3(d\tilde{v}_{(2)})_{lmn} & \delta_{mn}{}^{pq} \end{pmatrix} \quad (5.3.44)$$

while for the $SO(5, 5)$ theory it is

$$S^M{}_N = \begin{pmatrix} \delta^m{}_l & 0 & 0 \\ -3(d\tilde{v}_{(2)})_{lmn} & \delta_{mn}{}^{pq} & 0 \\ 15(d\tilde{v}_{(2)})_{l[mn}(d\tilde{v}_{(2)})_{pqr]} & -30\delta_{[mn}{}^{pq}\partial_r\tilde{v}_{st]} & \delta_{mnpqr}{}^{stuvw} \end{pmatrix} \quad (5.3.45)$$

and for the E_6 theory it is

$$S^M{}_N = \begin{pmatrix} \delta^m{}_l & 0 & 0 \\ -3(d\tilde{v}_{(2)})_{lmn} & \delta_{mn}{}^{pq} & 0 \\ -6(d\tilde{v}_{(5)})_{lmnpqr} + 15(d\tilde{v}_{(2)})_{l[mn}(d\tilde{v}_{(2)})_{pqr]} & -30\delta_{[mn}{}^{pq}\partial_r\tilde{v}_{st]} & \delta_{mnpqr}{}^{stuvw} \end{pmatrix}, \quad (5.3.46)$$

where $(d\tilde{v}_{(2)})_{rpq} = \partial_{[r}\tilde{v}_{pq]}$ and $(d\tilde{v}_{(5)})_{mnpqrs} = \partial_{[m}\tilde{v}_{npqrs]}$.

5.4. The generalized metric in DFT and EFT

For DFT and EFT, there is a duality group G (which is $O(D, D)$ for DFT and E_D for EFT) with maximal compact subgroup H . Remarkably, the fields (g_{mn}, B_{mn}) in DFT and $(g_{mn}, C_{mnp}, C_{mnpqrs})$ in EFT can be regarded as a field taking values in the coset G/H – the coset space can be locally parameterized by g_{mn}, B_{mn} or $g_{mn}, C_{mnp}, C_{mnpqrs}$ [21]. A field taking values in the coset G/H can be represented by a vielbein $\mathcal{V}^A{}_M(X) \in G$ transforming as

$$\mathcal{V} \rightarrow h\mathcal{V}g \quad (5.4.1)$$

under a rigid G transformation $g^N_M \in G$ and a local H transformation $h^A_B(X) \in H$. If k_{AB} is an H -invariant metric ($h^t k h = k$ for all $h \in H$) then the degrees of freedom can also be encoded in a generalized metric

$$\mathcal{H}_{MN} = k_{AB} \mathcal{V}^A_M \mathcal{V}^B_N \quad (5.4.2)$$

which by construction is invariant under H transformations.

We now show how the coset is parameterized by the fields (g_{mn}, B_{mn}) or $(g_{mn}, C_{mnp}, C_{mnpqrs})$. Let e^a_m be a vielbein for the metric g_{mn} , with

$$g_{mn} = \delta_{ab} e^a_m e^b_n \quad (5.4.3)$$

and inverse vielbein e_a^m . The indices a, b transform under the tangent space group $O(D)$. Then the vielbein \mathcal{V} can be written in terms of e^a_m and (g_{mn}, B_{mn}) or $(g_{mn}, C_{mnp}, C_{mnpqrs})$ as

$$\mathcal{V} = h \hat{R}(e^a_m) L \quad (5.4.4)$$

(This can be viewed as a consequence of the Iwasawa decomposition theorem.) Here $h \in H$ and can be chosen to be $h = 1$ by a local H transformation. The dependence on the gauge fields is given by the matrix $L(B)$ in DFT and $L(C_{(3)}, C_{(6)})$ in EFT; L is of the form $L = 1 + N$ where N is lower triangular. Finally $\hat{R}(e)$ is the matrix $\hat{R}(\lambda)$ given above, with $\lambda^a_m = e^a_m$ and serves to convert ‘curved’ indices m, n to ‘flat’ indices a, b .

Then the generalized metric is given by

$$\mathcal{H}(V, W) = \hat{\mathcal{H}}(\hat{V}, \hat{W}) \quad (5.4.5)$$

for any generalized vectors V, W , where

$$\hat{\mathcal{H}}(\hat{V}, \hat{W}) = k(\hat{R}(e) \hat{V}, \hat{R}(e) \hat{W}) \quad (5.4.6)$$

Explicitly,

$$\hat{\mathcal{H}}_{MN} = k_{AB} \hat{R}(e)^A_M \hat{R}(e)^B_N \quad (5.4.7)$$

and

$$\mathcal{H}_{MN} = \hat{\mathcal{H}}_{PQ} L^P_M L^Q_N \quad (5.4.8)$$

Then $\hat{\mathcal{H}}_{PQ}$ is the untwisted form of the generalized metric, and is the natural metric on generalized or extended tangent vectors induced by the metric g for ordinary vectors.

We now show how this works for the cases discussed here. For DFT, we recover the discussion

of [78]. An untwisted vector decomposes as

$$\hat{W}^M = \begin{pmatrix} w^m \\ \hat{w}_m \end{pmatrix}, \quad (5.4.9)$$

so

$$\hat{R}(e)\hat{W} = \begin{pmatrix} w^a \\ \hat{w}_a \end{pmatrix} \quad (5.4.10)$$

where as usual

$$w^a = e^a_m w^m, \quad \hat{w}_a = e_a^m \hat{w}_m \quad (5.4.11)$$

The metric k_{AB} decomposes under $O(D)$ to give

$$k_{AB} = \begin{pmatrix} \delta_{ab} & 0 \\ 0 & \delta^{ab} \end{pmatrix} \quad (5.4.12)$$

so

$$k(\hat{R}(e)\hat{V}, \hat{R}(e)\hat{W}) = \delta_{ab} v^a w^b + \delta^{ab} \hat{v}_a \hat{w}_b \quad (5.4.13)$$

This is equal to $\hat{\mathcal{H}}(\hat{V}, \hat{W}) = \hat{\mathcal{H}}_{MN} \hat{V}^M \hat{W}^N$ so

$$\hat{\mathcal{H}}_{MN} = \begin{pmatrix} g_{mn} & 0 \\ 0 & g^{mn} \end{pmatrix} \quad (5.4.14)$$

which is the standard metric on $T \oplus T^*$ induced by the metric g on T . Then the generalized metric is

$$\mathcal{H}_{MN} = \hat{\mathcal{H}}_{PQ} L^P_M L^Q_N \quad (5.4.15)$$

where $L(B)$ is given by (5.3.4). This gives the standard result

$$\mathcal{H}_{MN} = \begin{pmatrix} g_{mn} - B_{mk} g^{kl} B_{ln} & B_{mk} g^{kn} \\ -g^{mk} B_{kn} & g^{mn} \end{pmatrix} \quad (5.4.16)$$

Consider now EFT with $G = E_D$ for $D = 5, 6$. An untwisted vector decomposes as

$$\hat{W}^M = \begin{pmatrix} w^m \\ \hat{w}_{mn} \\ \hat{w}_{mnpqr} \end{pmatrix}, \quad (5.4.17)$$

so

$$\hat{R}(e)\hat{W} = \begin{pmatrix} w^a \\ \hat{w}_{ab} \\ \hat{w}_{abcde} \end{pmatrix}, \quad (5.4.18)$$

with all indices converted to tangent space indices using e^a_m . The metric k_{AB} decomposes under $O(D)$ to give

$$k(\hat{R}(e)\hat{V}, \hat{R}(e)\hat{W}) = v^a w_a + \frac{1}{2} \hat{v}_{ab} \hat{w}^{ab} + \frac{1}{5!} \hat{v}_{abcde} \hat{w}^{abcde} \quad (5.4.19)$$

where indices have been raised or lowered with δ_{ab} . The matrix can then be written as

$$k_{AB} = \begin{pmatrix} \delta_{ab} & 0 & \\ 0 & \delta^{ab,cd} & 0 \\ 0 & 0 & \delta^{a_1 \dots a_5, b_1 \dots b_5} \end{pmatrix} \quad (5.4.20)$$

where $\delta^{ab,cd} = \frac{1}{2} \delta^{c[a} \delta^{b]d}$ and a similar expression for $\delta^{a_1 \dots a_5, b_1 \dots b_5}$. Then (5.4.20) is equal to $\hat{\mathcal{H}}(\hat{V}, \hat{W}) = \hat{\mathcal{H}}_{MN} \hat{V}^M \hat{W}^N$ so

$$\hat{\mathcal{H}}_{MN} \hat{V}^M \hat{W}^N = v^m w_m + \frac{1}{2} \hat{v}_{mn} \hat{w}^{mn} + \frac{1}{5!} \hat{v}_{mnpqr} \hat{w}^{mnpqr} \quad (5.4.21)$$

where indices have been raised and lowered with g_{mn} . The corresponding matrix is

$$\hat{\mathcal{H}}_{MN} = \begin{pmatrix} g_{mn} & 0 & \\ 0 & g^{mn,pq} & 0 \\ 0 & 0 & g^{m_1 \dots m_5, n_1 \dots n_5} \end{pmatrix} \quad (5.4.22)$$

where $g^{lk,mn} = \frac{1}{2} g^{m[l} g^{k]n}$ and a similar expression for $g^{a_1 \dots a_5, b_1 \dots b_5}$. Then $\hat{\mathcal{H}}_{MN}$ is the standard metric on $T \oplus \Lambda^2 T^* \oplus \Lambda^5 T^*$ induced by the metric g on T .

The generalized metric is then the twisted from of this

$$\mathcal{H}_{MN} = \hat{\mathcal{H}}_{PQ} L^P_M L^Q_N \quad (5.4.23)$$

where $L(C_{(3)}, C_{(6)})$ is given by (5.3.36) or (5.3.37). This then gives explicit forms for the generalized metric, in agreement with [21, 41, 46, 48, 54, 57].

For $SL(5, \mathbb{R})$, there is no $C_{(6)}$ or $\hat{w}_{(5)}$, but similar formulae apply with

$$\hat{\mathcal{H}}_{MN} = \begin{pmatrix} g_{mn} & 0 \\ 0 & g^{kl,pq} \end{pmatrix}. \quad (5.4.24)$$

and (5.4.8) with $L(C_{(3)})$ given by (5.3.35) gives the

$$\mathcal{H}_{MN} = \begin{pmatrix} g_{mn} + C_m{}^{rs} C_{rsn} & \frac{1}{2} C_m{}^{pq} \\ \frac{1}{2} C^{kl}{}_n & g^{kl,pq} \end{pmatrix}, \quad (5.4.25)$$

recovering the generalized metric given in [21].

6. Degenerations of K3, Orientifolds and Exotic Branes

In this chapter, we consider the T-duality chain of the T^3 with H -flux background [80]. T-duality in one direction gives the nilfold background. A further T-duality gives the T-fold background [67]. Last T-duality will result in the R -flux background [64, 72]. However, these solutions do not define conformal field theories so cannot be used as string theory backgrounds.

These solutions, however, can arise in string theory as fibres in string backgrounds. The simplest case is that in which these spaces are fibred over a line. There is a hyperkähler metric in the case of nilfold fibred over a line. Doing T-duality in one direction of this background will result in a background T^3 with H -flux fibred over a line, while T-duality in another direction gives T-fold fibred over a line. The product of T^3 with H -flux fibred over a line and a Minkowski space can be identified as a smeared NS5-brane solution, which is dual to a D8-brane solution [98].

The D8-brane is a domain wall separating regions with different values of Roman mass [103]. The various duals of this considered above are then all domain wall solutions too, depending on a single transverse coordinate. The D8-branes can be consistently incorporated in a string theory background in the type I' string [85]. There is an O8-plane at either end of the interval and 16 D8-branes located at arbitrary positions on the interval in type I'. This solution is dual to type IIA string theory on $K3$. In this limit, $K3$ develops a long neck which is locally of the form of a nilfold fibred over a line [86].

6.1. The Nilfold and its and T-duals

Consider the 3-torus with H -flux given by an integer m . The metric and 3-form flux H are

$$ds_{T^3}^2 = dx^2 + dy^2 + dz^2 \quad H = m dx \wedge dy \wedge dz \quad (6.1.1)$$

with periodic coordinates

$$x \sim x + 2\pi \quad y \sim y + 2\pi \quad z \sim z + 2\pi$$

Here flux quantisation requires that (in our conventions) m is an integer.

Choosing the 2-form potential B with $H = dB$ as

$$B = m x dy \wedge dz \quad (6.1.2)$$

and T-dualising in the y direction gives the nilfold \mathcal{N}

$$ds_{\mathcal{N}}^2 = dx^2 + (dy - m x dz)^2 + dz^2 \quad H = 0 \quad (6.1.3)$$

This is a compact manifold that can be constructed as a quotient of the group manifold of the Heisenberg group by a cocompact discrete subgroup [64], [86]. It can be viewed as a circle bundle over a 2-torus, where the 2-torus has coordinates x, z while the fibre coordinate is y . Here the first Chern class is represented by $m dx \wedge dz$ and again m , which is the degree of the nilfold, is required to be an integer.

A complex structure modulus $\tau_1 + i\tau_2$ for the 2-torus with coordinates x, z and a radius R for the circle fibre can be introduced by choosing the identifications

$$(x, z) \sim (x + 2\pi, z), \quad (x, z) \sim (x + 2\pi\tau_1, z + 2\pi\tau_2)$$

so that for the nilfold we have

$$(x, y, z) \sim (x + 2\pi, y + 2\pi z, z), \quad (x, y, z) \sim (x + 2\pi\tau_1, y + 2\pi\tau_1 z, z + 2\pi\tau_2), \quad (x, y, z) \sim (x, y + 2\pi R, z) \quad (6.1.4)$$

To simplify our formulae, we will here display results for the simple case in which $R = 1, \tau_1 = 0, \tau_2 = 1$ but the generalisation of the results presented here to general values of these moduli is straightforward; see e.g. [81].

The nilfold can also be viewed as a 2-torus bundle over a circle, with a 2-torus parameterized by y, z and base circle parameterized by x . This viewpoint is useful in considering T-duality in the y or z directions, resulting in either case a fibration over the circle parameterized by x [64].

T-dualising this in the z -direction gives the T-fold \mathcal{T} with metric and B -field given by

$$ds_{\mathcal{T}-Fold}^2 = dx^2 + \frac{1}{1 + (mx)^2} (dy^2 + dz^2) \quad B = \frac{mx}{1 + (mx)^2} dy \wedge dz \quad (6.1.5)$$

which changes by a T-duality under $x \rightarrow x + 1$, and so has a T-duality monodromy in the x direction. A generalized T-duality [107] in the x direction gives something which is not locally geometric but which has a well-defined doubled geometry given in [63, 64].

These examples are instructive but have the drawback of not defining a CFT and so not giving a solution of string theory. However, these examples can arise in string theory in solutions in which these backgrounds appear as fibres over some base, related by a T-duality acting on

the fibres. The simplest case is that in which these solutions are fibred over a line, defining a solution that is sometimes referred to as a domain wall background. The cases with 3-torus or nilfold fibred over a line were obtained in [81] from identifications of suitable NS5-brane or KK-monopole solutions, and are dual to D8-brane solutions. We discuss these and their T-duals in the following section.

6.2. Domain wall solution from NS5-brane solution

6.2.1. Smeared NS5-brane solution

The NS5-brane solution has metric

$$ds_{10}^2 = V(x^i) ds^2(\mathbb{R}^4) + ds^2(\mathbb{R}^{1,5}), \quad (6.2.1)$$

where x^i are coordinates of the transverse space \mathbb{R}^4 . The function $V(x^i)$ is a harmonic function satisfying Laplace's equation

$$\nabla^2 V(x^i) = 0. \quad (6.2.2)$$

The H -flux is

$$H = *_4 dV^{-1}, \quad (6.2.3)$$

where $*_4$ is a Hodge dual on the space transverse to the world-volume of the NS5-brane. Explicitly, this gives

$$H_{ijk} = \epsilon_{ijkl} \delta^{lm} \partial_m V \quad (6.2.4)$$

where ϵ_{ijkl} is the alternating symbol with $\epsilon_{1234} = 1$. The dilaton is

$$e^{2\Phi} = V. \quad (6.2.5)$$

Therefore, the T-duality invariant dilaton is given by

$$e^{-2d} = e^{-2\Phi} \sqrt{g} = V. \quad (6.2.6)$$

If V is independent of one or more coordinates, the NS5-brane is said to be smeared in those directions. These directions can then be taken to be periodic and we can then T-dualise in them. In what follows, we shall review the various dual spaces that emerge in this way. In each case, we get a string background preserving half the supersymmetry.

We shall be particularly interested in the case in which $V = V(\tau)$ is independent of 3 coordinates, x, y, z , which can then be taken to be periodic. The NS5-brane is then said to

be smeared over the x, y, z directions, and there is H -flux on the 3-torus with coordinates x, y, z . The solution (6.2.1), (6.2.3), and (6.2.5) then represents the product of flat 6-dimensional space $\mathbb{R}^{1,5}$ with the 4-dimensional space given by a 3-torus with H -flux fibred over a line with coordinate τ . Successive T-dualities will then take the 3-torus with flux fibred over a line first to a nilfold fibred over a line, then to a T-fold fibred over a line and finally to a space with R -flux fibred over a line.

Consider first the case in which V is independent of one of the coordinates $x^i = (\tau, x, y, z)$, y say. We take y to be periodic (i.e. we can identify under $y \rightarrow y + 2\pi$) and T-dualise in the y direction to obtain the KK-monopole solution

$$ds_{10}^2 = ds^2(GH) + ds^2(\mathbb{R}^{1,5}), \quad (6.2.7)$$

where $ds^2(GH)$ is a Gibbons-Hawking metric

$$ds^2(GH) = V(d\tau^2 + dx^2 + dz^2) + V^{-1}(dy + \omega)^2 \quad (6.2.8)$$

with $V(\tau, x, z)$ a harmonic function on \mathbb{R}^3 and ω a 1-form on \mathbb{R}^3 satisfying

$$\vec{\nabla} \times \vec{\omega} = \vec{\nabla} V \quad (6.2.9)$$

and

$$H = 0, \quad \Phi = \text{constant} \quad (6.2.10)$$

The metric is hyperkähler.

If V is independent of y, z , so that $V = V(\tau, x)$ is a function on \mathbb{R}^2 , then the smeared NS5-brane solution can be taken to be periodic in the y, z directions and can be dualised in one or both directions. T-dualising in both the y and z directions is the same as T-dualising the KK-monopole solution (6.2.7) in the z direction and gives

$$ds^2 = ds^2(X) + ds^2(\mathbb{R}^{1,5}), \quad (6.2.11)$$

where X is a four-dimensional space with metric

$$ds^2(X) = V(d\tau^2 + dx^2) + \frac{V}{V^2 + w^2}(dy^2 + dz^2) \quad (6.2.12)$$

with $V(\tau, x)$ a harmonic function on \mathbb{R}^2 and w is a function on \mathbb{R}^2 which satisfies

$$\frac{\partial V}{\partial \tau} = \frac{\partial w}{\partial x}, \quad \frac{\partial V}{\partial x} = -\frac{\partial w}{\partial \tau} \quad (6.2.13)$$

and

$$B = \frac{w}{V^2 + w^2} dy \wedge dz. \quad (6.2.14)$$

To find the dilaton, we note that

$$e^{-2d} \equiv e^{-2\Phi} \sqrt{g} \quad (6.2.15)$$

is invariant under T-duality, so that if under T-duality

$$g_{\mu\nu} \rightarrow g'_{\mu\nu}, \quad \Phi \rightarrow \Phi' \quad (6.2.16)$$

we have that

$$\Phi' = \Phi + \frac{1}{2} \log(\sqrt{g'}/\sqrt{g}) \quad (6.2.17)$$

For the metric (6.2.8) we have $\sqrt{g} = V$ while for (6.2.12) we have

$$\sqrt{g'} = \frac{V^2}{V^2 + w^2}. \quad (6.2.18)$$

so the dilaton for the space X is

$$\Phi = \frac{1}{2} \log \left(\frac{V}{V^2 + w^2} \right) \quad (6.2.19)$$

Finally, if V is independent of x, y, z , so that $V = V(\tau)$, then we can take x, y, z as periodic and can T-dualise in one, two or three directions. For $V = V(\tau)$ to be a harmonic function on \mathbb{R} , $V'' = 0$, it must be a linear function. The simplest case is to take

$$V(\tau) = m\tau + c, \quad (6.2.20)$$

where m and c are constant. The form $V(\tau)$ implies the form of w (6.2.13) as

$$w = V'(\tau)x = mx. \quad (6.2.21)$$

Then for the NS5-brane solution we obtain the following conformally flat metric on $T^3 \times \mathbb{R}$

$$ds^2 = V(\tau)[d\tau^2 + dx^2 + dy^2 + dz^2] \quad (6.2.22)$$

together with the H -flux on T^3

$$H = *_4 dV^{-1} = -m dx \wedge dy \wedge dz. \quad (6.2.23)$$

and dilaton

$$e^{2\Phi} = V(\tau) \quad (6.2.24)$$

The usual flux quantisation condition implies that the coefficient m is quantized; we adopt conventions in which m is an integer. By changing coordinates $\tau \rightarrow \sigma(\tau) = \log V$, one could arrange for the dilaton to have linear dependence on the coordinate σ .

This 4-dimensional space has topology $\mathbb{R} \times T^3$. The geometry is a 3-torus with flux fibred over \mathbb{R} – the metric of the 3-torus and the dilaton depend on the coordinate τ , but the flux $\int_{T^3} H = m$ remains constant and is quantised.

More generally, V can be taken to be piecewise linear, e.g.

$$V(\tau) = \begin{cases} c + m'\tau, & \tau \leq 0 \\ c + m\tau, & \tau > 0 \end{cases} \quad (6.2.25)$$

This is continuous but not differentiable at $\tau = 0$. The singularity corresponds to a domain wall at $\tau = 0$ separating two ‘phases’ with fluxes m, m' . This can be thought of as a brane that has a tension proportional to $m - m'$. This can be understood [81] as a dual of the D8-brane solution [98], as we will discuss in section 6.3. A full string solution is then obtained by introducing O8-planes in the D8-brane solution and dualising.

A multi-brane solution with a domain wall at $\tau = \tau_1, \tau_2, \dots, \tau_n$ is given by

$$V(\tau) = \begin{cases} c_1 + m_1\tau, & \tau \leq \tau_1 \\ c_2 + m_2\tau, & \tau_1 < \tau \leq \tau_2 \\ \vdots \\ c_n + m_n\tau & \tau_{n-1} < \tau \leq \tau_n \\ c_{n+1} + m_{n+1}\tau & \tau > \tau_n. \end{cases} \quad (6.2.26)$$

for some constants c_1, m_i , and for continuity the constants c_r for $r > 1$ are given in terms of c_1, m_i by

$$c_{r+1} = c_r + (m_r - m_{r+1})\tau_r \quad (6.2.27)$$

The brane charge of the domain wall at τ_r is the integer

$$N_r = m_{r+1} - m_r. \quad (6.2.28)$$

Note that the derivative $M(\tau) \equiv V'(\tau)$ of $V(\tau)$ with respect to τ will be piecewise constant away from the domain wall points τ_r

$$M(\tau) = \begin{cases} m_1, & \tau < \tau_1 \\ m_2, & \tau_1 < \tau < \tau_2 \\ \vdots & \\ m_n, & \tau_{n-1} < \tau < \tau_n \\ m_{n+1} & \tau > \tau_n. \end{cases} \quad (6.2.29)$$

The solution is then given by the metric (6.2.1) and dilaton (6.2.5) with (6.2.26) and the H -flux on T^3 given by

$$H = *_4 dV^{-1} = -M dx \wedge dy \wedge dz. \quad (6.2.30)$$

Taking the product of the solution (6.2.22), (6.2.30) with 6-dimensional Minkowski space $\mathbb{R}^{1,5}$ gives a space $\mathbb{R} \times T^3 \times \mathbb{R}^{1,5}$ with NS5-branes smeared over the T^3 inserted at $\tau = \tau_i$. The transverse space for the NS5-branes is $\mathbb{R} \times T^3$.

6.2.2. Nilfold background

The background (6.2.22) has isometries in the x, y and z directions. Performing T-duality in the y -direction gives a background that is $\mathbb{R} \times \mathcal{N}$, with a metric dependent on the coordinate τ of \mathbb{R} , so that it is nilfold fibred over the real line. The metric is

$$ds^2 = V(\tau)(d\tau^2 + dx^2 + dz^2) + \frac{1}{V(\tau)}(dy + M(\tau)xdz)^2 \quad (6.2.31)$$

and the other fields are trivial

$$H = 0, \quad \Phi = \text{constant} \quad (6.2.32)$$

This 4-dimensional metric can be viewed as a Gibbons-Hawking metric with a harmonic function V depending linearly on a single coordinate. It preserves half the supersymmetry and so is hyperkähler. The three complex structures are given by

$$J^1 = d\tau \wedge (dy + M(\tau)xdz) + V(\tau)dx \wedge dz \quad (6.2.33)$$

$$J^2 = dx \wedge (dy + M(\tau)xdz) + V(\tau)d\tau \wedge dz \quad (6.2.34)$$

$$J^3 = dz \wedge (dy + M(\tau)xdz) + V(\tau)d\tau \wedge dx. \quad (6.2.35)$$

For V of the form (6.2.26) it represents a multi-domain wall solution and we will refer to this as the hyperkähler wall solution $\hat{\mathcal{N}}$. In the region between walls $\tau_i < \tau < \tau_{i+1}$ or for $0 < \tau < \tau_1$ or $\tau_n < \tau < \pi$ it has the topology $I \times \mathcal{N}$ where I is a line interval. The dilaton and antisymmetric tensor gauge fields are all constant but the metric depends on the coordinate τ . The space is singular at $\tau = 0, \tau_i, \pi$ and we will discuss the resolution of these singularities in later sections.

For fixed τ , the geometry is a nilfold, which can be viewed as a circle bundle over a 2-torus, where the circle fibre has coordinate y and the torus base has coordinates x, z . The geometry is warped by the factor of V : the circumference of the circle fibre is $1/V$ while the circumference of each of the circles of the 2-torus is V .

Taking the product of this with $\mathbb{R}^{1,5}$ gives a space which can be viewed as a background with Kaluza-Klein monopoles. The usual Kaluza-Klein monopole is given by the product of self-dual Taub-NUT space with $\mathbb{R}^{1,5}$ and has a transverse space which is \mathbb{R}^3 with a point removed, and the Taub-NUT space is a circle bundle over this space. We shall be interested later in the Kaluza-Klein monopole whose transverse space which is $\mathbb{R} \times T^2$ with a point removed, so that the Gibbons-Hawking circle bundle over the transverse space is $\mathbb{R} \times \mathcal{N}$ with a circle removed. Here, we are obtaining a version of this smeared over the coordinates of the transverse T^2 , and the geometry can be thought of as $\mathbb{R} \times \mathcal{N} \times \mathbb{R}^{1,5}$ with a smeared Kaluza-Klein monopole at each τ_i .

6.2.3. T-fold and R-fold backgrounds

Performing T-duality along the z -direction results in the T-fold \mathcal{T} fibred over a line. The metric and B -field of this background are given by

$$ds^2 = V(\tau)(d\tau)^2 + V(\tau)(dx)^2 + \frac{V(\tau)}{V(\tau)^2 + (M(\tau)x)^2}(dy^2 + dz^2), \quad (6.2.36)$$

$$B = \frac{M(\tau)x}{V(\tau)^2 + (M(\tau)x)^2} dy \wedge dz. \quad (6.2.37)$$

while the dilaton is

$$\Phi = \frac{1}{2} \log \left(\frac{V(\tau)}{V(\tau)^2 + (M(\tau)x)^2} \right) \quad (6.2.38)$$

For fixed τ we obtain the T-fold (6.1.5). In the region between walls $\tau_i < \tau < \tau_{i+1}$ the space is the product of the interval I with the T-fold (6.1.5) with the fields depending on τ through the warp by factor V .

Finally, a further T-duality in the x direction gives something which is not locally geometric but which has a well-defined doubled geometry which is given by the doubled configurations of [63, 64] fibred over a line. This will be discussed further elsewhere.

6.2.4. Exotic Branes and T-folds

T-dualising the NS5-brane on a transverse circle gives a Kaluza-Klein monopole [32]. The transverse space of the NS5 brane is taken to be $S^1 \times \mathbb{R}^3$ and the harmonic function determining the solution is taken to be independent of the circle coordinate, so that the NS5-brane is smeared over the transverse circle. Then both the KK-monopole and the NS5 solutions are given by a harmonic function on \mathbb{R}^3 .

If the NS5-brane solution is smeared over T^2 , then a further T-duality is possible. The transverse space of the NS5 brane is now taken to be $T^2 \times \mathbb{R}^2$ and the solution is given by a harmonic function on \mathbb{R}^2 . T-dualising on one circle gives a KK-monopole smeared over a circle, with transverse space $\mathbb{R}^2 \times S^1$ and the solution given by a harmonic function on \mathbb{R}^2 . The harmonic function in \mathbb{R}^2 leads to a monodromy round each source. A further T-duality on a transverse circle gives what has been termed an exotic brane, referred to in [60] as a 5_2^2 -brane and in [108] as a $(5, 2^2)$ -brane. This was interpreted in [70, 71] as a T-fold, with a T-duality monodromy round each source in the transverse \mathbb{R}^2 .

Here we are interested in an NS5-brane smeared over T^3 , so that the transverse space is $T^3 \times \mathbb{R}$ and the harmonic function is a linear function on \mathbb{R} . Then the first T-duality gives a KK-monopole smeared over T^2 with transverse space $T^2 \times \mathbb{R}$. The second T-duality gives an exotic 5_2^2 -brane or $(5, 2^2)$ -brane smeared over S^1 with transverse space $S^1 \times \mathbb{R}$. A third T-duality gives an exotic brane referred to in [108] as a $(5, 3^2)$ -brane. In the notation of [60], this would be a 5_3^3 -brane. This is the solution with R -flux.

The background (6.2.22), (6.2.24), (6.2.30) was interpreted as $\mathbb{R} \times T^3 \times \mathbb{R}^{1,5}$ with smeared NS5-branes inserted at $\tau = \tau_i$ and the geometry (6.2.31), (6.2.32) was thought of as $\mathbb{R} \times \mathcal{N} \times \mathbb{R}^{1,5}$ with a smeared Kaluza-Klein monopole at each τ_i . In the same spirit, the T-fold solution (6.2.36), (6.2.37), (6.2.38) can be thought of as the T-fold background $\mathbb{R} \times \mathcal{T} \times \mathbb{R}^{1,5}$ with a smeared exotic brane at each τ_i .

For the T-duality of the NS5-brane on a transverse circle, it is not necessary to assume that the NS5-brane is smeared over the circle. Instead, one can take an NS5-brane localized on $S^1 \times \mathbb{R}^3$. This can be constructed by taking a periodic array of NS5-branes on \mathbb{R}^4 located at points arranged on a line $(0, 0, 0, 2\pi Rm)$ for integers $m = 0, \pm 1, \pm 2, \dots$ and then identifying $x^4 \sim x^4 + 2\pi R$. The resulting harmonic function defining the NS5-brane solution depends explicitly on x^4 . The T-duality of this solution has been discussed in [109]; see also [107, 110–112]. The T-dual solution depends explicitly on the coordinate \tilde{x}^4 of the T-dual circle. This then gives a modification of the Kaluza-Klein monopole geometry with explicit dependence on the dual coordinate.

6.2.5. Single-sided domain walls and the Tian-Yau space

Consider a domain wall of the kind discussed in the previous subsections, with a profile given by a function $V(\tau)$ of the transverse coordinate.

$$V = c + m|\tau| \quad (6.2.39)$$

gives a domain wall at $\tau = 0$ of charge m that is invariant under the reflection

$$\tau \rightarrow -\tau \quad (6.2.40)$$

Orbifolding by this reflection identifies the half-line $\tau < 0$ with the half-line $\tau > 0$ resulting in a single-sided domain wall solution defined for $\tau \geq 0$, with a singular wall at $\tau = 0$ [113]. For the applications to string theory in the next section, we will be interested in the case in which the τ direction is a line interval with a single-sided domain wall at either end.

For the case of the nilfold fibred over a line with metric (6.2.31) with (6.2.39), this orbifold singularity has a remarkable resolution to give a smooth manifold, as was proposed in [113]. The Tian-Yau space [87] is a smooth four-dimensional hyperkähler manifold fibred over the half-line $\tau > 0$ such that for large τ it approaches the metric (6.2.31) for a nilfold fibred over a line, so that it can be regarded as a resolution of the single sided brane [86].

The Tian-Yau space [87] is a complete non-singular non-compact hyperkähler space that is asymptotic to a nilfold bundle over a line. The Tian-Yau space is of the form $M \setminus D$, where M is a del Pezzo surface and $D \subset M$ is a smooth anti-canonical curve. The del Pezzo surfaces are complex surfaces classified by their degree b , where $b = 1, 2, \dots, 9$. The del Pezzo surface of degree nine is a complex projective plane \mathbb{CP}^2 . A del Pezzo surface M_b of degree b can be constructed by blowing up a point in the del Pezzo surface of degree $b + 1$, M_{b+1} . That is, the degree b del Pezzo surface can be constructed from blowing up $9 - b$ points in \mathbb{CP}^2 , and there are restrictions on the positions of the points that can be blown up. There are two types of del Pezzo surface of degree eight, $\mathbb{CP}^1 \times \mathbb{CP}^1$ and the result of blowing up one point in \mathbb{CP}^2 .

The Tian-Yau space $X_b = M_b \setminus D$ is a non-compact space that is asymptotic to $\mathbb{R} \times \mathcal{N}_b$, where \mathcal{N}_b is a nilfold of degree $m = b$. In the asymptotic region, the Tian-Yau metric can be approximated by the metric (6.2.31) where $V(\tau)$ is a non zero linear function $V = c + m\tau$ so that $V(\tau) \rightarrow \infty$ as $\tau \rightarrow \infty$. The degree m of the nilfold is given by the degree b of the del Pezzo surface, so only degrees $m = 1, 2, \dots, 9$ can arise.

Note that starting from a del Pezzo surface of degree nine and blowing up nine points gives a rational elliptic surface M [114]. This space has the structure of elliptic fibration, $f : M \rightarrow \mathbb{CP}^1$ with the fiber being elliptic curve or 2-torus, D . For zero degree, the nilfold reduces to a 3-torus and the space $M \setminus D$ is a non compact space that is asymptotic to a cylinder $T^3 \times I$. It is then an ALH gravitational instanton. It will be convenient to refer to this case as a Tian-Yau space

of zero degree, so that we can take extend the range of the degree to $b = 0, 1, 2, \dots, 9$.

The $m \neq 0$ case gives a generalisation of the ALH case which is asymptotic to the product of a nilfold of degree m with an interval. Changing variables to $s = (m\tau)^{3/2}$, the metric takes the following asymptotic form for large s :

$$g \sim \frac{4}{9m^2} ds^2 + s^{2/3}(dx^2 + dz^2) + s^{-2/3}(dy + bxdz)^2 \quad (6.2.41)$$

Then for large s , the size of the S^1 fibres falls off as $s^{-1/3}$ while the size of each 1-cycle of the 2-torus base grows as $s^{1/3}$.

6.3. Embedding in string theory

6.3.1. The D8-brane and its duals

The D8-brane solution of the IIA string [98] has string-frame metric

$$ds^2 = V^{-1/2} ds^2(\mathbb{R}^{1,8}) + V^{1/2} d\tau^2 \quad (6.3.1)$$

with dilaton

$$e^\Phi = V(\tau)^{-\frac{5}{4}} \quad (6.3.2)$$

and RR field strength

$$\begin{aligned} F_{(10)} &= dt \wedge dx_1 \wedge \dots \wedge dx_8 \wedge d(V(\tau)^{-1}), \\ &= -\frac{M(\tau)}{V^2(\tau)} dt \wedge dx_1 \wedge \dots \wedge dx_8 \wedge d\tau, \\ &= -M(\tau) \Omega_{\text{Vol}}. \end{aligned} \quad (6.3.3)$$

where

$$\Omega_{\text{Vol}} = \sqrt{-g} dt \wedge dx_1 \wedge \dots \wedge dx_8 \wedge d\tau \quad (6.3.4)$$

is the volume form. The Hodge dual of $F_{(10)}$ is a zero-form,

$$F_{(0)} = -M(\tau). \quad (6.3.5)$$

This zero-form $F_{(0)}$ gives the mass parameter in the massive type IIA supergravity [103]. For our solution it is piecewise constant as in (6.2.29), so that the Romans mass parameter is different on either side of a domain wall. There is an 8+1 dimensional longitudinal space and a one-dimensional transverse space with coordinate τ . Here $V(\tau)$ is piecewise linear. Taking it to be of the form (6.2.25) gives a D8-brane of charge $m - m'$ at $\tau = 0$ while the multi-brane

solution (6.2.26) represents multi-D8-branes at positions τ_1, \dots, τ_n .

If 3 of the transverse dimensions are compactified to a 3-torus with coordinates x, y, z , the metric can be written

$$ds^2 = V^{-1/2}[dx^2 + dy^2 + dz^2 + ds^2(\mathbb{R}^{1,5})] + V^{1/2}d\tau^2 \quad (6.3.6)$$

T-dualising in the x, y, z directions gives a D5-brane IIB solution smeared over T^3

$$ds^2 = V^{-1/2}ds^2(\mathbb{R}^{1,5}) + V^{1/2}(d\tau^2 + dx^2 + dy^2 + dz^2) \quad (6.3.7)$$

$$F_{(7)} = dt \wedge dx_1 \wedge \dots \wedge dx_5 \wedge d(V(\tau)^{-1}) \quad (6.3.8)$$

$$e^{2\Phi} = V(\tau)^{-1} \quad (6.3.9)$$

where V is of the form (6.2.25), or (6.2.26). The Hodge dual of $F_{(7)}$ is the RR field strength three-form, which is given by

$$F_{(3)} = *F_{(7)} = -M(\tau)dx \wedge dy \wedge dz. \quad (6.3.10)$$

Next, S-duality gives a smeared NS5-brane solution, which is precisely the solution (6.2.22), (6.2.24), and (6.2.30) given in section 3, with a transverse space given by a T^3 bundle over a line.

6.3.2. The type I' string

The multi-D8 brane solution has a dilaton depending on the transverse coordinate in such a way that the dilaton becomes large and hence the string becomes strongly coupled in certain regions. A well-behaved solution of string theory with D8-branes arises in the type I' string [85], which arises from compactifying the type I string on a circle and T-dualising. This can be viewed as an orientifold of the type IIA string compactified on the dual circle, resulting in a theory on S^1/\mathbb{Z}_2 with O8 orientifold planes introduced at the fixed points, and with 16 D8-branes (together with their mirror images under the action of \mathbb{Z}_2) located at arbitrary locations. It can also be viewed as a theory on the interval I arising from the quotient S^1/\mathbb{Z}_2 with O8-planes at the end points, and with 16 D8-branes located at arbitrary points on the interval.

The supergravity solution corresponding to the type I' string with 16 D8-branes has string frame metric

$$ds^2 = V^{-1/2}ds^2(\mathbb{R}^{1,8}) + V^{1/2}d\tau^2, \quad (6.3.11)$$

where $V(\tau)$ is a harmonic function on the interval I with coordinate τ , $0 \leq \tau \leq \pi$. The dilaton

is given by

$$e^\Phi = V(\tau)^{-\frac{5}{4}}. \quad (6.3.12)$$

and the RR field strength is given by

$$F_{(0)} = -M(\tau). \quad (6.3.13)$$

where $M(\tau) \equiv V'(\tau)$.

The orientifold planes are located at $\tau = 0$ and $\tau = \pi$, while the 16 D8-branes are located at arbitrary points, τ_1, \dots, τ_{16} between $\tau = 0$ and $\tau = \pi$. The function $V(\tau)$ is piecewise linear and for general positions of the D8-branes, it is given by (6.2.29) with $n = 16$, $m_1 = -8$, $m_{i+1} = m_i + 1$ so that $m_i = i - 9$ and $m_{17} = 8$, so that the orientifold planes are treated as sources of charge -8 at the end-points of I . The gradient of V jumps by $+1$ at each D8-brane. Then

$$V(\tau) = \begin{cases} c_1 - 8\tau, & \tau \leq \tau_1 \\ c_2 - 7\tau, & \tau_1 < \tau \leq \tau_2 \\ \vdots & \\ c_i + (i - 9)\tau, & \tau_{i-1} < \tau \leq \tau_i \\ \vdots & \\ c_{16} + 7\tau & \tau_{15} < \tau \leq \tau_{16} \\ c_{17} + 8\tau & \tau > \tau_{16} \end{cases} \quad (6.3.14)$$

with

$$c_{r+1} = c_r - \tau_r \quad (6.3.15)$$

The slope of the function V is $M(\tau) \equiv V'(\tau)$ and is given by

$$M(\tau) = \begin{cases} -8, & \tau < \tau_1 \\ -7, & \tau_1 < \tau < \tau_2 \\ \vdots & \\ 7 & \tau_{15} < \tau \leq \tau_{16} \\ 8 & \tau > \tau_{16}. \end{cases} \quad (6.3.16)$$

For r coincident branes with $\tau_i = \tau_{i+1} = \dots = \tau_{i+r-1}$ the slope jumps from m_i for $\tau_{i-1} < \tau < \tau_i$ to $m_i + r$ for $\tau_i < \tau < \tau_{i+1}$. In general this leads to domain walls at $\tau = \tau_1, \tau_2, \dots, \tau_n$ with positive charges N_1, N_2, \dots, N_n corresponding to $N - i$ D8-branes at τ_i . If there are N_0 D8-branes coincident with the O8-plane at $\tau = 0$ and N_{n+1} D8-branes coincident with the

O8-plane at $\tau = \pi$ then

$$\sum_{i=0}^{n+1} N_i = 16 \quad (6.3.17)$$

Then V is given by (6.2.26) with $N_r = m_{r+1} - m_r$ and

$$m_1 = -(8 - N_0), \quad m_{n+1} = 8 - N_{n+1} \quad (6.3.18)$$

while M is (6.2.29).

At strong coupling, new effects can arise. Each orientifold plane can emit a further D8-brane, leading to a non-perturbative enhancement of the gauge symmetry [95].

6.4. Duals of the type I' string

The type I string can be obtained as an orientifold of the type IIB string

$$\text{I} = \frac{\text{IIB}}{\Omega} \quad (6.4.1)$$

where Ω is the world-sheet parity operator. This has 16 D9-branes to cancel the charge of the O9-plane. Compactifying on a circle in the X^9 direction and T-dualising gives the type I' string. This is now a quotient of the IIA string [85]

$$\text{I}' = \frac{\text{IIA}}{\Omega R_9} \quad (6.4.2)$$

where R_9 is reflection in X^9 . This results from the fact that the T-duality T_i in the X^i direction acts as [115]

$$T_i : \Omega \rightarrow \Omega R_i \quad (6.4.3)$$

The periodic coordinate $X^9 \sim X^9 + 2\pi$ is identified under the action of the reflection $I_9 : X^9 \rightarrow -X^9$ so that after the quotient, the X^9 circle becomes S^1/\mathbb{Z}_2 . An orientifold O8-plane is introduced at each of the fixed points $X^9 = 0$ and $X^9 = \pi$. These each have charge -8 (in units in which a single D8-brane has charge $+1$) which is cancelled by the charge of the 16 D8-branes arising from the T-dual of the 16 D9-branes of the type I string. The space S^1/\mathbb{Z}_2 can be viewed as the interval $0 \leq X^9 \leq \pi$ with an orientifold plane at either end of the interval.

Next compactifying on X^8 and T-dualising should give, from (6.4.2), an orientifold of the IIB string by ΩR_{89} where $R_{89} = R_8 R_9$. However, this leads to a problem, as this orientifold is not supersymmetric, but T-duality is expected to preserve supersymmetry. The resolution of this [116] results from the fact that the T-dualities T_8 and T_9 do not commute in the superstring.

The T-duality T_i in the X^i direction acts as a reflection on the left-moving bosonic world-sheet fields:

$$(X_L^i, X_R^i) \rightarrow (-X_L^i, X_R^i) \quad (6.4.4)$$

On the left-moving spin-fields S_L , this reflection acts through $S_L \rightarrow t_i S_L$ where $t_i = \Gamma^{11} \Gamma_i$. As t_i and t_j anticommute for $i \neq j$, it follows that $T_i T_j \neq T_j T_i$ when acting on fermions, the result of two T-dualities is only determined up to a factor of $(-1)^{F_L}$. Thus there is an ambiguity in how one takes two T-duals of the type I string: one gives IIB/ ΩR_{89} which is not supersymmetric, and one gives IIB/ $\Omega R_{89}(-1)^{F_L}$ which is supersymmetric. Here and in each case that follows, we will define T-duality to be the transformation that preserves supersymmetry. See [93, 116] for further discussion.

In this way, we obtain the standard chain of supersymmetric orientifolds by successive T-dualities for the type I string compactified on the 4-torus in the X^6, X^7, X^8, X^9 directions:

$$I = \frac{\text{IIB}}{\Omega} \xrightarrow{T_9} \frac{\text{IIA}}{\Omega R_9} \xrightarrow{T_8} \frac{\text{IIB}}{\Omega R_{89}(-1)^{F_L}} \xrightarrow{T_7} \frac{\text{IIA}}{\Omega R_{789}(-1)^{F_L}} \xrightarrow{T_6} \frac{\text{IIB}}{\Omega R_{6789}} \quad (6.4.5)$$

Here $R_{ij\dots k}$ denotes a reflection in the directions $X^i, X^j \dots X^k$. After $p \leq 4$ T-dualities, a p -torus T^p is identified under reflections so that the T^4 becomes $T^{4-p} \times T^p / \mathbb{Z}_2$ where $T^p / \mathbb{Z}_2 = T^p / R_{i_1 \dots i_p}$ identified under the reflection $R_{i_1 \dots i_p}$. This has 2^p fixed points with an $O(9-p)$ -plane at each fixed point of charge $-16/2^p$, which is cancelled by 16 D $(9-p)$ -branes.

The final case IIB/ ΩR_{6789} is an orientifold of type IIB compactified on T^4 / \mathbb{Z}_2 , which has 16 O5-planes at the 16 fixed points, each of charge -1 , together with 16 D5-branes to cancel the charge. Acting with S-duality takes [89]

$$S : \Omega \rightarrow (-1)^{F_L} \quad (6.4.6)$$

giving the orbifold IIB/ $(-1)^{F_L} R_{6789}$. The 16 D5-branes become 16 NS5-branes and the 16 O5-planes become 16 ON-planes [88, 89].

Next, acting with a T-duality in the X^6 direction gives the supersymmetric orbifold IIA/ R_{6789} giving the IIA string compactified on the $K3$ orbifold T^4 / \mathbb{Z}_2 .

$$\frac{\text{IIB}}{\Omega R_{6789}} \xrightarrow{S} \frac{\text{IIB}}{(-1)^{F_L} R_{6789}} \xrightarrow{T_6} \frac{\text{IIA}}{R_{6789}} \quad (6.4.7)$$

Combining (6.4.5) and (6.4.7) we have a duality between the type I string compactified on T^4 and the type IIA string compactified on the $K3$ orbifold T^4 / \mathbb{Z}_2 . As the type I string is dual to the heterotic string, this gives the duality between the heterotic string on T^4 and the type IIA string on $K3$ [32]. Each orbifold singularity of T^4 / \mathbb{Z}_2 can be resolved by gluing in an Eguchi-Hansen space, and the 16 NS5-branes of the IIB/ $(-1)^{F_L} R_{6789}$ theory can be thought of

as corresponding to the ALE spaces glued in to resolve the singularities.

6.4.1. Moduli spaces and dualities

In the orientifolds considered in the previous subsection, the total charge cancels between branes and orientifold planes and in general the dilaton is non-constant. However, in each case there is a particular configuration where the charge cancels locally and the dilaton is constant. In the type I' theory there are two orientifold planes of charge -8 and 16 D8-branes of charge $+1$, so that if there are 8 D8-branes at each orientifold plane, the charge at each plane is zero and the dilaton is constant. This configuration has an $SO(16) \times SO(16)$ gauge symmetry perturbatively, enhanced to $E_8 \times E_8$ in the non-perturbative theory [117, 118]. In this case, the function V is a constant, $V = c_1$, and the length of the interval with respect to the metric (6.3.1) is $L = \pi(c_1)^{1/4}$.

For the theory obtained from the type I string on T^p ($p \leq 4$) by performing a T-duality on each of the p circles, the charge-cancelling configuration has $16/2^p$ coincident $D(9-p)$ -branes at each of the 2^p $O(9-p)$ orientifold planes. For $p = 4$, there is a single D5-branes at each of the 16 fixed points, and S-dualising gives a single NS5-brane coincident with each of the 16 ON-planes.

The general configuration for each case is obtained by moving in the corresponding moduli space. The type I' string is defined on $I \times \mathbb{R}^{1,8}$ where I is the interval $0 \leq \tau \leq \pi$ with coordinate τ , with 16 D8-branes at positions τ_1, \dots, τ_{16} and O8-planes at the end-points $\tau = 0, \pi$. It has a 17-dimensional moduli space

$$O(1, 17; \mathbb{Z}) \backslash O(1, 17) / O(17) \quad (6.4.8)$$

which is the coset space $O(1, 17)/O(17)$ identified under the action of the discrete duality group $O(1, 17; \mathbb{Z})$. (Here and in each of the following cases, there is a further factor of \mathbb{R} corresponding to the dilaton; this factor will not be discussed explicitly.) The 17 moduli consist of the 16 D8-brane positions τ_1, \dots, τ_{16} and the length of the interval L . The duality group $O(1, 17; \mathbb{Z})$ (corresponding to the T-duality symmetry of the heterotic string compactified on S^1) acts on all 17 moduli.

For the theory obtained from the type I string on T^p ($p \leq 4$) by performing a T-duality on each of the p circles, the moduli space is

$$O(p, 16 + p; \mathbb{Z}) \backslash O(p, 16 + p) / O(p) \times O(16 + p) \quad (6.4.9)$$

The moduli consist of the $16p$ parameters determining the positions of the $D(9-p)$ -branes on T^p/\mathbb{Z}_2 and the p^2 moduli of constant metrics and RR 2-form gauge-fields on T^p/\mathbb{Z}_2 . The

moduli space for metrics and 2-form gauge-fields on T^p is

$$O(p, p; \mathbb{Z}) \backslash O(p, p) / O(p) \times O(p) \quad (6.4.10)$$

For $p = 4$, we obtain [119]

$$O(4, 20; \mathbb{Z}) \backslash O(4, 20) / O(4) \times O(20) \quad (6.4.11)$$

After S-duality, this becomes the 80-dimensional moduli space consisting of 4×16 positions of NS5-branes and 16 moduli for metrics and NS-NS 2-form gauge fields on T^4/\mathbb{Z}_2 . An $O(4, 4; \mathbb{Z})$ subgroup of $O(4, 20; \mathbb{Z})$ acts as T-duality on T^4 , while the remaining transformations mix the NS5-brane positions with torus moduli.

For $K3$, the moduli space is again (6.4.11) identified under $O(4, 20; \mathbb{Z})$, which is the automorphism group of $K3$ CFTs and contains large diffeomorphisms and mirror transformations. Moving in the moduli space away from the orbifold point blows up the singularities, and generic points in the moduli space correspond to smooth $K3$ manifolds.

6.5. Degenerations

We will start with the type I' string compactified on T^3 . With the charge-cancelling configuration of 8 D8-branes coincident with each O8-plane, the geometry is $I \times T^3 \times \mathbb{R}^{1,5}$ (where I is the interval $[0, \pi]$) with constant dilaton. For general positions of the D8-branes, the corresponding supergravity solution is (6.3.1), (6.3.2), (6.3.3) with x, y, z periodically identified.

Performing T-dualities in the x, y, z directions takes each D8-brane to a D5-brane smeared over the T^3 . In other words, instead of getting a D5-brane localized at a point on the 4-dimensional transverse space with a harmonic function $V(\tau, x, y, z)$, we get a harmonic function $V(\tau)$ depending only on τ . Applying the standard Buscher T-duality rules takes the solution (6.3.1), (6.3.2), (6.3.3) to the solution (6.3.7), (6.3.8), (6.3.9). This suggests that the O8-planes could behave like negative tension D8-branes under T-duality, transforming to O5-planes smeared over the T^3 . However, this picture is too naive, as we now discuss. Under T-duality, the transverse space $T^3 \times S^1/\mathbb{Z}_2$ transforms not to the product of the dual T^3 with S^1/\mathbb{Z}_2 but to T^4/\mathbb{Z}_2 .

The quotient S^1/\mathbb{Z}_2 is the circle with coordinate $\tau \sim \tau + 2\pi$ identified under the action of the \mathbb{Z}_2 acting as a reflection $\tau \rightarrow -\tau$, so that $\tau = 0$ and $\tau = \pi$ are fixed points. It can be represented by the line interval I with $0 \leq \tau \leq \pi$.

The orbifold T^4/\mathbb{Z}_2 can be realized similarly. First, T^4 has 4 periodic coordinates $x^\mu = (\tau, x, y, z)$ each identified with $x^\mu \sim x^\mu + 2\pi$. Then this is identified under the reflection acting as $x^\mu \rightarrow -x^\mu$. It can be viewed as a quotient of $I \times T^3$ where I is the interval with $0 \leq \tau \leq \pi$

and the T^3 has periodic coordinates x, y, z each with period 2π . There is then a further quotient at the end points of I , so that $I \times T^3$ is identified under the \mathbb{Z}_2 acting as

$$(0, x, y, z) \rightarrow (0, -x, -y, -z), \quad (\pi, x, y, z) \rightarrow (\pi, -x, -y, -z) \quad (6.5.1)$$

Thus we have T^3 ‘fibred’ over I with the fibre a T^3 for generic points τ with $0 < \tau < \pi$, but at the end points $\tau = 0, \pi$, the fibre becomes T^3/\mathbb{Z}_2 with 8 fixed points on each T^3/\mathbb{Z}_2 . If we take the length of the interval L to be large, then the naive supergravity solution can be a good approximation a long way away from the end points, but will need to be modified near $\tau = 0, \pi$.

Dualising the D8-brane configuration then gives a configuration that, for $|\tau - \pi/2| \ll \pi/2$, is well approximated by the supergravity solution (6.2.22), (6.2.24), (6.2.30) consisting of the three-torus with flux fibred over a line, but this will need modification near the end points $\tau = 0, \pi$. This then is a space with a long neck of the form $T^3 \times \mathbb{R}$ capped at the two ends. A further T-duality takes the fibres from a 3-torus with flux to a nilfold \mathcal{N} , giving the solution (6.2.31), (6.2.32). This then implies that $K3$ should have a limit in which it degenerates to a long neck of the form $\mathcal{N} \times \mathbb{R}$ capped off by suitable smooth geometries. Remarkably, such a limit of $K3$ has recently been found [86], with an explicit understanding of the geometries needed to cap off the neck and to resolve the domain wall singularities, as we review in the next subsection.

6.6. A degeneration of $K3$.

In [86], a family of hyperkähler metrics on $K3$ was constructed labelled by a parameter β in which the limit $\beta \rightarrow \infty$ gives a boundary of the $K3$ moduli space in which the $K3$ collapses to the one-dimensional line segment $[0, \pi]$. For large β , the metric is given to a good approximation at generic points by the multi-domain wall metric (6.2.31). The domain wall solution $\hat{\mathcal{N}}$ (6.2.31) has singularities at the end points $0, \pi$ where there are single-sided domain walls and at the domain wall locations τ_i , but the $K3$ metric of [86] smoothly resolves these singularities to give a smooth geometry. The geometry is obtained by gluing together a number of hyperkähler spaces, and these then give approximate metrics for different regions of $K3$. There is a long neck consisting of a number of segments, with each segment a product of a nilfold with a line interval, with metric of the form (6.2.31) with $V = c + m\tau$. The degree m of the nilfold jumps between segments. The domain wall connecting two segments is realized as a smooth Gibbons Hawking space corresponding to multiple Kaluza-Klein monopoles. At either end the geometry is capped with a Tian-Yau space, resolving the single-sided domain wall geometry, as discussed in section 6.2.5.

For large $\beta \gg 1$, there exists a continuous surjective map from $K3$ to the interval I

$$F_\beta : K3 \rightarrow [0, \pi]. \quad (6.6.1)$$

and a discrete set of points $\mathcal{S} = \{0, \tau_1, \dots, \tau_n, \pi\} \subset [0, \pi]$. It will be convenient to let $\tau_0 = 0$ and $\tau_{n+1} = \pi$. Let \mathcal{R}_ϵ^i be the interval

$$\mathcal{R}_\epsilon^i = \{\tau : \tau_{i-1} + \epsilon < \tau < \tau_i - \epsilon\} = (\tau_{i-1} + \epsilon, \tau_i - \epsilon) \quad (6.6.2)$$

for some small ϵ . Then for each $i = 1, \dots, n+1$, the region of $K3$ $F_\beta^{-1}(\mathcal{R}_\epsilon^i)$ projecting to \mathcal{R}_ϵ^i is diffeomorphic to the product of the interval \mathcal{R}_ϵ^i with a nilfold of degree m_i for some m_i . The metric is approximately given by the hyperkähler metric (6.2.31) with $V = c_i + m_i \tau$. The degree jumps at τ_i by

$$N_i = m_{i+1} - m_i \quad (6.6.3)$$

The degree of the nilfold fibres is piecewise constant:

$$M(\tau) = \begin{cases} m_1, & \tau < \tau_1 - \epsilon \\ m_2, & \tau_1 + \epsilon < \tau < \tau_2 - \epsilon \\ \vdots & \\ m_n, & \tau_{n-1} + \epsilon < \tau < \tau_n - \epsilon \\ m_{n+1}, & \tau > \tau_n + \epsilon. \end{cases} \quad (6.6.4)$$

and jumps across the domain walls at $\tau = \tau_1, \tau_2, \dots, \tau_n$.

For the end regions $F_\beta^{-1}(\mathcal{S}_\epsilon^-)$, $F_\beta^{-1}(\mathcal{S}_\epsilon^+)$ projecting to

$$\mathcal{S}_\epsilon^- = [0, \epsilon) \quad \mathcal{S}_\epsilon^+ = (\pi - \epsilon, \pi] \quad (6.6.5)$$

the singularities of the single-sided domain walls at the end points are resolved as in section 6.2.5 by Tian-Yau spaces. The region $F_\beta^{-1}(\mathcal{S}_\epsilon^+)$ is approximately a Tian-Yau space X_{b_+} of degree b_+ and the region $F_\beta^{-1}(\mathcal{S}_\epsilon^-)$ is a Tian-Yau space X_{b_-} of negative degree $-b_-$, where b_\pm are some integers $0 \leq b_\pm \leq 9$. Asymptotically, these give the product of a line with nilfolds of degree $-b_-$, b_+ and so to match with the solutions projecting to \mathcal{R}_ϵ^1 , $\mathcal{R}_\epsilon^{n+1}$ we must take $m_1 = -b_-$ and $m_{n+1} = b_+$. The case of zero degree gives an ALH Tian-Yau space X_0 that is asymptotically cylindrical, with fibres given by T^3 . Then the sum of the charges at the domain walls is

$$\sum_{i=1}^n N_i = b_- + b_+ \quad (6.6.6)$$

and so is an integer

$$0 \leq \sum_{i=1}^n N_i \leq 18 \quad (6.6.7)$$

The number n of domain walls then satisfies $0 \leq n \leq 18$ if all domain wall charges N_i are positive.

Consider now the interval

$$\mathcal{S}_\epsilon^i = (\tau_i - \epsilon, \tau_i + \epsilon) \quad (6.6.8)$$

In [86], the geometry in the region $F_\beta^{-1}(\mathcal{S}_\epsilon^i)$ is taken to be a Gibbons-Hawking metric (6.2.8) specified by a harmonic function $V(x, z, \tau)$ on $T^2 \times \mathcal{S}_\epsilon^i$ with N_i sources. This gives a hyperkähler space which is an S^1 fibration over the space given by removing the N_i singular points from $T^2 \times \mathcal{S}_\epsilon^i$. This can be constructed from a Gibbons-Hawking space on $\mathbb{R}^2 \times \mathcal{S}_\epsilon^i$ with a doubly periodic array of sources by taking the quotient by a lattice to obtain N_i sources on $T^2 \times \mathcal{S}_\epsilon^i$.

For the case $b_- = b_+ = 0$, both X_{b_+} and X_{b_-} are asymptotically cylindrical ALH spaces. A $K3$ surface can be constructed by glueing the two cylindrical ends of two ALH space together [120]. This means there are no domain walls, so that $n = 0$ for this case. This $K3$ surface is dual to the locally charge-cancelling type I' configuration, in which one end of the interval at $\tau = 0$ there is an O8-plane and 8 D8-branes, while at the other end, $\tau = \pi$, there is also an O8-plane and 8 D8-branes, so that the RR field strength $F_{(0)}$ is zero.

In the general case we have a geometry capped by two spaces X_{b_-} and X_{b_+} of degrees b_- , b_+ which are integers with $0 \leq b_\pm \leq 9$. For $b_\pm > 0$ these are Tian-Yau spaces asymptotic to the product of a nilfold of degree b_\pm and a line interval, while for $b_\pm = 0$ these are ALH spaces asymptotic to the product of a 3-torus and a line interval. These are joined by a neck region which can be thought of as a Gibbons-Hawking space on the product of a line interval and a nilfold with $b_- + b_+$ Kaluza-Klein monopoles inserted. For N_i Kaluza-Klein monopoles inserted at $\tau = \tau_i$, the degree of the nilfold jumps from m_i for $\tau < \tau_i$ to $m_i + N_i$ for $\tau > \tau_i$. The smooth $K3$ geometry is constructed by gluing together the Tian-Yau spaces, the product of the nilfold with a line interval and the Gibbons-Hawking spaces as shown in [86], and these various hyperkähler metrics provide good approximate metrics for the corresponding regions of the $K3$.

The form of the solution away from the domain walls is (6.2.31) with V given by (6.2.26) with

$$m_1 = -b_-, \quad m_{n+1} = b_+, \quad N_i = m_{i+1} - m_i \quad (6.6.9)$$

and the charges N_i satisfy (6.6.6).

The geometry is smooth if all the of the Kaluza-Klein monopoles are at distinct locations, so that the geometry is approximately that of self-dual Taub-NUT near each monopole. If k of the Kaluza-Klein monopoles are coincident, the $K3$ surface has an A_{k-1} orbifold singularity

and so there is a resulting A_{k-1} gauge symmetry. If $b_- = b_+ = 9$, there are 18 Kaluza-Klein monopoles and if these are coincident, there is a resulting $SU(18)$ gauge symmetry.

In the next section, the duals of the IIA string compactified on this degenerate $K3$ will be considered.

6.7. Duals of the degenerate $K3$.

In this section, we revisit the chain of dualities discussed in section 6.4 that led from the type I' string theory to the IIA string on $K3$. Starting from the local charge-cancelling configuration of the type I' string on $I \times \mathbb{R}^{1,8}$ with 8 D8-branes at each O8-plane, dualising took us to the type IIA string on the $K3$ -orbifold T^4/\mathbb{Z}_2 and to the quotient $\text{IIB}/(-1)^{F_L} R_{6789}$ of type IIB compactified on T^4/\mathbb{Z}_2 with one NS5-brane coincident with one ON-plane at each of the 16 fixed points. The equivalence of these theories at one point in moduli space then, in principle, should define an embedding of the moduli space $O(1, 17; \mathbb{Z}) \backslash O(1, 17)/O(17)$ of the type I' string theory into the moduli space of the IIA string on $K3$ and of the IIB quotient. The domain wall supergravity solutions provide a guide as to how this should work. Moving in the moduli space of the type I' string moves the 16 D8-branes away from the O8-planes to generic points τ_i on the interval, corresponding to the solution (6.3.1) for generic points away from the locations of the branes. Dualising the solution takes the solution (6.3.1) with D8-branes to the solution (6.2.22) with smeared NS5-branes and a T^3 fibration over a line or to the solution (6.2.31) of smeared KK-monopoles with a nilfold fibration over a line. The locations of domain walls arising from the smeared NS5-branes or KK-monopoles are at the same locations τ_i . The geometry discussed in the previous section then provides a non-singular $K3$ geometry that resolves the singularities of the domain-wall supergravity solution, and its existence supports the picture arising from duality arguments.

The type I' configuration with two O8-planes of charge -8 and 16 D8-branes then corresponds to the $K3$ geometry with end-caps given by Tian-Yau spaces with $b_+ = b_- = 8$ and with 16 Kaluza-Klein monopoles distributed over the interval. If $b_+ = 8 - n_+$ and $b_- = 8 - n_-$ with $16 - b_- - b_+$ Kaluza-Klein monopoles, this corresponds in the type I' string to having n_- D8-branes at the O8-plane at $\tau = 0$ and n_+ D8-branes at the O8-plane at $\tau = \pi$, with $16 - b_- - b_+$ D8-branes distributed over the interval.

However, the $K3$ geometry also allows $b_+ = 9$ and/or $b_- = 9$, which would lead to up to 17 or 18 Kaluza-Klein monopoles. This corresponds to the possibility in the type I' string at strong coupling for an O8-plane to emit a D8-brane leaving an O8*-plane of charge -9 [95], [121]. Then the $K3$ with $b_+ = b_- = 9$ and 18 Kaluza-Klein monopoles corresponds in the type I' string to two O8*-planes of charge -9 and 18 D8-branes. The configuration in which the 18 Kaluza-Klein monopoles are coincident corresponds to the one in which the 18 D8-branes are

coincident, and either picture gives an enhanced gauge group $SU(18)$ (together with a further $U(1)$ factor). For the type I' string, up to 16 D8-branes are possible at weak coupling and 17 or 18 D8-branes are only possible at strong coupling. However, for the $K3$ geometries, the IIA string on $K3$ can be taken at weak IIA string coupling and in particular 17 or 18 KK-monopoles and the gauge group $SU(18)$ can arise at weak coupling. The S-duality in the chain of dualities in section 6.4 has mapped strong coupling physics of the type I' string to weak coupling physics in the IIA string.

Consider now the type IIB dual of the $K3$ compactification. Compactifying the weakly-coupled I' string on T^3 and dualising on all three toroidal directions gave 16 D5-branes smeared over the T^3 16 O5-planes. The IIB theory is an orientifold on T^4/\mathbb{Z}_2 , which can be regarded as $I \times T^3$ with an identification of the 3-tori at the ends $\tau = 0, \pi$ of the interval to become T^3/\mathbb{Z}_2 , with an O5-plane at each fixed point. S-dualising gives the quotient $\text{IIB}/(-1)^{F_L} R_{6789}$ with 16 ON-planes and 16 NS5-branes smeared over the T^3 . This should be dual to the IIA string on $K3$, and for the orbifold T^4/\mathbb{Z}_2 they are related by a T-duality. However, the relation between the IIA and IIB pictures cannot be a conventional T-duality at generic points in the moduli space, as T-duality requires the geometry to have an isometry and a smooth $K3$ does not have any isometries.

The degenerating $K3$ geometry of [86] is constructed by gluing a number of hyperkähler segments. The segment with geometry $\hat{\mathcal{N}}$ (6.2.31) with a nilfold fibred over a line segment dualises to (6.2.22) with a 3-torus with flux fibred over a line segment. The Tian-Yau caps do not have the required isometries and so do not have conventional T-duals. However, from the duality with the type I' theory, they should be dual to the region around the 8 ON-planes. The Tian-Yau caps are asymptotic to the nilfold fibred over a line, and so their duals should be asymptotic to a T^3 with flux fibred over a line.

The segment near the domain wall of charge N_i at $\tau = \tau_i$ is realized in the $K3$ geometry as N_i Kaluza-Klein monopoles on $\mathcal{N} \times I$, realized as a Gibbons-Hawking metric with N_i sources on the base space $T^2 \times I$. T-dualising on the S^1 fibre of this Gibbons-Hawking space takes the N_i Kaluza-Klein monopoles on $\mathcal{N} \times I$ to N_i NS5-branes on $T^3 \times I$. This can be understood by first looking at the covering space $\mathbb{R}^2 \times I$ of $T^2 \times I$. A single Kaluza-Klein monopole in \mathbb{R}^3 is given by (6.2.7) in terms of the Gibbons-Hawking form of the Taub-NUT metric (6.2.8) with $V(\tau, x, z)$ a harmonic function on the \mathbb{R}^3

$$V = c + \frac{1}{|\mathbf{r} - \mathbf{r}_0|} \quad (6.7.1)$$

with 3-vector $\mathbf{r} = (\tau, x, z)$. T-dualising gives the NS5-brane (6.2.22) with harmonic function $V(\tau, x, y, z)$ on the transverse \mathbb{R}^4 given again by (6.7.1). It is independent of the coordinate y . T-dual to the Gibbons-Hawking fibre coordinate and so the solution is smeared in the y direction.

Taking a periodic array of such solutions in the x, z directions allows periodic identification of the x, z coordinates and gives the GH solution localized on $T^2 \times \mathbb{R}$. T-dualising in the y direction gives the NS5-brane on $T^3 \times \mathbb{R}$, smeared over one of the y direction. For the segment near $\tau = \tau_i$, one takes a superposition of N_i sources giving the Gibbons-Hawking solution with N_i sources on $T^2 \times I$.

At the level of supergravity solutions, the D8-brane domain wall supergravity solutions wrapped on T^3 map to the KK-monopole domain walls with Gibbons-Hawking metric smeared over two transverse directions (x, z) , so that $V(\tau, x, z)$ is independent of (x, z) . We have seen that these singular domain walls are smoothed out to give a Gibbons-Hawking metric with local sources at points (τ, x, z) in $T^2 \times I$. This suggests that the smeared NS5-brane domain walls too should be smoothed out to give an NS5-brane solution with sources localized at N_i points in the transverse $T^3 \times I$. Then S-dualising to D5-branes, this would lead to the D5-brane domain wall T-dual to a D8-brane of charge N_i realized as N_i localized D5-branes on $T^3 \times I$. T-dualising to Dp -branes, we would then expect the singularity of the smeared Dp -brane to be smoothed by having N_i local sources on the transverse $T^{8-p} \times I$.

6.8. Non-geometric string theory

Compactifying the type I string on T^4 and T-dualising in all four torus directions and then taking the S-dual gives the quotient $\text{IIB}/(-1)^{F_L} R_{6789}$ of the IIB string on T^4/\mathbb{Z}_2 . One could then in principle T-dualise this in one, two, three or four directions and this should lead to new string theory configurations.

The first T-duality works well, as has been discussed in the preceding sections. At the locally-charge-cancelling orbifold point, the T-duality takes this to the orbifold IIA/R_{6789} of the IIA string on the $K3$ orbifold T^4/\mathbb{Z}_2 . At generic points in the moduli space of configurations dual to the type I' string, this becomes a duality between a IIB configuration of NS5-branes and ON-planes and the IIA string on a smooth $K3$ manifold near the boundary of moduli space in which the $K3$ becomes a long neck capped with Tian-Yau spaces. As a smooth $K3$ has no isometries, this duality is not properly a T-duality but instead a dual form of the duality between IIA on $K3$ and the heterotic string on T^4 [32] (which is in turn dual to the type I string on T^4). However, in the long neck region, the geometry is well approximated by a Gibbons-Hawking space and the T-dual of this gives the appropriate configuration of NS5-branes, so dualising the corresponding supergravity solutions gives a good guide to how the duality works.

The first T-duality can be understood as taking NS5-branes to KK-monopoles, with NS5-branes on $T^3 \times I$ mapped to KK-monopoles on $\mathcal{N} \times I$. T-dualising in two or more directions takes the NS5-branes to exotic branes, so will result in string theory in a non-geometric background. We now explore this in more detail.

Consider first T-dualising $\text{IIB}/(-1)^{F_L} R_{6789}$ in two directions, taking an NS5-brane to an exotic 5_2^3 -brane or $(5, 2^2)$ -brane. In the last section, we have seen how the naive T-duality between the supergravity solutions representing KK-monopole domain walls and NS5-brane domain walls becomes a proper string theory duality between a smooth $K3$ geometry and an NS5-brane configuration. A T-duality in the z direction takes the nilfold (6.1.3) to the T-fold (6.1.5), and takes the solution (6.2.31) with the nilfold fibred over a line to the solution (6.2.36) with a T-fold fibred over a line. If this is subsumed in a proper string theory duality, then this suggests that the degenerate $K3$ solution with a long neck given by the nilfold fibred over a line should be dual to a non-geometric configuration with a long neck given by the T-fold fibred over a line. The ends of the neck should be capped off by configurations that can be thought of as the duals of the geometries that cap off the degenerate $K3$ or as the double T-dual of the ON-planes on the fixed points of the ends T^3/\mathbb{Z}_2 of $T^4/\mathbb{Z}_2 \sim I \times T^3$, and these are presumably non-geometric. The domain walls might then be expected to become localized configurations of exotic branes.

Similar remarks apply to T-dualising $\text{IIB}/(-1)^{F_L} R_{6789}$ in three directions, taking an NS5-brane to an exotic $(5, 3^2)$ -brane or 5_2^3 -brane. This takes the configuration $T^3 \times I$ with H -flux to the configuration with R-flux that is not geometric even locally. It cannot be formulated as a conventional background but can be formulated as a doubled geometry, with explicit dependence on the coordinates dual to string winding. The doubled geometry of this configuration will be discussed elsewhere.

7. Special Holonomy Domain Walls, Intersecting Branes and T-folds

In this chapter, we generalize a 3-dimensional nilmanifold to a higher-dimensional nilmanifold. In the previous chapter, taking a product of a nilfold with a real line gives a space admitting a hyperkähler metric. A special holonomy space can be obtained in the same way by replacing the 3-dimensional nilmanifold with higher-dimensional one [96]. In each case, the product of this space with Minkowski space gives a supersymmetric solution. This supersymmetric solution is T-dual to an intersecting brane solution [97].

7.1. Nilmanifolds as torus bundles over tori

In this section, we review certain generalizations of the 3-dimensional nilfold to higher dimensions [96]. The Heisenberg group is replaced by a higher dimensional nilpotent Lie group \mathcal{G} , and taking the quotient by a cocompact discrete subgroup gives a nilmanifold, which is a compact space which is a T^n bundle over T^m for some m, n .

A Lie algebra \mathfrak{g} is nilpotent if the lower central series terminates, that is

$$[X_1, [X_2, [\dots [X_p, Y] \dots]] = 0 \quad (7.1.1)$$

for all $X_1, \dots, X_p, Y \in \mathfrak{g}$, for some integer p . For a nilpotent Lie group \mathcal{G} , the smallest such p is known as the nilpotency class of \mathcal{G} and \mathcal{G} is called a p -step nilpotent Lie group. Note that an abelian group is 1-step nilpotent Lie group since

$$[X, Y] = 0 \quad (7.1.2)$$

for all $X, Y \in \mathfrak{g}$. The 3-dimensional Heisenberg group is a 2-step nilpotent Lie group since

$$[T_x, T_z] = mT_y \quad (7.1.3)$$

and T_y commutes with T_x and T_z .

For a general 2-step nilpotent Lie group \mathcal{G} , the commutator of any two generators X, Y of

the Lie algebra \mathfrak{g} must be in the centre $Z(\mathfrak{g})$ of \mathfrak{g} (consisting of generators commuting with all other generators):

$$[X, Y] \in Z(\mathfrak{g}) \quad (7.1.4)$$

In general, a 2-step nilpotent Lie group \mathcal{G} is non-compact. A compact space \mathcal{M} can be constructed from a nilpotent Lie group \mathcal{G} identified under the left action of a cocompact subgroup Γ :

$$\mathcal{M} = \mathcal{G}/\Gamma. \quad (7.1.5)$$

If the dimension of $Z(\mathfrak{g})$ is n and the dimension of the quotient $\mathfrak{g}/Z(\mathfrak{g})$ is m , the compact space \mathcal{M} can be regarded as a T^n bundle over T^m . For example, the Heisenberg group G_3 has a centre $Z(\mathfrak{g})$ of dimension $n = 1$ and the dimension of $\mathfrak{g}/Z(\mathfrak{g})$ is $m = 2$, and the nilfold is an S^1 bundle over T^2 .

Local coordinates z^a on the group manifold \mathcal{G} can be introduced by the exponential map giving a group element g as $g = \prod_a \exp(z^a T_a)$ where T_a are the Lie algebra generators. The general left-invariant metric on \mathcal{G} can be written as

$$ds^2 = x_{ab} P^a P^b, \quad (7.1.6)$$

where x_{ab} is a constant symmetric matrix. and P^a are the left-invariant one-forms

$$g^{-1} dg = P^a T_a, \quad (7.1.7)$$

In this paper, x_{ab} will be chosen as $x_{ab} = \delta_{ab}$ so the left-invariant metric is

$$ds^2 = \delta_{ab} P^a P^b. \quad (7.1.8)$$

The discrete subgroup Γ consists of group elements with integer coordinates, $g(n) = \prod_a \exp(n^a T_a)$ for integers n^a . Taking the quotient of \mathcal{G} by the left action of the discrete subgroup Γ gives the nilfold $\mathcal{M} = \mathcal{G}/\Gamma$ and (7.1.8) gives a metric on \mathcal{M} . Taking the quotient imposes identifications on the coordinates so that the space becomes a torus bundle over a torus.

7.1.1. S^1 bundle over T^4

Our first example is the five-dimensional nilpotent Lie algebra whose only non-vanishing commutators are

$$[T_2, T_3] = mT_1, \quad [T_4, T_5] = mT_1. \quad (7.1.9)$$

With coordinates z^1, \dots, z^5 , the left-invariant one-forms are then given by

$$\begin{aligned} P^1 &= dz^1 + m(z^3 dz^2 + z^5 dz^4), \\ P^2 &= dz^2, \quad P^3 = dz^3, \\ P^4 &= dz^4, \quad P^5 = dz^5. \end{aligned} \tag{7.1.10}$$

The metric (7.1.8) is then

$$ds^2 = \left(dz^1 + m(z^3 dz^2 + z^5 dz^4) \right)^2 + (dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2. \tag{7.1.11}$$

This is an S^1 bundle over T^4 with fibre coordinate z^1 and T^4 coordinates z^2, z^3, z^4, z^5 , with first Chern class represented by

$$m \left(dz^3 \wedge dz^2 + dz^5 \wedge dz^4 \right).$$

The metric is invariant under shifts of z^1 , z^2 , and z^4 so that T-dualising in these directions is straightforward, applying the standard Buscher rules [100, 101]. We will not give all dual backgrounds explicitly, but focus on some interesting examples.

T-duality in z^1 direction gives a T^5 with H -flux. The metric and H -flux of this space are given by

$$ds^2 = (dz^1)^2 + (dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2, \tag{7.1.12}$$

$$H = -mdz^1 \wedge dz^2 \wedge dz^3 - mdz^1 \wedge dz^4 \wedge dz^5. \tag{7.1.13}$$

T-dualising the metric (7.1.11) in the z^2 and z^4 directions gives a T-fold background. The metric and B -field are given by

$$\begin{aligned} ds^2 &= \frac{1}{1 + m^2 \left[(z^3)^2 + (z^5)^2 \right]} \left((dz^1)^2 + (dz^2)^2 + (dz^4)^2 \right) \\ &\quad + \frac{1}{1 + m^2 \left[(z^3)^2 + (z^5)^2 \right]} \left(mz^5 dz^2 - mz^3 dz^4 \right)^2 + (dz^3)^2 + (dz^5)^2, \end{aligned} \tag{7.1.14}$$

$$B = \frac{m}{1 + m^2 \left[(z^3)^2 + (z^5)^2 \right]} \left(z^3 dz^1 \wedge dz^2 + z^5 dz^1 \wedge dz^4 \right). \tag{7.1.15}$$

7.1.2. T^2 bundle over T^3

Next consider the five-dimensional nilpotent Lie algebra whose only non-vanishing commutators are

$$[T_3, T_4] = mT_1, \quad [T_3, T_5] = mT_2. \tag{7.1.16}$$

Introducing coordinates z^1, \dots, z^5 , the left-invariant one-forms are given by

$$\begin{aligned} P^1 &= dz^1 + mz^4 dz^3, \\ P^2 &= dz^2 + mz^5 dz^3, \\ P^3 &= dz^3, P^4 = dz^4, P^5 = dz^5. \end{aligned} \quad (7.1.17)$$

and the metric on the manifold is

$$ds^2 = \left(dz^1 + mz^4 dz^3\right)^2 + \left(dz^2 + mz^5 dz^3\right)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2. \quad (7.1.18)$$

This space is a T^2 bundle over T^3 with fibre coordinates z^1, z^2 .

T-duality in the z^1 direction gives an S^1 bundle over T^4 with H -flux; the metric and H -flux are

$$ds^2 = (dz^1)^2 + \left(dz^2 + mz^5 dz^3\right)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2, \quad (7.1.19)$$

$$H = -mdz^1 \wedge dz^3 \wedge dz^4. \quad (7.1.20)$$

A further T-duality in the z^2 direction gives T^5 with H -flux. The metric and H -flux of this space are

$$ds^2 = (dz^1)^2 + (dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2, \quad (7.1.21)$$

$$H = -mdz^1 \wedge dz^3 \wedge dz^4 - mdz^2 \wedge dz^3 \wedge dz^5. \quad (7.1.22)$$

After a change of coordinates, this is the same solution as (7.1.12),(7.1.13), establishing that the S^1 bundle over T^4 is T-dual to the T^2 bundle over T^3 .

Starting from the metric (7.1.18) and doing a T-duality in z^3 direction gives a T-fold with metric and B -field

$$ds^2 = \frac{1}{1 + m^2 \left[(z^4)^2 + (z^5)^2 \right]} \left[(dz^1)^2 + (dz^2)^2 + (dz^3)^2 \right] + \frac{1}{1 + m^2 \left[(z^4)^2 + (z^5)^2 \right]} \left(z^5 dz^1 - z^4 dz^2 \right)^2 + (dz^4)^2 + (dz^5)^2, \quad (7.1.23)$$

$$B = \frac{m}{1 + m^2 \left[(z^4)^2 + (z^5)^2 \right]} \left(z^4 dz^1 \wedge dz^3 + z^5 dz^2 \wedge dz^3 \right). \quad (7.1.24)$$

7.1.3. T^2 bundle over T^4

Consider the six-dimensional nilpotent Lie algebra whose only non-vanishing commutators are

$$\begin{aligned} [T_3, T_4] &= mT_1, & [T_3, T_5] &= mT_2, \\ [T_5, T_6] &= mT_1, & [T_4, T_6] &= -mT_2. \end{aligned} \quad (7.1.25)$$

Note that this six-dimensional Lie-algebra corresponding is the complexification of the standard 3-dimensional Heisenberg algebra. Introducing coordinates z^1, \dots, z^6 , the left-invariant one-forms are given by

$$\begin{aligned} P^1 &= dz^1 + m(z^4 dz^3 + z^6 dz^5), \\ P^2 &= dz^2 + m(z^5 dz^3 - z^6 dz^4), \\ P^3 &= dz^3, \quad P^4 = dz^4, \\ P^5 &= dz^5, \quad P^6 = dz^6. \end{aligned} \quad (7.1.26)$$

and the metric on the manifold is

$$ds^2 = \left(dz^1 + m(z^4 dz^3 + z^6 dz^5)\right)^2 + \left(dz^2 + m(z^5 dz^3 - z^6 dz^4)\right)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2 + (dz^6)^2. \quad (7.1.27)$$

This space is a T^2 bundle over T^4 with fibre coordinates z^1, z^2 .

T-duality in z^1 gives a S^1 bundle over T^5 with H -flux, given by

$$ds^2 = (dz^1)^2 + \left(dz^2 + m(z^5 dz^3 - z^6 dz^4)\right)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2 + (dz^6)^2 \quad (7.1.28)$$

$$H = -mdz^1 \wedge dz^3 \wedge dz^4 - mdz^1 \wedge dz^5 \wedge dz^6. \quad (7.1.29)$$

A further T-duality in the z^2 direction gives T^6 with H -flux, with a flat metric

$$ds^2 = (dz^1)^2 + (dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2 + (dz^6)^2, \quad (7.1.30)$$

and H -flux

$$\begin{aligned} H &= -mdz^1 \wedge dz^3 \wedge dz^4 - mdz^1 \wedge dz^5 \wedge dz^6 \\ &\quad -mdz^2 \wedge dz^3 \wedge dz^5 + mdz^2 \wedge dz^4 \wedge dz^6, \end{aligned} \quad (7.1.31)$$

Starting from the metric (7.1.27) and doing T-duality in the z^3 direction gives a T-fold with metric and B -field given by (C.0.1) and (C.0.2).

7.1.4. T^3 bundle over T^3

The next case is the six-dimensional nilpotent Lie algebra whose only non-vanishing commutators are

$$\begin{aligned} [T_5, T_6] &= mT_1, & [T_4, T_6] &= -mT_2, \\ [T_4, T_5] &= mT_3. \end{aligned} \quad (7.1.32)$$

The left-invariant one-forms are given by

$$\begin{aligned} P^1 &= dz^1 + mz^6 dz^5, \\ P^2 &= dz^2 - mz^6 dz^4, \\ P^3 &= dz^3 + mz^5 dz^4, \\ P^4 &= dz^4, \quad P^5 = dz^5, \quad P^6 = dz^6. \end{aligned} \quad (7.1.33)$$

The metric is

$$ds^2 = (dz^1 + mz^6 dz^5)^2 + (dz^2 - mz^6 dz^4)^2 + (dz^3 + mz^5 dz^4)^2 + (dz^4)^2 + (dz^5)^2 + (dz^6)^2. \quad (7.1.34)$$

This is a T^3 bundle over T^3 with fibre coordinates z^1, z^2, z^3 .

T-duality in the z^1 direction gives a T^2 bundle over T^4 with H -flux. The metric and H -flux are

$$ds^2 = (dz^1)^2 + (dz^2 - mz^6 dz^4)^2 + (dz^3 + mz^5 dz^4)^2 + (dz^4)^2 + (dz^5)^2 + (dz^6)^2, \quad (7.1.35)$$

$$H = -mdz^1 \wedge dz^5 \wedge dz^6. \quad (7.1.36)$$

A further T-duality in z^2 gives an S^1 bundle over T^5 with H -flux. The metric and H -flux are

$$ds^2 = (dz^1)^2 + (dz^2)^2 + (dz^3 + mz^5 dz^4)^2 + (dz^4)^2 + (dz^5)^2 + (dz^6)^2, \quad (7.1.37)$$

$$H = -mdz^1 \wedge dz^5 \wedge dz^6 + mdz^2 \wedge dz^4 \wedge dz^6. \quad (7.1.38)$$

A final T-duality in the z^3 direction gives a T^6 with H -flux

$$ds^2 = (dz^1)^2 + (dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2 + (dz^6)^2, \quad (7.1.39)$$

$$H = -mdz^1 \wedge dz^5 \wedge dz^6 - mdz^3 \wedge dz^4 \wedge dz^5 + mdz^2 \wedge dz^4 \wedge dz^6. \quad (7.1.40)$$

Starting with (7.1.34) and T-dualising in the z^4 direction gives a T-fold with metric and

B -field given by

$$ds^2 = (dz^1 + mz^6 dz^5)^2 + \frac{1}{1 + m^2[(z^5)^2 + (z^6)^2]} \left[(dz)^2 + (dz^3)^2 + (dz^4)^2 \right] \\ + \frac{1}{1 + m^2[(z^5)^2 + (z^6)^2]} \left(mz^5 dz^2 + mz^6 dz^3 \right)^2 + (dz^5)^2 + (dz^6)^2 \quad (7.1.41)$$

$$B = \frac{m}{1 + m^2[(z^5)^2 + (z^6)^2]} \left(z^5 dz^3 \wedge dz^4 - z^6 dz^2 \wedge dz^4 \right) \quad (7.1.42)$$

7.1.5. T^3 bundle over T^4

Next consider the seven-dimensional nilpotent Lie algebra whose only non-vanishing commutators are

$$\begin{aligned} [T_4, T_5] &= mT_1, & [T_6, T_7] &= mT_1, \\ [T_4, T_6] &= mT_2, & [T_5, T_7] &= -mT_2, \\ [T_4, T_7] &= mT_3, & [T_5, T_6] &= mT_3. \end{aligned} \quad (7.1.43)$$

The left-invariant one-forms are given by

$$\begin{aligned} P^1 &= dz^1 + m(z^5 dz^4 + z^7 dz^6), \\ P^2 &= dz^2 + m(z^6 dz^4 - z^7 dz^5), \\ P^3 &= dz^3 + m(z^7 dz^4 + z^6 dz^5), \\ P^4 &= dz^4, \quad P^5 = dz^5, \\ P^6 &= dz^6, \quad P^7 = dz^7. \end{aligned} \quad (7.1.44)$$

The metric is

$$ds^2 = \left(dz^1 + m(z^5 dz^4 + z^7 dz^6) \right)^2 + \left(dz^2 + m(z^6 dz^4 - z^7 dz^5) \right)^2 + \left(dz^3 + m(z^7 dz^4 + z^6 dz^5) \right)^2 \\ + (dz^4)^2 + (dz^5)^2 + (dz^6)^2 + (dz^7)^2. \quad (7.1.45)$$

T-duality in the z^1 direction gives a nilmanifold with H -flux. The metric and H -flux are given

$$ds^2 = (dz^1)^2 + \left(dz^2 + m(z^6 dz^4 - z^7 dz^5) \right)^2 + \left(dz^3 + m(z^7 dz^4 + z^6 dz^5) \right)^2 \\ + (dz^4)^2 + (dz^5)^2 + (dz^6)^2 + (dz^7)^2, \quad (7.1.46)$$

$$H = -mdz^1 \wedge dz^4 \wedge dz^5 - mdz^1 \wedge dz^6 \wedge dz^7. \quad (7.1.47)$$

T-duality again in z^2 direction gives a nilmanifold with H -flux. The metric and H -flux are given

$$ds^2 = (dz^1)^2 + (dz^2)^2 + \left(dz^3 + m(z^7 dz^4 + z^6 dz^5)\right)^2 + (dz^4)^2 + (dz^5)^2 + (dz^6)^2 + (dz^7)^2, \quad (7.1.48)$$

$$H = -mdz^1 \wedge dz^4 \wedge dz^5 - mdz^1 \wedge dz^6 \wedge dz^7 - mdz^2 \wedge dz^4 \wedge dz^6 \quad (7.1.49)$$

$$+mdz^2 \wedge dz^5 \wedge dz^7. \quad (7.1.50)$$

T-duality again in z^3 direction gives a T^7 with H -flux. The metric and H -flux are given

$$ds^2 = (dz^1)^2 + (dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2 + (dz^6)^2 + (dz^7)^2, \quad (7.1.51)$$

$$H = -mdz^1 \wedge dz^4 \wedge dz^5 - mdz^1 \wedge dz^6 \wedge dz^7 - mdz^2 \wedge dz^4 \wedge dz^6 - mdz^3 \wedge dz^4 \wedge dz^7 - mdz^3 \wedge dz^5 \wedge dz^6 + mdz^2 \wedge dz^5 \wedge dz^7. \quad (7.1.52)$$

While starting from the metric (7.1.45), and doing T-duality in z^4 direction gives T-fold with metric and B -field given by (C.0.3), (C.0.4).

7.1.6. S^1 bundle over T^6

The last case is the seven-dimensional nilpotent Lie algebra whose only non-vanishing commutators are

$$\begin{aligned} [T_2, T_3] &= mT_1, & [T_4, T_5] &= mT_1, \\ [T_6, T_7] &= mT_1. \end{aligned} \quad (7.1.53)$$

The left-invariant one-forms are given by

$$\begin{aligned} P^1 &= dz^1 + m(z^3 dz^2 + z^5 dz^4 + z^7 dz^6), \\ P^2 &= dz^2, \quad P^3 = dz^3, \\ P^4 &= dz^4, \quad P^5 = dz^5, \\ P^6 &= dz^6, \quad P^7 = dz^7. \end{aligned} \quad (7.1.54)$$

The metric is

$$ds^2 = \left(dz^1 + m(z^3 dz^2 + z^5 dz^4 + z^7 dz^6)\right)^2 + (dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2 + (dz^6)^2 + (dz^7)^2. \quad (7.1.55)$$

T-duality in the z^1 direction gives a T^7 with H -flux. The metric and H -flux are given

$$ds^2 = (dz^1)^2 + (dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2 + (dz^6)^2 + (dz^7)^2, \quad (7.1.56)$$

$$H = -mdz^1 \wedge dz^2 \wedge dz^3 - mdz^1 \wedge dz^4 \wedge dz^5 - mdz^1 \wedge dz^6 \wedge dz^7. \quad (7.1.57)$$

Starting with (7.1.55) and T-dualising in the z^2 , z^4 , and z^6 gives a T-fold with metric and B -field given by

$$\begin{aligned} ds^2 = & \frac{1}{1 + m^2[(z^3)^2 + (z^5)^2 + (z^7)^2]} \left[(dz^1)^2 + (dz^2)^2 + (dz^4)^2 + (dz^6)^2 \right] \\ & + (dz^3)^2 + (dz^5)^2 + (dz^7)^2 + \frac{1}{1 + m^2[(z^3)^2 + (z^5)^2 + (z^7)^2]} \left(mz^5 dz^2 - mz^3 dz^4 \right)^2 \\ & + \frac{1}{1 + m^2[(z^3)^2 + (z^5)^2 + (z^7)^2]} \left(mz^7 dz^2 - mz^3 dz^6 \right)^2 \\ & + \frac{1}{1 + m^2[(z^3)^2 + (z^5)^2 + (z^7)^2]} \left(mz^7 dz^4 - mz^5 dz^6 \right)^2, \end{aligned} \quad (7.1.58)$$

$$B = \frac{m}{1 + m^2[(z^3)^2 + (z^5)^2 + (z^7)^2]} \left(z^3 dz^1 \wedge dz^2 + z^5 dz^1 \wedge dz^4 + z^7 dz^1 \wedge dz^6 \right). \quad (7.1.59)$$

7.2. Supersymmetric domain wall solutions

In the previous chapter, taking the product of the 3-dimensional nilfold with the real line gave a space admitting a hyperkähler metric. Remarkably, a similar result applies for the nilmanifolds arising as higher dimensional analogues of the nilfold of section 7.1 [96]. Each of the spaces

$$\mathcal{M} = \mathcal{G}/\Gamma. \quad (7.2.1)$$

discussed in section 7.1 is a T^n bundle over T^m for some m, n . In each case, the space $\mathcal{M} \times \mathbb{R}$ admits a multi-domain wall type metric that has special holonomy [96], so that taking the product of the domain wall solution with Minkowski space gives a supersymmetric solution. The solutions all involve a piecewise linear function $V(\tau)$ given by (6.2.26) with derivative $M = V'$ given by (6.2.29), corresponding to domain walls at the points τ_i . We now discuss each case in turn.

7.2.1. 6-dimensional domain wall solutions with $SU(3)$ holonomy

6-dimensional solutions with $SU(3)$ holonomy can be constructed on $\mathcal{M} \times \mathbb{R}$ for the two cases of 5-dimensional nilfolds discussed in section 7.1, the S^1 bundle over T^4 and the T^2 bundle over T^3 .

Case 1 S^1 bundle over T^4

The six-dimensional domain wall metric for this case is given by

$$ds^2 = V^2(\tau)(d\tau)^2 + V(\tau)\left((dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2\right) + V^{-2}(\tau)\left(dz^1 + M(z^3 dz^2 + z^5 dz^4)\right)^2, \quad (7.2.2)$$

where τ is a coordinate on the real line, z^2, z^3, z^4 , and z^5 are coordinates on T^4 , and z^1 is a coordinate on S^1 . The harmonic function $V(\tau)$ is given by (6.2.26) and $M = V'$ (6.2.29).

This metric is Kähler Ricci-flat so it has $SU(3)$ holonomy and preserves $\frac{1}{4}$ supersymmetry. The Kähler form is given by

$$J = d\tau \wedge \left(dz^1 + M(z^3 dz^2 + z^5 dz^4)\right) - V(\tau) dz^2 \wedge dz^3 - V(\tau) dz^4 \wedge dz^5. \quad (7.2.3)$$

Case 2 T^2 bundle over T^3

The six-dimensional domain wall metric for this case is given by

$$ds^2 = V^2(\tau)(d\tau)^2 + V^2(\tau)(dz^3)^2 + V(\tau)\left((dz^4)^2 + (dz^5)^2\right) + V^{-1}(\tau)\left(dz^1 + Mz^4 dz^3\right)^2 + V^{-1}(\tau)\left(dz^2 + Mz^5 dz^3\right)^2, \quad (7.2.4)$$

where τ is a coordinate on the real line, z^3, z^4 , and z^5 are coordinates on T^3 , and z^1 , and z^2 are coordinates on T^2 . The harmonic function $V(\tau)$ is given by (6.2.26). This metric is also Kähler Ricci-flat with a Kähler form

$$J = V^2(\tau)d\tau \wedge dz^3 + \left(dz^1 + Mz^4 dz^3\right) \wedge dz^4 + \left(dz^2 + Mz^5 dz^3\right) \wedge dz^5. \quad (7.2.5)$$

7.2.2. 7-dimensional domain wall solutions with G_2 holonomy

7-dimensional solutions with G_2 holonomy can be constructed on $\mathcal{M} \times \mathbb{R}$ for the two cases of 6-dimensional nilfolds \mathcal{M} discussed in section 7.1, the T^2 bundle over T^4 and the T^3 bundle over T^3 . The G_2 holonomy implies the metrics are Ricci-flat metrics and preserve $\frac{1}{8}$ of the supersymmetry.

Case 1 T^2 bundle over T^4

The domain wall metric in this case is given by

$$\begin{aligned}
ds^2 = & V^4(\tau)(d\tau)^2 + V^2(\tau)\left((dz^3)^2 + (dz^4)^2 + (dz^5)^2 + (dz^6)^2\right) \\
& + V^{-2}(\tau)\left(dz^1 + M(z^4 dz^3 + z^6 dz^5)\right)^2 \\
& + V^{-2}(\tau)\left(dz^2 + M(z^5 dz^3 - z^6 dz^4)\right)^2,
\end{aligned} \tag{7.2.6}$$

where τ is a coordinate on the real line, z^3, z^4, z^5 , and z^6 are coordinates on T^4 , and z^1 , and z^2 are coordinates on T^2 .

Case 2 T^3 bundle over T^3

The domain wall metric in this case is given by

$$\begin{aligned}
ds^2 = & V^3(\tau)(d\tau)^2 + V^2(\tau)\left((dz^4)^2 + (dz^5)^2 + (dz^6)^2\right) + V^{-1}(\tau)\left(dz^1 + Mz^6 dz^5\right)^2 \\
& + V^{-1}(\tau)\left(dz^2 - Mz^6 dz^4\right)^2 + V^{-1}(\tau)\left(dz^3 + Mz^5 dz^4\right)^2,
\end{aligned} \tag{7.2.7}$$

where τ is a coordinate on the real line, z^4, z^5 , and z^6 are coordinates on T^3 base, and z^1, z^2 , and z^3 are coordinates on T^3 fibre. The harmonic function $V(\tau)$ is given by (6.2.26).

7.2.3. 8-dimensional domain wall solution with $Spin(7)$ holonomy

In this case, an 8-dimensional solution with $Spin(7)$ holonomy can be constructed on $\mathcal{M} \times \mathbb{R}$ with \mathcal{M} the 7-dimensional nilfold which is a T^3 bundle over T^4 . The domain wall metric is given by

$$\begin{aligned}
ds^2 = & V^6(\tau)(d\tau)^2 + V^3(\tau)\left((dz^4)^2 + (dz^5)^2 + (dz^6)^2 + (dz^7)^2\right) + V^{-2}(\tau)\left(dz^1 + M(z^5 dz^4 + z^7 dz^6)\right)^2 \\
& + V^{-2}(\tau)\left(dz^2 + M(z^6 dz^4 - z^7 dz^5)\right)^2 + V^{-2}(\tau)\left(dz^3 + M(z^7 dz^4 + z^6 dz^5)\right)^2,
\end{aligned} \tag{7.2.8}$$

where τ is a coordinate on the real line, z^4, z^5, z^6 and z^7 are coordinates on T^4 base, and z^1, z^2 , and z^3 are coordinates on T^3 fibre. The function $V(\tau)$ is given by (6.2.26). The $Spin(7)$ holonomy implies the metric is the Ricci-flat metric and preserves $\frac{1}{16}$ of the supersymmetry.

7.2.4. 8-dimensional domain wall solution with $SU(4)$ holonomy

In this case, an 8-dimensional solution with $SU(4)$ holonomy can be constructed on $\mathcal{M} \times \mathbb{R}$ with \mathcal{M} the 7-dimensional nilfold which is a S^1 bundle over T^6 . The domain wall metric is

given by

$$ds^2 = V^3(\tau)(d\tau)^2 + V(\tau)\left((dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2 + (dz^6)^2 + (dz^7)^2\right) + V^{-3}(\tau)\left(dz^1 + M(z^3 dz^2 + z^5 dz^4 + z^7 dz^6)\right)^2, \quad (7.2.9)$$

where τ is a coordinate on the real line, z^2, z^3, z^4, z^5, z^6 and z^7 are coordinates on T^6 base, and z^1 is a coordinate on S^1 fibre. The function $V(\tau)$ is given by (6.2.26). The $SU(4)$ holonomy implies the metric is the Ricci-flat metric and preserves $\frac{1}{8}$ of the supersymmetry.

7.3. Special holonomy domain walls and intersecting branes

In this section, we will T-dualise each of the special holonomy domain wall solutions of the last section to obtain a system of intersecting branes. In each case, we obtain a standard intersecting brane configuration and check that they preserve exactly the same fraction of supersymmetry as the corresponding special holonomy domain wall solutions.

7.3.1. S^1 fibred over T^4

The supersymmetric domain wall solution has ten-dimensional metric

$$ds_{10}^2 = ds^2(\mathbb{R}^{1,3}) + ds_6^2, \quad (7.3.1)$$

where $ds^2(\mathbb{R}^{1,n})$ is the flat metric of $(n+1)$ -dimensional Minkowski space and the 6-dimensional $SU(3)$ holonomy metric is

$$ds_6^2 = V^2(\tau)(d\tau)^2 + V(\tau)\left((dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2\right) + V^{-2}(\tau)\left(dz^1 + M(z^3 dz^2 + z^5 dz^4)\right)^2. \quad (7.3.2)$$

The H -flux and the dilaton are trivial,

$$H = 0, \quad \Phi = \text{constant}. \quad (7.3.3)$$

T-duality in the z^1 -direction gives the background with metric

$$ds_6^2 = V^2(\tau)\left((d\tau)^2 + (dz^1)^2\right) + V(\tau)\left((dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2\right), \quad (7.3.4)$$

H -flux

$$H = -Mdz^1 \wedge dz^2 \wedge dz^3 - Mdz^1 \wedge dz^4 \wedge dz^5, \quad (7.3.5)$$

and dilaton

$$e^\Phi = V(\tau). \quad (7.3.6)$$

These solutions describe two intersecting smeared NS5-branes.

	0	1	2	3	z^1	z^2	z^3	z^4	z^5	τ
NS5 1	\times	\times	\times	\times	\bullet	\bullet	\bullet	\times	\times	
NS5 2	\times	\times	\times	\times	\bullet	\times	\times	\bullet	\bullet	

Here and in what follows, a \times represents a world-volume direction and a \bullet represents a smeared direction. This then represents an NS5-brane lying in the $123z^4z^5$ directions and smeared over the $z^1z^2z^3$ directions intersecting an NS5-brane lying in the $123z^2z^3$ directions and smeared over the $z^1z^4z^4$ directions, with the intersection in the 123 directions. This intersection of two NS5-branes preserves 1/4 supersymmetry [97].

S-duality give a background with intersecting D5-branes with metric

$$ds_{10}^2 = V^{-1}(\tau)ds^2(\mathbb{R}^{1,3}) + V(\tau)\left((d\tau)^2 + (dz^1)^2\right) + \left((dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2\right), \quad (7.3.7)$$

and RR field strength

$$F_{(3)} = -Mdz^1 \wedge dz^2 \wedge dz^3 - Mdz^1 \wedge dz^4 \wedge dz^5, \quad (7.3.8)$$

and dilaton

$$e^\Phi = V^{-1}(\tau). \quad (7.3.9)$$

These solutions describe two intersecting D5-branes solutions.

	0	1	2	3	z^1	z^2	z^3	z^4	z^5	τ
D5 1	\times	\times	\times	\times	\bullet	\bullet	\bullet	\times	\times	
D5 2	\times	\times	\times	\times	\bullet	\times	\times	\bullet	\bullet	

T-duality in the z^1 , z^2 and z^3 directions gives a D4-brane inside a D8-brane with the metric

$$ds_{10}^2 = V^{-1}(\tau)ds^2(\mathbb{R}^{1,3}) + V(\tau)(d\tau)^2 + V^{-1}(\tau)(dz^1)^2 + \left((dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2\right), \quad (7.3.10)$$

and RR fluxes

$$F_{(0)} = -M, \quad F_{(4)} = -Mdz^2 \wedge dz^3 \wedge dz^4 \wedge dz^5. \quad (7.3.11)$$

and dilaton

$$e^\Phi = V^{-3/2}(\tau). \quad (7.3.12)$$

This solution consists of a number of parallel D8-branes with a D4-brane inside each. The D4-branes each lie in the $123z^1$ directions and are smeared over the $z^2z^3z^4z^5$ directions.

	0	1	2	3	z^1	z^2	z^3	z^4	z^5	τ
D8	\times	\times	\times	\times	\times	\times	\times	\times	\times	
D4	\times	\times	\times	\times	\times	\bullet	\bullet	\bullet	\bullet	

This is a standard example of a 1/4 supersymmetric brane configuration and is T-dual to a D0-brane inside a D4-brane.

7.3.2. T^2 fibred over T^3

The $SU(3)$ holonomy metric in this case is

$$\begin{aligned}
ds_6^2 = & V^2(\tau)(d\tau)^2 + V^2(\tau)(dz^3)^2 + V(\tau)\left((dz^4)^2 + (dz^5)^2\right) \\
& + V^{-1}(\tau)\left(dz^1 + Mz^4dz^3\right)^2 + V^{-1}(\tau)\left(dz^2 + Mz^5dz^3\right)^2.
\end{aligned} \tag{7.3.13}$$

The ten-dimensional metric is

$$ds_{10}^2 = ds^2(\mathbb{R}^{1,3}) + ds_6^2. \tag{7.3.14}$$

The H -flux and the dilaton are trivial,

$$H = 0, \quad \Phi = \text{constant}. \tag{7.3.15}$$

T-duality in the z^1 and z^2 directions gives the metric

$$\begin{aligned}
ds_6^2 = & V^2(\tau)\left((d\tau)^2 + (dz^3)^2\right) \\
& + V(\tau)\left((dz^1)^2 + (dz^2)^2 + (dz^4)^2 + (dz^5)^2\right)
\end{aligned}$$

and H -flux

$$H = -Mdz^1 \wedge dz^3 \wedge dz^4 - Mdz^2 \wedge dz^3 \wedge dz^5, \tag{7.3.16}$$

and dilaton

$$e^\Phi = V(\tau). \tag{7.3.17}$$

These solutions describe two intersecting NS5-branes.

	0	1	2	3	z^1	z^2	z^3	z^4	z^5	τ
NS5 1	\times	\times	\times	\times	\bullet	\times	\bullet	\bullet	\times	
NS5 2	\times	\times	\times	\times	\times	\bullet	\bullet	\times	\bullet	

S-duality give a background with intersecting D5-branes with metric

$$ds_{10}^2 = V^{-1}(\tau)ds^2(\mathbb{R}^{1,3}) + V(\tau)\left((d\tau)^2 + (dz^3)^2\right) + \left((dz^1)^2 + (dz^2)^2 + (dz^4)^2 + (dz^5)^2\right), \quad (7.3.18)$$

and RR field strength

$$F_{(3)} = -Mdz^1 \wedge dz^3 \wedge dz^4 - Mdz^2 \wedge dz^3 \wedge dz^5, \quad (7.3.19)$$

and dilaton

$$e^\Phi = V^{-1}(\tau). \quad (7.3.20)$$

This solution represents two intersecting D5-branes.

	0	1	2	3	z^1	z^2	z^3	z^4	z^5	τ
D5 1	\times	\times	\times	\times	\bullet	\times	\bullet	\bullet	\times	
D5 2	\times	\times	\times	\times	\times	\bullet	\bullet	\times	\bullet	

T-duality in the z^1 , z^3 and z^4 directions gives a metric

$$ds_{10}^2 = V^{-1}(\tau)ds^2(\mathbb{R}^{1,3}) + V(\tau)(d\tau)^2 + V^{-1}(\tau)((dz^3)^2 + ((dz^1)^2 + (dz^2)^2 + (dz^4)^2 + (dz^5)^2)), \quad (7.3.21)$$

and RR fluxes

$$F_{(0)} = -M, \quad F_{(4)} = -Mdz^1 \wedge dz^2 \wedge dz^4 \wedge dz^5. \quad (7.3.22)$$

and dilaton

$$e^\Phi = V^{-3/2}(\tau). \quad (7.3.23)$$

This solution again represents a D4-brane inside a D8-brane:

	0	1	2	3	z^1	z^2	z^3	z^4	z^5	τ
D8	\times	\times	\times	\times	\times	\times	\times	\times	\times	
D4	\times	\times	\times	\times	\bullet	\bullet	\times	\bullet	\bullet	

This was to be expected as the S^1 over T^4 case is T-dual to the T^2 over T^3 case.

7.3.3. T^2 fibred over T^4

The domain wall metric in this case is given by

$$\begin{aligned} ds_7^2 = & V^4(\tau)(d\tau)^2 + V^2(\tau)\left((dz^3)^2 + (dz^4)^2 + (dz^5)^2 + (dz^6)^2\right) \\ & + V^{-2}(\tau)\left(dz^1 + M(z^4dz^3 + z^6dz^5)\right)^2 \\ & + V^{-2}(\tau)\left(dz^2 + M(z^5dz^3 - z^6dz^4)\right)^2. \end{aligned} \quad (7.3.24)$$

The ten-dimensional metric is

$$ds_{10}^2 = ds^2(\mathbb{R}^{1,2}) + ds_7^2. \quad (7.3.25)$$

The H -flux and the dilaton are trivial,

$$H = 0, \quad \Phi = \text{constant}. \quad (7.3.26)$$

T-duality in the z^1 and z^2 directions followed by the coordinate transformations $z^4 \leftrightarrow z^5$ and $z^5 \leftrightarrow z^6$ gives the metric

$$ds_7^2 = V^4(\tau)(d\tau)^2 + V^2(\tau)\left((dz^1)^2 + (dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2 + (dz^6)^2\right), \quad (7.3.27)$$

and H -flux

$$\begin{aligned} H = & -Mdz^1 \wedge dz^3 \wedge dz^6 - Mdz^1 \wedge dz^4 \wedge dz^5 \\ & -Mdz^2 \wedge dz^3 \wedge dz^4 - Mdz^2 \wedge dz^5 \wedge dz^6, \end{aligned} \quad (7.3.28)$$

and dilaton

$$e^\Phi = V^2(\tau). \quad (7.3.29)$$

This solution represents four intersecting smeared NS5-branes:

	0	1	2	z^1	z^2	z^3	z^4	z^5	z^6	τ
NS5 1	×	×	×	•	×	•	×	×	•	
NS5 2	×	×	×	•	×	×	•	•	×	
NS5 3	×	×	×	×	•	•	•	×	×	
NS5 4	×	×	×	×	•	×	×	•	•	

This intersection of four NS5-branes preserves 1/8 supersymmetry [97].

S-duality give the metric

$$ds_{10}^2 = V^{-2}(\tau)ds^2(\mathbb{R}^{1,2}) + V^2(\tau)(d\tau)^2 + \left((dz^1)^2 + (dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2 + (dz^6)^2\right), \quad (7.3.30)$$

and RR field strength

$$\begin{aligned} F_{(3)} = & -Mdz^1 \wedge dz^3 \wedge dz^6 - Mdz^1 \wedge dz^4 \wedge dz^5 \\ & -Mdz^2 \wedge dz^3 \wedge dz^4 - Mdz^2 \wedge dz^5 \wedge dz^6, \end{aligned} \quad (7.3.31)$$

and dilaton

$$e^\Phi = V^{-2}(\tau). \quad (7.3.32)$$

These solutions describe four intersecting D5-branes.

	0	1	2	z^1	z^2	z^3	z^4	z^5	z^6	τ
D5 1	\times	\times	\times	\bullet	\times	\bullet	\times	\times	\bullet	
D5 2	\times	\times	\times	\bullet	\times	\times	\bullet	\bullet	\times	
D5 3	\times	\times	\times	\times	\bullet	\bullet	\bullet	\times	\times	
D5 4	\times	\times	\times	\times	\bullet	\times	\times	\bullet	\bullet	

T-duality in z^1, z^3 and z^6 directions gives the metric

$$ds_{10}^2 = V^{-2}(\tau)ds^2(\mathbb{R}^{1,2}) + V^2(\tau)(d\tau)^2 + \left((dz^1)^2 + (dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2 + (dz^6)^2\right), \quad (7.3.33)$$

and RR fluxes

$$F_{(0)} = -M, \quad (7.3.34)$$

$$F_{(4)} = -Mdz^3 \wedge dz^4 \wedge dz^5 \wedge dz^6 - Mdz^1 \wedge dz^2 \wedge dz^4 \wedge dz^6 - Mdz^1 \wedge dz^2 \wedge dz^3 \wedge dz^5. \quad (7.3.35)$$

and dilaton

$$e^\Phi = V^{-2}(\tau). \quad (7.3.36)$$

This solution represents 3 intersecting D4-branes inside a D8-brane.

	0	1	2	z^1	z^2	z^3	z^4	z^5	z^6	τ
D8	\times	\times	\times	\times	\times	\times	\times	\times	\times	
D4 1	\times	\times	\times	\times	\times	\bullet	\bullet	\bullet	\bullet	
D4 2	\times	\times	\times	\bullet	\bullet	\times	\bullet	\times	\bullet	
D4 3	\times	\times	\times	\bullet	\bullet	\bullet	\times	\bullet	\times	

T-dualising in the 1,2 directions and relabelling coordinates gives three mutually orthogonal D2-branes inside a D6-brane, with D2-branes in the 12, 34 and 56 planes all inside a D6-brane in the 123456 directions. This is a standard 1/8 supersymmetric brane intersection.

7.3.4. T^3 fibred over T^3

The G_2 holonomy metric in this case is given by

$$ds_7^2 = V^3(\tau)(d\tau)^2 + V^2(\tau)\left((dz^4)^2 + (dz^5)^2 + (dz^6)^2\right) + V^{-1}(\tau)\left(dz^1 + Mz^6dz^5\right)^2 + V^{-1}(\tau)\left(dz^2 - Mz^6dz^4\right)^2 + V^{-1}(\tau)\left(dz^3 + Mz^5dz^4\right)^2. \quad (7.3.37)$$

The ten-dimensional metric is

$$ds_{10}^2 = ds^2(\mathbb{R}^{1,2}) + ds_7^2. \quad (7.3.38)$$

The H -flux and the dilaton are trivial,

$$H = 0, \quad \Phi = \text{constant}. \quad (7.3.39)$$

T-duality in the z^1 , z^2 and z^3 directions followed by the coordinate transformation $z^2 \rightarrow -z^2$ gives the metric

$$\begin{aligned} ds_7^2 = & V^3(\tau)(d\tau)^2 + V(\tau)\left((dz^1)^2 + (dz^2)^2 + (dz^3)^2\right) \\ & + V^2(\tau)\left((dz^4)^2 + (dz^5)^2 + (dz^6)^2\right). \end{aligned} \quad (7.3.40)$$

and H -flux

$$H = -Mdz^1 \wedge dz^5 \wedge dz^6 - Mdz^3 \wedge dz^4 \wedge dz^5 - Mdz^2 \wedge dz^4 \wedge dz^6, \quad (7.3.41)$$

and dilaton

$$e^\Phi = V^{3/2}(\tau). \quad (7.3.42)$$

These solutions describe three intersecting smeared NS5-branes:

	0	1	2	z^1	z^2	z^3	z^4	z^5	z^6	τ
NS5 1	\times	\times	\times	\bullet	\times	\times	\times	\bullet	\bullet	
NS5 2	\times	\times	\times	\times	\times	\bullet	\bullet	\bullet	\times	
NS5 3	\times	\times	\times	\times	\bullet	\times	\bullet	\times	\bullet	

This intersection of three NS5-branes preserves 1/8 supersymmetry [97].

S-duality then gives the metric

$$\begin{aligned} ds_{10}^2 = & V^{-3/2}(\tau)ds^2(\mathbb{R}^{1,2}) + V^{3/2}(\tau)(d\tau)^2 + V^{-1/2}(\tau)\left((dz^1)^2 + (dz^2)^2 + (dz^3)^2\right) \\ & + V^{1/2}(\tau)\left((dz^4)^2 + (dz^5)^2 + (dz^6)^2\right). \end{aligned} \quad (7.3.43)$$

and RR field strength

$$F_{(3)} = -Mdz^1 \wedge dz^5 \wedge dz^6 - Mdz^3 \wedge dz^4 \wedge dz^5 - Mdz^2 \wedge dz^4 \wedge dz^6, \quad (7.3.44)$$

and dilaton

$$e^\Phi = V^{-3/2}(\tau). \quad (7.3.45)$$

These solutions describes three intersecting D5-branes.

	0	1	2	z^1	z^2	z^3	z^4	z^5	z^6	τ
D5 1	\times	\times	\times	\bullet	\times	\times	\times	\bullet	\bullet	
D5 2	\times	\times	\times	\times	\times	\bullet	\bullet	\bullet	\times	
D5 3	\times	\times	\times	\times	\bullet	\times	\bullet	\times	\bullet	

T-duality in the z^1 , z^5 and z^6 directions then gives the metric

$$ds_{10}^2 = V^{-3/2}(\tau)ds^2(\mathbb{R}^{1,2}) + V^{3/2}(\tau)(d\tau)^2 + V^{-1/2}(\tau)\left((dz^2)^2 + (dz^3)^2 + (dz^5)^2 + (dz^6)^2\right) + V^{1/2}(\tau)\left((dz^1)^2 + (dz^4)^2\right), \quad (7.3.46)$$

and RR fluxes

$$F_{(0)} = -M, \quad F_{(4)} = -Mdz^1 \wedge dz^3 \wedge dz^4 \wedge dz^6 - Mdz^1 \wedge dz^2 \wedge dz^4 \wedge dz^5, \quad (7.3.47)$$

and dilaton

$$e^\Phi = V^{-7/4}(\tau). \quad (7.3.48)$$

This solutions represents two intersecting D4-branes within a D8-brane:

	0	1	2	z^1	z^2	z^3	z^4	z^5	z^6	τ
D8	\times	\times	\times	\times	\times	\times	\times	\times	\times	
D4	\times	\times	\times	\bullet	\times	\bullet	\bullet	\times	\bullet	
D4	\times	\times	\times	\bullet	\bullet	\times	\bullet	\bullet	\times	

This is T-dual to two orthogonal D2-branes within a D6-brane, a standard 1/8 supersymmetric brane configuration.

7.3.5. T^3 fibred over T^4

This $Spin(7)$ holonomy metric is given by

$$ds_8^2 = V^6(\tau)(d\tau)^2 + V^3(\tau)\left((dz^4)^2 + (dz^5)^2 + (dz^6)^2 + (dz^7)^2\right) + V^{-2}(\tau)\left(dz^1 + M(z^5dz^4 + z^7dz^6)\right)^2 + V^{-2}(\tau)\left(dz^2 + M(z^6dz^4 - z^7dz^5)\right)^2 + V^{-2}(\tau)\left(dz^3 + M(z^7dz^4 + z^6dz^5)\right)^2. \quad (7.3.49)$$

The ten-dimensional metric is

$$ds_{10}^2 = ds^2(\mathbb{R}^{1,1}) + ds_8^2. \quad (7.3.50)$$

The H -flux and the dilaton are trivial,

$$H = 0, \quad \Phi = \text{constant}. \quad (7.3.51)$$

T-duality in the z^1 , z^2 and z^3 directions gives the metric

$$ds_8^2 = V^6(\tau)(d\tau)^2 + V^2\left((dz^1)^2 + (dz^2)^2 + (dz^3)^2\right) + V^3(\tau)\left((dz^4)^2 + (dz^5)^2 + (dz^6)^2 + (dz^7)^2\right). \quad (7.3.52)$$

and H -flux

$$H = -Mdz^1 \wedge dz^4 \wedge dz^5 - Mdz^1 \wedge dz^6 \wedge dz^7 - Mdz^2 \wedge dz^4 \wedge dz^6 - Mdz^3 \wedge dz^4 \wedge dz^7 - Mdz^3 \wedge dz^5 \wedge dz^6 + Mdz^2 \wedge dz^5 \wedge dz^7, \quad (7.3.53)$$

and dilaton

$$e^\Phi = V^3(\tau). \quad (7.3.54)$$

This solutions represents an intersection of five NS5-branes and one anti-NS5-brane

	0	1	z^1	z^2	z^3	z^4	z^5	z^6	z^7	τ
NS5 1	\times	\times	\bullet	\times	\times	\bullet	\bullet	\times	\times	
NS5 2	\times	\times	\bullet	\times	\times	\times	\times	\bullet	\bullet	
NS5 3	\times	\times	\times	\bullet	\times	\bullet	\times	\bullet	\times	
NS5 4	\times	\times	\times	\times	\bullet	\bullet	\times	\times	\bullet	
NS5 5	\times	\times	\times	\times	\bullet	\times	\bullet	\bullet	\times	
$\overline{\text{NS5}}$	\times	\times	\times	\bullet	\times	\times	\bullet	\times	\bullet	

This intersection of six NS5-branes is one of the cases given in [97] and preserves 1/16 supersymmetry .

S-duality then gives the metric

$$ds_{10}^2 = V^{-3}(\tau)ds^2(\mathbb{R}^{1,1}) + V^3(\tau)(d\tau)^2 + V^{-1}(\tau)\left((dz^1)^2 + (dz^2)^2 + (dz^3)^2\right) + \left((dz^4)^2 + (dz^5)^2 + (dz^6)^2 + (dz^7)^2\right). \quad (7.3.55)$$

and RR field strength

$$F_{(3)} = -Mdz^1 \wedge dz^4 \wedge dz^5 - Mdz^1 \wedge dz^6 \wedge dz^7 - Mdz^2 \wedge dz^4 \wedge dz^6 - Mdz^3 \wedge dz^4 \wedge dz^7 - Mdz^3 \wedge dz^5 \wedge dz^6 + Mdz^2 \wedge dz^5 \wedge dz^7, \quad (7.3.56)$$

and dilaton

$$e^\Phi = V^{-3}(\tau) \quad (7.3.57)$$

changing the NS5-branes to D5-branes:

	0	1	z^1	z^2	z^3	z^4	z^5	z^6	z^7	τ
D5 1	\times	\times	\bullet	\times	\times	\bullet	\bullet	\times	\times	
D5 2	\times	\times	\bullet	\times	\times	\times	\times	\bullet	\bullet	
D5 3	\times	\times	\times	\bullet	\times	\bullet	\times	\bullet	\times	
D5 4	\times	\times	\times	\times	\bullet	\bullet	\times	\times	\bullet	
D5 5	\times	\times	\times	\times	\bullet	\times	\bullet	\bullet	\times	
$\overline{\text{D5}}$	\times	\times	\times	\bullet	\times	\times	\bullet	\times	\bullet	

T-duality in the z^1, z^4 and z^5 directions then gives the metric

$$\begin{aligned}
ds_{10}^2 = & V^{-3}(\tau)(ds^2)(\mathbb{R}^{1,1}) + V^3(\tau)(d\tau)^2 + V(\tau)(dz^1)^2 + V^{-1}(\tau)\left((dz^2)^2 + (dz^3)^2\right) \\
& + \left((dz^4)^2 + (dz^5)^2 + (dz^6)^2 + (dz^7)^2\right).
\end{aligned} \tag{7.3.58}$$

and RR fluxes

$$F_{(0)} = -M, \tag{7.3.59}$$

$$\begin{aligned}
F_{(4)} = & -Mdz^4 \wedge dz^5 \wedge dz^6 \wedge dz^7 - Mdz^1 \wedge dz^2 \wedge dz^5 \wedge dz^6 - Mdz^1 \wedge dz^3 \wedge dz^5 \wedge dz^7 \\
& - Mdz^1 \wedge dz^3 \wedge dz^4 \wedge dz^6 + Mdz^1 \wedge dz^2 \wedge dz^4 \wedge dz^7,
\end{aligned} \tag{7.3.60}$$

and dilaton

$$e^\Phi = V^{-5/2}(\tau). \tag{7.3.61}$$

This then gives a 1/16 supersymmetric configuration of four D4-branes and one anti-D4-brane intersecting inside a D8-brane:

	0	1	z^1	z^2	z^3	z^4	z^5	z^6	z^7	τ
D8	\times	\times	\times	\times	\times	\times	\times	\times	\times	
D4 1	\times	\times	\times	\times	\times	\bullet	\bullet	\bullet	\bullet	
D4 2	\times	\times	\bullet	\bullet	\times	\times	\bullet	\bullet	\times	
D4 3	\times	\times	\bullet	\times	\bullet	\times	\bullet	\times	\bullet	
D4 4	\times	\times	\bullet	\times	\bullet	\bullet	\times	\bullet	\times	
$\overline{\text{D4}}$	\times	\times	\bullet	\bullet	\times	\bullet	\times	\times	\bullet	

7.3.6. S^1 fibred over T^6

This $SU(4)$ holonomy metric is given by

$$\begin{aligned}
ds_8^2 = & V^3(\tau)(d\tau)^2 + V(\tau)\left((dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2 + (dz^6)^2 + (dz^7)^2\right) \\
& + V^{-3}(\tau)\left(dz^1 + M(z^3dz^2 + z^5dz^4 + z^7dz^6)\right)^2.
\end{aligned} \tag{7.3.62}$$

The ten-dimensional metric is

$$ds_{10}^2 = ds^2(\mathbb{R}^{1,1}) + ds_8^2. \quad (7.3.63)$$

The H -flux and the dilaton are trivial,

$$H = 0, \quad \Phi = \text{constant}. \quad (7.3.64)$$

T-duality in the z^1 direction gives the metric

$$ds_8^2 = V^3(\tau) \left((d\tau)^2 + (dz^1)^2 \right) + V(\tau) \left((dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2 + (dz^6)^2 + (dz^7)^2 \right) \quad (7.3.65)$$

and H -flux

$$H = -Mdz^1 \wedge dz^2 \wedge dz^3 - Mdz^1 \wedge dz^4 \wedge dz^5 - Mdz^1 \wedge dz^6 \wedge dz^7, \quad (7.3.66)$$

and dilaton

$$e^\Phi = V^{3/2}(\tau). \quad (7.3.67)$$

These solutions describe three intersecting smeared NS5-branes:

	0	1	z^1	z^2	z^3	z^4	z^5	z^6	z^7	τ
NS5 1	×	×	•	•	•	×	×	×	×	
NS5 2	×	×	•	×	×	•	•	×	×	
NS5 3	×	×	•	×	×	×	×	•	•	

S-duality then gives the metric

$$ds_{10}^2 = V^{-3/2}(\tau) ds^2(\mathbb{R}^{1,1}) + V^{3/2}(\tau) \left((d\tau)^2 + (dz^1)^2 \right) + V^{-1/2}(\tau) \left((dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2 + (dz^6)^2 + (dz^7)^2 \right). \quad (7.3.68)$$

and RR field strength

$$F_{(3)} = -Mdz^1 \wedge dz^2 \wedge dz^3 - Mdz^1 \wedge dz^4 \wedge dz^5 - Mdz^1 \wedge dz^6 \wedge dz^7, \quad (7.3.69)$$

and dilaton

$$e^\Phi = V^{-3/2}(\tau), \quad (7.3.70)$$

changing the NS5-branes to D5-branes

	0	1	z^1	z^2	z^3	z^4	z^5	z^6	z^7	τ
D5 1	×	×	•	•	•	×	×	×	×	
D5 2	×	×	•	×	×	•	•	×	×	
D5 3	×	×	•	×	×	×	×	•	•	

T-duality in the z^1 , z^2 , and z^3 directions then gives the metric

$$ds_{10}^2 = V^{-3/2}(\tau)ds^2(\mathbb{R}^{1,1}) + V^{3/2}(\tau)(d\tau)^2 + V^{-3/2}(dz^1)^2 + V^{1/2}\left((dz^2)^2 + (dz^3)^2\right) + V^{-1/2}(\tau)\left((dz^4)^2 + (dz^5)^2 + (dz^6)^2 + (dz^7)^2\right) \quad (7.3.71)$$

and RR field strength

$$F_{(0)} = -M \quad (7.3.72)$$

$$F_{(4)} = -Mdz^2 \wedge dz^3 \wedge dz^4 \wedge dz^5 - Mdz^2 \wedge dz^3 \wedge dz^6 \wedge dz^7, \quad (7.3.73)$$

and dilaton

$$e^\Phi = V^{-7/4}(\tau). \quad (7.3.74)$$

This solutions represents two intersecting D4-branes within a D8-brane:

	0	1	z^1	z^2	z^3	z^4	z^5	z^6	z^7	τ
D8	\times	\times	\times	\times	\times	\times	\times	\times	\times	
D4	\times	\times	\times	\bullet	\bullet	\bullet	\bullet	\times	\times	
D4	\times	\times	\times	\bullet	\bullet	\times	\times	\bullet	\bullet	

This intersection of three NS5-branes preserves 1/8 supersymmetry [97].

7.4. T-folds fibred over line

7.4.1. T-fold from S^1 bundle over T^4 fibred over a line

Starting from the metric (7.3.2) and doing T-duality along the z^2 and z^4 directions gives T-fold background with the metric and B -field

$$ds_6^2 = V^2(\tau)(d\tau)^2 + \frac{V(\tau)}{V^3(\tau) + M^2[(z^3)^2 + (z^5)^2]}(dz^1)^2 + \frac{V^2(\tau)}{V^3(\tau) + M^2[(z^3)^2 + (z^5)^2]}(dz^2)^2 + \frac{M^2}{V(\tau)\left(V^3(\tau) + M^2[(z^3)^2 + (z^5)^2]\right)}(z^5dz^2 - z^3dz^4)^2 + \frac{V^2(\tau)}{V^3(\tau) + M^2[(z^3)^2 + (z^5)^2]}(dz^4)^2 + V(\tau)[(dz^3)^2 + (dz^5)^2], \quad (7.4.1)$$

$$B = \frac{M}{V^3(\tau) + M^2[(z^3)^2 + (z^5)^2]}(z^3dz^1 \wedge dz^2 + z^5dz^1 \wedge dz^4) \quad (7.4.2)$$

and the dilaton

$$e^{2\Phi} = \frac{V(\tau)}{V^3(\tau) + M^2[(z^3)^2 + (z^5)^2]}. \quad (7.4.3)$$

Since the S^1 bundle over T^4 is T-dual to the T^2 bundle over T^3 , doing T-duality in z^3 direction of the metric (7.3.13) will result in the same T-fold.

7.4.2. T-fold form T^3 bundle over T^3 fibred over a line

Starting from the metric (7.3.37) and doing T-duality in the z^4 direction gives T-fold background with the metric and B -field

$$\begin{aligned}
ds^2 = & V^3(\tau)(d\tau)^2 + \frac{1}{V(\tau)}(dz^1 + Mz^6dz^5)^2 + \frac{V^2(\tau)}{V^3(\tau) + M^2[(z^5)^2 + (z^6)^2]}[(dz^2)^2 + (dz^3)^2] \\
& + \frac{V(\tau)}{V^3(\tau) + M^2[(z^5)^2 + (z^6)^2]}(dz^4)^2 + \frac{M^2}{V(\tau)(V^3(\tau) + M^2[(z^5)^2 + (z^6)^2])}(z^5dz^2 + z^6dz^3)^2 \\
& + V^2(\tau)(dz^5)^2 + V^2(\tau)(dz^6)^2,
\end{aligned} \tag{7.4.4}$$

$$B = \frac{M}{V^3(\tau) + M^2[(z^5)^2 + (z^6)^2]}(z^5dz^3 \wedge dz^4 - z^6dz^2 \wedge dz^4). \tag{7.4.5}$$

and the dilaton

$$e^{2\Phi} = \frac{V(\tau)}{V^3(\tau) + M^2[(z^5)^2 + (z^6)^2]}. \tag{7.4.6}$$

7.4.3. T-fold form S^1 bundle over T^3 fibred over a line

Starting from the metric (7.3.62) and doing T-duality in the z^2, z^4 , and z^6 direction gives T-fold background with the metric and B -field

$$\begin{aligned}
ds^2 = & V^3(\tau)(d\tau^2) + V(\tau)((dz^3)^2 + (dz^5)^2 + (dz^7)^2) + \frac{V(\tau)}{V^4(\tau) + M^2[(z^3)^2 + (z^5)^2 + (z^7)^2]}(dz^1)^2 \\
& + \frac{V^3(\tau)}{V^4(\tau) + M^2[(z^3)^2 + (z^5)^2 + (z^7)^2]}[(dz^2)^2 + (dz^4)^2 + (dz^6)^2] \\
& + \frac{M^2}{V(\tau)(V^4(\tau) + M^2[(z^3)^2 + (z^5)^2 + (z^7)^2])}(z^5dz^2 - z^3dz^4)^2 \\
& + \frac{M^2}{V(\tau)(V^4(\tau) + M^2[(z^3)^2 + (z^5)^2 + (z^7)^2])}(z^7dz^2 - z^3dz^6)^2 \\
& + \frac{M^2}{V(\tau)(V^4(\tau) + M^2[(z^3)^2 + (z^5)^2 + (z^7)^2])}(z^7dz^4 - z^5dz^6)^2,
\end{aligned} \tag{7.4.7}$$

$$B = \frac{M}{V^4(\tau) + M^2[(z^3)^2 + (z^5)^2 + (z^7)^2]}(z^3dz^1 \wedge dz^2 + z^5dz^1 \wedge dz^4 + z^7dz^1 \wedge dz^6). \tag{7.4.8}$$

and the dilaton

$$e^{2\Phi} = \frac{V(\tau)}{V^4(\tau) + M^2 \left[(z^3)^2 + (z^5)^2 + (z^7)^2 \right]} \quad (7.4.9)$$

8. The Doubled Geometry of Nilmanifold Reductions

8.1. The doubled formalism

Dimensional reduction on a group G has been well-studied. Using a Scherk-Schwarz ansatz [79], the dependence of fields on the internal coordinates is given by a G transformation and leads to a consistent truncation to a lower dimensional field theory; see [122] for details of the Scherk-Schwarz dimensional reduction of the $\mathcal{N}=1$ supergravity action. However, for non-compact G this does not give a compactification, and considering the full theory on such a non-compact space gives a continuous spectrum without a mass gap. In [61] it was argued that in order to lift the Scherk-Schwarz dimensional reduction to a proper compactification of the full string theory, it is necessary to instead consider compactification on a compact space G/Γ with Γ a discrete subgroup. A discrete subgroup that gives a compact quotient is said to be cocompact, and so this construction in string theory is restricted to groups G that admit a cocompact subgroup.

Lie groups G that are nilpotent necessarily admit a cocompact subgroup and the resulting quotient $\mathcal{N} = G/\Gamma$ is referred to as a nilmanifold. A Lie group G is nilpotent if the Lie algebra \mathfrak{g} of G satisfies

$$[X_1, [X_2, [\dots [X_p, Y] \dots]] = 0 \quad (8.1.1)$$

for all $X_1, \dots, X_p, Y \in \mathfrak{g}$, for some integer p . For a nilpotent Lie group G , the smallest such p is known as the nilpotency class of G and \mathcal{G} is called a p -step nilpotent Lie group. A nilmanifold $\mathcal{N} = G/\Gamma$ is the compact space given by the quotient of a nilpotent group G by a cocompact discrete subgroup Γ . For a d -dimensional 2-step nilpotent Lie group G with centre of dimension n , the nilmanifold is a T^n bundle over T^m where $m = d - n$. For example, the 3-dimensional Heisenberg group G_3 has a centre of dimension $n = 1$ and the quotient is known as the nilfold and is an S^1 bundle over T^2 . Nilmanifolds are sometimes referred to as twisted tori.

The Lie algebra generators T_m satisfy an algebra

$$[T_m, T_n] = f_{mn}^p T_p \quad (8.1.2)$$

and the left-invariant one-forms P^m are

$$g^{-1}dg = P^m T_m, \quad (8.1.3)$$

Then the general left-invariant metric on G is

$$ds^2 = x_{mn} P^m P^n, \quad (8.1.4)$$

where x_{mn} is a constant symmetric matrix, and descends to a metric on the nilmanifold $\mathcal{N} = G/\Gamma$ if Γ is taken to act on the left. Here x_{mn} will be chosen as $x_{mn} = \delta_{mn}$ so the metric is

$$ds^2 = \delta_{mn} P^m P^n. \quad (8.1.5)$$

The Scherk-Schwarz reduction of the standard theory of gravity coupled to a 2-form gauge field B with field strength $H = dB$ and dilaton Φ in D dimensions

$$S = \int e^{-2\Phi} \left(R * 1 - \frac{1}{2} d\Phi \wedge * d\Phi - \frac{1}{2} H \wedge * H \right) \quad (8.1.6)$$

for a group G is given in [61, 122]. In the abelian case $G = U(1)^d$, this gives a theory with $2d$ gauge fields, d from the metric and d from the B -field, and gauge group $\mathcal{G} = U(1)^{2d}$. There are d^2 scalar fields taking values in the coset $O(d, d)/O(d) \times O(d)$ in addition to the dilaton and the field theory in $D - d$ dimensions has an $O(d, d)$ global symmetry. For non-abelian G , the reduction gives a gauging of this theory, with gauge group $G \ltimes U(1)^d$ with algebra

$$[T_m, T_n] = f_{mn}^p T_p, \quad [T_m, \tilde{T}^n] = f_{mp}^n \tilde{T}^p, \quad [\tilde{T}_m, \tilde{T}_n] = 0 \quad (8.1.7)$$

where the factor G generated by T_m comes from the isometries of the group manifold generated by left-invariant vector fields and the abelian factor generated by \tilde{T}^m comes from the B -field symmetries. This can be thought of as a gauging of a $2d$ -dimensional subgroup of $O(d, d)$. In addition, there is now a potential for the $d^2 + 1$ scalar fields, and for generic groups G , Minkowski space in $D - d$ dimensions will not be a solution. If G admits a cocompact subgroup Γ , compactification on G/Γ gives the same $D - d$ dimensional effective field theory but lifts to a compactification of the full supergravity or string theory. A nilmanifold is a T^n bundle over T^m , and this reduction can be regarded as compactification on T^n followed by a reduction with duality twists on T^m , with a monodromy round each circle in T^m that is a large diffeomorphism of T^n , in $SL(n, \mathbb{Z})$.

For superstring theory, $D = 10$ and (8.1.6) is the action for the massless graviton, dilaton and B -field of the type I, II or heterotic superstring. In this case, there is an interesting set of nilpotent groups G such that the resulting supergravity theory in $10 - d$ dimensions has no

Minkowski vacuum but has supersymmetric domain wall solutions [96]. These each then lift to 10-dimensional solutions on $\mathcal{B} \times \mathbb{R}^{1,r}$ where $\mathbb{R}^{1,r}$ is $(r+1)$ -dimensional Minkowski space with $r = 8 - d$ and \mathcal{B} is $G \times \mathbb{R}$ or $\mathcal{N} \times \mathbb{R}$ [96]. Remarkably, as the domain wall was supersymmetric, the metric on \mathcal{B} must have special holonomy, with the holonomy group determined by the number of supersymmetries [96]. For example for the Heisenberg group with \mathcal{N} the nilfold, the four-dimensional space \mathcal{B} is hyperkähler. These cases were further analysed in [3] and we will focus on these examples in this paper. The 10-dimensional space $\mathcal{B} \times \mathbb{R}^{1,r}$ then incorporates the nilmanifold into a string solution.

In [3], the T-duals of these solutions were considered. In each case, a set of T-dualities took the nilmanifold to a torus T^d with H -flux. These T-dualities acting on the special holonomy domain wall solution $\mathcal{B} \times \mathbb{R}^{1,r}$ resulted in a configuration of intersecting NS5-branes [97], preserving exactly the same amount of supersymmetry. Other T-dualities took these solutions to non-geometric backgrounds, including T-folds and spaces with R -flux.

These duals can be thought of as follows. If the dimension of the centre of G is n , the nilmanifold can be regarded as a T^n bundle over T^m , but in each of the cases we will consider it can also be regarded as a T^r bundle over T^s for some $r > n$ with $s = d - r$, and we will take the maximal choice of r . For example, the 3-dimensional nilfold can be regarded as a T^2 bundle over S^1 . The original nilmanifold compactification can be regarded as compactification on T^r followed by a reduction with duality twists on T^s , with a monodromy round each circle in T^s that is in $SL(r, \mathbb{Z})$. There is an $O(r, r, \mathbb{Z})$ group of T-dualities acting on the T^r fibres, and an $O(r, r, \mathbb{Z})$ transformation will take this to a twisted reduction in which the monodromy round each circle in T^s is now a transformation in $O(r, r, \mathbb{Z})$. On T-dualising to a torus T^d with H -flux, the monodromies all consist of shifts of the B -field. Other T-dualities will take it to cases in which the monodromies are T-dualities in $O(r, r, \mathbb{Z})$, giving a T-fold [65].

These duals can be represented in a doubled formalism, in which the torus fibres T^r are replaced with fibres that are given by a doubled torus T^{2r} , with an extra r coordinates conjugate to the string winding modes on T^r . The $O(r, r, \mathbb{Z})$ monodromies act geometrically as diffeomorphisms of the doubled torus T^{2r} , so that a geometric T^{2r} bundle over T^s is obtained. This doubled solution can be thought of as a universal space containing all T-duals: different T-dual solutions are obtained by choosing different polarisations, that is by choosing different splittings of the $2d$ coordinates into d coordinates that are to be regarded as the coordinates of a spacetime and d coordinates that are to be regarded as conjugate to winding numbers. T-duality can then be thought of as changing the polarisation [61]. This is worked out in detail for the nilfold in [61].

T-duality on the T^s base is less straightforward and gives results that are not geometric even locally and are sometimes said to have R -flux. The metric of the nilmanifold \mathcal{N} depends explicitly on the coordinates x^i of the T^s base. T-duality takes the coordinate x^i of the i 'th

circle to the coordinate \tilde{x}_i of the dual circle, and so takes the original x -dependent solution to one dependent on the dual coordinate \tilde{x}_i [64, 107]. The monodromy round the original circle transforms to monodromy round the dual circle [64, 107]. This was shown to give the correct T-duality in asymmetric orbifold limits in [107]. In such cases, the explicit dependence of the solution on \tilde{x}_i in general means there is no way of extracting a conventional background from the doubled one.

In [64], a doubled formulation of all these dualities was proposed. Instead of just doubling the torus fibres, all d dimensions were doubled to give a space which is a $2d$ dimensional nilmanifold. The Lie algebra (8.1.7) is that of $\mathcal{G} = G \ltimes \mathbb{R}^d$, with group manifold given by the cotangent bundle T^*G of G . This is itself a nilpotent group, and taking the quotient by a cocompact subgroup $\hat{\Gamma}$ gives a compact nilmanifold $\mathcal{M} = \mathcal{G}/\hat{\Gamma}$. This is sometimes referred to as the doubled twisted torus. The different choices of polarisation select the different dual backgrounds. This was checked in detail in [64] for the nilfold. Our purpose here is to extend that to each of the nilmanifolds of [96], constructing the doubled geometry on $\mathcal{M} = \mathcal{G}/\hat{\Gamma}$, and extracting the various dual solutions. Then the special holonomy space $\mathcal{B} = \mathcal{N} \times \mathbb{R}$ is doubled to $\hat{\mathcal{B}} = \mathcal{M} \times \mathbb{R}$ – there is no motivation to double the non-compact direction \mathbb{R} as there are no winding modes and there is no T-duality for this direction.

8.2. The doubled nilmanifold

The doubled group is $\mathcal{G} = G \ltimes \mathbb{R}^d$ with Lie algebra (8.1.7), which we write as

$$[T_M, T_N] = t_{MN}{}^P T_P. \quad (8.2.1)$$

where $M, N = 1, \dots, 2d$. Note that it preserves a constant $O(d, d)$ -invariant metric η_{MN} and so is a subgroup of $O(d, d)$.

On \mathcal{G} , there are two sets of globally-defined vector fields, the left-invariant vector fields, K_M , and the right-invariant vector fields, \tilde{K}_M . The left-invariant vector fields generate the right action \mathcal{G}_R , while the right-invariant vector fields generate left action \mathcal{G}_L . The left-invariant one-forms \mathcal{P}^M , dual to left-invariant vector field K_M , can be written as

$$g^{-1}dg = \mathcal{P}^M T_M \quad (8.2.2)$$

and satisfy the Maurer-Cartan equations

$$d\mathcal{P}^M + \frac{1}{2}t_{NP}{}^M \mathcal{P}^N \wedge \mathcal{P}^P = 0. \quad (8.2.3)$$

We introduce a left-invariant metric and three-form on \mathcal{G} constructed from the left-invariant

one-forms, given by

$$ds^2 = \mathcal{M}_{MN} \mathcal{P}^M \otimes \mathcal{P}^N, \quad (8.2.4)$$

$$\mathcal{K} = \frac{1}{3!} t_{MNP} \mathcal{P}^M \wedge \mathcal{P}^N \wedge \mathcal{P}^P, \quad (8.2.5)$$

where \mathcal{M}_{MN} is a constant symmetric positive definite matrix and $t_{MNP} = t_{MN}{}^Q \eta_{QP}$, and t_{MNP} is totally antisymmetric. The matrix \mathcal{M}_{MN} parameterizes the coset space $O(d, d)/O(d) \times O(d)$ and represents the moduli of the internal space which become scalar fields in $10 - d$ dimensions on compactification.

Taking the quotient by a cocompact subgroup Γ gives $\mathcal{M} = \mathcal{G}/\Gamma$. Taking Γ to have a left action $g \rightarrow \gamma g$, then the left-invariant 1-forms \mathcal{P}^M , the metric (8.2.4) and 3-form (8.2.5) descend to well-defined 1-forms, metric and 3-form on the quotient \mathcal{M} .

8.2.1. Sigma model

In [64], a doubled sigma model is formulated for maps from a 2-dimensional world-sheet Σ to \mathcal{M} . These maps pull back the one-forms \mathcal{P}^M to one-forms $\hat{\mathcal{P}}^M$ on Σ . Introducing a 3-dimensional space V with boundary $\partial V = \Sigma$ and extending the maps to V , the sigma model is given by

$$S_{\mathcal{M}} = \frac{1}{4} \oint_{\Sigma} \mathcal{M}_{MN} \hat{\mathcal{P}}^M \wedge * \hat{\mathcal{P}}^N + \frac{1}{2} \int_V \hat{\mathcal{K}}, \quad (8.2.6)$$

where $\hat{\mathcal{K}}$ is the pull-back of \mathcal{K} to V and $*$ is the Hodge dual on Σ . This theory is subjected to the constraint

$$\hat{\mathcal{P}}^M = \eta^{MP} \mathcal{M}_{PN} * \hat{\mathcal{P}}^N, \quad (8.2.7)$$

which implies that half the degrees of freedom are right-moving on Σ and half are left-moving. This constraint can be imposed in a number of ways; in [64] it was imposed by choosing a polarisation and then gauging.

8.2.2. Polarisation

A polarisation is a projector that projects the tangent space of \mathcal{G} into a physical subspace which is to be tangent to the spacetime. Different choices of polarisation select different dual backgrounds. We introduce a projector $\Pi^m{}_M$ (with $m, n = 1, \dots, d$) mapping onto a d -dimensional subspace of the $2d$ dimensional tangent space of \mathcal{M} , which is totally null (maximally isotropic) with respect to the metric η_{MN} , i.e.

$$\eta^{MN} \Pi^m{}_M \Pi^n{}_N = 0. \quad (8.2.8)$$

Introducing such a projector at the identity element of the group manifold then defines one everywhere; in a natural basis, the projector is constant over the manifold. The complementary projector $1 - \Pi$ is denoted by $\tilde{\Pi}_{mM}$. The polarisation splits the tangent space into two halves, and we will consider the case in which the frame components Π^m_M are locally constant, i.e. there is a constant matrix $\Pi^m_{(\alpha)M}$ in each patch U_α of \mathcal{G} , but there can be different polarisation matrices in different patches.

A vector V^M is then projected into

$$V^m = \Pi^m_M V^M, \quad V_m = \tilde{\Pi}_{mM} V^M. \quad (8.2.9)$$

It is useful to introduce the notation

$$V^{\hat{M}} = \begin{pmatrix} V^m \\ V_m \end{pmatrix} = \Theta^{\hat{M}}_N V^N, \quad (8.2.10)$$

where

$$\Theta^{\hat{M}}_N = \begin{pmatrix} \Pi^m_N \\ \tilde{\Pi}_{mN} \end{pmatrix}, \quad (8.2.11)$$

so that the polarisation can be seen as choosing a basis for the tangent space.

The polarisation projects the generators T_M into

$$Z_m \equiv \tilde{\Pi}_{mM} \eta^{MN} T_N, \quad X^m \equiv \Pi^m_M \eta^{MN} T_N. \quad (8.2.12)$$

The Lie algebra (8.2.1) will now take the form

$$[Z_m, Z_n] = f_{mn}^p Z_p + K_{mnp} X^p, \quad [X^m, X^n] = Q_p^{mn} X^p + R^{mnp} Z_p, \quad (8.2.13)$$

$$[X^m, Z_n] = f_{np}^m X^p - Q_n^{mp} Z_p, \quad (8.2.14)$$

for tensors $K_{mnp}, f_{mn}^p, Q_p^{mn}, R^{mnp}$, sometimes referred to as fluxes, obtained by projecting the structure constants t_{MN}^P with $\Pi, \tilde{\Pi}$. Different choices of polarisation will give different forms for these fluxes.

8.2.3. Recovering the physical space

For a given polarisation, we introduce coordinates $\mathbb{X}^M = (x^m, \tilde{x}_m)$ by writing a general group element as

$$g = \tilde{h} h, \quad (8.2.15)$$

where

$$h = \exp(x^m Z_m), \quad \tilde{h} = \exp(\tilde{x}_m X^m). \quad (8.2.16)$$

The action of $h(x)$ on the generators $T_M = (Z_m, X^m)$ defines an x -dependent vielbein $\mathcal{V}_M^N(x)$ by

$$h^{-1}T_M h = \mathcal{V}_M^N T_N. \quad (8.2.17)$$

Then defining

$$\Phi = \Phi^M T_M = \tilde{h}^{-1} d\tilde{h} + dh h^{-1}, \quad (8.2.18)$$

the left-invariant forms can be written as

$$\mathcal{P} = \mathcal{P}^M T_M = \Phi^M \mathcal{V}_M^N(x) T_N. \quad (8.2.19)$$

We define a generalized metric which depends on the coordinates x^i only by

$$\mathcal{H}_{MN}(x) = \mathcal{M}_{PQ} \mathcal{V}_P^M \mathcal{V}_N^Q. \quad (8.2.20)$$

With a polarisation tensor $\Theta_{\hat{M}}^{\hat{M}}$, we can define

$$\mathcal{H}_{\hat{M}\hat{N}}(x) = \Theta_{\hat{M}}^{\hat{M}} \mathcal{H}_{MN}(x) \Theta_{\hat{N}}^{\hat{N}}, \quad (8.2.21)$$

whose components define a metric g_{mn} and B -field B_{mn} by

$$\mathcal{H}_{\hat{M}\hat{N}}(x) = \begin{pmatrix} g_{mn} + B_{mp} g^{pq} B_{qn} & B_{mp} g^{pn} \\ g^{mp} B_{np} & g^{mn} \end{pmatrix}. \quad (8.2.22)$$

The metric $g_{mn}(x)$ and B -field $B_{mn}(x)$ depend only on the x^i coordinates. The physical metric is given by [64]

$$ds^2 = g_{mn}(x) r^m r^n. \quad (8.2.23)$$

The physical H -field strength is given by [64]

$$H = dB - \frac{1}{2} d(r^m \wedge \tilde{q}_m) + \frac{1}{2} \mathcal{K} \quad (8.2.24)$$

where

$$\tilde{h}^{-1} d\tilde{h} = \tilde{\ell}^m Z_m + \tilde{\ell}_m X^m, \quad dh h^{-1} = r^m Z_m + r_m X^m, \quad (8.2.25)$$

and

$$\tilde{q}_m = r_m + \tilde{\ell}_m. \quad (8.2.26)$$

If the R -tensor R^{mnp} vanishes, the X^m generate a subgroup $\tilde{G} \subset \mathcal{G}$

$$[X^m, X^n] = Q_p^{mn} X^p. \quad (8.2.27)$$

Then the generalized metric $\mathcal{H}_{MN}(x)$, the metric g_{mn} and B -field B_{mn} and H are all invariant under the left action of \tilde{G} . Then the reduction to the physical subspace is obtained by taking a quotient by the left action of \tilde{G} . In the sigma model, this is achieved by gauging the action of \tilde{G} . On eliminating the worldsheet gauge fields, one obtains a standard sigma model whose target space \mathcal{G}/\tilde{G} has coordinates x and the metric and H given above.

If the R -tensor is not zero, then the model will depend explicitly on both x and \tilde{x} . In this case, the expressions above give formal expressions for the metric and H that depend on both x and \tilde{x} , so that there is no interpretation in terms of a conventional d -dimensional spacetime.

8.2.4. Quotienting by the discrete group

The above structure was derived for the doubled group manifold \mathcal{G} . In the case of a vanishing R -tensor, the result is a conventional sigma model on \mathcal{G}/\tilde{G} . The next step is to consider the structure for the nilmanifold $\mathcal{M} = \mathcal{G}/\Gamma$.

A conventional background is obtained from double geometry by gauging \tilde{X}^m . The types of string theory background can be classified into three categories [63, 64]:

Type I: Geometric Backgrounds

If the generators X^m generate a subgroup $\tilde{G} \subset \mathcal{G}$ and this subgroup is preserved by Γ , so that

$$\gamma k \gamma^{-1} = k', \quad (8.2.28)$$

where $k, k' \in \tilde{G}$, $\gamma \in \Gamma$, then the quotient space \mathcal{M}/\tilde{G} is well-defined and gives a global description of a conventional geometric background.

Type II: T-fold Backgrounds

If the generators X^m generate a subgroup \tilde{G} but this subgroup is not preserved by Γ , then the quotient space \mathcal{M}/\tilde{G} is not well-defined. The conventional background can be recovered locally as a patch of \mathcal{G}/\tilde{G} . These patches are then glued together together with T-duality transition functions, resulting in a T-fold.

Type III: R -flux background

If the generators \tilde{X}^m do not close to form sub-algebra, then a conventional d -dimensional background cannot be recovered even locally, as there is dependence on both x^i and \tilde{x}_i . Such a background is sometimes called a R -flux background.

8.3. The nilfold example

The doubled formalism was developed for the 3-dimensional nilfold in [64]. Here we summarize the results and refer to [64] for the details. For this example, the nilpotent Lie group G is

taken to be the three-dimensional Heisenberg group with Lie algebra

$$[T_x, T_z] = mT_y \quad [T_y, T_z] = 0 \quad [T_x, T_y] = 0 \quad (8.3.1)$$

with m an integer. The quotient of G by a cocompact subgroup Γ gives the nilfold $\mathcal{N} = G/\Gamma$.

Then the corresponding 6-dimensional group $\mathcal{G} = G \ltimes \mathbb{R}^3$ has a Lie algebra whose only non-zero commutators are

$$[T_x, T_z] = mT_y \quad [T_x, \tilde{T}^y] = mX^z \quad [T_z, \tilde{T}^y] = -mX^x \quad (8.3.2)$$

8.3.1. The nilfold

Choosing the polarisation $\Theta = 1$, the algebra (8.3.2) is

$$[Z_x, Z_z] = mZ_y, \quad [Z_x, X^y] = mX^z, \quad [Z_z, X^y] = -mX^x. \quad (8.3.3)$$

The X^m generate an abelian subgroup \tilde{G} , and taking the quotient by \tilde{G} gives the nilfold $\mathcal{N} = G/\Gamma$ with metric

$$ds_{\mathcal{N}}^2 = dx^2 + (dy - mxdz)^2 + dz^2, \quad (8.3.4)$$

and $H = 0$. This can be viewed as a T^2 bundle over S^1 where the T^2 has coordinates y, z and the S^1 has coordinate x .

8.3.2. T^3 with H -flux

Choosing the polarisation

$$\Theta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

the algebra becomes

$$[Z_x, Z_z] = mX^y, \quad [Z_x, Z_y] = mX^z, \quad [Z_z, Z_y] = -mX^x,$$

where all other commutators vanish. The X^i generate an abelian subgroup $\tilde{G} = \mathbb{R}^3$. Taking the quotient gives the 3-torus with H -flux given by an integer m . The metric and 3-form flux H are

$$ds_{T^3}^2 = dx^2 + dy^2 + dz^2, \quad H = m dx \wedge dy \wedge dz. \quad (8.3.5)$$

The 2-form potential B with $H = dB$ can be chosen as

$$B = mxdy \wedge dz. \quad (8.3.6)$$

8.3.3. T-fold

The polarisation tensor for the T-fold is given by

$$\Theta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The algebra then becomes

$$[Z_x, X^z] = mZ_y, \quad [Z_x, X^y] = mZ_z, \quad [X^z, X^y] = -mX^x,$$

where all other commutators vanish. Now X^x, X^y, X^z generate a subgroup \tilde{G} which is isomorphic to the Heisenberg group.

The T-fold has metric and B -field given by

$$ds_{T-Fold}^2 = dx^2 + \frac{1}{1 + (mx)^2}(dy^2 + dz^2), \quad B = -\frac{mx}{1 + (mx)^2}dy \wedge dz, \quad (8.3.7)$$

which changes by a T-duality under $x \rightarrow x + 1$, and so has a T-duality monodromy in the x direction.

8.3.4. R -flux

In this case, the polarisation Θ is

$$\Theta = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The gauge algebra is now

$$[X^x, X^z] = -mZ_y, \quad [X^x, X^y] = -mZ_z, \quad [X^z, X^y] = mZ_x, \quad (8.3.8)$$

where all other commutators vanish. The X^i do not generate a subgroup, so there is no way of relating to a conventional theory on a 3-dimensional space, even locally.

8.4. Higher dimensional nilmanifolds

8.4.1. S^1 bundle over T^4

For this example, the five-dimensional nilpotent Lie group has non-vanishing commutators

$$[T_2, T_3] = mT_1, \quad [T_4, T_5] = mT_1. \quad (8.4.1)$$

The corresponding ten-dimensional group has a Lie algebra whose only non-zero commutators are

$$\begin{aligned} [T_2, T_3] &= mT_1, & [T_4, T_5] &= mT_1, \\ [\tilde{T}^1, T_2] &= m\tilde{T}^3, & [\tilde{T}^1, T_3] &= -m\tilde{T}^2, \\ [\tilde{T}^1, T_4] &= m\tilde{T}^5, & [\tilde{T}^1, T_5] &= -m\tilde{T}^4. \end{aligned} \quad (8.4.2)$$

The left-invariant one-form are given

$$\begin{aligned} P^1 &= dz^1 + mz^3dz^2 + mz^5dz^4, & Q_1 &= d\tilde{z}_1, \\ P^2 &= dz^2, & Q_2 &= d\tilde{z}_2 - mz^3d\tilde{z}_1, \\ P^3 &= dz^3, & Q_3 &= d\tilde{z}_3 + mz^2d\tilde{z}_1, \\ P^4 &= dz^4, & Q_4 &= d\tilde{z}_4 - mz^5d\tilde{z}_1, \\ P^5 &= dz^5, & Q_5 &= d\tilde{z}_5 + mz^4d\tilde{z}_1. \end{aligned} \quad (8.4.3)$$

Choosing the polarisation $\Theta = 1$, the algebra (8.4.2) is

$$\begin{aligned} [Z_2, Z_3] &= mZ_1, & [Z_4, Z_5] &= mZ_1, \\ [X^1, Z_2] &= mX^3, & [X^1, Z_3] &= -mX^2, \\ [X^1, Z_4] &= mX^5, & [X^1, Z_5] &= -mX^4. \end{aligned} \quad (8.4.4)$$

The left-invariant one-forms in this polarization are given The left-invariant one-form are given

$$\begin{aligned}
P^1 &= dz^1 + mz^3dz^2 + mz^5dz^4, & Q_1 &= d\tilde{z}_1, \\
P^2 &= dz^2, & Q_2 &= d\tilde{z}_2 - mz^3d\tilde{z}_1, \\
P^3 &= dz^3, & Q_3 &= d\tilde{z}_3 + mz^2d\tilde{z}_1, \\
P^4 &= dz^4, & Q_4 &= d\tilde{z}_4 - mz^5d\tilde{z}_1, \\
P^5 &= dz^5, & Q_5 &= d\tilde{z}_5 + mz^4d\tilde{z}_1.
\end{aligned} \tag{8.4.5}$$

Let $h = \prod \exp(z^m T_m)$ and $\tilde{h} = \prod \exp(\tilde{z}_m \tilde{T}^m)$, then the one-form (8.2.18) is

$$\Phi = \Phi^M{}_N T^N = \tilde{h}^{-1} d\tilde{h} + dh h^{-1}. \tag{8.4.6}$$

In this case, $dh h^{-1}$ and $\tilde{h}^{-1} d\tilde{h}$ are

$$dh h^{-1} = (dz^1 + mz^2dz^3 + mz^4dz^5)T_1 + (dz^2)T_2 + (dz^3)T_3 + (dz^4)T_4 + (dz^5)T_5, \tag{8.4.7}$$

$$\tilde{h}^{-1} d\tilde{h} = (d\tilde{z}_1)\tilde{T}^1 + (d\tilde{z}_2)\tilde{T}^2 + (d\tilde{z}_3)\tilde{T}^3 + (d\tilde{z}_4)\tilde{T}^4 + (d\tilde{z}_5)\tilde{T}^5. \tag{8.4.8}$$

The one-form Φ is given

$$\begin{aligned}
\Phi^1 &= dz^1 + mz^2dz^3 + mz^4dz^5, & \tilde{\Phi}_1 &= d\tilde{z}_1, \\
\Phi^2 &= dz^2, & \tilde{\Phi}_2 &= d\tilde{z}_2, \\
\Phi^3 &= dz^3, & \tilde{\Phi}_3 &= d\tilde{z}_3, \\
\Phi^4 &= dz^4, & \tilde{\Phi}_4 &= d\tilde{z}_4, \\
\Phi^5 &= dz^5, & \tilde{\Phi}_5 &= d\tilde{z}_5.
\end{aligned} \tag{8.4.9}$$

From the equation (8.2.19), one gets

$$\mathcal{V}^M{}_N = \begin{pmatrix} 1 & mz^3 & -mz^2 & mz^5 & -mz^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -mz^3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & mz^2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -mz^5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & mz^4 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{8.4.10}$$

From $\mathcal{V}^M{}_N$, one obtains the generalized metric \mathcal{H}_{MN} . The X^m generates an abelian subgroup

\tilde{G} and taking the quotient by \tilde{G} gives the S^1 bundle over T^4 with metric

$$ds^2 = \left(dz^1 + m(z^3 dz^2 + z^5 dz^4) \right)^2 + (dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2. \quad (8.4.11)$$

The H -flux is given in (8.2.24), which is

$$H = dB - \frac{1}{2}d(r^m \wedge \tilde{q}_m) + \frac{1}{2}\mathcal{K}. \quad (8.4.12)$$

In this case, \mathcal{K} and $r^m \wedge \tilde{q}_m$ is

$$\mathcal{K} = m\tilde{z}_1 \wedge dz^2 \wedge dz^3 + m d\tilde{z}_1 \wedge dz^4 \wedge dz^5. \quad (8.4.13)$$

$$r^m \wedge \tilde{q}_m = \left(dz^1 + mz^2 dz^3 + mz^4 dz^5 \right) \wedge d\tilde{z}_1 + dz^2 \wedge d\tilde{z}_2 + dz^3 \wedge d\tilde{z}_3 + dz^4 \wedge d\tilde{z}_4 + dz^5 \wedge d\tilde{z}_5, \quad (8.4.14)$$

and $d(r^m \wedge \tilde{q}_m)$ is

$$d(r^m \wedge \tilde{q}_m) = m\tilde{z}_1 \wedge dz^2 \wedge dz^3 + m d\tilde{z}_1 \wedge dz^4 \wedge dz^5. \quad (8.4.15)$$

In this case, $H = 0$. This construction will work in the following cases.

While choosing the polarization

$$\Theta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (8.4.16)$$

the algebra (8.4.2) is

$$\begin{aligned} [Z_2, Z_3] &= mX^1, & [Z_4, Z_5] &= mX^1, \\ [Z_1, Z_2] &= mX^3, & [Z_1, Z_3] &= -mX^2, \\ [Z_1, Z_4] &= mX^5, & [Z_1, Z_5] &= -mX^4. \end{aligned} \quad (8.4.17)$$

The X^m generates an abelian subgroup \tilde{G} . Taking the quotient gives the 5-torus with H -flux

$$ds^2 = (dz^1)^2 + (dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2, \quad (8.4.18)$$

$$H = -mdz^1 \wedge dz^2 \wedge dz^3 - mdz^1 \wedge dz^4 \wedge dz^5 \quad (8.4.19)$$

While choosing the polarization

$$\Theta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (8.4.20)$$

the algebra (8.4.2) is

$$\begin{aligned} [X^2, Z_3] &= mZ_1, & [X^4, Z_5] &= mZ_1, \\ [X^1, X^2] &= mX^3, & [X^1, Z_3] &= -mZ_2, \\ [X^1, X^4] &= mX^5, & [X^1, Z_5] &= -mZ_4. \end{aligned} \quad (8.4.21)$$

The X^m generates a subgroup \tilde{G} . Taking the quotient gives the T-fold with the metric and B -field

$$\begin{aligned} ds^2 &= \frac{1}{1 + m^2[(z^3)^2 + (z^5)^2]} \left((dz^1)^2 + (dz^2)^2 + (dz^4)^2 \right) \\ &\quad + \frac{1}{1 + m^2[(z^3)^2 + (z^5)^2]} \left(mz^5 dz^2 - mz^3 dz^4 \right)^2 + (dz^3)^2 + (dz^5)^2, \end{aligned} \quad (8.4.22)$$

$$B = \frac{m}{1 + m^2[(z^3)^2 + (z^5)^2]} \left(z^3 dz^1 \wedge dz^2 + z^5 dz^1 \wedge dz^4 \right). \quad (8.4.23)$$

8.4.2. T^2 bundle over T^3

For this example, the five-dimensional nilpotent Lie group has non-vanishing commutators

$$[T_3, T_4] = mT_1, \quad [T_3, T_5] = mT_2. \quad (8.4.24)$$

The corresponding ten-dimensional group has a Lie algebra whose only non-zero commutators are

$$\begin{aligned} [T_3, T_4] &= mT_1, & [T_3, T_5] &= mT_2, \\ [\tilde{T}^1, T_3] &= m\tilde{T}^4, & [\tilde{T}^1, T_4] &= -m\tilde{T}^3, \\ [\tilde{T}^2, T_3] &= m\tilde{T}^5, & [\tilde{T}^2, T_5] &= -m\tilde{T}^3. \end{aligned} \quad (8.4.25)$$

The left-invariant one-forms are given

$$\begin{aligned} P^1 &= dz^1 + mz^4 dz^3, & Q_1 &= d\tilde{z}_1, \\ P^2 &= dz^2 + mz^5 dz^3, & Q_2 &= d\tilde{z}_2, \\ P^3 &= dz^3, & Q_3 &= d\tilde{z}_3 + mz^4 d\tilde{z}_1 - mz^5 d\tilde{z}_2, \\ P^4 &= dz^4, & Q_4 &= d\tilde{z}_4 + mz^3 d\tilde{z}_1, \\ P^5 &= dz^5, & Q_5 &= d\tilde{z}_5 + mz^3 d\tilde{z}_2. \end{aligned} \quad (8.4.26)$$

Choosing the polarisation $\Theta = 1$, the algebra (8.4.25) is

$$\begin{aligned} [Z_3, Z_4] &= mZ_1, & [Z_3, Z_5] &= mZ_2, \\ [X^1, Z_3] &= mX^4, & [X^1, Z_4] &= -mX^3, \\ [X^2, Z_3] &= mX^5, & [X^2, Z_5] &= -mX^3. \end{aligned} \quad (8.4.27)$$

The X^m generates an abelian subgroup \tilde{G} and taking the quotient by \tilde{G} gives the T^2 bundle over T^3 with metric

$$ds^2 = \left(dz^1 + mz^4 dz^3\right)^2 + \left(dz^2 + mz^5 dz^3\right)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2. \quad (8.4.28)$$

Choosing the polarization

$$\Theta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (8.4.29)$$

the algebra (8.4.25) is

$$\begin{aligned}
[Z_3, Z_4] &= mX^1, & [Z_3, Z_5] &= mZ_2, \\
[Z_1, Z_3] &= mX^4, & [Z_1, Z_4] &= -mX^3, \\
[X^2, Z_3] &= mX^5, & [X^2, Z_5] &= -mX^3.
\end{aligned} \tag{8.4.30}$$

The X^m generates an abelian subgroup \tilde{G} . Taking the quotient gives the S^1 bundle over T^4 with H -flux.

$$ds^2 = (dz^1)^2 + \left(dz^2 + mz^5 dz^3\right)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2, \tag{8.4.31}$$

$$H = -mdz^1 \wedge dz^3 \wedge dz^4 \tag{8.4.32}$$

Choosing the polarization

$$\Theta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{8.4.33}$$

the algebra (8.4.25) is

$$\begin{aligned}
[Z_3, Z_4] &= mX^1, & [Z_3, Z_5] &= mX^2, \\
[Z_1, Z_3] &= mX^4, & [Z_1, Z_4] &= -mX^3, \\
[Z_2, Z_3] &= mX^5, & [Z_2, Z_5] &= -mX^3.
\end{aligned} \tag{8.4.34}$$

The X^m generates an abelian subgroup \tilde{G} . Taking the quotient gives the T^5 with H -flux.

$$ds^2 = (dz^1)^2 + (dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2, \tag{8.4.35}$$

$$H = -mdz^1 \wedge dz^3 \wedge dz^4 - mdz^2 \wedge dz^3 \wedge dz^5. \tag{8.4.36}$$

Choosing the polarization

$$\Theta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (8.4.37)$$

the algebra (8.4.25) is

$$\begin{aligned} [X^3, Z_4] &= mZ_1, & [X^3, Z_5] &= mZ_2, \\ [X^1, X^3] &= mX^4, & [X^1, Z_4] &= -mZ_3, \\ [X^2, X^3] &= mX^5, & [X^2, Z_5] &= -mZ_3. \end{aligned} \quad (8.4.38)$$

The X^m generates a subgroup \tilde{G} . Taking the quotient gives the T-fold with the metric and B -field

$$ds^2 = \frac{1}{1 + m^2 \left[(z^4)^2 + (z^5)^2 \right]} \left[(dz^1)^2 + (dz^2)^2 + (dz^3)^2 \right] + \frac{1}{1 + m^2 \left[(z^4)^2 + (z^5)^2 \right]} \left(z^5 dz^1 - z^4 dz^2 \right)^2 + (dz^4)^2 + (dz^5)^2, \quad (8.4.39)$$

$$B = \frac{m}{1 + m^2 \left[(z^4)^2 + (z^5)^2 \right]} \left(z^4 dz^1 \wedge dz^3 + z^5 dz^2 \wedge dz^3 \right). \quad (8.4.40)$$

8.4.3. T^2 bundle over T^4

Consider the six-dimensional nilpotent Lie algebra whose only non-vanishing commutators are

$$\begin{aligned} [T_3, T_4] &= mT_1, & [T_3, T_5] &= mT_2, \\ [T_5, T_6] &= mT_1, & [T_4, T_6] &= -mT_2. \end{aligned} \quad (8.4.41)$$

The corresponding twelve-dimensional group has a Lie algebra whose only non-zero commutators are

$$\begin{aligned}
[T_3, T_4] &= mT_1, & [T_3, T_5] &= mT_2, \\
[T_5, T_6] &= mT_1, & [T_4, T_6] &= -mT_2, \\
[\tilde{T}^1, T_3] &= m\tilde{T}^4, & [\tilde{T}^1, T_4] &= -m\tilde{T}^3, \\
[\tilde{T}^1, T_5] &= m\tilde{T}^6, & [\tilde{T}^1, T_6] &= -m\tilde{T}^5, \\
[\tilde{T}^2, T_3] &= m\tilde{T}^5, & [\tilde{T}^2, T_5] &= -m\tilde{T}^3, \\
[\tilde{T}^2, T_6] &= m\tilde{T}^4, & [\tilde{T}^2, T_4] &= -m\tilde{T}^6.
\end{aligned} \tag{8.4.42}$$

The left-invariant one-form are given

$$\begin{aligned}
P^1 &= dz^1 + mz^4 dz^3 + mz^6 dz^5, & Q_1 &= d\tilde{z}_1, \\
P^2 &= dz^2 + mz^5 dz^3 - mz^6 dz^4, & Q_2 &= d\tilde{z}_2, \\
P^3 &= dz^3, & Q_3 &= d\tilde{z}_3 - mz^4 d\tilde{z}_1 - mz^5 d\tilde{z}_2, \\
P^4 &= dz^4, & Q_4 &= d\tilde{z}_4 + mz^3 d\tilde{z}_1 + mz^6 d\tilde{z}_2, \\
P^5 &= dz^5, & Q_5 &= d\tilde{z}_5 - mz^6 d\tilde{z}_1 + mz^3 d\tilde{z}_2, \\
P^6 &= dz^6, & Q_6 &= d\tilde{z}_6 + mz^6 d\tilde{z}_1 - mz^4 d\tilde{z}_2.
\end{aligned} \tag{8.4.43}$$

Choosing the polarisation $\Theta = 1$, the algebra (8.4.42) is

$$\begin{aligned}
[Z_3, Z_4] &= mZ_1, & [Z_3, Z_5] &= mZ_2, \\
[Z_5, Z_6] &= mZ_1, & [Z_4, Z_6] &= -mZ_2, \\
[X^1, Z_3] &= mX^4, & [X^1, Z_4] &= -mX^3, \\
[X^1, Z_5] &= mX^6, & [X^1, Z_6] &= -mX^5, \\
[X^2, Z_3] &= mX^5, & [X^2, Z_5] &= -mX^3, \\
[X^2, Z_6] &= mX^4, & [X^2, Z_4] &= -mX^6.
\end{aligned} \tag{8.4.44}$$

The X^m generates an abelian subgroup \tilde{G} . Taking a quotient of \tilde{G} gives the metric

$$ds^2 = \left(dz^1 + m(z^4 dz^3 + z^6 dz^5)\right)^2 + \left(dz^2 + m(z^5 dz^3 - z^6 dz^4)\right)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2 + (dz^6)^2. \tag{8.4.45}$$

Choosing the polarization

$$\Theta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (8.4.46)$$

the algebra (8.4.42) is

$$\begin{aligned} [Z_3, Z_4] &= mX^1, & [Z_3, Z_5] &= mX^2, \\ [Z_5, Z_6] &= mX^1, & [Z_4, Z_6] &= -mX^2, \\ [Z_1, Z_3] &= mX^4, & [Z_1, Z_4] &= -mX^3, \\ [Z_1, Z_5] &= mX^6, & [Z_1, Z_6] &= -mX^5, \\ [Z_2, Z_3] &= mX^5, & [Z_2, Z_5] &= -mX^3, \\ [Z_2, Z_6] &= mX^4, & [Z_2, Z_4] &= -mX^6. \end{aligned} \quad (8.4.47)$$

The X^m generates an abelian subgroup \tilde{G} . Taking the quotient gives the T^6 with H -flux.

$$ds^2 = (dz^1)^2 + (dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2 + (dz^6)^2, \quad (8.4.48)$$

and H -flux

$$\begin{aligned} H &= -mdz^1 \wedge dz^3 \wedge dz^4 - mdz^1 \wedge dz^5 \wedge dz^6 \\ &\quad -mdz^2 \wedge dz^3 \wedge dz^5 + mdz^2 \wedge dz^4 \wedge dz^6. \end{aligned} \quad (8.4.49)$$

Choosing the polarization

$$\Theta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (8.4.50)$$

the algebra (8.4.42) is

$$\begin{aligned} [X^3, Z_4] &= mZ_1, & [X^3, Z_5] &= mZ_2, \\ [Z_5, Z_6] &= mZ_1, & [Z_4, Z_6] &= -mZ_2, \\ [X^1, X^3] &= mX^4, & [X^1, Z_4] &= -mZ_3, \\ [X^1, Z_5] &= mX^6, & [X^1, Z_6] &= -mX^5, \\ [X^2, X^3] &= mX^5, & [X^2, Z_5] &= -mZ_3, \\ [X^2, Z_6] &= mX^4, & [X^2, Z_4] &= -mX^6. \end{aligned} \quad (8.4.51)$$

The X^m generates a subgroup \tilde{G} . Taking the quotient gives a T-fold with metric and B -field given by (C.0.1) and (C.0.2).

8.4.4. T^3 bundle over T^3

Consider the six-dimensional nilpotent Lie algebra whose only non-vanishing commutators are

$$\begin{aligned} [T_5, T_6] &= mT_1, & [T_4, T_6] &= -mT_2, \\ [T_4, T_5] &= mT_3. \end{aligned} \quad (8.4.52)$$

The corresponding twelve-dimensional group has a Lie algebra whose only non-zero commutators are

$$\begin{aligned}
[T_5, T_6] &= mT_1, & [T_4, T_6] &= -mT_2, \\
[T_4, T_5] &= mT_3, & [\tilde{T}^1, T_5] &= m\tilde{T}^6, \\
[\tilde{T}^2, T_6] &= m\tilde{T}^4, & [\tilde{T}^2, T_4] &= -m\tilde{T}^6, \\
[\tilde{T}^3, T_4] &= m\tilde{T}^5, & [\tilde{T}^1, T_6] &= -m\tilde{T}^5, \\
[\tilde{T}^3, T_5] &= -m\tilde{T}^4.
\end{aligned} \tag{8.4.53}$$

The left-invariant one-form are given

$$\begin{aligned}
P^1 &= dz^1 + mz^6 dz^5, & Q_1 &= d\tilde{z}_1, \\
P^2 &= dz^2 - mz^6 dz^4, & Q_2 &= d\tilde{z}_2, \\
P^3 &= dz^3 + mz^5 dz^4, & Q_3 &= d\tilde{z}_3, \\
P^4 &= dz^4, & Q_4 &= d\tilde{z}_4 + mz^6 d\tilde{z}_2 + mz^5 d\tilde{z}_3, \\
P^5 &= dz^5, & Q_5 &= d\tilde{z}_5 - mz^6 d\tilde{z}_1 + mz^4 d\tilde{z}_3, \\
P^6 &= dz^6, & Q_6 &= d\tilde{z}_6 + mz^5 d\tilde{z}_1 - mz^4 d\tilde{z}_2.
\end{aligned} \tag{8.4.54}$$

Choosing the polarisation $\Theta = 1$, the algebra (8.4.53) is

$$\begin{aligned}
[Z_5, Z_6] &= mZ_1, & [Z_4, Z_6] &= -mZ_2, \\
[Z_4, Z_5] &= mZ_3, & [X^1, Z_5] &= mX^6, \\
[X^2, Z_6] &= mX^4, & [X^2, Z_4] &= -mX^6, \\
[X^3, Z_4] &= mX^5, & [X^1, Z_6] &= -mX^5, \\
[X^3, Z_5] &= -mX^4.
\end{aligned} \tag{8.4.55}$$

The X^m generates an abelian subgroup \tilde{G} . Taking a quotient of \tilde{G} gives the metric

$$ds^2 = \left(dz^1 + mz^6 dz^5\right)^2 + \left(dz^2 - mz^6 dz^4\right)^2 + \left(dz^3 + mz^5 dz^4\right)^2 + (dz^4)^2 + (dz^5)^2 + (dz^6)^2. \tag{8.4.56}$$

Choosing the polarization

$$\Theta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (8.4.57)$$

the algebra (8.4.53) is

$$\begin{aligned} [Z_5, Z_6] &= mX^1, & [Z_4, Z_6] &= -mX^2, \\ [Z_4, Z_5] &= mX^3, & [Z_1, Z_5] &= mX^6, \\ [Z_2, Z_6] &= mX^4, & [Z_2, Z_4] &= -mX^6, \\ [Z_3, Z_4] &= mX^5, & [Z_1, Z_6] &= -mX^5, \\ [Z_3, Z_5] &= -mX^4. \end{aligned} \quad (8.4.58)$$

The X^m generates an abelian subgroup \tilde{G} . Taking a quotient of \tilde{G} gives a T^6 with H -flux

$$ds^2 = (dz^1)^2 + (dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2 + (dz^6)^2, \quad (8.4.59)$$

$$H = -mdz^1 \wedge dz^5 \wedge dz^6 - mdz^3 \wedge dz^4 \wedge dz^5 + mdz^2 \wedge dz^4 \wedge dz^6. \quad (8.4.60)$$

Choosing the polarization

$$\Theta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad (8.4.61)$$

the algebra (8.4.53) is

$$\begin{aligned} [Z_5, Z_6] &= mZ_1, & [X^4, Z_6] &= -mZ_2, \\ [X^4, Z_5] &= mZ_3, & [X^1, Z_5] &= mX^6, \\ [X^2, Z_6] &= mZ_4, & [X^2, X^4] &= -mX^6, \\ [X^3, X^4] &= mX^5, & [X^1, Z_6] &= -mX^5, \\ [X^3, Z_5] &= -mZ_4. \end{aligned} \quad (8.4.62)$$

The X^m generates a subgroup \tilde{G} . Taking the quotient of \tilde{G} gives a T-fold with metric and B -field given by

$$\begin{aligned} ds^2 &= (dz^1 + mz^6 dz^5)^2 + \frac{1}{1 + m^2[(z^5)^2 + (z^6)^2]} \left[(dz)^2 + (dz^3)^2 + (dz^4)^2 \right] \\ &\quad + \frac{1}{1 + m^2[(z^5)^2 + (z^6)^2]} \left(mz^5 dz^2 + mz^6 dz^3 \right)^2 + (dz^5)^2 + (dz^6)^2 \end{aligned} \quad (8.4.63)$$

$$B = \frac{m}{1 + m^2[(z^5)^2 + (z^6)^2]} \left(z^5 dz^3 \wedge dz^4 - z^6 dz^2 \wedge dz^4 \right) \quad (8.4.64)$$

8.4.5. T^3 bundle over T^4

Consider the seven-dimensional nilpotent Lie algebra whose only non-vanishing commutators are

$$\begin{aligned} [T_4, T_5] &= mT_1, & [T_6, T_7] &= mT_1, \\ [T_4, T_6] &= mT_2, & [T_5, T_7] &= -mT_2, \\ [T_4, T_7] &= mT_3, & [T_5, T_6] &= mT_3. \end{aligned} \tag{8.4.65}$$

The corresponding fourteen-dimensional group has a Lie algebra whose only non-zero commutators are

$$\begin{aligned} [T_4, T_5] &= mT_1, & [T_6, T_7] &= mT_1, \\ [T_4, T_6] &= mT_2, & [T_5, T_7] &= -mT_2, \\ [T_4, T_7] &= mT_3, & [T_5, T_6] &= mT_3, \\ [\tilde{T}^1, T_4] &= m\tilde{T}^5, & [\tilde{T}^1, T_5] &= -m\tilde{T}^4, \\ [\tilde{T}^1, T_6] &= m\tilde{T}^7, & [\tilde{T}^1, T_7] &= -m\tilde{T}^6, \\ [\tilde{T}^2, T_4] &= m\tilde{T}^6, & [\tilde{T}^2, T_6] &= -m\tilde{T}^4, \\ [\tilde{T}^2, T_7] &= m\tilde{T}^5, & [\tilde{T}^2, T_5] &= -m\tilde{T}^7, \\ [\tilde{T}^3, T_4] &= m\tilde{T}^7, & [\tilde{T}^3, T_7] &= -m\tilde{T}^4, \\ [\tilde{T}^3, T_5] &= m\tilde{T}^6, & [\tilde{T}^3, T_6] &= -m\tilde{T}^5. \end{aligned} \tag{8.4.66}$$

The left-invariant one-form are given

$$\begin{aligned} P^1 &= dz^1 + mz^5dz^4 + mz^7dz^6, & Q_1 &= d\tilde{z}_1, \\ P^2 &= dz^2 + mz^6dz^4 - mz^7dz^5, & Q_2 &= d\tilde{z}_2, \\ P^3 &= dz^3 + mz^7dz^4 + mz^6dz^5, & Q_3 &= d\tilde{z}_3, \\ P^4 &= dz^4, & Q_4 &= d\tilde{z}_4 - mz^5d\tilde{z}_1 - mz^6d\tilde{z}_2 + mz^4d\tilde{z}_3, \\ P^5 &= dz^5, & Q_5 &= d\tilde{z}_5 + mz^4d\tilde{z}_1 + mz^7d\tilde{z}_2 - mz^6d\tilde{z}_3, \\ P^6 &= dz^6, & Q_6 &= d\tilde{z}_6 - mz^7d\tilde{z}_1 + mz^4d\tilde{z}_2 + mz^5d\tilde{z}_3, \\ P^7 &= dz^7, & Q_7 &= d\tilde{z}_7 + mz^6d\tilde{z}_1 - mz^5d\tilde{z}_2 + mz^4d\tilde{z}_3. \end{aligned} \tag{8.4.67}$$

Choosing the polarisation $\Theta = 1$, the algebra (8.4.66) is

$$\begin{aligned}
[Z_4, Z_5] &= mZ_1, & [Z_6, Z_7] &= mZ_1, \\
[Z_4, Z_6] &= mZ_2, & [Z_5, Z_7] &= -mZ_2, \\
[Z_4, Z_7] &= mZ_3, & [Z_5, Z_6] &= mZ_3, \\
[X^1, Z_4] &= mX^5, & [X^1, Z_5] &= -mX^4, \\
[X^1, Z_6] &= mX^7, & [X^1, Z_7] &= -mX^6, \\
[X^2, Z_4] &= mX^6, & [X^2, Z_6] &= -mX^4, \\
[X^2, Z_7] &= mX^5, & [X^2, Z_5] &= -mX^7, \\
[X^3, Z_4] &= mX^7, & [X^3, Z_7] &= -mX^4, \\
[X^3, Z_5] &= mX^6, & [X^3, Z_6] &= -mX^5.
\end{aligned} \tag{8.4.68}$$

The X^m generates an abelian subgroup \tilde{G} . Taking a quotient of \tilde{G} gives the metric

$$\begin{aligned}
ds^2 &= \left(dz^1 + m(z^5 dz^4 + z^7 dz^6)\right)^2 + \left(dz^2 + m(z^6 dz^4 - z^7 dz^5)\right)^2 + \left(dz^3 + m(z^7 dz^4 + z^6 dz^5)\right)^2 \\
&\quad + (dz^4)^2 + (dz^5)^2 + (dz^6)^2 + (dz^7)^2.
\end{aligned} \tag{8.4.69}$$

Choosing the polarization

$$\Theta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{8.4.70}$$

the algebra (8.4.66) is

$$\begin{aligned}
[Z_4, Z_5] &= mX^1, & [Z_6, Z_7] &= mX^1, \\
[Z_4, Z_6] &= mX^2, & [Z_5, Z_7] &= -mX^2, \\
[Z_4, Z_7] &= mX^3, & [Z_5, Z_6] &= mX^3, \\
[Z_1, Z_4] &= mX^5, & [Z_1, Z_5] &= -mX^4, \\
[Z_1, Z_6] &= mX^7, & [Z_1, Z_7] &= -mX^6, \\
[Z_2, Z_4] &= mX^6, & [Z_2, Z_6] &= -mX^4, \\
[Z_2, Z_7] &= mX^5, & [Z_2, Z_5] &= -mX^7, \\
[Z_3, Z_4] &= mX^7, & [Z_3, Z_7] &= -mX^4, \\
[Z_3, Z_5] &= mX^6, & [Z_3, T_6] &= -mX^5.
\end{aligned} \tag{8.4.71}$$

The X^m generates an abelian subgroup \tilde{G} . Taking the quotient of \tilde{G} gives the T^7 with H -flux.

$$ds^2 = (dz^1)^2 + (dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2 + (dz^6)^2 + (dz^7)^2, \tag{8.4.72}$$

$$\begin{aligned}
H = & -mdz^1 \wedge dz^4 \wedge dz^5 - mdz^1 \wedge dz^6 \wedge dz^7 - mdz^2 \wedge dz^4 \wedge dz^6 \\
& -mdz^3 \wedge dz^4 \wedge dz^7 - mdz^3 \wedge dz^5 \wedge dz^6 + mdz^2 \wedge dz^5 \wedge dz^7.
\end{aligned} \tag{8.4.73}$$

Choosing the polarization

$$\Theta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{8.4.74}$$

the algebra (8.4.66) is

$$\begin{aligned}
[X^4, Z_5] &= mZ_1, & [Z_6, Z_7] &= mZ_1, \\
[X^4, Z_6] &= mZ_2, & [Z_5, Z_7] &= -mZ_2, \\
[X^4, Z_7] &= mZ_3, & [Z_5, Z_6] &= -mZ_3, \\
[X^1, X^4] &= mX^5, & [X^1, Z_5] &= -mZ_4, \\
[X^1, Z_6] &= mX^7, & [X^1, Z_7] &= -mX^6, \\
[X^2, X^4] &= mX^6, & [X^2, Z_6] &= -mZ_4, \\
[X^2, Z_7] &= mX^5, & [X^2, Z_5] &= -mX^7, \\
[X^3, X^4] &= mX^7, & [X^3, Z_7] &= -mZ_4, \\
[X^3, Z_5] &= mX^6, & [X^3, T_6] &= -mX^5,
\end{aligned} \tag{8.4.75}$$

The X^m generates a subgroup \tilde{G} . Taking the quotient gives the T-fold with the metric and B -field given by (C.0.3), (C.0.4)

8.4.6. S^1 bundle over T^6

Consider the seven-dimensional nilpotent Lie algebra whose only non-vanishing commutators are

$$\begin{aligned}
[T_2, T_3] &= mT_1, & [T_4, T_5] &= mT_1, \\
[T_6, T_7] &= mT_1.
\end{aligned} \tag{8.4.76}$$

The corresponding fourteen-dimensional group has a Lie algebra whose only non-zero commutators are

$$\begin{aligned}
[T_2, T_3] &= mT_1, & [T_4, T_5] &= mT_1, \\
[T_6, T_7] &= mT_1, & [\tilde{T}^1, T_3] &= -m\tilde{T}^2, \\
[\tilde{T}^1, T_2] &= m\tilde{T}^3, & [\tilde{T}^1, T_5] &= -m\tilde{T}^4, \\
[\tilde{T}^1, T_4] &= m\tilde{T}^5, & [\tilde{T}^1, T_7] &= -m\tilde{T}^6, \\
[\tilde{T}^1, T_6] &= m\tilde{T}^7.
\end{aligned} \tag{8.4.77}$$

The left-invariant one-form are given

$$\begin{aligned}
P^1 &= dz^1 + mz^3dz^2 + mz^5dz^4 + mz^7dz^6, & Q_1 &= d\tilde{z}_1, \\
P^2 &= dz^2, & Q_2 &= d\tilde{z}_2 - mz^3d\tilde{z}_1, \\
P^3 &= dz^3, & Q_3 &= d\tilde{z}_3 + mz^2d\tilde{z}_1, \\
P^4 &= dz^4, & Q_4 &= d\tilde{z}_4 - mz^5d\tilde{z}_1, \\
P^5 &= dz^5, & Q_5 &= d\tilde{z}_5 + mz^4d\tilde{z}_1, \\
P^6 &= dz^6, & Q_6 &= d\tilde{z}_6 - mz^7d\tilde{z}_1, \\
P^7 &= dz^7, & Q_7 &= d\tilde{z}_7 + mz^6d\tilde{z}_1.
\end{aligned} \tag{8.4.78}$$

Choosing the polarisation $\Theta = 1$, the algebra (8.4.77) is

$$\begin{aligned}
[Z_2, Z_3] &= mZ_1, & [Z_4, Z_5] &= mZ_1, \\
[Z_6, Z_7] &= mZ_1, & [X^1, Z_3] &= -mX^2, \\
[X^1, Z_2] &= mX^3, & [X^1, Z_5] &= -mX^4, \\
[X^1, Z_4] &= mX^5, & [X^1, Z_7] &= -mX^6, \\
[X^1, Z_6] &= mX^7.
\end{aligned} \tag{8.4.79}$$

The X^m generates an abelian subgroup \tilde{G} . Taking a quotient of \tilde{G} gives the metric

$$ds^2 = \left(dz^1 + m(z^3dz^2 + z^5dz^4 + z^7dz^6) \right)^2 + (dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2 + (dz^6)^2 + (dz^7)^2. \tag{8.4.80}$$

Choosing the polarization

$$\Theta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{8.4.81}$$

the algebra (8.4.77) is

$$\begin{aligned}
[Z_2, Z_3] &= mX^1, & [Z_4, Z_5] &= mX^1, \\
[Z_6, Z_7] &= mX^1, & [Z_1, Z_3] &= -mX^2, \\
[Z_1, Z_2] &= mX^3, & [Z_1, Z_5] &= -mX^4, \\
[Z_1, Z_4] &= mX^5, & [Z_1, Z_7] &= -mX^6, \\
[Z_1, Z_6] &= mX^7.
\end{aligned} \tag{8.4.82}$$

The X^m generates an abelian subgroup \tilde{G} . Taking the quotient gives the T^7 with H -flux.

$$ds^2 = (dz^1)^2 + (dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2 + (dz^6)^2 + (dz^7)^2, \tag{8.4.83}$$

$$H = -mdz^1 \wedge dz^2 \wedge dz^3 - mdz^1 \wedge dz^4 \wedge dz^5 - mdz^1 \wedge dz^6 \wedge dz^7. \tag{8.4.84}$$

Choosing the polarization

$$\Theta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{8.4.85}$$

the algebra (8.4.77) is

$$\begin{aligned}
[X^2, Z_3] &= mZ_1, & [X^4, Z_5] &= mZ_1, \\
[X^6, Z_7] &= mZ_1, & [X^1, Z_3] &= -mZ_2, \\
[X^1, X^2] &= mX^3, & [X^1, Z_5] &= -mZ_4, \\
[X^1, X^4] &= mX^5, & [X^1, Z_7] &= -mZ_6, \\
[X^1, X^6] &= mX^7.
\end{aligned} \tag{8.4.86}$$

The X^m generates a subgroup \tilde{G} . Taking the quotient gives the T-fold with the metric and

B -field given by

$$\begin{aligned}
ds^2 = & \frac{1}{1+m^2[(z^3)^2+(z^5)^2+(z^7)^2]} \left[(dz^1)^2 + (dz^2)^2 + (dz^4)^2 + (dz^6)^2 \right] \\
& + (dz^3)^2 + (dz^5)^2 + (dz^7)^2 + \frac{1}{1+m^2[(z^3)^2+(z^5)^2+(z^7)^2]} \left(mz^5 dz^2 - mz^3 dz^4 \right)^2 \\
& + \frac{1}{1+m^2[(z^3)^2+(z^5)^2+(z^7)^2]} \left(mz^7 dz^2 - mz^3 dz^6 \right)^2 \\
& + \frac{1}{1+m^2[(z^3)^2+(z^5)^2+(z^7)^2]} \left(mz^7 dz^4 - mz^5 dz^6 \right)^2, \tag{8.4.87}
\end{aligned}$$

$$B = \frac{m}{1+m^2[(z^3)^2+(z^5)^2+(z^7)^2]} \left(z^3 dz^1 \wedge dz^2 + z^5 dz^1 \wedge dz^4 + z^7 dz^1 \wedge dz^6 \right). \tag{8.4.88}$$

8.5. Nilmanifolds fibred over a line.

In the previous section, the double geometry of the nilmanifold has been constructed. In this section, we will construct the double geometry of a special holonomy space. Generally, the special holonomy domain-wall solution will be in the form of $\mathcal{N} \times \mathbb{R}$, where \mathcal{N} is a nilmanifold. The doubled formulation of this space is constructed by doubling d -dimensional nilmanifold \mathcal{N} to $2d$ -dimensional nilmanifold \mathcal{M} . That is the space $\mathcal{N} \times \mathbb{R}$ is extended to $\mathcal{M} \times \mathbb{R}$.

Consider the non-linear sigma model of a special holonomy space $\mathcal{N} \times \mathbb{R}$

$$S_{\mathcal{N} \times \mathbb{R}} = \frac{1}{2} \oint_{\Sigma} \left(V^p(\tau) d\tau \wedge *d\tau + x_{mn}(\tau) P^m \wedge *P^n \right), \tag{8.5.1}$$

where $V(\tau)$ is a harmonic function of a line with a coordinate τ , p is a constant depending on a nilmanifold, x_{mn} is a symmetric matrix constructed from $V(\tau)$, and P^m is a left-invariant one-form on \mathcal{N} . This action can be generalized to the non-linear sigma model of $\mathcal{M} \times \mathbb{R}$, which is given by

$$S_{\mathcal{M} \times \mathbb{R}} = \frac{1}{2} \oint_{\Sigma} V^p(\tau) d\tau \wedge *d\tau + \frac{1}{4} \oint_{\Sigma} \mathcal{M}_{MN}(\tau) \hat{\mathcal{P}}^M \wedge *\hat{\mathcal{P}}^N + \frac{1}{2} \int_V \mathcal{K}, \tag{8.5.2}$$

where $\mathcal{M}_{MN}(\tau)$ is a symmetric matrix constructed from $V(\tau)$, $\hat{\mathcal{P}}^M$ is pull-back of a left-invariant one-form on \mathcal{M} to Σ , and \mathcal{K} is a pull-back 3-form on \mathcal{M} to V defined in (8.2.5). This non-linear sigma model includes all the detail about T-duality background of nilmanifold \mathcal{N} . To obtain each T-dual background, the polarization need to be specified as in the section 8.4 and gauging \tilde{G} will result in the T-dual background.

8.5.1. Example: 3-dimensional nilmanifold

The doubled space of the 3-dimensional nilmanifold \mathcal{N} is given by the 6-dimensional nilmanifold \mathcal{M} . The non-vanishing commutation relation is

$$[T_x, T_z] = mT_y \quad [T_x, \tilde{T}^y] = mX^z \quad [T_z, \tilde{T}^y] = -mX^x \quad (8.5.3)$$

The left-invariant one-form on this space is

$$\begin{aligned} P^x &= dx & P^y &= dy - mxdz & P^z &= dz \\ Q_x &= d\tilde{x} - mzd\tilde{y} & Q_y &= d\tilde{y} & Q_z &= d\tilde{z} + mxd\tilde{y} \end{aligned} \quad (8.5.4)$$

In this case, the non-linear sigma model on $\mathcal{M} \times \mathbb{R}$ is

$$S_{\mathcal{M} \times \mathbb{R}} = \frac{1}{2} \oint_{\Sigma} V(\tau) d\tau \wedge *d\tau + \frac{1}{4} \oint_{\Sigma} \mathcal{M}_{MN}(\tau) \hat{\mathcal{P}}^M \wedge * \hat{\mathcal{P}}^N + \frac{1}{2} \int_V \mathcal{K}, \quad (8.5.5)$$

where $\mathcal{M}_{MN}(\tau)$ is given

$$\mathcal{M}_{MN}(\tau) = \begin{pmatrix} V(\tau) & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/V(\tau) & 0 & 0 & 0 & 0 \\ 0 & 0 & V(\tau) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/V(\tau) & 0 & 0 \\ 0 & 0 & 0 & 0 & V(\tau) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/V(\tau) \end{pmatrix}. \quad (8.5.6)$$

To obtain a nilfold bundle over a line, the polarization is chosen as $\Theta = 1$. In this case, the generalized metric (8.2.20), which is the second term in (8.5.5), is given by

$$\mathcal{H} = \frac{1}{2} \mathcal{M}_{MN}(\tau) \mathcal{P}^M \otimes \mathcal{P}^N = \frac{1}{2} \mathcal{H}_{MN}(\tau, x^m) \Phi^M \otimes \Phi^N. \quad (8.5.7)$$

With polarization tensor, we define $\mathcal{H}_{\hat{M}\hat{N}}(\tau, x^m)$ as in (8.2.21). Its components can be used to define g_{mn} and B_{mn} as in (8.2.22). The metric (8.2.23) and the H -flux (8.2.24) can be obtained

$$ds_3^2 = V(\tau) (dx^2 + dz^2) + \frac{1}{V(\tau)} (dy - mxdz)^2, \quad (8.5.8)$$

$$H = 0. \quad (8.5.9)$$

This results in the metric

$$ds^2 = V(\tau) ((d\tau)^2 + (dx)^2 + (dz)^2) + \frac{1}{V(\tau)} (dy - mxdz)^2. \quad (8.5.10)$$

To obtain a T^3 with H -flux, the polarization is chosen as

$$\Theta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this polarization, the scalar moduli is given by

$$\mathcal{M}_{MN}(\tau) = \begin{pmatrix} V(\tau) & 0 & 0 & 0 & 0 & 0 \\ 0 & V(\tau) & 0 & 0 & 0 & 0 \\ 0 & 0 & V(\tau) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/V(\tau) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/V(\tau) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/V(\tau) \end{pmatrix}. \quad (8.5.11)$$

In this polarization, the left-invariant one-form is

$$\begin{aligned} P^x &= dx & P^y &= dy & P^z &= dz \\ Q_x &= d\tilde{x} - mzdy & Q_y &= d\tilde{y} - mxdz & Q_z &= d\tilde{z} + mxdy \end{aligned} \quad (8.5.12)$$

With this polarization tensor, we define $\mathcal{H}_{\hat{M}\hat{N}}(\tau, x^m)$ as in (8.2.21). Its components can be used to define g_{mn} and B_{mn} as in (8.2.22). The metric (8.2.23) and the H -flux (8.2.24) can be obtained

$$ds_3^2 = V(\tau) (dx^2 + dy^2 + dz^2) \quad (8.5.13)$$

$$H = m dx \wedge dy \wedge dz. \quad (8.5.14)$$

This results in the metric and three-form as

$$ds^2 = V(\tau) ((d\tau)^2 + (dx)^2 + (dy)^2 + (dz)^2), \quad (8.5.15)$$

$$H = m dx \wedge dy \wedge dz. \quad (8.5.16)$$

To obtain T-fold background, the polarization is chosen as

$$\Theta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

In this polarization, the scalar moduli is given by

$$\mathcal{M}_{MN}(\tau) = \begin{pmatrix} V(\tau) & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/V(\tau) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/V(\tau) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/V(\tau) & 0 & 0 \\ 0 & 0 & 0 & 0 & V(\tau) & 0 \\ 0 & 0 & 0 & 0 & 0 & V(\tau) \end{pmatrix}. \quad (8.5.17)$$

In this polarization, the left-invariant one-form is

$$\begin{aligned} P^x &= dx & P^y &= dy - mxd\tilde{z} & P^z &= dz + mxd\tilde{y} \\ Q_x &= d\tilde{x} - m\tilde{z}d\tilde{y} & Q_y &= d\tilde{y} & Q_z &= d\tilde{z} \end{aligned} \quad (8.5.18)$$

With this polarization tensor, we define $\mathcal{H}_{\hat{M}\hat{N}}(\tau, x^m)$ as in (8.2.21). Its components can be used to define g_{mn} and B_{mn} as in (8.2.22). The metric (8.2.23) and the B -field from the H -flux (8.2.24) can be obtained

$$ds_3^2 = V(\tau)dx^2 + \frac{V(\tau)}{V^2(\tau) + (mx)^2} \left((dy)^2 + (dz)^2 \right), \quad (8.5.19)$$

$$B = \frac{-mx}{V^2(\tau) + (mx)^2} dy \wedge dz. \quad (8.5.20)$$

This results in the metric and B -field as

$$ds^2 = V(\tau) \left((d\tau)^2 + (dx)^2 \right) + \frac{V(\tau)}{V^2(\tau) + (mx)^2} \left((dy)^2 + (dz)^2 \right), \quad (8.5.21)$$

$$B = \frac{-mx}{V^2(\tau) + (mx)^2} dy \wedge dz. \quad (8.5.22)$$

While choosing the polarization (8.3.4) will result in the R -flux background. This construction can be generalized to the higher dimensional nilmanifold.

9. Conclusion

In chapter 5, we generalize the finite transformation of double field theory to extended field theory. The cases of $E_4 = SL(5, \mathbb{R})$, $E_5 = Spin(5, 5)$, and E_6 have been studied. These finite transformations agree with the finite transformation for the metric and the 3-form gauge field and make explicit contact with generalized geometry.

In chapter 6, we have studied T-duality chain of famous example, which is a T^3 with H -flux. T-duality in one direction gives a nilfold background. A further T-duality in another direction gives a T-fold background. Last T-duality will result in an R -flux background. However, these solutions are not string backgrounds because they do not define conformal field theories. One can, however, construct string backgrounds from these solutions as fibres over some space. The simplest case is T^3 with H -flux fibred over a line. This solution can be identified as a smeared NS5-brane solution. For a general NS5-brane, transverse directions are \mathbb{R}^4 , while for a smeared case, transverse directions are $\mathbb{R} \times T^3$. T-duality transformation of a smeared NS5-brane will result in a KK-monopole, which has a hyperkähler metric. While S-duality of a smeared NS5-brane gives a smeared D5-brane over T^3 . T-duality of a smeared D5-brane in 3 directions of T^3 gives a D8-brane solution.

Normally a D8-brane solution will not give a good string solution because the harmonic function depends linearly on the transverse coordinate. The dilaton field, which is proportional to the harmonic function, will diverge at a large distance away from the D8-brane. To get a sensible solution, one starts from type I theory on a circle and does T-duality along this direction. This gives type I' theory on $I = S^1/\mathbb{Z}$ with O8-planes located at fixed points and 16 D8-branes. The type I' on $I \times T^3$ is dual to type I on T^4 , which is dual to type IIA on a $K3$ surface. This implies the duality between the type I' and the type IIA on a $K3$ surface.

The type I' configuration with two O8-planes at end points of I and 16 D8-branes distributed on the interval corresponds to the $K3$ geometry with end-caps given by Tian-Yau spaces with $b_+ = 8$ and $b_- = 8$ and 16 Kaluza-Klein monopoles distributed over the interval. If $b_+ = 8 - n_+$ and $b_- = 8 - n_-$ with $16 - n_+ - n_-$ Kaluza-Klein monopoles, this configuration corresponds to the type I' in which there are n_- D8-branes on top of O8-plane at $\tau = 0$ and n_+ D8-branes on top of O8-plane at $\tau = \pi$ and $16 - n_+ - n_-$ D8-branes distributed over an interval.

The $K3$ geometry also allows $b_+ = 9$ and/or $b_- = 9$, which will lead to up to 17 or 18 Kaluza-Klein monopoles. This configuration corresponds to the type I' at the strong coupling where

O8-plane can emit a D8-brane leaving O8* of charge -9. For the type I' up to 16 D8-branes are possible at the weak coupling and 17 or 18 D-branes will be possible at the strong coupling. However, for the type IIA on the $K3$ geometry 17 or 18 Kaluza-Klein monopoles can arise at the weak coupling since S-duality in the chain of dualities maps a strong coupling of the type I' to a weak coupling of type IIA on $K3$ surface.

In chapter 7, we generalize the domain-wall solution of the 3-dimensional nilfold to the higher dimensional nilfold. This higher dimensional nilfold, which can be constructed from the nilpotent Lie group identified with a left action of the cocompact subgroup, is in the form of T^n bundle over T^m for some n, m . Each domain-wall solution admits a special holonomy, namely, $SU(3)$, G_2 , $Spin(7)$, and $SU(4)$. The product of Minkowski space with these domain wall solutions give supersymmetric solutions, that preserve $1/4$, $1/8$, $1/16$, and $1/8$, respectively.

T-duality transformation of these domain-wall solutions will result in T^m with H -flux fibred over a line for some m . These solutions can be thought of as intersecting smeared NS5-brane solutions, which preserve the same amount of supersymmetry as T-dual domain-wall solutions. S-duality of intersecting NS5-brane solutions gives intersecting D5-brane solutions. A further T-duality of these solutions will result in intersecting D4-D8-brane solutions.

In this thesis, we have studied $K3$ surface in the limit that it becomes a 3-dimensional nilfold fibred over a line. It would be interesting to generalize this idea to find the analogues of $K3$ surface to other cases, such as Calabi-Yau 3-fold in the case of $SU(3)$ holonomy, G_2 manifold in the case of G_2 holonomy, $Spin(7)$ manifold in the case of $Spin(7)$ holonomy, or Calabi-Yau 4-fold in the case of $SU(4)$ holonomy. In other examples, one expects that compact special holonomy manifolds would have limits that they become higher dimensional nilmanifold fibred over a line.

It would be interesting to generalize our analysis to find the analogue of domain-wall solutions from the arbitrary configuration of intersecting branes. In our analysis, each of domain-wall solution is dual to the system of D4-D8 brane system. If one starts from the supersymmetric configuration, such as NS5-D4-D8, it would be interesting to see the supersymmetric solution that is dual to that configuration.

It will be interesting to study the configuration of a non-geometric background, such as T-fold. In our cases, a T-fold background appears in T-dual backgrounds of a supersymmetric domain-wall solution. It would be interesting to classify the configuration of T-fold. For example, it might be able to classify the configuration of T-fold in terms of intersecting exotic branes and NS5-branes.

A. Left-invariant one-forms of five-dimensional nilmanifold

Let \mathcal{G}_5 is a five dimensional nilpotent Lie group with non-vanishing commutators

$$[T_2, T_3] = mT_1, \quad [T_4, T_5] = mT_1. \quad (\text{A.0.1})$$

Let the element g be written as

$$g = \exp(z^1 T_1) \exp(z^2 T_2) \exp(z^3 T_3) \exp(z^4 T_4) \exp(z^5 T_5), \quad (\text{A.0.2})$$

where z^1, \dots, z^5 are local coordinates on \mathcal{G}_5 . The inverse of g can be written as

$$g^{-1} = \exp(-z^5 T_5) \exp(-z^4 T_4) \exp(-z^3 T_3) \exp(-z^2 T_2) \exp(-z^1 T_1). \quad (\text{A.0.3})$$

From (A.0.2), dg is given by

$$\begin{aligned} dg = & [dz^1 T_1] \exp(z^1 T_1) \exp(z^2 T_2) \exp(z^3 T_3) \exp(z^4 T_4) \exp(z^5 T_5) \\ & + \exp(z^1 T_1) [dz^2 T_2] \exp(z^2 T_2) \exp(z^3 T_3) \exp(z^4 T_4) \exp(z^5 T_5) \\ & + \exp(z^1 T_1) \exp(z^2 T_2) [dz^3 T_3] \exp(z^3 T_3) \exp(z^4 T_4) \exp(z^5 T_5) \\ & + \exp(z^1 T_1) \exp(z^2 T_2) \exp(z^3 T_3) [dz^4 T_4] \exp(z^4 T_4) \exp(z^5 T_5) \\ & + \exp(z^1 T_1) \exp(z^2 T_2) \exp(z^3 T_3) \exp(z^4 T_4) [dz^5 T_5] \exp(z^5 T_5). \end{aligned} \quad (\text{A.0.4})$$

The left-invariant one-form is given

$$g^{-1} dg = P^a T_a. \quad (\text{A.0.5})$$

$$\begin{aligned}
g^{-1}dg = & \exp(-z^5T_5) \exp(-z^4T_4) \exp(-z^3T_3) \exp(-z^2T_2) \exp(-z^1T_1) \times \\
& \left([dz^1T_1] \exp(z^1T_1) \exp(z^2T_2) \exp(z^3T_3) \exp(z^4T_4) \exp(z^5T_5) \right. \\
& + \exp(z^1T_1) [dz^2T_2] \exp(z^2T_2) \exp(z^3T_3) \exp(z^4T_4) \exp(z^5T_5) \\
& + \exp(z^1T_1) \exp(z^2T_2) [dz^3T_3] \exp(z^3T_3) \exp(z^4T_4) \exp(z^5T_5) \\
& + \exp(z^1T_1) \exp(z^2T_2) \exp(z^3T_3) [dz^4T_4] \exp(z^4T_4) \exp(z^5T_5) \\
& \left. + \exp(z^1T_1) \exp(z^2T_2) \exp(z^3T_3) \exp(z^4T_4) [dz^5T_5] \exp(z^5T_5) \right). \quad (\text{A.0.6})
\end{aligned}$$

Since T_1 commutes with every generator, the first term of (A.0.6) will be

$$dz^1T_1. \quad (\text{A.0.7})$$

The second term of (A.0.6) is

$$\exp(-z^5T_5) \exp(-z^4T_4) \exp(-z^3T_3) [dz^2T_2] \exp(z^3T_3) \exp(z^4T_4) \exp(z^5T_5). \quad (\text{A.0.8})$$

Since T_4 and T_5 commute with T_2 and T_3 , the second term will be

$$\exp(-z^3T_3) [dz^2T_2] \exp(z^3T_3). \quad (\text{A.0.9})$$

Consider Baker-Campbell-Hausdorff formula

$$e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2!} [X, [X, Y]] + \frac{1}{3!} [X, [X, [X, Y]]] + \dots. \quad (\text{A.0.10})$$

Therefore, the equation (A.0.9) will be

$$dz^2T_2 - z^3 dz^2 [T_3, T_2] + \dots. \quad (\text{A.0.11})$$

Since the group is 2-step nilpotent Lie group, the terms with $[T_3, [T_3, T_2]]$ and higher will be zero. The result is

$$dz^2T_2 + mz^3 dz^2T_1. \quad (\text{A.0.12})$$

The third term of (A.0.6) is

$$dz^3T_3. \quad (\text{A.0.13})$$

while the forth term is

$$\exp(-z^5T_5) [dz^4T_4] \exp(z^5T_5). \quad (\text{A.0.14})$$

With the same reason as (A.0.11), this term becomes

$$dz^4T_4 + mz^5 dz^4T_1. \quad (\text{A.0.15})$$

The last term is

$$dz^5 T_5. \quad (\text{A.0.16})$$

Therefore, the left-invariant one-form is

$$g^{-1}dg = P^a T_a = (dz^1 + mz^3 dz^2 + mz^5 dz^4)T_1 + (dz^2)T_2 + (dz^3)T_3 + (dz^4)T_4 + (dz^5)T_5. \quad (\text{A.0.17})$$

B. Cocompact subgroup of five-dimensional nilpotent Lie group

Let \mathcal{G}_5 is a five dimensional nilpotent Lie group with non-vanishing commutators

$$[T_2, T_3] = mT_1, \quad [T_4, T_5] = mT_1. \quad (\text{B.0.1})$$

The element g can be written as

$$g = \exp(z^1 T_1) \exp(z^2 T_2) \exp(z^3 T_3) \exp(z^4 T_4) \exp(z^5 T_5), \quad (\text{B.0.2})$$

where z^1, \dots, z^5 are local coordinates on \mathcal{G}_5 . Let Γ be a cocompact subgroup of \mathcal{G}_5 with group element

$$h = \exp(n^1 T_1) \exp(n^2 T_2) \exp(n^3 T_3) \exp(n^4 T_4) \exp(n^5 T_5), \quad (\text{B.0.3})$$

where $n^i \in \mathbb{Z}$, and $i = 1, \dots, 5$.

Consider the left action of h on g

$$\begin{aligned} h \cdot g &= \exp(n^1 T_1) \exp(n^2 T_2) \exp(n^3 T_3) \exp(n^4 T_4) \exp(n^5 T_5) \cdot \\ &\quad \left(\exp(z^1 T_1) \exp(z^2 T_2) \exp(z^3 T_3) \exp(z^4 T_4) \exp(z^5 T_5) \right). \end{aligned} \quad (\text{B.0.4})$$

Since T_1 commutes with every element and T_2 and T_3 commute with T_4 and T_5 , we get

$$\begin{aligned} h \cdot g &= \exp((z^1 + n^1) T_1) \left(\exp(n^2 T_2) \exp(n^3 T_3) \exp(z^2 T_2) \exp(z^3 T_3) \right) \\ &\quad \left(\exp(n^4 T_4) \exp(n^5 T_5) \exp(z^4 T_4) \exp(z^5 T_5) \right). \end{aligned} \quad (\text{B.0.5})$$

Consider a product

$$\exp(n^3 T_3) \exp(z^2 T_2). \quad (\text{B.0.6})$$

Using the product rule

$$e^X e^Y = e^{(Y + [X, Y] + \frac{1}{2!} [X, [X, Y]] + \frac{1}{3!} [X, [X, [X, Y]] + \dots)} e^X, \quad (\text{B.0.7})$$

the product (B.0.6) becomes

$$\exp(n^3 T_3) \exp(z^2 T_2) = \exp(z^2 T_2 - mn^3 z^2 T_1) \exp(n^3 T_3). \quad (\text{B.0.8})$$

Therefore, the product $h \cdot g$ becomes

$$\begin{aligned} h \cdot g = & \exp((z^1 + n^1 - mn^3 z^2 - mn^5 z^4) T_1) \exp((z^2 + n^2) T_2) \exp((z^2 + n^2) T_2) \exp((z^3 + n^3) T_3) \\ & \exp((z^4 + n^4) T_4) \exp((z^5 + n^5) T_5). \end{aligned} \quad (\text{B.0.9})$$

The quotient space \mathcal{G}_5/Γ is obtained by identifying g with $h \cdot g$,

$$g \sim h \cdot g. \quad (\text{B.0.10})$$

That is the global structure of \mathcal{G}_5/Γ required the following identification of local coordinates

$$\begin{aligned} z^1 & \sim z^1 + n^1 - mn^3 z^2 - mn^5 z^4, \\ z^2 & \sim z^2 + n^2, \\ z^3 & \sim z^3 + n^3, \\ z^4 & \sim z^4 + n^4, \\ z^5 & \sim z^5 + n^5. \end{aligned} \quad (\text{B.0.11})$$

C. T-fold

The metric and B -field of T-fold, which is T-dual to T^2 bundle over T^4 ,

$$g = \begin{pmatrix} \frac{m^2(z^5)^2+1}{f} & -\frac{m^2z^4z^5}{f} & 0 & \frac{m^3z^4z^5z^6}{f} & \frac{mz^6(m^2(z^5)^2+1)}{f} & 0 \\ -\frac{m^2z^4z^5}{f} & \frac{m^2(z^4)^2+1}{f} & 0 & -\frac{mz^6(m^2(z^4)^2+1)}{f} & -\frac{m^3z^4z^5z^6}{f} & 0 \\ 0 & 0 & \frac{1}{f} & 0 & 0 & 0 \\ \frac{m^3z^4z^5z^6}{f} & -\frac{mz^6(m^2(z^4)^2+1)}{f} & 0 & m^2(z^6)^2 - \frac{m^4(z^5)^2(z^6)^2}{f} + 1 & \frac{m^4z^4z^5(z^6)^2}{f} & 0 \\ \frac{mz^6(m^2(z^5)^2+1)}{f} & -\frac{m^3z^4z^5z^6}{f} & 0 & \frac{m^4z^4z^5(z^6)^2}{f} & m^2(z^6)^2 - \frac{m^4(z^4)^2(z^6)^2}{f} + 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\text{C.0.1})$$

$$B = \begin{pmatrix} 0 & 0 & 0 & \frac{mz^4}{f} & 0 & 0 \\ 0 & 0 & 0 & \frac{mz^5}{f} & 0 & 0 \\ -\frac{mz^4}{f} & -\frac{mz^5}{f} & 0 & \frac{m^2z^5z^6}{f} & -\frac{m^2z^4z^6}{f} & 0 \\ 0 & 0 & -\frac{m^2z^5z^6}{f} & 0 & 0 & 0 \\ 0 & 0 & \frac{m^2z^4z^6}{f} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\text{C.0.2})$$

where $f = 1 + m^2(z^4)^2 + m^2(z^5)^2$.

The metric and B -field of T-fold, which is T-dual to T^3 bundle over T^4 ,

$$g = \begin{pmatrix} 1 - \frac{m^2 (z^5)^2}{f} & -\frac{m^2 z^5 z^6}{f} & -\frac{m^2 z^5 z^7}{f} & 0 & 0 & m z^7 - \frac{m^3 (z^5)^2 z^7}{f} & 0 \\ -\frac{m^2 z^5 z^6}{f} & 1 - \frac{m^2 (z^6)^2}{f} & -\frac{m^2 z^6 z^7}{f} & 0 & -m z^7 & -\frac{m^3 z^5 z^6 z^7}{f} & 0 \\ -\frac{m^2 z^5 z^7}{f} & -\frac{m^2 z^6 z^7}{f} & 1 - \frac{m^2 (z^7)^2}{f} & 0 & m z^6 & -\frac{m^3 z^5 (z^7)^2}{f} & 0 \\ 0 & 0 & 0 & \frac{1}{f} & 0 & 0 & 0 \\ 0 & -m z^7 & m z^6 & 0 & m^2 (z^6)^2 + m^2 (z^7)^2 + 1 & 0 & 0 \\ m z^7 - \frac{m^3 (z^5)^2 z^7}{f} & -\frac{m^3 z^5 z^6 z^7}{f} & -\frac{m^3 z^5 (z^7)^2}{f} & 0 & 0 & m^2 (z^7)^2 - \frac{m^4 (z^5)^2 (z^7)^2}{f} + 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\text{C.0.3})$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{m z^5}{f} & -\frac{m z^6}{f} & -\frac{m z^7}{f} & 0 & 0 & -\frac{m^2 z^5 z^7}{f} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{m^2 z^5 z^7}{f} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\text{C.0.4})$$

where $f = 1 + m^2 (z^5)^2 + m^2 (z^6)^2 + m^2 (z^7)^2$.

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