Quantum black holes

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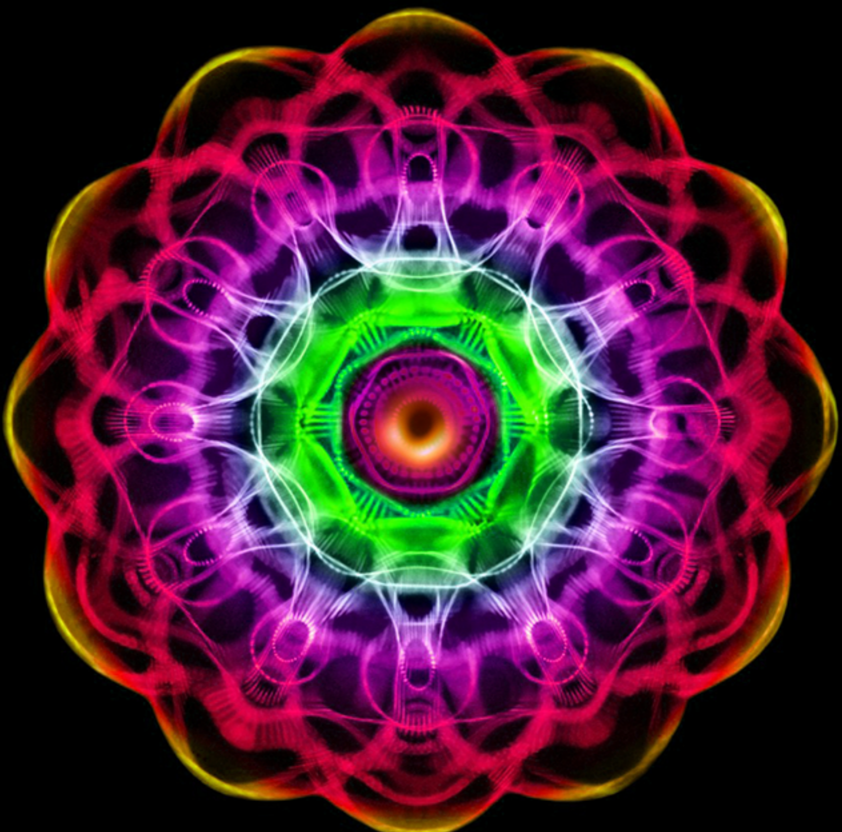

Victor Godet

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Cover: Image of vibrating water by James Stuart Reid, captured using a CymaScope (here with modified colors). The silhouette of the M87* black hole, as photographed by the Event Horizon Telescope, has been embedded at the center.

Quantum Black Holes

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[3] B. Freivogel, V. Godet, E. Morvan, J. Pedraza, A. Rotundo,
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Other papers by the author:

[4] J. de Boer, V. Godet, J. Kastikainen, E. Keski-Vakkuri,
"Quantum hypothesis testing in many-body systems",
arXiv:2007.11711 [quant-ph], submitted to SciPost

[5] V. Godet and C. Marteau, *"New boundary conditions for AdS_2"*,
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[6] V. Godet and C. Marteau, *"Gravitation in flat spacetime from entanglement"*,
arXiv:1908.02044 [hep-th], *JHEP* **12** (2019) 057

[7] V. Godet, *"The Monster, the Baby Monster and Traces of Singular Moduli"*,
arXiv:1705.05361 [hep-th]

À mes chers parents

Contents

J. S. Bach, *Violin Sonata No.1 in G minor*

1 General introduction

It is often said that quantum mechanics and gravity are incompatible. Recent progress in string theory has changed this perspective, showing that they are actually intertwined in a deep and surprising way, making them instead inseparable! This progress originated from the realization that black holes are quantum objects. This eventually led to the holographic principle which showed that gravity is not fundamental but emerges from quantum mechanics. A rich connection between spacetime and quantum information was then uncovered, in which fundamental aspects of quantum mechanics, such as entanglement, are equivalent to simple properties of spacetime, such as geometric connections. This thesis will develop some aspects of these ideas around three key topics.

The first topic is the holographic principle, in the form of the AdS/CFT correspondence, which is a precise equivalence between quantum gravity in AdS spacetimes and conformal field theories. The power of this duality comes from the fact that it gives a precise definition of quantum gravity while also providing a window into the strong coupling dynamics of field theories. Chapter 2, based on the paper [1], reviews the $\mathrm{nAdS}_2/\mathrm{nCFT}_1$ correspondence, and explains how to apply it to the Kerr black hole. One goal is to obtain simplified models of quantum gravity for realistic black holes. A motivating question can be: *What is the quantum mechanics of the Kerr black hole?*

The second topic is black hole thermodynamics. Developed in the seventies, it has been one of the main inspiration behind the more recent insights and developments. Chapter 3, based on the paper [2], investigates the properties of quantum corrections to black hole entropy in the context of string theory, focusing on a special class of corrections of logarithmic type, which are fully captured in the low energy theory. The driving question is: *What can semiclassical gravity tell us about the black hole microstates?*

The third topic is the connection between quantum information and spacetime. A powerful idea is that spacetime connectivity and entanglement are essentially the same thing, which is often referred to as "ER=EPR". This relationship was recently strengthened by showing that wormholes can be made traversable with a particular quantum teleportation protocol. Chapter 4, based on the paper [3], investigates the limits of this construction. The broad motivation is the question: *How large can quantum effects be in gravity?*

At the beginning of each chapter, we give an introduction to the relevant background. In this general introduction, we paint a broad overview of these topics, trying to highlight recent developments.

1.1 Black holes thermodynamics

In 1973, Bekenstein conjectured that black holes carry an entropy proportional to their horizon area in Planck units [8]. This proposal was put on firmer footing a year later, when Hawking demonstrated that quantum effects implied that black holes had a temperature [9]. This led to the Bekenstein-Hawking formula for black hole entropy

$$S = \frac{A}{4G\hbar} \ , \tag{1.1}$$

and the recognition that quantum black holes behave as standard thermodynamical systems. This formula shows that quantum gravity has very unexpected features such as the scaling of the number of degrees of freedom like the area instead of the volume. This counterintuitive fact has been at the root of the holographic principle, which is discussed in the next section.

1.1.1 Euclidean path integral

Hawking's derivation of black hole radiation was a difficult computation of quantum field theory in curved spacetime. A simpler and more conceptual approach was described later by Gibbons and Hawking [10]. They showed that the thermodynamical properties of general relativity can be derived using a Euclidean path integral, as for ordinary quantum systems.

The idea is that the partition function of quantum general relativity should be given by the Euclidean path integral

$$Z = \int Dg\, e^{-S[g]} \ , \tag{1.2}$$

with prescribed boundary conditions. For example, to obtain the thermal partition function at inverse temperature β, we should take Euclidean time to be a circle of length β. This path integral can then be evaluated in a saddle-point approximation. The leading saddle-point is the Euclidean black hole geometry and reproduces correctly the Bekenstein-Hawking formula.

This also provides a simple derivation of the Hawking temperature. The periodicity in Euclidean time of the saddle-point is fixed by demanding regularity at the horizon. The fact that black hole thermodynamics can be derived from the usual path integral approach suggests that black holes should be considered as ordinary quantum systems. However, black hole radiation and evaporation lead to an apparent conflict with quantum mechanics. This is Hawking's information paradox [11] which we will discuss in more details below.

The advantage of the Euclidean gravity approach is that it provides a tentative definition for the exact quantum entropy of the black hole. It suggests that the exact partition function is the full path integral, while the Bekenstein-Hawking formula arises in a saddle-point approximation. Of course, this definition is not really useful since the gravitational path integral is hard to compute and ambiguous because of UV divergences. Nonetheless, we will see that for a special type of quantum correction, of logarithmic type, the Euclidean path integral gives an unambiguous answer. This will be the subject of Chapter 3.

1.1.2 Microscopic counting

The understanding of black hole entropy that we have, from the Hawking temperature or from the Euclidean path integral, shed no light on the nature of the black hole microstates. This requires a complete theory of quantum gravity, such as string theory. One of the major successes of string theory was indeed the counting of black hole microstates for a 5d supersymmetric black hole, in perfect agreement with the Bekenstein-Hawking formula [12]. This counting was possible because the degeneracy is protected by supersymmetry and could be computed in a weakly coupled regime. This matching gives definitive evidence that black holes are ordinary quantum systems, although it doesn't directly shed light on the nature of the microstates in the black hole regime.

The Strominger-Vafa computation made it look like that the precise agreement was possible due to details of the configurations. This was rather unsatisfactory given the expected universality of the Bekenstein-Hawking formula. Strominger later realized [13] that the matching could be done more generally, by simply applying the Cardy formula using the Brown-Henneaux central charge, in the context of the AdS_3/CFT_2 correspondence.

We should mention that the problem of microscopic counting has lead to interesting connections with the theory of modular forms in number theory. As an illustration, we can mention that an exact counting formula for $\mathcal{N} = 4$ black hole microstates was proposed in [14, 15] using the Igusa cusp form and has lead to interesting connections with the theory of mock modular forms [16].

More recently, the microstates of BPS black holes in AdS_{d+1}, for $d \geq 3$, have been counted in terms of the dual CFT_d. This was first done for supersymmetric black holes solutions of M-theory on $AdS_4 \times S^7$ using supersymmetric localization in ABJM theory [17] and has now been generalized in many directions. In particular, this has lead to a resolution [18–20] of the long-standing mismatch between the superconformal index of $\mathcal{N} = 4$ super Yang-Mills and the entropy of BPS black holes in $AdS_5 \times S^5$ [21].

1.1.3 Logarithmic corrections

The exact black hole entropy is the logarithm of the number d_{micro} of microstates. The Bekenstein-Hawking formula arises as the leading term for large entropy, and has been successfully checked in the examples where d_{micro} is known, as was discussed above. Can this matching be extended beyond leading order?

The logarithmic correction is the coefficient C in an expansion of the form

$$\log d_{\text{micro}} = \frac{A_H}{4G} + \frac{C}{2} \log\left(\frac{A_H}{G}\right) + \dots \ , \tag{1.3}$$

where A_H is the area of the horizon. We consider here a regime where A_H is large and $\dots$ contains subleading terms. The logarithmic correction has a special status because it can be computed solely from the two-derivative low energy effective theory, depends only on the massless field content and thus constitutes a powerful "infrared window into the microstates" [22]. We review the details of this derivation in section 3.1.

To give a flavor of these corrections, we give the answer for the Schwarzschild black hole in Einstein gravity with minimally coupled fields:

$$C = \frac{1}{90}\left(2n_S - 26n_V + 7n_F - \frac{233}{2}n_\psi + 154\right) \ , \tag{1.4}$$

where n_S, n_V, n_F and n_ψ are respectively the number of massless scalar, vector, Dirac and Rarita-Schwinger fields [22]. We see that the logarithmic correction is highly sensitive of the specific theory of which the black hole is a solution. Notably, this was used to challenge loop quantum gravity which predicts an incorrect logarithmic correction for the Schwarzschild black hole.

To compare with microscopic counting, we need to compute the logarithmic correction in low energy effective theories coming from string theory. For extremal black holes, using the definition of quantum entropy of [23], the logarithmic corrections for $\mathcal{N}=4$ and $\mathcal{N}=8$ black holes were computed in [24] and for rotating black holes in [25]. They were successfully matched with the predictions of the corresponding microscopic formulas.

It has been of interest to compute these corrections in more general settings, for black holes for which we don't know the microscopic description. One motivation is that the structure of the logarithmic corrections could provide insight into the microscopic theory, in an analogous way to how low energy data can constrain the high energy spectrum, *e.g.* by modular invariance in a CFT_2. The computation of logarithmic corrections for the (non-extremal) Kerr-Newman black hole in general $\mathcal{N} \geq 2$ supergravities was done in [26] and an interesting pattern was found. It was observed that C is always independent on the black hole charges. This is not the generic expectation and requires delicate cancellations between bosons and fermions. The goal of the work presented in Chapter 3 is to challenge this pattern against a different class of black holes, in an effort to understand what it teaches us about the black hole microstates.

The above discussion takes place in asymptotically flat space, but logarithmic corrections have also been considered in AdS. In [27], the logarithmic correction to the partition function of supergravity in $\mathrm{AdS}_4 \times S^7$ was computed and matched with the prediction from the dual ABJM theory. More recently, the logarithmic correction to the entropy of AdS_4 black holes was obtained from a microscopic counting using supersymmetric localization in the CFT_3 and matched with the supergravity computation [28, 29].

1.2 The holographic principle

The Bekenstein-Hawking entropy formula shows that the number of true degrees of freedom is proportional to the area instead of the volume. A heuristic way to interpret this could be that all the information is encoded on the black hole horizon. Because black holes correspond to the maximum amount of energy that we can fit in a given region, this suggests the holographic principle: in quantum gravity, the information content of a region should be, in some sense, encoded in the area of this region [30, 31].

1.2.1 AdS/CFT dictionary

A concrete proposal of a holographic duality was put forward by Maldacena who suggested that string theory in an AdS spacetime was dual to a CFT living on its asymptotic boundary [32]. A precise dictionary was given [33,34]: it was proposed that the full quantum gravity path integral with fixed asymptotic values of the fields is equal to the CFT partition function with insertion of sources for dual CFT operators. This can be written as

$$Z_{\text{AdS}}\left[\phi_{\partial M}(x)\right] = \left\langle \exp\left(\int_{S^d} \phi_{\partial M}(x)\mathcal{O}(x)\right)\right\rangle_{\text{CFT}} \tag{1.5}$$

where x is a boundary coordinate. This proposal has been thoroughly tested and can be viewed as the first fully non-perturbative definition of quantum gravity, although it only works for asymptotically AdS spacetimes.

We have learned a lot about quantum black holes from AdS/CFT. According to the duality, an AdS black hole is dual to a CFT state that is "close" to a thermal state (in the sense of the eigenstate thermalization hypothesis). The black hole microstates are identified with high energy states of the dual CFT. Holography was used to show that black holes are the most extreme objects in nature, since they thermalize "as fast as possible" [35, 36] and are "fast scramblers", *i.e.* maximally chaotic quantum systems [37–39]. This gives a picture of a quantum black hole as a strongly coupled system made of many degrees of freedom. We will see that the $\text{nAdS}_2/\text{nCFT}_1$ correspondence provides a simple realization of this picture.

1.2.2 The $\text{nAdS}_2/\text{nCFT}_1$ correspondence

The case of the $\text{AdS}_2/\text{CFT}_1$ correspondence, which should naively be the simplest possible case, has been puzzling for a long time. It is particularly interesting because AdS_2 arises universally in the extremal limit of higher-dimensional black holes. As a result, it is expected that the $\text{AdS}_2/\text{CFT}_1$ correspondence would be a theory of the black hole microstates.

It is precisely this simplicity that has prevented the definition of a useful correspondence in the early days of AdS/CFT. It was understood that this correspondence is actually trivial, in a sense that it is just a theory of ground states and has no dynamics [40]. More recently, it was realized that the correspondence could be made non-trivial by slightly breaking the exact conformal symmetry, leading to the near-AdS_2/near-CFT_1 correspondence [41–44]. It was shown that near-AdS_2 physics is controlled by a universal pattern of spontaneous and explicit symmetry breaking.

This was sparked by the SYK model [45, 46], an ensemble of quantum mechanical theories with Hamiltonian

$$H = \sum_{i,j,k,\ell} J_{ijk\ell} \psi_i \psi_j \psi_k \psi_\ell \ , \tag{1.6}$$

where $\{\psi_i\}$ is a collection of N Majorana fermions in one dimension, and $J_{ijk\ell}$ are random couplings with a Gaussian distribution. Notably, this model can be analytically solved and shown to be maximally chaotic [46, 47]. At low energies, this model has an emergent $\mathrm{Diff}(S^1)$ symmetry of time reparametrization, which is broken to $\mathrm{SL}(2, \mathbb{R})$ by the Schwarzian effective action

$$I[f] = \int dt \, \{f(t), t\} \ . \tag{1.7}$$

This is also the structure appearing in near-AdS_2 gravity, such as in Jackiw-Teitelboim (JT) gravity reviewed in section 2.1. The duality between JT gravity and a subsector of the SYK model is an example of what we mean by the $\mathrm{nAdS}_2/\mathrm{nCFT}_1$ correspondence [47].

JT gravity is more than a toy model since it precisely captures the leading near-extremal dynamics of spherically symmetric black holes. For rotating black holes, JT gravity is not a consistent truncation [48]. In Chapter 2, we will see how to apply near-AdS_2 holography to the Kerr black hole.

The SYK model can be seen as a UV completion of JT gravity. It is not dual to Einstein gravity because the string scale in the bulk theory is comparable to the AdS_2 radius. It can nonetheless be viewed as a toy model of a quantum black hole, which has lead to many applications and concrete computations. For example, SYK was used to study the process of wormhole formation in real time [49].

JT gravity can also be considered as a UV complete theory in its own right. In fact, one can compute the full Euclidean path integral, and demonstrate that JT gravity is dual to a random matrix ensemble [50]. This has led to a better understanding of the discreteness and statistics of black hole microstates from the gravitational point of view. It has also shown that after including wormholes (connected contributions to the path integral), we obtain sensible answers to physical quantities (such as the spectral form factor). Notably, these wormholes generally imply that the gravitational theory is dual to an ensemble of theories instead of a unique one.

1.2.3 The information paradox

Holography has enabled new insights on the black hole information paradox, which we briefly review here. This involves many ideas coming from quantum information theory which are described in more details in the next section.

A direct consequence of AdS/CFT is the fact that information is actually preserved during black hole evaporation because the evolution with the CFT Hamiltonian is unitary. However, this doesn't provide a gravitational picture on how the information "gets out", which has been the motivation for many recent developments.

In 2013, a refinement of the information paradox was put forward by AMPS [51]. This "firewall paradox" suggested that the preservation of information and the equivalence principle are inconsistent with each other. The proposed solutions [52–55] involve a radical revision of locality, requiring a strong version of black hole complementarity: the interior and exterior degrees of freedom of black holes should, in some sense, be the same. We will come back to this in the next section.

The $\text{nAdS}_2/\text{nCFT}_1$ correspondence has played a central role in recent developments on the information paradox and the black hole interior. After introducing a coupling between an evaporating AdS_2 black hole and an external bath, the entropy of the bath was shown to follow the Page curve [56, 57] after using the holographic prescription to compute entanglement entropy as reviewed in the next section. This has led to the island prescription [58, 59] which was proven using replica wormholes [60, 61] appearing as saddle-points in the replicated Euclidean path integral. It was noticed that these new geometries could only make sense if some kind of average was taking place. It would be interesting to use the results of this thesis and explore whether these ideas could be applied in the more realistic context of the near-extreme Kerr black hole.

1.3 Spacetime and quantum information

We have seen that black holes can be thought of as complicated quantum systems with many degrees of freedom. The black hole description is powerful because it gives simple pictures for complicated effects, such as thermalization and chaos, which are usually out of the reach of analytical methods. However, it doesn't provide an understanding of *how* these degrees of freedom combine to produce such a simple geometrical description. Quantum information, and especially the idea of entanglement, has provided insights into this question, some of which will be discussed in this section.

1.3.1 Holographic entanglement entropy

Entanglement is a striking property of quantum systems which has no classical analog. For two systems A and B, respectively described by Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, a state of the combined system $\mathcal{H}_A \otimes \mathcal{H}_B$ is entangled if it cannot be written as a tensor product of a state in $\mathcal{H}_A$ and a state in $\mathcal{H}_B$. A natural measure of entanglement is the entanglement entropy, which is the von Neumann entropy of the density matrix obtained after tracing over the system B

$$S_{\text{EE}} = -\text{Tr}\,\rho_A \log \rho_A, \qquad \rho_A = \text{Tr}_{\mathcal{H}_B} |\psi\rangle\langle\psi| \ . \tag{1.8}$$

In 2001, Maldacena proposed [62] that the eternal AdS black hole is dual to an entangled state of two identical CFTs: the thermofield double state

$$|\text{TFD}\rangle = \sum_E e^{-\beta E/2} |E\rangle_L \otimes |E\rangle_R \ , \tag{1.9}$$

where the sum is over the CFT energy spectrum. The entanglement entropy, obtained after tracing over one of the two CFTs, is the thermal entropy on one side, which is equal here to the the Bekenstein-Hawking entropy. This gives, in this special setup, an interpretation of black hole entropy as entanglement entropy.

The central role of entanglement in the emergence of spacetime was appreciated seven years later, when Ryu and Takayanagi made a remarkable proposal [63, 64] which considerably generalizes the above observation. A covariant formulation was later given by Hubeny, Rangamani and Takayanagi [65]. The proposal is that the entanglement entropy associated to a subregion A of a holographic CFT, can be computed holographically by the formula

$$S(A) = \frac{\text{Area}(\Sigma_A)}{4G\hbar} \ , \tag{1.10}$$

where Σ_A is the bulk surface of extremal area that is homologous to A (*i.e.* it can be continuously deformed into A). Although this formula is similar to the Bekenstein-Hawking entropy, it really computes something different. The Bekenstein-Hawking entropy is a coarse-grained entropy, which is about the number of all states corresponding to the same black hole geometry. In contrast, the Ryu-Takayanagi formula computes fine-grained entropy, which depends in a precise way on the exact state. This relation between the quantum state and the spacetime geometry led to the proposal [66] that spacetime is actually built up from quantum entanglement. We should note that an important generalization including bulk quantum effects was proposed by Engelhardt and Wall [67].

This formula was proven in [68–70] by generalizing the original Gibbons-Hawking derivation of gravitational entropy, where the U(1) symmetry along the Euclidean time circle is generalized to a $\mathbb{Z}_n$ "replica" symmetry permuting n copies of the system, allowing to compute much more general gravitational entropies.

The connection between spacetime and entanglement has also dynamical content. Using the Ryu-Takayanagi formula, it was shown in [71, 72] that the linearized Einstein equation is equivalent to the first law of entanglement, which is a universal relation satisfied by entanglement entropy. In the paper [6], which will not be presented here, we have studied the flat space version of this derivation.

1.3.2 Entanglement and connectivity

Let's review the firewall paradox [51]. We consider an old evaporating black hole which started in a pure state. Two quanta A and B are pair produced close the horizon. A falls into the black hole while B comes out as Hawking radiation. Smoothness of the horizon requires A and B to be strongly entangled. Assuming that information is preserved, B should also be strongly entangled with an earlier Hawking quanta E, because unitarity requires the late radiation to purify the early radiation. However, this violates the monogamy of entanglement: quantum mechanics doesn't allow a particle to be strongly entangled with two different particles at the same time. Moreover, a single observer can collect the three qubits along a causal trajectory and observe, in his own hand, such a violation of quantum mechanics.

As mentioned above, the proposed solutions [52–55] relied on a strong form of black hole complementarity, where for a sufficiently old black hole, the interior is somehow completely encoded in the exterior. In the above setup, the resolution is roughly that A and E are secretly the same particle.

This idea doesn't contradict our observed locality because the experience of a semiclassical observer is described in a tiny subspace of the Hilbert space, called the small Hilbert space or code subspace, in which approximate locality holds. It's only for very fine-grained observables, such as the von Neumann entropy, that the effect of complementarity becomes important. This can be understood in the interpretation of AdS/CFT as a quantum error correcting code [73, 74].

The firewall paradox is particularly well-posed for the eternal AdS black hole, where it was resolved by Maldacena and Susskind [55]. It was argued that the extraction of E from the early radiation sends a particle through the wormhole, which disrupts the entanglement between A and B, so that the contradiction is avoided. This resolution suggested that a similar idea might work more generally. That is, entangled particles should always be connected by "quantum wormholes", which might not be semiclassical geometries, but still allow particles to be sent in. This is known as the ER=EPR proposal. We will see that the traversable wormhole protocol, discussed in the next section, gives evidence for this conjecture.

1.3.3 Traversable wormholes

Although traversable wormholes have always fascinated the general public, they were considered unphysical for a long time, because they require matter which violates the null energy condition. A related fact is that they are in tension with causality because they can be used to create time machines [75]. It turns out that quantum effects can violate the null energy condition and to preserve causality, it is sufficient to require a weaker energy condition: the achronal averaged null energy condition [76–79]. This allows traversable wormholes as long as they are not shortcuts: the path outside the wormhole must always be faster. This is discussed in more details in section 4.1.

Such traversable wormholes were in fact recently constructed. The first example came from a natural holographic setup, the Gao-Jafferis-Wall protocol [80]. Starting with the eternal AdS black hole, it was shown that simply introducing a coupling between the two CFTs makes the wormhole traversable. Although this coupling is non-local from the bulk perspective, it is a perfectly consistent procedure from the CFT perspective. This deformation creates negative energy in the bulk which allows a light ray to defocus and come out of the wormhole. We refer to Figure 4.1 for an illustration.

This protocol strengthens the ER=EPR proposal. It shows that an important property of entanglement, as a resource for quantum teleportation, is also present in the case of wormholes. It also gives a way to test this proposal in principle. An external experimenter, having two holographic CFTs in his possession, can put them in the thermofield double state and send an observer inside one of the dual black hole. He can then use the Gao-Jafferis-Wall protocol to make the observer come out on the other side. When asked how the trip was, the traveling observer can explain that he was feeling perfectly fine, being in free fall the entire time. Without this protocol, we can still test ER=EPR by sending two observers from each side and see whether they meet, but the result of the experiment can never be communicated to the outside, and dies with the infalling observers at the singularity. See [81] for a more detailed discussion along these lines.

In Chapter 4, we will explore the limits of the Gao-Jafferis-Wall protocol by attempting to construct an eternal traversable wormhole.

2 Quantum dynamics of near-extremal black holes

What is the quantum mechanics of the Kerr black hole?

Symmetries have played an important role in accounting for the quantum properties of black holes, and particularly the enhancement of symmetries that takes place for extremal and near-extremal black holes [13, 82, 83]. The extremal limit of a black hole achieves zero Hawking temperature, even though the entropy remains finite and large. More prominently, it exhibits conformal invariance in the near-horizon region and implies the existence of an AdS_2 factor [84–90]. Our understanding of (near-)extremal black holes is therefore tied to AdS_2 gravity, and our progress relies on our understanding of this instance of AdS/CFT.

2.1 *Introduction: near-AdS_2 holography*

In the early days of AdS/CFT, it was realized that AdS_2 holography is actually trivial, in the sense that it has trivial dynamics. It was shown in [40] that the backreaction is too strong in AdS_2: an excitation with non-zero energy destroys the AdS_2 asymptotics. This was argued by considering the Reissner-Nordström black hole and showing that there is no way to take a decoupling limit which leaves non-trivial excitations in the near-horizon region. This was also demonstrated explicitly using a dilaton model of 2d gravity. An alternative way to understand this is from the point of view of the dual CFT_1. There, conformal invariance implies that the trace of the stress-tensor vanishes, which in 1d implies that the Hamiltonian vanishes. Equivalently, $\rho(E) = e^{S_0}\delta(E)$ is the only normalizable density of states that is scale invariant. In summary, AdS_2 holography is a theory containing only ground states.

Although it doesn't have dynamics, the study of these ground states is still interesting. After all, they are the extremal black hole microstates! Indeed, the AdS_2/CFT_1 correspondence can be used to give a rigorous definition of the quantum entropy of extremal black holes [91, 92], as we will review in section 3.1.1.a. This allows a definition of the quantum corrections to black hole entropy from the "macroscopic" point of view, which can be matched against the exact counting formulas, known for black holes with enough supersymmetries. In particular, the leading logarithmic correction depends only on semiclassical data, thus offering an infrared window into the microstates. This direction will be explored in Chapter 3. In this Chapter, we will explain how AdS_2 holography can be made dynamical, and in particular, how to apply holography to the near-extreme Kerr black hole.

2.1.1 Near-AdS_2 dynamics in JT gravity

In 2016, Almheiri-Polchinski studied 2d dilaton-gravity theories as models of backreaction in AdS_2 [41]. Later in that year, inspired by the SYK model, the universal dynamics corresponding to the breaking of the conformal symmetry in AdS_2 was described by Jensen [42], Maldacena-Stanford-Yang [43] and Engelsöy-Mertens-Verlinde [44] in a model known as Jackiw-Teitelboim (JT) gravity. This was called *near-AdS_2* because it captures the leading effect that takes us away from AdS_2.

JT gravity is described by the action

$$\begin{aligned} I_{\rm JT} &= \frac{S_0}{2\pi}\left[\frac{1}{2}\int_{\mathcal{M}}\sqrt{-g}\,R+\int_{\partial\mathcal{M}}\sqrt{-h}\,K\right] \\ &\quad+\left[\frac{1}{2}\int_{\mathcal{M}}\sqrt{-g}\,\Phi(R+2)+\int_{\partial\mathcal{M}}\sqrt{-h}\,\Phi(K-1)\right] . \end{aligned} \tag{2.1}$$

The first line gives a topological term which in the black hole context, corresponds to the extremal black hole entropy S_0. The second line contains the leading deviation from extremality controlled by the dilaton Φ.

Let's now analyze this action. The equation of motion for the dilaton sets $R = -2$ as a constraint. This implies that the solution for the metric is a patch of AdS_2. In Fefferman-Graham gauge, the most general metric with $R = -2$ takes the form

$$ds^2 = -r^2\left(1+\frac{s(t)}{2r^2}\right)^2 dt^2+\frac{dr^2}{r^2} . \tag{2.2}$$

where we have imposed a Dirichlet boundary condition for the asymptotic metric. We see that the space of solutions for the metric is parametrized by an arbitrary function $s(t)$. The equation of motion for the dilaton is

$$\nabla_\mu \nabla_\nu \Phi - g_{\mu\nu} \Box \Phi + g_{\mu\nu} \Phi = 0 \ . \tag{2.3}$$

where we used the AdS$_2$ background (2.2). This equation implies that the solution takes the form

$$\Phi(t,r) = \nu(t) r + \frac{\mu(t)}{r} \ , \qquad \mu(t) = -\frac{1}{2}(s(t)\nu(t) + \nu''(t)) \tag{2.4}$$

where $s(t)$ and $\nu(t)$ are tied by the equation

$$\nu^{(3)}(t) + 2s(t)\nu'(t) + s'(t)\nu(t) = 0 \ . \tag{2.5}$$

This equation contains the dynamics of JT gravity: it ties the AdS$_2$ background, parametrized by $s(t)$, with the source $\nu(t)$ of the dilaton.

For Poincaré-AdS$_2$, which corresponds to $s(t) = 0$, the solution is

$$\nu(t) = c_1 + c_2 t + c_3 t^2 \ , \tag{2.6}$$

where c_1, c_2 and c_3 are arbitrary constants. It can be observed that the solution for a general $s(t)$ can be obtained from this one after acting with a large diffeomorphism of the form

$$\begin{aligned} t &\longrightarrow f(t) + O(r^{-2}) \ , \\ r &\longrightarrow \frac{r}{f'(t)} + O(r^{-1}) \ . \end{aligned} \tag{2.7}$$

We give here only the asymptotic form of this diffeomorphism, see (2.37) for its exact expression. Acting on the Poincaré-AdS$_2$ metric, this diffeomorphism gives the metric (2.2) with

$$s(t) = \{f(t), t\} \ , \tag{2.8}$$

which is the Schwarzian derivative of $f(t)$. The solution for the dilaton is then given by

$$\nu(t) = \frac{1}{f'(t)} \left(c_1 + c_2 f(t) + c_3 f(t)^2 \right) \ . \tag{2.9}$$

As we see, the different solutions of the theory are related by large diffeomorphisms, which correspond on the boundary to time reparametrizations. For example, the thermal solution corresponds to the choice $f(t) = e^{\varepsilon t}$ in the above equations. This gives $s(t) = -\varepsilon^2/2$ yielding the AdS$_2$-Rindler geometry (also known as the AdS$_2$ black hole) with inverse temperature $\beta = 2\pi/\varepsilon$.

The pure AdS$_2$ subsector corresponds to solutions with vanishing dilaton. It carries a full Diff(S^1) symmetry given by the diffeomorphism discussed above. Choosing the Poincaré vacuum, corresponding to $s(t) = 0$, spontaneously breaks this symmetry to its $\mathrm{SL}(2,\mathbb{R})$ subgroup. We can think of the reparametrization mode $f(t)$ as the associated Goldstone mode. The solutions are then labeled by $f(t)$ and are all degenerate, reflecting the fact that the pure AdS_2 subsector is topological.

Turning on the dilaton gives rise to non-trivial dynamics, by taking us to the so-called near-AdS_2 regime. This is achieved by the Schwarzian effective action

$$I_{\mathrm{bdy}}[f] = \frac{1}{8\pi G_2}\int \mathrm{d}t\,\nu(t)\{f(t),t\}\ , \tag{2.10}$$

which comes from the renormalized version of the Gibbons-Hawking-York boundary term, corresponding to the $K-1$ appearing in (2.1). This supplements the spontaneous symmetry breaking Diff$(S^1) \to \mathrm{SL}(2,\mathbb{R})$ by an explicit breaking of the same symmetry, and lifts the degeneracy by giving a non-trivial action to the different solutions. Moreover, the variation of I_{bdy} with respect to f reproduces the relation (2.9) which implies that the full dynamics of JT gravity is captured by the Schwarzian theory.

Although compelling, the above discussion (which follows closely [43]) is rather heuristic, and we refer to [93] for a more rigorous perspective on the Schwarzian action, from the study of the variational principle and asymptotic symmetries, the corresponding gravitational charges being analyzed in [5].

2.1.2 The Reissner-Nordström black hole

JT gravity is not just a toy model. It actually describes the leading near-extremal dynamics of spherically symmetric black holes. In this section, we will show how JT gravity arises in the near-extreme Reissner-Nordström black hole. It is customary to obtain JT gravity after performing a Kaluza-Klein reduction on the sphere [94–96] but we will take a different approach here: we will remain in four dimensions and describe the gravitational perturbation that controls the near-AdS_2 dynamics. This approach is more suited to generalization to situations where we cannot perform a Kaluza-Klein reduction, such as the near-extreme Kerr black hole that we consider next.

The Reissner-Nordström black hole is a solution of Einstein-Maxwell theory

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{16\pi}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)\ , \tag{2.11}$$

where we have set the 4d Newton constant $G=1$. The geometry and field strength

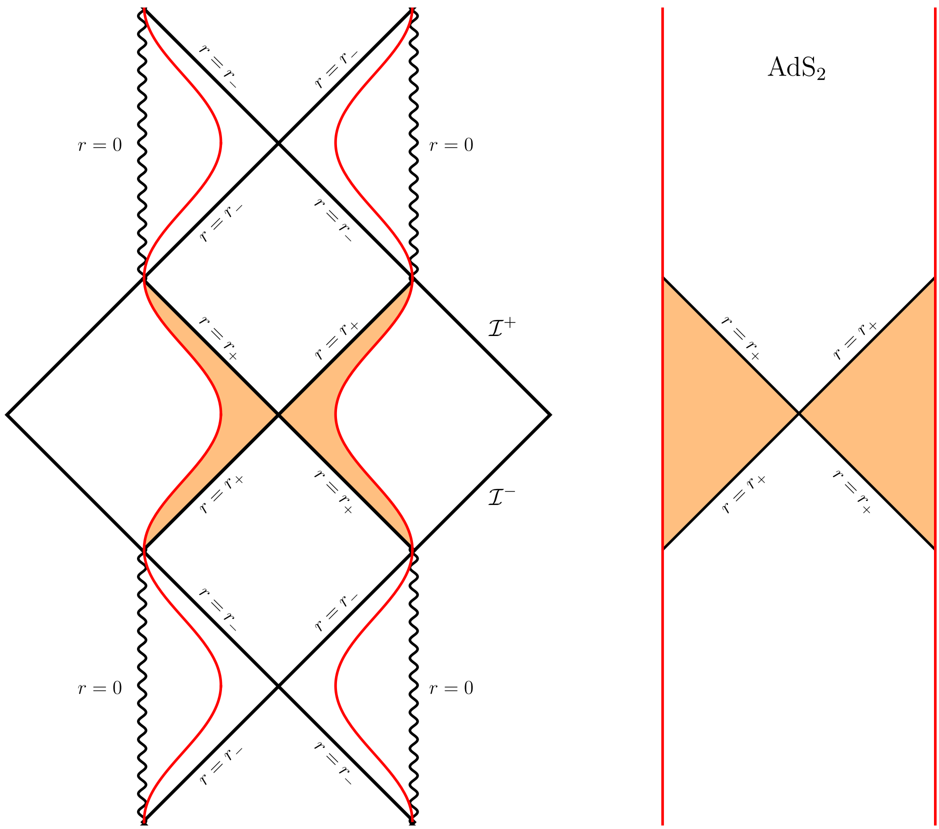

Figure 2.1: Penrose diagrams for a near-extremal black hole and its near-horizon geometry. In the left, we depict the full black hole geometry while in the right, we focus on the near-horizon region. The orange patch is the AdS$_2$-Rindler geometry.

are given by

$$
\begin{aligned}
ds^2 &= -\frac{(r-r_+)(r-r_-)}{r^2}dt^2 + \frac{r^2}{(r-r_+)(r-r_-)}dr^2 + r^2 d\Omega^2 \ , \\
F &= -\frac{Q}{\sqrt{\pi}} \sin\theta \, d\theta \wedge d\phi \ .
\end{aligned} \tag{2.12}
$$

We consider here the magnetically charged black hole. The inner and outer horizons are at $r_\pm = M \pm \sqrt{M^2 - M_0^2}$ and $M_0 = Q$ is the extremal mass. The entropy, temperature and magnetic potential are

$$
S_{\rm BH} = \pi r_+^2, \qquad \beta = \frac{1}{T_H} = \frac{4\pi r_+^2}{r_+ - r_-} \ , \qquad \Phi = \frac{Q}{r_+} \ , \tag{2.13}
$$

and the first law of thermodynamics reads

$$
dM = T_H dS_{\rm BH} + \Phi \, dQ \ . \tag{2.14}
$$

We consider a near-extremal limit where the horizons are separated according to

$$
r_\pm = M_0 \pm \varepsilon\lambda + O(\lambda^2) \ , \tag{2.15}
$$

where ε is a near-extremal parameter and λ is taken to be small. This corresponds to a change of mass M at fixed charge Q. The small Hawking temperature that is generated is

$$
T_H = \frac{\varepsilon\lambda}{2\pi M_0^2} + O(\lambda^2) \ . \tag{2.16}
$$

The near-extremal mass and entropy are

$$
M = M_0 + \frac{T_H^2}{M_{\rm gap}}, \qquad S = S_0 + \frac{2}{M_{\rm gap}} T_H \ , \tag{2.17}
$$

where $M_{\rm gap} \equiv 1/(2\pi^2 M_0^3)$ is a constant which can be interpreted as the smallest temperature at which the semiclassical description is valid, although this interpretation was recently challenged [96]. In fact, the behavior (2.83) is known to be universal in that *any* extremal black hole responds in this way when increasing the mass at fixed charge. As will be shown below, this is actually explained by the near-AdS$_2$ physics [43].

The near-horizon geometry is obtained by performing the change of coordinates

$$
t \to M_0^2 \frac{t}{\lambda}, \qquad r \to M_0 + \lambda r - \frac{1}{M_0}\lambda^2 r^2 \ , \tag{2.18}
$$

and taking the limit $\lambda \to 0$. Here, the λ^2 term is necessary to preserve the Fefferman-Graham gauge. This leads to $\mathrm{AdS}_2 \times S^2$ with the metric

$$\frac{ds^2}{M_0^2} = -\left(1 - \frac{\varepsilon^2}{4r^2}\right)^2 r^2 dt^2 + \frac{dr^2}{r^2} + d\Omega^2 \ . \tag{2.19}$$

The Penrose diagrams of the black hole and of its near-horizon geometry are depicted in Figure 2.1. We can consider instead a general AdS$_2$ background

$$\frac{ds^2}{M_0^2} = -\left(1 + \frac{\{f(t),t\}}{2r^2}\right)^2 r^2 dt^2 + \frac{dr^2}{r^2} + d\Omega^2 \ , \tag{2.20}$$

and the metric (2.33) is recovered for $f(t) = e^{\varepsilon t}$. Such a background can be obtained by starting with the Poincaré metric and acting with a large diffeomorphism (2.7) giving a general AdS$_2$ metric γ_{ab} in Fefferman-Graham gauge

$$\gamma_{ab} dx^a dx^b = -\left(1 + \frac{\{f(t),t\}}{2r^2}\right)^2 r^2 dt^2 + \frac{dr^2}{r^2} \ . \tag{2.21}$$

The previous section has shown that the near-AdS$_2$ physics depends crucially on the specific choice of background so it's important to consider a general one. Now, we would like to deform this geometry by adding a linearized gravitational perturbation. We use the following ansatz

$$\begin{aligned} \frac{ds^2}{M_0^2} &= -(1 - \lambda\, \psi(t,r)) \left(1 + \frac{\{f(t),t\}}{2r^2}\right)^2 r^2 dt^2 + \frac{dr^2}{r^2} + (1 + \lambda\, \Phi(t,r)) d\Omega^2 \ , \\ F &= -\frac{Q}{\sqrt{\pi}} \sin\theta \, d\theta \wedge d\phi \ , \end{aligned} \tag{2.22}$$

where we work at linear order in λ. The equation of motion for the gauge field $\partial_\mu F^{\mu\nu} = 0$ is trivially satisfied. This is an advantage of using the magnetic black hole instead of the electric one: the field strength has no backreaction at this order. The Einstein equation gives

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi \left(F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}\right) \ , \tag{2.23}$$

which leads to the following equation for $\psi(t,r)$ and $\Phi(t,r)$

$$\begin{aligned} \psi &= -2\Phi + 2\Box_2 \Phi + \frac{4}{\sqrt{-\gamma}} \partial_t \left(\sqrt{-\gamma}\, \gamma^{tt} \partial_t \Phi\right) \ , \\ \Phi &= \Phi_0 + \Phi_{\mathrm{JT}} \ , \qquad \nabla_a \nabla_b \Phi_{\mathrm{JT}} - \gamma_{ab} \Box_2 \Phi_{\mathrm{JT}} + \gamma_{ab} \Phi_{\mathrm{JT}} = 0 \ . \end{aligned} \tag{2.24}$$

We recognize Φ_{JT} to be the Schwarzian mode. The field ψ is the backreaction of the 2d metric and Φ_0 is a constant mode associated to an infinitesimal change of charge. As a result, we see that the mode Φ, which controls the area of the sphere (and hence the black hole entropy), satisfies the JT equations of motion. This demonstrates how the JT dynamics is captured in the near-extremal perturbations of the Reissner-Nordström black hole.

It's also possible to obtain the on-shell action on the background. Let's consider the action of Einstein-Maxwell theory

$$S = \int_M d^4x\sqrt{-g}\left(\frac{1}{16\pi}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) + \frac{1}{8\pi}\int_{\partial M} d^3x\sqrt{-h}\,K\ , \tag{2.25}$$

where we have added the Gibbons-Hawking-York. Using holographic renormalization, and integrating over the sphere, we obtain the renormalized action

$$S_{\text{ren}} = -\frac{\lambda M_0^2}{2}\int dt\,\nu(t)\{f(t),t\}\ , \tag{2.26}$$

which is the Schwarzian effective action. The details of the holographic renormalization procedure are explained in section 2.4, where we perform a similar analysis for the Kerr black hole.

As noted in [43], the Schwarzian action explains universal features of the near-extremal thermodynamics. The thermal background corresponds to $f(t) = e^{\varepsilon t}$ and we have

$$\{f(t),t\} = -\frac{\varepsilon^2}{2}, \qquad \nu(t) = \frac{2}{M_0}\ , \tag{2.27}$$

where ε is the near-extremal parameter and the value of $\nu(t)$ can be read from the gravitational perturbation. The Euclidean time circle has length $2\pi/\varepsilon$. This gives the Euclidean on-shell action

$$S_E = \pi M_0^2\lambda\varepsilon\ . \tag{2.28}$$

The variation of the Bekenstein-Hawking entropy due to the perturbation is then given by

$$\delta S_{\text{BH}} = (1+\varepsilon\partial_\varepsilon)(-S_E) = 2\pi M_0^2\lambda\varepsilon\ , \tag{2.29}$$

which matches the near-extremal thermodynamics (2.83). The argument presented in this paragraph only followed from the Schwarzian effective action. Since it is believed to be the universal description of near-AdS$_2$ physics, the Schwarzian action actually explains the linear dependence in temperature in the near-extremal entropy of black holes.

2.1.3 *Introduction to our work*

The near-AdS_2 physics described by the Schwarzian action is completely controlled by symmetries. We thus expect it to be valid universally for near-extremal black holes. However, the JT gravity discussion only demonstrates this for spherically symmetric black holes from which JT gravity is obtained by Kaluza-Klein on the sphere. For rotating black holes, the dynamics is *not* described by JT gravity, as was first demonstrated in [48] for Kerr-AdS_5. This begs the question of how the near-AdS_2 dynamics is realized for the four-dimensional Kerr geometry. This is what we address in our work.

For Kerr, dimensional reduction on the sphere is not a good approach because of the complicated angular dependence of the solution. Instead, we will take the approach given in the previous section for Reissner-Nordström. That is, we will look for the gravitational perturbation of the near-horizon geometry of extreme Kerr which captures the Schwarzian mode.

An important motivation for this work is the fact that the Kerr black hole is the generic black hole solution observed in our universe. When formed from collapsing stars, it is often the case that the black hole ends up spinning rapidly, because the collapse reduces the moment of inertia while conserving angular momentum. The geometry should then be well-approximated by near-extreme Kerr. Recently, gravitational wave astronomy has opened a new window into the universe and routinely observes black hole mergers. Also, the direct image of a Kerr black hole has been captured. This offers the exciting prospect that AdS/CFT, in its near-AdS_2/near-CFT_1 incarnation, could be, in some sense, observed in the sky.

2.2 Near-extreme Kerr geometry

In this section we review properties of the near-extreme Kerr geometry, with particular emphasis on its near-horizon geometry. We start by considering the general Kerr solution,

$$\begin{aligned} ds^2 = & -\frac{\Sigma\,\Delta}{(\tilde{r}^2+a^2)^2-\Delta\,a^2\sin^2\theta}\mathrm{d}\tilde{t}^2+\Sigma\left(\frac{\mathrm{d}\tilde{r}^2}{\Delta}+\mathrm{d}\theta^2\right) \\ & +\frac{\sin^2\theta}{\Sigma}((\tilde{r}^2+a^2)^2-\Delta\,a^2\sin^2\theta)\left(\mathrm{d}\tilde{\phi}-\frac{2aM\tilde{r}}{(\tilde{r}^2+a^2)^2-\Delta\,a^2\sin^2\theta}\mathrm{d}\tilde{t}\right)^2 , \end{aligned} \tag{2.30}$$

with

$$\Delta = (\tilde{r}-r_-)(\tilde{r}-r_+)\ , \quad r_\pm = M\pm\sqrt{M^2-a^2}\ , \quad \Sigma = \tilde{r}^2+a^2\cos^2\theta\ . \tag{2.31}$$

Here r_- and r_+ are the inner and outer horizons. We are using conventions where $G_4 = 1$. M is the mass and $J = aM$ is the angular momentum of the black hole.

The extreme Kerr solution is obtained as the confluence of the inner and outer horizon: $r_+ = r_-$. We are interested in describing the dynamics of Kerr slightly above extremality. In this context, *near-extremality* is defined as a deviation from the extreme limit which keeps J fixed. Implementing it as a limit, we have

$$r_\pm = M_0 \pm \varepsilon\lambda + \frac{\varepsilon^2\lambda^2}{4M_0} + O(\lambda^3) \ , \tag{2.32}$$

where λ is a small parameter that controls deviations away from extremality. M_0 is the value of the mass at extremality, and ε is a constant that controls the deviation of the mass above extremality. Under these conditions, we can identify a near-horizon region. Redefining the coordinates in (2.30) as

$$\tilde{r} = \frac{r_+ + r_-}{2} + \lambda\left(r + \frac{\varepsilon^2}{4r}\right), \qquad \tilde{t} = 2M_0^2\frac{t}{\lambda} \ , \qquad \tilde{\phi} = \phi + M_0\frac{t}{\lambda} \ , \tag{2.33}$$

and taking the limit $\lambda \to 0$ –with other parameters fixed– leads to the line element

$$\begin{aligned} ds^2 \quad &= \quad M_0^2(1+\cos^2\theta)\left[-r^2\left(1-\frac{\varepsilon^2}{4r^2}\right)^2 \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2} + \mathrm{d}\theta^2\right] \\ &\quad + M_0^2\frac{4\sin^2\theta}{1+\cos^2\theta}\left[\mathrm{d}\phi + r\left(1+\frac{\varepsilon^2}{4r^2}\right)\mathrm{d}t\right]^2 \ . \end{aligned} \tag{2.34}$$

For $\varepsilon = 0$, this is the near-horizon geometry of Extreme Kerr (NHEK) [97,98]. For $\varepsilon \neq 0$, we will call this background the near-NHEK geometry.

It is instructive to discuss some properties of (2.34). For $\varepsilon = 0$, we have

$$ds^2 \quad = \quad M_0^2(1+\cos^2\theta)\left(-r^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2} + \mathrm{d}\theta^2\right) + M_0^2\frac{4\sin^2\theta}{1+\cos^2\theta}\left(\mathrm{d}\phi + r\,\mathrm{d}t\right)^2 \ . \tag{2.35}$$

This geometry has four Killing vectors:

$$\xi_{-1} = \partial_t \ , \quad \xi_0 = t\partial_t - r\partial_r \ , \quad \xi_1 = \left(\frac{1}{r^2} + t^2\right)\partial_t - 2rt\partial_r - \frac{2}{r}\partial_\phi \ , \qquad k = \partial_\phi \ . \tag{2.36}$$

These vectors generate an $sl(2) \times u(1)$ algebra which corresponds to the enhanced conformal symmetry of the near-horizon geometry. One can also impose asymptotic boundary conditions on (2.35). In particular, the set of diffeomorphisms preserving the asymptotic metric is [99]

$$\begin{aligned} t &\longrightarrow f(t) + \frac{2f''(t)f'(t)^2}{4r^2f'(t)^2 - f''(t)^2} \ , \\ r &\longrightarrow \frac{4r^2f'(t)^2 - f''(t)^2}{4r\,f'(t)^3} \ , \\ \phi &\longrightarrow \phi + \log\left(\frac{2rf'(t) - f''(t)}{2rf'(t) + f''(t)}\right) \ , \end{aligned} \tag{2.37}$$

where $f(t)$ is an arbitrary function that reflects the freedom of reparametrization the boundary metric. Acting on (2.34), this diffeomorphism gives

$$\begin{aligned} ds^2 &= M_0^2(1+\cos^2\theta)\left[-r^2\left(1+\frac{\{f(t),t\}}{2r^2}\right)^2 \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2} + \mathrm{d}\theta^2\right] \\ &\quad + \frac{4M_0^2\sin^2\theta}{1+\cos^2\theta}\left[\mathrm{d}\phi + r\left(1-\frac{\{f(t),t\}}{2r^2}\right)\mathrm{d}t\right]^2 \ , \end{aligned} \tag{2.38}$$

where

$$\{f(t),t\} = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 \ , \tag{2.39}$$

is the Schwarzian derivative. It is important to note that for $f(t) = e^{\varepsilon t}$, (2.38) reduces to the near-NHEK metric (2.34). At this stage, this implies that NHEK and near-NHEK are just one diffeomorphism away. It is also worth noting that the shift of ϕ in (2.37) is the large gauge transformation discussed in [100].

2.3 Gravitational perturbations

In this section we will study the response of NHEK to a small amount of energy: how the metric responds when we deviate from extremality. Our goal is to find a consistent truncation of the perturbations that captures the Schwarzian mode which is believed to be universal in the response to black hole near extremality. Our strategy is rather simple: we will propose an ansatz for the metric perturbations of NHEK and solve the linearized Einstein equations.

A deviation from extremality is a correction due to the near-horizon parameter λ introduced in (2.33). By inspection of the full on-shell Kerr geometry (2.30), which would correspond to stationary perturbations, it is clear that a suitable ansatz for metric perturbations needs to account for non-trivial θ-dependence. With the insight on the behavior of Kerr, we will consider the following deviation of the

NHEK geometry

$$\begin{aligned} ds^2 &= -M_0^2 \frac{(1+\cos^2\theta + \lambda\tilde{\chi}(t,r))}{1+\lambda\psi(t,r)} r^2 \mathrm{d}t^2 + M_0^2 \left(1+\cos^2\theta + \lambda\chi(t,r)\right)\left(\frac{\mathrm{d}r^2}{r^2} + \mathrm{d}\theta^2\right) \\ &\quad + 4M_0^2 \frac{\sin^2\theta\,(1+\lambda\Phi(t,r))}{1+\cos^2\theta+\lambda\chi(t,r)} \left(\mathrm{d}\phi + r\mathrm{d}t + \lambda A\right)^2 , \end{aligned} \tag{2.40}$$

where the one-form A is supported in the (t,r) subspace

$$A = A_t(t,r,\theta)\mathrm{d}t + A_r(t,r,\theta)\mathrm{d}r \ , \tag{2.41}$$

and captures the angular dependence of the ansatz. We treat the metric at linear order in λ. The metric perturbation $\Phi(t,r)$ parametrizes the change of the volume of the squashed sphere; $\chi(t,r)$ characterizes the squashing parameter that breaks spherical symmetry; $\psi(t,r)$ and $\tilde{\chi}(t,r)$ are introduced for consistency of the ansatz. At this stage it is a guess that χ, $\tilde{\chi}$ and ψ have no θ-dependence, and we will show that this is compatible with the equations of motion. We are not introducing ϕ-dependence since it seems consistent, for the purpose of capturing deviations from extremality, to focus on solutions which respect the isometry due to the Killing vector $k = \partial_\phi$.

We now proceed to solve the linearized Einstein equations

$$R_{\mu\nu} = 0 \ , \tag{2.42}$$

where $R_{\mu\nu}$ is the 4D Ricci tensor, and look at the first correction due to λ in (2.40). The θ-components of this equation are the simplest to solve first. From $R_{t\theta}$ and $R_{\theta\phi}$ we can determine that the one-form can be written as

$$A = \alpha + \varepsilon_{ab}\partial^a\Psi\,\mathrm{d}x^b \ , \qquad \Psi = \frac{1}{2\sin^2\theta}\left[\left(1+\frac{\sin^4\theta}{4}\right)\Phi(t,r) - \chi(t,r)\right] \ , \tag{2.43}$$

with

$$\alpha = \alpha_t(t,r,\theta)\mathrm{d}t + \alpha_r(t,r)\mathrm{d}r \ , \qquad \alpha_t(t,r,\theta) = a_1(t,r) + a_2(r,\theta) \ . \tag{2.44}$$

The components of α are arbitrary functions at this stage. In (2.43) we introduced an auxiliary 2D metric, defined as

$$\gamma_{ab}\mathrm{d}x^a\mathrm{d}x^b = -r^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2} \ , \tag{2.45}$$

and ε_{ab} is the Levi-Civita tensor of this space, with $\varepsilon_{tr} = \sqrt{-\det\gamma_{ab}}$. This is the

AdS$_2$ space appearing in the NHEK geometry (2.35). Using (2.43) in $R_{r\theta}$ and $R_{\theta\theta}$, we can see that $a_2 = 0$, and that $\tilde{\chi} = \chi$. In addition $R_{\theta\theta} = 0$ implies

$$\Box_2 \chi = 2\chi \ , \tag{2.46}$$

where $\Box_2$ is the Laplacian for the AdS$_2$ background (2.45), and therefore χ is an operator of conformal dimension $\Delta = 2$. With this input in place, setting $R_{\phi\phi} = 0$ leads to

$$\psi(t, r) = -\Phi + \Box_2 \Phi - 2\,\varepsilon^{ab}\partial_a \alpha_b \ . \tag{2.47}$$

We have five components left to solve: R_{tt}, R_{tr}, $R_{t\phi}$, R_{rr} and $R_{r\phi}$. Using the previous equations, one of these components is redundant. After some simple manipulations, we find

$$\Phi(t, r) = \Phi_0 + \Phi_{\rm JT}(t, r) \ . \tag{2.48}$$

Here Φ_0 is a constant: this is the degree of freedom that changes the value of M_0, since it can be reabsorbed as a rescaling of the angle ϕ. The field $\Phi_{\rm JT}$ satisfies

$$\nabla_a \nabla_b \Phi_{\rm JT} - \gamma_{ab} \Box_2 \Phi_{\rm JT} + \gamma_{ab} \Phi_{\rm JT} = 0 \ , \tag{2.49}$$

which is the equation of motion of the scalar field in Jackiw-Teitelboim gravity [101, 102]. Finally, we also have

$$\alpha = -\varepsilon_{tr} \partial^t \Phi \, \mathrm{d}r + \tilde{\alpha} \ . \tag{2.50}$$

There is also a constraint on $\tilde{\alpha}$, but this makes it pure gauge: we can remove $\tilde{\alpha}$ via a trivial diffeomorphism. The details are given in Appendix 2.5.

In summary, the linearized perturbations are captured by two fields: χ and Φ. By solving the dynamics of these two fields, dictated by (2.46) and (2.49) one can reconstruct consistently the metric near NHEK. At this stage it is important to make some technical remarks:

1. Our analysis is also a consistent truncation of the linearized Einstein equations around the locally NHEK background (2.38) where we take the ansatz for the perturbations to have the same form as in (2.40). The explicit form of the perturbed metric can be found in (2.60). The solution is given by (2.43)-(2.50), with the modification that the auxiliary 2D metric in (2.45) is

changed to a locally AdS_2 metric:[1]

$$\gamma_{ab}\mathrm{d}x^a\mathrm{d}x^b = -r^2\left(1+\frac{\{f(t),t\}}{2r^2}\right)^2\mathrm{d}t^2+\frac{\mathrm{d}r^2}{r^2}\ . \tag{2.51}$$

In particular, the solutions to (2.49) on this background are of the form

$$\Phi_{\mathrm{JT}} = \nu(t)r+\frac{\mu(t)}{r}\ , \qquad \mu(t) = -\frac{1}{2}(s(t)\nu(t)+\nu''(t)) \tag{2.52}$$

where ν obeys

$$\left(\frac{1}{f'}\left(\frac{(f'\nu)'}{f'}\right)'\right)' = 0\ . \tag{2.53}$$

This equation, whose solutions are given in (2.9), relates the explicit breaking of symmetries in NHEK, due to $\nu(t)$, with the diffeomorphism (2.37) on its boundary, parametrized by $f(t)$. It can also be obtained from the Schwarzian effective action (2.10), as reviewed in the section 2.1.

2. It is instructive to match the perturbations derived in this section with the stationary configuration that would match the behavior of the Kerr black hole. Applying the limit (2.33) to the Kerr geometry (2.30), and comparing the linear order in λ with the perturbations (2.40) for near-NHEK, we obtain

$$\chi_{\mathrm{Kerr}} = \Phi_{\mathrm{Kerr}} = \frac{2}{M_0}\left(r+\frac{\varepsilon^2}{4r}\right)\ , \tag{2.54}$$

and the one-form α is zero. Hence both modes are non-trivial for the Kerr solution.

3. We constructed a consistent truncation of the linearized problem that captures the deviations away from the AdS_2 throat of the extremal Kerr solution. We do not expect (2.40) to be the most general ansatz for gravitational dynamics near the NHEK geometry: additional angular dependence could be added. We have been exploring this question in current work [103]. Using the Teukolsky formalism applied in [104, 105] for NHEK, we have found the most general axisymmetric perturbation of NHEK and are studying how it glues to a perturbation of the full Kerr geometry.

The $nAdS_2$ analysis of the Kerr black hole shares one similarity with the charged counterparts studied in [106, 107]: there is one gravitational mode Φ which satisfies the JT equations of motion (2.49). For Reissner-Nordström black holes, it was consistent to only focus on the dynamics of Φ as the leading effect in devia-

[1] Although the formula (2.50) is not covariant with respect to the 2D metric γ_{ab}, it still holds for a linearized perturbation around near-NHEK.

tions away from extremality. But there are some important differences for Kerr. First, the θ-dependence in (2.43) prevents us from building a 2D effective theory that describes these modes. This is mostly a technical barrier, since it is more cumbersome to keep track of the dynamics of the system. Nonetheless, we expect to be able to quantify, for example, correlation functions of these gravitational perturbations in future work.

The second, and most important, difference relative to Reissner-Nordström black holes is the additional degree of freedom χ that we have found. This is similar to the 5D rotating black holes studied in [48]: there is a squashing mode χ that influences the gravitational perturbations. Remarkably, χ and Φ are both irrelevant operators of conformal dimension $\Delta = 2$. While the dynamics of Φ is restricted by the large diffeomorphism of NHEK, via (2.53), the field χ is a dynamical mode. As indicated by (2.54), the source for χ is turned on for the Kerr solution: this a strong indication that although (2.53) captures some important aspects of the deviations away from extremality, a complete characterization needs to take into account the interactions of Φ with χ.

Large diffeomorphisms play a prominent role in our analysis, which begs for a comparison with Kerr/CFT. A crucial difference is that the asymptotic symmetry group used in [98] had arbitrary functions of ϕ, while here we are considering generators that reparametrize the boundary time.[2] It would be interesting to investigate whether there is a deformation of NHEK that ties the explicit breaking of the conformal symmetry by an irrelevant deformation to the conformal anomaly in the Virasoro algebra of Kerr/CFT. This will require searching for gravitational perturbations that have non-trivial ϕ-dependence, which we have ignored in this work. We hope to pursue this direction in future work.

2.4 On-shell action and thermodynamics

It is instructive to discuss the thermodynamics near extremality, and its ties to the gravitational perturbation Φ. This follows closely the corresponding discussion for Reissner-Nordström in section 2.1.2. The thermodynamic properties of the near-NHEK geometry are as follows [109]: implementing (2.32) on the standard thermodynamic variables, the energy above extremality is

$$E = M - M_0 = \frac{\varepsilon^2\lambda^2}{4M_0} + O(\lambda^3) \ . \tag{2.55}$$

[2] In the context of Kerr/CFT, our symmetry group follows more closely the analysis in [108].

The near-extremal entropy at linear order in λ is

$$S_{\rm BH} = \frac{A_H}{4} = 2\pi M_0^2 + 2\pi M_0\,\varepsilon\lambda + O(\lambda^2)\ , \tag{2.56}$$

and in this limit the Hawking temperature is given by

$$T = \frac{r_+ - r_-}{8\pi M r_+} = \frac{\varepsilon\lambda}{4\pi M_0^2} + O(\lambda^2)\ . \tag{2.57}$$

This allows us to write

$$E = CT^2 + O(T^3)\ , \qquad S = 2\pi M_0^2 + 2CT + O(T^2)\ , \tag{2.58}$$

where $C = 4\pi^2 M_0^3$.

We will see that these thermodynamical properties can be understood using the renormalized on-shell action, along the lines of [43]. Let's consider

$$I_{\rm 4D} = \frac{1}{16\pi}\int_M d^4x\sqrt{|g|}\,R + \frac{1}{8\pi}\int_{\partial M} d^3x\sqrt{|h|}\,K\ , \tag{2.59}$$

which is the standard Einstein-Hilbert action with the addition of the Gibbons-Hawking-York term. We would like to evaluate $I_{\rm 4D}$ on the general perturbation of the locally NHEK background. The on-shell solution is

$$\begin{aligned} {\rm d}s^2 \ =\ & -M_0^2\frac{(1+\cos^2\theta+\lambda\tilde{\chi}(t,r))}{1+\lambda\psi(t,r)}r^2\left(1+\frac{\{f(t),t\}}{2r^2}\right)^2{\rm d}t^2 \\ & +M_0^2(1+\cos^2\theta+\lambda\chi(t,r))\left(\frac{{\rm d}r^2}{r^2}+{\rm d}\theta^2\right) \\ & +4M_0^2\frac{\sin^2\theta\,(1+\lambda\Phi(t,r))}{1+\cos^2\theta+\lambda\chi(t,r)}\left({\rm d}\phi + r\left(1-\frac{\{f(t),t\}}{2r^2}\right){\rm d}t+\lambda A\right)^2\ , \end{aligned} \tag{2.60}$$

which we treat at linear order in λ, and the fields obey (2.43)-(2.50) with background metric (2.51). Replacing (2.60) in the 4D action (2.59) leads to divergences that are common for on-shell gravitational actions. To remove them, we will take a standard route: after specifying a set of boundary conditions, we will build a renormalized action by requiring that its variation is finite. Our setup follows closely the rules of holographic renormalization in AdS gravity, with [48] being the closest example, and any deviation from these rules will be highlighted.

To start, it is convenient to rewrite (2.60) as an asymptotic solution with arbitrary sources for the fields:

$$\begin{aligned} ds^2 &= M_0^2 \frac{(1+\cos^2\theta + \lambda\tilde{\chi}(t,r))}{1+\lambda\psi(t,r)} \gamma_{tt}(t,r)\mathrm{d}t^2 \\ &\quad + M_0^2 (1+\cos^2\theta+\lambda\chi(t,r)) \left(\frac{\mathrm{d}r^2}{r^2} + \mathrm{d}\theta^2\right) \\ &\quad + 4M_0^2 \frac{\sin^2\theta\,(1+\lambda\Phi(t,r))}{1+\cos^2\theta+\lambda\chi(t,r)} \left(\mathrm{d}\phi + a_t(t,r)\mathrm{d}t + \lambda A\right)^2 \ , \end{aligned} \tag{2.61}$$

For $\tilde{\chi}$, ψ, and A we will be using the on-shell values determined by γ_{tt}, Φ and χ as described in section 2.3. For the additional fields, we have

$$\begin{aligned} \sqrt{-\gamma_{tt}} &= \alpha(t)\,r + \frac{\beta(t)}{r} \ , \qquad & a_t &= \alpha(t)\,r - \frac{\beta(t)}{r} + \zeta(t) \ , \\ \Phi &= \nu(t) r + \frac{\mu(t)}{r} \ , \qquad & \chi &= \sigma(t) r + \cdots + \frac{\kappa(t)}{r^2} + \cdots \ . \end{aligned} \tag{2.62}$$

Here we identify α, ν, σ as sources for γ_{tt}, Φ and χ, respectively; the functions β, μ and κ are the corresponding vevs. ζ is the source for a_t, while its charge is one in our conventions.[3] Note that for χ we are only highlighting its source and vev: the dots are subleading terms in the large r expansion that are determined by imposing its equation of motion. In this notation, the solution to equation (**??**) reads

$$\beta(t) = \frac{\alpha(t)\mu'(t)}{\nu'(t)} \ , \qquad \mu(t) = \frac{c_0}{\nu(t)} - \frac{\nu'(t)^2}{4\alpha(t)^2\nu(t)} \ , \tag{2.63}$$

where c_0 is a constant.

The renormalized action is of the form

$$I_{\mathrm{ren}} = I_{\mathrm{4D}} + I_{\mathrm{ct}} \ , \tag{2.64}$$

where I_{4D} is specified above and I_{ct} is a counterterm action. We want to cast our variational problem with respect to the 2D variables in (2.62). Leaving the gauge field fixed, for reasons explained below, we set up the variation of the action as follows:

$$\begin{aligned} \delta I_{\mathrm{ren}} &= \int_\Sigma d^3x\, \pi^{\mu\nu}\delta h_{\mu\nu} \\ &= \int_\Sigma d^3x \left(\Pi_\Phi \delta\Phi + \Pi^{tt}\delta\gamma_{tt} + \Pi_\chi \delta\chi\right) \end{aligned}$$

[3]For a 2D Maxwell field we are simply identifying the electric charge Q from $F_{rt} = Q\sqrt{|\gamma|}$.

$$= \int dt \left(\pi_\alpha \delta\alpha(t) + \pi_\nu \delta\nu(t) + \pi_\sigma \delta\sigma(t)\right) \ , \tag{2.65}$$

where Σ is a cutoff surface of constant r with induced metric $h_{\mu\nu}$. From the first to the second line we are simply casting the variation of the 3D boundary metric $h_{\mu\nu}$ in terms of the 2D fields. In the last line we are specifying the variations of the 2D fields in terms of their sources, and we have integrated over the angular variables (θ, ϕ). Fixing the variation of the gauge field in this notation means that we do not vary the sources appearing in a_t and A. The task is now to build $I_{\rm ct}$ such that the momenta π_α, π_ν, and π_σ are finite as we approach the boundary at $r \to \infty$.

In terms of the 3D variables, the momenta $\pi^{\mu\nu}$ receives a contribution from $I_{\rm 4D}$ which is the usual Brown-York stress tensor:

$$\pi^{\mu\nu}_{\rm 4D} = \frac{\delta I_{\rm 4D}}{\delta h_{\mu\nu}} = -\frac{1}{16\pi}\sqrt{-h}\left(K^{\mu\nu} - K h^{\mu\nu}\right) \ . \tag{2.66}$$

This term will lead to divergences in π_α, π_ν, and π_σ as we take $r \to \infty$; in particular we get

$$\begin{aligned}
\pi_{\alpha,\rm 4D} &= \frac{M_0^2}{2}\left(\nu(t)\, r^2 - \mu(t)\right)\lambda - \frac{M_0^2}{8}\nu(t)(4\,\nu(t) - \pi\sigma(t))\lambda^2 r^3 + \cdots \\
\pi_{\nu,\rm 4D} &= \frac{M_0^2}{2}\left(\alpha(t)\, r^2 - \beta(t)\right)\lambda - \frac{M_0^2}{8}\alpha(t)\left(2\,\nu(t) - (\pi-2)\sigma(t)\right)\lambda^2 r^3 + \cdots \\
\pi_{\sigma,\rm 4D} &= \frac{M_0^2}{32}\alpha(t)\left(4(\pi-2)\,\nu(t) - (4+3\pi)\,\sigma(t)\right)\lambda^2 r^3 + \cdots \ ,
\end{aligned} \tag{2.67}$$

where the dots are higher-order terms in λr, and we have integrated over the angular variables (θ, ϕ). It is important to emphasize that our perturbative expansion is only meaningful at leading order in the deformations we turn on, which implies that $\lambda r \ll 1$ as $r \to \infty$.

The leading divergences in the canonical momenta π_α, π_ν and π_σ can be cancelled using the following counterterms

$$I_{\rm ct} = \frac{M_0^2}{8}\int dt \sqrt{-\gamma_{tt}}\left(c_1\lambda\Phi + c_2\lambda^2\Phi^2 + c_3\lambda^2\chi^2 + c_4\lambda^2\Phi\chi\right) \ , \tag{2.68}$$

where the coefficients are found to be

$$\begin{aligned}
c_1 &= -4, \qquad c_2 = 1, \\
c_3 &= \frac{1}{8}(4+3\pi), \qquad c_4 = 2-\pi \ .
\end{aligned} \tag{2.69}$$

Note that the counterterms used here are very similar to those in [48] which also displays similar equations of motion. Adding the contribution from these counterterms to (2.67), the renormalized momenta are

$$\begin{aligned}\pi_\alpha &= \pi_{\alpha,4\mathrm{D}} + \pi_{\alpha,\mathrm{ct}} = -M_0^2\,\mu(t)\,\lambda + O(\lambda^2 r)\ ,\\ \pi_\nu &= \pi_{\nu,4\mathrm{D}} + \pi_{\nu,\mathrm{ct}} = -M_0^2\,\beta(t)\,\lambda + \frac{3M_0^2}{4}\alpha(t)\kappa(t)\,\lambda^2 + O(\lambda^2 r)\ ,\\ \pi_\sigma &= \pi_{\sigma,4\mathrm{D}} + \pi_{\sigma,\mathrm{ct}} = \frac{3M_0^2}{32}(\pi+4)\,\alpha(t)\kappa(t)\,\lambda^2 + O(\lambda^2 r)\ .\end{aligned} \tag{2.70}$$

We have retained some subleading terms in conformal perturbation theory: this is to illustrate the different behavior of χ compared to Φ. Because the momenta for Φ is influenced by the large diffeomorphism of the background metric, the finite contribution appears at $O(\lambda)$. In contrast, χ behaves as a more traditional propagating field in AdS, and hence the term $\kappa(t)\,\delta\sigma(t)$ appears at $O(\lambda^2)$.

Using (2.70) in (2.65), the renormalized variation is

$$\delta I_{\mathrm{ren}} = -M_0^2\lambda \int dt\,(\mu(t)\delta\alpha(t) + \beta(t)\delta\nu(t)) + O(\lambda^2)\ , \tag{2.71}$$

which can be integrated using the relations (2.63) and evaluated on-shell to give the effective action

$$I_{\mathrm{ren}} = -\frac{M_0^2\lambda}{2}\int dt\left(\nu(t)\{f(t),t\} + \frac{4c_0}{\nu(t)}\right) + O(\lambda^2)\ . \tag{2.72}$$

We can compare with the near-extremal entropy by evaluating this action on the near-extremal black hole. Using (2.34) and (2.54) we have

$$\{f(t),t\} = -\frac{\varepsilon^2}{2}, \qquad \nu(t) = \frac{2}{M_0}, \qquad c_0 = 0\ . \tag{2.73}$$

Going to Euclidean signature by taking $t\to -it_E$, we can derive the near-extremal entropy from the Euclidean renormalized action $I_E = -iI_{\mathrm{ren}}$ on a circle of size $2\pi/\varepsilon$ according to

$$\delta S_{\mathrm{BH}} = (1+\varepsilon\partial_\varepsilon)(-I_E) = 2\pi M_0\varepsilon\lambda\ . \tag{2.74}$$

This matches the linear response of the thermodynamics in (2.56).

Finally, we return to the role of the gauge field in our variational problem. The treatment of this field is more delicate since the source $\zeta(t)$ in (2.62) is subleading compared to its electric charge and the backreaction in (2.43). This is a known effect in 2D theories with a Maxwell field, and how to properly treat this is discussed in detail in [48, 110]. Following that discussion, one simple way to circumvent the issues related to the gauge field is to freeze it in the variational problem, and focus on the remaining variables. This would not be the most general variational problem, but it suffices to capture the Schwarzian effective action as illustrated by our computations.

2.5 Generalization to Kerr-Newman

We have described above the linearized perturbation of near-extreme Kerr that captures the near-AdS$_2$ physics and in particular the Schwarzian mode. This section contains a generalization to the Kerr-Newman black hole.

The Kerr-Newman black hole is a solution of Einstein-Maxwell theory (2.11) corresponding to a rotating charged black hole. In Boyer-Lindquist coordinates (t, r, θ, ϕ), the metric and gauge field are

$$\begin{aligned} ds^2 &= -\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta\, d\phi\right)^2 + \rho^2\left(\frac{dr^2}{\Delta} + d\theta^2\right) + \frac{\sin^2\theta}{\rho^2}\left(a\, dt - (r^2+a^2)d\phi\right)^2 \ , \\ A^{\text{EM}} &= \frac{1}{\sqrt{\pi}}\frac{1}{\rho^2}\left(-Qr(dt - a\sin^2\theta\, d\phi) + P\cos\theta\,(a dt - (r^2+a^2)d\phi\right) \ , \end{aligned} \tag{2.75}$$

where

$$a = \frac{J}{M}, \qquad \rho^2 = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 - 2Mr + a^2 + Q^2 + P^2 \ , \tag{2.76}$$

and M, J, Q and P are respectively the mass, angular momentum, electric and magnetic charge of the black hole. The gauge field is denoted A^{EM} to distinguish it from the 2d gauge field A appeing in the perturbation. The outer and inner horizons are at

$$r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2 - P^2} \ . \tag{2.77}$$

The entropy and inverse temperature are

$$S_{\text{BH}} = \pi(a^2 + r_+^2), \qquad \beta = 4\pi\frac{a^2 + r_+^2}{r_+ - r_-} \ , \tag{2.78}$$

and the angular velocity, electric and magnetic potential at the horizon are

$$\Omega = \frac{a}{a^2 + r_+^2}, \qquad \Phi_E = \frac{Qr_+}{a^2 + r_+^2}, \qquad \Phi_M = \frac{Pr_+}{a^2 + r_+^2} \ . \tag{2.79}$$

The first law gives

$$dM = T_H dS_{\text{BH}} + \Omega\, dJ + Q\, d\Phi_E + P\, d\Phi_M \ . \tag{2.80}$$

We focus below on the magnetic case so we set $Q = 0$.

We consider a near-extremal limit where the horizons are separated according to

$$r_\pm = M_0 \pm \varepsilon\lambda + O(\lambda^2) \ , \tag{2.81}$$

where ε is a near-extremal parameter and λ is taken to be small. This corresponds to a change of mass M at fixed J and P. The small Hawking temperature that is generated is given by

$$T_H = \frac{\varepsilon\lambda}{2\pi(M_0^2 + a_0^2)} + O(\lambda^2) \ . \tag{2.82}$$

The near-extremal mass and entropy take the form

$$M = M_0 + \frac{T_H^2}{M_{\text{gap}}}, \qquad S = S_0 + \frac{2}{M_{\text{gap}}} T_H \ , \tag{2.83}$$

where $M_{\text{gap}} \equiv 1/(2\pi^2 M_0(a_0^2 + M_0^2))$.

To analyze the near-AdS$_2$ dynamics, we focus on the exactly extremal geometry corresponding to $\varepsilon = 0$.[4] The near-horizon geometry is obtained with

$$t \to (M_0^2 + a_0^2)\frac{t}{\lambda}, \qquad r \to M_0 + \lambda, \qquad \phi \to \phi + \frac{a_0}{\lambda} t \ , \tag{2.84}$$

which gives

$$\begin{aligned} ds^2 &= \rho_0^2 \left(-r^2 \mathrm{d}t^2 \frac{\mathrm{d}r^2}{r^2} + \mathrm{d}\theta^2 \right) + \frac{4\sin^2\theta}{\rho_0^2} \left[\frac{a_0^2 + M_0^2}{2} \mathrm{d}\phi + a_0 M_0 r \, \mathrm{d}t \right]^2 \ , \\ A^{\text{EM}} &= -\frac{P}{\sqrt{\pi}} \frac{2\cos\theta}{\rho_0^2} \left[\frac{a_0^2 + M_0^2}{2} \mathrm{d}\phi + a_0 M_0 r \, \mathrm{d}t \right] \ , \end{aligned} \tag{2.85}$$

where we define $\rho_0^2 = M_0^2 + a_0^2 \cos^2\theta, P = \sqrt{M_0^2 - a_0^2}$.

To go to the near-AdS$_2$ regime, we consider a linearized perturbation of this geometry. We will take the following ansatz

$$\begin{aligned} ds^2 = & -\frac{\rho_0^2 + \lambda\chi(t,r)}{1 + \lambda\psi(t,r)} r^2 \mathrm{d}t^2 + (\rho_0^2 + \lambda\chi(t,r)) \left(\frac{\mathrm{d}r^2}{r^2} + \mathrm{d}\theta^2 \right) \\ & + \frac{4\sin^2\theta \, (1 + \lambda\Phi(t,r))}{\rho_0^2 + \lambda\chi(t,r)} \left[\frac{a_0^2 + M_0^2}{2} \mathrm{d}\phi + a_0 M_0 r \mathrm{d}t + \lambda A(t,r,\theta) \right]^2 \ , \end{aligned} \tag{2.86}$$

[4] As for Kerr, a non-zero ε can be obtained by applying the large diffeomorphism (2.7) with $f(t) = e^{\varepsilon t}$.

$$A^{\rm EM} = -\frac{P}{\sqrt{\pi}}\frac{\cos\theta\,(2+\lambda\Phi(t,r))}{\rho_0^2+\lambda\chi(t,r)}\left[\frac{a_0^2+M_0^2}{2}\mathrm{d}\phi + a_0M_0r\mathrm{d}t+\lambda U(t,r)\right] \ ,$$

where we work at linearized order in λ and with the one-forms

$$\begin{aligned} A(t,r,\theta) &= A_t(t,r,\theta)a\mathrm{d}t + A_r(t,r,\theta)\mathrm{d}r \ , \\ U(t,r) &= U_t(t,r)\mathrm{d}t + U_r(t,r)\mathrm{d}r \ . \end{aligned} \tag{2.87}$$

The perturbation corresponding to the black hole geometry is obtained by keeping the $O(\lambda)$ term in the near-horizon limit (2.84) of the extremal black hole. This gives a perturbation which fits into the ansatz (2.86) with

$$\begin{aligned} &\Phi=\psi=\frac{4M_0r}{a_0^2+M_0^2}, && \chi = 2M_0 r \ , \\ &A=-\frac{a_0r^2}{a_0^2+M_0^2}\left((M_0^2-a_0^2)+\frac{1}{2}a_0^2\sin^2\theta\right)\mathrm{d}t, && U=-\frac{a_0(3M_0^2-a_0^2)\,r^2}{2(a_0^2+M_0^2)}\mathrm{d}t \ . \end{aligned} \tag{2.88}$$

This gives a particular solution for the ansatz, corresponding to going away from the near-horizon in the Kerr-Newman geometry.

To find the general solution, we need to solve the equations of motion. Drawing lessons from the Kerr case, we assume the following θ-dependence

$$A(t,r,\theta) = A_0(t,r)+\sin^2\theta\, A_1(t,r)+\frac{1}{\sin^2\theta}A_2(t,r) \ , \tag{2.89}$$

where A_0, A_1, A_2 are one-forms supported on the (t,r) subspace. The equation of motion takes the form

$$E_{\mu\nu}=R_{\mu\nu}-8\pi T^{\rm EM}_{\mu\nu}=0, \qquad T^{\rm EM}_{\mu\nu}=F_{\mu\rho}F_\nu{}^\rho-\frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \ , \tag{2.90}$$

where $F=dA^{\rm EM}$ is the field strength. From $E_{r\theta}, E_{\theta\phi}$ and $E_{t\theta}$, we can solve A in terms of the other fields. The solution is given by

$$A_a = U_a+\varepsilon_{ab}\partial^b\widetilde{\Psi}(t,r), \qquad \widetilde{\Psi}(t,r)=\frac{\rho_0^2}{8a_0M_0\sin^2\theta}(\rho_0^2\Phi(t,r)-2\chi(t,r)) \ , \tag{2.91}$$

where we have introduced the auxiliary AdS_2 metric

$$\gamma_{ab}\mathrm{d}x^a\mathrm{d}x^b = -r^2\mathrm{d}t^2+\frac{\mathrm{d}r^2}{r^2} \ . \tag{2.92}$$

The component $E_{\theta\theta}$ implies that

$$\Box_2\chi = 2\chi \ , \tag{2.93}$$

where $\Box_2$ is taken with respect to (2.92). We also obtain the solution for $\psi(t,r)$ which can be written

$$\psi(t,r) = -\Phi + \frac{1}{M_0^2}\chi - \frac{2}{a_0 M_0}\left(\partial_r U_t - \partial_t U_r\right) \, . \tag{2.94}$$

The other components determine that

$$\Phi(t,r) = \Phi_0 + \Phi_{\rm JT}(t,r) \, , \tag{2.95}$$

where $\Phi_{\rm JT}$ satisfies the JT equation of motion

$$\nabla_a \nabla_b \Phi_{\rm JT} - g_{ab}\Box_2 \Phi_{\rm JT} + g_{ab}\Phi_{\rm JT} = 0 \, . \tag{2.96}$$

Lastly, the component E_{tt} gives a constraint on U, written as

$$\partial_r U_u - \partial_t U_r = -a_0 M_0 \left(\frac{1}{r^2}\partial_t^2 \Phi + \Box_2 \Phi\right) + \frac{a_0}{2M_0}\chi + \frac{F(t)}{2r^2} + G(t) \tag{2.97}$$

The arbitrary pieces F and G are solutions of the homogeneous equations

$$\partial_r \left(r^3 \partial_r(\partial_r U_t - \partial_t U_r)\right) = 0 \tag{2.98}$$

Hence, following the discussion in the Appendix 2.5, it should be possible to remove them by trivial diffeomorphisms. We will thus set $F = G = 0$. The piece proportional to $\partial_t^2\Phi$ is familiar and corresponds to the α of Kerr. To extract it, we define

$$U = \alpha + V, \qquad \alpha = -\varepsilon_{tr}\partial^t \Phi \, \mathrm{d}r \tag{2.99}$$

Then, the equation becomes

$$\partial_r V_t - \partial_t V_r = -\frac{a_0 M_0}{2}\Box_2 \Phi + \frac{a_0}{2M_0}\chi \tag{2.100}$$

From this, we can rewrite the solution for $\psi(t,r)$ as

$$\psi(t,r) = -\Phi + \Box_2 \Phi - 2\varepsilon^{ab}\partial_a \alpha_b \tag{2.101}$$

which is exactly the same expression as in Kerr. We can now solve the constraint on V and we obtain

$$V_a = \varepsilon_{ab}\partial^b \Lambda. \qquad \Lambda = -a_0 M_0 \Phi + \frac{a_0}{4M_0}\chi \, . \tag{2.102}$$

This allows us to rewrite the solution for A. We see that the V corresponds to a shift of $\widetilde{\Psi}$ in A, defining a new field Ψ.

Collecting everything, the solution of the ansatz (2.86) with (2.89) takes the form

$$\begin{aligned}
\psi(t,r) &= -\Phi + 2\Box_2\Phi + 2\varepsilon^{ab}\partial_a\alpha_b \qquad \alpha = -\varepsilon_{tr}\partial^t\Phi\,\mathrm{d}r \\
A_a &= \alpha_a + \varepsilon_{ab}\partial^b\Psi, \qquad \Psi = \frac{1}{2}\left(a_0M_0 + \frac{\rho_0^4}{4a_0M_0\sin^2\theta}\right)\Phi - \frac{a_0^2+M_0^2}{4a_0M_0\sin^2\theta}\chi \\
U_a &= \alpha_a + \varepsilon_{ab}\partial^b\Lambda, \qquad \Lambda = -a_0M_0\Phi + \frac{a_0}{4M_0}\chi\ ,
\end{aligned} \tag{2.103}$$

so that everything is expressed in terms of two fields $\chi(t,r)$ and $\Phi(t,r)$ which further satisfy

$$\Box_2\chi = 2\chi\ , \tag{2.104}$$

and

$$\Phi(t,r) = \Phi_0 + \Phi_{\rm JT}(t,r)\ , \qquad \nabla_a\nabla_b\Phi_{\rm JT} - g_{ab}\Box_2\Phi_{\rm JT} + g_{ab}\Phi_{\rm JT} = 0$$

This is similar to the Kerr case, which is recovered when setting $a_0 = M_0$. We note that the Reissner-Nordström limit $a_0 \to 0$ is singular, corresponding to the fact that the mode χ doesn't exist for Reissner-Nordström.

The near-AdS_2 regime of the Kerr-Newman black hole was studied in [111] where the authors make a comparison between its thermodynamics and that of an SYK-like model with marginal deformations. The perturbation described here allows us to go further because it captures the dynamics. It would be interesting to look for quantum mechanical models capturing some of its features.

Appendices

2.A Redundancies due to diffeomorphisms

In this appendix we determine which components of the metric fluctuations in (2.40) correspond to pure diffeomorphisms. First consider an arbitrary infinitesimal diffeomorphism

$$\delta x^\mu = \xi^\mu(t,r,\theta,\phi)\,, \tag{2.105}$$

which leads to a perturbation

$$\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}\ , \tag{2.106}$$

where $g_{\mu\nu}$ is the NHEK metric (2.35). Demanding that the perturbation $\delta g_{\mu\nu}$ fits in the ansatz (2.40) gives some constraints on ξ^μ which can be solved explicitly. From this analysis, we can show that Φ and χ are physical fields and that the one-form $\tilde{\alpha}$ is pure gauge.

To see that $\tilde{\alpha}$ can be removed by a diffeomorphism, we first need to solve the following constraint which comes from the (t,t) component of the linearized Einstein equation. Using (2.43)-(2.49) on $R_{tt}=0$ gives[5]

$$\partial_r \left(r^3 \partial_r (\partial_t \tilde{\alpha}_r - \partial_r \tilde{\alpha}_t) \right) = 0\,. \tag{2.107}$$

This constraint can be integrated explicitly and we can write the result as follows

$$\begin{aligned} \tilde{\alpha}_r(t,r) &= \partial_r F(t,r)\,, \\ \tilde{\alpha}_t(t,r) &= \partial_t F(t,r) + \frac{G^{(3)}(t)}{2r} + H'(t) r\,, \end{aligned} \tag{2.108}$$

where $F(t,r), G(t)$ and $H(t)$ are arbitrary functions. The infinitesimal diffeomorphism that we are looking for is then given by

$$\xi = \left(-H + G(t) + \frac{G''(t)}{2r^2} \right) \partial_t - r G'(t)\, \partial_r - \left(F(t,r) + G''(t) \right) \partial_\phi\,. \tag{2.109}$$

Indeed, the corresponding perturbation takes the form

$$\mathcal{L}_\xi g = 2M_0^2 (1+\cos^2\theta)(\partial_t \tilde{\alpha}_r - \partial_r \tilde{\alpha}_t)\, r^2 \mathrm{d}t^2 + \frac{8M_0^2 \sin^2\theta}{1+\cos^2\theta} \left(\tilde{\alpha}_t \mathrm{d}t + \tilde{\alpha}_r \mathrm{d}r \right)(\mathrm{d}\phi + r\mathrm{d}t)\,, \tag{2.110}$$

and precisely cancels the contribution of $\tilde{\alpha}$ in the solution of our ansatz (2.40). We have also noticed that the perturbations associated with the gravitational mode Φ are related to some large diffeomorphisms of the NHEK with non-trivial ϕ-dependence. We hope to investigate them in future work.

[5] Solving $R_{rr}=0$ gives the same constraint as $R_{tt}=0$ after using (2.43)-(2.49).

3 Quantum corrections to black hole entropy

What can semiclassical gravity tells us about the black hole microstates?

3.1 *Introduction: logarithmic corrections to black holes*

A landmark result in string theory is the derivation of the Bekenstein-Hawing entropy from black hole microstates by Strominger and Vafa [12]. They computed the number of states $d_{\text{micro}}(q_\alpha)$ of a system of strings and branes in type IIB string theory, labeled by the charges q_α. Although the computation was done at weak coupling, supersymmetry ensures that the answer is independent on the coupling. At strong coupling, this system becomes a 5d black hole in type IIB supergravity with the same charges q_α. They were able to show that

$$d_{\text{micro}}(q_\alpha) = \frac{A(q_\alpha)}{4G} + o(A(q_\alpha)) \ , \tag{3.1}$$

where $A(q_\alpha)$ is the area of the black hole horizon. This was the first microscopic account of the Bekenstein-Hawking formula.

This matching was done at leading order in a large charge expansion, *i.e.* up to terms denoted $o(A(q_\alpha))$. This begs the question: can we push this matching beyond the leading order term?

This naively requires knowledge about details of the UV completion. For example, higher-derivative corrections can be computed and matched with microscopic counting formulas [83, 112]. Rather surprisingly, it was shown [24, 113, 114] that a particular class of correction, of logarithmic type, doesn't require *any* knowledge about the UV: depending only on the two-derivative part of the low-energy effective action for massless fields. As such, these corrections can be computed in semiclassical gravity and put strong constraints on possible microscopic formulas, constituting a powerful "*infrared window into the microstates*".

The claim that logarithmic corrections computed from the IR theory agree with results for the UV completion has been successfully tested in many cases where string theory provides a microscopic counting formula for black hole microstates. We refer to [115, 116] for a broad overview and [27, 117–119] for more recent developments in AdS_4/CFT_3. Logarithmic corrections have also been evaluated for a plethora of other black holes [22, 26] where a microscopic account still awaits.[1]

3.1.1 Euclidean quantum gravity

We review the general framework used to compute logarithmic corrections to black hole entropy [22, 25, 114]. We consider theories of Einstein gravity in D dimensions coupled to massless matter (abelian gauge fields and neutral scalar, Dirac and Rarita-Schwinger fields). We restrict to theories with a scaling property so that purely bosonic terms have two derivatives, terms with two fermions have one derivative and terms with four fermions have no derivative. This covers a wide range of theories, such as Einstein gravity with minimally coupled scalars, fermions and gauge fields, but also a variety of supergravity theories at a generic point in the moduli space. This excludes theories with a cosmological constant, which can be considered but require a separate discussion.

We now consider a (charged and rotating) black hole solution in this theory. The scaling symmetry implies that we have a whole family of solutions under

$$g_{\mu\nu} \to \lambda^2 g_{\mu\nu}, \qquad A_\mu^{(\alpha)} \to \lambda A_\mu^{(\alpha)}, \qquad \varphi_s \to \varphi_s \ , \tag{3.2}$$

where α labels the gauge fields and s the scalar fields. This black hole appears as a saddle-point of the Euclidean path integral

$$Z(\beta, \mu_\alpha) = \int D\Psi \, e^{-\mathcal{S}_E(\Psi)} \ , \tag{3.3}$$

where $\mathcal{S}_E$ is the Euclidean action and the integration is done while fixing the temperature β and the chemical potentials μ^α associated to the charges q_α. The

[1] In certain cases the logarithm can be accounted for very simply by using thermodynamics [22, 120]: the measure that controls the change from, for example, the microcanonical to the canonical ensemble correctly reproduces the gravitational result without leading to new insight in the microscopic theory.

black hole entropy is given by the Legendre transform

$$S = \log Z + \beta M + \mu^\alpha q_\alpha \ , \tag{3.4}$$

where a sum over α is implied.

At leading order, we have the classical approximation

$$Z(\beta, \mu_\alpha) \sim e^{-\mathcal{S}_E^{\text{class.}}} \tag{3.5}$$

where $\mathcal{S}_E^{\text{class.}}$ is the Euclidean on-shell action. As was shown in [10], this gives the Bekenstein-Hawking formula

$$S = \frac{\text{Area}(q_\alpha)}{4G} + \dots \ . \tag{3.6}$$

At one-loop around the saddle-point, we obtain

$$Z(\beta, \mu_\alpha) \sim \frac{1}{\sqrt{\det \Lambda}} e^{-\mathcal{S}_E^{\text{class.}}} \tag{3.7}$$

where $\Lambda = \frac{\delta^2 \mathcal{S}_E}{\delta \Psi^2}$ is the quadratic operator for the fluctuating fields on the background. This expression is divergent and needs to be regulated. This gives the one-loop correction to the black hole entropy

$$\delta S = -\frac{1}{2} \log \det \Lambda \tag{3.8}$$

We are interested in the correction proportional to $\log(\text{Area}(q_\alpha))$ in the entropy. To isolate it, the strategy explained in [121] is to consider a reference configuration with length scale L_0 and a rescaled one with length scale L obtained using (3.2). The scaling symmetry implies that the Euclidean action rescales as

$$\mathcal{S}_E = \left(\frac{L}{L_0}\right)^{D-2} \mathcal{S}_E^{(0)} \ . \tag{3.9}$$

The prescription is then to compute the one-loop correction to the difference $\log Z - \log Z_0$ and keep only the piece proportional to $\log L$. This has the effect of removing the thermal gas and the spurious terms appearing after regulating infrared divergences. This is a way to isolate the contribution from the black hole microstates and to obtain the quantum correction to the black hole entropy. We refer to [121] for a detailed discussion.

We have only discussed the one-loop contribution. It can be shown that higher loops don't contribute to the logarithmic correction as they are suppressed by positive powers of L, as discussed in details in [22]. After introducing the heat kernel below, we will also show that massive fields don't give a contribution. At the end, the logarithmic correction to the entropy arises only at one-loop from the two-derivative Lagrangian of massless fields. Hence, it can be unambiguously computed in the low-energy effective theory.

3.1.1.a The quantum entropy function

The above prescription doesn't work for extremal black holes because the thermal circle is infinite which makes the Euclidean on-shell action divergent. A well-defined procedure is to do the computation for the non-extremal black hole and take the extremal limit of the answer.

An alternative, and perhaps more rigorous, procedure exists for extremal black holes, known as the quantum entropy function [23]. The idea is to focus only on the near-horizon geometry which is of the form $\mathrm{AdS}_2 \times M$ where M is some compact space.

We use the following metric for Euclidean AdS_2

$$ds^2 = (r^2-1)dt^2 + \frac{dr^2}{r^2-1} \ . \tag{3.10}$$

The absence of singularity at $r=1$ implies that t must be periodic $t \sim t+2\pi$. From the point of view of AdS_2, the black hole background often implies that we have 2d abelian gauge fields of the form

$$A_\alpha = q_\alpha (r-1)dt \ , \tag{3.11}$$

where $q_\alpha \sim \int_K *(dA_\alpha)$ is the corresponding electric charge. The fact that $r-1$ appears here ensures that A_i is well-defined at $r=1$.

Let's now consider the Euclidean path integral Z_{AdS_2} over all string fields configurations on the AdS_2 background. To define the path integral, we introduce a cutoff r_0 so that we only consider the part of the geometry $r \leq r_0$. The asymptotic metric is

$$ds^2_{\mathrm{bdy}} = r_0^2 dt^2 \equiv d\tau^2 \ , \tag{3.12}$$

where we have defined a boundary time coordinate $\tau = r_0 t$ so that $\tau \sim 2\pi r_0 \tau$. We also need to fix the asymptotic behavior of A_α, which corresponds here to fixing the electric charge q_α. Then, according to the AdS/CFT dictionary, we have

$$Z_{\mathrm{AdS}_2}(q_i) = \mathrm{Tr}\left(e^{-2\pi r_0 (H - i q_\alpha A_\alpha^{\mathrm{bdy}})}\right) \ , \tag{3.13}$$

where H is the Hamiltonian of the CFT_1 (associated to τ), A_α^{bdy} is the operator dual to A_α in the CFT_1 and the trace is over the states with charges q_α. To remove the additional term, we insert a Wilson line

$$\widehat{Z}_{\mathrm{AdS}_2}(q_\alpha) = \left\langle \exp\left(-i q_\alpha \oint A_\alpha\right)\right\rangle_{\mathrm{AdS}_2} , \tag{3.14}$$

which is taken to be along the boundary. This leads to

$$\widehat{Z}_{\mathrm{AdS}_2}(q_\alpha) = \mathrm{Tr}\, e^{-2\pi r_0 H} . \tag{3.15}$$

In the large r_0 limit, we project onto the ground states of H and we have

$$\widehat{Z}_{\mathrm{AdS}_2}(q_\alpha) = d(q_\alpha)\, e^{-2\pi r_0 E_0} , \tag{3.16}$$

where $d(q_\alpha)$ is the number of ground states with charges q_α and E_0 is their energy.

The AdS_2 path integral also has an infinite factor $e^{2\pi r_0 C}$ which comes from the infinite volume of AdS_2. The $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence suggests that this divergence is the contribution from the ground state energy. This leads to an unambiguous definition of the degeneracy as

$$d(q_\alpha) = \left\langle \exp\left(-i q_\alpha \oint A_\alpha\right)\right\rangle_{\mathrm{AdS}_2}^{\mathrm{finite}} , \tag{3.17}$$

where the superscript "finite" corresponds to the procedure where we remove the divergent term $e^{2\pi r_0 C}$. This gives a gravitational (or macroscopic) definition of the exact degeneracy of black hole microstates, known as the quantum entropy function [23].

This can be used to compute quantum corrections to black hole entropy [113]. In the path integral, the calculation reduces to the computation of a determinant as explained above in the non-extremal case. This prescription has been used to compute the corrections to black hole entropy for $\mathcal{N} = 4$ and $\mathcal{N} = 8$ black holes, which were successfully matched with the known microscopic formulas [24, 25].

3.1.2 Heat kernel expansion

We will now describe the main technical tool which makes possible the exact computation of the logarithmic correction for a variety of black holes: the heat kernel expansion [122].

The one-loop correction to the partition function decomposes as a contribution Z_{nz} from the non-zero modes and a contribution Z_{zm} from the zero modes, so that we have

$$Z_{\text{1-loop}}(\beta,\mu_\alpha) = Z_{\text{nz}} Z_{\text{zm}}\, e^{-\mathcal{S}_E^{\text{class.}}} \ , \tag{3.18}$$

and the one-loop corrected Bekenstein-Hawking entropy is

$$S_{\text{1-loop}} = \frac{A}{4G} + \log Z_{\text{nz}} + \log Z_{\text{zm}} \ . \tag{3.19}$$

We would like to focus on the logarithmic correction

$$S = \frac{A}{4G} + (C_{\text{local}} + C_{\text{zm}}) \log L + \dots \ , \tag{3.20}$$

where C_{local} and C_{zm} are respectively the contribution from the non-zero and zero modes. We will now explain how to compute them.

Non-zero mode contribution. Let's denote by κ_n the eigenvalues of the quadratic operator Λ. The contribution of the non-zero modes takes the form

$$\log Z_{\text{nz}} = -\frac{1}{2}\sum_n{}' \log \kappa_n \ , \tag{3.21}$$

where the primed sum runs only over the non-zero modes $\kappa_n \neq 0$. To compute this, we introduce the heat kernel

$$K(x,s) = \sum_n e^{-\kappa_n s} f_n^\ell(x) f_n^{\ell'}(x) G_{\ell\ell'} \ , \tag{3.22}$$

where $\{f_n^\ell\}$ are the normalized eigenfunctions of Λ with eigenvalues $\{\kappa_n\}$ and $G_{\ell\ell'}$ is the metric on field space. In particular, we have

$$\int_{\mathcal{M}} d^D x \sqrt{g}\, K(x,s) = \sum_n e^{-s\kappa_n} = \sum_n{}' e^{-s\kappa_n} + N_{\text{zm}} \ , \tag{3.23}$$

where N_{zm} is the number of zero modes. As explained above, we are considering a configuration with length scale L obtained by rescaling of a reference configuration with length scale L_0. The eigenvalues of massless fields rescale according to

$$\kappa_n = \left(\frac{L}{L_0}\right)^2 \kappa_n^{(0)} \ . \tag{3.24}$$

We will make use of the relation

$$\log\kappa - \log\kappa^{(0)} = -\lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{ds}{s}\left(e^{-s\kappa} - e^{-s\kappa^{(0)}}\right) \ , \tag{3.25}$$

which implies that we have

$$\log Z_{\rm nz} - \log Z_{\rm nz}^{(0)} = \frac{1}{2}\int_{\epsilon}^{\epsilon L^2/L_0^2} \frac{ds}{s}\left(\int_{\mathcal{M}} d^D x\sqrt{g}\, K(x,s) - N_{\rm zm}\right) . \tag{3.26}$$

The above expression makes it clear that only the range of very small s contributes.

The heat kernel expansion is the statement that we have a small s expansion of the form

$$K(x,s) = \sum_{n\geq 0} s^{n-D/2} a_{2n}(x) \tag{3.27}$$

where D is the dimension of spacetime. The coefficients $a_{2n}(x)$ are known as Seeley-DeWitt coefficients. For smooth manifolds, $a_{2n}(x)$ is a sum of $2n$-derivative terms constructed from the fields appearing in the action [122].

We are mainly interested in $D = 4$ for which we have

$$K(x,s) = s^{-2}a_0(x) + s^{-1}a_2(x) + s^0 a_4(x) + O(s) . \tag{3.28}$$

We only want to compute the $\log L$ contribution in $\log Z_{\rm nz}$. The integral (3.26) makes it clear that this comes from the a_4 coefficient and we have

$$\log Z_{\rm nz} = C_{\rm local} \log L + \dots , \tag{3.29}$$

where we have defined

$$C_{\rm local} \equiv \int d^D x\sqrt{g}\, a_4(x) . \tag{3.30}$$

In spacetime dimension D, $a_4(x)$ must be replaced by $a_D(x)$. Note that this vanishes when D is odd so that there is no contribution from the non-zero mode in odd dimensions.

The power of the heat kernel expansion comes from the fact that there is a general expression for $a_4(x)$ given in [122]. This allows to compute $C_{\rm local}$ in a simple way without ever computing any eigenvalue. To describe this formula, we write the operator of quadratic fluctuations canonically as

$$\Lambda^n_m = (\Box){\rm id}^n_m + 2(\omega^\mu D_\mu)^n_m + P^n_m , \tag{3.31}$$

where the index m, n refers to the different fields and D_μ is the covariant derivative. We define $\mathcal{D}_\mu = D_\mu + \omega_\mu$ to complete the square so that

$$\Lambda^n_m = (\mathcal{D}^\mu \mathcal{D}_\mu)^n_m + E^n_m, \qquad E \equiv P - \omega^\mu\omega_\mu - (D^\mu\omega_\mu) . \tag{3.32}$$

The Seeley-DeWitt coefficient $a_4(x)$ is then given explicitly by the formula

$$
\begin{aligned}
(4\pi)^2 a_4(x) \quad &= \quad \mathrm{Tr}\left[\frac{1}{2}E^2 + \frac{1}{6}RE + \frac{1}{12}\Omega_{\mu\nu}\Omega^{\mu\nu}\right. \\
&\qquad \left. + \frac{1}{360}(5R^2 + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu})\right] ,
\end{aligned}
\tag{3.33}
$$

where $\Omega_{\mu\nu} = [D_\mu + \omega_\mu, D_\nu + \omega_\nu]$ is the curvature associated to the connection $\mathcal{D}_\mu$. For simple enough examples, it is possible to write $a_4(x)$ as

$$
(4\pi)^2 a_4(x) = -a\,\mathrm{Euler} + c\,\mathrm{Weyl}^2 , \tag{3.34}
$$

in terms of the Euler density and the Weyl tensor squared whose expressions are

$$
\begin{aligned}
\mathrm{Euler} \quad &= \quad R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 , \\
\mathrm{Weyl}^2 \quad &= \quad R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2 .
\end{aligned}
\tag{3.35}
$$

This is always possible for the examples we consider below although additional four-derivative terms can appear for more complicated black holes.

Massive fields. We have restricted above to massless fields. Let us argue that massive fields don't contribute to the logarithmic correction. For a field of mass m, the scaling relation (3.24) for the eigenvalue becomes

$$
\kappa_n - m^2 = \left(\frac{L}{L_0}\right)^2 (\kappa_n^{(0)} - m^2) . \tag{3.36}
$$

From this, it can be shown that massive fields don't contribute to the logarithmic correction. This is a consequence of the fact that we get an additional factor of $e^{-m^2 s}$ in (3.26) which prevents the appearance of a $\log L$ term. We refer to [123] for a more detailed discussion. This implies that the logarithmic correction depends only on the massless spectrum of the theory. In particular, it can be computed in the low-energy theory.

Zero mode contribution. The zero modes need to be treated separately. They are associated to asymptotic symmetries: gauge transformations with parameters that do not vanish at infinity. In the path integral, we can treat them by making a change of variable to the parameters of the asymptotic symmetry group. For a field Ψ, the Jacobian of this change of variable introduces a factor

$$
\left(\frac{L}{L_0}\right)^{\beta_\Psi} , \tag{3.37}
$$

which contributes a logarithmic correction $\beta_\Psi \log L$ to the entropy. As a result, the total contribution from the zero modes is

$$C_{\text{zm}} = \sum_{\Psi} (\beta_\Psi - 1) n_\Psi^0 \tag{3.38}$$

where we are summing over all fields Ψ (including ghosts) and we denote by n_Ψ^0 the number of zero modes for Ψ. There is a -1 because we include here the $-N_{\text{zm}}$ which was in the non-zero mode contribution (3.26) (and not included in C_{local}). The value of β_Ψ can be computed by normalizing correctly the path integral measure. We refer to [25] for a more detailed discussion. As an illustration, we report below the values of β_Ψ for a gauge field, a Rarita-Schwinger field and the graviton in D spacetime dimensions

$$\beta_A = \frac{D}{2} - 1, \qquad \beta_\psi = D - 1, \qquad \beta_g = \frac{D}{2} \; . \tag{3.39}$$

3.1.3 Minimally coupled fields

As a simple example of the above procedure, we will compute the logarithmic correction to the entropy of a black hole in a theory of Einstein gravity with minimally coupled fields. For each of the fields, we report the value of ω^μ, P and gives the Seeley-DeWitt coefficient $a_4(x)$. These results are well-known (see for example section 4.2.2 of [122]).

For a massless scalar field, we have the Lagrangian

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}\left(\partial_\mu \varphi \partial^\mu \varphi + \xi R \phi^2\right) \; , \tag{3.40}$$

where we also include a conformal-type coupling to the Ricci curvature. The quadratic operator is

$$\Lambda = \Box - \xi R \; , \tag{3.41}$$

so we have $\omega^\mu = 0$ and $P = \xi R$. This then implies that $E = \xi R$ and $\Omega_{\mu\nu} = 0$. Hence, applying the formula gives

$$(4\pi)^2 a_4^{\text{scalar}} = -\frac{1}{360}\text{Euler} + \frac{1}{120}\text{Weyl}^2 + \frac{1}{2}\left(\xi - \frac{1}{6}\right)^2 R^2 \; . \tag{3.42}$$

Let's now consider a Majorana spinor described by the Lagrangian is

$$\mathcal{L} = \bar{\chi}\gamma^\mu D_\mu \chi \tag{3.43}$$

The fermionic fluctuation operator is $\gamma^\mu D_\mu$. This is a first order operator so we apply the heat kernel to its square and divide the final result by two. The identity $(\gamma^\mu D_\mu)^2 = -\Box + \frac{1}{4}R$ gives $\omega^\mu = 0$ and $P = -\frac{1}{4}R$. This gives

$$(4\pi)^2 a_4^{\text{spinor}} = -\frac{11}{720}\text{Euler} + \frac{1}{40}\text{Weyl}^2 \ , \tag{3.44}$$

where we have included the additional multiplication by -1 due to the Grassmann statistics. The result for Weyl spinors is the same. The result for Dirac spinors needs to be multiplied by two.

Let's now consider a gauge field with Lagrangian

$$\mathcal{L} = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} \ , \tag{3.45}$$

where we use $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. Integrating by part gives

$$\mathcal{L} = \frac{1}{2}\left(a^\nu \Box a_\nu - a^\nu R_{\mu\nu} a^\mu - (D^\mu a_\mu)^2\right) \ . \tag{3.46}$$

The last term is removed by adding a gauge-fixing term $\mathcal{L}_{\text{g.f.}} = \frac{1}{2}(D^\mu a_\mu)^2$ which introduces a minimally coupled scalar ghost. The contribution of this ghost is just the one written in (3.42) with $\xi = 0$ and an overall minus sign due to the opposite statistics. For the gauge field, we obtain $\omega_\mu = 0$ and $P = -R_{\mu\nu}$. As a result, we obtain

$$(4\pi)^2 a_4^{\text{vector}} = -\frac{31}{180}\text{Euler} + \frac{1}{10}\text{Weyl}^2 \ . \tag{3.47}$$

A similar analysis can be done for Rarita-Schwinger fields and gravitons, being careful of taking into account the ghosts which appear after gauge-fixing.

We can then evaluate $a_4(x)$ on the background we are interested in and obtain the result. For example, the logarithmic correction to the entropy of the Schwarzschild or Kerr black hole takes the form

$$\delta S = \left[\frac{1}{90}\left(2n_S - 26n_V + 7n_F - \frac{233}{2}n_\psi + 424\right) + C_{\text{zm}}\right]\log L \ , \tag{3.48}$$

where n_S, n_V, n_F and n_ψ are respectively the number of massless scalar, vector, Dirac and Rarita-Schwinger fields [22]. The contribution C_{zm} of the zero modes depends on the symmetries preserved by the black hole. It can be summarized by the formula [26]

$$C_{\text{zm}} = -(3 + K) + 2N_{\text{SUSY}} + 3\delta_{\text{non-ext}} \ , \tag{3.49}$$

where $K=1$ for rotating black hole, $K=3$ for spherical symmetry, $N_{\text{SUSY}}=4$ if the black hole is supersymmetric, $N_{\text{SUSY}}=0$ otherwise, $\delta_{\text{non-ext}}=1$ if the black hole is non-extremal and $\delta_{\text{non-ext}}=1$ otherwise.

For example, the Schwarzschild black hole in pure Einstein gravity has

$$\delta S = \frac{77}{45}\log L\ . \tag{3.50}$$

This correction should be reproduced by a theory that claims any microscopic counting of this entropy. This was used in [22] to challenge loop quantum gravity which predicts the incorrect answer $\delta S = -2\log L$.

3.1.4 BPS branch: Kerr-Newman in $\mathcal{N}\geq 2$ supergravity

The above computation in minimally coupled environments is not expected to correspond to any microscopic counting in string theory. This is because the low-energy theory is supergravity and has non-minimal couplings.

The computation of the logarithmic corrections for the Kerr-Newman black hole embedded in $\mathcal{N}\geq 2$ supergravity was done in [26]. The Seeley-DeWitt coefficient can be written as

$$(4\pi)^2 a_4(x) = -a\,\text{Euler} + c\,\text{Weyl}^2\ , \tag{3.51}$$

and it was observed that the c coefficient vanishes multiplet by multiplet due to cancellations between bosons and fermions:

$$c = 0\ . \tag{3.52}$$

The Weyl^2 term contains the dependence on the black hole charges while the Euler term is topological. Hence, these cancellations imply the following fact: for a non-extremal Kerr-Newman black hole embedded in $\mathcal{N}\geq 2$ supergravity, the logarithmic correction is always independent of the black hole charges.

This is not true in general and relies on delicate cancellations between bosons and fermions. For example, we have $c\neq 0$ for the Kerr-Newman black hole in Einstein-Maxwell theory. This shows that supergravity gives simpler logarithmic corrections than pure Einstein-Maxwell theory, even for black holes that don't preserve any supersymmetry. This was later understood from symmetry considerations and formulated as a non-renormalization theorem [124].

Integrating over the Euclidean geometry, the result finally takes the form

$$C_{\text{local}} = \int d^4x\sqrt{g}\,a_4(x) = -\frac{1}{6}(-11+11(\mathcal{N}-2)+n_V-n_H)\ . \tag{3.53}$$

The zero mode contribution is given by the formula (3.49). It is interesting to report some explicit results below.

Theory	Result
$\mathcal{N}=2$	$C_{\text{local}}=\frac{1}{6}(11-n_V+n_H)=2-\frac{\chi}{24}$
$\mathcal{N}=4 \quad (n_V=n_H+1)$	$C_{\text{local}}=-2$
$\mathcal{N}=6 \quad (n_V=7, n_H=4)$	$C_{\text{local}}=-6$
$\mathcal{N}=8, \quad (n_V=15, n_H=10)$	$C_{\text{local}}=-10$

In the $\mathcal{N}=2$ case, we have written the result for C_{local} in terms of $\chi=2(n_V-n_H+1)$: the Euler characteristic of a corresponding Calabi-Yau. For $\mathcal{N}\geq 4$, we note that we have $C_{\text{local}}=6-2\mathcal{N}$. The simplicity of this answer suggests that an index theorem could be at play here.

3.1.5 *Introduction to our work*

As explained above, for Kerr-Newman black holes embedded in $\mathcal{N}\geq 2$ supergravity [26], the c-anomaly vanishes. This leads to a remarkable simplification since the logarithmic correction becomes universal in the sense that it does not depend on details of the black hole background; it is determined entirely by the content of massless fields.

The class of backgrounds considered in [26] was constructed such that, in the extremal limit, they continuously connect to BPS solutions. For this reason we denote this class as the *BPS branch*. The black holes on the BPS branch are not generally supersymmetric, but their couplings to matter are arranged such that supersymmetry is attained in the extremal limit. One of the motivations for the present article is to study universality of logarithmic corrections outside of the BPS branch.

Supergravity (with $\mathcal{N}\geq 2$) also allows for black holes that do not approach BPS solutions in the extremal limit. We refer to such solutions as the *non-BPS branch*. In their minimal incarnation, they correspond to solutions of the $D=4$ theory obtained by a Kaluza-Klein reduction of five dimensional Einstein gravity [125]. In a string theory setup it is natural to identify the compact Kaluza-Klein dimension with the M-theory circle, and then these solutions are charged with respect to electric $D0$-brane charge and magnetic $D6$-brane charge. Such configurations break supersymmetry even in the extremal limit. Therefore, they offer an interesting arena for studying logarithmic corrections and their possible universality.

The minimal Kaluza-Klein theory needed to describe the non-BPS branch is a four dimensional Einstein-Maxwell-dilaton theory where the couplings are dictated by the reduction from five dimensions. We will refer to the black hole solutions of this theory as "Kaluza-Klein black holes." These solutions can be embedded in supergravity, as we will discuss in detail. In particular, we will consider the embedding of the Kaluza-Klein theory in $\mathcal{N} = 4, 6, 8$ supergravity and for $\mathcal{N} = 2$ we consider $ST(n)$ models [2], which include the well-known STU-model as a special case.

Multiplet	Block content
KK block	1 graviton, 1 vector, 1 scalar
Vector block	1 vector and 1 (pseudo)scalar
Scalar block	1 real scalar
Gravitino block	2 gravitini and 2 gaugini
Gaugino block	2 gaugini

Table 3.1: Decomposition of quadratic fluctuations.

Our technical goal is to evaluate the Seeley-DeWitt coefficient $a_4(x)$ for the Kaluza-Klein black hole when it is embedded in one of the supergravities. This involves the study of quadratic fluctuations around the background, potentially a formidable task since there are many fields and generally they have non-minimal couplings to the background and to each other. Fortunately we find that, in the cases we consider, global symmetries of supergravity organize the quadratic fluctuations into manageable groups of fields that are decoupled from one another. We refer to such groups of fields as "blocks". There are only five distinct types of blocks, summarized in Table 3.1.

The KK block comprises the quadratic fluctuations in the seed theory, *i.e.* the Kaluza Klein theory with no additional matter fields. The scalar block is a single minimally coupled spectator scalar field. The remaining matter blocks have unfamiliar field content and their couplings to the background are non-standard. The great simplification is that the spectrum of quadratic fluctuations of each supergravity theory we consider can be characterized by the number of times each type of block appears. We record those degeneracies in Tables 3.4 and 3.8.

[2] We work out the bosonic fluctuations for $\mathcal{N} = 2$ with any prepotential. It is only for fermionic fluctuations that we restrict our attention to the $ST(n)$ models.

Once the relevant quadratic fluctuations are identified it is a straightforward (albeit cumbersome) task to evaluate the Seeley-DeWitt coefficient $a_4(x)$. We do this for every block listed above and so determine their contribution to C_{local} in (**??**). Having already computed the degeneracies of the blocks, it is elementary algebra to find the values of c and a for each supergravity theory. Our results for individual blocks are given in Table 3.7 and those for theories are given in Table 3.8.

One of our main motivation is to identify theories where $c = 0$ since for those the coefficient of the logarithm is universal. We find that the non-trivial cancellations on the BPS branch reported in [26] are much rarer on the non-BPS branch. For example, on the non-BPS branch the c coefficient does not vanish for any $\mathcal{N} = 2, 4$ supergravity we consider, whatever their matter content. Therefore, as we discuss in section 3.4, this implies that the logarithmic correction to the entropy depends on black hole parameters in a combination different from the horizon area.

In contrast, for $\mathcal{N} = 6, 8$ we find that $c = 0$. The vanishing of c on the non-BPS branch is rather surprising, since it is apparently due to a different balance among the field content and couplings than the analogous cancellation on the BPS-branch. It would be very interesting to understand the origin of this cancellation from a more fundamental principle.

3.2 The Kaluza-Klein Black Hole

Our starting point is a black hole solution to Kaluza-Klein theory. It is sufficient for our purposes to consider the original version of Kaluza-Klein theory: the compactification to four spacetime dimensions of Einstein gravity in five dimensions. In this section, we briefly present the theory and its black hole solutions. In the following sections we embed the theory and its solutions into supergravity and study perturbations around the Kaluza-Klein black holes in the framework of supergravity.

The Lagrangian of Kaluza-Klein theory is given by[3]

$$e^{-1}\mathcal{L}_{\text{KK}} = \frac{1}{16\pi G}\left(R - 2D_\mu\Phi D^\mu\Phi - \frac{1}{4}e^{-2\sqrt{3}\Phi}F_{\mu\nu}F^{\mu\nu}\right) . \tag{3.54}$$

The scalar field Φ parametrizes the size of the compact fifth dimension and the field strength $F_{\mu\nu}$ is the 4D remnant of the metric with one index along the fifth

[3] We use e and $\sqrt{-g}$ interchangeably, to denote the square root of the determinant of the metric.

dimension. The Lagrangian (3.54) gives the equations of motion

$$D^2\Phi + \frac{\sqrt{3}}{8}e^{-2\sqrt{3}\Phi}F_{\mu\nu}F^{\mu\nu} = 0 \ , \tag{3.55}$$

$$D_\mu\left(e^{-2\sqrt{3}\Phi}F^{\mu\nu}\right) = 0 \ , \tag{3.56}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = (2D_\mu\Phi D_\nu\Phi - g_{\mu\nu}D^\rho\Phi D_\rho\Phi) + \frac{1}{2}e^{-2\sqrt{3}\Phi}\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right) \ . \tag{3.57}$$

Some of our considerations will apply to any solution of the Kaluza-Klein theory (3.54) but our primary interest is in asymptotically flat black holes. We therefore focus on the general Kaluza-Klein black hole [125–127]. It is characterized by the black hole mass M and angular momentum J, along with the electric/magnetic charges (Q, P) of the Maxwell field. Its 4D metric is given by

$$ds_4^2 = g_{\mu\nu}^{(\mathrm{KK})}dx^\mu dx^\nu = -\frac{H_3}{\sqrt{H_1H_2}}(dt-B)^2 + \sqrt{H_1H_2}\left(\frac{dr^2}{\Delta} + d\theta^2 + \frac{\Delta}{H_3}\sin^2\theta\, d\phi^2\right) \ , \tag{3.58}$$

where

$$\begin{aligned} H_1 &= r^2 + a^2\cos^2\theta + r(p-2m) + \frac{p}{p+q}\frac{(p-2m)(q-2m)}{2} \\ &\quad - \frac{p}{2m(p+q)}\sqrt{(q^2-4m^2)(p^2-4m^2)}a\cos\theta \ , \end{aligned} \tag{3.59}$$

$$\begin{aligned} H_2 &= r^2 + a^2\cos^2\theta + r(q-2m) + \frac{q}{p+q}\frac{(p-2m)(q-2m)}{2} \\ &\quad + \frac{q}{2m(p+q)}\sqrt{(q^2-4m^2)(p^2-4m^2)}a\cos\theta \ , \end{aligned} \tag{3.60}$$

$$H_3 = r^2 - 2mr + a^2\cos^2\theta \ , \tag{3.61}$$

$$\Delta = r^2 - 2mr + a^2 \ , \tag{3.62}$$

and the 1-form B is given by

$$B = \sqrt{pq}\frac{(pq+4m^2)r - m(p-2m)(q-2m)}{2m(p+q)H_3}a\sin^2\theta\, d\phi \ . \tag{3.63}$$

The matter fields are the gauge field

$$A^{(\mathrm{KK})} = -\left[2Q\left(r + \frac{p-2m}{2}\right) + \sqrt{\frac{q^3(p^2-4m^2)}{4m^2(p+q)}}a\cos\theta\right]H_2^{-1}dt$$

$$- \left[2P \left(H_2 + a^2 \sin^2\theta \right) \cos\theta + \sqrt{\frac{p(q^2-4m^2)}{4m^2(p+q)^3}} \right. \tag{3.64}$$

$$\left. \times \left[(p+q)(pr - m(p-2m)) + q(p^2-4m^2) \right] a \sin^2\theta \right] H_2^{-1} d\phi \ ,$$

and the dilaton

$$e^{-4\Phi^{(\mathrm{KK})}/\sqrt{3}} = \sqrt{\frac{H_2}{H_1}} \ . \tag{3.65}$$

The superscript "KK" on $g_{\mu\nu}^{(\mathrm{KK})}$, $A^{(\mathrm{KK})}$, and $\Phi^{(\mathrm{KK})}$ refers to the Kaluza-Klein black hole. These background fields should be distinguished from the exact fields in (3.54-3.57) which generally include fluctuations around the background.

The four parameters m, a, p, q appearing in the solution determine the four physical parameters M, J, Q, P as

$$2GM = \frac{p+q}{2} \ , \tag{3.66}$$

$$GJ = \frac{\sqrt{pq}(pq+4m^2)}{4(p+q)} \frac{a}{m} \ , \tag{3.67}$$

$$Q^2 = \frac{q(q^2-4m^2)}{4(p+q)} \ , \tag{3.68}$$

$$P^2 = \frac{p(p^2-4m^2)}{4(p+q)} \ . \tag{3.69}$$

Note that $q, p \geq 2m$, with equality corresponding to the absence of electric or magnetic charge, respectively.

The spectrum of quadratic fluctuations around the general black hole solution to Kaluza-Klein theory is complicated. In section 3.3.4 we start with a general solution to the equations of motion (3.55-3.57) such as the Kaluza-Klein black hole $g_{\mu\nu}^{(\mathrm{KK})}$, $A_\mu^{(\mathrm{KK})}$, and $\Phi^{(\mathrm{KK})}$ presented above. We construct an embedding into $\mathcal{N} = 2$ SUGRA with arbitrary cubic prepotential and study fluctuations around the background. Although we make some progress in this general setting it proves notable that the analysis simplifies greatly when the background dilaton is constant $\Phi^{(\mathrm{KK})} = 0$.

In the predominant part of the paper we therefore focus on the simpler case from the outset and assume $\Phi^{(\mathrm{KK})} = 0$. We arrange this by considering the non-rotating black hole $J = 0$ with $P^2 = Q^2$. In this special case the metric $g_{\mu\nu}^{(\mathrm{KK})}$ is (3.58) with

$$H_1 = H_2 = \left(r + \frac{q-2m}{2}\right)^2 ,$$
$$H_3 = \Delta = r^2 - 2mr , \tag{3.70}$$

and the gauge field (3.64) becomes

$$A^{(\mathrm{KK})} = -2Q \left(r + \frac{q-2m}{2}\right)^{-1} dt - 2P\cos\theta d\phi . \tag{3.71}$$

In the simplified setting it is easy to eliminate the parameters m, q in favor of the physical mass $2GM = q$ and charges $P^2 = Q^2 = \frac{1}{8}(q^2 - 4m^2)$ but we do not need to do so.

When $\Phi^{(\mathrm{KK})} = 0$ the geometry of the Kaluza-Klein black hole is in fact the same as the Reissner-Nordström black hole. Indeed, they both satisfy the standard Einstein-Maxwell equations

$$R^{(\mathrm{KK})}_{\mu\nu} = \frac{1}{2}\left(F^{(\mathrm{KK})}_{\mu\rho}F^{(\mathrm{KK})\rho}_{\nu} - \frac{1}{4}g_{\mu\nu}F^{(\mathrm{KK})}_{\rho\sigma}F^{(\mathrm{KK})\rho\sigma}\right) , \tag{3.72}$$
$$D_\mu F^{(\mathrm{KK})\mu\nu} = 0 . \tag{3.73}$$

However, whereas the Reissner-Nordström solution can be supported by any combination of electric and magnetic charges (Q, P) with the appropriate value of $Q_{\text{eff}} = \sqrt{P^2 + Q^2}$, for the Kaluza-Klein black hole we must set $P^2 = Q^2$ so

$$F^{(\mathrm{KK})}_{\mu\nu}F^{(\mathrm{KK})\mu\nu} = 0 , \tag{3.74}$$

or else the dilaton equation of motion (3.55) is inconsistent with a constant dilaton $\Phi^{(\mathrm{KK})}$. This difference between the two cases is closely related to the fact that, after embedding in supergravity, the Kaluza-Klein black hole does not preserve supersymmetry in the extremal limit.

3.3 Embedding in $\mathcal{N} \geq 2$ supergravity

3.3.1 The KK Black Hole in $\mathcal{N} = 8$ supergravity

In this section, we review $\mathcal{N} = 8$ SUGRA and show how to embed a solution of $D = 4$ Kaluza-Klein theory with constant dilaton into $\mathcal{N} = 8$ SUGRA.

3.3.1.a $\mathcal{N}=8$ Supergravity in Four Dimensions

The matter content of $\mathcal{N}=8$ SUGRA is a spin-2 graviton $g_{\mu\nu}$, 8 spin-3/2 gravitini $\psi_{A\mu}$ (with $A=1,...,8$), 28 spin-1 vectors B_μ^{MN} (antisymmetric in $M,N=1,...,8$), 56 spin-1/2 gaugini λ_{ABC} (antisymmetric in $A,B,C=1,...,8$), and 70 spin-0 scalars. The Lagrangian can be presented as [128][4]

$$
\begin{aligned}
e^{-1}\mathcal{L}^{(\mathcal{N}=8)} \;=\; & \frac{1}{4}R-\frac{1}{2}\bar{\psi}_{A\mu}\gamma^{\mu\nu\rho}D_\nu\psi_{A\rho}-\frac{i}{8}G^{MN}_{\mu\nu}\widetilde{H}^{(\mathrm{F})\mu\nu}_{MN}-\frac{1}{12}\bar{\lambda}_{ABC}\gamma^\mu D_\mu\lambda_{ABC} \\
& -\frac{1}{24}P_{\mu ABCD}\bar{P}^{\mu ABCD}-\frac{1}{6\sqrt{2}}\bar{\psi}_{A\mu}\gamma^\nu\gamma^\mu\left(\bar{P}^{ABCD}_\nu+\hat{\bar{P}}^{ABCD}_\nu\right)\lambda_{BCD} \\
& +\frac{1}{8\sqrt{2}}\left(\bar{\psi}_{A\mu}\gamma^\nu\hat{\mathcal{F}}_{AB}\gamma^\mu\psi_{\nu B}-\frac{1}{\sqrt{2}}\bar{\psi}_{C\mu}\hat{\mathcal{F}}_{AB}\gamma^\mu\lambda_{ABC}\right. \\
& \qquad\left.+\frac{1}{72}\epsilon^{ABCDEFGH}\bar{\lambda}_{ABC}\hat{\mathcal{F}}_{DE}\lambda_{FGH}\right)\,, \qquad (3.75)
\end{aligned}
$$

in conventions where all fermions are in Majorana form, the metric is "mostly plus", and Hodge duality is defined by

$$
\widetilde{H}^{(\mathrm{F})\mu\nu}_{MN}=-\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}H^{(\mathrm{F})}_{MN\rho\sigma}\,,\quad \epsilon_{0123}=e\,. \qquad (3.76)
$$

Below we also use (R/L) superscripts on fermions, to denote their right- and left-handed components.

We include all the glorious details of $\mathcal{N}=8$ SUGRA to facilitate comparison with other references. The symmetry structure is the most important aspect for our applications so we focus on that in the following. The starting point is the 56-bein

$$
\mathcal{V}=\begin{pmatrix} U_{AB}{}^{MN} & V_{ABMN} \\ \bar{V}^{ABMN} & \bar{U}^{AB}{}_{MN}\end{pmatrix}\,, \qquad (3.77)
$$

that is acted on from the left by a local $SU(8)$ symmetry (with indices $A,B,\ldots$) and from the right by a global $E_{7(7)}$ duality symmetry (with indices M,N). The connection

$$
\partial_\mu\mathcal{V}\mathcal{V}^{-1}=\begin{pmatrix} 2Q_{\mu[A}{}^{[C}\delta_{B]}{}^{D]} & P_{\mu ABCD} \\ \bar{P}^{ABCD}_\mu & 2\bar{Q}_\mu{}^{[A}{}_{[C}\delta^{B]}{}_{D]}\end{pmatrix}\,, \qquad (3.78)
$$

defines an $SU(8)$ gauge field $Q_{\mu A}{}^B$ that renders the $SU(8)$ redundant. We therefore interpret $P_{\mu ABCD}$ as covariant derivatives of scalar fields that belong to the coset $E_{7(7)}/SU(8)$ with dimension $133-63=70$. The term in (3.75) that is quadratic

[4] To match with the conventions of many authors, when discussing $\mathcal{N}=8$ supergravity, we set Newton constant to $\kappa^2=8\pi G=2$. In section 3.3.4, we will restore the explicit κ dependence.

in $P_{\mu ABCD}$ is therefore a standard kinetic term for the physical scalars. The terms linear in $P_{\mu ABCD}$, including

$$\hat{P}_{\mu ABCD} = P_{\mu ABCD} + 2\sqrt{2}\left(\bar{\psi}^{(L)}_{\mu[A}\lambda^{(R)}_{BCD]} + \frac{1}{24}\epsilon_{ABCDEFGH}\bar{\psi}^{(R)E}_{\mu}\lambda^{(L)FGH}\right) , \tag{3.79}$$

do not contribute to quadratic fluctuations around a background with constant scalars. The covariant derivatives D_μ that act on fermions are $SU(8)$ covariant so at this point the Lagrangian is manifestly invariant under the local $SU(8)$.

The gauge fields and their duals are

$$G^{MN}_{\mu\nu} = \partial_\mu B^{MN}_\nu - \partial_\nu B^{MN}_\mu , \tag{3.80}$$

$$\widetilde{H}^{(\mathrm{F})\mu\nu}_{MN} = \frac{4i}{e}\frac{\partial\mathcal{L}}{\partial G^{MN}_{\mu\nu}} . \tag{3.81}$$

They enter the Lagrangian (3.75) explicitly. Their Pauli couplings are written in terms of

$$\hat{\mathcal{F}}_{AB} = \gamma^{\mu\nu}\mathcal{F}_{AB\mu\nu} , \tag{3.82}$$

where

$$\begin{aligned}\mathcal{F}_{AB\mu\nu} \quad = \quad & \mathcal{F}^{(\mathrm{F})}_{AB\mu\nu} + \sqrt{2}\left(\bar{\psi}^{(R)}_{[A[\mu}\psi^{(L)}_{[B[\nu} - \frac{1}{\sqrt{2}}\bar{\psi}^{(L)C}_{[\mu}\gamma_{\nu]}\lambda^{(L)}_{ABC}\right. \\ & \left. - \frac{1}{288}\epsilon_{ABCDEFGH}\bar{\lambda}^{CDE}_{(L)}\gamma_{\mu\nu}\lambda^{FGH}_{(R)}\right) ,\end{aligned} \tag{3.83}$$

with

$$\begin{pmatrix}\mathcal{F}^{(\mathrm{F})}_{AB\mu\nu} \\ \bar{\mathcal{F}}^{(\mathrm{F})AB}_{\mu\nu}\end{pmatrix} = \frac{1}{\sqrt{2}}\mathcal{V}\begin{pmatrix}G^{MN}_{\mu\nu} + iH^{(\mathrm{F})}_{MN\mu\nu} \\ G^{MN}_{\mu\nu} - iH^{(\mathrm{F})}_{MN\mu\nu}\end{pmatrix} . \tag{3.84}$$

These relatives of the gauge fields encode couplings and $E_{7(7)}$ duality symmetries. They satisfy the self-duality constraint

$$\mathcal{F}_{\mu\nu AB} = \widetilde{\mathcal{F}}_{\mu\nu AB} . \tag{3.85}$$

This self-duality constraint is a complex equation that relates the real fields $G^{MN}_{\mu\nu}$, $H^{(\mathrm{F})}_{MN\mu\nu}$ and their duals linearly, with coefficients that depend nonlinearly on scalar fields. It has a solution of the form

$$\widetilde{H}^{(\mathrm{F})}_{MN\mu\nu} = -\, i\left(\mathcal{N}_{MNPQ}G^{-PQ}_{\mu\nu} + \mathrm{h.c.}\right) + (\text{terms quadratic in fermions}) , \tag{3.86}$$

where the self-dual (anti-self-dual) parts of the field strengths are defined as

$$G^{\pm MN}_{\mu\nu} = \frac{1}{2}\left(G^{MN}_{\mu\nu} \pm \widetilde{G}^{MN}_{\mu\nu}\right) \ , \tag{3.87}$$

and the gauge coupling function is

$$\mathcal{N}_{MNPQ} = \left(U_{AB}{}^{MN} - V_{ABMN}\right)^{-1}\left(U_{AB}{}^{MN} + V_{ABPQ}\right) \ . \tag{3.88}$$

Using (3.86) for $\widetilde{H}^{(\mathrm{F})}_{MN\mu\nu}$ and (3.82–3.84) for $\hat{\mathcal{F}}_{AB}$ we can eliminate these fields from the Lagrangian (3.75) in favor of the dynamical gauge field $G^{MN}_{\mu\nu}$, embellished by scalar fields and fermion bilinears.

The relatively complicated classical dynamics of $\mathcal{N} = 8$ SUGRA is due to the interplay between fermion bilinears, duality, and the scalar coset. These disparate features are all important in our considerations but they largely decouple. For example, although we need the Pauli couplings of fermions, we need them only for trivial scalars.

In our explicit computations it is convenient to remove the $SU(8)$ gauge redundancy by writing the 56-bein (3.77) in a symmetric gauge

$$\mathcal{V} = \exp\begin{pmatrix} 0 & W_{ABCD} \\ \bar{W}^{ABCD} & 0 \end{pmatrix} \ , \tag{3.89}$$

where the 70 complex scalars W_{ABCD} are subject to the constraint

$$\bar{W}^{ABCD} = \frac{1}{24}\epsilon^{ABCDEFGH}W_{EFGH} \ . \tag{3.90}$$

After fixing the local $SU(8)$ symmetry, the theory still enjoys a global $SU(8)$ symmetry. Moreover, it is linearly realized when compensated by $SU(8) \subset E_{7(7)}$. We identify this residual global $SU(8)$ as the R-symmetry $SU(8)_R$. This identification proves useful repeatedly. For example, it is according to this residual symmetry that W_{ABCD} transforms as an antisymmetric four-tensor.

3.3.1.b The Embedding into $\mathcal{N} = 8$ SUGRA

The embedding of the Kaluza-Klein black hole (3.58, 3.70, 3.71) in $\mathcal{N} = 8$ SUGRA is implemented by

$$\begin{aligned} \mathring{g}^{(\mathrm{SUGRA})}_{\mu\nu} &= g^{(\mathrm{KK})}_{\mu\nu} \ , \\ \mathring{G}^{MN}_{\mu\nu} &= \frac{1}{4}\Omega^{MN}F^{(\mathrm{KK})}_{\mu\nu} \ , \end{aligned}$$

$$\mathring{W}_{ABCD} = 0 \ ,$$
$$(\text{All background fermionic fields}) = 0 \ , \tag{3.91}$$

where

$$\Omega^{MN} = \text{diag}(\epsilon, \epsilon, \epsilon, \epsilon) \ , \qquad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ . \tag{3.92}$$

In this section (and beyond) we shall often declutter formulae by omitting the superscript "KK" when referring to fields of the seed solution.

To establish the consistency of our embedding, in the following we explicitly check that the $\mathcal{N} = 8$ SUGRA equations of motion are satisfied by the background (3.91). Vanishing fermions satisfy trivially their equations of motion, because they appear at least quadratically in the action. The equations of motion for the scalars W_{ABCD} take the form

$$\begin{aligned}
&(\text{Terms at least linear in } \mathring{W}_{ABCD} \text{ or quadratic in fermions}) \\
&= \ 3\,\mathring{G}^{+[AB}_{\mu\nu}\mathring{G}^{+CD]\mu\nu} + \frac{1}{8}\epsilon_{ABCDEFGH}\mathring{G}^{-EF}_{\mu\nu}\mathring{G}^{-GH\mu\nu} \ .
\end{aligned} \tag{3.93}$$

The scalars $\mathring{W}_{ABCD}$ and the fermions vanish so the right-hand side of the equation must also vanish. Inserting $\mathring{G}^{MN}_{\mu\nu}$ from our embedding (3.91), we find the condition $F^{(\text{KK})}_{\mu\nu}F^{(\text{KK})\mu\nu} = 0$. This condition is satisfied by the seed solution (3.74) because the electric and magnetic charges are equal $P = Q$. Therefore it is consistent to take all scalars $\mathring{W}_{ABCD} = 0$ in $\mathcal{N} = 8$ SUGRA.

The $\mathcal{N} = 8$ Einstein equation is given by

$$\begin{aligned}
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \ = \ & \frac{1}{6}P_{ABCD\{\mu}\bar{P}^{ABCD}_{\nu\}} - \frac{1}{12}g_{\mu\nu}P_{\rho ABCD}\bar{P}^{\rho ABCD} \\
& + \text{Re}(\mathcal{N}_{MNPQ})\left(G^{MN}_{\mu\rho}G_{\nu}{}^{\rho PQ} - \frac{1}{4}g_{\mu\nu}G^{MN}_{\rho\sigma}G^{\rho\sigma PQ}\right) \ .
\end{aligned} \tag{3.94}$$

The vanishing of the scalars $\mathring{W}_{ABCD} = 0$ implies

$$\mathring{\mathcal{V}} = \begin{pmatrix} \delta_{[A}{}^{[M}\delta_{B]}{}^{N]} & 0 \\ 0 & \delta^{[A}{}_{[M}\delta^{B]}{}_{N]} \end{pmatrix} \ , \qquad \mathring{\mathcal{N}}_{MNPQ} = \mathbf{1}_{MNPQ} \ , \tag{3.95}$$

so the Einstein equation simplifies to

$$\mathring{R}_{\mu\nu} - \frac{1}{2}\mathring{g}_{\mu\nu}\mathring{R} = \mathring{G}^{MN}_{\mu\rho}\mathring{G}_{\nu MN}^{\ \rho} - \frac{1}{4}\mathring{g}_{\mu\nu}\mathring{G}^{MN}_{\rho\sigma}\mathring{G}^{\rho\sigma}_{MN} \ . \tag{3.96}$$

The embedding (3.91) reduces the right-hand side so that these equations coincide with the Einstein equation (3.72) satisfied by the seed solution.

Finally, the equations of motion for the vector fields in $\mathcal{N} = 8$ SUGRA are

$$D_\mu \left(\mathcal{N}_{MNPQ} G^{-\mu\nu PQ} + \bar{\mathcal{N}}_{MNPQ} G^{+\mu\nu PQ} \right) = 0 \ . \tag{3.97}$$

The embedding (3.91) and the simplifications (3.95) reduce these equations to the Maxwell equation $D_\mu F^{(\mathrm{KK})\mu\nu} = 0$, consistent with the seed equation of motion (3.73).

In summary, the equations of motion in $\mathcal{N} = 8$ SUGRA are satisfied by the embedding (3.91). Therefore, for any seed solution that satisfies (3.72-3.74), the embedding (3.91) gives a solution to $\mathcal{N} = 8$ SUGRA. Our primary example is the Kaluza-Klein black hole with dilaton $\Phi^{(\mathrm{KK})} = 0$.

3.3.2 Quadratic Fluctuations in $\mathcal{N} = 8$ SUGRA

In this section we expand the Lagrangian (3.75) for $\mathcal{N} = 8$ SUGRA to quadratic order around the background (3.91). We reparametrize the fluctuation fields so that they all transform in representations of the global $USp(8)$ symmetry group preserved by the background. We then partially decouple the quadratic fluctuations into different blocks corresponding to different representations of $USp(8)$.

3.3.2.a Global Symmetry of Fluctuations

The $\mathcal{N} = 8$ SUGRA theory has a global $SU(8)$ symmetry, as discussed at the end of section 3.3.1. The graviton, gravitini, vectors, gaugini, and scalars transform in the representations **1**, **8**, **28**, **56** and **70** of this $SU(8)$ group. The **28**, **56**, and **70**, are realized as antisymmetric combinations of the fundamental representation **8**.

A generic background solution does not respect all the symmetries of the theory, so the global $SU(8)$ symmetry is not generally helpful for analyzing fluctuations around the background. Our embedding (3.91) into $\mathcal{N} = 8$ SUGRA indeed breaks the $SU(8)$ symmetry since $\mathring{G}^{MN}_{\mu\nu} = \frac{1}{4}\Omega^{MN} F^{(\mathrm{KK})}_{\mu\nu}$ is not invariant under the $SU(8)$ group. However, the matrix Ω^{MN} (3.92) can be interpreted as a canonical symplectic form so our embedding respects most of the global $SU(8)$, it preserves a $USp(8)$ subgroup. Therefore, different $USp(8)$ representations cannot couple at quadratic order and it greatly simplifies the analysis to organize fluctuations around the background as representations of $USp(8)$. In the following we analyze one $USp(8)$ representation at a time.

- *Graviton*

  The graviton $h_{\mu\nu} = \delta g_{\mu\nu} = g_{\mu\nu} - \mathring{g}_{\mu\nu}$ is a singlet of $SU(8)$ and remains a singlet of $USp(8)$.

- *Vectors*

  The fluctuations of the gauge fields $\delta G^{MN}_{\mu\nu} = G^{MN}_{\mu\nu} - \mathring{G}^{MN}_{\mu\nu}$ transform in the **28** of $SU(8)$ which has the branching rule to $USp(8)$ $\mathbf{28} \to \mathbf{1} \oplus \mathbf{27}$. We realize this decomposition directly on the fluctuations by defining

$$f_{\mu\nu} = \Omega_{MN}\delta G^{MN}_{\mu\nu} \ , \qquad f^{MN}_{\mu\nu} = \delta G^{MN}_{\mu\nu} - \frac{1}{8}\Omega^{MN} f_{\mu\nu} \ . \tag{3.98}$$

  The $f^{MN}_{\mu\nu}$ are Ω-traceless $f^{MN}_{\mu\nu}\Omega_{MN} = 0$ by construction so they have only $2 \times (28-1)$ degrees of freedom which transform in the **27** of $USp(8)$. The remaining 2 degrees of freedom are in $f_{\mu\nu}$, which transforms in the **1** of $USp(8)$. This decomposition under the global symmetry shows that the graviton can only mix with the "overall" gauge field $f_{\mu\nu}$ and not with $f^{MN}_{\mu\nu}$.

- *Scalars*

  The scalars transform in **70** of $SU(8)$ and the branching rule to $USp(8)$ is $\mathbf{70} \to \mathbf{1} \oplus \mathbf{27} \oplus \mathbf{42}$. We realize this decomposition by defining

$$\begin{aligned} &W' = W_{ABCD}\Omega^{AB}\Omega^{CD} \ , \quad W'_{AB} = W_{ABCD}\Omega^{CD} - \frac{1}{8}W'\Omega_{AB} \ , \\ &W'_{ABCD} = W_{ABCD} - \frac{3}{2}W'_{[AB}\Omega_{CD]} - \frac{1}{16}W'\Omega_{[AB}\Omega_{CD]} \ . \end{aligned} \tag{3.99}$$

  W'_{ABCD} is antisymmetric in all indices and Ω-traceless on any pair or pairs, so it is in the **42** of $USp(8)$. W'_{AB} is antisymmetric, Ω-traceless, and hence in the **27** of $USp(8)$. The remainder W' has no index and is in the **1** of $USp(8)$. The obvious construction of an antisymmetric four-tensor representation of $SU(8)$ has 70 *complex* degrees of freedom, but the scalars W_{ABCD} in $\mathcal{N} = 8$ SUGRA have 70 *real* degrees of freedom that realize an irreducible representation, as implemented by the reality constraint (3.90). The decomposition of this reality constraint under $SU(8) \to USp(8)$ shows that the scalar W' that couples to gravity is real $\overline{W}' = W'$, as expected from Kaluza-Klein theory. It also implies the reality condition on the four-tensor

$$\overline{W}'^{ABCD} = \frac{1}{24}\epsilon^{ABCDEFGH}W'_{EFGH} \ , \tag{3.100}$$

  and an analogous condition on the two-tensor W'^{AB}. An interesting aspect of these reality conditions is that, just like the KK block must couple to a scalar (as opposed to a pseudoscalar), the condition on the $USp(8)$ four-tensor demonstrates that the scalar moduli must comprise exactly 22 scalars and 20 pseudoscalars. The vector multiplet couples vectors and scalars/pseudoscalars precisely so that it restores the overall balance between scalars and pseudoscalars required by $\mathcal{N} = 8$ SUGRA, with 12 scalars and 15 pseudoscalars.

The distinctions between scalars and pseudoscalars are interesting because these details must be reproduced by viable microscopic models of black holes. Extrapolations far off extremality of phenomenological models that are motivated by the BPS limit lead to entropy formulae [129–131] with moduli dependence that is very similar but not identical to the result found here. It would be interesting to construct a model for non-extremal black holes that combines the features of the BPS and the non-BPS branch.

- *Gravitini*

The gravitini $\psi_{A\mu}$ transform in the fundamental **8** of $SU(8)$. The gravitini only carry one $SU(8)$ index which cannot be contracted with the symplectic form Ω^{AB}. Therefore, the gravitini also transform in the **8** of $USp(8)$.

- *Gaugini*

The gaugini λ_{ABC} of $\mathcal{N} = 8$ SUGRA transform in the **56** of the global $SU(8)$. The branching rule to $USp(8)$ is $\mathbf{56} \to \mathbf{8} \oplus \mathbf{48}$. We can realize this decomposition by introducing

$$\lambda'_A = \frac{1}{\sqrt{12}} \lambda_{ABC} \Omega^{BC} , \tag{3.101}$$

and

$$\lambda'_{ABC} = \lambda_{ABC} - \frac{1}{8} (\lambda_{ADE} \Omega^{DE}) \Omega_{BC} . \tag{3.102}$$

The gaugini λ'_A transform in the **8** of $USp(8)$. We will find that these gaugini are coupled to the gravitini. This is allowed because they have the same quantum numbers under the global $USp(8)$. The normalization $1/\sqrt{12}$ introduced in (3.101) ensures that the gaugini retain a canonical kinetic term after the field redefinition.

The gaugini λ'_{ABC} introduced in (3.102) satisfy the constraint $\lambda'_{ABC}\Omega^{BC} = 0$. This ensures that they transform in the **48** of $USp(8)$. No other fields transform in the same way under the global symmetry so these gaugini decouple from other fields. They can of course mix among themselves and we will find that they do in fact have nontrivial Pauli couplings. However, the normalization of the fields is inconsequential and we have retained the normalization inherited from the full $\mathcal{N} = 8$ SUGRA.

Table 3.2 summarizes the decomposition of quadratic fluctuations according to their representations under the global $USp(8)$ that is preserved by the background.

Representations	Fields
1	$h_{\mu\nu},\ f_{\mu\nu},\ W'$
8	$\psi_{A\mu},\ \lambda'_A$
27	$f^{\mu\nu}_{AB},\ W'_{AB}$
42	W'_{ABCD}
48	λ'_{ABC}

Table 3.2: The $USp(8)$ representation content of the quadratic fluctuations.

3.3.2.b The Decoupled Fluctuations

The quadratic fluctuations around any bosonic background decouple into a bosonic part $\delta^2\mathcal{L}_{\text{bosons}}$ and a fermionic part $\delta^2\mathcal{L}_{\text{fermions}}$ because fermions always appear quadratically in the Lagrangian. As we expand the Lagrangian (3.75) around the background (3.91) to quadratic order, these parts further decouple into representations of the preserved $USp(8)$ global symmetry.

The bosonic fluctuations therefore decouple into three blocks

$$\delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{bosons}} = \delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{KK}} + \delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{vector}} + \delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{scalar}} \ . \tag{3.103}$$

- *KK block*

  The first block $\delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{KK}}$, which we call the "KK block", consists of all fields that are singlets of $USp(8)$: the graviton $h_{\mu\nu}$, 1 vector with field strength $f_{\mu\nu}$, and 1 scalar W'. The Lagrangian for this block is given by

$$\begin{aligned} e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{KK}} = \ & \bar{h}^{\mu\nu}\Box\bar{h}_{\mu\nu} - \frac{1}{4}h\Box h + 2\bar{h}^{\mu\nu}\bar{h}^{\rho\sigma}R_{\mu\rho\nu\sigma} - 2\bar{h}^{\mu\nu}\bar{h}_{\mu\rho}R^{\rho}{}_{\nu} - h\bar{h}^{\mu\nu}R_{\mu\nu} \\ & - F_{\mu\nu}F_{\rho\sigma}\bar{h}^{\mu\rho}\bar{h}^{\nu\sigma} + a^{\mu}\left(\Box g_{\mu\nu} - R_{\mu\nu}\right)a^{\nu} + 2\sqrt{2}F_{\nu}{}^{\rho}f_{\mu\rho}\bar{h}^{\mu\nu} \\ & - 4\partial_{\mu}\phi\partial^{\mu}\phi + 2\sqrt{3}F^{\mu\nu}f_{\mu\nu}\phi - 4\sqrt{6}R_{\mu\nu}\bar{h}^{\mu\nu}\phi \ , \end{aligned} \tag{3.104}$$

after the fields were redefined as $h_{\mu\nu} \to \sqrt{2}h_{\mu\nu}$, $f_{\mu\nu} \to 4f_{\mu\nu}$, and $\phi = \frac{1}{16\sqrt{3}}W'$. We also decomposed the graviton into its trace $h = g^{\rho\sigma}h_{\rho\sigma}$ and its traceless part $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{4}g_{\mu\nu}g^{\rho\sigma}h_{\rho\sigma}$, and further included the gauge-fixing term

$$e^{-1}\mathcal{L}_{\text{g.f.}} = -\left(D^{\mu}\bar{h}_{\mu\rho} - \frac{1}{2}D_{\rho}h\right)\left(D^{\nu}\bar{h}_{\nu}{}^{\rho} - \frac{1}{2}D^{\rho}h\right) - \left(D^{\mu}a_{\mu}\right)^{2} \ . \tag{3.105}$$

The rather complicated Lagrangian (3.104) represents the theory of fluctuations around any solution of Kaluza-Klein theory (3.54) with constant dilaton. The fields $f_{\mu\nu}$ and ϕ correspond to the fluctuations of the field strength and the dilaton. The gauge-fixed theory (3.104) must be completed with additional ghost terms. We discuss those in Appendix 3.5.

- *Vector blocks*

The second block $\delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{vector}}$ consists of all fields that transform in the **27** of $USp(8)$: $f^{\mu\nu}_{AB}$ and W'_{AB}. We use $f^{\mu\nu}_{a}$ and W'_a to denote the 27 independent vectors and scalars respectively. It includes two slightly different parts. One part has 12 copies of a vector coupled to a scalar $W'^{(\text{R})}_a$ with the Lagrangian

$$\begin{aligned} e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=8)(\text{R})}_{\text{vector}} &= -\frac{1}{2}\partial^{\mu}W'^{(\text{R})}_a\partial_{\mu}W'^{(\text{R})}_a - f^{\mu\nu}_a f_{a\mu\nu} \\ &\quad - W'^{(\text{R})}_a f_{a\mu\nu}F^{\mu\nu} \ , \qquad a = 1, ..., 12 \ , \end{aligned} \tag{3.106}$$

and the other has 15 copies of a vector coupled to a pseudoscalar $W'^{(\text{P})}_a$ given by

$$\begin{aligned} e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=8)(\text{P})}_{\text{vector}} &= -\frac{1}{2}\partial^{\mu}W'^{(\text{P})}_a\partial_{\mu}W'^{(\text{P})}_a - f^{\mu\nu}_a f_{a\mu\nu} \\ &\quad - iW'^{(\text{P})}_a f_{a\mu\nu}\widetilde{F}^{\mu\nu} \ , a = 13, ..., 27. \end{aligned} \tag{3.107}$$

Although these two Lagrangians are distinct, they give equations of motion that are equivalent under a duality transformation. This is consistent with the fact that $SU(8)$ duality symmetry is the diagonal combination of local $SU(8)$ and global $E_{7(7)}$ duality symmetry, where the latter is not realized at the level of the Lagrangian.

- *Scalar blocks*

The last bosonic block $\delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{scalar}}$ consists of the remaining 42 scalars, transforming in the **42** of $USp(8)$. There are no other bosonic fields with the same quantum numbers so, these fields can only couple to themselves. The explicit expansion around the background (3.72-3.74) shows that all these scalars are in fact minimally coupled

$$e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{scalar}} = -\frac{1}{24}\partial^\mu W'_{ABCD}\partial_\mu \overline{W}'^{ABCD} \ . \tag{3.108}$$

We now turn to the quadratic fluctuations for the fermions. Since they appear at least quadratically in the Lagrangian the bosonic fields can be fixed to their background values. In this case, the $\mathcal{N}=8$ SUGRA Lagrangian (3.75) simplifies to

$$\begin{aligned} e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{fermions}} &= -\frac{1}{2}\bar{\psi}_{A\mu}\gamma^{\mu\nu\rho}D_\nu\psi_{A\rho} - \frac{1}{12}\bar{\lambda}_{ABC}\gamma^\mu D_\mu\lambda_{ABC} \\ &\quad +\frac{1}{4\sqrt{2}}\bar{\psi}_{A\mu}\gamma^\nu\mathring{\mathcal{F}}_{AB}\gamma^\mu\psi_{\nu B} - \frac{1}{8}\bar{\psi}_{C\mu}\mathring{\mathcal{F}}_{AB}\gamma^\mu\lambda_{ABC} \\ &\quad +\frac{1}{288\sqrt{2}}\epsilon^{ABCDEFGH}\bar{\lambda}_{ABC}\mathring{\mathcal{F}}_{DE}\lambda_{FGH} \ , \end{aligned} \tag{3.109}$$

where all fermions are in Majorana form and

$$\mathring{\mathcal{F}}_{AB} = \frac{1}{\sqrt{2}}\left(\mathring{G}_{AB\mu\nu} + \gamma_5\mathring{\tilde{G}}_{AB\mu\nu}\right)\gamma^{\mu\nu} = \frac{1}{2\sqrt{2}}\Omega_{AB}F_{\mu\nu}\gamma^{\mu\nu} \ . \tag{3.110}$$

The field redefinitions introduced in section 3.3.2.a decouple this Lagrangian as

$$\delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{fermions}} = \delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{gravitino}} + \delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{gaugino}} \ . \tag{3.111}$$

- *Gravitino blocks*

  The first block $\delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{gravitino}}$ consists of the 8 gravitini $\psi_{A\mu}$ and the 8 gaugini λ'_A singled out by the projection (3.101). The gravitini and the gaugini both transform in **8** of $USp(8)$ and couple through the Lagrangian

$$\begin{aligned} e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{gravitino}} &= -\bar{\psi}_{A\mu}\gamma^{\mu\nu\rho}D_\nu\psi_{A\rho} - \bar{\lambda}'_A\gamma^\mu D_\mu\lambda'_A + \frac{1}{4}\Omega^{AB}\bar{\lambda}'_A F_{\rho\sigma}\gamma^{\rho\sigma}\lambda'_B \\ &\quad +\frac{1}{4}\Omega^{AB}\bar{\psi}_{A\mu}\left(F^{\mu\nu} + \gamma_5\widetilde{F}^{\mu\nu}\right)\psi_{B\nu} \\ &\quad -\frac{\sqrt{6}}{8}\bar{\psi}_{A\mu}F_{\rho\sigma}\gamma^{\rho\sigma}\gamma^\mu\lambda'_A \ . \end{aligned} \tag{3.112}$$

  The indices take values $A, B = 1, \dots 8$. However, this block actually decouples into 4 identical pairs, with a single pair comprising two gravitini and two gaugini. The canonical pair is identified by restricting the indices to $A, B = 1, 2$ and so $\Omega_{AB} \to \epsilon_{AB}$. The other pairs correspond to $A, B = 3, 4$, $A, B = 5, 6$, and $A, B = 7, 8$.

- *Gaugino blocks*

  The second block $\delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{gaugino}}$ consists of the 48 gaugini (3.102) that transform in the **48** of $USp(8)$. These 48 gaugini decompose into 24 identical groups that decouple from one another. Each group has 2 gaugini and a Lagrangian given by

$$e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{gaugino}} = -\,\bar{\lambda}_a\gamma^\mu D_\mu\lambda_a - \frac{1}{8}\epsilon^{ab}\bar{\lambda}_a F_{\mu\nu}\gamma^{\mu\nu}\lambda_b\ , \tag{3.113}$$

  where $a, b = 1, 2$ denote the 2 different gaugini in one group. It is interesting that *no* fermions in the theory are minimally coupled. Moreover, the numerical strength of the Pauli couplings to black holes on the non-BPS branch are different from the corresponding Pauli couplings for fermions on the BPS branch [26].

3.3.2.c Summary of Quadratic Fluctuations

In the previous sections we defined a seed solution (3.72-3.74) of Kaluza-Klein theory with vanishing dilaton and embedded it into $\mathcal{N} = 8$ SUGRA through (3.91). In this section, we have studied fluctuations around the background by expanding the $\mathcal{N} = 8$ SUGRA Lagrangian (3.75) to quadratic order. In section 3.3.2.a, we decomposed the fluctuations in representations of the $USp(8)$ symmetry preserved by the background. In section 3.3.2.b, we have decoupled the quadratic fluctuations into blocks corresponding to distinct representations of $USp(8)$. They are summarized in Table 3.3.

Degeneracy	Multiplet	Block content	$USp(8)$	Lagrangian
1	KK block	1 graviton, 1 vector, 1 scalar	**1**	(3.104)
27	Vector block	1 vector and 1 (pseudo)scalar	**27**	(3.106)
42	Scalar block	1 real scalar	**42**	(3.108)
4	Gravitino block	2 gravitini and 2 gaugini	**8**	(3.112)
24	Gaugino block	2 gaugini	**48**	(3.113)

Table 3.3: Decoupled quadratic fluctuations in $\mathcal{N} = 8$ supergravity around the KK black hole.

3.3.3 Consistent Truncations of $\mathcal{N} = 8$ SUGRA

In this section we present consistent truncations from $\mathcal{N} = 8$ SUGRA to $\mathcal{N} = 6$, $\mathcal{N} = 4$, $\mathcal{N} = 2$ and $\mathcal{N} = 0$. These truncations are well adapted to the KK black hole in that all its nontrivial fields are retained. In other words, the truncations amount to removal of fields that are trivial in the background solution.

It is easy to analyse the spectrum of quadratic fluctuations around the KK black hole in the truncated theories. In each case some of the fluctuating fields are removed, but always consistently so that blocks of fields that couple to each other are either all retained or all removed. Therefore, the fluctuation spectrum in all these theories can be described in terms of the same simple blocks that appear in $\mathcal{N} = 8$ supergravity. For these truncations the entire dependence on the theory is encoded in the degeneracy of each type of block. They are summarized in Table 3.4.

Multiplet \ Theory	$\mathcal{N} = 8$	$\mathcal{N} = 6$	$\mathcal{N} = 4$	$\mathcal{N} = 2$	$\mathcal{N} = 0$
KK block	1	1	1	1	1
Gravitino block	4	3	2	1	0
Vector block	27	15	$n + 5$	n_V	0
Gaugino block	24	10	$2n$	$n_V - 1$	0
Scalar block	42	14	$5n - 4$	$n_V - 1$	0

Table 3.4: The degeneracy of multiplets in the spectrum of quadratic fluctuations around the KK black hole embedded in various theories. For $\mathcal{N} = 4$, the integer n is the number of $\mathcal{N} = 4$ matter multiplets. For $\mathcal{N} = 2$, the integer n_V refers to the $ST(n_V - 1)$ model.

All the truncations in this section heavily utilize the $SU(8)_R$ global symmetry of $\mathcal{N} = 8$ supergravity. We therefore recall from the outset that the gravitons, gravitini, vectors, gaugini, and scalars transform in the irreducible representations **1**, **8**, **28**, **56**, **70** of $SU(8)_R$.

3.3.3.a The $\mathcal{N} = 6$ Truncation

The $\mathcal{N} = 6$ truncation restricts $\mathcal{N} = 8$ SUGRA to fields that are *even* under the $SU(8)_R$ element $\text{diag}(I_6, -I_2)$. This projection preserves $\mathcal{N} = 6$ local supersymmetry since the 8 gravitini of $\mathcal{N} = 8$ SUGRA are in the fundamental **8** of $SU(8)_R$ and so exactly two gravitini are odd under $\text{diag}(I_6, -I_2)$ and projected out. The branching rules of the matter multiplets under $SU(8)_R \to SU(6)_R \times SU(2)_{\text{matter}}$

are

$$\begin{aligned} \mathbf{70} &\rightarrow (\mathbf{15},\mathbf{1}) \oplus (\overline{\mathbf{15}},\mathbf{1}) \oplus (\mathbf{20},\mathbf{2}) \ , \\ \mathbf{56} &\rightarrow (\mathbf{20},\mathbf{1}) \oplus (\mathbf{15},\mathbf{2}) \oplus (\mathbf{6},\mathbf{1}) \ , \\ \mathbf{28} &\rightarrow (\mathbf{15},\mathbf{1}) \oplus (\mathbf{6},\mathbf{2}) \oplus (\mathbf{1},\mathbf{1}) \ . \end{aligned} \tag{3.114}$$

These branching rules follow from decomposition of the $SU(8)_R$ four-tensor T_{ABCD} (**70**), the three-tensor T_{ABC} (**56**), and the two-tensor T_{AB} (**28**), by splitting the $SU(8)_R$ indices as $A, B, ... \rightarrow (\alpha, a), (\beta, b), ...$ where the lower case indices refer to $SU(2)_{\text{matter}}$ (greek) and $SU(6)_R$ (latin). The truncation to $\mathcal{N} = 6$ SUGRA retains only the fields that are invariant under $SU(2)_{\text{matter}}$ so fields in the **2** are removed. Therefore the truncated theory has 30 scalar fields, 26 gaugini, and 16 vector fields. Taking the 6 gravitini and the graviton into account as well, the total field content comprises 64 bosonic and 64 fermionic degrees of freedom.

The claim that the truncation is *consistent* means that the equations of motion *of the retained fields* are sufficient to guarantee that *all* equations of motion are satisfied, as long as the removed fields vanish. In general, the primary obstacle to truncation is that the equations of motion for the omitted fields may fail. This is addressed here because the equations of motion for fields in the **2** of $SU(2)_{\text{matter}}$ only involve terms in the **2**. Therefore their equations of motion are satisfied when all fields in the **2** vanish.

Our interest in the consistent truncation of $\mathcal{N} = 8$ SUGRA to $\mathcal{N} = 6$ SUGRA is the application to the KK black hole. The embedding (3.91) of the Kaluza-Klein black hole into $\mathcal{N} = 8$ SUGRA turns on the four field strengths on the skew-diagonal of the **28** (which is realized by an antisymmetric 8×8 matrix of field strengths F_{AB}). The entries on the skew diagonal are all contained in the $SU(6)_R \times SU(2)_{\text{matter}}$ subgroup of $SU(8)_R$, because the antisymmetric representation of $SU(2)$ is trivial. The embedding of the KK black hole in $\mathcal{N} = 8$ SUGRA therefore defines an embedding in $\mathcal{N} = 6$ SUGRA as well. In other words, the truncation and the embedding are *compatible*.

We can find the spectrum of quadratic fluctuations in $\mathcal{N} = 6$ SUGRA either by truncating the spectrum determined in the $\mathcal{N} = 8$ SUGRA context, or by directly analyzing the spectrum of fluctuations around the $\mathcal{N} = 6$ solution. Consistency demands that these procedures agree.

We begin from the $SU(6)$ content of $\mathcal{N} = 6$ SUGRA: **1** graviton, **6** gravitini, $\mathbf{15} \oplus \mathbf{1}$ vectors, $\mathbf{20} \oplus \mathbf{6}$ gaugini, and 2(**15**) scalars. The KK black hole in $\mathcal{N} = 6$ SUGRA breaks the global symmetry $SU(6) \rightarrow USp(6)$. Therefore, the quadratic fluctuations around the background need not respect the $SU(6)$ symmetry, but they must respect the $USp(6)$. Their $USp(6)$ content is: **1** graviton, **6** gravitini,

$\mathbf{14} \oplus 2(\mathbf{1})$ vectors, $\mathbf{14} \oplus 2(\mathbf{6})$ gaugini, $2(\mathbf{14} \oplus \mathbf{1})$ scalars. The black hole background breaks Lorentz invariance so the equations of motion for fluctuations generally mix Lorentz representations, as we have seen explicitly in section 3.3.2, but they always preserve global symmetries. In the present context the mixing combines the fields into $\mathbf{1}$ KK block (gravity + 1 vector + 1 scalar), 3 gravitino blocks (1 gravitino + 1 gaugino) (transforming in the $\mathbf{6}$), $\mathbf{14} \oplus \mathbf{1}$ vector blocks (1 vector + 1 scalar), 10 gaugino blocks (transforming in the $\mathbf{14} \oplus \mathbf{6}$), and $\mathbf{14}$ (minimally coupled) scalars.

To verify these claims and find the specific couplings for each block, we could analyze the equations of motion for $\mathcal{N} = 6$ SUGRA using the methods of section 3.3.2. However, no new computations are needed because it is clear that the fields in the truncated theory are a subset of those in $\mathcal{N} = 8$ SUGRA. In that context we established that the fluctuations decompose into $\mathbf{1}$ (KK block), $\mathbf{8}$ (gravitini mixing with gaugini), $\mathbf{27}$ (vectors mixing with scalars), $\mathbf{24}$ (gaugini with Pauli couplings to the background), and $\mathbf{42}$ (minimal scalars) of the $USp(8)$ that is preserved by the background. The consistent truncation to $\mathcal{N} = 6$ SUGRA removes some of these fluctuations as it projects the global symmetry $USp(8) \to USp(6)$. This rule not only establishes the mixing claimed in the preceding paragraph but also shows that all couplings must be the same in the $\mathcal{N} = 8$ and $\mathcal{N} = 6$ theories. It is only the degeneracy of each type of block that is reduced by the truncation.

3.3.3.b The $\mathcal{N} = 4$ Truncation

The $\mathcal{N} = 4$ truncation restricts $\mathcal{N} = 8$ SUGRA to fields that are even under the $SU(8)_R$ element $\mathrm{diag}(I_4, -I_4)$. This projection breaks the global symmetry $SU(8)_R \to SU(4)_R \times SU(4)_{\mathrm{matter}}$. It preserves $\mathcal{N} = 4$ local supersymmetry since the 8 gravitini of $\mathcal{N} = 8$ SUGRA are in the $\mathbf{4}$ of $SU(4)_R$. The branching rules of the matter multiplets under the symmetry breaking are

$$\begin{aligned} \mathbf{70} &\to 2(\mathbf{1},\mathbf{1}) \oplus (\mathbf{6},\mathbf{6}) \oplus (\mathbf{4},\bar{\mathbf{4}}) \oplus (\bar{\mathbf{4}},\mathbf{4}) \ , \\ \mathbf{56} &\to (\bar{\mathbf{4}},\mathbf{1}) \oplus (\mathbf{6},\mathbf{4}) \oplus (\mathbf{4},\mathbf{6}) \oplus (\mathbf{1},\bar{\mathbf{4}}) \ , \\ \mathbf{28} &\to (\mathbf{1},\mathbf{6}) \oplus (\mathbf{6},\mathbf{1}) \oplus (\mathbf{4},\mathbf{4}) \ . \end{aligned} \tag{3.115}$$

The consistent truncation preserving $\mathcal{N} = 4$ supersymmetry is defined by omission of all fields in the $\mathbf{4}$ (or $\bar{\mathbf{4}}$) of $SU(4)_{\mathrm{matter}}$.

There is a unique supergravity with n $\mathcal{N} = 4$ matter multiplets. It has a global $SU(4)_R$ symmetry that acts on its supercharges and also a global $SO(n)_{\mathrm{matter}}$ that reflects the equivalence of all matter multiplets. The consistent truncation of $\mathcal{N} = 8$ by the element $\mathrm{diag}(I_4, -I_4)$ retains a $SU(4)_R \times SU(4)_{\mathrm{matter}}$ symmetry so, recalling that $SO(6)$ and $SU(4)$ are equivalent as Lie algebras, the truncated theory must be $\mathcal{N} = 4$ SUGRA with $n = 6$ matter multiplets.

Several important features of $\mathcal{N}=4$ SUGRA are succinctly summarized by the scalar coset

$$\frac{SU(1,1)}{U(1)} \times \frac{SO(6,n)}{SO(6)\times SO(n)}\ . \tag{3.116}$$

It has dimension $6n+2$ with scalars transforming in $2(\mathbf{1},\mathbf{1})\oplus(\mathbf{6},\mathbf{n})$ under $SU(4)_R\times SO(n)_{\text{matter}}$. It also encodes the $SU(1,1)\simeq SL(2)$ electromagnetic duality of the $6+n$ vector fields in the fundamental of $SO(6,n)$. The representation content obtained by removal of $\mathbf{4}$ (and $\bar{\mathbf{4}}$) from the branchings (3.115) is consistent with these expectations when $n=6$.

The $\mathcal{N}=4$ truncation has a natural interpretation in perturbative Type II string theory. There is a simple duality frame where the diagonal element $\text{diag}(I_4,-I_4)$ changes the sign on the RR sector and interchanges the RNS and NSR sectors; so the consistent truncation projects on to the common sector of Type IIA and Type IIB supergravity. The complete string theory orbifold includes twisted sectors as well. It is conveniently implemented by a flip of the GSO projection and is equivalent to T-duality between Type IIA and Type IIB string theory.

The embedding of the KK black hole into $\mathcal{N}=8$ SUGRA is compatible with the truncation to $\mathcal{N}=4$ SUGRA: the four field strengths on the skew-diagonal of the $\mathbf{28}$ are all contained in the $SU(4)_R\times SU(4)_{\text{matter}}$ subgroup of $SU(8)_R$ and therefore retained in the truncation to $\mathcal{N}=4$ SUGRA. The embedding of the KK black hole in $\mathcal{N}=8$ SUGRA therefore defines an embedding in $\mathcal{N}=4$ SUGRA as well. The consistent truncation just removes fields that are not excited by the KK black hole in $\mathcal{N}=8$ SUGRA.

The quadratic fluctuations around the KK black hole in $\mathcal{N}=8$ SUGRA similarly project on to the $\mathcal{N}=4$ setting. As discussed in section 3.3.2, the KK black hole in $\mathcal{N}=8$ SUGRA breaks the global symmetry $SU(8)_R\to USp(8)$ and this symmetry breaking pattern greatly constrains the spectrum of fluctuations around the black hole. Moreover, the symmetry breaking pattern is largely preserved by the consistent truncation: the analogous breaking pattern in $\mathcal{N}=4$ SUGRA is $SU(4)_R\times SU(4)_{\text{matter}}\to USp(4)_R\times USp(4)_{\text{matter}}$. For example, the entire KK block (with a graviton, a vector, and a scalar), identified as the $\mathbf{1}$ of $USp(8)$, is unchanged by the consistent truncation.

The 27 vector blocks (3.106-3.107), each with a vector coupled to a scalar, are perturbations of the 8×8 matrix of field strengths F_{AB} after its symplectic trace is removed. The branching (3.115) of the $\mathbf{28}$ under $SU(4)_R\times SU(4)_{\text{matter}}$ shows that 16 vector blocks are projected out by the truncation. None of these are affected by the symplectic trace so $27-16=11$ vector blocks remain in $\mathcal{N}=4$ SUGRA. Among the 38 scalars from the coset (3.116) with $n=6$ there is 1 coupled to gravity and 11 that couple to the vectors, so 26 minimally coupled scalars remain. They parametrize the coset

$$SU(1,1) \times \frac{SO(5,5)}{USp(4) \times USp(4)} \ . \tag{3.117}$$

The fermionic sector is simpler because the truncation removes exactly one half of the fermions. The retained fermions are essentially identical to those that are projected away, they differ at most in their chirality and the KK black holes is insensitive to this distinction. The quadratic fluctuations for the fermions in $\mathcal{N} = 8$ SUGRA are 4 gravitino pairs (with each pair including two gravitini coupled to two Weyl fermions, a total of 32 degrees of freedom) and 24 gaugino pairs with Pauli couplings to the background field strength. In $\mathcal{N} = 4$ SUGRA with 6 matter multiplets there are 4 gravitino pairs and 12 gaugino pairs.

There is a simple extension of these results to the case of $\mathcal{N} = 4$ SUGRA with $n \neq 6$ matter multiplets. For this generalization, we recast the symmetry breaking by the field strengths that have been designated $\mathcal{N} = 4$ matter as $SO(6)_{\text{matter}} \to SO(5)_{\text{matter}}$ using the equivalences $SU(4) = SO(6)$ and $USp(4) = SO(5)$ as Lie algebras. In this form the symmetry breaking just amounts to picking the direction of a vector on an S^5. We can equally consider any number n of matter fields and break the symmetry $SO(n)_{\text{matter}} \to SO(n-1)_{\text{matter}}$ by picking a vector on S^{n-1}. The only restriction is $n \geq 1$ in order to ensure that there is a direction to pick in the first place. This more general construction gives the scalar manifold

$$SU(1,1) \times \frac{SO(5,n-1)}{SO(5) \times SO(n-1)} \ . \tag{3.118}$$

In particular, it has $5n-4$ dimensions, each corresponding to a minimally coupled scalar field. The duality group read off from the numerator correctly indicates $n+5$ vector fields, not counting the one coupling to gravity. Each of these vector fields couples to a scalar field, as in (3.106-3.107).

The black hole attractor mechanism offers a perspective on the scalar coset (3.118). The attractor mechanism is usually formulated in the context of extremal black holes in $\mathcal{N} \geq 2$ supergravity where it determines the value of some of the scalars at the horizon in terms of black hole charges. Importantly, the attractor mechanism generally leaves other scalars undetermined. Such undetermined scalars can take any value, so they are moduli. The hyper-scalars in $\mathcal{N} = 2$ BPS black hole backgrounds are well-known examples of black hole moduli.

In the case of extremal (but non-supersymmetric) black holes in $\mathcal{N} \geq 2$ supergravity the moduli space is determined by the centralizer remaining after extremization of the black hole potential over the full moduli space of the theory. The result for non-BPS black holes in $\mathcal{N} = 4$ supergravity was obtained in [132] and agrees with (3.118). Our considerations generalize this result to a moduli space of non-extremal KK black holes. The exact masslessness of moduli is protected by the breaking of global symmetries so supersymmetry is not needed.

3.3.3.c The $\mathcal{N} = 2$ Truncation

Starting from $\mathcal{N} = 4$ SUGRA with n $\mathcal{N} = 4$ matter multiplets, there is a consistent truncation to $\mathcal{N} = 2$ SUGRA with $n + 1$ $\mathcal{N} = 2$ vector multiplets that respects the KK black hole background. It is defined by keeping only fields that are even under the $SU(4)_R$ element $\mathrm{diag}(I_2, -I_2)$.

All fermions, both gravitini and gaugini are in the fundamental $\mathbf{4}$ of $SU(4)_R$ so the consistent truncation retains exactly 1/2 of them. In particular, the SUSY is reduced from $\mathcal{N} = 4$ to $\mathcal{N} = 2$. The bosons are either invariant under $SU(4)_R$ or they transform as an antisymmetric tensor $\mathbf{6}$. The branching rule $\mathbf{6} \to 2(1, 1) \oplus (2, 2)$ under $SU(4)_R \to SU(2)^2$ determines that its truncation retains only the 2 fields on the skew-diagonal of the antisymmetric 4×4 tensor.

The truncated theory has $2(2n + 4)$ fermionic degrees of freedom and the same number of bosonic ones. We can implement the truncation directly on the $\mathcal{N} = 4$ coset (3.116) and find that scalars of the truncated theory parametrize

$$\frac{SU(1,1)}{U(1)} \times \frac{SO(2,n)}{SO(2) \times SO(n)} . \tag{3.119}$$

This theory is known as the $ST(n)$ model. In the special case $n = 2$ the $ST(2)$ model is the well-known STU model. This model has enhanced symmetry ensuring that its 3 complex scalar fields are equivalent and similarly that its 4 field strengths are equivalent. The STU model often appears as a subsector of more general $\mathcal{N} = 2$ SUGRA theories, such as those defined by a cubic prepotential. These in turn arise as the low energy limit of string theory compactified on a Calabi-Yau manifold, so the STU model may capture some generic features of such theories.

The consistent truncation to the $ST(n)$ model in $\mathcal{N} = 2$ SUGRA is compatible with the embedding of the KK black hole in $\mathcal{N} = 8$ SUGRA. The embedding (3.91) in $\mathcal{N} = 8$ excites precisely the field strengths on the skew-diagonal, breaking $SU(8)_R \to USp(8)$. As discussed in (3.3.3.b), they were retained by the truncation to $\mathcal{N} = 4$ SUGRA. The further truncation of the antisymmetric representation to $\mathcal{N} = 2$ SUGRA projects $\mathbf{6} \to 2(\mathbf{1}, \mathbf{1})$ and so it specifically retains field strengths on the skew diagonal. Moreover, the gauge fields that are projected out are in the $\mathbf{2}$ of an $SU(2)$ so they are not coupled to other fields at quadratic order.

It can be shown that the $\mathcal{N} = 4$ embedding identifies the "dilaton" of the KK black hole with the scalar (as opposed to the pseudoscalar) in the coset $SU(1,1)/U(1)$. This part of the scalar coset is untouched by the truncation to $\mathcal{N} = 2$ SUGRA. Therefore, the truncation to $\mathcal{N} = 2$ does not remove any of the fields that are turned on in the background, nor any of those that couple to them at quadratic order. This shows that the consistent truncation to $\mathcal{N} = 2$ SUGRA, like other truncations considered in this section, removes only entire blocks of fluctuations: the fields that remain have the same couplings as they do in the $\mathcal{N} = 8$ context.

The breaking pattern determines the moduli space of scalars for the black hole background as

$$SU(1,1) \times \frac{SO(1,n-1)}{SO(n-1)} \ . \tag{3.120}$$

In particular this confirms that, among the $2n+2$ scalars of the $ST(n)$ model, exactly n are moduli and so are minimally coupled massless scalars.

3.3.3.d More Comments on Consistent Truncations

The natural endpoint of the consistent truncations is $\mathcal{N} = 0$ SUGRA, *i.e.* the pure Kaluza-Klein theory (3.54). We constructed our embedding (3.91) into $\mathcal{N} = 8$ SUGRA so that the Kaluza-Klein black hole would remain a solution also to the full $\mathcal{N} = 8$ SUGRA. Thus we arranged that all the additional fields required by $\mathcal{N} = 8$ supersymmetry would be "unimportant", in the sense that they can be taken to vanish on the Kaluza-Klein black hole. It is therefore consistent to remove them again, and that is the content of the "truncation to $\mathcal{N} = 0$ SUGRA".

From this perspective, the truncations considered in this section are intermediate stages between $\mathcal{N} = 8$ and $\mathcal{N} = 0$ in that only some of the "unimportant" fields are included. For each value of $\mathcal{N} = 6, 4, 2$, the requirement that the Kaluza-Klein black hole is a solution largely determines the truncation. The resulting embedding of the STU model into $\mathcal{N} = 8$ SUGRA is very simple, and possibly simpler than others that appear in the literature, in that symmetries between fields in the STU model are manifest even without performing any electromagnetic duality.

Having analyzed the spectrum of fluctuations around Kaluza-Klein black holes in the context of SUGRA with $\mathcal{N} = 8, 6, 4, 2$ (and even $\mathcal{N} = 0$), it is natural to inquire about the situation for SUGRA with odd $\mathcal{N}$. Our embeddings in $\mathcal{N} = 6, 4, 2$ rely on the skew-diagonal nature of the embedding in $\mathcal{N} = 8$ so they do not have any generalizations to odd $\mathcal{N}$. This fact is vacuous for $\mathcal{N} = 7$ SUGRA which automatically implies $\mathcal{N} = 8$. Moreover, it is interesting that $\mathcal{N} = 3, 5$ SUGRA do not have any non-BPS branch at all: all extremal black holes in these theories must be BPS (they preserve supersymmetry) [132]. This may indicate that our examples exhaust a large class of non-BPS embeddings.

3.3.4 The General KK Black Hole in $\mathcal{N}=2$ SUGRA

In this section, we start afresh with an *arbitrary* solution to the $D=4$ Kaluza-Klein theory (3.54), such as the general Kaluza-Klein black hole (3.55-3.57). We embed this solution into $\mathcal{N}=2$ SUGRA with a *general cubic prepotential* and analyze the quadratic fluctuations around the background in this setting. Along the way we make additional assumptions that further decouple the fluctuations, and ultimately specialize to a constant background dilaton and $ST(n)$ prepotential. In this case the final results of the direct computations will be consistent with those found in section 3.3.3.c, by truncation from $\mathcal{N}=8$ SUGRA, and summarized in section 3.3.2.c.

The setup in this section complements our discussion of the Kaluza-Klein black hole in $\mathcal{N}=8$ SUGRA and its truncations to $\mathcal{N}<8$ SUGRA. Here we do not assume vanishing background dilaton $\Phi^{(\mathrm{KK})}=0$ from the outset and we consider more general theories.

3.3.4.a $\mathcal{N}=2$ SUGRA with Cubic Prepotential

We first introduce $\mathcal{N}=2$ SUGRA. We allow for matter in the form of n_V $\mathcal{N}=2$ vector multiplets with couplings encoded in a cubic prepotential

$$F=\frac{1}{\kappa^2}\frac{d_{ijk}X^iX^jX^k}{X^0}\ , \tag{3.121}$$

where d_{ijk} is totally symmetric. We also include n_H $\mathcal{N}=2$ hypermultiplets. The theory is described by the $\mathcal{N}=2$ SUGRA Lagrangian

$$\begin{aligned} e^{-1}\mathcal{L}^{(\mathcal{N}=2)} &= \kappa^{-2}\left(\frac{R}{2}-\bar{\psi}_{i\mu}\gamma^{\mu\nu\rho}D_\nu\psi^i_\rho\right)-g_{\alpha\bar{\beta}}\partial^\mu z^\alpha\partial_\mu z^{\bar{\beta}}-\frac{1}{2}h_{uv}\partial_\mu q^u\partial^\mu q^v \\ &\quad+\left(-\frac{1}{4}i\mathcal{N}_{IJ}F^{+I}_{\mu\nu}F^{+\mu\nu J}+F^{-I}_{\mu\nu}\,\mathrm{Im}\mathcal{N}_{IJ}Q^{\mu\nu-J}-\frac{1}{4}g_{\alpha\bar{\beta}}\bar{\chi}^\alpha_i \not{D}\chi^{i\bar{\beta}}\right. \\ &\quad\left.-\bar{\zeta}_A\not{D}\zeta^A+\frac{1}{2}g_{\alpha\bar{\beta}}\bar{\psi}_{i\mu}\not{\partial}z^\alpha\gamma^\mu\chi^{i\bar{\beta}}+\mathrm{h.c.}\right)\ , \end{aligned} \tag{3.122}$$

where

$$F^{\pm}_{\mu\nu} = \frac{1}{2}\left(F_{\mu\nu}\pm\widetilde{F}_{\mu\nu}\right),\ \text{with}\ \widetilde{F}_{\mu\nu}=-\frac{i}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\ , \tag{3.123}$$

$$Q^{\mu\nu-J} \equiv \overline{\nabla}_{\bar{\alpha}}\bar{X}^J\left(\frac{1}{8}g^{\beta\bar{\alpha}}C_{\beta\gamma\delta}\bar{\chi}^\gamma_i\gamma^{\mu\nu}\chi^\delta_j\epsilon^{ij}+\bar{\chi}^{\bar{\alpha}i}\gamma^\mu\psi^{\nu j}\epsilon^{ij}\right) \tag{3.124}$$

$$+X^J \left(\bar{\psi}_i^\mu \psi_j^\nu \epsilon^{ij} + \frac{1}{2}\kappa^2 \bar{\zeta}^A \gamma^{\mu\nu} \zeta^B C_{AB} \right) \, .$$

We follow the notations and conventions from [133]. In particular, the $\chi_i^\alpha = P_L \chi_i^\alpha, \alpha = 1, \ldots, n_V$ denote the physical gaugini and $\zeta^A = P_L \zeta^A, A = 1, \ldots, 2\,n_H$ denote the hyperfermions. The Kähler covariant derivatives are

$$\nabla_\alpha X^I = \left(\partial_\alpha + \frac{1}{2}\kappa^2 \partial_\alpha \mathcal{K} \right) X^I \, , \tag{3.125}$$

$$\overline{\nabla}_{\bar{\alpha}} X^I = \left(\partial_{\bar{\alpha}} - \frac{1}{2}\kappa^2 \partial_{\bar{\alpha}} \mathcal{K} \right) X^I \, , \tag{3.126}$$

where the Kähler potential $\mathcal{K}$

$$e^{-\kappa^2 \mathcal{K}} = -i(X^I \bar{F}_I - F_I \bar{X}^I) \, , \tag{3.127}$$

with $F_I = \partial_I F = \frac{\partial F}{\partial X^I}$.

The projective coordinates X^I (with $I = 0, \ldots, n_V$) are related to physical coordinates as $z^i = X^i / X^0$ (with $i = 1, \ldots, n_V$). We split the complex scalars z^i into real and imaginary parts

$$z^i = x^i - i y^i \, . \tag{3.128}$$

With cubic prepotential (3.121) we have

$$g_{i\bar{j}} = \partial_I \partial_{\bar{J}} \mathcal{K} = \kappa^{-2} \left(-\frac{3 d_{ij}}{2d} + \frac{9 d_i d_j}{4 d^2} \right) \, , \tag{3.129}$$

where we define

$$d_{ij} \equiv d_{ijk} y^k \, , \qquad d_i \equiv d_{ijk} y^j y^k \, , \qquad d \equiv d_{ijk} y^i y^j y^k \, . \tag{3.130}$$

Finally, the scalar-vector coupling are encoded in

$$\mathcal{N}_{IJ} = \mu_{IJ} + i \nu_{IJ} \, , \tag{3.131}$$

with

$$\mu_{IJ} = \kappa^{-2} \begin{pmatrix} 2 d_{ijk} x^i x^j x^k & -3 d_{ijk} x^j x^k \\ -3 d_{ijk} x^j x^k & 6 d_{ijk} x^k \end{pmatrix} \, , \tag{3.132}$$

and

$$\nu_{IJ} = \kappa^{-2} \begin{pmatrix} -d + 6d_{\ell m}x^\ell x^m - \frac{9}{d}(d_\ell x^\ell)^2 & \frac{9}{d}(d_\ell x^\ell)d_i - 6d_{i\ell}x^\ell \\ \frac{9}{d}(d_\ell x^\ell)d_i - 6d_{i\ell}x^\ell & 6d_{ij} - \frac{9}{d}(d_i d_j) \end{pmatrix} . \qquad (3.133)$$

3.3.4.b The Embedding into $\mathcal{N} = 2$ SUGRA

We want to embed our seed solution into $\mathcal{N} = 2$ SUGRA. The starting point is a solution to the equations of motion (3.55, 3.56, 3.57) of the Kaluza-Klein theory. We denote the corresponding fields $g_{\mu\nu}^{(\mathrm{KK})}$, $F_{\mu\nu}^{(\mathrm{KK})}$ and $\Phi^{(\mathrm{KK})}$. The fields of $\mathcal{N} = 2$ SUGRA are then defined to be

$$\begin{aligned}
&g_{\mu\nu}^{(\mathrm{SUGRA})} = g_{\mu\nu}^{(\mathrm{KK})} \ , \\
&F_{\mu\nu}^0 = \frac{1}{\sqrt{2}} F_{\mu\nu}^{(\mathrm{KK})}, \quad F_{\mu\nu}^i = 0, \quad \text{for } 1 \le i \le n_V \\
&x^i = 0, \quad \text{for } 1 \le i \le n_V \ , \\
&y^i = c^i y_0, \text{ with } y_0 = \frac{\exp\left(-2\Phi^{(\mathrm{KK})}/\sqrt{3}\right)}{(d_{ijk}c^i c^j c^k)^{1/3}} \ , \\
&(\text{All other bosonic fields in } \mathcal{N} = 2 \text{ SUGRA}) = 0 \ , \\
&(\text{All fermionic fields in } \mathcal{N} = 2 \text{ SUGRA}) = 0 \ . \qquad (3.134)
\end{aligned}$$

This field configuration solves the equations of motion of $\mathcal{N} = 2$ SUGRA for any seed solution to the Kaluza-Klein theory. In the following, we will often declutter formulae by omitting the superscript "KK" when referring to fields in the seed solution.

The embedding (3.134) is really a family of embeddings parameterized by the n_V constants c^i (with $i = 1, \ldots, n_V$). They are projective coordinates on the moduli space parametrized by the n_V scalar fields y_i with the constraint

$$d = d_{ijk}y^i y^j y^k = \exp\left(-2\sqrt{3}\Phi^{(\mathrm{KK})}\right) \ . \qquad (3.135)$$

In the special case of the non-rotating Kaluza-Klein black hole with $P = Q$, we have $\Phi^{(\mathrm{KK})} = 0$ and so the constraint is $d = 1$. More generally, d is the composite field defined through the constraints (3.130) and related to the Kaluza-Klein dilaton by (3.135).

3.3.4.c Decoupled Fluctuations: General Case

The Lagrangian for quadratic fluctuations around a bosonic background always decouples into a bosonic sector and fermionic sector,

$$\delta^2\mathcal{L}^{(\mathcal{N}=2)} = \delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{bosons}} + \delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{fermions}} \ . \tag{3.136}$$

With the above embedding into $\mathcal{N}=2$, each sector further decouples into several blocks.

The bosonic sector decomposes as the sum of three blocks

$$\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{bosons}} = \delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{gravity}} + \delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{vectors}} + \delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{scalars}} \ . \tag{3.137}$$

The "gravity block" $\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{gravity}}$ consists of the graviton $\delta g_{\mu\nu}$, the gauge field δA^0_μ, and the n_V real scalars δy^i:

$$e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{gravity}} = \frac{1}{\sqrt{-g}}\delta^2\left[\sqrt{-g}\left(\frac{R}{2\kappa^2} - g_{ij}\partial_\mu y^i\partial^\mu y^j + \frac{d}{4\kappa^2}F^0_{\mu\nu}F^{\mu\nu 0}\right)\right] \ . \tag{3.138}$$

Generically, the fields $\delta g_{\mu\nu}, \delta A^0_\mu$ and δy^i all mix together. This block can nonetheless be further decoupled with simplifying assumptions, as we will discuss later.

The block $\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{vectors}}$ consists of the n_V vector fields δA^i_μ and the n_V real pseudoscalars δx^i:

$$\begin{aligned} e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{vectors}} = g_{ij}\Big(&-\partial_\mu\delta x^i\partial^\mu\delta x^j - \frac{1}{2}dF_{\mu\nu}F^{\mu\nu}\delta x^i\delta x^j \\ &+ \sqrt{2}dF_{\mu\nu}\delta x^i\delta F^{\mu\nu j} - d\delta F^i_{\mu\nu}\delta F^{\mu\nu j}\Big) \ . \end{aligned} \tag{3.139}$$

The Kähler metric g_{ij} can be diagonalized and we obtain n_V identical decoupled copies, that we call "vector block", each consisting in one vector field and one real scalar. Denoting the fluctuating field $f_{\mu\nu}$, one such copy has the Lagrangian

$$e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{vector}} = -\frac{1}{2}\partial_\mu x\partial^\mu x - \frac{d}{4}F_{\mu\nu}F^{\mu\nu}x^2 + \frac{d}{2}F_{\mu\nu}f^{\mu\nu}x - \frac{d}{4}f_{\mu\nu}f^{\mu\nu} \ , \tag{3.140}$$

using conventional normalizations for the scalar fields.

The last bosonic block contains the hyperbosons:

$$e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{scalars}} = -\frac{1}{2}h_{uv}\partial_\mu\delta q^u\partial^\mu\delta q^v \ . \tag{3.141}$$

The quaternionic Kähler metric h_{uv} is trivial on the background. Hence, this block decouples at quadratic order into $4\,n_H$ independent minimally coupled massless scalars.

We next turn to the fermions. The Lagrangian (3.122) is the sum of the decoupled Lagrangians

$$\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{fermions}} = \delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{hyperfermions}} + \delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{gravitino-gaugino}}\,. \tag{3.142}$$

The hyperfermions consist of n_H identical copies, that we call "hyperfermion block", each containing two hyperfermions. For any two such fermions we can take $C_{AB} = \epsilon_{AB}$ with $A, B = 1, 2$. The resulting Lagrangian is

$$e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{hyperfermion}} = -2\bar{\zeta}_A \not{D}\zeta^A + \left(\frac{\kappa^2}{2}F^{-I}_{\mu\nu}\nu_{IJ}X^J\bar{\zeta}^A\gamma^{\mu\nu}\zeta^B\epsilon_{AB} + \text{h.c.}\right)\,. \tag{3.143}$$

In our background, we use (3.134, 3.133) to find

$$e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{hyperfermion}} = -2\bar{\zeta}_A \not{D}\zeta^A - \left(\frac{d^{\frac{1}{2}}}{8}F^{-}_{\mu\nu}\bar{\zeta}^A\gamma^{\mu\nu}\zeta^B\epsilon_{AB} + \text{h.c.}\right)\,. \tag{3.144}$$

We used the T-gauge [133] to fix the projective coordinates X^I resulting in $X^0 = (8d)^{-1/2}$.

The "gravitino-gaugino block" contains two gravitini and n_V gaugini and has Lagrangian

$$\begin{aligned} e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{gravitino-gaugino}} &= -\frac{1}{\kappa^2}\bar{\psi}_{i\mu}\gamma^{\mu\nu\rho}D_\nu\psi^i_\rho \\ &+ \Bigg(-\frac{d^{\frac{1}{2}}}{4\kappa^2}F^{-}_{\mu\nu}\bar{\psi}^\mu_i\psi^\nu_j\epsilon^{ij} + \frac{9}{256\kappa^2 d^{\frac{3}{2}}}F^{-}_{\mu\nu}d_{\bar{\alpha}}g^{\beta\bar{\alpha}}d_{\beta\gamma\delta}\bar{\chi}^\gamma_i\gamma^{\mu\nu}\chi^\delta_j\epsilon^{ij} \\ &-\frac{3i}{8\kappa^2 d^{\frac{1}{2}}}F^{-}_{\mu\nu}d_{\bar{\alpha}}\bar{\chi}^{\bar{\alpha}i}\gamma^\mu\psi^{\nu j}\epsilon^{ij} - \frac{1}{4}g_{\alpha\bar{\beta}}\bar{\chi}^\alpha_i\not{D}\chi^{i\bar{\beta}} + \frac{1}{2}g_{\alpha\bar{\beta}}\bar{\psi}_{ia}\not{\partial}z^\alpha\gamma^a\chi^{i\bar{\beta}} + \text{h.c.}\Bigg)\,. \end{aligned} \tag{3.145}$$

Generally, all the gravitini and gaugini couple nontrivially but they can be further decoupled in simpler cases, as we will discuss later.

Summarizing so far: given any Kaluza-Klein solution, the embedding (3.134) provides solutions of $\mathcal{N} = 2$ SUGRA. We have expanded the $\mathcal{N} = 2$ Lagrangian around this background to quadratic order and observed that the fluctuations can be decoupled as shown in Table 3.5.

Degeneracy	Multiplet	Block content	Lagrangian
1	Gravity block	1 graviton, 1 vector, n_V scalars	(3.138)
n_V	Vector block	1 vector and 1 (pseudo)scalar	(3.140)
$4n_H$	Scalar block	1 real scalar	(3.141)
1	Gravitino-gaugino block	2 gravitini and $2\,n_V$ gaugini	(3.145)
n_H	Hyperfermion block	2 hyperfermions	(3.144)

Table 3.5: Decoupled quadratic fluctuations in $\mathcal{N} = 2$ SUGRA around a general KK black hole.

These results are reminiscent of the analogous structure for $\mathcal{N} = 8$ SUGRA, summarized in (3.91). However, with the more general assumptions made here, there are more scalars in the $\mathcal{N} = 2$ gravity block than in the analogous $\mathcal{N} = 8$ KK block and these additional scalars do not generally decouple from gravity. Similarly, the $\mathcal{N} = 2$ gravitino-gaugino block here includes more gaugini than the analogous $\mathcal{N} = 8$ gravitino block.

3.3.4.d Decoupled Fluctuations: Constant Dilaton

So far, we have been completely general about the underlying Kaluza-Klein solution. In this section, we further decouple the quadratic fluctuations by assuming that the scalar fields of $\mathcal{N} = 2$ SUGRA are constant

$$y^i = \text{constant}, \ i = 1, ..., n_V \ . \tag{3.146}$$

From the embedding (3.134), this is equivalent to taking the Kaluza-Klein dilaton to vanish

$$\Phi^{(\text{KK})} = 0 \ , \tag{3.147}$$

since we can always rescale the field strengths to arrange for $d = d_{ijk}y^i y^j y^k = 1$. As noted previously, this is satisfied by the non-rotating Kaluza-Klein black hole with $P = Q$. This is the simplified background that we already studied in $\mathcal{N} = 8$ SUGRA, but it is embedded here in $\mathcal{N} = 2$ SUGRA with arbitrary prepotential. As in the $\mathcal{N} = 8$ case, we will use that the background satisfies

$$R = 0 \ , \qquad F_{\mu\nu}F^{\mu\nu} = 0 \tag{3.148}$$

to decouple further the quadratic fluctuations.

- *Gravity*

The gravity block decouples as

$$\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{gravity}} = \delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{KK}} + \delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{relative}} \ , \tag{3.149}$$

where $\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{KK}}$ is the "KK block", consisting of the graviton $\delta g_{\mu\nu}$, the graviphoton δA^0_μ and the center-of-mass scalar $\delta y'^1$. $\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{relative}}$ denotes $n_V - 1$ free massless scalars $\delta y'^i$, $i = 2, \dots n_V$. This decoupling is obtained by center-of-mass diagonalization: the $\delta y'^i$ are linear combinations of δy^i such that $\delta y'^1$ is precisely the combination that couples to the graviton and graviphoton at quadratic order. Then, the "relative scalars" $\delta y'^i$, $i = 2, \dots, n_V$ are minimally coupled to the background

$$e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{relative}} \quad = \quad -\frac{2}{\kappa^2}\partial_\mu\delta y'^i\partial^\mu\delta y'^i \quad (\text{for } i = 2, \dots, n_V) \ , \tag{3.150}$$

The center-of-mass Lagrangian turns out to be exactly the same as the $\mathcal{N} = 8$ KK block (3.104)

$$\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{KK}} = \delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{KK}} \ , \tag{3.151}$$

with the identifications

$$\bar{h}_{\mu\nu} = \frac{1}{\sqrt{2}}\left(\delta g_{\mu\nu} - \frac{1}{4}g_{\mu\nu}g^{\rho\sigma}\delta g_{\rho\sigma}\right) \ , \quad h = \frac{1}{\sqrt{2}}g^{\rho\sigma}\delta g_{\rho\sigma} \ , \tag{3.152}$$

$$a_\mu = \sqrt{2}\delta A^0_\mu \ , \quad f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \ , \tag{3.153}$$

$$\phi = \delta y'^1 = -\frac{\sqrt{3}d_i}{2d}\delta y^i = \delta\Phi \ . \tag{3.154}$$

The equality between $\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{KK}}$ and $\delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{KK}}$ is expected because the KK block is the same for any $\mathcal{N} = 2$ SUGRA and in particular for the $\mathcal{N} = 2$ truncations of $\mathcal{N} = 8$ SUGRA.

The $n_V - 1$ minimally coupled massless scalars $\delta y'^i, i = 2, \dots, n_V$ parameterize flat directions in the moduli space, at least at quadratic order. In important situations with higher symmetry, including homogeneous spaces constructed as coset manifolds, it can be shown that these $n_V - 1$ directions are exactly flat at all orders. This implies that, in particular, these models are stable [134, 135]. In such situations the "relative" coordinates $\delta y'^i$ are Goldstone bosons parameterizing symmetries of the theories.

- *Vector block*

  Using the fact that $F_{\mu\nu}F^{\mu\nu} = 0$, the vector block becomes

$$e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{vector}} = -\frac{1}{2}\partial_\mu x \partial^\mu x + \frac{1}{2}F_{\mu\nu}f^{\mu\nu}x - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} \ . \tag{3.155}$$

  Again, we find that $\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{vector}} = \delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{vector}}$ after proper normalization of the field strength.

- *Scalar block*

  The Lagrangian for hyperbosons $\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{scalars}}$ consists of $4n_H$ minimally coupled scalars. In addition, the center-of-mass diagonalization has brought $n_V - 1$ minimally coupled "relative" scalars $\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{relative}}$. This gives a total of $n_V + 4n_H - 1$ minimally coupled scalars.

We now turn to fermions. The interactions between gravitini and gaugini simplify greatly when scalars are constant. However, they still depend on the prepotential through the structure constants $d_{\alpha\beta\gamma}$. The fermionic fluctuations in $\mathcal{N} = 2$ SUGRA are therefore qualitatively different from the bosonic fluctuations which, as we just saw, reduce to the form found in $\mathcal{N} = 8$ SUGRA.

For fermions we need to further specialize and study the $ST(n)$ model. This model already appeared in section 3.3.3.c, as a truncation of $\mathcal{N} = 8$ SUGRA to $\mathcal{N} = 2$. Presently, we introduce it as the model with $n_V = n + 1$ vector multiplets and prepotential

$$F = \frac{1}{\kappa^2}\frac{X^1(X^2X^2 - X^\alpha X^\alpha)}{2X^0} \qquad (\alpha = 3, \ldots, n_V) \ . \tag{3.156}$$

We take the background scalars

$$y^1 = 1, \quad y^2 = \sqrt{2}, \quad y^\alpha = 0 \quad (\alpha = 3, \ldots, n_V) \ , \tag{3.157}$$

such that the normalization is $d = 1$ and therefore $\Phi^{(\text{KK})} = 0$. As mentioned already in section 3.3.3.c, this model generalizes the STU model which is equivalent to $ST(2)$.

- *Gravitino-gaugino block*

  The Lagrangian for the gravitino-gaugino block decouples as

$$\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{gravitino-gaugino}} \quad = \quad \delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{gravitino}} + \delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{gaugino}} \ , \tag{3.158}$$

  after using center-of-mass diagonalization. We call χ'^{i1} the center-of-mass gaugini, *i.e.* the gaugini that couples to the gravitini. More precisely, we define

$$\chi'^{i1} = \frac{1}{4}\left(\frac{\sqrt{3}}{3}\chi^{i1} + \frac{\sqrt{6}}{3}\chi^{i2}\right) \ , \quad \chi'^{i2} = \frac{1}{4}\left(\frac{\sqrt{6}}{3}\chi^{i1} - \frac{\sqrt{3}}{3}\chi^{i2}\right) \ ,$$
$$\chi'^{i\alpha} = \frac{1}{4}\chi^{i\alpha} \quad \text{for } \alpha = 3, \ldots, n_V \ . \tag{3.159}$$

We find a center-of-mass multiplet that we call "gravitino block"

$$e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{gravitino}} = -\frac{1}{\kappa^2}\bar{\psi}_{i\mu}\gamma^{\mu\nu\rho}D_\nu\psi^i_\rho + \frac{1}{\kappa^2}\left(-\bar{\chi}'^1_i \not{D}\chi'^{i1} - \frac{1}{4}\bar{\psi}^\mu_i F^-_{\mu\nu}\psi^\nu_j\epsilon^{ij}\right.$$
$$\left. + \frac{1}{4}\bar{\chi}'^1_i F^-_{\mu\nu}\gamma^{\mu\nu}\chi'^1_j\epsilon^{ij} - \frac{\sqrt{3}i}{2}\bar{\chi}'^{i1}\gamma^\mu F^-_{\mu\nu}\psi^{\nu j}\epsilon^{ij} + \text{h.c.}\right) \ , \tag{3.160}$$

This Lagrangian couples the two gravitini to two center-of-mass gaugini. The "relative" multiplets are $n_V - 1$ identical copies of a "gaugino block"

$$e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{gaugino}} = -\frac{2}{\kappa^2}\bar{\chi}'^\alpha_i \not{D}\chi'^i_\alpha - \left(\frac{1}{8\kappa^2}\bar{\chi}'^\alpha_i F^-_{\mu\nu}\gamma^{\mu\nu}\chi'_{j\alpha}\epsilon^{ij} + \text{h.c.}\right) \ , \tag{3.161}$$

where $\alpha = 2, \ldots, n_V$.

- *Hyperfermion block*

  The hyperfermion Lagrangian is given in (3.144). We notice that

$$\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{hyperfermion}} = \delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{gaugino}} \ , \tag{3.162}$$

  The fluctuations of "relative" gaugini are therefore the same as the fluctuations of hyperfermions. Therefore, we call both of them "gaugino block".

The Lagrangians (3.160) and (3.161) are written in terms of Weyl fermions. If we rewrite them with Majorana fermions, we find that

$$\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{gravitino}} = \delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{gravitino}} \ , \tag{3.163}$$
$$\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{gaugino}} = \delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{gaugino}} \ , \tag{3.164}$$

where the right-hand sides were defined in (3.112) and (3.113). The agreement between our explicit computations of the fermionic blocks for the $ST(n)$ model in $\mathcal{N} = 2$ SUGRA and the analogous results in $\mathcal{N} = 8$ SUGRA is an important consistency check on the truncations discussed in section 3.3.3.c. This also explains the agreement (3.162) between fermionic fluctuations that are in different $\mathcal{N} = 2$ multiplets. $\mathcal{N} = 2$ gaugini and hyperfermions becomes equivalent when embedded into some larger structure, ultimately furnished by $\mathcal{N} = 8$ SUGRA.

In summary, taking the dilaton to be constant has further decoupled the fluctuations in $\mathcal{N} = 2$ SUGRA around the KK background, as shown in Table 3.6. For bosons, we recover the results of $\mathcal{N} = 8$ SUGRA as expected, although we are more general here since we allow for an arbitrary prepotential. For fermions, we have to specialize to the $ST(n)$ model to be able to further decouple the fluctuations. The resulting fermionic fluctuations also reproduce the fluctuations of $\mathcal{N} = 8$ SUGRA.

Degeneracy	Multiplet	Block content	Lagrangian
1	KK block	1 graviton, 1 vector, 1 scalar	(3.151)
n_V	Vector block	1 vector and 1 (pseudo)scalar	(3.155)
$n_V + 4n_H - 1$	Scalar block	1 real scalar	(3.141, 3.150)
1	Gravitino block	2 gravitini and 2 gaugini	(3.160)
$n_V + n_H - 1$	Gaugino block	2 spin 1/2 fermions	(3.144, 3.161)

Table 3.6: Decoupled fluctuations in $\mathcal{N} = 2$ SUGRA around the KK black hole with constant dilaton. The decoupling in the bosonic sector holds for an arbitrary prepotential. The fermionic sector has been further decoupled by specializing to the $ST(n)$ model.

3.4 Logarithmic Corrections to Black Hole Entropy

The logarithmic correction controlled by the size of the horizon in Planck units is computed by the functional determinant of the quadratic fluctuations of light fields around the background solution. The arguments establishing this claim for non-extremal black holes are made carefully in [22]. In this section we give a brief summary of the steps needed to extract the logarithm using the heat kernel approach. It follows the discussion in [26] and we refer to [122] for background literature on technical aspects.

Naturally, we apply the procedure to the Kaluza-Klein black holes on the non-BPS branch. This gives our final results for the coefficients of the logarithmic corrections, summarized in Table 3.8.

3.4.1 General Framework: Heat Kernel Expansion

In Euclidean signature, the effective action W for the quadratic fluctuations takes the schematic form

$$e^{-W} = \int \mathcal{D}\phi \exp\left(-\int d^4x\sqrt{g}\,\phi_n\Lambda^n_m\phi^m\right) = \det{}^{\mp 1/2}\Lambda \ , \tag{3.165}$$

where Λ is a second order differential operator that characterizes the background solution, and ϕ_n embodies the entire field content of the theory. The sign $\mp$ is $-$ for bosons and $+$ for fermions. The formal determinant of Λ diverges and a canonical way to regulate it is by introducing a heat kernel: if $\{\lambda_i\}$ is the set of eigenvalues of Λ, then the heat kernel $D(s)$ is defined by

$$D(s) = \mathrm{Tr}\, e^{-s\Lambda} = \sum_i e^{-s\lambda_i} \ , \tag{3.166}$$

and the effective action becomes

$$W = \mp\frac{1}{2}\int_\epsilon^\infty \frac{ds}{s}D(s) \ . \tag{3.167}$$

Here ϵ is an ultraviolet cutoff, which is typically controlled by the Planck length, *i.e.* $\epsilon \sim \ell_P^2 \sim G$.

In our setting it is sufficient to focus on the contribution of massless fields in the two derivative theory. For this part of the spectrum, the scale of the eigenvalues λ_i is set by the background size which in our case is identified with the size of the black hole horizon, denoted by A_H. The integral (3.167) is therefore dominated by the integration range $\epsilon \ll s \ll A_H$, and there is a logarithmic contribution

$$\int_\epsilon^\infty \frac{ds}{s}D(s) = \cdots + C_{\mathrm{local}}\log(A_H/G) + \cdots \ . \tag{3.168}$$

with coefficient denoted by C_{local}. This term comes from the constant term in the Laurent expansion of the heat kernel $D(s)$. Introducing the heat kernel density $K(x,x;s)$ which satisfies

$$D(s) = \int d^4x\sqrt{g}\,K(x,x;s) \ , \tag{3.169}$$

it is customary to cast the perturbative expansion in s as

$$K(x,x;s) = \sum_{n=0}^{\infty} s^{n-2}a_{2n}(x) \ , \tag{3.170}$$

and we identify

$$C_{\rm local} = \int d^4x\sqrt{g}\, a_4(x) \ . \tag{3.171}$$

The functions $\{a_{2n}(x)\}$ are known as the Seeley-DeWitt coefficients. The logarithmic term that we need is controlled by $a_4(x)$. The omitted terms denoted by ellipses in (3.168) are captured by the other Seeley-DeWitt coefficients. For example, the term $a_0(x)$ induces a cosmological constant at one-loop and the term $a_2(x)$ renormalizes Newton constant.

There is a systematic way to evaluate the Seeley-DeWitt coefficients in terms of the background fields and covariant derivatives appearing in the operator Λ [122]. The procedure assumes that the quadratic fluctuations can be cast in the form

$$-\Lambda^n_m = (\Box)I^n_m + 2(\omega^\mu D_\mu)^n_m + P^n_m \ . \tag{3.172}$$

Here, I^n_m is the identity matrix in the space of fields, ω^μ and P are matrices constructed from the background fields, and $\Box = D_\mu D^\mu$. From this data, the Seeley-DeWitt coefficient $a_4(x)$ is given by the expression

$$\begin{aligned}(4\pi)^2 a_4(x) \quad &= \quad \mathrm{Tr}\left[\frac{1}{2}E^2 + \frac{1}{6}RE + \frac{1}{12}\Omega_{\mu\nu}\Omega^{\mu\nu} \right. \\ &\qquad \left. + \frac{1}{360}(5R^2 + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu})\right] \ ,\end{aligned} \tag{3.173}$$

where

$$E = P - \omega^\mu\omega_\mu - (D^\mu\omega_\mu) \ , \qquad \Omega_{\mu\nu} = [D_\mu + \omega_\mu, D_\nu + \omega_\nu] \ . \tag{3.174}$$

This is the advantage of the heat kernel approach: after explicitly expanding the action around the background to second order, we have a straightforward formula to compute the Seeley-DeWitt coefficients from Λ (3.172).

The preceding discussion is based on the operator Λ (3.172) that is second order in derivatives. For fermions, the quadratic fluctuations are described by a first order operator H so the discussion must be modified slightly. We express the quadratic Lagrangian as

$$\delta^2\mathcal{L} = \bar{\Psi}H\Psi \ . \tag{3.175}$$

Following the conventions in [26], we always cast the quadratic fluctuations for the fermions in terms of Majorana spinors. The one-loop action is obtained by applying heat kernel techniques to the operator $H^\dagger H$ and using

$$\log\det H = \frac{1}{2}\log\det H^\dagger H \ . \tag{3.176}$$

Fermi-Dirac statistics also gives an additional minus sign. Thus, the fermionic contribution is obtained by multiplying (3.173) with an additional factor of $-1/2$.

3.4.2 Local Contributions

It is conceptually straightforward to compute $a_4(x)$ via (3.173). However, it can be cumbersome to decompose the differential operators, write them in the form (3.172) and compute their traces. The main complication is that our matter content is not always minimally coupled, as emphasized in sections 3.3.2 and 3.3.4.

To overcome these technical challenges we automated the computations using Mathematica with the symbolic tensor manipulation package `xAct`[5]. In particular, we used the subpackage `xPert` [136] to expand the bosonic Lagrangian to second order. We created our own package for treatment of Euclidean spinors. The computation proceeds as follows:

1. Expand the Lagrangian to second order.
2. Gauge-fix and identify the appropriate ghosts.
3. Reorganize the fluctuation operator Λ^n_m and extract the operators ω_μ and P from (3.172).
4. Compute the Seeley-DeWitt coefficient $a_4(x)$ using formula (3.173).
5. Simplify $a_4(x)$ using the background equations of motion, tensor and gamma matrix identities.

The results of the expansion to second order with `xPert` match with the bosonic Lagrangians summarized in Table 3.3. In Appendix 3.5 we elaborate on the intermediate steps and record the traces of E and $\Omega_{\mu\nu}$ for each of the blocks encountered in our discussion.

A priori, the Seeley-DeWitt coefficient $a_4(x)$ is a functional of both the geometry and the matter fields. The fact that the dilaton $\Phi^{(\mathrm{KK})}$ is constant on our background simplifies the situation greatly. By using the equations of motion, $a_4(x)$ can be recast as a functional of the geometry alone. We list the equations that we use to simplify $a_4(x)$ explicitly in Appendix 3.5.

As a result, for our background, the Seeley-DeWitt coefficient at four derivative

[5] `http://www.xact.es/`www.xact.es

order can be arranged in the canonical form

$$a_4(x) = \frac{c}{16\pi^2} W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma} - \frac{a}{16\pi^2} E_4 \ , \tag{3.177}$$

where a and c are constants governed by the couplings and field content of the theory and the curvature invariants are defined in (3.189) and (3.190). The values of c and a are summarized in Tables 3.7 and 3.8.

Multiplet \ Properties	Content	d.o.f.	c	a	$c-a$
Minimal boson	1 real scalar	1	$\frac{1}{120}$	$\frac{1}{360}$	$\frac{1}{180}$
Gaugino block	2 gaugini	4	$\frac{13}{960}$	$-\frac{17}{2880}$	$\frac{7}{360}$
Vector block	1 vector and 1 (pseudo)scalar	3	$\frac{1}{40}$	$\frac{11}{120}$	$-\frac{1}{15}$
Gravitino block	2 gravitini and 2 gaugini	8	$-\frac{347}{480}$	$-\frac{137}{1440}$	$-\frac{113}{180}$
KK block	1 graviton, 1 vector, 1 scalar	5	$\frac{37}{24}$	$\frac{31}{72}$	$\frac{10}{9}$

Table 3.7: Contributions to $a_4(x)$ decomposed in the multiplets that are natural to the KK black hole.

Multiplet / Theory	$\mathcal{N}=8$	$\mathcal{N}=6$	$\mathcal{N}=4$	$\mathcal{N}=2$	$\mathcal{N}=0$
KK block	1	1	1	1	1
Gravitino block	4	3	2	1	0
Vector block	27	15	$n+5$	n_V	0
Gaugino block	24	10	$2n$	n_V+n_H-1	0
Scalar block	42	14	$5n-4$	n_V+4n_H-1	0
a	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{1}{32}(22+3n)$	$\frac{1}{192}(65+17n_V+n_H)$	$\frac{31}{72}$
c	0	0	$\frac{3}{32}(2+n)$	$\frac{3}{64}(17+n_V+n_H)$	$\frac{37}{24}$

Table 3.8: The degeneracy of multiplets in the spectrum of quadratic fluctuations around the KK black hole embedded in to various theories, and their respective values of the c and a coefficients defined in (3.177). For $\mathcal{N}=4$, the integer n is the number of $\mathcal{N}=4$ matter multiplets. For $\mathcal{N}=2$, the recorded values of c and a for the gravitino and the gaugino blocks were only established for $ST(n_V-1)$ models.

It is worth making a few remarks.

1. The value of $c-a$ in each case is independent of the couplings of the theory. In other words, $c-a$ can be reproduced by an equal number of minimally coupled fields on the same black hole background. This property is due to the fact that none of the non-minimal couplings appearing in our blocks involve the Riemann tensor $R_{\mu\nu\rho\sigma}$. Therefore, the coefficient of $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ is insensitive to the non-trivial couplings.

2. The values of c for blocks recorded in Table 3.7 do not have any obvious regularity, they are not suggestive of any cancellations. The vanishing of the c-anomaly for the $\mathcal{N}=6$ and $\mathcal{N}=8$ theories, exhibited in Table 3.8, seems therefore rather miraculous. Somehow these embeddings with large supersymmetry have special properties that are not shared by those with lower supersymmetry.

3.4.3 Quantum Corrections to Black Hole Entropy

The logarithmic terms in the one-loop effective action of the massless modes correct the entropy of the black hole as

$$\delta S_{\rm BH} = \frac{1}{2}(C_{\rm local} + C_{\rm zm}) \log \frac{A_H}{G} \ . \tag{3.178}$$

In this subsection we gather our results and evaluate the quantum contribution for the Kaluza-Klein black hole.

The local contribution is given by the integrated form of the Seeley-DeWitt coefficient $a_4(x)$:

$$C_{\rm local} = \frac{c}{16\pi^2}\int \sqrt{g}\, d^4x\, W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} - \frac{a}{16\pi^2}\int \sqrt{g}\, d^4x\, E_4 \ . \tag{3.179}$$

The second term is essentially the Euler characteristic

$$\chi = \frac{1}{32\pi^2}\int d^4x \sqrt{g}\, E_4 = 2 \ , \tag{3.180}$$

for any non-extremal black hole. It is a topological invariant so it does not depend on black hole parameters. In contrast, the first integral in (3.179) depends sensitively on the details of the black hole background. Using the KK black hole presented in section 3.2 with $J=0$ and $P=Q$ we find

$$\frac{1}{16\pi^2}\int d^4x \sqrt{g}\, W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} = 4 + \frac{8}{5\,\xi(1+\xi)} \ , \tag{3.181}$$

where $\xi \geq 0$ is a dimensionless parameter related to the black hole parameters as

$$\frac{Q}{GM} = \frac{P}{GM} = \frac{\sqrt{2(1+\xi)}}{2+\xi} \ . \tag{3.182}$$

In this parametrization the extremal (zero temperature) limit corresponds to $\xi \to 0$ and the Schwarzschild (no charge) limit corresponds to $\xi \to \infty$.

We also need to review the computation of $C_{\rm zm}$, the integer that captures corrections to the effective action due to zero modes. In our schematic notation zero modes $\lambda_i = 0$ are included in the heat kernel (3.166) and therefore contribute to the local term $C_{\rm local}$. However, the zero mode contribution to the effective action is not computed correctly by the Gaussian path integral implied in (3.165) and should instead be replaced by an overall volume of the symmetry group responsible for the zero mode. It is the combination of removing the zero-mode from the heat kernel and adding it back in again as a volume factor that gives the correction $C_{\rm zm}$.

Additionally, the effective action defined by the Euclidean path integral with thermal boundary conditions is identified with the free energy in the canonical ensemble whereas the entropy is computed in the microcanonical ensemble where mass and charges are fixed. The Legendre transform relating these ensembles gives a logarithmic contribution to the entropy that we have absorbed into $C_{\rm zm}$, for brevity.

The various contributions to $C_{\rm zm}$ are not new, they were analyzed in [22]. The result can be consolidated in the formula [26]

$$C_{\rm zm} = -(3+K) + 2N_{\rm SUSY} + 3\,\delta_{\text{non-ext}} \ . \tag{3.183}$$

Here K is the number of rotational isometries of the black hole, $N_{\rm SUSY}$ is the number of preserved real supercharges. $\delta_{\text{non-ext}}$ is 0 if the black hole is extremal and 1 otherwise. The non-extremal KK black hole with $J = 0$ is spherically symmetric and has $K = 3$, $N_{\rm SUSY} = 0$ and $\delta_{\text{non-ext}} = 1$. Therefore, $C_{\rm zm} = -3$ for all the non-extremal black holes we consider in this paper but $C_{\rm zm} = -6$ in the extreme limit.

Combining all contributions, our final result for the coefficient of the logarithmic correction to the non-extreme black hole entropy is

$$\frac{1}{2}(C_{\rm local} + C_{\rm zm}) = 2(c-a) - \frac{3}{2} + \frac{4}{5\,\xi(1+\xi)}\,c \ , \tag{3.184}$$

where the values of c and a for the theories discussed in this paper are given in Table 3.8. The expression manifestly shows that when $c \neq 0$, which is the case for $\mathcal{N} = 0, 2, 4$, the quantum correction to the entropy depends on black hole parameters through ξ or, by the relation (3.182), through the physical ratio Q/GM. The cases with very high supersymmetry are special since $c = 0$ when $\mathcal{N} \geq 6$ and then the coefficient of the logarithm is purely numerical. For example, we find the quantum corrections

$$\delta S^{(\mathcal{N}=6)}_{\text{non-ext}} = -\frac{9}{2} \log \frac{A_H}{G} \ , \qquad \delta S^{(\mathcal{N}=8)}_{\text{non-ext}} = -\frac{13}{2} \log \frac{A_H}{G} \ , \tag{3.185}$$

to the non-extremal black holes on the non-BPS branch.

As we have stressed, the KK black hole on the non-BPS branch is not intrinsically exceptional. In the non-rotating case with $P = Q$ that is our primary focus, the geometry is the standard Reissner-Nordström black hole. However, Kaluza-Klein theory includes a scalar field, the dilaton, and this dilaton couples non-minimally to gravity and to the gauge field. According to Table 3.8 we find $c = \frac{37}{24}$ for the KK black hole that is, after all, motivated by a higher dimensional origin.

An appropriate benchmark for this result is the minimally coupled Einstein-Maxwell theory, which has Reissner-Nordström as a solution, with an additional minimally coupled scalar field. The KK theory and the minimal theory both have $c - a = \frac{10}{9}$, because these theories have the same field content, and the zero-mode content of the black holes in the two theories is also identical, because the geometries are the same. However, $c = \frac{55}{24}$ for the minimally coupled black hole, a departure from the KK black holes. Thus, as one would expect, the quantum corrections to the black hole entropy depend not only on the field content but also on the couplings to low energy matter.

Although the focus in this paper has been on the non-extreme case, and specifically whether the logarithmic corrections to the black hole entropy depend on the departure from extremality, it is worth highlighting the extremal limit since in this special case a detailed microscopic model is the most realistic. In the extremal case we find the quantum correction on the non-BPS branch

$$\delta S_{\text{ext}} = -\mathcal{N} \log \frac{A_H}{G} \ , \tag{3.186}$$

for $\mathcal{N} = 6, 8$. The surprising simplicity of this result is inspiring.

3.5 Discussion

In summary, we have shown that the spectrum of quadratic fluctuations around static Kaluza-Klein black holes in four dimensional supergravity partially diagonalizes into blocks of fields. Tables 3.7 and 3.8 give the c and a coefficients that control the Seeley-DeWitt coefficient $a_4(x)$ for each block and, taking into account appropriate degeneracies, for each supergravity theory. These coefficients directly yield the logarithmic correction to the black hole entropy via (3.178-3.179).

The detailed computations are quite delicate since any improper sign or normalization can dramatically change our conclusions. We therefore proceeded with extreme care, devoting several sections to explain the embedding of the Kaluza-Klein black hole into a range of supergravities and carefully record the action for quadratic fluctuations of the fields around the background. Moreover, we allowed for considerable redundancy, with indirect symmetry arguments supporting explicit computations and also performing many computations both analytically and using Mathematica. These steps increase our confidence in the results we report.

The prospect that interesting patterns in these corrections could lead to novel insights into black hole microstates is our main motivation for computing these quantum corrections in supergravity theories. Our discovery that $c = 0$ for $\mathcal{N} = 6, 8$ on the non-BPS branch is therefore gratifying. Recall that when c vanishes, the quantum correction is universal, it depends on the matter content of the theory but not on the parameters of the black hole. This property therefore holds out promise for a detailed microscopic description of these corrections. Such progress would be welcome since our current understanding of, for example, the $D0 - D6$ system leaves much to be desired [137–140] for the non-BPS branch.

Conversely, our analysis shows that on the non-BPS branch $c \neq 0$ for $\mathcal{N} \leq 4$. On the BPS-branch not only has it been found that $c = 0$ for all $\mathcal{N} \geq 2$ but this fact has also been shown to be a consequence of $\mathcal{N} = 2$ supersymmetry [124]. It would be interesting to similarly understand why $c = 0$ requires $\mathcal{N} \geq 6$ on the non-BPS branch.

To date, there is no known microstate counting formula that, when compared to the black hole entropy, accounts for terms that involve $c \neq 0$. For example, in *all* cases considered in [115, 116, 141], the object of interest is an index, or a closely related avatar, and the resulting logarithmic terms nicely accommodate quantum corrections when C_{local} is controlled by a alone. The challenge of reproducing the logarithmic correction when c is non-vanishing comes from the intricate dependence on the black hole parameters that the Weyl tensor gives to C_{local}. It would be interesting to understand which properties a partition function must possess in order that the logarithmic correction to the thermodynamic limit leads to $c \neq 0$.

An interesting concrete generalization of the present work would be to increase the scope of theories considered. In section 3.3.4 our main obstacle to covering all $\mathcal{N} = 2$ theories is the complicated structure of fermion couplings for a generic prepotential, and hence we restrict the discussion in section 3.3.4.d to the $ST(n)$ models. Nevertheless, we suspect that for a generic prepotential our conclusions would not be significantly different. In particular, we predict that $c \neq 0$ on the non-BPS branch for any $\mathcal{N} = 2$ supergravity. It would of course be desirable to confirm this explicitly.

A more ambitious generalization would be to consider more general black hole solutions, specifically those where the dilaton $\Phi^{(\mathrm{KK})}$ is not constant. Our assumption that $\Phi^{(\mathrm{KK})} = 0$ simplified our computations greatly by sorting quadratic fluctuations into blocks that are decoupled from one another. By addressing the technical complications due to relaxation of this assumption and so computing $a_4(x)$ for black holes with non-trivial dilaton we could, in particular, access solutions with non-zero angular momentum $J \neq 0$. The rotating black holes on the non-BPS branch are novel since they never have constant dilaton, even in the extremal limit [142]. Therefore, they offer an interesting contrast to the Kerr-Newman black hole, their counterparts on the BPS branch [26]. Rotation is quite sensitive to microscopic details so any differences or similarities between the quantum corrections to rotating black holes on the BPS and non-BPS branches may well provide valuable clues towards a comprehensive microscopic model. A non-constant dilaton is also the linchpin to connections with the new developments in AdS_2 holography for rotating black holes such as in [48, 143].

Appendix

In this appendix, we give the details on the computation of the Seeley-DeWitt coefficients for Kaluza Klein black holes and their embeddings in $\mathcal{N} \geq 2$ supergravity. Most of the computations were done using the Mathematica package `xAct`. We present our results according to the organization of quadratic fluctuations into blocks that was introduced in section 3.3.2.

The basic steps of our implementation are:

1. We expand the Lagrangian to second order.[6] This was done in sections 3.3.2 and 3.3.4 for the supergravity theories of interest. The bosonic Lagrangian can also be expanded using `xPert`.

2. We gauge-fix and add the corresponding ghosts. The gauge-fixing and the ghosts were detailed for each block in sections 3.3.2 and 3.3.4. In this appendix, we highlight and record their contributions to the heat kernel.

[6] For fermions we always write the quadratic fluctuations with Majorana spinors, following the conventions of [26].

3. We rearrange the fluctuation operator Λ_m^n so that it takes the canonical form (3.172). We then read off the operators ω_μ and P and compute the operators E and $\Omega_{\mu\nu}$. These are the most cumbersome steps so they are executed primarily using Mathematica. Since some expressions are rather lengthy for the matrix operators due to the non-minimal couplings, we mostly present the traces of these operators.

4. We compute the Seeley-DeWitt coefficient $a_4(x)$ using formula (3.173). This also includes the ghosts from the second step.

5. We simplify $a_4(x)$ using the equations of motion, tensor and gamma matrix identities. This brings $a_4(x)$ to its minimal form (3.177), where we can read off the coefficients c and a.

3..1 Preliminaries

We use the following formula to compute the Seeley-DeWitt coefficient

$$(4\pi)^2 a_4(x) = \mathrm{Tr}\Big[\frac{1}{2}E^2 + \frac{1}{6}RE + \frac{1}{12}\Omega_{\mu\nu}\Omega^{\mu\nu} + \frac{1}{360}(5R^2 + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu})\Big] , \tag{3.187}$$

This object further simplifies due to the equations of motion, Bianchi, and Schouten identities. These simplifications imply that we can cast (3.187) in the form

$$a_4(x) = \frac{c}{16\pi^2}W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} - \frac{a}{16\pi^2}E_4 , \tag{3.188}$$

where the square of the Weyl tensor is

$$W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2 , \tag{3.189}$$

and the Euler density is

$$E_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 . \tag{3.190}$$

For each block, as summarized in Table 3.3, we will report both (3.187) and (3.188). The identities used to simplify (3.187) to its minimal form (3.188) are listed below. For fermionic fluctuations, we also use many gamma matrix identities which are well known and not repeated here.

On-shell conditions: The equations of motion background with constant dilaton are

$$F_{\mu\alpha}F_{\nu}{}^{\alpha} = 2R_{\mu\nu}\ , \qquad R = 0\ , \tag{3.191}$$
$$F_{\mu\nu}F^{\mu\nu} = 0\ , \qquad D_{\mu}F^{\mu\nu} = 0\ .$$

Bianchi identities: Starting from

$$\nabla_{\mu}\tilde{F}^{\mu\nu} = 0\ , \qquad R_{\mu[\nu\alpha\beta]} = 0\ , \tag{3.192}$$

where $\tilde{F}_{\mu\nu} = -\frac{i}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$ we find

$$\begin{aligned} R_{\mu\nu\alpha\beta}R^{\mu\alpha\nu\beta} &= \frac{1}{2}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\ , \\ (D_{\alpha}F_{\mu\nu})(D^{\nu}F^{\mu\alpha}) &= \frac{1}{2}(D_{\alpha}F_{\mu\nu})(D^{\alpha}F^{\mu\nu})\ , \\ F^{\alpha\nu}(D_{\alpha}F_{\mu\nu}) &= \frac{1}{2}F^{\nu\alpha}(D_{\mu}F_{\nu\alpha})\ , \\ R_{\mu\alpha\nu\beta}F^{\mu\nu}F^{\alpha\beta} &= \frac{1}{2}R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}\ , \\ \epsilon_{\mu\nu\alpha\beta}D^{\alpha}F^{\rho\beta} &= \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}D^{\rho}F^{\alpha\beta}\ . \end{aligned} \tag{3.193}$$

Schouten identities: The Schouten identity is $g^{\mu[\nu}\epsilon^{\rho\sigma\tau\lambda]} = 0$. From this, we can derive

$$\tilde{F}_{\mu\alpha}F_{\nu}{}^{\alpha} = \frac{1}{4}g_{\mu\nu}\tilde{F}_{\alpha\beta}F^{\alpha\beta} \tag{3.194}$$

Derivative relations: The following identity is also useful

$$(D_{\alpha}F_{\mu\nu})(D^{\alpha}F^{\mu\nu}) = -2R_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha} + R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} \tag{3.195}$$

and holds up to a total derivative.

3..2 KK Block

The quadratic Lagrangian is given in (3.104). To evaluate the Seeley-DeWitt coefficient, the kinetic term of $h_{\mu\nu}$ is analytically continued to

$$h^{\text{new}}_{\mu\nu} = -\frac{i}{2}h_{\mu\nu}\ , \tag{3.196}$$

for the kinetic term to have the right sign. In addition, in order to project onto the traceless part of a symmetric tensor, we define

$$G^{\mu\nu}_{\rho\sigma} = \frac{1}{2}\left(\delta^{\mu}{}_{\rho}\delta^{\nu}{}_{\sigma} + \delta^{\mu}{}_{\sigma}\delta^{\nu}{}_{\rho} - \frac{1}{2}g^{\mu\nu}g_{\rho\sigma}\right) . \tag{3.197}$$

Traces of operators must be taken after contraction with this tensor. For example, for a four index operator O we use

$$\mathrm{Tr}\, O = G^{\mu\nu}_{\rho\sigma} O^{\rho\sigma}_{\mu\nu} . \tag{3.198}$$

The relevant traces that appear in (3.187) for the KK block are

$$\begin{aligned}
\mathrm{Tr}\, E &= 3F_{\mu\nu}F^{\mu\nu} - 7R , \\
\mathrm{Tr}\, E^2 &= \frac{33}{16}F^{\mu}{}_{\rho}F^{\nu\rho}F_{\mu\sigma}F_{\nu}{}^{\sigma} + \frac{21}{16}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} - 5R_{\mu\nu}R^{\mu\nu} \\
&\quad -\frac{5}{2}R_{\mu\nu}F^{\mu}{}_{\rho}F^{\nu\rho} - \frac{1}{2}RF_{\mu\nu}F^{\mu\nu} + 5R^2 + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\
&\quad +2R_{\mu\rho\nu\sigma}R^{\mu\nu\rho\sigma} - 2F^{\mu\nu}{}_{;\mu}F_{\nu}{}^{\rho}{}_{;\rho} + \frac{1}{2}F_{\mu\rho;\nu}F^{\mu\nu;\rho} + \frac{1}{2}F_{\mu\nu;\rho}F^{\mu\nu;\rho} , \\
\mathrm{Tr}\, \Omega_{\mu\nu}\Omega^{\mu\nu} &= -\frac{7}{8}F^{\mu}{}_{\rho}F^{\nu\rho}F_{\mu\sigma}F_{\nu}{}^{\sigma} - \frac{23}{8}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + 2R_{\mu\nu}F^{\mu}{}_{\rho}F^{\nu\rho} \\
&\quad +RF_{\mu\nu}F^{\mu\nu} + 3R_{\mu\rho\nu\sigma}F^{\mu\nu}F^{\rho\sigma} - 7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - F^{\mu\nu}{}_{\mu}F_{\nu}{}^{\rho}{}_{;\rho} \\
&\quad +4F_{\mu\rho;\nu}F^{\mu\nu;\rho} - 8F_{\mu\nu;\rho}F^{\mu\nu;\rho} .
\end{aligned} \tag{3.199}$$

The gauge-fixing also introduces ghosts with the Lagrangian

$$e^{-1}\mathcal{L}_{\mathrm{ghosts}} = 2b_{\mu}\left(\Box g^{\mu\nu} + R^{\mu\nu}\right)c_{\nu} + 2b\Box c - 4bF^{\mu\nu}D_{\mu}c_{\nu} , \tag{3.200}$$

where b_{μ}, c_{μ} are vector ghosts associated to the graviton and b, c are scalar ghosts associated to the graviphoton. The contribution of the ghosts are

$$\begin{aligned}
\mathrm{Tr}\, E &= 2R , \\
\mathrm{Tr}\, E^2 &= 2R_{\mu\nu}R^{\mu\nu} , \\
\mathrm{Tr}\, \Omega_{\mu\nu}\Omega^{\mu\nu} &= -2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} .
\end{aligned} \tag{3.201}$$

The total ghost contribution is

$$(4\pi)^2 a_4^{\mathrm{ghost}}(x) = \frac{1}{9}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{17}{18}R_{\mu\nu}R^{\mu\nu} - \frac{17}{36}R^2 . \tag{3.202}$$

Combining the contributions (3.199) and (3.202) gives

$$\begin{aligned}(4\pi)^2 a_4(x) &= \frac{23}{24}F^{\mu}_{\ \rho}F^{\nu\rho}F_{\mu\sigma}F_{\nu}^{\ \sigma} + \frac{5}{12}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} - \frac{127}{36}R_{\mu\nu}R^{\mu\nu} \\ &\quad -\frac{13}{12}R_{\mu\nu}F^{\mu}_{\ \rho}F^{\nu\rho} + \frac{1}{3}RF_{\mu\nu}F^{\mu\nu} + \frac{77}{72}R^2 + \frac{1}{4}R_{\mu\rho\nu\sigma}F^{\mu\nu}F^{\rho\sigma} \\ &\quad +\frac{11}{18}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + R_{\mu\rho\nu\sigma}R^{\mu\nu\rho\sigma} - \frac{13}{12}F^{\mu\nu}{}_{;\mu}F_{\nu}{}^{\rho}{}_{;\rho} + \frac{7}{12}F_{\mu\rho;\nu}F^{\mu\nu;\rho} \\ &\quad -\frac{5}{12}F_{\mu\nu;\rho}F^{\mu\nu;\rho} \ .\end{aligned}$$

We use the identities listed in (3.191-3.194) to obtain

$$(4\pi)^2 a_4(x) = \frac{10}{9}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{49}{36}R_{\mu\nu}R^{\mu\nu} \ , \tag{3.203}$$

and from here we find

$$a_{\rm KK} = \frac{31}{72} \ , \qquad c_{\rm KK} = \frac{37}{24} \ . \tag{3.204}$$

3..3 Vector Block

The vector block in its minimal form is described by the quadratic Lagrangian (3.155) and for the matter content of $\mathcal{N}=8$ by (3.106). The matrices that appear in the quadratic fluctuation operator are

$$\begin{aligned} E &= \begin{pmatrix} \frac{1}{4}F_{\mu}{}^{\rho}F_{\nu\rho} - R_{\mu\nu} & \frac{1}{2}F_{\nu}{}^{\rho}{}_{;\rho} \\ \frac{1}{2}F_{\mu}{}^{\rho}{}_{;\rho} & -\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma} \end{pmatrix} , \\ \Omega_{\rho\sigma} &= \begin{pmatrix} R_{\mu\nu\rho\sigma} + \frac{1}{4}F_{\mu\sigma}F_{\nu\rho} - \frac{1}{4}F_{\mu\rho}F_{\nu\sigma} & \frac{1}{2}F_{\mu\sigma;\rho} - \frac{1}{2}F_{\mu\rho;\sigma} \\ -\frac{1}{2}F_{\nu\sigma;\rho} + \frac{1}{2}F_{\nu\rho;\sigma} & 0 \end{pmatrix} ,\end{aligned} \tag{3.205}$$

where the first row/column corresponds to the vector field and the second row/column to the scalar field. The relevant traces are

$$\begin{aligned} \mathrm{Tr}\,E &= -R \ , \\ \mathrm{Tr}\,E^2 &= \frac{1}{16}F^{\mu}_{\ \rho}F^{\nu\rho}F_{\mu\sigma}F_{\nu}^{\ \sigma} + \frac{1}{16}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + R_{\mu\nu}R^{\mu\nu} \\ &\quad -\frac{1}{2}R_{\mu\nu}F^{\mu}_{\ \rho}F^{\nu\rho} - \frac{1}{2}F^{\mu\nu}{}_{;\mu}F_{\nu}{}^{\rho}{}_{;\rho} \ , \\ \mathrm{Tr}\,\Omega_{\mu\nu}\Omega^{\mu\nu} &= \frac{1}{8}F^{\mu}_{\ \rho}F^{\nu\rho}F_{\mu\sigma}F_{\nu}^{\ \sigma} - \frac{1}{8}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + R_{\mu\rho\nu\sigma}F^{\mu\nu}F^{\rho\sigma} \\ &\quad -R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + F_{\mu\rho;\nu}F^{\mu\nu;\rho} - F_{\mu\nu;\rho}F^{\mu\nu;\rho} \ . \end{aligned} \tag{3.206}$$

The ghosts for the vector block are two minimally coupled scalars with fermionic statistics. Their contribution to the Seeley-DeWitt coefficient is

$$(4\pi)^2 a_4^{\text{ghost}}(x) = -\frac{1}{180}(2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + 5R^2) \ . \tag{3.207}$$

We combine the contributions of the vector block and its associated ghosts and get

$$\begin{aligned}(4\pi)^2 a_4(x) \quad = \quad & \frac{1}{24}F^{\mu}{}_{\rho}F^{\nu\rho}F_{\mu\sigma}F_{\nu}{}^{\sigma} + \frac{1}{48}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + \frac{29}{60}R_{\mu\nu}R^{\mu\nu} \\ & -\frac{1}{4}R_{\mu\nu}F^{\mu}{}_{\rho}F^{\nu\rho} - \frac{1}{8}R^2 + \frac{1}{12}R_{\mu\rho\nu\sigma}F^{\mu\nu}F^{\rho\sigma} - \frac{1}{15}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\ & -\frac{1}{4}F^{\mu\nu}{}_{;\mu}F_{\nu}{}^{\rho}{}_{;\rho} + \frac{1}{12}F_{\mu\rho;\nu}F^{\mu\nu;\rho} - \frac{1}{12}F_{\mu\nu;\rho}F^{\mu\nu;\rho}\end{aligned} \tag{3.208}$$

After using the identities (3.191-3.194), we obtain

$$(4\pi)^2 a_4(x) = -\frac{1}{15}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \frac{19}{60}R_{\mu\nu}R^{\mu\nu} \ . \tag{3.209}$$

This leads to

$$a_{\text{vector}} = \frac{11}{120} \ , \qquad c_{\text{vector}} = \frac{1}{40} \ . \tag{3.210}$$

When the vector block contains a pseudoscalar instead of a scalar, such as in (3.107), the result remains the same because of simplifications due to our background.

3..4 Gravitino Block

The gravitino block is characterized by the quadratic Lagrangian (3.112). After using gamma matrix identities, the relevant traces are

$$\begin{aligned}\mathrm{Tr}\, E \quad = \quad & \frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} - 10R \ , \\ \mathrm{Tr}\, E^2 \quad = \quad & -\frac{105}{128}F^{\mu}{}_{\rho}F^{\nu\rho}F_{\mu\sigma}F_{\nu}{}^{\sigma} + \frac{81}{128}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + \frac{43}{64}F^{\mu\nu}F^{\rho\sigma}\tilde{F}_{\mu\rho}\tilde{F}_{\nu\sigma} \\ & -\frac{13}{32}F^{\mu}{}_{\rho}F^{\nu\rho}\tilde{F}_{\mu}{}^{\sigma}\tilde{F}_{\nu\sigma} + \frac{7}{128}\tilde{F}^{\mu}{}_{\rho}\tilde{F}^{\nu\rho}\tilde{F}_{\mu}{}^{\sigma}\tilde{F}_{\nu\sigma} - \frac{21}{64}F_{\mu\nu}F^{\mu\nu}\tilde{F}_{\rho\sigma}\tilde{F}^{\rho\sigma} \\ & +\frac{9}{128}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}\tilde{F}_{\rho\sigma}\tilde{F}^{\rho\sigma} - \frac{1}{4}RF_{\mu\nu}F^{\mu\nu} - \frac{1}{4}R\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{5}{2}R^2 \\ & -\frac{3}{2}R_{\mu\rho\nu\sigma}F^{\mu\nu}F^{\rho\sigma} + \frac{3}{2}R_{\mu\rho\nu\sigma}\tilde{F}^{\mu\nu}\tilde{F}^{\rho\sigma} + 4R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\ & -\frac{7}{2}F_{\mu\rho;\nu}F^{\mu\nu;\rho} + 3F_{\mu\nu;\rho}F^{\mu\nu;\rho} + \frac{3}{2}\tilde{F}_{\mu\rho;\nu}\tilde{F}^{\mu\nu;\rho} - 2\tilde{F}_{\mu\nu;\rho}\tilde{F}^{\mu\nu;\rho} \ ,\end{aligned}$$

$$\begin{aligned}\mathrm{Tr}\,\Omega_{\mu\nu}\Omega^{\mu\nu} &= \frac{185}{64}F^{\mu}{}_{\rho}F^{\nu\rho}F_{\mu\sigma}F_{\nu}{}^{\sigma} - \frac{185}{64}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} - \frac{27}{32}F^{\mu\nu}F^{\rho\sigma}\tilde{F}_{\mu\rho}\tilde{F}_{\nu\sigma}\\ &-\frac{3}{16}F^{\mu}{}_{\rho}F^{\nu\rho}\tilde{F}_{\mu}{}^{\sigma}\tilde{F}_{\nu\sigma} + \frac{9}{64}\tilde{F}^{\mu}{}_{\rho}\tilde{F}^{\nu\rho}\tilde{F}_{\mu}{}^{\sigma}\tilde{F}_{\nu\sigma} + \frac{33}{32}F_{\mu\nu}F^{\mu\nu}\tilde{F}_{\rho\sigma}\tilde{F}^{\rho\sigma}\\ &-\frac{9}{64}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}\tilde{F}_{\rho\sigma}\tilde{F}^{\rho\sigma} + 7R_{\mu\rho\nu\sigma}F^{\mu\nu}F^{\rho\sigma} - 3R_{\mu\rho\nu\sigma}\tilde{F}^{\mu\nu}\tilde{F}^{\rho\sigma}\\ &-13R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 7F_{\mu\rho;\nu}F^{\mu\nu;\rho} - 7F_{\mu\nu;\rho}F^{\mu\nu;\rho} - 3\tilde{F}_{\mu\rho;\nu}\tilde{F}^{\mu\nu;\rho}\\ &+3\tilde{F}_{\mu\nu;\rho}\tilde{F}^{\mu\nu;\rho}\ . \end{aligned} \tag{3.211}$$

The gauge-fixing produces fermionic ghosts b_A, c_A, e_A with Lagrangian

$$e^{-1}\mathcal{L}_{\text{ghost}} = \bar{b}_A\gamma^{\mu}D_{\mu}c_A + \bar{e}_A\gamma^{\mu}D_{\mu}e_A\ , \tag{3.212}$$

where $A = 1, 2$ is the flavor index. This simply corresponds to six minimally coupled Majorana fermions which contribute with an opposite sign. Their Seeley-DeWitt contribution is

$$(4\pi)^2 a_4^{\text{ghost}}(x) = -\frac{1}{120}\left(7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 8R_{\mu\nu}R^{\mu\nu} - 5R^2\right)\ . \tag{3.213}$$

Combining (3.211) and (3.213) gives

$$\begin{aligned}(4\pi)^2 a_4(x) &= \frac{65}{768}F^{\mu}{}_{\rho}F^{\nu\rho}F_{\mu\sigma}F_{\nu}{}^{\sigma} - \frac{29}{768}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} - \frac{17}{128}F^{\mu\nu}F^{\rho\sigma}\tilde{F}_{\mu\rho}\tilde{F}_{\nu\sigma}\\ &+\frac{7}{64}F^{\mu}{}_{\rho}F^{\nu\rho}\tilde{F}_{\mu}{}^{\sigma}\tilde{F}_{\nu\sigma} - \frac{5}{256}\tilde{F}^{\mu}{}_{\rho}\tilde{F}^{\nu\rho}\tilde{F}_{\mu}{}^{\sigma}\tilde{F}_{\nu\sigma} + \frac{5}{128}F_{\mu\nu}F^{\mu\nu}\tilde{F}_{\rho\sigma}\tilde{F}^{\rho\sigma}\\ &-\frac{3}{256}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}\tilde{F}_{\rho\sigma}\tilde{F}^{\rho\sigma} + \frac{2}{45}R_{\mu\nu}R^{\mu\nu} + \frac{1}{48}RF_{\mu\nu}F^{\mu\nu} + \frac{1}{48}R\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}\\ &-\frac{1}{36}R^2 + \frac{1}{12}R_{\mu\rho\nu\sigma}F^{\mu\nu}F^{\rho\sigma} - \frac{1}{4}R_{\mu\rho\nu\sigma}\tilde{F}^{\mu\nu}\tilde{F}^{\rho\sigma} - \frac{113}{180}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\\ &+\frac{7}{12}F_{\mu\rho;\nu}F^{\mu\nu;\rho} - \frac{11}{24}F_{\mu\nu;\rho}F^{\mu\nu;\rho} - \frac{1}{4}\tilde{F}_{\mu\rho;\nu}\tilde{F}^{\mu\nu;\rho} + \frac{3}{8}\tilde{F}_{\mu\nu;\rho}\tilde{F}^{\mu\nu;\rho}\ . \end{aligned}$$

Using the identities (3.191-3.194) gives

$$(4\pi)^2 a_4(x) = -\frac{113}{180}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \frac{767}{720}R_{\mu\nu}R^{\mu\nu}\ , \tag{3.214}$$

and this leads to

$$a_{\text{gravitino}} = -\frac{137}{1440}, \qquad c_{\text{gravitino}} = -\frac{347}{480}\ . \tag{3.215}$$

3..5 Gaugino Block

The gaugino block is given by the Lagrangian (3.113). In this case, the relevant traces are

$$\begin{aligned}
\mathrm{Tr}\,E &= \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - 2R\ , \\
\mathrm{Tr}\,E^2 &= -\frac{1}{32}F^{\mu}{}_{\rho}F^{\nu\rho}F_{\mu\sigma}F_{\nu}{}^{\sigma} + \frac{3}{128}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} - \frac{1}{8}RF^{\mu\nu}F_{\mu\nu} + \frac{1}{2}R^2 \\
&\quad -\frac{1}{2}F_{\mu\rho;\nu}F^{\mu\nu;\rho} + \frac{1}{4}F_{\mu\nu;\rho}F^{\mu\nu;\rho}\ , \\
\mathrm{Tr}\,\Omega_{\mu\nu}\Omega^{\mu\nu} &= \frac{1}{8}F^{\mu}{}_{\rho}F^{\nu\rho}F_{\mu\sigma}F_{\nu}{}^{\sigma} - \frac{1}{8}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + R_{\mu\rho\nu\sigma}F^{\mu\nu}F^{\rho\sigma} \\
&\quad -R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + F_{\mu\rho;\nu}F^{\mu\nu;\rho} - F_{\mu\nu;\rho}F^{\mu\nu;\rho}\ .
\end{aligned} \tag{3.216}$$

The Seeley-DeWitt coefficient is

$$\begin{aligned}
(4\pi)^2 a_4(x) &= \frac{1}{384}F^{\mu}{}_{\rho}F^{\nu\rho}F_{\mu\sigma}F_{\nu}{}^{\sigma} - \frac{1}{1536}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + \frac{1}{45}R_{\mu\nu}R^{\mu\nu} \\
&\quad +\frac{1}{96}RF^{\mu\nu}F_{\mu\nu} - \frac{1}{72}R^2 - \frac{1}{24}R_{\mu\rho\nu\sigma}F^{\mu\nu}F^{\rho\sigma} + \frac{7}{360}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\
&\quad +\frac{1}{12}F_{\mu\rho;\nu}F^{\mu\nu;\rho} - \frac{1}{48}F_{\mu\nu;\rho}F^{\mu\nu;\rho}\ .
\end{aligned} \tag{3.217}$$

and gives after simplification

$$(4\pi)^2 a_4(x) = \frac{7}{360}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{73}{1440}R_{\mu\nu}R^{\mu\nu}\ , \tag{3.218}$$

which leads to

$$a_{\text{gaugino}} = -\frac{17}{2880}\ , \qquad c_{\text{gaugino}} = \frac{13}{960}\ . \tag{3.219}$$

4 Quantum teleportation via traversable wormholes

How large can quantum effects be in gravity?

The broad question we want to ask is: how large can quantum effects be in gravity? In this chapter, which is based on [3], we consider this question for traversable wormholes in the context of AdS/CFT. Recently, it was shown by Gao, Jafferis and Wall that a small quantum effect can be used to make a wormhole traversable for a very short time. Can we make this quantum effect larger? We will attempt to use the same procedure to construct wormholes that are traversable for all time, *i.e.* that are *eternal*. We will learn that, assuming Poincaré symmetry in the transverse directions, it does not seem possible to do so. This leads to interesting lessons about what is possible or not in quantum gravity.

4.1 *Introduction: traversable wormholes*

4.1.1 Wormholes in general relativity

The idea of wormholes dates back to the paper by Einstein and Rosen [144], who observed that general relativity has solutions corresponding to bridges connecting two different universes (or two distant regions of the same universe). One of these solutions is the maximally extended Schwarzschild solution. These wormholes are not traversable, because a light ray sent from one side does not make it to the other side, but falls into the black hole singularity. This is related to the fact that nothing can escape from a black hole in classical general relativity.

Morris and Thorne described the geometry of a traversable wormhole in [145]. They showed that it could be traversed by human beings. They observed that it had to be supported by exotic matter which violates all the energy conditions that ordinary matter is expected to satisfy. At that time, they didn't think of it as physical (they actually presented it as a tool to teach general relativity). They acknowledged however that quantum field theory provides tantalizing hints that these energy conditions could in fact be violated in nature. More than thirty years later, this expectation was fulfilled: Maldacena, Milekhin and Popov have shown that a macroscopic and long-lived traversable wormhole is a solution of the accepted low energy theory of nature (semi-classical Einstein gravity coupled to the Standard Model) [146].

We now review why traversable wormholes cannot exist in classical gravity. Physical matter is required to satisfy suitable energy conditions. This is necessary to have a well-defined notion of causality. For example, we don't want to allow closed timelike curves. This is to avoid causality violation such as the grandfather paradox: one should not be able to travel to the past and kill his own progenitor. Traversable wormholes are problematic because one can use them to build time machines allowing this [75].

Suitable energy conditions are also a fundamental ingredient in the singularity theorems of Penrose and Hawking. The usual classical requirement is the *null energy condition* (NEC) which says that the matter stress tensor $T_{\mu\nu}$ should satisfy

$$T_{\mu\nu}k^{\mu}k^{\nu} \geq 0 \; , \tag{4.1}$$

for any future-directed null vector k^{μ}. The inclusion of quantum effects, in a semiclassical context, showed that this energy condition could be violated. For example, the Hawking evaporation process gives a local violation of the NEC. Simpler examples are the Casimir energy between two plates [147] or moving mirrors [148, 149]. It is now believed that the condition on quantum matter should be the *achronal averaged null energy condition* (ANEC), which says that

$$\int_{-\infty}^{+\infty} ds\, T_{\mu\nu}k^{\mu}k^{\nu} \geq 0 \; , \tag{4.2}$$

over a complete achronal null geodesic, where achronal means that no two points should be connected by a timelike curve. This statement turns out to be sufficient to preserve causality and the singularity theorems [76–79].

Let's comment on the requirement that the geodesic is achronal. Dropping this requirement, there are simple counterexamples: a null geodesic on the cylinder doesn't satisfy the ANEC because the Casimir energy can be negative everywhere. At the same time, this geodesic necessarily wraps around the cylinder and is thus not achronal: there are points on the geodesic that are timelike separated. Importantly, the achronal requirement implies that although traversable wormholes are possible, they can never be used as shortcuts: the outside path must always be faster. As put in [80], *"traversable wormholes are like getting a bank loan: you can only get one if you are rich enough not to need it"*.

The achronal ANEC seems to be the weakest form of energy condition that ensures a well-defined notion of causality. It is believed to be true in semiclassical general relativity, although no proof is known. It has been proven for linearized perturbations of Minkowski spacetime[1] using ideas from quantum information theory [150] and from the conformal bootstrap [151].

4.1.2 The Gao-Jafferis-Wall protocol

In 2017, Gao, Jafferis and Wall proposed a protocol to create a traversable wormhole in AdS/CFT [80]. Let's consider an eternal AdS black hole, such as BTZ in three dimensions. It is dual to a state living in the tensor product of two CFTs: the thermofield double state [62]

$$|\mathrm{TFD}\rangle = \sum_E e^{-\beta E/2} |E\rangle \otimes |E\rangle \in \mathcal{H}_L \otimes \mathcal{H}_R \tag{4.3}$$

The large amount of entanglement is responsible for the connectivity in spacetime. The entanglement entropy obtained after tracing over one of the side corresponds to the Bekenstein-Hawking entropy on the other side. This geometry has an Einstein-Rosen wormhole which is not traversable because any signal crossing the horizon hits the singularity. In some sense, the wormhole is barely traversable, because a signal sent early enough almost makes it to the side.

The idea of Gao, Jafferis and Wall is to introduce a small coupling between the two CFTs for a short time. This takes the form of a double-trace coupling with interaction Lagrangian

$$\mathcal{L}_{\text{int}} = g \int_{t_i}^{t_f} dt d\vec{x}\, \mathcal{O}_L(t,\vec{x}) \mathcal{O}_R(t,\vec{x}) \ . \tag{4.4}$$

The effect of this coupling can be computed in perturbation theory. It generates a stress tensor in the bulk, which, for the correct sign of g, gives a contribution

[1] Around Minkowski spacetime, the achronal ANEC is equivalent to the ANEC since any null geodesic is achronal.

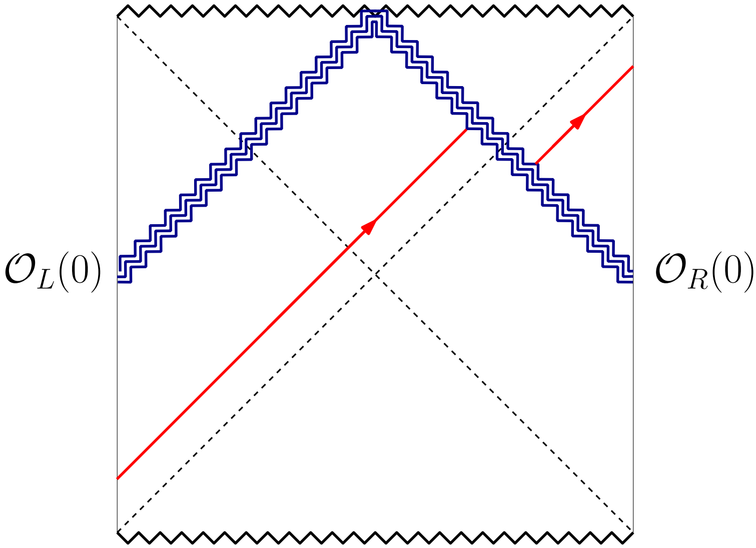

Figure 4.1: The Gao-Jafferis-Wall protocol: a non-local coupling between the two side of an eternal AdS black hole creates a negative energy shockwave (in blue), which allows a signal (in red) to travel from one side to the other.

that violates the ANEC for the geodesic shown in red in Figure 4.1. This allows the geodesic to defocus and escape from the singularity: the wormhole is now traversable! An equivalent picture is that the coupling creates a negative energy shockwave in the bulk, whose effect is to give a time advance to the signal [152,153], allowing it to escape to the other side. This is illustrated in Figure 4.1.

Even though this protocol leads to a non-local coupling in the bulk, which is why such a spectacular effect is possible, this coupling is perfectly consistent and natural in AdS/CFT. This is because given two holographic CFTs in the thermofield states, one can always couple the two CFTs in the laboratory, *e.g.* by connecting them through a wire. In the lab, the two systems can be close to each other so that the coupling appears instantaneous from the point of view of the bulk. Actually, concrete proposals to realize this protocol using atom arrays and trapped ions have recently appeared [154].

We must stress that it is not surprising that a signal can be sent from one side to the other, since we have explicitly introduced a coupling between the two sides. What is surprising is the way the signal gets through, namely in free fall through a wormhole. A bulk traveler going through the protocol would feel nothing in particular and will remain in good health. This cannot be said for a traveler which explicitly uses a coupling of the form (4.4) to go from one side to another.

He will be annihilated on one side and recreated on the other side, which doesn't appear to be a very safe travel route.

JT gravity provides a nice framework to study traversable wormholes, where everything is solvable analytically. The dynamics of JT gravity can be formulated as that of a "boundary particle". In the thermofield double, we have two boundary particles corresponding to the two sides of the AdS_2-Rindler geometry. The Gao-Jafferis-Wall coupling introduces an attractive force between the two boundary particles which changes the positions of the horizons and makes the wormhole traversable. We refer to [155] for a detailed discussion.

Quantum teleportation is a process by which the full state of a quantum system can be transmitted using shared entanglement and classical communication [156]. This protocol makes it possible to send the full quantum state of a system, even though naively any measurement would destroy it. The idea is to make a joint measurement also involving a shared entangled system. Using a classical channel to communicate the result of this measurement allows the other party to reconstruct the original quantum state. The Gao-Jafferis-Wall protocol implements quantum teleportation in AdS/CFT. The coupling (4.4) plays the role of the classical communication and the Einstein-Rosen wormhole is the required shared entanglement. As discussed in section 1.3.3, this protocol strengthens the ER=EPR proposal.

4.1.3 Some traversable wormhole solutions

We now describe two long-lived traversable solutions that have appeared in the literature following [80]. These solutions are relevant for the question we are asking because they are example where the quantum effect is relatively large.

In the context of JT gravity, it was shown in [157] that one can obtain an eternal traversable wormhole in two dimensions, by introducing a static coupling between the two sides. This is actually the 2d version of the construction that we will study. As we will see, a higher-dimensional version of this wormhole doesn't actually exist because the quantum effect that supports it cannot be made strong enough in higher dimensions. This is related to the fact that 2d gravity is rather peculiar and can behave differently compared to more realistic gravity theories.

In [146], a traversable wormhole solution was constructed in the accepted theory of nature, namely semiclassical gravity coupled to the Standard Model. This relied on a subtle balance between the large magnetic field of near-extremal Reissner-Nordström black holes and the small negative Casimir energy of fermions looping through the wormhole. We must note that although this wormhole was shown to be a solution, no natural process to produce it (apart from quantum tunneling), was proposed. Hence, it is unclear whether this solution has actually been realized in our universe. As discussed in [158], it is possible that primordial black hole carried a large magnetic charge (coming from magnetic monopoles). The evaporation process is greatly enhanced for these black holes and they would have quickly decayed to extremality. Then, a binary system of these black holes could eventually have tunneled into the traversable wormhole solution.

4.1.4 *Introduction to our work*

The basic question behind our work is how much can we push the Gao-Jafferis-Wall protocol. More precisely, we will ask whether this protocol can be used to construct eternal traversable wormholes connecting two asymptotically AdS regions. We attempt to do this by introducing a static coupling between their dual CFTs. Assuming Poincaré invariance in the boundary directions, we prove that there are no semiclassical traversable wormholes in higher than two spacetime dimensions. We critically examine the possibility of evading our result by coupling a large number of bulk fields. Static, traversable wormholes with less symmetry may be possible, and could be constructed using the ingredients we develop here.

To understand better what types of traversable wormholes are possible, we attempt to construct static traversable wormholes in the controlled setting of asymptotically Anti-de Sitter spacetime, within a regime where the semiclassical approximation is valid. In the context of the AdS/CFT correspondence, a traversable wormhole should be dual to two conformal field theories (CFTs) which live on the two asymptotic boundaries of the spacetime. Traversable wormholes can be constructed by introducing an appropriate coupling between these two theories, CFT_L and CFT_R.

Bulk solutions that correspond to traversable wormholes require a violation of the NEC. This is possible in standard quantum field theory, e.g. through the Casimir effect, but a coupling between the two CFTs is unavoidable in our setting. In the decoupled system, no operator in CFT_L can influence CFT_R, which implies that no signal can be transmitted through the bulk. The existence of a traversable wormhole solution in the decoupled system would violate this "no-transmission principle" [159] which follows from basic postulates of the holographic dictionary.

Hence, we violate the NEC by introducing a double trace deformation that couples the two CFTs [160] and has the form $h\,\mathcal{O}_L(x)\mathcal{O}_R(x)$. Our major assumption which makes the analysis tractable is that the solution preserves Poincaré invariance in the field theory directions. Assuming this symmetry, we can pick a gauge where the metric takes the form

$$ds^2 = a(z)^2\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right)\ . \tag{4.5}$$

We look for solutions with two asymptotic regions, so that the 'scale factor' $a(z)$ diverges at two locations while remaining nonzero in between. Furthermore, we assume that the bulk field providing the NEC violation preserves Weyl invariance. This allows us to compute explicitly the NEC violating stress-energy tensor by using a conformal map to flat spacetime.

This is a crucial simplification: in general we need to solve the Einstein equation using the expectation value of the stress-energy tensor as a source. The stress-energy tensor should be computed by doing quantum field theory in the background metric defined by $a(z)$. However, the expectation value of the stress-energy tensor $\langle T_{\mu\nu}(z)\rangle$ depends non-locally on the metric function $a(z)$, rendering the problem apparently intractable. Weyl invariance allows us to package the non-local dependence of the stress-energy tensor on the metric in terms of a single parameter encoding the 'width' of the geometry.

Within these assumptions, we demonstrate a no-go result: the effect of the double trace deformation is too small to support a semiclassical wormhole. In order to establish this result, we consider various strategies for enhancing the NEC violation and show that they cannot work.

First, we argue that increasing the coupling does not help because the "quantum inequalities" [161] bound the amount of NEC violation. It is an interesting open problem to demonstrate a more general and more rigorous bound on NEC violation for a quantum field theory in a geometry with two asymptotically AdS regions when couplings between the boundaries are allowed.

Second, we try to add conventional matter in the bulk. We present the detailed analysis of an additional bulk scalar field with a quartic potential, as well as establishing a general result showing that adding any additional matter satisfying the NEC does not allow for a semiclassical wormhole with Poincaré invariance in the field theory directions. Our result is rigorously true when the NEC violating fields are Weyl invariant, allowing for an explicit calculation, but we suspect that adjusting the field content will not change the result.

Finally, we try to increase the number of species contributing to the NEC violation. Although this allows one to make the curvature small in Planck units, this strategy is problematic because a large number of species is believed to lower the UV cutoff as

$$M_{\mathrm{UV}}^{D-2} \leq \frac{1}{N} M_p^{D-2}\,, \tag{4.6}$$

where N is the number of species and D is the number of spacetime dimensions in the bulk [162–165]. We show that although a large number of species can reduce the curvature of the wormhole, the radius of curvature is always at or below the UV cutoff. We discuss the possibility of violating the lore (4.6) by choosing appropriately the field content, making use of cancellations in the one-loop renormalization of Newton's constant. However, we argue that (4.6) can never be softened because it would imply the existence of traversable wormholes between two asymptotically AdS boundaries without a coupling between the two CFTs, in contradiction with the "no-transmission principle". This also agrees with non-perturbative arguments regarding the renormalization of Newton's constant.

The failure to construct a controlled solution within our assumptions can be partially understood heuristically as follows. In the absence of the coupling between the two boundaries, the ground state is simply two unentangled CFTs in their ground state, and the corresponding geometry is simply two copies of vacuum AdS. Turning on the coupling $h\mathcal{O}_L(x)\mathcal{O}_R(x)$ will lead to an amount of entanglement of order the coupling h- in other words, the entanglement is of order one if the coupling is perturbative. On the other hand, a controlled traversable wormhole should have a smooth geometry, leading to an entanglement of order N^2. This heuristic argument, however, leaves open the possibility that the construction can succeed by going to strong coupling or increasing the number of fields that are coupled. Our more detailed arguments rule out these possibilities.

We have shown that there is no semiclassical solution with Poincaré invariance along the boundary directions and a Weyl invariant field in the non-local coupling. This suggests avenues for future constructions based on a less symmetric ansatz. One could try to import the recent construction of long-lived traversable wormholes in flat space [146] to the AdS setting. This construction makes use of magnetic fields, which break the transverse Poincaré invariance. More generally, we expect that a static traversable wormhole should look like an AdS-Schwarzschild black hole or black brane near the two asymptotic boundaries. These metrics do not preserve Poincaré invariance, which further motivates reducing the amount of symmetry. We could also consider NEC violating matter that is not conformally invariant but we do not expect our results to change dramatically.

We must note that many constructions of traversable wormholes in general relativity exist in the literature but they involve either exotic matter [75,145,166–170] or higher-derivative theories [171–173] which seem to lack a UV completion [174]. On the other hand, introducing a coupling between two CFTs should be perfectly physical in the context of AdS/CFT.

4.2 Poincaré wormholes in AdS

We look for traversable wormholes connecting two asymptotically AdS_{d+1} spacetimes. In order to make the problem analytically tractable, we assume Poincaré invariance in the boundary directions. Using this symmetry, we can pick a gauge such that the metric takes the form

$$ds^2 = a^2(z)\left(-dt^2 + d\vec{x}^2 + dz^2\right) , \tag{4.7}$$

where $\vec{x} = (x_1, \ldots, x_{d-1})$ are boundary coordinates and z is the radial coordinate in the bulk.[2] This metric is foliated by flat $\mathbb{R}^{1,d-1}$ slices and is similar to the Poincaré patch of AdS. The geometry is completely determined by one function, the conformal factor $a(z)$. For solutions with two asymptotically AdS boundaries, this means that $a(z)$ should have two simple poles, say at $z = \pm\frac{L}{2}$, and be positive in the range $-\frac{L}{2} < z < \frac{L}{2}$.

4.2.1 Setup

We consider a theory of gravity with negative cosmological constant coupled to matter,

$$S = \frac{1}{16\pi G}\int d^{d+1}x\sqrt{-g}\,\left(R - 2\Lambda\right) + S_{\text{matter}} \, , \tag{4.8}$$

where $\Lambda = -\frac{d(d-1)}{2\ell_{\text{AdS}}^2}$. In order to find traversable wormholes in such a theory, we need to violate the null energy condition (NEC) in the bulk. This is possible in the framework of semiclassical gravity, where the matter fields are treated quantum mechanically, but the geometry is kept classical. Following [80], we do this by introducing a non-local coupling between the two boundaries

$$\delta S = h\int d^dx\, \phi_L(x)\phi_R(x)\, . \tag{4.9}$$

Here $\phi_{L,R}(x)$ corresponds to a bulk field and the subindex L/R means that it is evaluated at the left/right boundary. In AdS/CFT, such a deformation is achieved by coupling together the two CFTs with a double trace operator.

In [80], the deformation (4.9) was activated for a short time on an eternal AdS black hole, i.e. a non-traversable wormhole. The resulting quantum stress-energy tensor made the wormhole traversable but only in a very small time window. Our work differs from [80] in two aspects. First, we start from the vacuum state, which consists of two unentangled copies of the same CFT. Second, we are interested in finding *eternal* traversable wormholes so we turn on the deformation for all times.

[2]For later convenience we also define coordinates $x^\mu = (t, \vec{x})$ and $y^m = (x^\mu, z)$.

Finally, a word on the methodology is in order. Since our ansatz (4.7) is conformally flat, we compute the quantum stress-energy tensor in flat spacetime and map the result to our wormhole background by means of a Weyl transformation. For this to be possible, we require S_{matter} to be Weyl invariant. A simple choice is a conformally coupled scalar field. The boundary conditions for the scalar field are chosen as follows. Near the two asymptotic boundaries the behavior is

$$\phi(z) \sim \alpha_\pm \left(\tfrac{L}{2} \pm z\right)^{\Delta_+} + \beta_\pm \left(\tfrac{L}{2} \pm z\right)^{\Delta_-} \,, \tag{4.10}$$

where

$$\Delta_\pm = \frac{d \pm 1}{2} \,. \tag{4.11}$$

which follow by performing a Weyl transformation to the flat space solution $\phi(z) \sim \alpha_\pm \left(\tfrac{L}{2} \pm z\right) + \beta_\pm$ near the boundaries.[3] We choose the boundary condition

$$\alpha_\pm = 0 \,, \tag{4.12}$$

which, in the alternate quantization, implies that the dimension of the dual operator is given by Δ_-.[4] Upon a Weyl transformation, this condition corresponds to imposing Neumann boundary conditions on the plates in Minkowski space.

4.2.2 Minkowski configuration

We compute the stress-energy tensor by a Weyl transformation from Minkowski spacetime. In the following, we will specialize to $3+1$ dimensions. The generalization to other dimensions is described in Appendix 4.5.

The configuration in flat space consists of two infinite plates located at $z = -L/2$ and $z = L/2$, respectively, and a massless scalar field living in the region between the plates. The non-local coupling then takes the form

$$\delta S = h \int d^3x\, \phi\left(x, -\tfrac{L}{2}\right) \phi\left(x, \tfrac{L}{2}\right) \,, \tag{4.13}$$

where $x = (t, x_1, x_2)$ denote the transverse coordinates. We will denote $y = (x, z)$ the coordinate of a point between the plates. The stress-energy tensor generated by the non-local coupling can be computed by point splitting

$$\langle T_{\mu\nu}(y)\rangle = \lim_{y' \to y} \left(\frac{\partial}{\partial y^\mu} \frac{\partial}{\partial y'^\nu} \delta G(y, y') - \frac{1}{2} \eta_{\mu\nu} \eta^{\rho\sigma} \frac{\partial}{\partial y^\rho} \frac{\partial}{\partial y'^\sigma} \delta G(y, y') \right) \,, \tag{4.14}$$

[3] This can also be derived from the formula $\Delta(d-\Delta) = -m^2\ell^2_{AdS}$ since the conformal coupling $\xi R \phi^2$ with $\xi = \frac{d-1}{4d}$ gives an effective mass $m^2\ell^2_{\text{AdS}} = -\frac{d^2-1}{4}$ close to the boundaries.

[4] In the standard quantization the dimension of the double trace term would be $2\Delta_+ = d+1 > d$ which would make the coupling irrelevant.

where $\delta G(y, y)$ is the correction to the Feynman propagator due to the non-local coupling. As explained above, we are imposing Neumann boundary conditions on the plates. Note that imposing instead Dirichlet boundary conditions would make the deformation (4.13) vanish. The Feynman propagator with Neumann boundary conditions has a simple form in a mixed representation where we go to momentum space in the transverse directions

$$G(x, z; x', z') = \int \frac{d^3k}{(2\pi)^3} e^{ik(x-x')} G_{\text{mixed}}(z, z'; k)\,, \tag{4.15}$$

where $k = (\omega, k_1, k_2)$ is the momentum associated to the transverse directions (t, x_1, x_2). The propagator with Neumann boundary conditions takes the form

$$G_{\text{mixed}}(z, z'; k) = \frac{1}{\kappa \sin(\kappa L)} \cos\big(\kappa \left(z_- + \tfrac{L}{2}\right)\big) \cos\big(\kappa \left(z_+ - \tfrac{L}{2}\right)\big)\,, \tag{4.16}$$

with $z_- = \min(z, z')$, $z_+ = \max(z, z')$ and $\kappa = \sqrt{\omega^2 - k_1^2 - k_2^2}$. We will perform the computation in Euclidean signature where the propagator takes the form

$$G_{\text{mixed}}(z, z'; k) = \frac{1}{|k| \sinh(|k|L)} \cosh\big(|k| \left(z_- + \tfrac{L}{2}\right)\big) \cosh\big(|k| \left(z_+ - \tfrac{L}{2}\right)\big)\,, \tag{4.17}$$

where $|k| = \sqrt{\omega^2 + k_1^2 + k_2^2}$.

The correction to the two-point function due to the non-local coupling (4.13) is given by

$$\delta G(y, y') = h \int d^3\tilde{x}\, G\left(\tilde{x}, -\tfrac{L}{2}; y\right) G\left(y'; \tilde{x}, \tfrac{L}{2}\right) + (y \leftrightarrow y')\,. \tag{4.18}$$

Using the mixed representation, we can rewrite this as

$$\delta G(y, y') = h \int \frac{d^3k}{(2\pi)^3} \frac{1}{|k|^2 \sinh^2(|k|L)} \cosh\big(|k| \left(z + \tfrac{L}{2}\right)\big) \cosh\big(|k| \left(z' - \tfrac{L}{2}\right)\big)\, e^{ik(x-x')} + (y \leftrightarrow y')\,. \tag{4.19}$$

From the above expression it can be seen that

$$\lim_{y' \to y} \eta^{\rho\sigma} \partial_\rho \partial'_\sigma \delta G(y, y') = 0\,. \tag{4.20}$$

Weyl invariance and transverse Lorentz symmetry imply that the stress-energy tensor has the form

$$\langle T^{\text{flat}}_{\mu\nu} \rangle = -\rho \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -3 \end{pmatrix} . \tag{4.21}$$

The parameter ρ can in principle depend on z, but the conservation of the stress-energy tensor requires ρ to be a constant. In order to determine this constant it suffices to compute only one of the components. For instance, we can compute

$$\langle T^{\text{flat}}_{zz} \rangle = \lim_{y' \to y} \partial_z \partial'_z \delta G(y, y') . \tag{4.22}$$

The calculation further simplifies by going to the midpoint $z = 0$, which gives

$$\langle T^{\text{flat}}_{zz} \rangle = -h \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\cosh^2\left(\frac{|k|L}{2}\right)} = -\frac{h}{6L^3} . \tag{4.23}$$

Hence, the full stress-energy tensor is given by

$$\langle T^{\text{flat}}_{\mu\nu} \rangle = \frac{h}{18L^3} \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -3 \end{pmatrix} . \tag{4.24}$$

4.2.3 Wormhole solution

The conformal mapping allows us to compute the expectation value of the stress-energy tensor in the conformally flat geometry (4.7). This is given by

$$\langle T^{\text{NL}}_{\mu\nu} \rangle = \frac{1}{a(z)^2} \langle T^{\text{flat}}_{\mu\nu} \rangle . \tag{4.25}$$

The superscript 'NL' is a reminder that this component of the stress-energy tensor is generated by the non-local coupling, but generically there can be other contributions.

An important issue here is that the above Weyl transformation is anomalous in even dimensions. The anomaly generates a higher-derivative term in $\langle T^{\text{NL}}_{\mu\nu} \rangle$ which prevents us from solving Einstein's equation. For the time being we will assume that the anomaly term is negligible, but we will come back to this issue in section 4.3.2. We can also go to odd spacetime dimensions where there is no anomaly.

We define λ to be the dimensionless parameter measuring the amount of negative energy generated by the non-local coupling. It is defined by

$$8\pi\langle T_{00}^{\text{flat}}\rangle = -\frac{\lambda}{L^4}\,, \tag{4.26}$$

so that $\lambda \sim hL$ up to a numerical factor of order one. The stress-energy tensor in the geometry (4.7) is then given by

$$8\pi\langle T_{\mu\nu}^{\text{NL}}\rangle = \frac{\lambda}{a^2L^4}\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -3 \end{pmatrix}. \tag{4.27}$$

We will solve the semiclassical Einstein equations

$$G_{\mu\nu} - \frac{3}{\ell_{\text{AdS}}^2}g_{\mu\nu} = 8\pi G\langle T_{\mu\nu}^{\text{NL}}\rangle\,. \tag{4.28}$$

The zz component can be written as

$$\frac{1}{2}a'(z)^2 + V(a) = 0\,, \tag{4.29}$$

which can be thought as a particle in the potential

$$V(a) = \frac{G\lambda}{2L^4} - \frac{a^4}{2\ell_{\text{AdS}}^2}\,. \tag{4.30}$$

The other components can be obtained from the z-derivative of (4.29).

There is a diffeomorphism

$$y^\mu \to \zeta\, y^\mu\,, \qquad L \to \zeta\, L\,, \qquad h \to \zeta^{-1}h\,, \qquad a(z) \to \zeta^{-1}a(z)\,, \tag{4.31}$$

which allows us to set $a(0) = 1$. Furthermore, we focus on solutions with reflection symmetry so we have $a'(0) = 0$ and can restrict the domain of integration to $z \geq 0$. Evaluating (4.29) at $z = 0$ gives us the relation

$$\frac{1}{\ell_{\text{AdS}}^2} = \frac{G\lambda}{L^4}\,, \tag{4.32}$$

This allows us to eliminate λ in (4.29) and to determine the value for the range L

$$\frac{L}{2} = \int_0^{L/2} dz = \int_1^\infty \frac{da}{a'} = \ell_{\text{AdS}}\int_1^\infty \frac{da}{\sqrt{a^4-1}}\,. \tag{4.33}$$

Performing the integral gives

$$L = c\,\ell_{\text{AdS}}, \qquad c = \frac{2\sqrt{\pi}\,\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})} \sim 2.62\,. \tag{4.34}$$

This defines a one-parameter family of wormhole solutions. The potential (4.30) and a typical wormhole solution are shown in Figure 4.2.

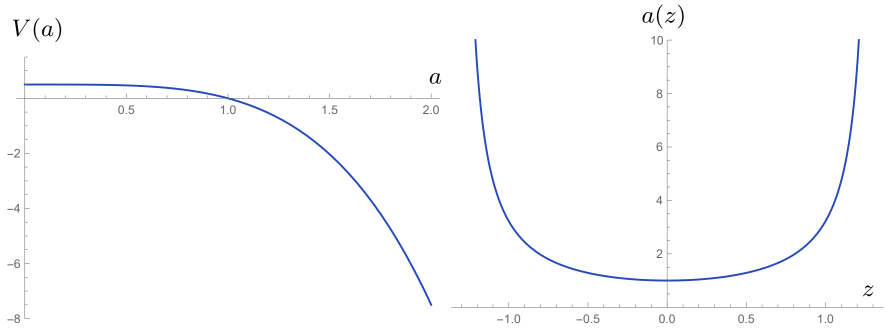

Figure 4.2: Typical shape of the potential $V(a)$ and the conformal factor for a wormhole solution $a(z)$. For the plots we have set $\ell_{\text{AdS}} = 1$ and we have set $a(0) = 1$.

4.3 Some challenges

4.3.1 Planckian curvature

The most important issue of our solutions is that they generically have large curvatures. Combining the equations (4.32) and (4.34) implies that

$$\left(\frac{\ell_{\text{AdS}}}{\ell_p}\right)^2 \sim \lambda\,, \tag{4.35}$$

where ℓ_p is the Planck length and $\sim$ means proportionality up to an order one numerical factor. Since the computation of the stress-energy tensor is valid only in the perturbative regime, $\lambda \ll 1$, this leads to a wormhole with super-Planckian curvature. Hence, the wormhole solutions described in the previous section are outside the regime where semiclassical gravity can be trusted. In the following sections we will explore various potential ideas to try to resolve this issue, but we first discuss two more challenges.

4.3.2 Weyl anomaly

We have used a Weyl transformation from flat space to compute the stress-energy tensor in the geometry (4.7). For this to be possible, we used a Weyl invariant field in the non-local coupling. However, Weyl invariance can be anomalous at the quantum level. The anomaly is problematic in our setup because, as we will see below, it can be of the same order as the effect that we use to support the wormholes.

In four dimensions, there is always an anomaly. The stress-energy tensor on the conformally flat metric (4.7) is related to the stress-energy tensor in flat space by [175]

$$\langle T_{\mu\nu}\rangle = \frac{1}{a^2}\langle T^{\text{flat}}_{\mu\nu}\rangle + \frac{1}{16\pi^2}\left(2\alpha F_{\mu\nu} - \frac{\beta}{9}H_{\mu\nu}\right) . \tag{4.36}$$

The anomalous piece is the second term and it is expressed in terms of four-derivative terms

$$F_{\mu\nu} = R_\mu^{\ \rho}R_{\rho\nu} - \frac{2}{3}RR_{\mu\nu} - \frac{1}{2}R_{\rho\sigma}R^{\rho\sigma}g_{\mu\nu} + \frac{1}{4}R^2 g_{\mu\nu} , \tag{4.37}$$

$$H_{\mu\nu} = 2R_{;\mu\nu} - 2g_{\mu\nu}\Box R - \frac{1}{2}g_{\mu\nu}R^2 + 2RR_{\mu\nu} . \tag{4.38}$$

The rational numbers α and β can be extracted from the tables of [175]. For massless free fields, they are given by

$$\alpha = \frac{1}{360}(n_S + 11n_F + 62n_V) , \tag{4.39}$$

$$\beta = \frac{1}{20}(n_S(1-5\xi) + n_F + 2n_V) , \tag{4.40}$$

with n_S conformally coupled scalars, n_F Dirac fermions and n_V vectors. The parameter ξ is the coupling to R which should be taken to be $\xi = \frac{1}{6}$ because we want a Weyl invariant theory. From the above expression, we can see that α is strictly positive. In fact, the trace of (4.36) shows that α is the a-anomaly of the 4d theory which does not vanish for a unitary theory [176–178]. This implies that there is no way to make the anomaly vanish in four dimensions.

The main problem with the anomalous piece is that it contains four-derivative terms which prevent us from solving Einstein's equation. For a given field, the size of the anomaly is of the order of

$$|T^{\text{anomaly}}_{\mu\nu}| \sim \frac{1}{\ell^4_{\text{AdS}}} . \tag{4.41}$$

In the configuration with a single non-locally coupled field, the anomaly can be ignored in the regime $L < \ell_{\rm AdS}$ because we have $\lambda \ll 1$. This condition is already necessary as will be derived later in (4.61). In the configuration where we have a large number of fields, discussed in section 4.4.3, the anomaly dominates over the cosmological constant term which prevents us from obtaining a semiclassical solution.

In odd dimensions, there is no Weyl anomaly: the Weyl transformation of the stress-energy tensor does not contain an anomalous piece. Thus, the stress-energy tensor in the wormhole spacetime (4.7) can be obtained from the stress-energy tensor in flat space by the classical formula

$$\langle T_{\mu\nu} \rangle = \frac{1}{a^2} \langle T_{\mu\nu}^{\rm flat} \rangle \,. \tag{4.42}$$

Hence, when using a large number of fields, we will focus only on odd dimensions.

4.3.3 Casimir energy

There is another problem for building eternal traversable wormholes between asymptotically AdS regions. Negative energy can already be present in the wormhole geometry without the need to turn on a non-local coupling. Indeed, in the flat space configuration described in section 4.2.2, the Casimir effect [179] generates a stress-energy tensor of the form [175]

$$T_{\mu\nu}^{\rm Casimir} \sim \frac{1}{L^4} \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -3 \end{pmatrix} . \tag{4.43}$$

This negative energy is more important than the one generated by the non-local coupling, which is multiplied by $\lambda \ll 1$. Hence, it seems that the non-local coupling is unnecessary! However, it should not be possible to build a semiclassical wormhole between two asymptotically AdS geometries without coupling the two dual CFTs. This would violate the "no-transmission principle" [159] because signals could be sent from one asymptotic boundary to the other without any coupling between the two CFTs. We note that the sign of the Casimir energy can be modified by changing the boundary conditions of the fields. However, having a positive Casimir energy is also a problem because it would overwhelm the non-local coupling.

The issue of Casimir energy should be present in any attempt to build eternal traversable wormholes. In the asymptotically AdS_2 version [157] (see also [180, 181]), this is avoided because the Weyl anomaly precisely cancels the Casimir energy. As explained there, this is enforced by $SL(2,\mathbb{R})$ invariance. In higher dimensions, the wormhole is less symmetric which makes such a cancellation unlikely. From the analysis of section 4.3.2, we see that this cancellation happens neither in four dimensions nor in odd dimensions where the Weyl anomaly vanishes. Thus, if our conformally flat wormholes are to be consistent, some mechanism has to ensure that the Casimir energy is negligible. For example, a supersymmetric spectrum with supersymmetric boundary conditions leads to a vanishing Casimir energy. This holds despite the fact that the wormhole geometry breaks supersymmetry[5] as can be seen by making a Weyl transformation from the flat space configuration.

4.4 Attempts and lessons

4.4.1 Increasing the non-local coupling

We might hope that a solution with ℓ_{AdS} large in Planck units can be obtained in the strong coupling regime $\lambda \gg 1$. In fact, we will show that increasing the coupling cannot lead to a very large negative energy. This can be done by adapting the "quantum inequalities" [161]. The authors proved that for any state $|\psi\rangle$ of a free massless scalar in Minkowski spacetime, there is a bound

$$\hat{\rho} \geq -\frac{c}{t_0^4}\,, \tag{4.44}$$

where $\hat{\rho}$ is the energy density averaged over a time interval of characteristic length t_0,

$$\hat{\rho} \equiv \int_{-\infty}^{+\infty} dt\, f(t) \langle\psi|T_{00}|\psi\rangle\,, \tag{4.45}$$

and f is a smearing function which determines the number c. This shows that the smeared energy cannot get "too negative". In their proof, the smearing function is a Lorentzian but the same argument can be repeated for a more general smearing function as long as its Fourier transform decays sufficiently fast. This is because the bound is proportional to the integral

$$\int_0^{\infty} d\omega\, \omega^3 \hat{f}(\omega)\,, \tag{4.46}$$

where $\hat{f}(\omega)$ is the Fourier transform of f. In particular, it is possible to obtain a bound when f being compactly supported. This follows from a Theorem by Ingham [182] which determines how fast the Fourier transform of a compactly supported function can decay. This theorem guarantees that there are compactly

[5] This can be checked explicitly by showing that the geometry (4.7) does not have a covariantly constant spinor, except when it is flat space or Poincaré-AdS.

supported functions whose Fourier transform decays exponentially, e.g. as $e^{-|\omega|^{1/2}}$, which is fast enough to make (4.46) converge.

In our Minkowski configuration, we can consider a causal diamond centered at $z = 0$ which is as large as possible without touching the plates. Because the diamond is not in contact with the plates, the quantum state inside this causal diamond is that of a free massless scalar field. Hence, we expect that the quantum inequalities should be applicable if the smearing function is supported in this diamond. This is possible by taking a compactly supported function on a time interval of length L. The resulting bound is

$$\langle T_{00}\rangle \gtrsim -\frac{1}{L^4}\,, \tag{4.47}$$

up to an order one numerical factor. Thus, the best we can achieve by increasing the coupling would lead to a wormhole with Planckian curvature

$$\left(\frac{\ell_{\rm AdS}}{\ell_p}\right)^2 \sim 1\,. \tag{4.48}$$

This shows that increasing the non-local coupling does not help in making the wormhole semiclassical.

4.4.2 Adding conventional matter

In the previous sections, we have shown that the negative energy generated by the non-local coupling is too small to support the wormhole. From (4.32), we can see that to have $\lambda \ll 1$ we need a hierarchy of scales $\ell_p \ll L \ll \ell_{\rm AdS}$. However, with only the non-local coupling and the cosmological constant we have $L \sim \ell_{\rm AdS}$ as shown in (4.34). We can attempt to solve this problem by adding a new *classical* source in the Einstein equation. This will introduce a new scale which in principle could be used to separate L from $\ell_{\rm AdS}$ or remove the necessity of this hierarchy altogether. In this section, we prove a no-go theorem showing that this is not possible: adding matter that satisfies the null energy condition cannot make the wormhole semiclassical.

4.4.2.a Scalar field

Before going to the general situation, we consider a bulk scalar field minimally coupled to gravity, described by the action

$$S_{\rm m} = -\int \sqrt{-g}\, d^4x \left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right). \tag{4.49}$$

We assume that the matter preserves Poincaré invariance in the transverse directions so that ϕ depends only on z. In our geometry (4.7), the matter stress-energy tensor is

$$\begin{aligned} T^{\mathrm{m}}_{zz} &= a^2\left(\frac{\phi'^2}{2a^2} - V\right) , \\ T^{\mathrm{m}}_{xx} &= T^{\mathrm{m}}_{yy} = -T^{\mathrm{m}}_{tt} = -a^2\left(\frac{\phi'^2}{2a^2} + V\right) . \end{aligned} \tag{4.50}$$

The scalar field does not violate the NEC at classical level because $T^{\mathrm{m}}_{zz} + T^{\mathrm{m}}_{tt} = \phi'^2 \geq 0$. Its equation of motion is

$$2a'\phi' + a\phi'' = a^3\,\partial_\phi V . \tag{4.51}$$

Einstein equation gives

$$\begin{aligned} a'^2 &= -\frac{G\lambda}{L^4} + \frac{8\pi G}{3}a^4\left(\frac{\phi'^2}{2a^2} - V\right) , \\ \frac{a''}{a^3} &= -\frac{8\pi G}{3}\left(\frac{\phi'^2}{2a^2} + 2V\right) . \end{aligned} \tag{4.52}$$

The term generated by the non-local coupling is the negative term proportional to λ in the first equation. We can explicitly see that this term is necessary by considering the expression

$$\left(\frac{a'}{a^2}\right)' = \frac{2G\lambda}{L^4a^3} - 4\pi G\frac{\phi'^2}{a} . \tag{4.53}$$

In a wormhole solution, a'/a^2 goes from 0 at the throat, to $1/\ell_{\mathrm{AdS}}$ at the boundary. Therefore, its derivative needs to be positive somewhere, implying that λ cannot be zero. More explicitly, integrating the equation between $z = 0$ and $z = L/2$ gives

$$\frac{1}{\ell_{\mathrm{AdS}}} = \int_0^{L/2} dz\left(\frac{a'}{a^2}\right)' = \int_0^{L/2} dz\left(\frac{2G\lambda}{a^3L^4} - 4\pi G\frac{\phi'^2}{a}\right) \leq \frac{G\lambda}{L^3} . \tag{4.54}$$

This leads to the following lower bound

$$\lambda \geq \left(\frac{\ell_{\mathrm{AdS}}}{\ell_p}\right)^2\left(\frac{L}{\ell_{\mathrm{AdS}}}\right)^3 . \tag{4.55}$$

We are in a regime where $\lambda \ll 1$ which implies that we must have $L \ll \ell_{\mathrm{AdS}}$. We see that the scalar field does not modify the required hierarchy we pointed out at the beginning of the section. This is a special case of a more general statement

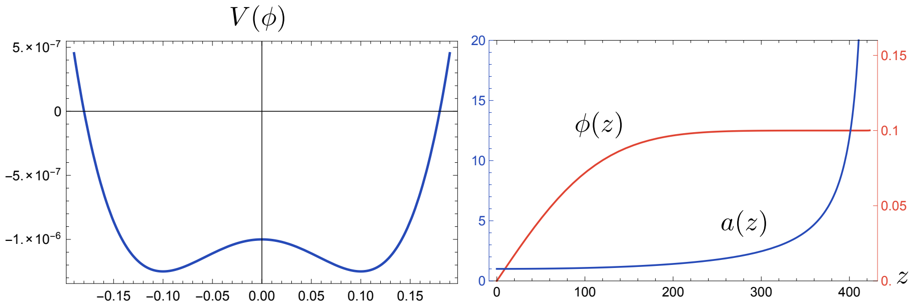

Figure 4.3: **Left:** Shape of the potential $V(\phi)$. **Right:** The corresponding solution. We are using Planck units for the two plots.

(4.61) which will be described later. Note that this requirement guarantees that the Weyl anomaly is negligible, as discussed in section 4.3.2.

We solve numerically the coupled ODEs for $a(z)$ and $\phi(z)$ using Mathematica. The solution is obtained by integrating the second order Einstein equation which leads to more stable numerics.[6] We consider a Higgs-like potential

$$V = -\frac{m_0^2}{2}\phi^2 + \frac{c^2}{4}\phi^4 + V_0 \,. \tag{4.56}$$

which is illustrated in Figure 4.3.

We look for solutions of ϕ that interpolate between the two minima of V at the two asymptotically AdS regions. These solutions are odd so we can restrict the range of integration to $0 \le z \le \frac{L}{2}$. We consider the following boundary conditions

$$\begin{aligned} a(0) &= 1\,, \quad a'(0) = 0\,; \\ \phi(0) &= 0\,, \quad \phi(\tfrac{L}{2}) = \phi_L\,, \end{aligned} \tag{4.57}$$

where ϕ_L is the value corresponding to the minimum of the potential.[7]

[6]The two Einstein equations in (4.52) are not independent. We can obtain the second order differential equation by taking the derivative of first one and using the ϕ equation of motion to remove the ϕ'' terms.

[7]The value of L is determined dynamically because it corresponds to location at which $a(z)$ diverges. For this reason, imposing the condition at the boundary is a bit tricky. In practice we impose a second boundary condition at the throat $\phi'(0) = \phi'_0$. The correct value of ϕ'_0 is determined through algorithmically to ensure that ϕ approaches the right value at $z = L/2$.

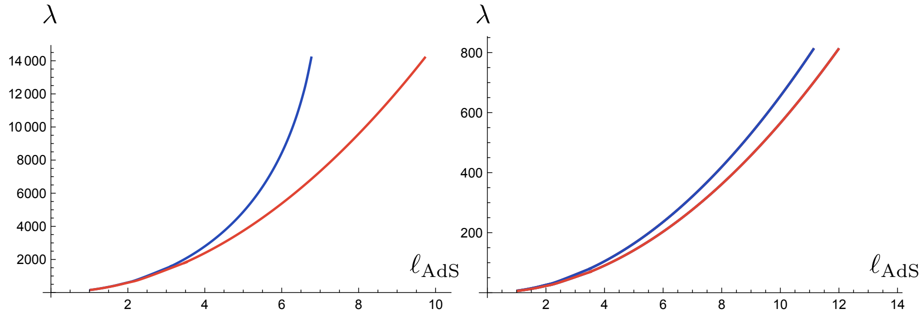

Figure 4.4: We plot λ as a function of ℓ_{AdS} while varying V_0 in the interval $-1 \leq V_0 \leq -10^{-4}$ and we use Planck units. The different curves correspond to different choices for the other parameters in the potential. **Left:** $m_0 = c = 10^{-1}$ (blue), $m_0 = c = 10^{-2}$ (red). **Right:** $c = 10^{-i}$ with $i = 2, 3$ and $m_0 = 10^{-1}c$ (blue) and $m_0 = 10^{-2}c$ (red). The series corresponding to different values of i are indistinguishable. It's impossible to have $\lambda \ll 1$ if we want ℓ_{AdS} to be large in Planck units.

From the numerical solutions, we can find the value of λ and ℓ_{AdS} using

$$\lambda = \frac{8\pi L^4}{3}\left(\frac{\phi'(0)^2}{2} - V(\phi(0))\right), \quad \ell_{\text{AdS}} = \sqrt{-\frac{3}{8\pi G V(\phi_L)}}\,. \tag{4.58}$$

An example of solution is given in Figure 4.3. In general we notice that λ can be made small only at the cost of making ℓ_{AdS} small in Planck units. In Figure 4.4, we plot λ as a function of ℓ_{AdS} for a large sample of parameters. We only keep solutions which leads to $\ell_{\text{AdS}} > 1$. In all cases, even for ℓ_{AdS} close to ℓ_p we do not find solutions consistent with $\lambda \ll 1$. This means that the addition of the bulk scalar field does not help in making the wormhole semiclassical.

4.4.2.b No-go theorem

We will now show that any kind of conventional matter in the bulk does not help in making the wormhole semiclassical, assuming that the matter respects Poincaré invariance in the transverse directions. We can model the addition of bulk matter by the addition of a term $f(a)$ in Einstein equation

$$a'^2 = \frac{a^4}{\ell_{\text{AdS}}^2} - \frac{G\lambda}{L^4} + \frac{f(a)}{\ell_{\text{m}}^2}\,, \tag{4.59}$$

where ℓ_m is a characteristic length scale of the additional matter. We show below that $f(1)$ needs to be positive so ℓ_{m} can be chosen so that $f(1) = 1$. We are also using conventions in which $a(0) = 1$. Evaluating the equation at $z = 0$ gives

$$\lambda = \frac{L^4}{G}\left(\frac{1}{\ell_{\rm AdS}^2} + \frac{1}{\ell_{\rm m}^2}\right) . \tag{4.60}$$

From the above formula, we see that $\lambda \ll 1$ implies that

$$\frac{L}{\ell_{\rm AdS}} \ll \frac{\ell_p}{L} \qquad \text{and} \qquad \frac{L}{\ell_m} \ll \frac{\ell_p}{L} . \tag{4.61}$$

The Einstein equation can be rewritten

$$a'^2 = \frac{f(a) - f(1)}{\ell_m^2} + \frac{a^4 - 1}{\ell_{\rm AdS}^2} . \tag{4.62}$$

We require the asymptotically AdS boundary condition

$$a(z)\big|_{z \to L/2} \sim \frac{\ell_{\rm AdS}}{\frac{L}{2} - z} . \tag{4.63}$$

In other words, the cosmological constant should dominate close to the boundary $z = \frac{L}{2}$. For the additional matter to be helpful, we would like to dominate close to the wormhole throat $z = 0$. We can define the transition point z_* and the corresponding conformal factor $a_* \equiv a(z_*)$ by

$$\frac{f(a_*) - f(1)}{\ell_m^2} = \frac{a_*^4 - 1}{\ell_{\rm AdS}^2} . \tag{4.64}$$

We assume that the additional matter dominates below z_* while the cosmological constant dominates above z_*. Hence, we have

$$\frac{a^4 - 1}{\ell_{\rm AdS}^2} \leq \frac{f(a) - f(1)}{\ell_{\rm m}^2} , \qquad 0 \leq z \leq z_* , \tag{4.65}$$

$$\frac{a^4 - 1}{\ell_{\rm AdS}^2} \geq \frac{f(a) - f(1)}{\ell_{\rm m}^2} , \qquad z_* \leq z \leq \frac{L}{2} . \tag{4.66}$$

We also assume that $a(z)$ is monotonically increasing close to $z = 0$.[8] In Figure 4.5 we show a schematic representation of the two regimes described above.

[8]Relaxing these two assumptions cannot help in making the wormhole more semiclassical. Indeed, the above discussion shows that in order to be useful, the conventional matter needs to make L as small as possible. It can be seen that relaxing these assumptions will only make things worse.

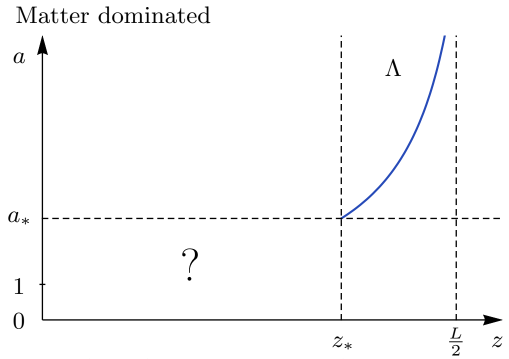

Figure 4.5: We assume that the new term in the Einstein equation dominates up to some $z = z_*$, for which we have $a(z_*) = a_*$. After this value the cosmological constant dominates.

Null energy condition. We impose the null energy condition on the additional matter. The zz-component can be directly read off from (4.59)

$$8\pi G\, T^{\mathrm{m}}_{zz} = \frac{3}{a^2}\frac{f(a)}{\ell_{\mathrm{m}}^2}\,. \tag{4.67}$$

We consider matter that respects the transverse Lorentz symmetry. This implies that the stress-energy tensor is diagonal and satisfies $T^{\mathrm{m}}_{xx} = T^{\mathrm{m}}_{yy} = -T^{\mathrm{m}}_{tt}$. Its conservation then implies that

$$\frac{d}{dz}T^{\mathrm{m}}_{zz} + \frac{a'}{a}T^{\mathrm{m}}_{zz} + \frac{3a'}{a}T^{\mathrm{m}}_{tt} = 0\,. \tag{4.68}$$

The resulting stress-energy tensor is

$$8\pi G\, T^{\mathrm{m}}_{\mu\nu} = \frac{1}{a^2\ell_{\mathrm{m}}^2}\begin{pmatrix} f(a) - af'(a) & 0 & 0 & 0 \\ 0 & af'(a) - f(a) & 0 & 0 \\ 0 & 0 & af'(a) - f(a) & 0 \\ 0 & 0 & 0 & 3f(a) \end{pmatrix}.$$

The null energy condition applied to the vector $\partial_t + \partial_z$ implies

$$f'(a) \leq \frac{4f(a)}{a}\,. \tag{4.69}$$

In particular, this implies that $f(1) > 0$. Indeed, dividing (4.65) by $a - 1$ and taking the limit $a \to 1$ implies that $f'(1) > 4\ell_{\mathrm{m}}^2/\ell_{\mathrm{AdS}}^2$.

Contradiction. First, integrating the NEC gives

$$f(a) \leq a^4 \,. \tag{4.70}$$

Intuitively, this means that the fastest function of a which satisfies the NEC is a cosmological constant. Since our original problem was that the cosmological constant does not grow fast enough, no other conventional matter should help us in making the wormhole more semiclassical.

More precisely, from (4.62) and (4.65), we see that

$$a'^2 \leq \frac{2(f(a)-1)}{\ell_{\mathrm{m}}^2} \,, \qquad 0 \leq z \leq z_* \,, \tag{4.71}$$

$$a'^2 \leq \frac{2(a^4-1)}{\ell_{\mathrm{AdS}}^2} \,, \qquad z_* \leq z \leq \frac{L}{2} \,. \tag{4.72}$$

Integrating the first equation implies that

$$z_* = \int_1^{a_*} \frac{da}{a'} \geq \frac{\ell_{\mathrm{m}}}{\sqrt{2}} \int_1^{a_*} \frac{da}{\sqrt{f(a)-1}} \geq \frac{\ell_{\mathrm{m}}}{\sqrt{2}} \int_1^{a_*} \frac{da}{\sqrt{a^4-1}} \,. \tag{4.73}$$

Next, we can obtain a bound on a_* by integrating the second equation,

$$\frac{L}{2} - z_* = \int_{a_*}^{\infty} \frac{da}{a'} \geq \frac{\ell_{\mathrm{AdS}}}{\sqrt{2}} \int_{a_*}^{\infty} \frac{da}{\sqrt{a^4-1}} \geq \frac{\ell_{\mathrm{AdS}}}{\sqrt{2}} \int_{a_*}^{\infty} \frac{da}{a^2} = \frac{\ell_{\mathrm{AdS}}}{\sqrt{2}} \frac{1}{a_*} \,. \tag{4.74}$$

This gives

$$a_* \geq \frac{\sqrt{2}\ell_{\mathrm{AdS}}}{L} \gg 1 \,, \tag{4.75}$$

where the second inequality follows from (4.61). Then, equation (4.73) gives $z_* > O(1)\ell_{\mathrm{m}}$ where $O(1)$ is an order one number (which is bigger than ~ 0.5 already for $a_* = 2$). Using again (4.61), this implies

$$z_* \gg L/2 \,, \tag{4.76}$$

which is a contradiction. This shows that adding conventional matter does not help in making the wormhole semiclassical.

4.4.3 Coupling a large number of fields

In the previous section we have shown quite generally that we cannot build a traversable wormhole with just a perturbative amount of negative energy, assuming Poincaré invariance in the boundary directions. Increasing the coupling or adding conventional matter do not help. Another strategy is to use a large number N of fields in the non-local coupling to enhance its effect. This approach was exploited in previous constructions of traversable wormholes [146,155,157]. For our wormholes, the estimate (4.35) gets replaced by

$$\left(\frac{\ell_{\text{AdS}}}{\ell_p}\right)^{D-2} \sim N\lambda\,, \tag{4.77}$$

where N is the number of fields and $D = d + 1$ is the number of spacetime dimensions in the bulk. Thus, it seems that by taking N large enough, we can obtain a large AdS radius while keeping $\lambda \ll 1$. However, a large number of fields implies a lowering of the UV cutoff of the theory, which can be interpreted as the renormalization of Newton's constant. On general grounds [162–165], we expect that

$$M_{\text{UV}}^{D-2} \leq \frac{1}{N} M_p^{D-2}\,. \tag{4.78}$$

This implies that the solution cannot be made semiclassical

$$\left(\frac{\ell_{\text{AdS}}}{\ell_{\text{UV}}}\right)^{D-2} \sim \lambda \ll 1\,. \tag{4.79}$$

4.4.3.a Renormalization of Newton's constant

The perturbative renormalization of Newton's constant can be computed from the one-loop effective action. We will use the heat kernel expansion which provides a canonical way to regulate the divergences [183]. In D dimensions, the effective Lagrangian at one-loop is

$$\mathcal{L}_{\text{eff}} \;=\; -\frac{1}{D}\frac{a_0(x)}{\ell_{\text{UV}}^D} - \frac{1}{D-2}\frac{a_2(x)}{\ell_{\text{UV}}^{D-2}} + \dots\,, \tag{4.80}$$

where ℓ_{UV} is a UV cutoff and $a_n(x)$ are the so-called Seeley-DeWitt coefficients which are obtained from the heat kernel expansion. The term $a_0(x)$ gives a renormalization of the cosmological constant and the term $a_2(x)$ gives the renormalization of Newton's constant.

Because of the Weyl anomaly discussed in section 4.3.2, we will focus on odd-dimensional theories. We want a Weyl invariant theory so we consider n_S massless scalars with the conformal coupling $\xi = \frac{D-2}{4(D-1)}$ and n_F Dirac fermions. The computation is detailed in the Appendix 4.A.2 and gives

$$(4\pi)^{D/2} a_2(x) = \left(\frac{4-D}{12(D-1)} n_S + \frac{1}{12} 2^{\lfloor D/2 \rfloor} n_F\right) R\,. \tag{4.81}$$

In special cases of interest, this is

$$(4\pi)^{D/2}a_2(x) = \begin{cases} \frac{1}{24}(n_S + 4n_F)R\,, & D = 3\,, \\ \frac{1}{48}(-n_S + 16n_F)R\,, & D = 5\,, \\ \frac{1}{24}(-n_S + 16n_F)R\,, & D = 7\,. \end{cases} \tag{4.82}$$

We see that in 5d and 7d, we can make $a_2(x)$ vanish by choosing appropriately the field content. Note that this is possible because of the negative contribution from the conformally coupled scalars. In such cases, the perturbative lowering of the UV cutoff will be given by two-loop diagrams. This leads to a softer lowering of the UV cutoff, (4.78) becomes

$$M_{\rm UV}^{D-2} \leq \frac{1}{\sqrt{N}} M_p^{D-2}\,. \tag{4.83}$$

Assuming that no other effects lower the UV cutoff, we obtain

$$\left(\frac{\ell_{\rm AdS}}{\ell_{\rm UV}}\right)^{D-2} \sim \sqrt{N}\lambda\,. \tag{4.84}$$

From this analysis, it seems that taking large enough N allows for semiclassical wormholes. However, we argue in the next section that non-perturbatively, the UV cutoff is always lowered according to (4.78), preventing the possibility of having a semiclassical wormhole in this way. For related discussions on the renormalization of Newton's constant and the meaning of the negative contribution, we refer to [184–189].

4.4.3.b Non-perturbative considerations

Non-perturbative arguments based on black hole physics suggest that the UV cutoff of a gravity theory with N species is always lowered according to (4.78). These arguments are based on the rate of black hole evaporation, or entropy bounds [162, 163, 165]. These arguments suggest that the one-loop cancellations in (4.82) are not sufficient to lower the cutoff at the non-perturbative level.

Another consideration is the "no-transmission principle" which implies that a traversable wormhole between asymptotically AdS regions supported only by Casimir energy is inconsistent. Indeed, we should not be able to send signals from one asymptotic boundary to another if the two dual CFTs are decoupled [159].

For example, let us consider a theory in 5d with a large number of scalar and spinor fields such that $n_S = 16n_F$. This is chosen so that $a_2(x) = 0$ so that according to the above computation, Newton's constant is not renormalized at one-loop.

Assuming that the UV cutoff is not lowered by other effects, (4.84) would imply that we can have a semiclassical wormhole by taking the number of fields large enough.

If the above scenario is really possible, we could also construct a traversable wormhole without the non-local coupling, using only Casimir energy. For this to be true, we need to make sure that the Casimir energy is non-zero and negative. Let us consider the above setup with $n_S = 16n_F$ but without a non-local coupling. We are free to change the boundary conditions of the fields because the computation of $a_2(x)$ is insensitive to them (up to boundary terms which are irrelevant here). In particular, we can choose the boundary conditions so that the n_F spinors and the $4n_F$ scalar fields are in a supersymmetric configuration in the flat space setup. As discussed in section 4.3.3, this implies that the Casimir energy of these fields will compensate. The remaining $12n_F$ scalar fields can be chosen to have Dirichlet boundary conditions in flat space. The Casimir energy of 5d massless scalars between two plates with Dirichlet boundary conditions is computed in [179] and is indeed negative. Following the discussion in 4.3.3, this would give a traversable wormhole supported only by Casimir energy. More generally, we expect that it should always be possible to make the Casimir energy negative by choosing adequate boundary conditions.

Thus, we obtain a configuration where a traversable wormhole connects two decoupled CFTs which is inconsistent because of the “no-transmission principle” of AdS/CFT. This strongly suggests, in agreement with the black hole arguments, that the non-perturbative UV cutoff is still lowered according to (4.78) despite the perturbative cancellations at one loop. The wormhole cannot be made semiclassical by using a large number of fields.

4.5 Discussion

In this paper we have investigated the possibility of constructing eternal traversable wormholes connecting two asymptotically AdS regions by coupling the two dual CFTs. We focused on gravity solutions preserving Poincaré invariance along the field theory directions and used a Weyl invariant field in the non-local coupling.

Under these assumptions, the stress-energy tensor can be computed analytically. Although it violates the null energy condition, it does not provide enough negative energy to support a semiclassical wormhole. As argued from the “quantum inequalities” [161], increasing the coupling does not help. We also proved a no-go theorem saying that adding conventional matter in the bulk cannot make the wormhole semiclassical. Another strategy is to use a large number of fields in the non-local coupling. This increases the negative energy but lowers the UV cutoff by a compensating amount, disallowing any semiclassical traversable wormholes. A one-loop computation suggests that this lowering of the UV cutoff, interpreted as a renormalization of Newton's constant, can be soften by adequately choosing the field content. However, non-perturbative arguments suggest that this cannot work [162,163,165]. In particular, this would lead to a traversable wormhole solely supported by Casimir energy, without a non-local coupling. This contradicts the "no-transmission principle" which follows from basic postulates of the AdS/CFT duality [159].

This argument suggests that any mechanism that enhances the effect of the non-local coupling should always make the Casimir energy negligible, as to prevent the possibility of a semiclassical wormhole without a non-local coupling. We expect this requirement to provide some guidance in the construction of eternal traversable wormholes in AdS/CFT.

There are many avenues for future research. We can consider changing the conformal dimensions of the field, going away from the conformally coupled case. The effect becomes more difficult to compute but the numerics in [80] suggests that the negative energy can be increased in this way. We could also investigate situations with less symmetry. This would provide more room to produce the large hierarchy between the small quantum effect induced by the non-local coupling and the large semiclassical geometry. Adding rotation has been shown to enhance the effect of the non-local coupling [190]. Also, it should be possible to import in AdS/CFT the recent construction of long-lived traversable wormhole in Minkowski spacetime [146].

Appendices

4.A General dimensions

4.A.1 Setup

The computation done in section 4.2.2 can be generalized to any dimension. In D spacetime dimensions, the zz component of the stress-energy tensor is

$$\begin{aligned}\langle T_{zz}^{\text{flat}}\rangle &= -h\int \frac{d^{D-1}k}{(2\pi)^{D-1}}\frac{1}{2\cosh^2\left(\frac{|k|L}{2}\right)}\,, \\ &= -\frac{h}{L^{D-1}}\frac{\text{vol}(S^{D-2})}{(2\pi)^{D-1}}\int_0^{+\infty}\frac{x^{D-2}dx}{2\cosh^2\left(\frac{x}{2}\right)}\,.\end{aligned} \tag{4.85}$$

The full stress-energy tensor has the general form

$$\langle T^{\text{flat}}_{\mu\nu} \rangle \sim \frac{\lambda}{L^D} \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1-D \end{pmatrix}, \tag{4.86}$$

where $\lambda \sim hL$ is the perturbative parameter and $\sim$ means up to an order one numerical factor.

After the conformal transformation to the metric (4.7), the Einstein equation has the same form (4.29) as in 4d but with the potential

$$V(a) = \frac{G\lambda}{2L^D} - \frac{a^4}{2\ell_{\text{AdS}}^2}. \tag{4.87}$$

We can redo the computation done in section 4.2.3 and we obtain

$$\left(\frac{\ell_{\text{AdS}}}{\ell_p}\right)^{D-2} \sim \lambda. \tag{4.88}$$

We are in the perturbative regime $\lambda \ll 1$ so the wormhole cannot be semiclassical.

4.A.2 No-go theorem

The no-go theorem presented in the main text can be generalized to any dimension. In D dimensions, the Einstein equation takes the form

$$a'^2 = -\frac{G\lambda}{L^D} + \frac{a^4}{\ell_{\text{AdS}}^2}. \tag{4.89}$$

We consider a modified Einstein equation

$$\begin{aligned} a'^2 &= -\frac{G\lambda}{L^D} + \frac{f(a)}{\ell_{\text{m}}^2} + \frac{a^4}{\ell_{\text{AdS}}^2}, \\ &= \frac{a^4-1}{\ell_{\text{AdS}}^2} + \frac{f(a)-1}{\ell_{\text{m}}^2}, \end{aligned} \tag{4.90}$$

where $f(a)$ is subject to the same assumptions as in the main text. We will show that $f(1)$ is positive which allows us to fix ℓ_{m} by requiring that $f(1) = 1$. We also use conventions where $a(0) = 1$. Evaluating Einstein equation at $z = 0$ leads to

$$\lambda \sim \frac{L^d}{G}\left(\frac{1}{\ell_{\rm AdS}^2} + \frac{1}{\ell_{\rm m}^2}\right) . \tag{4.91}$$

The null energy condition is obtained as in the four dimensional case. We first compute the stress-energy tensor corresponding to the new term we added in the Einstein equation. The zz component can be read off the Einstein equation

$$T^{\rm m}_{zz} = \frac{(D-1)(D-2)}{16\pi G}\frac{f(a)}{a^2\ell_{\rm m}^2} . \tag{4.92}$$

We can determine $T^{\rm m}_{tt}$ by solving at the conservation equation $\nabla_\mu T^{\mu\nu} = 0$. This gives

$$\partial_z T^{\rm m}_{zz} + (D-3)\,\frac{a'}{a}T^{\rm m}_{zz} + (D-1)\frac{a'}{a}T^{\rm m}_{tt} = 0 . \tag{4.93}$$

From this equation, we can determine

$$T^{\rm m}_{tt} = \frac{(D-2)}{16\pi G}\frac{(5-D)f(a) - af'(a)}{a^2\ell_{\rm m}^2} . \tag{4.94}$$

Evaluating the NEC, we obtain as in four dimensions

$$T^{\rm m}_{zz} + T^{\rm m}_{tt} \geq 0 \implies 4f\left(a\right) - f'\left(a\right)a \geq 0 . \tag{4.95}$$

This bound implies that $f(1)$ is positive and leads to

$$f(a) \leq a^4 . \tag{4.96}$$

The remainder of the proof is unchanged.

4.B Heat kernel expansion

We consider fluctuations of quantum fields around a given classical background. The effective action can be written

$$S_{\rm eff} = S_0 + S_{\text{1-loop}} , \tag{4.97}$$

where S_0 is the action of the classical background. The effective action is computed by a Euclidean path integral

$$e^{-S_{\rm eff}} = e^{-S_0}\int D\phi\, e^{-\phi\Lambda\phi} , \tag{4.98}$$

where Λ is the operator of quadratic fluctuations. The heat kernel expansion [183] provides a way to regularize and compute the effective action. The term that renormalizes G_N is

$$(4\pi)^2 a_2(x) = \frac{1}{6}\mathrm{Tr}(6E + R)\,, \tag{4.99}$$

where the trace is over the components of the fields and E is defined by

$$-\Lambda = g^{\mu\nu} D_\mu D_\nu + E\,, \tag{4.100}$$

and $D_\mu = \nabla_\mu + \omega_\mu$ is a suitable covariant derivative. We compute below $a_2(x)$ for fields of interest in d spacetime dimensions. These results can also be found in [191] except that the conformal coupling is not considered there.

Scalar. We consider a massless scalar field. The Lagrangian is $\mathcal{L} = (\partial\varphi)^2 + \xi R\phi^2$. This gives $E = -\xi R$. Hence,

$$(4\pi)^{d/2} a_2^{\mathrm{scalar}}(x) \quad = \quad \left(\frac{1}{6} - \xi\right) R\,. \tag{4.101}$$

Fermion. We consider a Dirac spinor. The Lagrangian is $\mathcal{L} = \bar{\psi}\gamma^\mu D_\mu \psi$. The fermionic fluctuation operator is thus $\gamma^\mu D_\mu$. This is a first order operator so we apply the heat kernel to its square. The identity $(\gamma^\mu D_\mu)^2 = g^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{1}{4}R$ implies that $E = -\frac{1}{4}R$. This gives

$$(4\pi)^{d/2} a_2^{\mathrm{spinor}}(x) \quad = \quad \frac{1}{12} 2^{\lfloor d/2 \rfloor}\,. \tag{4.102}$$

Maxwell vector. The Lagrangian is $\mathcal{L} = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = -D^\mu a^\nu D_\mu a_\nu + D^\nu a^\mu D_\mu a_\nu$. We integrate the two terms by parts and swap the two derivatives in the second term to obtain

$$\mathcal{L} = a^\nu \Box a_\nu - a^\nu R_{\mu\nu} a^\mu - (D^\mu a_\mu)^2\,. \tag{4.103}$$

As usual, the last term is removed by adding a gauge-fixing term $\mathcal{L}_{\mathrm{g.f.}} = (D^\mu a_\mu)^2$. This introduces two scalar ghosts which are minimally coupled. The contribution of these ghosts is two times the one written in (4.101) with $\xi = 0$ and an overall minus sign due to the opposite statistics. For the gauge field, we obtain $E = -R_{\mu\nu}$. This gives

$$(4\pi)^{d/2} a_2^{\mathrm{vector}}(x) \quad = \quad \frac{d-8}{6} R\,. \tag{4.104}$$

5 Future directions

This final chapter is a brief outlook on the work presented in this thesis and sketches some possible, sometimes speculative, future research directions.

Near-extreme Kerr holography

In Chapter 2, we reviewed the universal near-AdS_2 dynamics governed by the Schwarzian mode and its realization in near-extremal black holes. The main new result, based on [1], is the description of the gravitational perturbation that captures the deviation from extremality for the Kerr black hole. This perturbation can be described in a rather simple fashion, using AdS_2 holography as an organizing principle. This might have many possible applications, some of them being listed below.

Our work was restricted to the near-horizon region. It will be important to understand the corresponding perturbation in the full Kerr geometry, and this is work in progress [103]. This can be done by using the method of asymptotic matching. Understanding better the gluing offers the prospect of using the near-AdS_2 physics as a solvable subsector of the Kerr perturbations.

Perturbations of Kerr can be studied systematically in the Teukolsky formalism [192]. Interestingly, this formalism applied to the near-horizon region misses the Schwarzian mode. This is because the Schwarzian mode can be generated by a large diffeomorphism of the NHEK while the Teukolsky formalism is diffeomorphism-invariant by design. This shows that it's important to understand the near-AdS_2 picture as it might shed light on puzzling aspects of the near-horizon dynamics [104, 105] related to the Kerr/CFT correspondence [98].

It would be exciting to use near-AdS_2 holography in astrophysical applications. For example, we might try to study, from this perspective, the ringdown phase after the merger of two black holes into a rapidly spinning one. Also, analyzing the results from the Event Horizon Telescope requires the study of the dynamics close to the horizon. The present analysis would need to be generalized to involve matter fields but it seems probable that the near-AdS_2 perspective will be useful here, for a black hole that spins sufficiently fast.

Another direction would be to look for a quantum mechanical dual of the Kerr black hole. As the SYK model contains JT gravity, we could try to find a 1d model which contains the near-AdS_2 physics of the Kerr black hole. This would require a better understanding of the additional mode, that we denoted χ, and which we are currently investigating [103].

We would also like to study interactions in the near-extreme Kerr black hole. This would be obtained by studying gravitational perturbations at second order. This is intractable in general but the simplicity of AdS_2 dynamics allows one to obtain analytical solutions, which compute holographic three-point functions in the NHEK. In JT gravity, the Schwarzian mode is responsible for the maximal chaos [43]. It would be interesting to see in a similar way whether the near-extreme Kerr dynamics is maximally chaotic.

In the paper [5], which was not presented in this thesis, we have explored new boundary conditions for AdS_2. This enhances the asymptotic symmetry group to the warped Virasoro group $\mathrm{Diff}(S^1) \ltimes C^\infty(S^1)$ which gets broken down to $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{U}(1)$ by a generalization of the Schwarzian theory. As a result, this theory captures the full $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{U}(1)$ symmetry of the near-horizon geometry of black holes (the Schwarzian theory capturing only the $\mathrm{SL}(2,\mathbb{R})$ part). This suggests that our boundary conditions will be useful in the study of near-extremal black holes. We demonstrated how this dynamics was embedded in near-extreme Kerr, following the same steps as described in Chapter 2, using our new boundary conditions. This gives a phase space of linearized perturbations of the NHEK geometry, on which the warped Virasoro group acts. It would be interesting to explore this further, for example by computing the gravitational charges.

Logarithmic corrections

In Chapter 3, we have explained that a special class of quantum corrections, growing as the logarithm of the black hole entropy, is computable in the low energy theory while being sensitive to details of the microscopic counting, constituting an "infrared window into the black hole microstates". It seems important to understand how much semiclassical gravity knows about the UV theory. In the context of black hole entropy, this requires a detailed understanding of the logarithmic corrections. This also echoes some recent progress on the information paradox suggesting that semiclassical gravity has more to offer than previously thought.

As discussed in this thesis, an interesting pattern was observed in [26]: the logarithmic corrections in effective theories coming from string theory have a simpler structure than in the generic case: it doesn't depend on the details of the black hole. In Chapter 3, we have tested this pattern against a different class of black holes, the so-called non-BPS branch of $\mathcal{N}=2$ supergravity. There, we showed that the universality property continues to hold for enough supersymmetries ($\mathcal{N}=6,8$) but fails for a smaller amount ($\mathcal{N}=2,4$). These statements were made for non-extremal black holes, whose microscopic counting is presently unknown and is expected to be complicated. In the extremal limit, the universality property is restored for all these cases.

In known examples of microscopic counting, which all involve extremal black holes, the logarithmic correction is indeed universal: it is a pure number. This begs the question: *do we expect that the logarithmic correction for extremal black holes is universal in a consistent semiclassical theory?*

Such a statement echoes the swampland program, where low energy criteria for UV consistency are proposed, drawing inspiration from string theory examples. An affirmative answer to the above question has powerful implications. For example, it would imply that pure Einstein-Maxwell theory doesn't exist as a consistent theory, because the extreme Kerr-Newman black hole has charge-dependent logarithmic corrections. Although answering this question seems out of reach, we might hope to make partial progress in restricted cases. For example, it might be possible to show that if the microscopic counting is realized by a Jacobi form, the logarithmic corrections are always charge-independent.

Actually, by considering more general black holes in the non-BPS branch, with a non-trivial profile for the dilaton, we have found that logarithmic corrections can be charge-dependent even in the extremal case and with maximal supersymmetry. This suggests that the answer to the above question might sometimes be negative, and it would be interesting to understand when it fails. This could also be related to the near-AdS_2 physics of these black holes, and it would be interesting to make a connection with the ideas discussed in the previous section.

Using similar techniques, we could also investigate logarithmic corrections to the entanglement entropy of holographic CFTs. From [70], we expect that the logarithmic correction will come from the bulk entanglement entropy piece. This suggests two different ways of doing the computation, either from a Euclidean perspective involving the heat kernel, as explained in section 3.1, or from a Lorentzian perspective where we quantize the bulk fields with appropriate boundary conditions, as was done in [193], and extract the logarithmic piece from the bulk entanglement entropy. It would be interesting to see whether such a match can be achieved. This might give insights on how to properly define bulk entanglement entropy, involving issues such as the correct choice of boundary conditions and the way to treat gauge fields and gravitons.

Traversable wormholes

In Chapter 4, we reviewed the Gao-Jafferis-Wall protocol [80] for traversable wormholes in AdS/CFT and described some wormhole solutions [146, 157]. To see how much this idea can be pushed, we tried to construct an eternal traversable wormhole in higher than two dimensions. Assuming Poincaré symmetry in the boundary directions, we argued that no traversable wormhole can be constructed.

The mechanisms that naively would support such a wormhole, all ended up breaking semiclassical gravity in some way. This "conspiracy" can be understood as follows. Let's assume that we have managed, using some ingredients, to build an eternal traversable wormhole with Poincaré symmetry. In our construction, we showed that the stress tensor generated by the Gao-Jafferis-Wall protocol takes the same form, and with a smaller magnitude, as the stress tensor due solely to Casimir energy in the wormhole. Thus, if a Poincaré traversable wormhole can be constructed using the Gao-Jafferis-wall protocol, such a wormhole can also be constructed without it, using instead Casimir energy of the bulk. However this cannot be, because it would be in direct contradiction with the AdS/CFT correspondence. Indeed, in a setup involving two decoupled CFTs, it should not be possible to send a signal from one CFT to the other through the bulk, a fact that has been called the "no-transmission principle" [159].

An interesting future direction is to take the above argument in reverse, as a way to constrain what is possible or not in quantum gravity. Any mechanism which enhances significantly quantum effects, sufficiently to allow the existence of a traversable wormhole between two decoupled CFTs, has to be prohibited. For example, our Poincaré wormholes gives a new argument for the renormalization of the UV cutoff (the effective Planck scale) when many species are present. This says that if we have N different fields in a D-dimensional bulk, the effective UV cutoff M_{UV} of semiclassical gravity gets lowered according to the formula

$$M_{\text{UV}}^{D-2} \leq \frac{1}{N} M_p^{D-2}\,, \tag{5.1}$$

where M_p is the Planck scale. It was explained in section 4.4.3 that if this was not true, we could indeed build a traversable wormhole. This construction would also work if the traversable wormhole is supported by Casimir energy only, without using the Gao-Jafferis-Wall coupling. This would then violate the no-transmission principle and should be ruled out, showing that the bound (5.1) has to hold. Other heuristic arguments for this bound are known [162–165] involving perturbative analysis of loop diagrams or black hole entropy bounds. For many reasons, these arguments can be seen as not completely satisfactory. The traversable wormhole argument, in the particular cases where it applies, seems to be more solid, the only assumption being the validity of AdS/CFT. An important restriction is that our argument relies on the Poincaré wormhole solution presented in Chapter 4, which assumes conformal matter in the bulk and has issues related to conformal anomalies. It is nonetheless likely that this argument could be made general by showing that with a large negative Casimir energy, a traversable wormhole solution always exists, even if it cannot be described analytically.

JT gravity allows a simple description of traversable wormholes [155, 157]. A flat space analog of JT gravity was introduced in [194] and shown to be dual to a scaling limit of the complex SYK model. We have further studied this model in [5] and showed that, as in JT gravity, this model can be formulated as a "boundary particle" moving in 2d Minkowski spacetime. It should be possible to create traversable wormholes in this setup, mirroring the AdS_2 construction. The Gao-Jafferis-Wall coupling would give an attractive force between the two boundary particles of Rindler space, rendering the wormhole traversable. One could also build the analog of the eternal traversable wormhole of [157], which corresponds here to 2d Minkowski spacetime. It would be interesting to study these configurations as they may offer some insights into flat space holography [195].

We have also showed in [5] that this 2d model of flat holography is dual to an ensemble average, of a similar but much simpler nature than the matrix ensemble dual to JT gravity [50]. The cylinder geometry, which is responsible for this ensemble interpretation, is the Euclidean version of global Minkowski, which is the eternal traversable wormhole of this theory. It will be interesting, using this model, to study the relation between ensemble averages and traversable wormholes.

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Samenvatting

Quantum zwarte gaten

Er wordt vaak gezegd dat kwantummechanica en zwaartekracht onverenigbaar zijn. Recente ontwikkelingen in de snaartheorie hebben dit perspectief veranderd: eigenlijk zijn kwantummechanica en zwaartekracht op een diepe en verrassende manier met elkaar verweven, en blijken ze onafscheidelijk te zijn! Deze vooruitgang kwam voort uit het besef dat zwarte gaten kwantumobjecten zijn, wat uiteindelijk leidde tot het holografische principe en de opvatting dat zwaartekracht voortkomt uit de kwantummechanica. In dit proefschrift onderzoeken we enkele aspecten van deze ideeën door kwantumzwarte gaten te onderzoeken.

We bestuderen eerst het holografische principe, in de vorm van de AdS/CFT-correspondentie, wat een precieze gelijkwaardigheid is tussen kwantumzwaartekracht in AdS-ruimtetijden en conforme veldentheorieën. De kracht van deze dualiteit komt voort uit het feit dat het een precieze definitie geeft van kwantumzwaartekracht, terwijl het ook een venster biedt op de sterke koppelingsdynamiek van veldtheorieën. Hoofdstuk 2, gebaseerd op de paper [1], bespreekt de $\mathrm{nAdS}_2/\mathrm{nCFT}_1$ correspondentie, en legt uit hoe deze toe te passen op het Kerr zwarte gat.

Vervolgens onderzoeken we de thermodynamica van zwarte gaten. Dit vakgebied werd in de jaren zeventig ontwikkeld en was een van de belangrijkste inspiratiebronnen achter de meer recente inzichten en ontwikkelingen. Hoofdstuk 3, gebaseerd op de paper [2], onderzoekt de eigenschappen van kwantumcorrecties op zwart gat-entropie in de context van snaartheorie, met de nadruk op een speciale klasse van logaritmische correcties, die volledig zijn gevangen in de lage-energietheorie.

Ten slotte onderzoeken we het verband tussen kwantuminformatie en ruimtetijd. Een krachtig idee is dat ruimtetijdconnectiviteit en verstrengeling in wezen hetzelfde zijn, waarnaar vaak wordt verwezen als "ER=EPR". Deze relatie werd onlangs versterkt door aan te tonen dat wormgaten doorkruisbaar kunnen worden gemaakt met een bepaald kwantumteleportatieprotocol. Hoofdstuk 4, gebaseerd op het artikel [3], onderzoekt de grenzen van deze constructie door te proberen een eeuwig doorkruisbaar wormgat te construeren.

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