Testing Gravity In The Local Universe

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Abstract

General relativity (GR) has stood as the most accurate description of gravity for the last 100 years, weathering a barrage of rigorous tests. However, attempts to derive GR from a more fundamental theory or to capture further physical principles at high energies has led to a vast number of alternative gravity theories. The individual examination of each gravity theory is infeasible and as such a systematic method of examining modified gravity theories is a necessity.

Studying generic classes of gravity theories allows for general statements about observables to be made independent of explicit models. Take, for example, those models described by the Horndeski action, the most general class of scalar-tensor theory with at most second-order derivatives in the equations of motion, satisfying theoretical constraints. But these constraints alone are not enough for a given modified gravity model to be physically viable and hence worth studying. In particular, observations place incredibly tight constraints on the size of any deviation in the solar system. Hence, any modified gravity would have to mimic GR in such a situation. To accommodate this requirement, many models invoke screening mechanisms which suppress deviations from GR in regions of high density. But these mechanisms really upon non-linear effects and so studying them in complex models is mathematically complex.

To constrain the space of actions of Horndeski type to those which pass solar-system tests, a set of conditions on the four free functions of the Horndeski action are derived which indicate whether a specific model embedded in the action possesses a GR limit. For this purpose, a new and surprisingly simple scaling method is developed, identifying dominant terms in the equations of motion by considering formal limits of the couplings that enter through the new terms in the modified gravity action. Solutions to the dominant terms identify regimes where nonlinear terms dominate and Einstein's field equations are recovered to leading order. Together with an efficient approximation of the scalar field profile, one can

determine whether the recovery of Einstein's field equations can be attributed to a genuine screening effect.

The parameterised post-Newtonian (PPN) formalism has enabled stringent tests of static weak-field gravity in a theory-independent manner. This is through parameterising common perturbations of the metric found when performing a post-Newtonian expansion. The framework is adapted by introducing an effective gravitational coupling and defining the PPN parameters as functions of position. Screening mechanisms of modified gravity theories can then be incorporated into the PPN framework through further developing the scaling method into a perturbative series. The PPN functions are found through a combination of the scaling method with a post-Newtonian expansion within a screened region.

For illustration, we show that both a chameleon and cubic galileon model have a limit where they recover GR. Moreover, we find the effective gravitational constant and all PPN functions for these two theories in the screened limit. To examine how the adapted formalism compares to solar-system tests, we also analyse the Shapiro time delay effect for these two models and find no deviations from GR insofar as the signal path and the perturbing mass reside in a screened region of space. As such, tests based upon the path light rays such as those done by the Cassini mission do not constrain these theories.

Finally, gravitational waves have opened up a new regime where gravity can be tested. To this end, we examine how the generation of gravitational waves are affected by theories of gravity with screening to second post-Newtonian (PN) order beyond the quadrupole. This is done for a model of gravity where the black hole binary lies in a screened region, while the space between the binary's neighbourhood and the detector is described by Brans-Dicke theory. We find deviations at both 1.5 and 2 PN order. Deviations of this size can be measured by the Advanced LIGO gravitational wave detector highlighting that our calculation may allow for constraints to be placed on these theories. We model idealised data from the black hole merger signal GW150914 and perform a best fit analysis. The most likely value for the un-screened Brans-Dicke parameter is found to be $\omega = -1.42$, implying on large scales gravity is very modified, incompatible with cosmological results.

Lay Summary

The scientific study of gravity spans centuries, with work stretching from Galileo to Newton resulting in the development of the classical theory of gravity. The classical theory stood strong until the discovery of electrodynamics, highlighting problems with Newtonian gravity, namely the absolute nature of space and time.

These problems were overcome by Einstein. First, the discovery of special relativity describing how both time and space where intertwined. This is through different observers, who when traveling at different speeds would measure the length of a given rod and the time taken for a given clock to tick differently from each other. However, they would both agree on a function of both length and time, called the proper time. The proper time is the time taken for the clock to tick if one was not moving relative to it, hence the name.

This conception of space-time proved fundamental to formulating a theory of gravity which melded with special relativity, called general relativity (GR). The intrinsic idea of GR is that space is curved and observers travel along the straight lines of curved space. But an observer at cannot tell if they are in free fall or accelerating due to gravity without comparison to other regions of space. Gravity, too, is relative.

However, the ideas that form general relativity are not unique to general relativity. One can find other models of gravity which satisfy the many of the assumptions of GR but introduce new gravitational forces, exotic types of matter or extra dimensions. This thesis concerns itself with the study models which contain new gravitational forces.

The motivation for this choice is that it has recently been found that the rate that the universe is expanding is accelerating. This would imply that there exists a "dark energy" in the universe causing this acceleration. In GR, this is explained through the use of a cosmological constant; a ubiquitous energy which is constant

across all of space-time. However, this energy has to be much smaller than predicted from any fundamental theory of physics to match observations. A common aim of modern modified gravity physics is to explain this late time acceleration dynamically without a cosmological constant.

However, gravitational effects span from laboratory scales up to the size of the universe, and so any theory would need to satisfy all existing gravity measurements. This thesis examines which modified gravity theories can be made to pass solar system tests but allow for deviations on cosmological scales, and effects such theories are expected to have in the solar system not found in GR.

Finally, extreme gravitational events such as the mergers of black holes emit gravitational radiation similar to how light is emitted by the motion of charged particles. With the recent detection of gravitational waves, new tests of gravity are available. We look at a particular class of gravity theories and examine how the gravitational radiation is changed relative to GR, and if such a change is potentially measurable.

Declaration

I declare that this thesis was composed by myself, that the work contained herein is my own except where explicitly stated otherwise in the text, and that this work has not been submitted for any other degree or professional qualification except as specified.

Parts of this work have been published in [127].

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(*Ryan McManus, May 2018*)

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Chapter 1

Introduction

1.1 Gravity in non-relativistic physics

We begin with a discussion of the history of gravity. While no doubt people pondered why things fell towards the ground since time immemorial, we start with Galileo Galilei who can be considered the first scientist to turn their attention to the systematic study of gravity.

The classic tale of him dropping balls from the tower of Pisa, whether or not apocryphal, demonstrates a wonderful contribution to the theory of gravity. The experiment is meant to demonstrate that under the force of gravity, the composition and mass of objects do not determine their acceleration, rather both fall at the same rate. We will see this idea still holds a special place in the theory of gravity in section 1.4.1, forming one of the pillars of General Relativity. It is with some wonder that his work in astronomy is also fundamental to the development of the field, discovering the moons of Jupiter and proving evidence of the heliocentric model of the solar system. As we now explain this as an effect of gravity, this connection is amazingly fortuitous. Moreover, he discovered the symmetry group for classical, non-relativistic physics, dubbed Galilean symmetry, although this description is distinctly modern and tied to Newtonian physics. These symmetries are that physics should be invariant under translation, rotation, and the addition of constant speed, [74].

From this, we move on to Issac Newton, wherein his Principia Mathematica, [136], he outlined his three laws of motion in order to describe the movements of any material body, from the motion of pendulums to that of the stars and planets. Moreover, he included the inverse square law of gravitation, explaining that all masses exert a force upon each other proportional to the product of the masses of each body,

$$F = -\frac{GM_1M_2}{r^2}\,. \tag{1.1}$$

With this, the motions of the solar system can be found and Kepler's laws are consequential. Moreover, this equation holds true for the motion of all but the most extreme objects in the Milky Way, the motion of orbiting dwarf galaxies and laboratory Cavendish experiments. This is an incredible range of scales and so it is no surprise that Newtonian gravity stood unchallenged for hundreds of years.

However, discrepancies began to appear around the end of the 19th century, leading to the birth of modern physics. The orbits of planets processes due to the influence of the other planets; only a single spherical source allows for a closed orbit. Astronomers calculated the precession of the perihelion of the planets but found an excess in Mercury's procession. Further, electrodynamics describes the motion of charged particles but breaks Galilean symmetry, leading to special relativity. Together, this drove theorists to find a relativistic theory of gravity culminating in Einstein's theory of General Relativity, [66].

1.2 Special relativity

The laws of electromagnetism where being constructed during the 1800s, culminating in Maxwell's equations. Within these equations, it was shown that the electric and magnetic fields where not only functions the spatial derivatives of sources but also upon their time derivatives. Moreover, it was found that changing magnetic fields sourced electric fields, and vice versa, the basis of electrodynamics. The presence of both derivatives allows for Maxwell's equations to be written as a wave equation in contrast to gravity's Poisson equation, wherein a constant appeared, the speed of light. A consequence was the discovery that light is electromagnetic waves. However, this discovery leads to several problems with non-relativistic physics.

That the speed of light appeared in the field equations implied that the equations themselves were only valid in one frame of reference, as other frames would have a different speed of light. This idea follows from the intuitive understanding we have for the Galilean group of non-relativistic physics. But it was also found that the equations were not invariant under the Galilean group, rather, a new symmetry was found, that of the Poincaré group, $SO(1,3)$.

Einstein's contribution was the radical departure from non-relativistic physics with the development of special relativity, [65]. An important axiom of special relativity is that the speed of light is a constant for all observers regardless of the speed they travel relative to each other. In order to keep a speed constant, different observers will have to disagree about both distances and times. This leads to the effects of time dilation and length contraction.

This effect is not present in non-relativistic physics as such waves such as sound or water waves travel through a medium. A privileged reference frame exists, that which is still with respect to the medium, and so there is no expectation for the speed to be the same for both observers. In special relativity, there is no aether which light travels through.

However, while there is no agreement about the time an event occurs or the length of a path, there is an invariant measurement in special relativity, the proper time,

$$ds^2 = dx^{\mathrm{T}} \eta dx \,. \tag{1.2}$$

This quantity introduces the concept of the four-vector

$$dx = (dt, d\overline{x}) \,, \tag{1.3}$$

and the Minkowski metric,

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{1.4}$$

Hence, given a Lorentz transformation $\Lambda \in SO(1,3)$, the proper-time remains the same,

$$\begin{aligned} ds^2 =& d\overline{x}^{\mathrm{T}} \Lambda^{\mathrm{T}} \eta \Lambda d\overline{x} \\ =& d\overline{x}^{\mathrm{T}} \eta d\overline{x} \,. \end{aligned} \tag{1.5}$$

While electromagnetism fits perfectly into the mathematics of special relativity, the inclusion of a viable gravity theory eluded scientists until the invention of general relativity. This is as special relativity is still the physics of flat space, while general relativity is intrinsically the physics of curved space. To properly understand why we now look at the mathematical basis of general relativity.

1.3 Mathematical groundwork

To allow for a comprehensive understanding of this thesis, we present a crash course on the mathematical foundations of general relativity. This section is intended to help understand and enlighten the bulk of the thesis for the reader unfamiliar in the mathematical background of general relativity, being a concise amalgamation of [41, 87, 134, 180]. Ultimately, the only part of this section that we deem completely necessary is (1.22) as from it we find Einstein's equations in section 1.4. In section 1.3.1 we present a discussion of what manifolds are and why we wish to use them. How one can build vectors on curved space in a coordinate independent manner is built from elementary ideas and extended to tensors with the introduction of the metric tensor is in section 1.3.2. Finally how one can go from the abstract construction of curved space to being able to find the amount of curvature at a point with the Riemann tensor is shown in 1.3.3.

1.3.1 Manifolds

In both non-relativistic and special relativity, we use four continuous coordinates to describe spacetime: three for space and one for time. Even though the relationship between space and time is changed in the transition between non-relativistic and special relativity, it is taken for granted that spacetime is can be globally described by these four numbers, each with the range $-\infty$ to ∞ denoted $\mathbb{R}^4$. With these four numbers, we can give a unique value to each point of space at a fixed time, and track the motion of each body through a function $\overline{x}(t)$, which describes how the location of an object changes in time.

But to perform physics, we need more structure than just the abstract vector space $\mathbb{R}^4$. Physical laws depend upon derivatives in order to describe how systems evolve across spacetime such as in Newton's second law of motion [136],

$$\overline{F} = m\frac{\mathrm{d}^2\overline{x}}{\mathrm{d}t^2}. \tag{1.6}$$

As such, we need the ability to describe how objects change across small regions of space or time, as such we need a method to determine whether two points in space are near. Such concerns lead one to the study of topology, [87], but this still is not enough to build derivatives. From the usual calculus of several variables, we know how to perform derivatives within real vector spaces. Hence in order to have derivatives, we also need the small regions of space around the point we intend to find the derivative in look like $\mathbb{R}^4$. This requirement, with a few more technical points, leads to the necessity of differentiable manifolds in our description of space.

That in both non-relativistic and special relativity space can be described by $\mathbb{R}^4$ seems to render the previous discussion as overly generic. We end by saying that space needs to locally look like $\mathbb{R}^4$, but is it not globally? No, that space is not globally $\mathbb{R}^4$ is the crux of general relativity and leads to a plethora of interesting effects. We will see that spacetime is best described as being curved through physical arguments in section 1.4.1. To allow for a discussion of these concepts, we now provide a short, simplified introduction to differentiable manifolds and various aspects of Riemannian geometry in section 1.3.2.

We have discussed open balls around a point p in $\mathbb{R}^4$ as the method to describe points near p which we need to generalise. As the general goal of this section is to take complex ideas and place them into a real vector space, the points near p should look in some sense look like a real vector space. To do so one builds a topological space, but the important concept from such a construction is that for every point in the space, there exists a neighbourhood of the point which is homeomorphic, that is continuously mapped, to an open ball in $\mathbb{R}^n$ for $n \in \mathbb{N}$. This allows us the describe the topological space as being n-dimensional.

Let $\mathcal{M}$ be the set of all spacetime points. There is a collection of subsets of $\mathcal{M}$, which we will label as $\{U_i\}$ such that the union of $\{U_i\}$ covers the space. As each point in $\mathcal{M}$ lies within some U_i, then each point has a neighbourhood that we can work with, but we need to map them onto a real vector space.

Together with $\{U_i\}$, there are a collection of one-to-one functions, $\{\psi_i\}$, called a coordinate charts such that $\psi_i : U_i \to \mathbb{R}^4$ with the image of ψ_i an open set in $\mathbb{R}^4$

for all i. We require the chart to be one-to-one so as to not label two points in spacetime with the same coordinates after the mapping. That reason the image is an open set is more opaque but will eventually allow for differentiation. We could work within a region of spacetime U_i using the usual tools for calculus in $\mathbb{R}^4$ as we have mapped the space to something usable.

However, any point $p \in \mathcal{M}$ may lie in multiple open sets U_i. This poses a problem as we perform physics in each subset U_i by using its coordinate chart, but the physics at any point must be independent of the subset, or specifically the coordinates we chose. This would not be a problem if physics were not continuous, but we demand more than even that, we want it to be differentiable. To accommodate this, we need to understand how overlapping subsets and their coordinate chart's interplay.

We need a function Ψ that describes how one takes the coordinates defined in one chart to the coordinates in another, which can just be considered a coordinate redefinition on the intersection. Ψ being smooth guarantees that as we perform calculus in either subset via the use of a chart, the results in one coordinate system are compatible with those of the other. More rigorously, given two subsets U_i and U_j such that $U_i \cap U_j \neq /0$, then the function $\Psi = \psi_i \circ \psi_j^{-1}$, which maps from the image of $\psi_j(U_i \cap U_j) \subset \mathbb{R}^4$ to that of $\psi_i(U_i \cap U_j) \subset \mathbb{R}^4$, is smooth and has an open image. We work on the overlap as this is where the worry about assigning multiple values to a point in $\mathcal{M}$ takes place. Any function defined on the two subsets U_i and U_j using the coordinates defined by their respective charts can then be compared to the overlap through this coordinate redefinition. This then allows for functions to be smooth across spacetime as they can be smooth on each subset and the smoothness of the overlap guarantees the function is smooth between subsets.

Finally, we want to include all such charts; the set of which we call the atlas. This allows us to change coordinates freely, but the subset of spacetime the coordinates cover may change.

Together, we have listed the requirements for a differential manifold and explained why we physically need all this structure. With the machinery manifolds provide, we can generalise the tools for needed to perform physics: distances, vectors and derivatives with Riemannian geometry.

1.3.2 Tensors

In the previous section, we described how one sets up a differential manifold in order to describe the coordinates on curved spaces. While doing this, we justified our steps through references to wanting to perform calculus, but our tools so far only allow for scalar fields. In this section, we roughly show how one can also include vectors, tensors and define the metric tensor. For a more thorough introduction, we refer to standard textbooks such as [41, 180].

The concept of a vector requires some refinement from the usual understanding from flat space. In particular, vectors on curved spaces are only defined at a point in that point's tangent space, T_p. The concept of a tangent space can intuitively be understood by thinking of a sphere. Then a plane that touches the sphere tangentially defines the tangent space at that point. But this definition is not generic enough: it relies upon embedding the manifold in a higher dimensional space in order to find such a tangent space, which one can see as being dependent on the embedding chosen. We will construct the tangent plane from objects intrinsic to the manifold.

As a rough outline, we can consider curves $\mathbb{R} \to M$ that pass through a point p. Different curves through p can pass through this point in different directions, and so we wish to identify the direction curves take at p to vectors at p. To talk about the direction a curve takes, we use the directional derivative defined on the space of functions on M, $f : M \to \mathbb{R}$. The space of directional derivatives form a vector space, and it is this space we identify with the tangent space.

In more detail, we let $\mathcal{F}$ be the set of all smooth functions on M. Consider $p \in M$, and ϕ be a coordinate chart, then each curve $\gamma : \mathbb{R} \to M$ defines an action on this space through directional derivatives such that for $f \in \mathcal{F}$,

$$\frac{\mathrm{d}f}{\mathrm{d}\lambda} = \frac{\mathrm{d}}{\mathrm{d}\lambda}(f \circ \gamma)\bigg|_{\lambda=p}. \tag{1.7}$$

We identify the tangent space at p to the space of directional derivatives. It is easy to see this is a vector space and is closed because the definition is linear in derivatives, which is also why the space is closed.

As a vector space, we are free to pick a basis to write any directional derivatives with respect to. Consider the following:

- Manifold M,
- Coordinate chart $\phi : M \to \mathbb{R}^4$,
- Curve $\gamma : \mathbb{R} \to M$,
- Function $f : M \to \mathbb{R}$.

Then we can express any vector as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}f =&\frac{\mathrm{d}}{\mathrm{d}\lambda}(f \circ \gamma)\bigg|_p \\ =&\frac{\mathrm{d}}{\mathrm{d}\lambda}(\phi \circ \gamma)^\mu\bigg|_p \frac{\mathrm{d}}{\mathrm{d}x^\mu}(f \circ \phi^{-1})\bigg|_p \\ =&\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\bigg|_p \frac{\mathrm{d}}{\mathrm{d}x^\mu}f,\end{aligned} \tag{1.8}$$

where we have used the definition, chain rule and inserted an identity map. Thus we can express any vector as linear combination of the vectors (directional derivatives) $\frac{\mathrm{d}}{\mathrm{d}x^\mu}$.

As the definition of the directional derivative is invariant under a change in the coordinate map, a corresponding change in basis must leave the vector invariant, too. So given a vector V, we have

$$\begin{aligned}V =&V^\mu \partial_\mu \\ =&V^{\mu'} \partial_{\mu'} \\ =&V^{\mu'} \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu,\end{aligned} \tag{1.9}$$

where $V^\mu \in \mathbb{R}^4$. Hence we see that the vector components must transform as

$$V^\mu \frac{\partial x^{\mu'}}{\partial x^\mu} = V^{\mu'}. \tag{1.10}$$

In practice, we do not use the coordinate-independent vector, but rather the vector coefficients and use the transformation (1.10) upon a change of basis as the definition of a vector. We will drop the strict reference to a change of basis and rather refer to the process as a change of coordinates, leaving the change of the basis from the previous coordinate basis to the new implicit.

With the definition of a vector in hand, we now turn to dual vectors. A dual vector maps vectors to the real numbers, and we identify these with the gradient df of a function f on M.

Again, a basis can be defined, dx^μ such that

$$dx^\mu \frac{\partial}{\partial x^\nu} = \delta^\mu_\nu, \tag{1.11}$$

from which we can write out any dual vector as

$$df = f_\mu dx^\mu. \tag{1.12}$$

The coefficients again must transform as

$$f_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} f_\mu \tag{1.13}$$

in order to ensure that the dual vector is coordinate independent. As with calling the vector components the vector, we call the dual vector components the dual vector.

With the transformations of vectors and dual vectors, we can now write down the transformation law defining a tensor. $T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n}$ is a tensor if under a coordinate change, it transforms as

$$T^{\mu'_1 \dots \mu'_m}{}_{\nu'_1 \dots \nu'_n} = T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n} \frac{\partial x^{\mu'}}{\partial x^\mu} \cdots \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \cdots \frac{\partial x^\nu}{\partial x^{\nu'}}. \tag{1.14}$$

One can alternatively define a tensor as a map from the tensor product of vectors and dual vectors to the real numbers, in line with the definition of the dual vector. We have chosen this definition as it requires less machinery to define. We will build all of relativistic physics from tensorial objects so that we can express the laws of physics in a coordinate independent manner. With objects that have both upper and lower indices, we can contract the tensor in an analogous method to taking the trace of a matrix,

$$\sum_\alpha T^{\alpha \mu'_1 \dots \mu'_m}{}_{\alpha \nu'_1 \dots \nu'_n} = T^{\alpha \mu'_1 \dots \mu'_m}{}_{\alpha \nu'_1 \dots \nu'_n}. \tag{1.15}$$

On the right hand side, we have implemented the Einstein summation convention by dropping the summation symbol, leaving it implied by the upper and lower index having the same symbol, α.

With the concept of a tensor defined, we can now explain the metric $g_{\mu\nu}$, a tensor field defined across the manifold which is symmetric in its two indices with a non-vanishing determinant. As such, it is invertible with inverse $g^{\mu\nu}$ such that $g_{\mu\alpha}g^{\mu\beta} = \delta^{\alpha}_{\beta}$. The metric tensor is a special tensor in that we will use it to measure distances across the manifold, define light cones, world lines, derivatives, gravity and a method of measuring proper time. Moreover, the metric carries the information about the curvature of space. The study of various aspects of the metric is the subject of the next section.

1.3.3 The metric connection and the Riemann tensor

As we are trying to build the machinery needed to describe physics on a curved spacetime, a description of how curved spacetime is needed. As claimed, the metric will be crucial to this, so some further discussion of it is enlightening.

As the components of the metric are dependent upon coordinates, it is worth defining local Lorentz coordinates. Through changing coordinate, at a point $p \in M$, one can diagonalize the metric such that $g_{\mu\nu} = \eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric. Moreover, through coordinate redefinitions, one can also make the first derivative of the components vanish at the point p. However, second and higher order derivatives cannot be made to vanish at a point. Coordinates such that $g_{\mu\nu} = \eta_{\mu\nu}$ and $\partial_\alpha g_{\mu\nu} = 0$ are called local Lorentz coordinates.

The existence of these coordinates implies that for a coordinate independent measure of curvature, we need the second derivative of the metric. However, partial derivatives of tensorial objects are not tensorial. Hence, to proceed we need to develop a tensorial version of the derivative, called the covariant derivative.

The covariant derivative ∇_α must satisfy various rules that seem intuitive such as linearity and obeying the Leibnitz rule, but there are subtleties as well. In particular, we require that the second covariant derivative of any scalar ϕ leads to a symmetric 2 tensor, $\nabla_\mu \nabla_\nu \phi = \nabla_\nu \nabla_\mu \phi$. This means that the covariant derivative is torsion free. Recall that we defined vectors in terms of directional derivatives of scalar fields along curves: we also require that the covariant derivative is compatible with this derivative as the usual derivative is in multi-variable calculus.

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\phi = \lambda^{\alpha}\nabla_{\alpha}\phi\,. \tag{1.16}$$

Finally, we need the covariant derivative to be metric compatible, that is,

$$\nabla_\alpha g_{\mu\nu} = 0\,. \tag{1.17}$$

It can be shown, [180], that these conditions uniquely define the covariant derivative of a vector field to be

$$\nabla_\mu V^\nu = \partial_\mu t^\nu + \Gamma^\nu_{\mu\sigma} t^\sigma\,, \tag{1.18}$$

where ${\Gamma^\nu}_{\mu\sigma}$ are the Christoffel symbols defined by

$$\Gamma^\nu_{\mu\sigma} = \frac{1}{2} g^{\nu\alpha} \left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}\right) \tag{1.19}$$

Note that while the Christoffel symbols carry indices they are not tensors, they do not transform as (1.14), while the covariant derivative does transform as a tensor.

Recalling that it is the second derivative of the metric which contains information about the deviation away from flat space regardless of coordinate choices, we will examine the second derivative of a dual vector field V_σ multiplied by a scalar field, ϕ. In particular, it can be shown that

$$\nabla_\mu \nabla_\nu (\phi V_\sigma) - \nabla_\nu \nabla_\mu (\phi V_\sigma) = \phi(\nabla_\mu \nabla_\nu V_\sigma - \nabla_\nu \nabla_\mu V_\sigma)\,, \tag{1.20}$$

implying that the combination $\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu$ when acting on a dual vector is not a derivative, but rather a tensor field its self,

$$\nabla_\mu \nabla_\nu V_\sigma - \nabla_\nu \nabla_\mu V_\sigma = {R_{\mu\nu\sigma}}^\gamma V_\gamma\,. \tag{1.21}$$

On the right hand side we have introduced an important tensor, ${R_{\mu\nu\sigma}}^\gamma$ called the Riemann tensor.

The physical interpretation of what we have calculated is that we have taken the vector V_γ and moved it around an infinitesimal loop such that the angle between the path and vector is held constant. On flat space we do this naturally, which is why we can both talk about vectors is real vector spaces as pointing between two points, from the origin or placed end to end. However, on curved space, this is no longer true. The visual example is to consider taking a vector from the earth's equator to the north pole, turning 90 degrees and walking back to the equator,

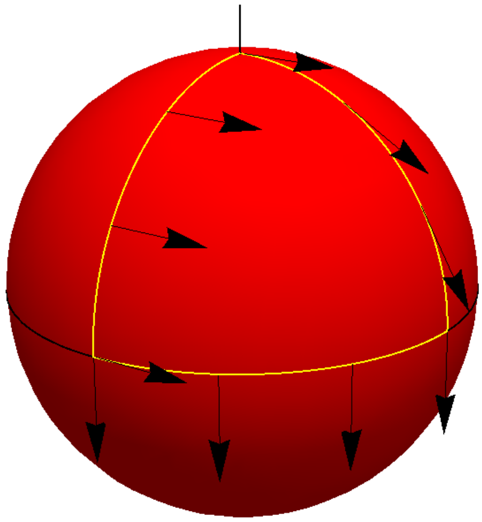

Figure 1.1 *Illustration of the change in vector when it is moved through a closed loop on a curved space. The vector begins at the equator (bottom left), moves to the north pole before returning to the initial point via an alternative route. In this demonstration, the vector has become orthogonal to its original state.*

then along the equator to back to the start while ensuring that the vector's angle to each path remains constant, as shown in figure 1.1. The resulting vector is no longer parallel to the initial vector. However, demonstrating this with the formalism we have built would be a substantial detour and so we refer the reader to chapter 3.2 of [180].

The Riemann tensor satisfies the Bianchi identity, $\nabla_{[\alpha} R_{\mu\nu]\sigma}{}^{\rho} = 0$, which can be shown by parallel transporting the covariant derivative of a vector field around an infinitesimal square. Contracting the second and fourth index of the Riemann tensor in the identity leads to the identity

$$\nabla^{\mu}(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = 0\,, \tag{1.22}$$

where $R_{\mu\nu} = R_{\mu\rho\sigma}{}^{\rho}$ is the Ricci tensor and $R = R^{\mu}_{\mu}$ is the Ricci scalar.

This identity will play a critical role in the development of Einstein's field equations, the subject of the next section.

1.4 Einstein's General Relativity

Having built the mathematical groundwork needed for General Relativity, in section 1.4.1 we discuss the equivalence principle in order to justify treating spacetime as curved. Following this, we derive Einsteins equations through the use of the Bianchi identity and an understanding of how to define an invariant stress-energy density in 1.4.2. We then demonstrate in section 1.4.3 that Einstein's theory of gravity passes all Newtonian tests of gravity through as the low energy static limit recovers Newtonian gravity. We close with a discussion of the Einstein-Hilbert action and how this allows for a method of constructing new actions describing modified gravity theories in section 1.4.4. We will work throughout with $c = 1$.

1.4.1 The equivalence principle

In Newtonian gravity, gravitational mass is the property of objects which determines how much gravitational force the object causes and experiences within a gravitational field. Inertial mass is the property which measures an object's resistance to acceleration according to Newton's second law. The equivalence of these two versions of mass is not guaranteed by theory but rather is observed in experiment. The equivalence was first noted by the experiments of Galileo, but made more concrete by Newton in the Principica Mathmatica. The equivalence of gravitational and inertial mass is called the weak equivalence principle. The formulation of the weak equivalence principle as stated by C. Will [191] which we will find useful is

> "...the trajectory of a freely falling 'test' body (one not acted upon by such forces as electromagnetism and too small to be affected by tidal gravitational forces) is independent of its internal structure and composition."

The weak equivalence principle is one of the building blocks of general relativity, but Einstein uses an even stricter form. This is as the weak equivalence principle is a statement about gravitational forces, but makes no claim about the physics of non-gravitational forces.

The Einstein (or strong) equivalence principle is the sum of three ideas: the weak equivalence principle, local Lorentz invariance and local position invariance. Local Lorentz invariance is the principle that the laws of non-gravitational physics are independent of the velocity of the observer. Local position invariance is the principle that the laws of non-gravitational physics are independent of the position of the observer. In particular, the laws of physics as seen in a small enough region of spacetime take their usual form found in special relativity.

For an examination of the effect of the Einstein equivalence principle, consider the following thought experiment of an elevator containing an observer with experimental equipment in a gravitational field. Einstein's equivalence principle would indicate that the observer cannot detect the presence of the gravitational field; falling due to gravity and floating freely are indistinguishable. This is non-trivial as one could imagine that the results of some experiment may change depending on the velocity of the elevator with respect to some other observer. However, the principle asserts the observer in the elevator would not detect any acceleration as the elevator and its contents all follow the same path, the weak equivalence principle in action.

A secondary observer outside the elevator on earth would detect the acceleration of the elevator due to a gravitational field, balls fall down when dropped off towers. When doing this, the external observer would have to measure the trajectory of the elevator with respect to coordinates not tied to the elevator, but rather with respect to another object, such as the ground. Moreover, any experiment performed by the observer in the elevator would proceed to give the same result as done without the presence of a gravitational field because of local Lorentz invariance and local position invariance. Strictly, the field being inhomogeneous would allow for the detection of tidal effects across the elevator, but we assume that the region is sufficiently small so as to make such effects negligible.

Importantly, being unable to locally detect gravitational fields implies that test bodies follow geodesics, the equivalent to straight lines on curved spaces,[190, 191]. This is as the observer experiences special relativity within their own reference frame. As such, they appear to move along straight lines, the tangent vector to their path remains parallel with the path as they move through spacetime. But this is just the definition of a geodesic as described in section 1.3.3! For the observer within the elevator, the only motion they can detect is that their clock ticks, they do not move relative to the elevator itself, and so their tangent vector is only temporal in their reference frame. That test bodies follow geodesics is important to our discussion as it shows that in order to find gravitational motion in general relativity, all we need to find is the metric.

1.4.2 Einstein's field equations

Newtonian gravity provides the gravitational field due to a body itself, which together with the his second law describes the dynamics of bodies in a gravitational field. While we have that test particles move along geodesics, analogous to Newton's first and second laws, we have no method of finding the metric, analogous to the gravitational field. Because of the existence of local Lorentz frames, it is clear that we need an equation which relates the second derivative of the metric to the energy content of spacetime. The field equation should be of second order in derivatives to avoid Ostrogradski ghosts, [145]. Moreover, the field equation will need to be generally covariant in order to allow physics to not depend upon coordinate choice.

The Riemann tensor was shown to describes curvature in section 1.3.3, but more importantly, we had the identity

$$\nabla^{\mu}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\right) = 0\,. \tag{1.23}$$

The term in brackets is often referred to as the Einstein tensor. That the quantity has no divergence implies some conservation, a backbone of physical theories. We wish to describe how energy causes spacetime to curve, and so we need a conservation law for energy. But as energy is a frame dependent quantity, we will also need a tensorial way to describe the amount of energy at a point. To this end, we introduce the stress-energy tensor.

The stress-energy tensor, $T^{\alpha\beta}$, is a tensor field which describes the 4-momentum density held in each (infinitesimal) volume across spacetime. To do this, the stress-energy must be a rank two tensor and when contracted with a volume one-form, what remains must be the total 4-momentum of the volume. The contraction with the total 4-momentum with the volume's four-velocity, in turn, gives the energy within the volume. Through mass-energy equivalence, we can roughly associate the stress-energy tensor with the mass contained within the volume, implying that the stress-energy tensor should source our relativistic theory of gravity.

We impose physics upon the stress-energy tensor by insisting that its covariant divergence vanishes, $\nabla_\alpha T^{\alpha\beta} = 0$. The reason for this can be understood by considering the flow of 4-momentum through a closed 4-volume $\mathcal{V}$ whose surface is a 3-volume, $\delta\mathcal{V}$. The conservation of energy-momentum would then be that the integral of the flux across the surface would be zero,

$$\int_{\delta\mathcal{V}} dp^\mu = 0, \tag{1.24}$$

but the momentum flow through each infinitesimal part of the surface, described by the 1-form $d\Sigma_\mu$, is

$$\int_{\delta\mathcal{V}} T^{\mu\nu} d\Sigma_\mu = 0. \tag{1.25}$$

Through use of Gausses' theorem, we arrive at the conclusion $\partial_\mu T^{\mu\nu} = 0$. The frame independent reformulation of this statement is to promote the partial derivate into a covariant derivative, finally arriving at $\nabla_\mu T^{\mu\nu} = 0$ For a more in-depth discussion of the stress-energy tensor, see chapter 5 of [134]. That the stress-energy tensor is covariantly conserved can be shown to be a consequence of general covariance from the examination of a matter action as demonstrated in chapter 12 of [183], in contrast to our assertion that it should be.

As the stress-energy tensor is analogous to mass-density in Newtonian physics, we use it as the source of gravity. Using that it has zero divergence, we have that it is proportional to the Einstein tensor,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{1.26}$$

We have also introduced the term $g_{\mu\nu}\Lambda$ as it too has vanishing divergence due to the metric comparability of the covariant derivative. In general, Λ is the cosmological constant. The proportionality constant is chosen to be $\frac{8\pi G}{c^4}$ in order to recover Newtonian gravity in the low energy, static limit as in section 1.4.3. These are Einstein's equations, [66]. The derivation given is similar to that of Einstein, but using modern notation and reasoning.

With Einstein's equations found, a natural question is whether these equations have anything particularly special about them, in particular, is there anything unique about the condition (1.23) which allows us to equate Einsteins tensor to the stress-energy tensor. There is a well-known theorem by Lovelock, [123, 135], which states that the only rank 2 tensors constructed from the metric, its first and second derivatives and symmetric with vanishing covariant derivatives in four dimensions are the Einstein tensor and the metric itself. Thus Einstein's equations, (1.26), are the unique field equations containing only a rank-2 tensor in a vacuum ($T^{\mu\nu} = 0$). We will see that when modifying gravity, we will always break one of Lovelock's assumptions.

1.4.3 The post-Newtonian expansion of General Relativity

Having found a relativistic theory of gravity, the natural question is whether or not it recovers the results of Newtonian gravity. In order to check this, we will need to apply general relativity to the non-relativistic regime. This consists of two parts: that such physics works for speeds much smaller than the speed of light, and that there is little curvature as the theory is linear. Together they produce the low-energy static limit.

The low-energy static limit has been used extensively to test GR in the Solar System, see sections chapter 3 and ref. [191]. For a comprehensive discussion of the expansion and on how to proceed for a range of gravitational theories other than GR, we refer to section 1.7 and ref. [190].

We assume that the asymptotic metric far from the system under consideration is Minkowski for some period of time where the cosmological evolution of the metric can be neglected. We can make this assumption as far from the system there will be no energy but still small on cosmological scales and so the metric will be that of flat space to a high precision. The metric is then expanded about its asymptotic form in orders of v/c,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + h^{(2)}_{00} + h^{(2)}_{ij} + h^{(3)}_{0j} + h^{(4)}_{00} \,, \tag{1.27}$$

where the indices i and j run from 1 to 3. The perturbations to the metric are $h^{(k)}_{\mu\nu}$, where we will see the superscript indicates that the field components are of the order $(v/c)^k$. We will denote the order as $\mathcal{O}_{PN}(k)$ and let $c = 1$. Note that in chapter 4, we will use a different method of counting. We will keep the gravitational constant G explicit as we make use of it throughout this thesis.

The virial relation indicates that for the Newtonian potential U, $U \approx v^2$ and so is of the order $\mathcal{O}_{PN}(2)$. Further, the Poisson equation, $\nabla^2 U = -4\pi G\rho$, indicates that the matter density ρ is of the same order as U. We consider the matter content as a perfect non-viscous fluid. The pressure in the Solar System is comparable to the total gravitational energy ρU, and so we have $p/\rho \approx \mathcal{O}_{PN}(2)$.

Similarly, the specific energy density Π is also of the order of the Newtonian potential. Hence we can count orders using

$$U \approx v^2 \approx p/\rho \approx \Pi \approx \mathcal{O}_{PN}(2)\,. \tag{1.28}$$

The stress-energy tensor for the fluid to $\mathcal{O}_{PPN}(4)$ is

$$T_{00} = \rho[1 + \Pi + v^2 - h_{00}^{(2)}]\,, \tag{1.29}$$

$$T_{0i} = -\rho\, v^i\,, \tag{1.30}$$

$$T_{ij} = \rho\, v^i v^j + p\, \delta^{ij}\,. \tag{1.31}$$

As we consider a system that evolves slowly in time, we can approximate $d/dt \approx 0$. Writing the total derivative in terms of partial derivatives, it is clear that $\partial_t + \overline{v} \cdot \overline{\nabla} \approx 0$. This relation indicates that time derivatives are of one order higher than the spatial derivatives.

For computational convenience, we adopt the standard post-Newtonian gauge where the ij components of the metric are diagonal and isotropic. For GR this gauge choice is specified by

$$h^{\mu}_{i,\mu} - \frac{1}{2} h^{\mu}_{\mu,i} = 0\,, \tag{1.32}$$

$$h^{\mu}_{0,\mu} - \frac{1}{2} h^{\mu}_{\mu,0} = -\frac{1}{2} h_{00,0}\,, \tag{1.33}$$

where i runs from 1 to 3 and indices are raised and lowered with the Minkowski metric. In employing these conditions, it is important to remember that the terms compared must be of the same order and that time derivatives increase the order. These gauge conditions may change in modified gravity theories to ensure that the metric is diagonal and isotropic.

The Ricci tensor to quadratic order for the 00 component and linear order for the ij and $0j$ components is

$$\begin{aligned} R_{00} =& -\frac{1}{2}\overline{\nabla}^2 h_{00} - \frac{1}{2}(h_{jj,00} - 2h_{j0,j0}) + \frac{1}{2} h_{00,j}(h_{jk,k} - \frac{1}{2} h_{kk,j}) \\ & - \frac{1}{4}|\overline{\nabla} h_{00}|^2 + \frac{1}{2} h_{jk} h_{00,jk}\,, \end{aligned} \tag{1.34}$$

$$R_{0j} = -\frac{1}{2}(\overline{\nabla}^2 h_{0j} - h_{k0,jk} + h_{kk,0j} - h_{kj,0k})\,, \tag{1.35}$$

$$R_{ij} = -\frac{1}{2}(\overline{\nabla}^2 h_{ij} - h_{00,ij} + h_{kk,ij} - h_{ki,kj} - h_{kj,ki})\,. \tag{1.36}$$

The v/c expansion is algebraically easiest to perform on the contracted Einstein equations,

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)\,. \tag{1.37}$$

To $\mathcal{O}_{PN}(2)$ we have $R_{00} = -\frac{1}{2}\overline{\nabla}^2 h^{(2)}_{00}$, $T_{00} = -T = \rho$ and $\eta_{00} = -1$. The Einstein equations for the 00 component reduce to

$$\begin{aligned} -\frac{1}{2}\overline{\nabla}^2 h^{(2)}_{00} =& 4\pi G\rho\,, \\ h^{(2)}_{00} =& 2GU\,, \end{aligned} \tag{1.38}$$

where we have identified the Newtonian potential U with the perturbation $h^{(2)}_{00}$. At this point, we have shown that general relativity recovers Newtonian gravity in the low-energy static limit. However, we may carry on performing perturbation theory to find the rest of the metric and the relativistic corrections to Newtonian gravity

For R_{ij} we employ the gauge condition (1.32) to render the equation diagonal and isometric. Using that $T_{ij} = 0$ to $\mathcal{O}_{PN}(2)$, the field equation then becomes

$$\begin{aligned} 4\pi G\rho\delta_{ij} =& -\frac{1}{2}(\overline{\nabla}^2 h^{(2)}_{ij} - h^{(2)}_{00,ij} + h^{(2)}_{kk,ij} - h^{(2)}_{ki,ki} - h^{(2)}_{kj,ki}) \\ =& -\frac{1}{2}\overline{\nabla}^2 h^{(2)}_{ij}\,. \end{aligned} \tag{1.39}$$

Note that the gauge condition is used twice, once on the differentiation with respect to the i and once with respect to the j coordinate. Hence

$$h^{(2)}_{ij} = 2GU\delta_{ij}\,. \tag{1.40}$$

The metric components h_{0j} are $\mathcal{O}_{PN}(3)$ as can be seen from eq. (1.30). The equation of motion for $h^{(3)}_{0j}$ is given by

$$\begin{aligned} 8\pi G\rho v_j =& -\frac{1}{2}(\overline{\nabla}^2 h^{(3)}_{0j} - h^{(3)}_{k0,jk} + h^{(2)}_{kk,0j} - h^{(2)}_{kj,0k}) \\ =& -\frac{1}{2}\overline{\nabla}^2 h^{(3)}_{0j} - \frac{1}{4}h^{(2)}_{00,0j} \\ =& -\frac{1}{2}\overline{\nabla}^2 h^{(3)}_{0j} - \frac{1}{2}GU_{,0j}\,, \end{aligned} \tag{1.41}$$

for which both gauge conditions (1.32) and (1.33) are used. To evaluate the time derivative to the correct PN order, we make use of the identity

$$\frac{\partial}{\partial t}\int \rho' f(\overline{x},\overline{x}')d^3x = \int \rho'\overline{v}'\cdot\overline{\nabla}' f(\overline{x},\overline{x}')d^3x(1+\mathcal{O}_{PN}(2))\,, \tag{1.42}$$

where $\rho' \equiv \rho(\overline{x}', t)$ and $v'_i = \frac{\partial x'_i}{\partial t}$. This identity follows from the vanishing of the total derivative. We can thus solve eq. (1.41) by using the Greens function for the Poisson equation and defining the post-Newtonian potentials

$$V_i \equiv \int \frac{\rho' v'_i}{|\overline{x}-\overline{x}'|}d^3x'\,, \tag{1.43}$$

$$W_i \equiv \int \frac{\rho'[\overline{v}'\cdot(\overline{x}-\overline{x}')](x-x')_i}{|\overline{x}-\overline{x}'|^3}d^3x'\,, \tag{1.44}$$

to recover the solution

$$h^{(3)}_{0j} = -\frac{7}{2}GV_j - \frac{1}{2}GW_j\,. \tag{1.45}$$

For the $\mathcal{O}_{PN}(4)$ component of g_{00}, we need to use the Ricci tensor up to quadratic order. Using the gauge conditions, the Ricci tensor becomes

$$\begin{aligned} R^{(4)}_{00} &= -\frac{1}{2}\overline{\nabla}^2 h^{(4)}_{00} - \frac{1}{2}(\nabla h^{(2)}_{00})^2 + \frac{1}{2}h^{(2)}_{jk}h^{(2)}_{00,jk} \\ &= -\frac{1}{2}\overline{\nabla}^2(h^{(4)}_{00} + 4(\overline{\nabla}^2)^{-1}[(\overline{\nabla}U)^2 - U\overline{\nabla}^2 U]) \\ &= -\frac{1}{2}\overline{\nabla}^2(h^{(4)}_{00} + 4(\overline{\nabla}^2)^{-1}[\frac{1}{2}\overline{\nabla}^2 U^2 - 2U\overline{\nabla}^2 U]) \\ &= -\frac{1}{2}\overline{\nabla}^2(h^{(4)}_{00} + 2U^2 - 8\Phi_2)\,. \end{aligned} \tag{1.46}$$

In the second line we have factored out the Laplacian and inserted the potentials defined earlier. In the third line we have employed the product rule and in the last line we have used the linearity of the inverse of the Laplacian, defining the PPN potential

$$\Phi_2 = \int \frac{\rho' U'}{|\overline{x}-\overline{x}'|}d^3x'\,. \tag{1.47}$$

Finally, for the matter contribution, we have the 00 component

$$T_{00} - \frac{1}{2}g_{00}T = \rho\left(v^2 - \frac{1}{2}h^{(2)}_{00} + \frac{1}{2}\Pi + \frac{3}{2}p/\rho\right)\,. \tag{1.48}$$

We define new potentials for the scalar sources through the use of the Green's function for the Laplacian to find

$$h_{00}^{(4)} = -2G^2U^2 + 4G\Phi_1 + 4G^2\Phi_2 + 2G\Phi_3 + 6G\Phi_4 \,. \tag{1.49}$$

We refer to ref. [190] for the expressions of the potentials Φ_i.

The benefit of this calculation is that while any gravitational theory must obviously recover Newtonian gravity to be considered viable, the relativistic corrections offer further tests of gravity, [191].

1.4.4 The Einstein-Hilbert action

The idea that actions can be used in physics is fascinating as the global statement that an action is minimised across space seems counter-intuitive to ideas of local physics. Never the less, the use of actions provides many insights into physics, from conservation laws, [140], particle interactions, [69], path integrals, [61], and geodesics, section 1.3.3.

Shortly after Einstein's publication of his field equations, the Einstein-Hilbert action was discovered, from which one can derive Einstein's field equations using the variation principle as opposed to the method presented in section 1.4.2. A benefit of the Einstein-Hilbert action over the field equations themselves is that it allows for modifications of gravity to be made easily, as we will see in section 1.6.

The Einstein-Hilbert action is

$$S = \int d^4x\sqrt{-g}R + S_m[g_{\mu\nu}, \Phi] \,. \tag{1.50}$$

The term $\sqrt{-g}$ is required in order to make the volume measure transform as a scalar under coordinate changes, leaving the Ricci scalar as the only non trivial term. The additional action $S_m[g_{\mu\nu}, \Phi]$ is a diffeomorphic matter action where the metric is coupled minimally in the sense that it only enters through the volume element and the contraction of tensor indices. This is possibly the simplest action one could write down given diffeomorphism invariance and that the second derivative of the metric includes curvature information.

Importantly, this method also provides an alternative route to finding the stress-energy tensor as opposed to the opaque description given in 1.4.2. As the matter action is also a function of the metric, it too can be varied leading to the definition

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}\,. \tag{1.51}$$

At this point, we again mention Lovelock's theorem. While the equation of motion is unique for the metric, the action clearly is not. This is as we can always add total derivatives to the action, but the variation would still result in Einstein's equations.

We also mention as an aside that while we vary the action with respect to the metric, an alternative route exists. If we use the definition of the Riemann tensor in terms of the connection, the connection need not be the metric connection. This would imply that both are independent fields in the Einstein-Hilbert action. This is known as the Palatini formalism. In four dimensions the resulting field equations for the connection imply that it is the metric connection, hence the theory is the same [180]. While both assuming the connection is the metric connection and the Palatini formalism give equivalent results for general relativity, the often diverge when considering modified gravity theories.

1.5 Open questions in gravity

Having traced the history of gravity from Galileo Galilei to development of general relativity by Einstein and having outlined the mathematics needed and discussed some of the phenomenology of the theory, one may wonder why we intend to dedicate the rest of this thesis to the study of modified gravity theories. To justify this switch, we will discuss the open questions in General Relativity and how modifying gravity may be the solution to these problems. In particular, many problems are associated to the cosmological constant, Λ in eq.(1.26).

The presence of Λ in the field equations may not be obvious from the outlined derivation, but mathematically it seems natural to include it as our argument was that based on a vanishing divergence. Intuitively, one can think of the cosmological constant in this regard as an integration constant parameterizing the families of possible manifolds. Physically, one can also view the cosmological constant as the energy held in the vacuum, being a constant contribution in the matter action. From the geometric point of view, the value of the cosmological constant is arbitrary leaving an additional free parameter of the theory. However, using the interpretation as a vacuum energy, it can be sourced both classically and quantum mechanically.

The value of the cosmological constant can be found through the combination of two tests of cosmology. Analysis of the cosmological microwave background (CMB) place the first peak in the CMB angular power spectrum at $l \approx 200$, suggesting that the universe is spatially flat, [51], resulting in a degeneracy that $\Omega_m + \Omega_\Lambda \approx 0$ where Ω_Λ (Ω_m) is the ratio of the energy density due to the cosmological constant (matter content) to the critical density. The luminosity distance of an object will depend on the evolution of the scale factor between emission and observation, and hence a series of observations across varying redshift can show the evolution of the scale factor. The deceleration parameter q_0 describes quadratic corrections to the luminosity distance with respect to redshift and was measured the with use of Type Ia supernovae to be approximately -0.67, [148, 158], implying the universe's expansion is accelerating. This parameter is related to both Ω_Λ and Ω_m as $q_0 \approx \Omega_m/2 - \Omega_\Lambda$, breaking the degeneracy resulting in $\Omega_M \approx 0.3$ and $\Omega_\Lambda \approx 0.7$. Measurements made by the Planck collaboration, [151], place the constraint

$$\Omega_\Lambda = 0.6911 \pm 0.0062 \tag{1.52}$$

$$\implies \rho_\Lambda \approx 10^{-120} M_p^4, \tag{1.53}$$

where $M_p = \frac{1}{\sqrt{8\pi G}}$ is the Planck mass in natural units.

This is vanishingly small when compared to the value one would find from simple quantum field theory (QFT) considerations. In QFT, we know that loop Feynman diagrams contribute to physical observables, but as QFT takes place on a flat background, there is no dependence on the ground state energy. As such little attention is paid to the diagrams which have no external legs, called vacuum diagrams, see [149] for a discussion.

The simplest diagram is just a single loop, corresponding to the sum of ground state energies for a quantum field. For a scalar field with mass m, the contribution would be,

$$\begin{aligned} \rho_\Lambda =& \frac{1}{16\pi^3} \int_0^{k_{cutoff}} d^3k E_k \\ =& \frac{1}{4\pi^2} \int_0^{k_{cutoff}} dk \sqrt{k^2 + m^2} \approx k_{cutoff}^4. \end{aligned} \tag{1.54}$$

We have introduced the UV cutoff k_{cutoff} as we do not expect the theory to be true for all energy scales, high energies would induce significant curvature and QFT is no longer viable. At the simplest level, we expect gravity to not be important until energies the Planck scale as it is the energy scale set by parameters of general relativity and so $k_{cutoff} \approx M_p$. This would lead to an absurd difference of 10^{-120} orders of magnitude. One may wish to weaken the UV cutoff to the TeV scale that we have access to from experiments such as the those performed on the LHC at CERN, and rely on a supersymmetry to reduces the contribution from higher energies. But even so. the estimate for the cosmological constant is $\Lambda \approx 10^{-60} M_p^4$, still a massive clash between observation and theory, [185].

One may wish to find solace in classical effects. In particular, classical physics allows for the value of the minima of a potential to be arbitrarily set, but this is no longer true in general relativity. When classical fields rest at the minima of their potentials, V_{min}, the stress-energy tensor contributes a term $-V_{min} g_{\mu\nu}$, so classical fields would also contribute to the cosmological constant. However, explaining why their contribution is so small is non-trivial. Attempts to have classical potentials explain the cosmological constant and late time acceleration leads to quintessence theories of dark energy, [40, 160, 173]. For a review of the problem of the cosmological constant being small, see [126]

Related to the small cosmological constant, there is also the problem of the energy density of the cosmological constant being of the same order as the matter density today. This is referred to as the coincidence problem, [120, 178], but will not drive much of this work. Also, there are explanations for the size of the cosmological constant using anthropic arguments, [184], and through the existence of multiple universes with differing vacuum energy in string theory, [153].

Finally, in the cosmological standard model, inflation requires the existence of an addition scalar field. The only known fundamental scalar field, the Higgs boson [43], could be this inflation field, [26], but so too could a modification to gravity, [19, 169, 174].

In the preceding discussion, we had made reference to quantum effects while trying to explain the cosmological constant. But this is trying to mix quantum effects with the classical realm that general relativity resides within. Intuitively, general relativity needs to break down at high energy scales in order to allow these two cornerstones of modern physics to meld. Such a theory of quantum gravity would necessarily have a classical limit where it recovers general relativity.

But as we have seen when recovering Newtonian gravity from general relativity, higher order corrections to the theory should exist. Such an approach is called effective field theory [62, 63]. These corrections take the form of additional self-interactions suppressed by a cutoff scale where a full theory would be needed. In the context of general relativity, one would be inclined to add additional terms to the Einstein-Hilbert action to mimic such an expansion,

$$S = \int d^4xR + m_2R^2 + \dots . \tag{1.55}$$

This action is an example of a modified gravity model called $f(R)$ gravity, and as such we now move on to a discussion of different modified gravity models.

1.6 Modified gravity

A major goal of modified gravity is to explain late time acceleration without resorting to a cosmological constant, called self-acceleration, [121, 122]. We can also expect gravity to be modified when considered as an effective theory for some quantum gravity, [62, 63]. Also, attempts to incorporate further physical principles can also lead to modification to gravity. For example, wishing to make the gravitational constant a function, which we will see in section 1.6.1. Another example is the introduction of extra dimensions, which arises in many high energy theories, introducing a spectrum of new fields. The effect extra dimensions have on gravity has been studied from shortly after the discovery of General Relativity. Two notable examples are those of the Kaluza-Klein model which attempts to geometrically unify electromagnetism with General Relativity, resulting in extra vector and scalar degrees of freedom, [146], or the Randall-Sundrum model which intended to change spacetime geometry to solve the hierarchy problem, [156]. In section 1.6.3, we will see how modifying gravity through the use of extra dimensions can lead to interesting phenomenologies with the DGP model.

For the body of this work, we will focus exclusively upon scalar-tensor gravity theories, and so we begin our discussion of modified gravity with a Brans-Dicke theory, the prototypical scalar-tensor theory in section 1.6.1, before moving on to a discussion of screening mechanism which many modern models rely upon to pass solar system tests in 1.6.2. We look in detail at two types of scalar-tensor theory, Galileon gravity in section 1.6.3 and chameleon gravity in section 1.6.4.

While we focus entirely on scalar-tensor gravity, many more types of modified gravity exist: TeVeS [165], MOND [129], Bi-Metric [78], $R_{\mu\nu}R^{\mu\nu}$ [28], infinite derivative gravity [49], string theory [152] and many more.

1.6.1 Brans-Dicke theory

The prototypical scalar-tensor theory of gravity is Brans-Dicke theory [34]. As a historical note, the development of Brans-Dicke theory was not driven by the problems modern relativists try to solve, but rather by the philosophical desire to incorporate Mach's principle into general relativity. The objective of the model is to replace the gravitational constant with the inverse of a scalar field, ϕ.

The action of Brans-Dicke theory is

$$S = \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}\left[\phi R + \frac{2\omega}{\phi}X\right] + S_m[g]\,, \tag{1.56}$$

where $X = \partial_\mu\phi\partial^\mu\phi$, the constant G is a bare gravitational constant (not present in the original formulation) and the constant ω is called the Brans-Dicke parameter. The addition of this scalar field can be considered a modification to gravity as through a conformal transformation, the presence of the scalar field manifests as a fifth force and so breaks the strong equivalence principle, [93].

Many modified gravity theories are built from this action and so an examination of its properties will be beneficial when faced with more complex models. This is our first example of a scalar-tensor modified gravity action and is indicative of how modified gravity actions are in general modified: additional degrees of freedom are included and couple non-minimally to the metric.

The equations of motion are found from the variation of the action (1.56), giving

$$\Box\phi = \frac{8\pi G}{3+2\omega}T^\mu_\mu, \tag{1.57}$$

$$G_{\mu\nu} = \frac{8\pi G}{\phi}T_{\mu\nu} + \frac{\omega}{\phi^2}(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\gamma\phi\partial^\gamma\phi) + \frac{1}{\phi}(\nabla_\mu\partial_\nu\phi - g_{\mu\nu}\Box\phi) \tag{1.58}$$

It is important to note that general relativity is exactly recovered when the Brans-Dicke parameter tends to infinity, $\omega \to \infty$. This can be seen as $\partial\phi \approx \omega^{-1}$ and so ϕ tends to a constant, hence all modifications to the metric field equation vanish.

It is clear from the field equations that we should expect a different phenomenology than general relativity. Later in this section, we will examine the low energy limit of Brans-Dicke theory showing how it deviates from that of general relativity, in particular, the deflection of light and Shapiro time delay differ. The effect on cosmology having been examined in, for example, [14]. Hawking showed that black-hole solutions in Brans-Dicke theory are the same as in general relativity, [79], that is that the black holes have no hair. It has been conjectured that the no-hair theorem may extend to the radiation emitted by a black-hole binary being as in general relativity, but as we will see in section 4, this isn't true.

The additional scalar field can be introduced through the argument that as a scalar, it is covariant and so the action remains mathematically valid. But one may wonder if this is not already excessive as a function of only the Ricci scalar, $f(R)$, would also be valid and so the action

$$S = \int d^4x\sqrt{-g}f(R)\,, \tag{1.59}$$

should be considered. Such a gravity model is called $f(R)$ gravity. However, any $f(R)$ gravity model is equivalent to a Brans-Dicke theory with a scalar field potential and $\omega = 0$. It should also be noted that in the Palatini formalism, $f(R)$ gravity is still equivalent to a Brans-Dicke theory, but now $\omega = -3/2$, [141, 167]. Interestingly, when $\omega = -1$, Brans-Dicke theory is related to low-energy-effective superstring theory, [32].

While the scalar field can be justified through it being invariant under diffeomorphisms, the exact modification to the action is not unique. Without much thought, one could easily add a generic potential $V(\phi)$ and allow for the Brans-Dicke parameter to be a function, $\omega = \omega(\phi)$, examples of such are the chameleon fields, [95, 96]. Alternatively, one could consider a non-canonical kinetic term, $K(\phi, X)$, such as in k-mouflage gravity, [17]. But both of these modifications can also be done for quintessence theories, albeit as an exotic matter component, so we now look at what modifying gravity can provide. Uniquely, we could also include a generic coupling to the Ricci scalar or introduce a non-minimal conformal coupling to the scalar field in the matter action, or even a disformal coupling, [161]. In doing this, we have drifted away from Brans-Dicke theory and towards the full freedom of the Horndeski action, [83], and beyond-Horndeski theories, [75]. We will delay full discussion of the Horndeski action until chapter 2, but for now, we will leave the remark that it is in some sense the most general scalar-tensor theory.

As the prototypical scalar-tensor theory of modified gravity, the low energy limit is enlightening as it will highlight where differences from general relativity appear. As the low energy expansion of general relativity was performed in detail in section 1.4.3, we only highlight the differences. For a full discussion, see [142, 190].

To begin, there are two field and both need to be expanded,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + h^{(2)}_{00} + h^{(2)}_{ij} + h^{(3)}_{0j} + h^{(4)}_{00}\,, \tag{1.60}$$

$$\phi = \phi_0(1 + \psi^{(2)} + \psi^{(4)})\,, \tag{1.61}$$

where we will see that the superscript denotes the post-Newtonian order. The scalar field perturbations $\psi^{(i)}$ should all decay away from the source as the field takes its cosmological (or galactic) background value, ϕ_0. The background value is taken to be a constant as the size of any spatial and temporal variance is much larger than that of any low-energy, local experiment. Hence, the background field is effectively constant across the region of interest. This is analogous to neglecting any variance in the gravitational potential due to the galaxy in solar system dynamics.

Inserting the expansion into the scalar field equation to leading order we find

$$\overline{\nabla}^2 h^{(2)}_{00} = -8\pi G\phi_0\rho + \overline{\nabla}^2\psi\,, \tag{1.62}$$

$$\overline{\nabla}^2 \psi^{(2)}_{00} = -\frac{8\pi G\phi_0^{-1}}{3+2\omega}\rho\,, \tag{1.63}$$

which, for a coupled system of equations, can be trivially solved to find

$$h^{(2)}_{00} = 2G\phi_0^{-1}\frac{4+2\omega}{3+2\omega}U\,, \tag{1.64}$$

$$\phi^{(2)} = \frac{2G\phi_0^{-1}}{3+2\omega}U. \tag{1.65}$$

We do this explicitly to show the prefactor to the Newtonian potential in $h^{(2)}_{00}$. As Newtonian gravity is found from the 00 component of the metric, the additional factors may seem a problem. However, the bare gravitational coupling G introduced in the action (1.56) is not the same gravitational coupling introduced in Newtonian gravity (in contrast, the gravitational coupling in general relativity is the same as Newton's constant). Rather, we identify the observed Newtons constant, G_{obs} with the prefactors of $2U$,

$$G_{obs} = \frac{G}{\phi_0}\frac{4+2\omega}{3+2\omega}. \tag{1.66}$$

This identification is a step needed by most modified gravity theories to identify the Newtonian limit, they must recover $h_{00}^{(2)} = 2G_{obs}U$.

We end by stating the solution for $h_{ij}^{(2)}$,

$$h_{ij}^{(2)} = \frac{1+\omega}{2+\omega}2G_{obs}U\,. \tag{1.67}$$

The full expansion up to order $(v/c)^4$ follows a smiler path than that of general relativity, with the main differences being highlighted. That $h_{ij}^{(2)}$ contains a different prefactor than $h_{00}^{(2)}$ has an effect on lensing, which is dependent on the ratio and difference between these two metric components.

1.6.2 Screening

There was much theoretical interest shortly after the discovery of general relativity due to its new interpretation of the physical world. However, beyond the classical tests of general relativity, many of the phenomena predicted, such as black holes and gravitational waves, where considered too extreme to be found and rather were of mathematical interest only. With the wealth of alternative theories developed, interest in precise measurements of gravity started to increase in order to test whether general relativity was the correct theory of gravity, [189]. As we have seen in the case of Brans-Dicke theory, many of the initial solar system tests of gravity, such as the deflection of light, are predicted too, but with differing values.

We have examined the low energy limit of gravity in section 1.4.3, and it is needless to say that general relativity passes all tests that have been made of this expansion, also see section 1.7. Of particular interest to this body of work are solar system tests of gravity. Due to the immediate access to the solar system, very highly accurate tests have been made, [191]. This is a death sentence to many theories of gravity as in order to satisfy them, they become indistinguishable from general relativity on all scales and hence theoretically unfavored. For example, tests have placed the constraint $\omega > 40,000$ for the Brans-Dicke parameter, and as discussed in section 1.6.1, $\omega \to \infty$ recovers general relativity.

The constraint that general relativity is all but recovered in the solar system can be satisfied through the use of "screening mechanisms". These are non-linear effects that occur in regions of large ambient matter density. By this, we mean densities consistent with the solar system or the Milky Way galaxy, but not such that there is no modification on intergalactic scales, or for some mechanisms even within dwarf galaxies, [39]. The actual calculation of the static low energy limit in screened modified gravity theories is mathematically challenging and will be the main topic of chapter 3.

In scalar-tensor theories, screening mechanisms can be split into two categories:

- Those that depend on the scalar-field value such as the chameleon mechanism [96] or symmetron mechanism [80].
- Those that depend on derivative self-interactions of the scalar field such as k-moflage models [17] and the Vainshtein mechanism, [176], which occurs in galileon models [139].

Well will look in detail at galileon models as they are important to the space of Horndeski models, and at the chameleon mechanism, as it naturally occurs in $f(R)$ gravity theories. However, the existence of such interactions do not in and of themselves give rise to a limit where the low energy expansion of general relativity is recovered. What they do provide is a method to decouple the large-scale theory of gravity from that of solar system scales. In chapter 2, we will examine those theories of Horndeski type that have a screening mechanism and recover general relativity when screening dominates.

1.6.3 Galileon gravity

The study of galileon gravity grew out of string theory demonstrating the connection between gravity and high-energy particle physics. In [64], Dvali et al. considered a 3-brane embedded in a 5D flat Minkowski spacetime, called the Dvali-Gabadadze-Porrati (DGP) model. This model contains a fifth force which contributes a potential on the brane which goes as $1/r$ around a source, and $1/r^2$ beyond some cut of scale r_0. The existence of this scale allows for the theory to have a much smaller effect than Newtonian gravity near a body while being comparable far away. This model is of particular interest as it allows for self-acceleration, [58]. As the main drive for modifications to gravity come from cosmology, this is a huge boon for the model. However, the perturbations around this solution are unstable, [77, 100].

It can be shown that this 5D theory is equivalent to a massive 4D gravity, [53]. A problem with massive gravity is that when taking the mass of the gravitational wave to zero, one does not recover the predictions of general relativity in the linear approximation, this is known as the vDVZ discontinuity. The discontinuity has the effect of causing theories to not pass solar system tests, in particular, the measurement of the perihelion of Mercury [54, 177] due to the existence of additional forces. This can be seen as a result of massive theories carrying more degrees of freedom than the equivalent massless theory, and the additional degrees not vanishing in the massless limit, so continuing to carry forces. This reason is analogous to the case of the weak force; the masses of the gauge fields are caused by the Higgs mechanism, while the Higgs field is needed to absorb the massive degree of freedom in the unbroken phase.

A. Vainshtein suggested that the linear approximation was not appropriate for the study of the massless limit of massive gravity due to the introduction of divergences in higher order corrections. Importantly, these divergences happened for radii smaller than what is now called the Vainshtein radius, while the linear theory remains valid at large distances. Moreover, when the full non-linear theory was considered for a spherical solution, there was no discontinuity, [176]. This led to the idea that non-linear interactions can allow for modifications to gravity at large distances but to recover general relativity near sources due to extra degrees of freedom becoming "kinetically heavy" preventing their propagation, [16]. It is this mechanism that allows for the DGP model to be suppressed below r_0 but not beyond.

Galileon gravity theories are scalar-tensor theories which contain non-linear self-interactions in order to induce a Vainshtein mechanism, [139]. This is as they are generalizations of the effective theory of DGP gravity modifying general relativity in the infrared limit. The reason for the name is that when considered on flat space, an interesting symmetry arises. The action of the galileon scalar field is invariant under shift of its amplitude and of its derivative, $\phi(x) \rightarrow \phi(x)+a+b_\mu x^\mu$. The name arises due to the similarity to Galilean invariance of coordinates.

Imposing that the scalar field exists in four flat space-time dimensions, has only second order derivatives in the equations of motion and satisfies the galileon symmetry reduces the space of all possible field theories to four lagrangians [139], denoted $\mathcal{L}_i$. Each $\mathcal{L}_i$ includes self interactions of order i, such as $\mathcal{L}_2 = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi$, where ϕ is is the galileon scalar field. This additional galileon symmetry these scalar fields exhibit is responsible for a non-renormalisation theorem [125], stating that the coefficients of the theory are stable under quantum corrections.

The simplest term with screening is the so-called cubic galileon,

$$\mathcal{L}_3 = -\frac{1}{2}\Box\phi\partial_\mu\phi\partial^\mu\phi. \tag{1.68}$$

This highlights the unexpected nature of these models, despite enforcing that the equations of motion are at most second order, the Lagrangian contains a non-linear term with a second derivative of the field. This is surprising as such a term would intuitively give rise to third order derivatives in the field equations and hence Ostrogradski instabilities, [193]. This is a common feature of cubic, quartic and quintic galileon lagrangians.

However, generalizing the four Galilean Lagrangians to curved space is non-trivial. If one follows the usual procedure of minimally coupling the classical action to the metric, one finds higher order derivatives in the equations of motion and so reintroduces Ostrogradski instabilities. This can be averted however by introducing non-minimal couplings between the metric and the galileon field, leading to the covariant gelileons [59].

1.6.4 Chameleon gravity

The chameleon mechanism is an example of a screening mechanism that depends on the value the scalar field. In particular, the presence of a matter density drives the scalar field to take a value that suppresses its dynamics. The mechanism was developed by J. Khory and A. Weltman in [95, 96].

The idea of the chameleon mechanism is that the scalar field has an effective mass which is not constant across spacetime, but rather is a function of the matter density. The chameleon mechanism is often expressed in the Einstein frame, where the metric in the matter action includes a conformal factor $A(\phi)$ for ϕ the chameleon field, while the metric part of the action is just the Einstein-Hilbert action. This direct coupling to the matter density gives rise to an effective potential in the chameleon's field equation which is both a function of the field value and matter density. Through a suitable choice of the scalar potential and conformal factor, the effective mass m_{eff} then grows in regions of high density, Yukawa suppressing the scalar field.

The chameleon mechanism also arises in $f(R)$ gravity. Examples which exhibit the chameleon mechanism are those in [84], which consider a class of broken power law models. We further discuss this paper and give a criticism of the use of the Cassini mission constraint in section 3.5.

For a brief explanation of the chameleon mechanism, consider the flat space action

$$S = \int d^4x \partial_\mu \phi \partial^\mu \phi + V(\phi) + A(\phi)\rho \,. \tag{1.69}$$

Here we model the non-minimal coupling of the scalar field to the matter action through the conformal factor as $A(\phi)\rho$. Then the scalar field equation is easily found to be

$$\Box\phi - (V_{,\phi} + A_{,\phi}\rho) = 0 \,. \tag{1.70}$$

We see the effective potential is a function of ϕ.

Assuming the scalar field rests at a minima of the effective potential with value $\overline{\phi}$, then the effective mass of the scalar can be found to be

$$m^2_{eff}(\overline{\phi}) = V_{,\phi\phi}(\overline{\phi}) + A_{,\phi\phi}(\overline{\phi})\rho \,. \tag{1.71}$$

For the effective potential to have a minima, we have to impose the constraints:

- $A(\phi)$ is monotonically increasing.
- $V(\phi)$ is monotonically decreasing.
- m^2_{eff} is positive

Following [96], we pick the potential to be an inverse power law, $V(\phi) = M^{4+n}\phi^{-n}$ and the conformal factor $A(\phi) = exp(\beta\phi/M_p) \approx 1 + \beta\phi/M_p$. Note that in the approximation, the expansion is valid as the field should not vary beyond orders of the Planck mass. The effective mass can be directly found to be

$$m_{eff} = n(n+1)M^{-\frac{4+n}{1+n}} \left(\frac{\beta\rho}{nM_p}\right)^{\frac{n+2}{n+1}} \,. \tag{1.72}$$

The effective mass satisfies the screening properties we want, the mass increases as the ambient density increases, and hence the scalar field experiences a large

Yukawa suppression in regions of high density.

Using that the scalar field rests in the minimum of its effective potential, the full solution can be found. Consider a massive body or radius R embedded in a much lower cosmic mass density. Due to the high density, the effective mass is very large so the scalar field is approximately constant within the body due to the large energy needed to excite it, $\phi \approx \phi_{obj}$ for $r < R$. Far from the mass, the field lies in the cosmic background effective minima. Near the massive body, the field changes from the local minima to the cosmic minima, where the field will Yukawa suppressed by the cosmic effective mass.

However, there will be a discontinuity of the scalar fields gradient at the boundary [91, 154] if the mass contrast is large enough, leading to what is known as a thin-shell effect. This is where only a thin shell of mass under the surface of the body contributes to the extra scalar force. The deep interior is heavily Yukawa suppressed, so the external solution does not receive a contribution from these locations. The existence of the thin shell is what gives rise to the screening effect.

Because the field is continuous, at the thin shell there is a gradient as the field changes minima which costs energy. The scalar field aims to minimise the energy cost of its profile, and so the thin shell will only form provided the energy gain from lying in the bottom of the effective potential across a screened region counteracts the loss of changing minima. This has the effect of the interior value being dependent upon the value of the chameleon field in its surroundings. This gradient also has the effect of changing the force profile of the chameleon field, which could potentially be detected, [119].

1.7 The parameterised post-Newtonian formalism

In the post-Newtonian expansion of general relativity, performed in section 1.4.3, we found the metric perturbations around flat space up to $(v/c)^4$. In doing so we have introduced new potentials that solve a range of Poisson-like equations. The prefactors to these potentials are theory dependent, as can be seen in the expansion of Brans-Dicke theory, cf. (1.40) and (1.67), or [142] for higher orders.

A parameterisation of the metric expansion would allow for a theory-independent method of testing gravity. The parameterised post-Newtonian formalism provides such an expansion, built upon several assumptions. First, the metric can be split into a scalar part g^{00}, a vector part g^{0i} and a tensor part g^{ij}, which transform under Galilean group. As the metric expansion is for a compact system in the Newtonian limit, we also assert that as the distance from the system tends to infinity the metric tends to the Minkowski metric, $\eta^{\mu\nu}$. Moreover, a coordinate system is used such that $\eta^{\mu\nu}$ takes its usual diagonal form. The expansion takes the form of the Minkowski metric with the metric components written in terms of several potentials.

That the Minkowski metric is recovered implies that all potentials which appear in the expansion vanish far from the source. None of the potentials should depend upon the distance to a arbitrary origin, $\overline{x}$, but rather they should depend on the distance to a source, $\overline{x}'$. As we assert the validity Galilean group, the $\overline{x}$ dependence of potentials will only appear in the combination $\overline{x} - \overline{x}'$. Moreover, for the potentials to transform under the Galilean group correctly, they must be constructed from variables which transform appropriately, e.g. $\overline{x}^i - \overline{x}'^i$, v^i, ∂^i together with tensor products and contractions of these quantities. Finally, there is a standard gauge condition such that g^{ij} is diagonal and isotropic and simplifies g^{00} at order $(v/c)^4$. It was found that the metric expansion of many modified gravity theories satisfied these conditions, and form the basis of this formalism.

The PPN metric is

$$\begin{aligned} g_{00} =& -1 + 2GU - 2\beta G^2U^2 - 2\xi G\Phi_W + (2\gamma + 2 + \alpha_3 + \zeta_1 - 2\xi)G\Phi_1 \\ &+ 2(3\gamma - 2\beta + 1 + \zeta_2 + \xi)G^2\Phi_2 + 2(1 + \zeta_2)G\Phi_3 + 2(\gamma + 3\zeta_4 - 2\xi)G\Phi_4 \\ &- (\zeta_1 - 2\xi)G\mathcal{A} - (\alpha_1 - \alpha_2 - \alpha_3)Gw^2U - \alpha_2 w_i w_j GU_{ij} + (2\alpha_3 - \alpha_1)w^i GV_i, \end{aligned} \tag{1.73a}$$

$$\begin{aligned} g_{0i} =& -\frac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1 - 2\xi)GV_i - \frac{1}{2}(1 + \alpha_2 - \zeta_1 + 2\xi)GW_i \\ &- \frac{1}{2}(\alpha_1 - 2\alpha_2)Gw_iU - \alpha_2 w_j GU_{ij}, \end{aligned} \tag{1.73b}$$

$$g_{ij} = (1 + 2\gamma GU)\delta_{ij}, \tag{1.73c}$$

where the PPN parameters are γ, β, ξ, α_i and ζ_i, and w^i is the velocity of the system with respect to a universal rest frame. For a definition of the potentials used in eqs. (1.73a) to (1.73c), we refer to ref. [190]. There are several potentials that did not appear in the PPN expansion of GR, such as $\mathcal{A}$. These potentials are found, for example, in the expansion of vector-tensor or bimetric theories, [190].

Should any new potentials be discovered in the post-Newtonian expansion of a modified gravity theory, they can easily be added to the PPN formalism through inclusion in the metric expansion (1.73) together with a new parameter.

The PPN parameters from various theories are presented in [190, 191]. For example, one finds the values $\gamma = \beta = 1$ with all other parameters vanishing in GR, while in Brans-Dicke theory, $\gamma = \frac{1+\omega}{2+\omega}$, $\beta = 1$ and all other parameters are zero. One of the strengths of the PPN formalism is that it directly relates physical effects to each parameter (see, for example, table 3.1). The measurement of the PPN parameters can be compared to predictions of a given theory and used to constrain them, such as with the change in the Shapiro time delay (section 3.5). It should be noted, however, that traditionally these parameters are constants, which differs from the results presented in sections 3.3.3 and 3.4 due to the presence of screening effects.

The first tests of relativity where the classical tests of General Relativity, gravitational redshift, the excess precession of Mercuries' perihelion, and light deflection about the sun. However, it can be argued that gravitational redshift is not a true test of general relativity, but rather of the equivalence principle, [190], so we will not say more about it. But these three tests did much to cement general relativity as the most accurate description of gravitational phenomena in the universe.

But as shown, other relativistic theories of gravity have since been developed and so we now turn to experimental tests meant to discriminate between different models of gravity. One of the strictest constraints is that by the Cassini mission, a radio echo experiment testing the Shapiro time delay as opposed to the deflection angle, [24], placing the constraint $|\gamma - 1| < 10^{-6}$. Other solar system tests place constraints on the remaining PPN parameters, [137, 191], such as tests of the equivalence principle [13], and high precision numerical simulation of the orbits of planets, [70]. So far, all constraints are consistent with general relativity and spell the death of theories that cannot mimic General Relativity in the solar system, hence the interest in screening mechanism.

While the traditional tests of gravity all take place in the solar system, our understanding of cosmology has exploded in the time since general relativity was developed (unsurprising as the FRW metric is a solution to Einsteins Equations). However, this allows for cosmological tests of gravity to be made too. The CMB and the growth of structure together allow for strong tests of gravity, such as in

[21, 115, 164]. But we will not consider cosmology further.

A new avenue of testing gravity recently opened with the detection of gravitational waves, [3]. This will be the focus of the next section.

1.8 Gravitational waves in General Relativity

When examining the low energy static limit in section 1.4.3, we linearised Einstein's field equations and used Euler's equation to argue that time derivatives of the metric are of an order v/c smaller than spacial derivatives. This is because the objects of interest in the Newtonian limit are solar-system bodies, where the velocities are significantly smaller than the speed of light. In this section, we undo the static limit. The wealth of new phenomena this gives rise to are analogous to the step from electrostatics to electrodynamics, an analogy that will occur throughout the section. One discovery of electrodynamics was that of electromagnetic waves, and as we will see, undoing the static limit gives rise to gravitational waves.

However, unlike electromagnetic waves, gravitational waves are small perturbations around a background metric. This is as electrodynamics is a linear theory and so the amplitude of the wave does not change the physics of the wave, while GR is non-linear and so the wave is affected by the energy it carries changing the space it moves through. We will see that gravitational waves are massless and have two polarisations and gravitational waves are sourced by the mass quadrupole. This is in contrast to electromagnetic waves being sourced by dipoles and is entirely due to the gravitational field being a rank 2 tensor, while the electromagnetic field is a vector.

We will consider perturbations $h_{\mu\nu}$ on a flat Minkowski background,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{1.74}$$

We consider a flat background for simplicity, but one can consider perturbations about a curved background which brings all the subtleties of curved space with it, [134].

Linearising Einstein's tensor with respect to $h_{\mu\nu}$, one finds

$$G_{\mu\nu} = \partial^\alpha \partial_{(\mu} h_{\nu)\alpha} - \frac{1}{2}\partial^\alpha \partial_\alpha h_{\mu\nu} - \frac{1}{2}\partial_\mu \partial_\nu h^\alpha_\alpha - \frac{1}{2}\eta_{\mu\nu}(\partial^\alpha \partial^\beta h_{\alpha\beta} - \partial^\alpha \partial_\alpha h^\beta_\beta). \quad (1.75)$$

This seems complicated when compared to Maxwell's wave equation, but we can already see the wave operator on the right hand side.

A field redefinition simplifies the calculation and so we define,

$$\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h^\alpha_\alpha \eta_{\mu\nu}. \quad (1.76)$$

In [134], $\overline{h}_{\mu\nu}$ is identified with the gravitational field as opposed to a metric perturbation. Einstein's tensor becomes

$$G_{\mu\nu} = -\frac{1}{2}\partial^\alpha \partial_\alpha \overline{h}_{\mu\nu} + \partial^\alpha \partial_{(\mu} \overline{h}_{\nu)\alpha} - \frac{1}{2}\eta_{\mu\nu}\partial^\alpha \partial^\beta \overline{h}_{\alpha\beta}. \quad (1.77)$$

We can then impose the gauge condition

$$\partial^\alpha \overline{h}_{\alpha\beta} = 0, \quad (1.78)$$

analogous to the Lorentz gauge. This is often called the Harmonic gauge or the De Donder gauge. But as in electrodynamics, this does not uniquely specify the gauge and we can use residual freedoms to make the trace vanish and $\overline{h}_{0\beta} = 0$

Using the Harmonic gauge, Einstein's linearised field equations reach their final form,

$$\partial^\alpha \partial_\alpha \overline{h}_{\mu\nu} = -16\pi G T_{\mu\nu}. \quad (1.79)$$

For the rest of this section, we will write the flat space d'Alembertian operator as $\partial^\alpha \partial_\alpha = \Box$.

As our analysis takes place on flat space, we are in the regime of special relativity. As such, the gravitational field $\overline{h}_{\alpha\beta}$ should be a representation of the Poincaré group. As one may expect, it indeed is, the gravitational field takes the form of a spin two field as described by Fierz and Pauli [12, 55].

To find the gravitational wave solution to the wave equation, we first examine the vacuum solution, $T = 0$. As we expect a wave, consider the gravitational field as a single wave

$$\overline{h}_{\mu\nu} = H_{\mu\nu} exp(ik_\alpha x^\alpha). \quad (1.80)$$

In the wave solution we have defined several objects, the wave vector k_α, scalar amplitude $\mathcal{H} = (\frac{1}{2}H^*_{\mu\nu}H^{\mu\nu})^{\frac{1}{2}}$ and wave vector $e_{\mu\nu} = H_{\mu\nu}/\mathcal{H}$.

We have placed several conditions on the gravitational field: that it is transverse,

$$k^\mu e_{\mu\nu} = 0, \tag{1.81}$$

that the trace vanishes,

$$H^\mu_\mu = 0, \tag{1.82}$$

and that the gravitational field is spatial,

$$H_{0\nu} = 0. \tag{1.83}$$

While this appears to be 9 constraints, that the gravitational field is spatial satisfies one of the transverse requirements, and so of the 10 components of the tensor $H_{\mu\nu}$, there are two independent degrees of freedom corresponding to the cross and plus polarisations.

But what of the generation of gravitational waves sourced by a non-zero stress-energy tensor? The process of solving the wave equation (1.79) again takes the same route as in electrodynamics. The retarded solution for the wave equation immediately gives

$$\overline{h}_{ij} = 4\int \frac{T_{ij}(x')}{|\overline{x} - \overline{x}'|} dS(x'), \tag{1.84}$$

where we integrate over the past light cone enforced through $dS(x') = \theta(-t + |\overline{x} - \overline{x}'|)d^3x'dt$. One may wonder if this satisfies the gauge condition (1.78), but it is automatically satisfied by the linear conservation equation for the stress-energy tensor, $\partial^\mu T_{\mu\nu} = 0$.

Taking the Fourier transform with respect to time of the wave equation, (1.79), one can solve a Poisson equation to find

$$\hat{\overline{h}}_{ij} = 4\int \frac{\hat{T}_{ij}}{|\overline{x} - \overline{x}'|} exp(i\omega|\overline{x} - \overline{x}'|)d^3x. \tag{1.85}$$

let the source to be far from the detector, $R = |\overline{x}| \gg \max|\overline{x}'|$, which also implies R is much greater than the wavelength of the gravitational waves. Thus we can simplify the integral to

$$\hat{\overline{h}}_{ij} = 4\frac{exp(i\omega R)}{R}\int \hat{T}_{ij}d^3x. \tag{1.86}$$

Through repeated integrations by parts, use of Gauss's theorem and the conservation of stress energy, the solution becomes

$$\hat{\bar{h}}_{ij} = 4\frac{exp(i\omega R)}{R}\int \hat{T}_{00}x^i x^j d^3x, \tag{1.87}$$

where the integral is just the quadrapole moment, q_{ij}. Finally, the inverting the Fourier transform gives the solution

$$\bar{h}_{ij} = \frac{2}{3R}\frac{d^2 q_{ij}}{dt^2}\bigg|_{ret}. \tag{1.88}$$

Note that the gravitational wave is sourced by the quadrapole and not the dipole as in electromagnetism.

Thus we have the solution for the linearised Einstein's equations showing that radiation is emitted from a dynamical gravitational system with a quadrupole. This radiation carries with it angular momentum and energy. For a binary system, the emission of angular momentum circularises the orbit, while the energy loss causes the system in inspiral. Such an inspiral was first detected for the binary pulsar PSR B1913+16, [85] which is in close agreement with the predictions of general relativity, [187, 188]. In fact, the constraints on the quadrupole approximation from the binary pulsar are much greater than they are from gravitational wave detection, [171], due to the long observation time.

This provides a strong test of gravity at leading order, but higher orders in the expansion are needed for the detection of gravitational waves. From a practical point of view, this increases the signal to noise in the data analysis of the LIGO project. Moreover, to break degeneracies between intrinsic parameters of the theory, higher order perturbations are needed.

However, from the point of view of testing general relativity, the difference in field equations caused by modifications to gravity will give rise to changes in the energy emitted, the decay of the orbit and the gravitational waveform. For the theories we consider, this will be through how the wave interacts with itself, with the background and with any additional gravitational fields present in the theory. This analysis will be performed in chapter 4. There are also other changes induced by modifying gravity, such as introducing a graviton mass, [53], or propagation speed, [122, 157, 162].

1.8.1 Detection

The binary pulsar PSR B1913+16, [85] provided strong evidence for the existence of gravitational waves through the orbital decay being consistent with the predicted energy loss due to the emission of gravitational waves. However, the first direct detection of gravitational waves was made by the Laser Interferometric Gravitational Wave Observatory (LIGO) project in September 2015 in the in Hanford and Livingston detectors, [3]. The signal GW150914 matched gravitational wave templates for a binary black hole merger, also providing direct evidence for the existence of black holes. Five other mergers of black holes have been detected[2, 5–7]. However, the merger of black hole binaries so far have not produced an electromagnetic counterpart. Traditionally, they were not expected to produce such a counterpart. However, the coincident signal detected by the Fermi satellite for GW150914 motivated the construction of models where a electromagnetic counterpart is produced, [48, 112].

A possible electromagnetic counterpart has a large impact on modified gravity. This is as the cosmological distances involved place tight constraints on any deviations between the speed of light and gravitational waves, should both signal be detected on a simultaneously [122]. This reduces the space of viable theories which can also be responsible for late time acceleration. Such a counterpart was found with the detection of a binary Neutron star merger, [8], constraining various gravity models using the effective equality between the speed of light and gravity, [10, 22, 50, 89].

As the LIGO detector is more sensitive to changes in frequencies than changes in amplitudes, [171] used the TIGER framework, [109], to place constraints on deviations in the waveform's phase during the inspiral, merger and ringdown. Unsurprisingly, general relativity is consistent with this analysis. This provides a strong test of modified gravity theories provided the gravitational waveform for them can be found. With the wealth of new data soon to be available, finding the gravitational waveform in screened modified gravity theories will be the subject of chapter 4.

As we enter the era of gravitational wave astronomy, new tests of gravity are opening for the first time. Moreover, the detections will only become more numerous and cover a wider range of wavelengths as further ground-based experiments begin such as the Kamioka Gravitational wave detector, Advanced

Virgo and LIGO India, [9, 166, 175], as well as space-based detectors such as the Evolved Laser Interferometer Space Antenna [11].

1.9 Outline of Chapters 2 - 5

The main body of the thesis represents the research undertaken by the author.

In chapter 2, I develop a new scaling method for finding the screened and unscreened limits of gravity theories and apply the method to the Horndeski action, obtaining conditions on the four free functions of the Horndeski action which when satisfied imply the existence of a limit where general relativity is recovered. The research presented in this chapter was performed under the supervision of Dr. Jorge Penarrubia and Dr. Lucas Lombriser, and was published in the Journal of Cosmology and Astroparticle Physics, [127].

Chapter 3 further develops the scaling method to include an expansion around the screened and unscreened limits. The expansion is used to find the post-screening effects on the metric for the cubic galileon and chameleon models. The PPN formalism is expanded to incorporate the new metric components and we show that existing tests have to be treated with subtlety when constraining the new modifications. The research presented in this chapter was performed under the supervision of Dr. Jorge Penarrubia and Dr. Lucas Lombriser, and was published in the Journal of Cosmology and Astroparticle Physics, [128].

Chapter 4 examines the generation of gravitational waves in screened theories after the scaling method is applied. This is done to 2PN orders beyond the quadrupole contribution. With the modified waveform, an analysis of an idealised LIGO signal is performed to place constraints upon the size of the modification. This work represents ongoing research and was performed under the supervision of Dr. Jorge Penarrubia and Dr. Lucas Lombriser with valuable suggestions from Dr. Jonathan Gair.

Finally, in chapter 5, the conclusions of this body of work are presented together with potential directions for future work.

Chapter 2

Finding Horndeski theories with Einstein gravity limits

This chapter was published in [127]. The research contained within was performed by the author and was also the lead author of the paper under the supervision of Dr Jorge Penarrubia and Dr Lucas Lombriser.

2.1 Introduction

The late-time accelerated expansion of our Universe has been confirmed by a wealth of observational evidence from the measured distances of type Ia supernovae [148, 158] to measurements of the secondary anisotropies of the cosmic microwave background [151]. The simplest explanation for the effect is the contribution of a positive cosmological constant Λ to the Einstein field equations. The cosmological constant is a crucial constituent of the standard model of cosmology, Λ Cold Dark Matter (ΛCDM), where it dominates the present energy budget of the Universe. One would expect the Planck mass M_p to set a natural scale for Λ as it defines the relevant scale for matter interactions in the field equations, but the measured cosmological constant, when expressed in terms of M_p, is inexplicably small, $\Lambda_{\rm obs} \approx (10^{-30} M_p)^4$ [159]. This calls into question whether Einstein's Theory of General Relativity (GR) with the observed small value of $\Lambda_{\rm obs}$ is a fundamental description of gravity. One may hope for a theoretical justification for $\Lambda_{\rm obs}$ from quantum-field-theory, but calculations from Standard-Model vacuum diagrams put the expected value in the order of $\Lambda_{\text{theory}} \approx (10^{-15} M_{\text{pl}})^4$ by assuming an ultra-violet cut-off at the scales probed with the Large Hadron Collider [185]. The difference of 60 orders of magnitude begs for an explanation.

To overcome such issues, much work has been performed to expand the ΛCDM model through modifying gravity directly instead of trying to solve the problem within the standard particle and cosmological paradigms. This has often been done through the introduction of new degrees of freedom. Postulating the presence of an extra degree of freedom is not a new concept in cosmology and is, for instance, done to facilitate inflation in the early universe. The most common addition is the introduction of a minimally coupled scalar field with an appropriate potential. The discovery of the Standard-Model Higgs particle has seemingly confirmed the existence of fundamental scalar fields [43]. Furthermore, quantum corrections to the Einstein-Hilbert action give rise to an effective field theory containing terms such as R^2, where R is the Ricci scalar; this can be shown to be equivalent to adding a non-minimally coupled scalar field [141]. In general, one can expect the appearance of new degrees of freedom from such effective field theory considerations because of Lovelock's theorem. These corrections often cause non-minimal couplings of gravity to these new degrees of freedom, giving rise to a wide variety of gravity models. As such, it seems that gravity must have a more accurate description than Einstein's theory at high energies but still well below the Planck scale, so that a quantum theory of gravity is not needed and we can use classical fields.

The easiest modification is to consider the addition of a non-minimally coupled scalar field in the Einstein-Hilbert action, adding only one more degree of freedom. In general, one may also want to consider higher-derivative actions of the scalar field beyond first derivatives such as in kinetic terms. The instinctive problem with a higher-derivative theory is that they can contain Ostrogradsky instabilities, ghost degrees of freedom due to third or higher time derivatives in the equations of motion. To avoid it, one may require the equations of motion to be of at most second order in derivatives. This consideration leads to the Horndeski action [83], describing the most general four-dimensional, local, second-derivative theory of gravity with the only gravitational degrees of freedom being the metric and the scalar field [60, 98]. Higher-derivative theories arise, for instance, as effective scalar-tensor description in the decoupling limit of braneworld scenarios [64] or massive gravity [56] (also see Refs. [92, 139]). Healthy theories beyond the

Horndeski action may also be formulated [75, 196], where the equations of motion contain third-order time derivatives but the existence of hidden constraints prevents the appearance of ghost degrees of freedom [107].

Despite the theoretical justifications to expect a modification of GR in the high-energy, strong-gravity regime, so far no observations are inconsistent with the theory. Moreover, precision measurements in the Solar System put very tight constraints on potential remnant infrared (IR) deviations [191]. A good example of this is the measured value for γ_{PPN}, the ratio of the time–time and space–space components of the metric in a low-energy static expansion, with a constraint of $|\gamma_{PPN} - 1| \lesssim 10^{-5}$ set by the Cassini mission [191]. From this, one may infer that any modification to gravity in the IR is too small to account for deviations at large scales causing effects like late-time acceleration.

However, there exist screening mechanisms such that modifications to gravity are naturally suppressed in regions of high ambient mass density such as the Solar System or galaxies on larger scales, separating the IR regimes (see Ref. [92] for a review). As such, the local experiments of gravity set the strength of screening required but do not immediately place tight constraints on gravity on large scales. This motivates the use of modified gravity theories in cosmology (for reviews see Refs. [46, 93, 101]). Scalar-tensor modifications have long been considered as a potential alternative explanation to the problem of cosmic acceleration. However, it was recently shown in Refs. [117, 122] that Horndeski theories cannot provide a self-acceleration genuinely different from the contribution of dark energy or a cosmological constant if gravitational waves propagate at the speed of light. Nevertheless, the dark energy field may couple non-minimally to the metric and modify gravity.

The screening mechanisms that can suppress the effect of this coupling in high-density regions fall into two main categories: (i) Screening by the scalar field's local value such as in chameleon [96] or symmetron [80] models. These models have a canonical kinetic term in the Einstein frame and an effective potential of the additional scalar field that depends on environment, making the field static in deep gravitational potential wells. (ii) Derivative screening such as in the Vainshtein mechanism [176] or in k-mouflage models [17], where derivatives of the scalar field dominate the equations of motion. Both classes of screening mechanisms rely on nonlinear terms in the equations of motion. When these are dominant, they cause the effect of the scalar field on the metric to become sub-dominant and hence suppress the impact on the motion of matter (when matter is minimally coupled to the metric, which we shall assume throughout the chapter). Gravitational models that employ these screening mechanisms can reduce to GR in the Solar System, passing the stringent local constraints, while yielding significant modifications of gravity on large, cosmological scales. A further, linear shielding mechanism can additionally cause these large-scale modifications to cancel [121, 122].

Importantly, in screened regions, gravitational modifications from scalar field contributions in the action can still cause higher-order corrections to GR that can be tested against observations. However, a linear expansion of the scalar field cannot describe these corrections as the nonlinear terms that give rise to the screening become should remain small in the series. Hence, higher-order deviations from the expansion of GR in the low-energy static limit should not be used to describe the expansion of a screened theory. To correctly describe the screened regime, a perturbative expansion can be conducted using a dual Lagrangian instead, which can be obtained from the original Lagrangian with a Legendre transformation [73] or using Lagrange multipliers [147]. While the dual Lagrangian describes the same physics, its equations of motion allow for a perturbative series that is valid in the nonlinear regime of the original Lagrangian. These procedures are mathematically involved and may not be suited for an application to the large variety of gravity theories or a generic gravitational action.

In this chapter, we present an alternative, scaling method that finds a perturbative expansion for gravity theories in nonlinear regimes but does not rely upon finding a dual Lagrangian. We demonstrate the operability of this method on a galileon and chameleon model, whereby we easily recover some known results for the Vainshtein and chameleon mechanisms. We then apply the approach to the full Horndeski action and derive a number of conditions on the modifications that guarantee the existence of a limit where Einstein's field equations are recovered. We use these results to examine several known gravity theories and their different limits. Finally, we complement the scaling method with a technique that enables an efficient approximation of the scalar field's radial profile for a symmetric mass distribution, which is used to assess whether the Einstein gravity limit obtained reflects a screening mechanism, where the recovery of GR holds near a massive body but not far from it.

The outline of the chapter is as follows. In section 2.2, we revisit the Horndeski action and cast the equations of motion in a form that is more useful to the application of the scaling method. We introduce the novel method in section 2.3, where we also provide examples using both a galileon and a chameleon model to demonstrate its applicability. In section 2.4, we then use the method to derive the conditions on the free Horndeski functions that ensure the existence of an Einstein gravity limit. We apply our findings to a range of different models to illustrate their screening effects with an approximation of the radial scalar field profile. We close with a discussion of our results and an outlook of their application to observational tests of gravity in section 2.5. For completeness, we provide the Horndeski field equations in the appendix, where we also present an alternative, coordinate-dependent scaling method.

2.2 Horndeski gravity

The effective four-dimensional scalar-tensor theory of a string-theory-inspired braneworld scenario [64] that self-accelerates [58] was observed to contain a second derivative of the scalar field in the action. Naïvely this would yield problematic third time derivatives in the equations of motion, but instead it was found to only yield second time derivatives, therefore avoiding ghosts due to any Ostrogradsky instabilities. It was later noted, however, that the self-accelerated branch of the model is perturbatively not stable (e.g., [77, 100]). Furthermore, the extra gravitational force exerted from the scalar field near massive bodies was shown to be suppressed with respect to the Newtonian force [138]. Another interesting aspect of the effective action is its Galilean symmetry: invariance under the transformation of the field $\phi(x) \to \phi(x) + a + b_\mu x^\mu$ for constants a and b_μ. This motivated the extension of the action to the most general scalar-tensor theories invariant under this symmetry in flat space, dubbed galileon gravity [139]. Note that this symmetry is of relevance to the non-renormalisation theorem [125].

In four dimensions there are five flat-space galileon Lagrangians with three of them containing higher derivatives of the scalar field but still yielding second-order equations of motion invariant under the symmetry. Should one try to naïvely covariantise the flat-space Lagrangians by making the metric dynamical and promoting partial to covariant derivatives, the resulting equations of motion would produce third-order time derivatives. Ref. [59] showed that counterterms consisting of non-minimal couplings between the scalar field and the metric remove the higher derivatives in the equations of motion, but at the expense of explicitly breaking the Galilean symmetry in these actions due to the couplings to gravity. The symmetry breaking caused by these interactions can be considered weak [150] provided that in a decoupling limit, distinct from the metric being Minkowski, Galileon symmetry is recovered. As a result loop corrections have a small impact on the theory and the non-renomalisation properties are preserved to the maximal extent. A construction of the most general scalar-tensor theory with a single scalar field on a curved space-time, not adhering to the Galilean shift symmetry, and with the derivatives in the equations of motion being of second order at most was performed in Ref. [60]. The resulting action was found to be equivalent [98] to the scalar-tensor action derived earlier by Horndeski [83]. Meanwhile, it was shown that the covariantisation of the flat-space galileons, while introducing third derivatives, does not necessarily yield any Ostrogradsky instabilities, which allows to formulate healthy theories beyond Horndeski gravity [75] through evading the non-degeneracy conditions of the Ostrogradsky theorem [107].

We briefly review the equations of motion produced by the Horndeski action, whereby we focus on a strongly condensed form following the notation of Ref. [98]. The full expressions can be found in appendix A.1. We then point out an important pattern that appears in these equations that will become very useful in section 2.4 to reduce the number of terms that need to be considered when exploring limits wherein a theory recovers Einstein gravity.

The Horndeski action, including a minimally coupled matter action $S_m[g]$, is given by

$$\begin{aligned} S_H =& \frac{M_p^2}{2} \int d^4x \sqrt{-g} \Big\{ G_2(\phi, X) - G_3(\phi, X)\Box\phi \\ & + G_4(\phi, X)R + G_{4X}(\phi, X)[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2] \\ & + G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{G_{5X}(\phi, X)}{6}[(\Box\phi)^3 - 3\Box\phi(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3] \Big\} \\ & + S_m[g] \,, \end{aligned} \tag{2.1}$$

where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor, G_2, G_3, G_4 and G_5 are free functions of a unitless scalar field ϕ and $X \equiv -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi$, and subscripts of X or ϕ denote functional derivatives with respect to X or ϕ. Variation of the action with respect to the metric and scalar field yields the equations of motion [98]

$$\sum_{i=2}^{5} \mathcal{G}^{(i)}_{\mu\nu} = \frac{T_{\mu\nu}}{M_p^2}\,, \tag{2.2}$$

$$\sum_{i=2}^{5} \nabla^\mu J^{(i)}_\mu = \sum_{i=2}^{5} P^{(i)}_\phi\,, \tag{2.3}$$

respectively, where $\mathcal{G}^{(i)}$, $J^{(i)}_\mu$, $P^{(i)}_\phi$ are functions of G_i, their derivatives and functional derivatives with respect to ϕ and X (see Ref. [98] or appendix A.1 for $J^{(i)}_\mu$ and $P^{(i)}_\phi$), and $T_{\mu\nu}$ is the stress-energy tensor from the matter action.

Note that $G_4 G_{\mu\nu}$ appears within $\mathcal{G}^{(4)}_{\mu\nu}$ and $-G_{5\phi} X G_{\mu\nu}$ appears in $\mathcal{G}^{(5)}_{\mu\nu}$. We define $\overline{\mathcal{G}}^{(4)}_{\mu\nu} \equiv \mathcal{G}^{(4)}_{\mu\nu} - G_4 G_{\mu\nu}$ and $\overline{\mathcal{G}}^{(5)}_{\mu\nu} \equiv \mathcal{G}^{(5)}_{\mu\nu} + G_{5\phi} X G_{\mu\nu}$ as well as a $\Gamma \equiv G_4 - G_{5\phi}X$. One can then find the trace-reversed form of eq. (2.2) to get the metric field equations

$$\Gamma R_{\mu\nu} = -\sum_{i=2}^{5} R^{(i)}_{\mu\nu} + (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T)/M_p^2\,, \tag{2.4}$$

where for convenience, we have also defined the trace-reversed tensors

$$R^{(i)}_{\mu\nu} \equiv \mathcal{G}^{(i)}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \mathcal{G}^{(i)}_{\alpha\beta}\,,$$
$$R^{(j)}_{\mu\nu} \equiv \overline{\mathcal{G}}^{(j)}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \overline{\mathcal{G}}^{(j)}_{\alpha\beta}\,,$$

for $i = 2, 3$ and $j = 4, 5$. Note that for $R^{(i)}_{\mu\nu} = 0\ \forall i$ and a constant Γ, the metric field equations (2.4) reduce to Einstein's equations.

We wish to source the scalar field equation (2.3) by the matter density. To do this, we select the terms: $RG_{4\phi}$ in $P^{(4)}_\phi$, $-RG_{4X}\Box\phi$ in $\nabla^\mu J^{(4)}_\mu$, $\frac{1}{2} RG_{5X}(\Box\phi)^2$ and $G_{5\phi} R \Box\phi$ both in $\nabla^\mu J^{(5)}_\mu$. These identify all possible contributions where the Ricci scalar enters the scalar field equation. We define $\overline{P^{(i)}_\phi}$ and $\overline{\nabla^\mu J^{(i)}_\mu}$ for $i = 4, 5$ by removing these terms, so that we may write the scalar field equation (2.3) as

$$\left(\sum_{i=2,3} (\nabla^\mu J^{(i)}_\mu - P^{(i)}_\phi) + \sum_{i=4,5} (\overline{\nabla^\mu J^{(i)}_\mu} - \overline{P^{(i)}_\phi}) \right) - R\Xi = 0\,, \tag{2.5}$$

where $\Xi \equiv G_{4\phi} + (G_{4X} - G_{5\phi})\Box\phi - \frac{1}{2} G_{5X} (\Box\phi)^2$. Inserting the trace of eq. (2.4),

	2n+m+1	2n+m	2n+m-1
i even	-	$R^{(i)}_{\mu\nu}$	$P^{(i)}_\phi$, $\nabla^\mu J^{(i)}_\mu$
i odd	$R^{(i)}_{\mu\nu}$	$P^{(i)}_\phi$, $\nabla^\mu J^{(i)}_\mu$	-

Table 2.1 *The number of factors of the form $\partial^s\phi$ that multiply G_i within the functions $R^{(i)}$, $\nabla^\mu J^{(i)}_\mu$ and $P^{(i)}$, where m is the number of functional derivatives with respect to ϕ and n with respect to X that act upon the G_i.*

we rewrite the scalar field equation as

$$\left(\sum_{i=2,3}(\nabla^\mu J^{(i)}_\mu - P^{(i)}_\phi) + \sum_{i=4,5}(\overline{\nabla^\mu J^{(i)}_\mu} - \overline{P^{(i)}_\phi})\right)\Gamma + \Xi\sum_{i=2}^{5} R^{(i)} = -\frac{T}{M_p^2}\Xi\,. \tag{2.6}$$

The equations of motion (2.4) and (2.6) are now in the form that we will use in the rest of the chapter.

There is an important pattern to notice in $R^{(i)}_{\mu\nu}$, $P^{(i)}_\phi$ and $\nabla^\mu J^{(i)}_\mu$ relating the functional derivatives of G_i to their pre-factors of $\partial^s\phi$ for integers s. For any term within one of these objects let m be the number of functional derivatives with respect to ϕ acting on the G_i contained in it. Correspondingly, let n be the number of functional derivatives with respect to X. Consider the example

$$-\frac{1}{2}G_{2X}\nabla_\mu\phi\nabla_\nu\phi \in \mathcal{G}^{(2)}_{\mu\nu} \tag{2.7}$$

with $n = 1$ and $m = 0$. We can see that $2n + m = 2$ derivatives of ϕ appear multiplying G_{2X}. We find that this relation holds for all terms in $R^{(2)}_{\mu\nu}$. Furthermore, we can examine all of the $R^{(i)}_{\mu\nu}$, $P^{(i)}_\phi$ and $\nabla^\mu J^{(i)}_\mu$ and find the relations listed in table 2.1. These observations will become very useful when studying the Einstein gravity limits of Horndeski theory in section 2.4.

It should be noted that the scalar field equation will also contain disformal terms such as $R^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$ which are able to source the scalar field equation through terms such as $T^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$. Such terms have had their screening capabilities investigated in cosmological contexts (e.g., Ref. [161]). However, the most general case of replacing all metric sources in favour of the stress energy tensor cannot be achieved, [25], disformal terms will remain. We do not consider such sources and only consider conformal couplings to the metric which can always be replaced.

Finally, we note that a cosmological propagation speed of gravitational waves at the speed of light places very tight constraints on non-vanishing contributions of

G_{4X} and non-constant G_5 and that with the direct detection of tensor waves [3] a corresponding measurement may soon be realized.

2.3 Scaling to describe nonlinear regimes

Screening mechanisms rely on nonlinear terms in gravitational actions. These mechanisms usually introduce different regimes such that when near massive bodies or in high-density environments, the non-linear terms dominate, screening is active and GR is recovered, thus passing local tests of gravity. When they do not dominate, there is no screening effect and gravity is modified, allowing for deviations on large scales.

Due to this reliance on nonlinear contributions, a linearisation through a low-energy, static perturbative expansion of the equations of motion cannot be applied in the screened regime. Hence in the presence of screening, one cannot use such an expansion to test gravity in the local universe.

To find a perturbative expansion in the nonlinear regime for galileon gravity models, where Vainshtein screening operates, Refs. [73, 147] proposed the use of dual Lagrangians obtained from Laplace transforms or Lagrange multipliers. A dual Lagrangian is physically equivalent to the original Lagrangian but written in terms of auxiliary fields. The benefit of this dual description is that when a low-energy, static expansion is performed, the natural regime which the expansion describes is within the nonlinear, and hence screened, regime of the original Lagrangian. This expansion for the dual breaks down as one approaches a regime where the linear terms of the original Lagrangian dominate. The expansion of the dual is therefore complimentary to the expansion of the original Lagrangian. As a result, these dual methods allow for a comparison between the predictions of modified gravity theories in screened regions and observables in the local universe. Ref. [15] demonstrates how the Lagrange multiplier method can be used to perform a parametrised post-Newtonian expansion [190] for derivatively coupled theories and gives an example of the expansion to second order for the cubic galileon model in the Jordan frame. However, the dual methods become increasingly involved when applied to more complex Horndeski models and the transform is not always obvious. A more concise method, enabling an expansion in the nonlinear, screened regime, should therefore be very useful to facilitate the analysis of deviations from GR in the local region.

In section 2.3.1, we present a new, simpler method enabling this expansion that is based on the scaling of the scalar field within the metric equations of motion and scalar field equation and reproduces the known results from the dual approach of the cubic galileon. We first demonstrate its operability with the explicit example of the cubic galilieon coupled to gravity in section 2.3.2. We then also show how this method applies to screening through a scalar field potential by examining the chameleon model in section 2.3.3, which has eluded these dual methods.

2.3.1 A scaling method

Let us first consider the heuristic form of a scalar field equation

$$\alpha^s F_1(\phi, X) + \alpha^t F_2(\phi, X) = T/M_p^2 \tag{2.8}$$

for free functions $F_{1,2}$, the trace of the stress-energy tensor T of a given matter distribution, and arbitrary real numbers s and t. Let α be an arbitrary coupling constant that controls the scale at which the different terms become important. Now, consider the expansion of the scalar field

$$\phi = \phi_0(1 + \alpha^q \psi)\,, \tag{2.9}$$

where we have separated out a constant part ϕ_0 and a varying part ψ; q is a real number which is determined by the gravitational model under consideration. Note that ψ is not dimensionless as, in general, α may carry dimensions. If α is large in comparison to ψ, then clearly for a sensible expansion, q should be negative, whereas if α is small, $q \geq 0$. This ensures that the second term will be small provided $\psi < \alpha^{-q}$. We will see that the values that q can take are restricted, and this will be the crux of the scaling method we propose here.

Let $F_{1,2}(\phi, X)$ scale homogeneously in α^q with respect to the expansion (2.9), so that we get

$$\alpha^{s+mq} F_1(\psi, (\partial\psi)^2) + \alpha^{t+nq} F_2(\psi, (\partial\psi)^2) = T/M_p^2\,, \tag{2.10}$$

for real numbers m, n. We now have the original equation (2.8) cast as a function of q. Equation (2.10) needs to hold for arbitrary α but the right-hand side is not a function of α, which implies that there must be a term on the left-hand side that is not a function of α either. As a result, q can only take on certain values, namely

$$q \in \left\{ -\frac{s}{m}, -\frac{t}{n} \right\} . \tag{2.11}$$

Next we wish to examine the case where $\alpha^{-q} \gg \psi$ or $\alpha^{-q} \ll \psi$, and so we take the formal limits of $\alpha \to \infty$ or $\alpha \to 0$, respectively. It should be stressed that the physical value for such constants are given when one writes down a specific action, and that being constants, such limits do not involve changing the value for α, rather the scale of α changes with respect to ψ.

In order for these limits to be meaningful, we need to ensure that no terms in eq. (2.10) diverge. For simplicity, we let $-\frac{s}{m} > 0 > -\frac{t}{n}$. So if we consider the case when $\alpha \to \infty$, we need to take the smaller of the two values for q such that all powers of α that appear in eq. (2.10) are less than or equal to zero, preventing any divergences. This leads to the equations

$$\begin{aligned} \alpha^{s+m(-\frac{t}{n})} F_1(\psi, (\partial\psi)^2) + F_2(\psi, (\partial\psi)^2) &= T/M_p^2 , \\ \alpha \to \infty : \quad F_2(\psi, (\partial\psi)^2) &= T/M_p^2 . \end{aligned} \tag{2.12}$$

Conversely, if we let $\alpha \to 0$, then all powers must be positive. Hence, we must take the largest possible value for q. In this case we get

$$\begin{aligned} F_1(\psi, (\partial\psi)^2) + \alpha^{t+n(-\frac{s}{m})} F_2(\psi, (\partial\psi)^2) &= T/M_p^2 , \\ \alpha \to 0 : \quad F_1(\psi, (\partial\psi)^2) &= T/M_p^2 . \end{aligned} \tag{2.13}$$

Hence, we have found two different equations of motion governing the dynamics of ψ in the two different limits. This allows us to perform a simplified perturbative expansion in each of the two limits which is valid in one region but not in the other and breaks down near $\psi \approx \alpha$, where one may want to impose some matching condition. One can easily see that we can continue to add additional functions to eq. (2.8) to extract the dominating terms in the different limits.

Note that in general, the background of the scalar field ϕ_0 could be a function of the coordinates. This would have applications to metric backgrounds such as Friedmann-Robertson-Walker or anti-de Sitter, where considering a constant background field may not be suitable.

Besides the scalar field equation, we also need the equation of motion for the metric to be consistent in these limits. We apply the same expansion in eq. (2.9) to rewrite the metric field equation as a function of $g_{\mu\nu}$ and ψ, making it dependent on q. Upon taking a limit of α, the value of q used in the metric field equation must be the same as that in the scalar field equation as it describes the same field. For a sensible limit, the metric field equation should not diverge in the same limiting process described above.

The prescription given here applies broadly to the equations of motion that appear in different gravity theories. There are, however, two special scenarios that we have to consider in more detail: (i) one term is not a function of α after the expansion in eq. (2.9); and (ii) the power of α is not a function of q. We have insisted that in the limit of concern, q must take a value that ensures a non-vanishing contribution to the left-hand side of the equation of motion (2.10). In the first case, the term independent of α already provides a non-vanishing term in both limits, so we are left with the condition that no terms diverge. Hence, the set of feasible values of q in either limit creates an inequality condition for q, which requires that q is equal to or less (greater) than the minimum (maximum) of the analogous set to eq.(2.11) for $\alpha \to \infty(0)$; we are then free to pick a value that satisfies this condition. However, the consideration of further equations of motion may still provide a requirement for an exact value of q. The second scenario poses a problem when taking either one or the other limits of α. If for instance, we have a contribution of the form α^m with $m > 0$, then this term will diverge in the limit of $\alpha \to \infty$ regardless of the value of q, but not for $\alpha \to 0$, and viceversa for $m < 0$. We can then only solve for ψ in the limit where there are no divergences.

The extension of the scaling method to a full, higher-order expansion $\phi = \phi_0(1 + \sum \alpha^i \psi_i)$ will be presented in separate work. In summary, the prescription for the first-order scaling approach is as follows:

(i) expand the scalar field in the equations of motion according to eq.(2.9): $\phi = \phi_0(1 + \alpha^q \psi)$;

(ii) for a field equation, find all values of q where an exponent of α becomes 0;

(iii) if taking the limit $\alpha \to 0$, q takes at least the maximum value of this set;

(iv) if taking the limit $\alpha \to \infty$, q takes at most the minimum value of this set;

(v) check that the complementary field equations do not diverge with this limit and value of q;

(vi) should no terms diverge and should at least one non-vanishing term exist in all field equations for this q, the resulting equations of motion describe the fields in the corresponding limit.

In order to demonstrate the applicability of the scaling method introduced here for the description of the different screening mechanisms operating in Horndeski theory (see section 2.2), we start by providing two simple examples: we first discuss the application of the scaling method to the derivative screening of the cubic galileon model in section 2.3.2 and then apply it to the scalar field screening in the chameleon model in section 2.3.3. In section 2.4, we then discuss its application to the full Horndeski theory.

2.3.2 Scaling with derivative screening

Derivative terms in a field theory can be approximated by the energy of the system, which in the low-energy limit is generally smaller than their coefficients. Thus, these contributions are generally suppressed and one would not expect terms involving derivatives other than the kinetic term to be relevant for low-energy physics. However, it has been found that derivative terms can give rise to screening mechanisms, caused either by powers of $\partial\phi$ as in kinetic screening like k-mouflage, or by higher-derivative terms of the form $\partial^2\phi$ as in the Vainshtein mechanism.

As a demonstration of the scaling method introduced in section 2.3.1, we now apply it to the cubic galileon model, which employs the Vainshtein screening mechanism. The model is defined by the action

$$S_{cubic} = \frac{M_p^2}{2}\int d^4x\sqrt{-g}\left[\phi R + \frac{2\omega}{\phi}X - \frac{\alpha}{4}\frac{X}{\phi^3}\Box\phi\right] + S_m[g]\,, \tag{2.14}$$

where S_m denotes the minimally coupled matter action, ω is the Brans-Dicke parameter and α is the coupling strength with units $mass^{-2}$. Note that the model is embedded in the Horndeski action, eq. (2.1), which can easily be seen by setting $G_2 = 2\omega\phi^{-1}X$, $G_3 = \alpha\phi^{-3}X/4$, $G_4 = \phi$, and $G_5 = 0$. With this choice of the G_i functions, the metric field equations become

$$\phi R_{\mu\nu} = M_p^{-2}\left[T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right] + \frac{\omega}{\phi}\nabla_\mu\phi\nabla_\nu\phi + \frac{1}{2}\Box\phi g_{\mu\nu} + \nabla_\mu\nabla_\nu\phi + \frac{\alpha}{8}\left[\phi^{-3}\mathcal{M}^{(3)}_{\mu\nu} + \phi^{-4}\mathcal{M}^{(4)}_{\mu\nu}\right]\,, \tag{2.15}$$

where we have introduced the rank-2 tensors

$$\mathcal{M}^{(3)}_{\mu\nu} \equiv -X\Box\phi g_{\mu\nu} - \Box\phi\nabla_\mu\phi\nabla_\nu\phi - \nabla_\mu X\nabla_\nu\phi - \nabla_\mu\phi\nabla_\nu X\,, \tag{2.16}$$

$$\mathcal{M}^{(4)}_{\mu\nu} \equiv 6X\nabla_\mu\phi\nabla_\nu\phi\,. \tag{2.17}$$

The scalar field equation becomes

$$(3+2\omega)\Box\phi + \frac{\alpha}{4}\left[\phi^{-2}\mathcal{S}^{(2)} + \phi^{-3}\mathcal{S}^{(3)} + \phi^{-4}\mathcal{S}^{(4)}\right] = M_p^{-2}T\,, \tag{2.18}$$

where we have defined the scalar quantities

$$\mathcal{S}^{(2)} \equiv -(\Box\phi)^2 - \nabla^\mu\phi\nabla_\mu\Box\phi - \Box X\,, \tag{2.19}$$

$$\mathcal{S}^{(3)} \equiv 5\nabla_\mu\phi\nabla^\mu X - X\Box\phi\,, \tag{2.20}$$

$$\mathcal{S}^{(4)} \equiv 18X^2\,. \tag{2.21}$$

The introduction of $\mathcal{M}$ and $\mathcal{S}$ facilitates the analysis of the equations of motion; the superscripts describe the power to which the scalar field appears inside of them, so that $\mathcal{M}$ and $\mathcal{S}$ are homogeneous polynomials with respect to $\partial^n\phi$ for $n = 1, 2$.

As we want to describe the cases where the interaction terms are dominant or vanishing, the scaling parameter in eq. (2.9) is given by the coupling strength α. This makes the $\mathcal{S}^{(i)}$ and $\mathcal{M}^{(i)}_{\mu\nu}$ become functions of ψ that scale homogeneously with respect to α^q with degree of their superscript. Performing the expansion, terms on the left-hand side of eq. (2.18) become functions of α, while the right-hand side remains a function of the stress-energy tensor only. From eq. (2.18) one can easily identify the exponents of α that appear in the expansion. For example, $\alpha\mathcal{S}^{(2)}(\phi) \to \alpha^{1+2q}\phi_0^2\mathcal{S}^{(2)}(\psi)$ gives the exponent $1+2q$. The remaining exponents from $\mathcal{S}$ are q, $1+3q$, and $1+4q$. In the limits of large or small α, one term on the left-hand side of eq. (2.18) must balance the right-hand side by being independent of α. This puts restrictions on the values of q as at least one of the exponents must be zero, namely,

$$q \in \left\{0, -\frac{1}{4}, -\frac{1}{3}, -\frac{1}{2}\right\} = Q_{\mathcal{S}}\,. \tag{2.22}$$

We expect that the limit of $\alpha \to \infty$ corresponds to the limit where screening dominates and hence where Einstein gravity is recovered. To prevent any terms from diverging in this regime, all powers of α in eq. (2.18) should be less than or equal to zero. This requirement and the restriction that $q \in Q_{\mathcal{S}}$ implies that we must adopt the smallest value in $Q_{\mathcal{S}}$, $q = \min(Q_{\mathcal{S}}) = -\frac{1}{2}$.

Next, we must also find the corresponding exponents of α in the metric field equation, which are

$$\left\{0, -\frac{1}{4}, -\frac{1}{3}\right\} = Q_{\mathcal{M}} \,. \tag{2.23}$$

However, eq. (2.15) contains the term $\phi_0 R_{\mu\nu}$ which is not a function of α, and so we do not require that the value of q must be in $Q_{\mathcal{M}}$. The condition that no terms diverge implies that $q \leq \min(Q_{\mathcal{M}}) = -\frac{1}{3}$. Hence the value $q = -\frac{1}{2}$ is allowed by both field equations.

Adopting this value for q and taking the limit of large α, eq. (2.15) and (2.18) become

$$\phi_0 R_{\mu\nu} = M_p^{-2}\left[T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right] \,, \tag{2.24}$$

$$\frac{1}{4}\mathcal{S}^{(2)}(\psi) = M_p^{-2} T \,, \tag{2.25}$$

where the galileon field scales as

$$\phi = \phi_0(1 + \alpha^{-1/2}\psi) \,. \tag{2.26}$$

One can see that the metric field equation has reduced to Einstein's field equations (when setting $\phi_0 = 1$), indicating a screening effect. Hence, we have recovered the known result of Vainshtein screening in cubic galileon gravity when the self-interaction term dominates, and we have also found the scalar field equation that ψ and thus ϕ satisfies in this limit.

In contrast, for the opposite limit of $\alpha \to 0$, we expect no screening effect. In order to prevent divergences in this limit, the powers of α must be greater than or equal to zero, and thus $q = \max(Q_{\mathcal{S}}) = 0$. The field equations then become

$$\phi R_{\mu\nu} = M_p^{-2}\left[T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right] + \frac{\phi_0^2}{\phi}\omega \nabla_\mu \psi \nabla_\nu \psi + \frac{\phi_0}{2}\Box\psi g_{\mu\nu} + \phi_0 \nabla_\mu \nabla_\nu \psi \,, \tag{2.27}$$

$$\phi_0 \Box\psi = M_p^{-2} T \,, \tag{2.28}$$

and the galileon field scales as

$$\phi = \phi_0(1 + \psi) \,. \tag{2.29}$$

We recognise these relations as the equations of motion of Brans-Dicke gravity in the Jordan frame [34]. Hence, we have recovered the metric and scalar field equations describing the cubic galileon model in the deeply screened and unscreened limits. All steps taken in the process were trivial, demonstrating the simplicity and efficiency of our scaling method.

Strictly speaking, one must also Taylor-expand the negative powers of ϕ that appear in the equation of motion. This contributes extra values of q into the sets (2.22) and (2.23). However, these will come from terms $\alpha^{a+bq+iq}$ with positive integers i, and for simplicity $a, b > 0$, contributing extra q values of $-a/(b+i)$. The minimum of these additional values is assumed when $i = 0$. Hence, adopting this minimum corresponds to only taking the first, constant, term in a Taylor expansion of the negative powers of ϕ, i.e., the power of ϕ_0, and neglecting higher orders. The other relevant value is the maximum, which is $q \to 0$ when $i \to \infty$ and is already included in the sets. We have omitted these terms for simplicity in the calculations above.

2.3.3 Scaling with local field value screening

Besides adding powers of derivatives to an action, one may add a self-interaction potential such as a mass term. In applying the expansion in eq. (2.9), we have so far relied on derivatives such that the constant part vanishes and ψ is separated out with a factor of α to some power in each term. This allowed the direct manipulation of the equations of motion. However, screening with a scalar field potential relies upon the field assuming a particular value that minimises an effective potential of scalar field and matter density. In doing so, the scalar field acquires an effective mass that is dependent of the ambient mass density. We shall consider here the chameleon mechanism, which operates by forcing the field to become more massive in high-density regions, and so satisfies stringent Solar System tests.

To demonstrate how chameleon screening emerges in our framework, we consider the action

$$S_{cham} = \frac{M_p^2}{2} \int d^4x \sqrt{-g} \left[\phi R + \frac{2\omega}{\phi} X - \alpha(\phi - \phi_{min})^n \right] + S_m[g] \,, \tag{2.30}$$

where n is a constant, α describes a coupling constant, and ϕ_{min} denotes the value that minimises the potential (for $n > 0$). The action can be embedded in

Horndeski theory by setting $G_2 = -2\omega\phi^{-1}X - \alpha(\phi - \phi_{min})^n$, $G_3 = 0$, $G_4 = \phi$, and $G_5 = 0$. The equations of motion are

$$(3+2\omega)\Box\phi = M_p^{-2}T + \alpha(\phi - \phi_{min})^{n-1}(2(\phi - \phi_{min}) - n\phi) \equiv V'_{eff}(\phi)\,, \quad (2.31)$$

$$\phi R_{\mu\nu} = M_p^{-2}[T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T] + \frac{\omega}{\phi}\nabla_\mu\phi\nabla_\nu\phi + \frac{1}{2}\Box\phi g_{\mu\nu} + \nabla_\mu\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\alpha(\phi - \phi_{min})^n\,. \quad (2.32)$$

We again use the coupling constant in the expansion of ϕ. However the potential provides powers of α that are dependent on the value of ϕ_0. We consider two cases: (i) $\phi_0 \approx \phi_{min}$; and (ii) $|\phi_0 - \phi_{min}| \gg \phi_0\alpha^q\psi$.

For (i), $\phi_0 - \phi_{min} \approx 0$ approximately minimises the self-interaction potential. This results in $\alpha^q\psi$ becoming the argument of the potential terms and thus dependent on q. We obtain the factors $\alpha^{1+(n-1)q}$ and α^{1+nq} in the scalar field equation (2.31), which together with the powers from the kinetic term implies that

$$q \in \left\{0, -\frac{1}{n}, \frac{1}{1-n}\right\} = Q_{\mathcal{S}}\,. \quad (2.33)$$

The maximum and minimum values in $Q_{\mathcal{S}}$ thus depend on the value of n. For $q = 0$ and $\alpha \to 0$, the scalar field equation describes a free scalar field sourced by the trace of the stress-energy tensor regardless of the value of n.

Consider the case when $q = \frac{1}{1-n} > 0$ and $\alpha \to 0$; the field is no longer dynamical as no gradients appear in the scalar field equation. Rearranging for ψ gives the value which minimises V_{eff} in this limit,

$$\psi = \left(-\frac{M_p^{-2}T}{n\,\phi_0^n}\right)^{\frac{1}{n-1}}\,. \quad (2.34)$$

We observe that for $n < 1$, ψ is suppressed for large $|T|$ but relevant when $|T|$ is small, which is as expected for the chameleon screening mechanism. However, note that $n < 1$ represents the limit of $\alpha \to 0$, not ∞ as in the screened limit for the Vainshtein mechanism (section 2.3.2).

In the metric field equation (2.32), derivatives contribute to the powers of α as q and $2q$ whereas the potential carries the power $1 + nq$. Upon examination, one finds that the $n < 1$ chameleon case requires an additional restriction. In order for the metric field equation to not diverge in the limit of $\alpha \to 0$ with $q = \frac{1}{1-n} > 0$, the term proportional to $\alpha^{1+n/(1-n)}$ must have a positive exponent.

This is true for $n > 0$, reproducing that for chameleon screening the potential must have an exponent $0 < n < 1$ (see, e.g., [116]). One can also see this from $Q_{\mathcal{S}}$ as we examined the case when $\frac{1}{1-n} > -\frac{1}{n}$, which requires that $n \in (0, 1)$.

For (ii), the potential is a power of $(\Delta\phi + \phi_0\alpha^q\psi)$ with $\Delta\phi \equiv \phi_0 - \phi_{min}$. Here, $\Delta\phi$ is the dominant term as by definition $\phi_0\alpha^q\psi$ is a small perturbation, so a Taylor expansion of the potential in ψ then gives

$$\alpha((\Delta\phi)^n + n(\Delta\phi)^{n-1}\alpha^q\phi_0\psi + ...)\,. \tag{2.35}$$

From this we see that the relevant set of values that q must take for the equation of motion (2.31) to have a term that goes as α^0 is

$$\left\{0, -\frac{1}{i}\right\} \tag{2.36}$$

for positive integers i.

However, we now have a power of α that is not a function of q. As such, we cannot take the $\alpha \to \infty$ limit without causing a divergence in the field equations. We are left to consider the limit of $\alpha \to 0$, where the maximum of the set in eq. (2.36) implies $q = 0$. In this limit the scalar field equation becomes one of a free field sourced by the stress-energy tensor and the metric field equation becomes that of Brans-Dicke theory.

2.4 Einstein limits in Horndeski gravity

A viable theory of gravity must have the capability of reducing to Einstein gravity in the Solar System, so the Horndeski-type actions of interest must provide such a limit. In this section, we describe a novel method that allows an efficient assessment of whether an action can assume an Einstein limit or not. This is achieved by examining the powers of α, our limiting parameter, that appear in the action and employ the scaling procedure introduced in section 2.3. Insisting that the metric field equations become Einstein's equations in a given limit and that the related scalar field equation does not diverge amounts to a set of two inequalities on the value of q, the exponent of α in the expansion (2.9). To find these conditions, we examine the form of the equations of motion in Horndeski gravity written in terms of $\alpha^q\psi$. Recurring patterns in these equations, identified in table 2.1, allow a construction of the sets of all powers of α that the field equations contain. The inequalities check for consistency between the extrema of these sets and allow us to determine whether the gravity theory of concern possesses a limit where the Einstein field equations are recovered.

We emphasise, however, that the consistency of the Einstein limit alone does not guarantee that Einstein gravity is recovered due to the operation of a screening effect. We demonstrate this with examples of known screened and non-screened gravity theories which possess an Einstein gravity limit. As we will show, one can, however, assess whether the recovery of Einstein gravity can be attributed to a screening effect or not by assuming a radial profile of the scalar field and examining the range of validity of the limit adopted.

There have been previous attempts to capture the actions of the Horndeski type that give rise to Vainshtein screening on a Minkowski background such as in Refs. [94, 102] or on a FRW background as in Ref. [97], which has also been examined for chameleon screening in the Horndeski action [35]. A benefit of our method is that we incorporate both screening mechanisms simultaneously. Moreover, we do not assume a specified background metric initially and instead find conditions that can be checked for any background which is a solution to Einstein's field equations. A difference, however, is in our definition of screening; we find that to leading order Einstein's equations are recovered, and hence, in a low-energy limit, so is Newtonian gravity. This is a stronger condition than what is required in these papers that only examine if Newton's equations are recovered, which enables them to capture effects we cannot in our formalism as currently presented.

In section 2.4.1 we outline the expansion of the Horndeski functions G_i adopted and set up the tools needed to identify the embedded Einstein limits. Section 2.4.2 focuses on finding all powers of α that can appear in the field equations given the free G_i and hence determine the conditions on q. We then use these conditions in section 2.4.3 to find the limits of Einstein gravity. Finally, we close with the discussion on how to assess whether these can be attributed to a screening effect in section 2.4.4.

Note that while our analysis is performed on the Horndeski action, one could also extend it to beyond-Horndeski theories [75, 196].

2.4.1 Expansion of G_i functions and implications

In order to apply the scaling method to Horndeski gravity, we first need to find a sensible description of the four generic G_i function in the action (2.1). We adopt an expansion of the form

$$F(\phi, X) = \sum_{(m,n)\in I} \alpha^{p_{mn}} M_p^{-2m} X^m \prod_{\phi_i \in \mathcal{P}_{mn}} (\phi - \phi_i)^{p_{\phi_i}} , \tag{2.37}$$

which embeds the galileon, chameleon, and k-mouflage actions. Hereby, I denotes a set of indices and the $\alpha^{p_{mn}}$ are coefficients that determine when the terms they multiply become important. Further, $\mathcal{P}_{mn}$ are sets of constants ϕ_i which only appear at most once per set and p_{ϕ_i} indicates the corresponding exponent of the scalar field potential for this ϕ_i. Note that we do not consider different parameters to describe the couplings, e.g., a combination of α and β; this is because these parameters are constants with the particular relationship between them set by the action, hence, limits in these couplings are taken simultaneously. Additional powers of M_p are needed in the expansion of G_i as unlike F, these are not unitless.

In our scaling method, we only need to consider the powers of α that appear in the expansion of F with respect to eq. (2.9), so we define $\alpha[\cdot]$ to be the set of powers of α which prefactor all terms in the equations of motion after performing the expansion. More specifically, we have

$$\alpha[F] = \bigcup_{(m,n)\in I} \left\{ \alpha^{p_{mn} + 2mq + \sum_{\phi_i \in \mathcal{P}_{mn}} p_{\phi_i} q \delta(\phi_0 - \phi_i)} \right\} , \tag{2.38}$$

where δ denotes the Kronecker δ-function, evaluating to unity when the argument is zero and vanishing otherwise. In addition, several functional derivatives of F need to be known to describe the equations of motion. Again we only need to consider the powers of α that can appear. Applying $\alpha[\cdot]$, we get

$$\alpha[F_X] = \bigcup_{(m,n)\in I} \left\{ m\alpha^{p_{mn}+2(m-1)q+\sum_{\phi_i\in\mathcal{P}_{mn}} p_{\phi_i} q\delta(\phi_0-\phi_i)} \right\}, \tag{2.39}$$

$$\alpha[F_\phi] = \bigcup_{(m,n)\in I} \bigcup_{\phi_i\in\mathcal{P}_{mn}} \left\{ p_{\phi_i}\alpha^{p_{mn}+2mq+\sum_{\phi_j\in\mathcal{P}_{mn}} q(p_{\phi_j}-\delta(\phi_j-\phi_i))\delta(\phi_0-\phi_j)} \right\}, \tag{2.40}$$

$$\alpha[F_{XX}] = \bigcup_{(m,n)\in I} \left\{ m(m-1)\alpha^{p_{mn}+2(m-2)q+\sum_{\phi_i\in\mathcal{P}_{mn}} p_{\phi_i} q\delta(\phi_0-\phi_i)} \right\}, \tag{2.41}$$

$$\alpha[F_{X\phi}] = \bigcup_{(m,n)\in I} \bigcup_{\phi_i\in\mathcal{P}_{mn}} \left\{ mp_{\phi_i}\alpha^{p_{mn}+2(m-1)q+\sum_{\phi_j\in\mathcal{P}_{mn}} q(p_{\phi_j}-\delta(\phi_j-\phi_i))\delta(\phi_0-\phi_j)} \right\}. \tag{2.42}$$

For our discussion, we shall define the multiplication of a set by α^s as the multiplication of all elements of the set by α^s, which yields a new set.

To find an Einstein gravity limit for a general Horndeski theory, we apply $\alpha[\cdot]$ to the equations of motion to extract a set of values of q a given gravity theory can assume to prevent divergences. The minima and maxima of this set then determine whether the theory possesses an Einstein gravity limit. To do this, we separate the equations of motion into suitable sub-components. For instance, consider a single collection of terms from the metric field equation (2.4) with the simplest case of $\alpha[R^{(2)}_{\mu\nu}]$. Using the expansion of $R^{(2)}_{\mu\nu}$ given in appendix A.1, we find that the powers of α arise from combinations of G_2 and G_{2X}. Using the relations in table 2.1 we can thus write

$$\alpha[R^{(2)}_{\mu\nu}] \subset \alpha^{2q}\alpha[G_{2X}] \cup \alpha[G_2] = \alpha[G_2]\,. \tag{2.43}$$

The reverse inclusion is the subject of the next section. This shows that not all functional derivatives in the equations of motion need to be considered to determine q since, in general, the values found from a functional derivative of F are not independent of those found from F itself. We can summarise this conclusion as

$$\alpha[F] \supset \alpha^{2q}\alpha[F_X] \supset \alpha^{4q}\alpha[F_{XX}] \supset \alpha^{6q}\alpha[F_{XX}]\,, \tag{2.44}$$

$$\alpha[F_\phi] \supset \alpha^{2q}\alpha[F_{X\phi}] \supset \alpha^{4q}\alpha[F_{XX\phi}]\,, \tag{2.45}$$

$$\alpha[F_{\phi\phi}] \supset \alpha^{2q}\alpha[F_{X\phi\phi}]\,. \tag{2.46}$$

No further derivatives appear in the Horndeski equations of motion and are thus not required to determine q. Hence, we only need to consider the functional derivatives which are highest on these chains.

Unlike with $\alpha^{2q}\alpha[F_X]$, we generally do not have $\alpha^q\alpha[F_\phi] \subset \alpha[F]$ as can be demonstrated with the counterexample $F = X^m(\phi - \phi_{min})^n$. The relevant set of powers then are $\alpha^q\alpha[F_\phi] = \{\alpha^{(2m+1+(n-1)\delta(\phi_0-\phi_{min})q}\}$ in one case and $\alpha[F] = \{\alpha^{(2m+n\delta(\phi_0-\phi_{min}))q}\}$ in the other. In the limit of $\phi_0 = \phi_{min}$ these two sets are equivalent: $\{\alpha^{(2m+n)q}\}$. However, when $\phi_0 \neq \phi_{min}$, the Kronecker δ-functions vanish such that the two sets become $\alpha^q\alpha[F_\phi] = \{\alpha^{(2m+1)q}\}$ and $\alpha[F] = \{\alpha^{2qm}\}$, which differs in general. This implies that we must have multiple inclusion chains for the functional derivatives with respect to ϕ.

2.4.2 Screening conditions

We now discuss the powers of α that appear in the Horndeski equations of motion. We first focus on the metric field equation in section 2.4.2 and then on the scalar field equation in section 2.4.2. Next, we utilise the identification of those powers to directly infer a set of conditions on the four free function in the Horndeski action that they must satisfy in order to provide a limit where Einstein's field equations are recovered. We summarise those in section 2.4.2. For clarity of the discussion, as there are two field equations, we define q_{metric} and q_{scalar} to be the values of q dictated by the metric and scalar field equation, respectively. But because there is just one scalar field, we ultimately require that $q_{metric} = q_{scalar}$.

Metric field equation

For the metric field equations (2.4) to be considered screened, we require Einstein's field equations to be recovered up to a rescaled, effective Planck mass, which should be obtained after applying the expansion (2.9) and taking one of the limits in α. Examining eq. (2.4), we see that this corresponds to the requirement that $\Gamma \to \epsilon$ and $\sum_{i=2}^{5} R^{(i)}_{\mu\nu} \to 0$ for some constant ϵ.

Recall that $\Gamma = G_4 - XG_{5X}$. There are two scenarios for which $\Gamma \to \epsilon$: (i) when Γ contains a constant term with all other terms vanishing upon taking the limit; and (ii) when Γ contains a potential term of the form $(\phi_{min} - \phi)^n$ which is not minimised such that the term $(\phi_{min} - \phi_0)^n$ remains in the equations after taking the limit. Both scenarios recover Einstein's equations up to an effective Planck mass and can be expressed as

$$\Gamma \rightarrow \sum_{n} \alpha^0 \prod_{\phi_i \in \mathcal{P}_{mn}} (\phi_0 - \phi_i)^{p_{\phi_i}} = \epsilon < \infty \,, \tag{2.47}$$

where n is an integer index and p_{ϕ_i} denotes an exponent in the potential. Note that this term is independent of α before the expansion (2.9) and hence is the leading term when performing the expansion. We shall denote the value of the maximum or minimum q found from $\alpha[\Gamma]$ as q_Γ. The requirement that only terms which scale as α^0 do not vanish when taking a limit becomes an inequality on the value of q_{metric}. Choosing any value for q_{metric} beyond q_Γ ensures that all terms which are functions of q_{metric} will go to zero, and as such we are only left with $\Gamma \rightarrow \epsilon$. Thus we must have $q_{metric} < q_\Gamma$ for $\alpha \rightarrow \infty$ (or $q_{metric} > q_\Gamma$ for $\alpha \rightarrow 0$). The set of coefficients we need to consider to check these conditions is

$$\alpha[\Gamma] = \alpha[G_4 - XG_{5X}]. \tag{2.48}$$

One must remove all terms which go as α^0 from this set in order to find the minimum or the maximum value for q_Γ.

The requirement that $\sum_{i=2}^{5} R^{(i)}_{\mu\nu} \rightarrow 0$ removes the contribution of an effective stress-energy component attributed to the scalar field from the metric field equation. In principle, one could also allow for this limit to tend to a cosmological constant, which, however, we assume to be negligible in regions where screening operates. Further, this implies that we insist that no terms in this sum scales as α^0 after our expansion.

Our aim is to use the chains of inclusion, eqs. (2.44)-(2.46), to reduce the number of terms we need to consider in the equations of motion (2.4) and (2.6). A collection of such chains exists for each of the free functions G_i. By examining the highest sets in those chains, we can identify all powers of α that appear in the metric and scalar field equations. These highest sets will also contain degenerate terms, which further reduces the number of terms that we need to consider. By degenerate we mean that the resulting powers of α found from examining these particular terms are the same (see table 2.1).

However, we are not interested in each $\alpha[G_i]$ individually, rather in the powers of α that arise in the full equations of motion. For the metric field equation specifically, we are interested in

$$\mathcal{M} \equiv \alpha[\sum_{i=2}^{5} R_{\mu\nu}^{(i)}] \,. \tag{2.49}$$

The minimum or maximum values of q in $\mathcal{M}$, which we shall denote as $q_{\mathcal{M}}$, will allow us to directly assess if the choice of gravitational action has an Einstein gravity limit or not. In specific, we demand that the value of q from the metric field equation satisfies $q_{metric} < q_{\mathcal{M}}$ in the $\alpha \to \infty$ limit (or $q_{metric} > q_{\mathcal{M}}$ when $\alpha \to 0$).

We now determine the relation between $\mathcal{M}$ and the highest elements of the chains in eq. (2.44)-(2.46) so that our analysis can be simplified through the arguments outlined above. This is non-trivial as while it is clear that $\mathcal{M}$ is included in the union of these sets, e.g., as in eq. (2.43), terms from different $R_{\mu\nu}^{(i)}$ may mutually cancel out when taking the sum. Hence, it is not a priori clear whether this inclusion can be reversed; all we know is that $\mathcal{M} \subset \bigcup_{i=2}^{5} \alpha[\, R_{\mu\nu}^{(i)}\,]$. Without the reversal, we cannot be sure that the powers of α^q we are using will remain in the sum, even though should they exist within any $R_{\mu\nu}^{(i)}$ individually.

To proceed, we draw the analogy to a vector space over the free functions G_i and their functional derivatives. The basis vectors are multiples of $\Box\phi$, $\nabla_\mu\nabla_\nu\phi$, the Ricci scalar, the Ricci tensor, and the Riemann tensor; $\sum_{i=2}^{5} R_{\mu\nu}^{(i)}$ is then an element of this space. Our method for showing equality between the sets is then to identify terms with coefficients that are highest in the chains of G_i and linearly independent such that they cannot vanish unless they are already set to zero to begin with. In doing so, any power of α that appears in a linearly independent term will not vanish in the sum. Finding such terms for all the highest sets in our chains of each G_i will show that every power in the union appears in the sum, and therefore in $\mathcal{M}$, showing the reverse inclusion.

In the contribution of $R_{\mu\nu}^{(5)}$ to the metric field equations, one can see that the only terms with basis vectors that do not contribute to $R_{\mu\nu}^{(i\neq 5)}$ are

$$G_{5X} R^{\alpha\beta} \nabla_\alpha\phi \nabla_\beta \nabla_{(\mu}\phi \nabla_{\nu)}\phi \,, \tag{2.50}$$

$$G_{5\phi} R_{\alpha(\mu\nu)\beta} \nabla^\alpha\phi \nabla^\beta\phi \,, \tag{2.51}$$

which cannot vanish unless $G_{5X} = 0$ or $G_{5\phi} = 0$, respectively. Thus we have that

$\mathcal{M} \supset \alpha^{3q}\alpha[G_{5X}] \cup \alpha^{2q}\alpha[G_{5\phi}]$. Similarly for $R^{(4)}_{\mu\nu}$, we find that the terms

$$G_{4X} R \nabla_\mu \phi \nabla_\nu \phi \,, \tag{2.52}$$

$$G_{4\phi} \nabla_\mu \nabla_\nu \phi \tag{2.53}$$

are linearly independent from the rest of the summands in $\sum R^{(i)}_{\mu\nu}$ such that the exponents of α are given by $\mathcal{M} \supset \alpha^{q}\alpha[G_{4\phi}] \cup \alpha^{2q}\alpha[G_{4X}]$. In $R^{(3)}_{\mu\nu}$, we can isolate the term

$$G_{3X} \nabla_{(\mu} X \nabla_{\nu)} \phi \,. \tag{2.54}$$

Thus, the functions G_{5X}, $G_{5\phi}$, G_{4X}, $G_{4\phi}$, and G_{3X} are all sole coefficient of a linearly-independent vector in the operator vector space. However, the remaining terms in $R^{(i)}_{\mu\nu}$ do not appear alone in independent terms: G_{2X}, G_2, $G_{3\phi}$, $G_{4\phi\phi}$, and $G_{5\phi\phi}$ cannot be considered in isolation. Hence, the powers of α we would get by including them individually may not be present upon taking the sum due to possible mutual cancellations.

Let us first examine $G_{5\phi\phi}$ and consider, for instance, the terms proportional to (or in direction of) $\nabla_\mu \phi \nabla_\nu \phi \Box \phi$, which are

$$\nabla_\mu \phi \nabla_\nu \phi \Box \phi \left\{ \frac{1}{2} G_{5\phi\phi} - 2G_{4X\phi} + \frac{1}{2} G_{3X} \right\} . \tag{2.55}$$

If any term in $\frac{1}{2}G_{5\phi\phi}$ within the combination (2.55) is cancelled, this implies that the same term must also appear in $-2G_{4X\phi} + \frac{1}{2}G_{3X}$. As can be seen from eqs. (2.52) and (2.54), $\alpha[G_{4X\phi}]$ and $\alpha[G_{3X}]$ are contained in coefficients of independent terms found in $R^{(4)}_{\mu\nu}$ and $R^{(3)}_{\mu\nu}$, respectively. Therefore, any term that could be canceled in $\frac{1}{2}G_{5\phi\phi}$ must still contribute to $\mathcal{M}$ through these independent terms and we can simply include the whole set $\alpha[G_{5\phi\phi}]$ in $\mathcal{M}$. Thus we have that $\mathcal{M} \supset \alpha^{3q}\alpha[G_{5X}] \cup \alpha^{2q}\alpha[G_{5\phi}] \cup \alpha^{3q}\alpha[G_{5\phi\phi}] \supset \alpha[R^{(5)}_{\mu\nu}]$.

Now consider the remaining non-independent functions in $R^{(i)}_{\mu\nu}$, i.e., G_{2X}, G_2, $G_{3\phi}$, and $G_{4\phi\phi}$. There are two equations that mix the four:

$$\nabla_\mu \phi \nabla_\nu \phi (G_{3\phi} - \frac{1}{2} G_{2X} - G_{4\phi\phi}) \,, \tag{2.56}$$

$$- \frac{1}{2} g_{\mu\nu} (X G_{2X} - G_2 - 2X G_{4\phi\phi}) \,. \tag{2.57}$$

Hence, it is possible that terms in G_{2X}, G_2, $G_{3\phi}$, and $G_{4\phi\phi}$ cancel in their contribution to $R^{(i)}_{\mu\nu}$, which if they do should be excluded when determining q.

For the general scenario and for simplicity, we therefore impose that no terms contained in the four functions G_{2X}, G_2, $G_{3\phi}$, and $G_{4\phi\phi}$, nor any combination thereof, cancel in their contribution to eqs. (2.56) and (2.57). Note, however, that one can avoid this condition if dealing with each choice of these functions individually. Moreover, as we are only interested in the terms that provide the largest and smallest values of q in these sets, in principle, it is only those terms that need to be non-vanishing. But to simplify the analysis done here, we are including the set of all possible q and so more restrictively insist that no terms shall vanish.

When this condition applies, we can include G_2, $G_{3\phi}$, and $G_{4\phi\phi}$ in $\mathcal{M}$. With these final sets, we have reversed the inclusion and shown $\mathcal{M} \supset \bigcup \alpha[R^{(i)}_{\mu\nu}]$ as we have the highest elements that appear in the chains, eqs. (2.44)-(2.46), being included in $\mathcal{M}$. In this case the full expression for $\mathcal{M}$ becomes

$$\begin{aligned}\mathcal{M} =&\alpha[\sum_{i=2}^{4} R^{u}_{\mu\nu}]\\ =&\alpha[G_2] \cup \alpha^q \alpha[G_{2\phi}] \cup \alpha^q \alpha[G_{4\phi}] \cup \alpha^{2q} \alpha[G_{4\phi\phi}] \cup \alpha^{2q} \alpha[G_{4X}]\\ &\cup \alpha^{3q} \alpha[G_{5\phi\phi}] \bigcup_{i=3,5} (\alpha^{2q} \alpha[G_{i\phi}] \cup \alpha^{3q} \alpha[G_{iX}]).\end{aligned} \tag{2.58}$$

With all powers of α that can appear in $\mathcal{M}$ accounted for, $q_{\mathcal{M}}$ can easily be extracted for specified G_i functions when initially defining the action. For the term $\sum R^{(i)}_{\mu\nu}$ to vanish, as required to recover Einstein's field equations, there must be no terms independent of α (i.e., α^0) in eq. (2.58) and we must have $q_{metric} < q_{\mathcal{M}}$ in the limit $\alpha \to \infty$, or $q_{metric} > q_{\mathcal{M}}$ for $\alpha \to 0$. These inequalities prevent divergences in the equation of motion. Further, in order for $\Gamma \to \epsilon$, we must have $q_{metric} < q_\Gamma$ $(\alpha \to \infty)$ or $q_{metric} > q_\Gamma$ $(\alpha \to 0)$ as explained above.

Scalar field equation

In the scalar field equation (2.6), P_ϕ, $J^{(i)}_\mu$, and $R^{(i)}_{\mu\nu}$ are functions of ϕ. The stress-energy tensor is also multiplied by the function $\Xi(\phi)$, which complicates the analysis since the contribution may disappear from the equation when taking a limit. This allows for the possibility of the scalar field to be sourced by self-interactions rather than T. Even should the metric field equation reduce to Einstein's equations in such a limit, in order to attribute this to a screening effect, we shall require that the matter density should be the source of the scalar field equation. This implies that the right-hand side of eq. (2.6) must not vanish (or diverge) when taking the limit $\alpha \to \infty$ (or 0), hence,

$$\Xi \not\to 0\,. \tag{2.59}$$

Moreover, with a non-vanishing term on the right-hand side of eq. (2.6), at least one term on the left must also remain to balance it. The relevant set for these conditions is

$$\begin{aligned}\alpha[\Xi] =&\alpha[G_{4\phi} + (G_{4X} - G_{5\phi})\Box\phi - \frac{1}{2}G_{5X}(\Box\phi)^2] \\ =&\alpha[G_{4\phi}] \cup \alpha^q\alpha[G_{4X} - G_{5\phi}] \cup \alpha^{2q}\alpha[G_{5X}]\end{aligned} \tag{2.60}$$

In the heuristic examples given in section 2.3.1, the first requirement was satisfied by letting a term in eq. (2.59) scale as α^0 (such as $G_4 = \phi$) so it would not vanish when taking one of the limits. This meant that we only had to examine the left-hand side of the equation. For simplicity, let us in the following only consider the limit of $\alpha \to \infty$. Analogous results, however, also apply for $\alpha \to 0$. As both sides are now functions of α^q, one must also consider the right-hand side and find the smallest value of q for which eq. (2.59) holds. We shall denote it as q_Ξ. More specifically, for (2.59), the value of q_{scalar} required in the scalar field equations is $q_{scalar} \leq q_\Xi$, where we have equality when eq. (2.59) contains no terms that scale as α^0 akin to Γ. Next, the left-hand side of eq. (2.6) must be checked to ensure that the equation is balanced.

We first consider the second series of terms on the left-hand side of the scalar field equation (2.6), which is

$$(\sum_{i=2}^{5} R^{(i)})\Xi\,. \tag{2.61}$$

As we have required $\sum_{i=2}^5 R^{(i)}_{\mu\nu} \to 0$ for screening in the metric field equation as well as that Ξ does not diverge, eq. (2.61) vanishes when taking the limit. We are left with the first term in (2.6),

$$\left(\sum_{i=2,3}(\nabla^\mu J^{(i)}_\mu - P^{(i)}_\phi) + \sum_{i=4,5}(\overline{\nabla^\mu J^{(i)}_\mu} - \overline{P^{(i)}_\phi})\right)\Gamma = -\frac{T}{M_p^2}\Xi\,. \tag{2.62}$$

Analogous to the set $\mathcal{M}$ used in the metric field equation, let us now define the set of α powers

$$\mathcal{S} \equiv \alpha[\sum_{i=2,3}(\nabla^\mu J_\mu^{(i)} - P_\phi^{(i)}) + \sum_{i=4,5}(\overline{\nabla^\mu J_\mu^{(i)}} - \overline{P_\phi^{(i)}})]\,. \tag{2.63}$$

There are fewer unique terms in the scalar field equation than what we encountered in the metric field equation; the only one being

$$G_{5X}R_{\alpha\mu\beta\nu}\nabla^\mu\nabla^\nu\phi\nabla^\alpha\nabla^\beta\phi\,. \tag{2.64}$$

When isolating the highest terms in the chains defined in eqs. (2.44)-(2.46), we again insist that specific combinations contain no terms that can cancel, ensuring all possible terms remain in the final field equations. More specifically, for the remaining terms we impose for generality that

$$(-2G_{5\phi\phi} + 2G_{4\phi X})R_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi\,, \tag{2.65}$$
$$(G_{3X} - 2G_{4\phi X})(\Box\phi)^2\,, \tag{2.66}$$
$$(2G_{3X\phi} - 2G_{4\phi\phi X})\nabla^\mu X\nabla_\mu\phi\,, \tag{2.67}$$
$$(2G_{3\phi} - G_{2X})\Box\phi\,, \tag{2.68}$$
$$-4G_{3\phi\phi}X - G_{2\phi} \tag{2.69}$$

contain no terms that cancel. When satisfied, the set of all relevant powers of α in the scalar field equation becomes

$$\begin{aligned}\mathcal{S} =& \alpha[G_{2\phi}\Gamma] \cup \alpha^q\alpha[G_{2X}\Gamma] \cup \alpha^{2q}\alpha[G_{4\phi X}\Gamma] \cup \alpha^q\alpha[G_{4X}\Gamma] \\ & \bigcup_{i=3,5}(\alpha^{2q}\alpha[G_{iX}\Gamma] \cup \alpha^q\alpha[G_{i\phi}\Gamma])\,,\end{aligned} \tag{2.70}$$

which allows us to determine the values $q_\mathcal{S}$ takes from the minimum or maximum of $\mathcal{S}$. Then the value of q in the limit of $\alpha \to \infty$ must obey $q_{scalar} \leq q_\mathcal{S}$, where again we have equality when there are no terms independent of α. This inequality then ensures that the left-hand side of eq. (2.6) has at least one term that does not vanish.

Einstein gravity limit

We have formulated the requirements on the metric and scalar field equations for an Einstein gravity limit to exist, which translate directly onto conditions on G_i

that must be satisfied. This determines the powers q_{metric} and q_{scalar} that we must adopt in the scaling equation (2.9). In order for the limit to be self-consistent, we further require that $q_{metric} = q_{scalar}$.

The metric field equation puts the only strict inequality on the value of q_{metric} and to recover Einstein's equations, we insisted that (see section 2.4.2)

$$\begin{aligned} \sum R^{(i)}_{\mu\nu} \to 0 \implies q_{metric} < q_{\mathcal{M}} \,, \\ \Gamma \to \epsilon = const. \implies q_{metric} < q_{\Gamma} \,. \end{aligned}$$

Complimentary to those conditions, for the scalar field equation we required that (see section 2.4.2)

$$\begin{aligned} \Xi \not\to 0 \implies q_{scalar} \leq q_{\Xi} \,, \\ \left(\sum_{i=2,3} (\nabla^{\mu} J^{(i)}_{\mu} - P^{(i)}_{\phi}) + \sum_{i=4,5} (\overline{\nabla^{\mu} J^{(i)}_{\mu}} - \overline{P^{(i)}_{\phi}}) \right) \Gamma \not\to 0 \implies q_{scalar} \leq q_{\mathcal{S}} \,, \end{aligned}$$

where the first condition guaranteed that the scalar field is sourced only by the trace of the stress-energy tensor and the second condition ensured that the right-hand side of the scalar field equation (2.6) is balanced by a contribution on the left.

In summary, for a self-consistent Einstein gravity limit, the gravitational model must satisfy the conditions:

$$q_{metric} < q_{\mathcal{M}} \,,\; q_{\Gamma} \,, \tag{2.71}$$

$$q_{scalar} \leq q_{\mathcal{S}} \,,\; q_{\Xi} \,, \tag{2.72}$$

$$q_{metric} = q_{scalar} \,. \tag{2.73}$$

Recall that these conditions apply for the limit of $\alpha \to \infty$ and the inequalities flip when taking the limit of $\alpha \to 0$ instead. These conditions can easily be checked for a given gravitational model, and in the next section we provide a few examples. In the case of having no terms independent of α in the equations of motion, the inequalities become

$$q_{metric} < q_{\mathcal{M}} \,, q_{\Gamma} \,, \tag{2.74}$$

$$q_{scalar} = q_{\Xi} = q_{\mathcal{S}} \,, \tag{2.75}$$

$$q_{metric} = q_{scalar} \,. \tag{2.76}$$

2.4.3 Examples

Let us examine a few example Lagrangians and apply the procedure laid out in sections 2.4.2 through 2.4.2 to determine whether they contain a self-consistent Einstein gravity limit. We start by re-examining the cubic galileon and chameleon models discussed in sections 2.3.2 and 2.3.3, respectively.

Applying eqs. (2.58) and (2.48) to the cubic galileon action (2.14), we can directly identify the sets of α that are relevant in the the metric field equation,

$$\mathcal{M} = \alpha[2\omega\phi^{-1}X] \cup \alpha^q\alpha[-2\omega\phi^{-2}X] \cup \alpha^q\alpha[1] \cup \alpha^{2q}\alpha[3\alpha\phi^{-4}X/4] \cup \alpha^{3q}\alpha[\alpha\phi^{-3}/4]\,, \tag{2.77}$$

$$\alpha[\Gamma] = \alpha[\phi] = \alpha[\alpha^0\phi_0 + \alpha^q\phi_0\psi]\,. \tag{2.78}$$

We consider the limit of $\alpha \to \infty$, for which from condition (2.74), we find

$$q_\Gamma \in \{0\}\,, \quad q_{\mathcal{M}} = \min\left\{0, -\frac{1}{4}, -\frac{1}{3}\right\} \implies q_{metric} < -\frac{1}{3}\,. \tag{2.79}$$

From the applying eqs. (2.70) and (2.60) to the action, one finds the relevant sets of α in the scalar field equation as

$$\Xi = \alpha[1]\,, \tag{2.80}$$

$$\mathcal{S} = \alpha[-2\omega\phi^{-1}X] \cup \alpha^q\alpha[2\omega] \cup \alpha^{2q}\alpha[\alpha\phi^{-2}/4] \cup \alpha^q\alpha[-3\alpha\phi^{-3}X]\,. \tag{2.81}$$

The condition (2.75) implies that

$$q_\Xi \in \mathbb{R}\,, \quad q_{\mathcal{S}} = \min\left\{-\frac{1}{2}, -\frac{1}{3}\right\} \implies q_{scalar} = -\frac{1}{2}\,, \tag{2.82}$$

where q_Ξ is undetermined as there are no powers of α in Ξ. Finally we must check for consistency between the two equations of motion, condition (2.76),

$$q_{scalar} = q_{metric} = -\frac{1}{2} < -\frac{1}{3}\,, \tag{2.83}$$

which shows that the theory contains a consistent Einstein gravity limit when $\alpha \to \infty$. Note that we directly arrive at this conclusion from analysing the free functions G_i in the action only without the need of considering the galileon equations of motion directly.

In the case of the chameleon action (2.30), we let $\phi_0 = \phi_{min}$ in the expansion (2.9) which minimises the scalar field potential for $n > 0$. For the limit $\alpha \to 0$, eqs. (2.58), (2.48) and the conditions (2.74) imply that

$$q_\Gamma \in \{0\} , \quad q_\mathcal{M} = \max\left\{0, -\frac{1}{n}\right\} \implies q_{metric} > 0 \,. \tag{2.84}$$

Examination of eqs. (2.70), (2.60) and condition (2.75) yields

$$q_\Xi \in \mathbb{R} , \quad q_\mathcal{S} = \max\left\{-\frac{1}{n-1}, 0\right\} \implies q_{scalar} = -\frac{1}{n-1} \tag{2.85}$$

when $n < 1$. For consistency between the two equations of motion, we require that

$$q_{scalar} = q_{metric} = -\frac{1}{n-1} > 0 \,, \tag{2.86}$$

which holds for $n \in (0,1)$ and provides an Einstein gravity limit. In the limit of $\alpha \to \infty$, we now use the minimum of these sets and the condition (2.86) switches signs. Hence, we must have $-1/(n-1) < -1/n$ and $n > 1$ for a consistent Einstein gravity limit.

The most trivial example of a gravity theory without an Einstein limit is $G_4 = X$, in which case the condition (2.47) is violated as G_4 will not tend to a constant. Another, more involved example is $G_4 = \phi$, mimicking our other examples in satisfying conditions (2.74) and (2.75), but where we further set $G_3 = \alpha\phi X^{-2}$ so that $\mathcal{M}$ and $\mathcal{S}$ are non-trivial,

$$\mathcal{M} = \left\{\alpha^q, \alpha^{-2q+1}, \alpha^{-3q+1}\right\} , \tag{2.87}$$

$$\mathcal{S} = \left\{\alpha^{-4q+1}, \alpha^{-3q+1}, \alpha^{-3q+1}, \alpha^{-2q+1}\right\} , \tag{2.88}$$

which follows from eqs. (2.58) and (2.70). For $\alpha \to \infty$, we get $q_\mathcal{M} = 0$ and $q_\mathcal{S} = \frac{1}{4}$. Thus, we cannot recover Einstein gravity since this would require $q_\mathcal{M} > q_\mathcal{S}$ instead. Similarly, if we take the limit $\alpha \to 0$, we obtain $q_\mathcal{M} = q_\mathcal{S} = \frac{1}{2}$, which does not satisfy the requirement for an Einstein limit, $q_\mathcal{M} < q_\mathcal{S}$, either.

As our last example, we consider Brans-Dicke theory which possesses no screening mechanism unless we add a scalar field potential. The model is embedded in the Horndeski action by setting $G_4 = \phi$, $G_2 = 2\alpha\phi^{-1}X$ with all other G_i vanishing. The usual Brans-Dicke parameter ω becomes the scaling parameter ($\omega \to \alpha$). We then find from eqs. (2.58) and (2.70) that

$$\mathcal{M} = \{\alpha^{1+2q}, \alpha^{q}\}, \tag{2.89}$$

$$\mathcal{S} = \{\alpha^{1+q}, \alpha^{1+2q}\}. \tag{2.90}$$

Let us first consider the case where $\alpha \to \infty$. Then for the metric field equation to reproduce the Einstein field equations, we must have $q_{metric} < -1/2$. The scalar field equation demands that $q_{scalar} = -1$. Hence, conditions (2.74) and (2.75) are satisfied for $q = -1$ and we recover an Einstein gravity limit. A possible recovery of GR in this model is not surprising as it is well known to succeed when ω becomes large. However, this limit is different as in the scaling method the value of α is a given constant and we are taking the limits of its comparable magnitude with respect to the scalar field (see section 2.3.1). But the example of large ω illustrates that an Einstein gravity limit may not necessarily be attributed to a screening effect. Considering the limit $\alpha \to 0$ instead, we find $q_{\mathcal{M}} = 0$ and $q_{\mathcal{S}} = -\frac{1}{2}$ and hence, we do not recover Einstein gravity in this limit, which would require $q_{\mathcal{M}} < q_{\mathcal{S}}$. Thus, the model provides both a limit of modified gravity and a recovery of GR whereas it is well known to not possess a screening mechanism. We therefore need an auxiliary method to determine whether a particular Einstein gravity limit is due to a screening effect or not. This will be the focus of the next section.

2.4.4 Radial dependence for screening

When the metric field equations reduce to the Einstein field equations for a limit of α, they become independent of the variation of the scalar field. The dynamics of the scalar field is then solely determined by the scalar field equation and the metric can be considered a background field. However, the equation of motion for the perturbation ψ in the expansion (2.9) may, in general, be a complicated differential equation. For the expansion to be valid, we must have $\psi < \alpha^{-q}$ in the regime we wish to describe and we must check that the solution to the scalar field equation satisfies this condition. For the regime in question to show a recovery of Einstein gravity due to a screening effect, we furthermore want it to describe a region of high ambient density or in proximity of a massive body.

Consider, for instance, a spherical matter source. We now must find the profile of ψ around this mass to determine where the expansion breaks down. For

Vainshtein screening, we expect that the Einstein gravity limit becomes invalid once the distance from the source is large, at which point the modified gravity model becomes unscreened. For chameleon screening, there will be a thin shell interpolating the scalar field between the minima of the effective potential set by the different ambient matter densities in the interior and exterior of the source as discussed in section 2.3.3, where both regions are described by the same limit of $\alpha \to 0$.

Here, we outline a method for crudely approximating the radial profile $\phi(r)$ of a modified gravity model with a radial matter distribution, which can then be used to evaluate whether the Einstein gravity limits are associated with a screening mechanism or not. As a demonstration, we apply the method to the cubic galileon with a cylindrical mass distribution and a chameleon model with a spherical mass distribution, where we show that the Einstein limit is attributed to screening. Finally, we also consider Brans-Dicke theory with a spherical matter source and show that its Einstein limit described in section 2.4.3 is simply associated with large distances from the mass distribution and hence is not attributed to a screening effect. For simplicity, we shall adopt a Minkowski metric as an approximation in all scenarios. One could also consider a Schwarzschild background, where we assume that we are describing distances large with respect to the Schwarzschild radius (but small with respect to any screening radius) so that our approximation holds. This assumption can easily be dropped however, and in the Einstein gravity limit, one would then be working around a non-trivial solution to the metric field equation such as Anti-de Siter or Friedmann-Robertson-Walker. In such a case, the scalar field background may cause problems due to being an assumed constant.

In general, suppose that ϕ satisfies some partial differential equation

$$F(\phi, \partial\phi, \partial^2\phi) = \rho M_p^{-2}. \tag{2.91}$$

Writing out the derivatives in the coordinate choice suitable for the symmetry of the problem, such as spherical or cylindrical, yields $F(\phi, \partial_r\phi, \partial_r^2\phi) = \rho(r) M_p^{-2}$. Approximating the radial derivatives as

$$\partial_r \approx r^{-1}\,, \tag{2.92}$$

and the mass distribution (for a spherical symmetry) as $\rho \approx M r^{-3}$ changes the differential equation to a polynomial equation for $\phi(r)$. We justify this approximation on both dimensional and symmetry grounds. While such a mass distribution is unphysical as the integral will diverge, we are avoiding performing integrals in the approximation and so we smear out the mass source over the space we consider to get the radial dependence. In general, there will not be an analytical solution to this equation and a numeric solution will have to be found, which is still simpler than finding a solution to the potentially nonlinear differential equation.

As our first example, consider the cubic galileon model with equation of motion

$$\Box\phi + \alpha\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right] = \frac{\rho}{M_p^2}\,. \tag{2.93}$$

This is a simplified version of the equation of motion (2.18) that schematically contains the terms of interest. Solutions for different symmetric mass distributions of this equation can, for instance, be found in Ref. [33]. Consider a cylindrical geometry and mass distribution with line element $ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + dz^2$ and scalar field profile $\phi = \phi(r)$. The equation of motion then becomes

$$\phi'' + \frac{\phi'}{r} + \alpha\frac{2\phi'\phi''}{r} = \frac{\rho}{M_p^2}\,, \tag{2.94}$$

which after our approximations $\rho(r) \approx CMr^{-2}$, for some constant C with units *mass*, and $\partial_r \approx r^{-1}$ becomes

$$\frac{2\phi}{r^2} + \alpha\frac{2\phi^2}{r^4} \approx \frac{CM}{M_p^2 r^2}\,. \tag{2.95}$$

A solution of this quadratic equation is

$$\phi(r) = \frac{r^2}{2\alpha}\left(\sqrt{1 + \frac{r_v^2}{r^2}} - 1\right)\,, \tag{2.96}$$

where we have defined the Vainshtein radius $r_v^2 \equiv 2\alpha CM/M_p^2$. Note that this solution differs from the exact solution found for a cylindrical top-hat mass by an overall factor of 1/2 [33]. However, the functional form of this simple approximation agrees with the full solution.

Applying the expansion (2.9) with $q = -\frac{1}{2}$ and taking the limit of $\alpha \to \infty$, the equation of motion (2.94) for the perturbation ψ becomes

$$\frac{2\psi'\psi''}{r} \approx \frac{2\psi^2}{r^4} \approx \frac{CM}{M_p^2 r^2} \tag{2.97}$$

with $\psi < \alpha^{1/2}$. Hence, the scalar field profile is $\psi \propto r$, which needs to be small compared to $\alpha^{-q} = \alpha^{1/2} \approx r_v$. Thus, the solution is valid within the Vainshtein radius, demonstrating that the Einstein gravity limit in the cubic galileon model can be attributed to a screening effect.

The chameleon model of section 2.3.3 has the equation of motion

$$(3+2\omega)\Box\phi = V'_{eff}(\phi) = M_p^{-2}T + \alpha(\phi-\phi_m)^{n-1}(2(\phi-\phi_m) - n\phi)\,, \tag{2.98}$$

where we will consider its approximation on Minkowski space. We showed in section 2.4.3 that when $\alpha \to 0$ and $\phi_0 \approx \phi_{min}$ the equation of motion gives the solution for ψ as

$$\psi = \left(-\frac{M_p^{-2}T}{n\,\phi_0^n}\right)^{\frac{1}{n-1}}. \tag{2.99}$$

If we approximate $T \approx -\rho \approx -Mr^{-3}$, then the exponent of r is positive for $n \in (0,1)$. Hence, as ψ is small with respect to $\alpha^{\frac{1}{1-n}}$ for small r, the effect can be attributed to screening in the vicinity of a source. In contrast, for the case $n > 1$ discussed in section 2.4.3, the exponent of r is negative and the recovery of GR only occurs at large distances from the source, which does not correspond to a screening effect. Setting $\psi \lesssim \alpha^{\frac{1}{1-n}}$ gives a screening scale $r \lesssim \sqrt[3]{\alpha M / n M_p^2 \phi_0^n}$.

Finally, we study the example of Brans-Dicke gravity, for which in section 2.4.3, we found an Einstein limit when $q = -1$ and $\alpha = \omega \to \infty$. The scalar field equation can be approximated as

$$\frac{1}{r^2}\psi \approx \Box\psi = \frac{\rho}{M_p^2} \approx \frac{M}{M_p^2 r^3}\,, \tag{2.100}$$

from which we find that $\psi \propto 1/r$. As ψ needs to be small compared to $\alpha^{-q} = \alpha$ in order for the expansion to be valid, the solution only applies to scales of $r \gtrsim \alpha M / M_p^2$. Thus, the Einstein gravity limit is obtained at large distances from the matter source and cannot be attributed to a screening mechanism. Rather, we find that far from the source, the scalar field ϕ decouples from the metric field equation and Einstein gravity is recovered to highest order.

A caveat with this approximate method is that it does not work for all differential equations. Take for example $\partial_r \psi = \psi$ which results in an absurdity when the approximation $\partial_r \approx \frac{1}{r}$ is made. Furthermore, in the above approximation, we assume that the symmetry for the mass distribution is either cylindrical or spherical. But it is known that the morphology of the mass distribution can affect whether Vainshtein screening is operating or not (see, e.g., Ref. [33]). In essence, terms in the equations of motion disappear when a coordinate symmetry is imposed on the fields and hence the equations found after taking a limit of α may not be consistent. The result is that the value of q chosen is no longer valid as it does not provide non-vanishing terms in the equations of motion. There may, however, still be a screening effect if, for instance, a quartic galileon term vanishes but the cubic contributes. Conversely, our method may not indicate a screening effect with the scalar field equation becoming inconsistent despite a screening mechanism operating in the covariant equations of motion such as for the quntic galileon where screening would also require a time dependence of the source. Further work is needed to examine these scenarios. In this respect, it should be noted that mass distributions do not have perfect symmetry in reality. Even in highly symmetric cases, they will contain perturbations. Working without covariance, one might then consider the coordinate dependence of the field to also contain a dependence on α. Using the chain rule to extract powers of α, one can then proceed with our limiting arguments. This alternative scaling method may provide an approach to addressing morphology dependent screening, and we shall briefly outline such a method in appendix A.2. Also, it should be noted that our method does not cover all possible screening effects. This is as we insist that the metric field equation reduces to Einstein's field equations in some limit, from which it is guaranteed that a low-energy limit recovers Newtonian gravity. However, this is a stronger condition than only requiring that Newtonian gravity is recovered. In this case, when non-linear terms are included, a scale where Einstein gravity is recovered may not exist.

2.5 Conclusion

In the century since Einstein's discovery of GR, a plethora of modified gravity models have been proposed to address a variety of problems in physics, ranging from the quantum nature of gravity to cosmic acceleration. Powerful observational tests of gravity have placed tight bounds on deviations from GR in the Solar System. To pass these constraints and allow modifications on cosmological scales, screening mechanisms have been invoked that suppress these modifications in high-density regions or near massive bodies, where, however, small deviations from GR remain. Screening effects are predominantly dependent on nonlinear terms in the equations of motion, making the calculation of these small deviations mathematically challenging.

To overcome some of these challenges, we introduce a method for efficiently finding the relevant equations of motion in regimes of the strong or weak coupling of extra terms in the modified gravitational action. It works through introducing these coupling parameters to some power in an expansion of the scalar field. The power becomes fixed when considering formal limits of the parameters, facilitating the individual study of the two opposing regimes. This greatly aids the examination of gravity in screened regions as we can efficiently and consistently determine when terms dominate or become irrelevant in the equations of motion. We provide two explicit examples with the cubic galileon and chameleon models, illustrating the applicability of the scaling method to both screening by derivative interactions and local field values respectively.

Thanks to the simplicity of our method, it can be applied to the general Horndeski action to find embedded models of physical interest by insisting that there exists a limit where the metric field equations become Einstein's equations. From this, we derive a set of conditions on the four free functions of the Horndeski action which, when satisfied, ensure this limit exists for the covariant equations of motion. These conditions relate both the metric and scalar field equations through the exponent of the coupling entering in the scalar field expansion. Again, we illustrate the use of these conditions by re-examining the cubic galileon and chameleon models as well as two models that do not employ any screening mechanisms.

Importantly, while an Einstein limit may exist for a given gravitational action, it is not guaranteed that it is due to a screening effect. To determine whether the limit can be attributed to screening, further information on the scalar field profile is required. By adopting an appropriate coordinate symmetry and turning the differential equation describing the dynamics of the field into a polynomial whose coefficients are functions of the radius, the scalar field profile can be approximately derived. If the metric field equations recover Einstein's field equations in a region of high matter density or close to a matter source, we associate the Einstein limit with a screening effect. We demonstrate that in the cubic galileon and chameleon models, the Einstein limits can be attributed to screening mechanisms whereas in Brans-Dicke theory, we find that the Einstein limit is associated with large distances from the matter source and so there is no screening effect.

Importantly, our scaling method combined with a low-energy static limit in principle allows for a parameterised post-Newtonian (PPN) expansion of screened theories to be performed in regimes where screening mechanisms operate. Previous work has preformed a PPN expansion for Vainshtein screening [15] but the approach adopted is mathematically involved and has only been applied to the cubic galileon model. Given the simplicity of our scaling procedure, it becomes feasible to develop a PPN expansion of the Horndeski action which simultaneously takes into account the wide variety of screening mechanisms, testing all models that can be embedded in the action. This will be presented in separate work. One could also consider the case where the scalar field background is promoted to a function of time. This would allow for an investigation of screening in a cosmological context [130] and generalize our results.

Further, we plan to use our method to inspect gravitational waves in screened theories (see, e.g., [20, 57]). As gravitational waves can be treated as perturbations of the metric at large distances from the source far from their sources, our method naturally compliments this analysis in theories with screening mechanisms. Given the recent direct measurement of gravitational waves with the LIGO detectors [3], a comparison between waves in modified gravities and the observed post-Newtonian parameters [171] potentially allows for tests to be made on this phenomenon covering all regimes of gravity: strong and weak, screened and unscreened. The development of a suitable theoretical framework is of great interest and can be done in a model independent manner analogous to the PPN formalism (e.g., [194]) and our work may helps facilitate such an analysis.

Finally, one may also consider the scaling method we developed outside of the context of gravity to study other classical systems that adhere to a hierarchy in the contributing terms.

Chapter 3

The parameterised post-Newtonian expansion in screened regions

This chapter was published in [128]. The research contained within was performed by the author and was also the lead author of the paper under the supervision of Dr Jorge Penarrubia and Dr Lucas Lombriser.

3.1 Introduction

Modified gravity theories have gained interest among other reasons because of their cosmological use in explaining early universe inflation or late-time acceleration [46, 92, 93, 101] (however, see refs. [118, 122, 182]). In contrast, a wide range of gravitational phenomena on scales from the Solar System to galaxies can be described by the low-energy static regime of gravity. Because of the immediate access to this regime, low-energy tests, such as light deflection or the precession of the perihelion of Mercury, have probed relativistic gravity since the inception of General Relativity (GR). Modern tests of gravity in the Solar System have since put very tight constraints on deviations from GR as described in section 1.7. This makes many modified gravity theories incapable of introducing significant effects on cosmological scales while simultaneously passing these stringent constraints. However, a range of screening mechanisms allow for these theories to mimic general relativity, typically are due to non-linear interactions as described in section 1.6.2 (see refs. [121, 163] for linear shielding effects). However, the modifications in the low-energy static regime do not vanish completely and can still be used to constrain the range of possible modifications on cosmological scales. The difficulty thereby lies in inferring constraints that are applicable to the wide scope of gravitational theories proposed and independent of the specifics of any theory.

In order to conduct theory independent tests of gravity in the low-energy static limit, the parameterised post-Newtonian (PPN) formalism was developed as described in section 1.7. The formalism has traditionally been constructed through a linear post-Newtonian (PN) expansion of the metric field and stress-energy contributions in powers of v/c, where v is of the order of the velocity of planets. More specifically, the PPN formalism decomposes the metric components in terms of scalar and vector potentials which are parameterised by linear combinations of 10 constant parameters and the gravitational constant, usually set to unity. The measured values of these parameters then capture the effects of gravity in a theory-independent way (see [191] for a list of measurements). Calculating the predicted values for the PPN parameters, in principle, directly sets constraints on the theory from their measured values. However, modified gravity theories with non-linear screening mechanisms cannot naïvely be mapped onto the PPN formalism because the linearisation of the field equations removes contributions from the non-linear interactions which are fundamental to the screening effects. Therefore, a simple comparison to the theory independent Solar-System tests becomes infeasible. Moreover, screened theories do not exhibit a single low-energy limit due to the dependence of the screening effect on ambient density, giving rise to both a screened and unscreened low-energy static limit.

The PPN formulation for theories with screening has been examined previously for several cases. Ref. [81], for instance, performed a low-energy static expansion for the Horndeski action with minimally coupled matter. This is achieved through expanding the four free functions of the Horndeski action, which depend only on the scalar field and its first derivatives, as a Taylor series. Only the terms in the expansion linear in field perturbations are then kept. The two PPN parameters γ, β and the effective gravitational strength for the Horndeski action are then found in terms of Newtonian and Yukawa potentials for a spherical static system. Due to the linearisation and hence removal of the non-linear effects, this expansion does not incorporate screening mechanisms. A similar expansion was performed in ref. [195] for scalar-tensor theories in the Einstein frame with arbitrary potentials and conformal coupling functions. Their parameterisation embeds a multitude of theories that exhibit scalar field value screening. These mechanisms are incorporated through an effective potential that is a function of the ambient mass density. The minimum of this potential is used as the background value for the scalar field and is perturbed about, giving rise to an environment dependent mass. The PPN parameters γ, β and the effective gravitational strength are then found from a calculation that assumes a static spherical mass distribution, which again involves solutions in terms of Newtonian and Yukawa potentials. The static mass distribution removes the effects of the vector potentials in the expansion so that no prediction is made for several of the PPN parameters. Furthermore, while the spherical assumption allows for an exact solution, it limits the generic nature of the expansion. Refs. [73] and [147] used Legendre transforms and Lagrange multipliers, respectively, to find the low-energy limit in the screened regions of derivatively shielded theories. The purpose of constructing these dual theories is to change the action into one where the expansion becomes more natural for the screened regions. The Lagrange multiplier method was then implemented in ref. [15] to find the expansion of a Vainshtein-screened cubic galileon model [139] to order $(v/c)^2$. However, the calculations are mathematically involved and are not easily generalised to more complex derivatively screened theories. Further, this method has not been extended to include field value screening.

In order to provide a unified but systematic and efficient method for deriving the effective field equations in the limits of screening or no screening, for which one can then perform a low-energy static expansion, we have developed a scaling approach in chapter 2. The method allows one to find the dominant terms in the field equations for both screening with large field values and derivatives through the same algorithmic process. It relies on the expansion of the scalar field around a constant background and its scaling with some exponent of the coupling constant that controls the strength of the screening term in the action. The effective field equations describing either screening or no screening are then found by taking a formal limit of the coupling constant in the expansion of the full field equations. This extracts the terms in the field equations that are relevant for the given limit.

In this chapter, we extend our method to find a perturbative series in powers of the coupling constant around such limits. This extended scaling method is then combined with a low-energy static expansion to produce a PPN formulation for the screened modified gravity theories. The results for the metric perturbations can be directly parameterised to extend the PPN formalism with new potentials. But alternatively to this more traditional formalism, we also present the PPN parameters as functions of both time and space, as in refs. [81, 195]. The functional forms of these parameters are found from the series of corrections in the coupling constant. In particular, the PPN functions reproduce the GR parameters when evaluated in screened regions to leading order, while deviations are captured in higher-order corrections. We illustrate the method by finding the expansion for a cubic galileon and chameleon model. As an application of our results, we re-examine the implication of the Shapiro time delay measurement made by the Cassini mission [24] in the context of the two screening mechanisms. We find that the measurement remains unaffected by the two gravitational modifications as long as the Solar System can be considered screened. Hence, it cannot be used as a direct constraint on the models.

We extend the method of chapter 2 to a higher-order perturbative series around the screened and unscreened limits in section 3.2.1, and apply it for illustration to a cubic galileon on flat space in section 3.2.2. In section 3.3, we then combine the corrections about the screened limit with a low-energy static expansion to find the metric to order $(v/c)^4$ for a curved-space cubic galileon model. To demonstrate that our method also applies to screening by large field values, in section 3.4, we also perform an expansion for a chameleon model about the screening limit to order $(v/c)^4$. The re-examination of the Shapiro time delay for both models is presented in section 3.5. Concluding remarks are made in section 3.6. Finally, the appendices B.1 and B.2 provide a number of technical remarks about the higher-order corrections in the scaling method.

3.2 Scaling method

The scaling method developed in chapter 2 allows one to find the effective equations of motion that describe a modified gravity model in the screened regime. The method works through consistently identifying the dominant terms in the field equations when a formal limit of the coupling parameter is taken. The key to the scaling method is that within the expansion of a field, corrections are scaled by the coupling parameter. We define the screening limit to be a limit of the coupling parameters such that the metric field equations reproduce those of GR up to a constant effective gravitational strength. As GR is recovered in the screened limit, higher-order corrections around this limit are required to capture the non-vanishing effects of modifying gravity. The PPN parameters of a gravity theory in a screened region can then be found through performing a PN expansion of the effective field equations obtained at each scaling order (sections 3.3 and 3.4).

In section 3.2.1, we expand upon the method of chapter 2 to find the required perturbative series around these limits. We provide an example of this expansion with the flat-space cubic galileon in section 3.2.2. Note that for simplicity, in this section, we shall only consider the application of the scaling method to a scalar field equation. For a complete discussion that does not restrict to this flat-space limitation, we refer to chapter 2 (also see ref. [114] for an application of the scaling method to further modified gravity models).

3.2.1 Higher-order screening-corrections

The expansion (2.9) only describes a scalar field in the strict limits of $\alpha \to 0, \infty$. The two field equations obtained in these limits are the leading-order descriptions of the theory in the regions where the approximations $\alpha^{-q} \gg \psi$ or $\alpha^{q} \ll \psi$ hold respectively. To include higher-order corrections, we perform the expansion

$$\phi = \phi_0 \left(1 + \sum_{i=1}^{\infty} \alpha^{q_i} \psi^{(q_i)} \right) , \tag{3.1}$$

where $\psi^{(q_i)}$ represent the scalar field perturbations and q_i are real numbers. For this series to be perturbative, we require that the i^{th} term is a smaller correction to the field than the j^{th} for $i > j$, but note that this requirement alone is not sufficient for convergence. The condition is translated into our formalism by requiring that higher-order terms in the series vanish more quickly than the lower orders when taking the limits in α. Hence for the limit of $\alpha \to 0$, we must insist on the ordering $q_i < q_{i+1}$ for all i, and similarly for $\alpha \to \infty$ we have $q_i > q_{i+1}$. We impose that $q_1 \geq 0$ for the $\alpha \to 0$ limit and $q_1 \leq 0$ for $\alpha \to \infty$. We denote the set of values that q_i can take as Q_i.

To find the equations of motion for $\psi^{(q_i)}$, we again consider a generic homogeneous function $F_k(\phi, \partial\phi)$ with order k. Importantly, the leading-order term obtained from performing the expansion (2.9) must recover the same result as that found with the expansion (2.9) in order to preserve the relevant non-linear features. Applying the expansion (3.1) to F_k and performing a Taylor expansion in all

$\alpha^{q_i}\psi^{(q_i)}$ except for $\alpha^{q_1}\psi^{(q_1)}$ gives

$$\begin{aligned}F_k(\phi,\partial\phi) =&F_k(\phi,\partial\phi)|_{\bar{\psi}=0}\\&+\sum_{i=2}^{\infty}\alpha^{q_i}\left[\delta_{\alpha^{q_i}\psi^{(q_i)}}F_k(\phi,\partial\phi)|_{\bar{\psi}=0}\psi^{(q_i)}+\delta_{\alpha^{q_i}\partial\psi^{(q_i)}}F_k(\phi,\partial\phi)|_{\bar{\psi}=0}\partial\psi^{(q_i)}\right]+\dots\ ,\end{aligned} \tag{3.2}$$

where the ellipses contain terms quadratic in ψ, $\partial\psi$ or higher and $\bar{\psi} = (\psi^{(q_2)},\psi^{(q_3)},\dots)$. At this point we recognise that $F_k(\phi,\partial\phi)|_{\bar{\psi}=0} = \alpha^{kq_1}F_k(\psi^{(q_1)},\partial\psi^{(q_1)})$. Applying this expansion to the field equation provides all terms that may contribute in the α limit. For eq. (2.8), one finds

$$\begin{aligned}\frac{T}{M_p^2} =& \alpha^{s+uq_1}F_u(\psi^{(q_1)},\partial\psi^{(q_1)})+\alpha^{t+vq_1}F_v(\psi^{(q_1)},\partial\psi^{(q_1)})\\&+\sum_{k=u,v}\sum_{i=2}^{\infty}\alpha^{q_i}\left[\delta_{\alpha^{q_i}\psi^{(q_i)}}F_k(\phi,\partial\phi)|_{\bar{\psi}=0}\psi^{(q_i)}+\delta_{\alpha^{q_i}\partial\psi^{(q_i)}}F_k(\phi,\partial\phi)|_{\bar{\psi}=0}\partial\psi^{(q_i)}\right]+\dots\ .\end{aligned} \tag{3.3}$$

To find the equation of motion for the leading-order field perturbation $\psi^{(q_1)}$, the extremal values of q_1 are needed so that there exists a term independent of α upon taking the appropriate limit. However, the set Q_1 of such values now contains values that are functions of $q_{i>1}$. In appendix B.1 we show that the extremal values of Q_1 are not functions of $q_{i>1}$ and that generally for all j and k such that $j \geq k$, the extremal value of Q_j are only functions of q_k. This implies that the values of q_i can be found iteratively. Intuitively, this follows from the ordering we have imposed on the exponents. Thus the value for q_1 is found from the first line of eq. (3.3), which is equivalent to the expansion (2.9). Hence, the leading-order solution for the full expansion coincides with the case where only one term is considered, cf. eq. (2.9).

For illustration, let us assume that $q_1 = -s/u$ so that the equation for $\psi^{(-s/u)}$ is

$$F_u\big(\psi^{(-s/u)},\partial\psi^{(-s/u)}\big) = \frac{T}{M_p^2}\,. \tag{3.4}$$

Next we aim to find the field equation for the correction $\psi^{(q_2)}$. Using the result in appendix B.1, we see that terms that contain only functional derivatives with respect to $\psi^{(q_2)}$ or $\partial\psi^{(q_2)}$ and terms that are only functions of $\psi^{(-s/u)}$ are needed. Inserting the solution for the leading-order term, eq. (3.4), the field equation (3.3) for the second-order correction reduces to

$$
\begin{aligned}
0 &= \alpha^{t-vs/u}F_v(\psi,\partial\psi) \\
&\quad + \alpha^{q_2}\sum_{k=u,v}\left[\delta_{\alpha^{q_2}\psi^{(q_2)}}F_k(\phi,\partial\phi)|_{\bar{\psi}=0}\psi^{(q_2)} + \delta_{\alpha^{q_2}\partial\psi^{(q_2)}}F_k(\phi,\partial\phi)|_{\bar{\psi}=0}\partial\psi^{(q_2)}\right] + \dots\ .
\end{aligned}
\tag{3.5}
$$

Note that there no longer exists a term that goes as α^0, which is to be expected as such a term is associated with the leading order. Rather, the source term for the higher order corrections must be vanishing in the strict limit as the corrections must vanish too. Therefore, the set Q_2 contains the values that q_2 must take for a term to have the same α dependence as that of the slowest vanishing term. In the second-order equation (3.5) the slowest vanishing term goes as $\alpha^{t-vs/u}$. We show in appendix B.2 that only terms linear in $\psi^{(q_2)}$ or $\partial\psi^{(q_2)}$ can give the extremal values of the set Q_2. As such we may linearise the field equation (3.5) with respect to $\psi_{(q_2)}$ and its derivatives,

$$
-\alpha^{t-vs/u}F_v(\psi,\partial\psi) = \alpha^{q_2}\sum_{k=u,v}\left[\delta_{\alpha^{q_2}\psi^{(q_2)}}F_k(\phi,\partial\phi)|_{\bar{\psi}=0}\psi^{(q_2)} + \delta_{\alpha^{q_2}\partial\psi^{(q_2)}}F_k(\phi,\partial\phi)|_{\bar{\psi}=0}\partial\psi^{(q_2)}\right] .
\tag{3.6}
$$

In general, all higher-order corrections obey linear field equations as discussed in appendix B.2. Non-linear equations are therefore restrained to the leading order, $\psi^{(q_1)}$, and how these source the higher-order corrections.

Finally, when including the tensor field, it too must be expanded in α-corrections around the limits $\alpha \to \infty, 0$. More specifically, we will employ the expansion

$$
g_{\mu\nu} = g^{(0)}_{\mu\nu} + \sum_{i=1}^{\infty}\alpha^{p_i}g^{(p_i)}_{\mu\nu}
\tag{3.7}
$$

for the metric. We again impose that $0 < p_i < p_j\ \forall i < j$ in the limit $\alpha \to \infty$ and $0 > p_i > p_j\ \forall i < j$ for $\alpha \to 0$. The same procedure applies for finding the metric field equations as described here for the scalar field.

3.2.2 Example: flat-space cubic galileon

To illustrate how to include higher-order α-corrections in the field equations, we will examine the example of a cubic galileon model, for simplicity considered here in flat space. We chose this example as it involves an expansion in a screened regime and also because we examine the extension to non-negligible curvature in the next section. The field equation of the flat-space cubic galileon is given by

$$6\Box\phi + \alpha[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2] = -\frac{T}{M_p^2}\,. \tag{3.8}$$

Applying the scaling method of chapter 2 at first order, one finds for the limit $\alpha \to \infty$ that $\phi = \phi_0(1 + \alpha^{-\frac{1}{2}}\psi^{(-\frac{1}{2})})$, where $\psi^{(-\frac{1}{2})}$ is a solution to

$$\phi_0^2[(\Box\psi^{(-\frac{1}{2})})^2 - (\nabla_\mu\nabla_\nu\psi^{(-\frac{1}{2})})^2] = -\frac{T}{M_p^2}\,. \tag{3.9}$$

Next, we adopt the expansion (3.1) to find the second-order correction to this field equation, i.e., $\phi = \phi_0(1 + \alpha^{-\frac{1}{2}}\psi^{(-\frac{1}{2})} + \alpha^{q_2}\psi^{(q_2)})$ with $q_2 < -\frac{1}{2}$. As discussed in section 3.2.1, after performing a Taylor expansion of eq. (3.8) in $\alpha^{q_2}\psi^{(q_2)}$, we need only examine the linear terms. Inserting eq. (3.9), the field equation for $\psi^{(q_2)}$ becomes

$$\begin{aligned} 0 = {} & 3(\alpha^{-\frac{1}{2}}\Box\psi^{(-\frac{1}{2})} + \alpha^{q_2}\Box\psi^{(q_2)}) \\ & + \phi_0\alpha^{\frac{1}{2}+q_2}[(\Box\psi^{(-\frac{1}{2})})(\Box\psi^{(q_2)}) - (\nabla_\mu\nabla_\nu\psi^{(-\frac{1}{2})})(\nabla^\mu\nabla^\nu\psi^{(q_2)})]\,. \end{aligned} \tag{3.10}$$

We identify that the next-order source goes as $\alpha^{-\frac{1}{2}}$ and so to balance the counter-term in eq. (3.10), the exponent must be $q_2 = \min\left\{-\frac{1}{2}, -1\right\} = -1$. Using this value and taking the limit of $\alpha \to \infty$, the equation of motion for $\psi^{(-1)}$ becomes

$$3\Box\psi^{(-\frac{1}{2})} + \phi_0[(\Box\psi^{(-\frac{1}{2})})(\Box\psi^{(-1)}) - (\nabla_\mu\nabla_\nu\psi^{(-\frac{1}{2})})(\nabla^\mu\nabla^\nu\psi^{(-1)})] = 0\,. \tag{3.11}$$

Similarly, we can determine the second-order correction for the limit of $\alpha \to 0$. Briefly, to leading order, one finds $q_1 = 0$ and the field equation

$$6\Box\psi^{(0)} = -\frac{T}{M_P^2}\,. \tag{3.12}$$

With the expansions (3.1) and (3.12) of the field equation, one finds that $q_2 = \max\{1, 0\} = 1$ and the scalar field equation for $\psi^{(1)}$,

$$6\phi_0\Box\psi^{(0)} + \phi_0^2[(\Box\psi^{(1)})^2 - (\nabla_\mu\nabla_\nu\psi^{(1)})^2] = 0\,. \tag{3.13}$$

3.3 Post-Newtonian expansion of the cubic galileon model

To illustrate how a low-energy expansion can be performed in the screened limit of a modified gravity theory, we will first adopt the cubic galileon in the Jordan frame [139]. The model exhibits a Vainshtein screening mechanism [176] and is the simplest of the galileon models. The combination of the PN expansion with our scaling method enables a systematic low-energy expansion of the model in its screened regime. First, we employ the scaling method to obtain the screened metric field equations at both the leading and second order in α. Due to screening, the leading-order metric field equation is just that of GR with a constant effective gravitational coupling. Thus, its PN expansion can be found, e.g., in ref. [190] or section 1.4.3. At second order, the screened metric field equation contains deviations from GR and so its PN expansion gives the corrections to the PPN parameters. A benefit to adopting the cubic galileon model is that its low-energy expansion of the screened regime has previously been studied using dual Lagrangians up to PN order 2 in ref. [15] and that we can directly compare our results to that source while extending the expansion to PN order 4.

The curved space extension of the cubic galileon in the Jordan frame has the action

$$S_{cubic} = \frac{M_p^2}{2} \int d^4x \sqrt{-g} \left[\phi R + \frac{2\omega}{\phi} X - \frac{\alpha}{4} \frac{X}{\phi^3} \Box \phi \right] + S_m[g] \,, \tag{3.14}$$

where S_m denotes the minimally coupled matter action, ω is the Brans-Dicke parameter [34], $X \equiv -\frac{1}{2}\partial_\mu \phi \partial^\mu \phi$, $M_p^2 = (8\pi G)^{-1}$ and α is the coupling strength with units $mass^{-2}$. The metric field equation is

$$\phi R_{\mu\nu} = 8\pi G \left[T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right] + \frac{\omega}{\phi} \nabla_\mu \phi \nabla_\nu \phi + \frac{1}{2} \Box \phi g_{\mu\nu} + \nabla_\mu \nabla_\nu \phi + \frac{\alpha}{8} \left[\phi^{-3} \mathcal{M}^{(3)}_{\mu\nu} + \phi^{-4} \mathcal{M}^{(4)}_{\mu\nu} \right] \,, \tag{3.15}$$

and the scalar field equation is

$$(3 + 2\omega) \Box \phi + \frac{\alpha}{4} \left[\phi^{-2} \mathcal{S}^{(2)} + \phi^{-3} \mathcal{S}^{(3)} + \phi^{-4} \mathcal{S}^{(4)} \right] = 8\pi G T \,. \tag{3.16}$$

For convenience, we have made use of the rank 2 tensors

$$\mathcal{M}^{(3)}_{\mu\nu} \equiv -X\Box\phi g_{\mu\nu} - \Box\phi\nabla_\mu\phi\nabla_\nu\phi - \nabla_\mu X\nabla_\nu\phi - \nabla_\mu\phi\nabla_\nu X\,,$$
$$\mathcal{M}^{(4)}_{\mu\nu} \equiv 6X\nabla_\mu\phi\nabla_\nu\phi\,,$$

and scalar quantities,

$$\mathcal{S}^{(2)} \equiv -(\Box\phi)^2 + (\nabla_\mu\nabla_\nu\phi)^2 - R^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\ ,$$
$$\mathcal{S}^{(3)} \equiv 5\nabla_\mu\phi\nabla^\mu X - X\Box\phi\,,$$
$$\mathcal{S}^{(4)} \equiv 18X^2\,,$$

as defined in eq. (2.16) and (2.19).

We will use the metric signature $(-,+,+,+)$. To label both α and PN orders, we will write $A^{(i,j)}$ where i denotes the α order and j the PN order. As we will find the relevant α order before the PN order, we will use $A^{(i)}$ to label to the i^{th} α order. This labelling should not be confused with the tensors defined above, such as $\mathcal{S}^{(2)}$, in which case the notation will read $\mathcal{S}^{(2)(i,j)}$. This convention will hold for all composite objects such as the Ricci tensor, scalar and the d'Alembertian. Greek indices will run from 0 to 3, while Latin ones will run from 1 to 3. We will denote spatial derivatives as both $\overline{\nabla}_i$ or as an index with a comma. We let $c = 1$. Finally, we will refer to an object of PN order $(v/c)^i$ as $\mathcal{O}_{PN}(i)$.

In section 3.3.1 we recover the leading-order PN expansion for the 00 and ij components of the metric, and in section 3.3.2, we find the leading-order PN expansion for the $0j$ components and the second order for the 00 component. In section 3.3.3, we solve for the metric components in terms of potentials and find the PPN parameters for this theory.

3.3.1 Second-order expansion for the 00 and ij components

Using the method outlined in chapter 2, one finds for the $\alpha \to \infty$ limit of eq. (3.16) that $q_1 = -\frac{1}{2}$ (see chapter 2 for more details). This limit corresponds to a screened regime with field equations

$$\phi_0 R^{(0)}_{\mu\nu} = 8\pi G \left[T^{(0)}_{\mu\nu} - \frac{1}{2} T^{(0)} g^{(0)}_{\mu\nu} \right] , \tag{3.17}$$

$$\frac{1}{4}\mathcal{S}^{(2)(0)}(\psi^{(-\frac{1}{2})}) = 8\pi G T^{(0)} . \tag{3.18}$$

We now wish to perform the PN expansion of these equations in v/c. To do this we will expand each metric correction coming from eq. (3.7) in PN order. For the leading-order term we use

$$g^{(0)}_{\mu\nu} \to \eta_{\mu\nu} + \sum h^{(0,i)}_{\mu\nu} . \tag{3.19}$$

As the metric field equation is just Einstein's equation with an effective gravitational constant, the field perturbations $h^{(0,i)}_{\mu\nu}$ are the same as those for the PN expansion of GR (see section 1.4.3). This is as expected as the leading-order term should recover GR to be considered screened. However, the gauge conditions imposed differ due to the additional scalar field:

$$g^{\mu}_{i,\mu} - \frac{1}{2} g^{\mu}_{\mu,i} = \psi_{,i} , \tag{3.20}$$

$$g^{\mu}_{0,\mu} - \frac{1}{2} g^{\mu}_{\mu,0} = \psi_{,0} - \frac{1}{2} g_{00,0} , \tag{3.21}$$

where indices are raised and lowered with the Minkowski metric $\eta_{\mu\nu}$. It is required that these conditions match terms of the same order in both α and PN, and so have no effect on the expansion of $g^{(0)}_{\mu\nu}$.

We are left to find the PN expansion of the leading-order scalar field equation (3.18). Therefor, we use that the trace of the stress-energy tensor defined in eqs. (1.29)-(1.31), $T = -\rho$, is $\mathcal{O}_{PPN}(2)$. As it sources the scalar field equation, there must be at least one other term which is also $\mathcal{O}_{PPN}(2)$. Using that $R^{(0)}_{\mu\nu} \approx \mathcal{O}_{PN}(\geq 2)$ and $\partial_t \approx \mathcal{O}_{PN}(1)$, to lowest order the scalar field equation (3.18) is

$$[(\overline{\nabla}^2 \psi^{(-\frac{1}{2},i)})^2 - (\overline{\nabla}_m \overline{\nabla}_n \psi^{(-\frac{1}{2},i)})^2] = 32\pi G\rho . \tag{3.22}$$

To balance the PN orders across the equation, the field $\psi^{(-\frac{1}{2},i)}$ must be $\mathcal{O}_{PN}(1)$ and so $i = 1$. Note that with equation (3.22), we have recovered with a simple and systematic procedure the same result as presented in equation (3.54) of ref. [15], which was obtained from consideration of a dual Lagrangian.

To determine the deviations in the metric coming from the modification of gravity, we must first find the field equations for the next order α-correction. Inserting the first two terms from the expansions of the scalar field, eq. (3.1), and metric, eq. (3.7), into the metric field equation (3.15), we obtain

$$\begin{aligned}
&\phi_0\big(1+\alpha^{-\frac{1}{2}}\psi^{(-\frac{1}{2})}\big)\big(R^{(0)}_{\mu\nu}+\alpha^{p_1}R^{(p_1)}_{\mu\nu}\big)=8\pi G(\mathcal{T}^{(0)}_{\mu\nu}+\mathcal{T}^{(p_1)}_{\mu\nu})+\frac{\alpha^{-1}\omega\phi_0^2}{\phi_0}\partial_\mu\psi^{(-\frac{1}{2})}\partial_\nu\psi^{(-\frac{1}{2})}\\
&+\frac{1}{2}\alpha^{-\frac{1}{2}}\phi_0\big(\Box^{(0)}+\alpha^{p_1}\Box^{(p_1)}\big)\psi^{(-\frac{1}{2})}\big(g^{(0)}_{\mu\nu}+\alpha^{p_1}g^{(p_1)}_{\mu\nu}\big)+\alpha^{-\frac{1}{2}}\phi_0\big(\nabla^{(0)}_\mu+\alpha^{p_1}\nabla^{(p_1)}_\mu\big)\partial_\nu\psi^{(-\frac{1}{2})}\\
&+\frac{\alpha}{8}\Big[\alpha^{-\frac{3}{2}}\big(M^{(3)(0)}_{\mu\nu}(\psi^{(-\frac{1}{2})})+\alpha^{p_2}M^{(3)(p_1)}_{\mu\nu}(\psi^{(-\frac{1}{2})})\big)\\
&+\alpha^{-2}\big(M^{(4)(0)}_{\mu\nu}(\psi^{(-\frac{1}{2})})+\alpha^{p_1}M^{(4)(p_1)}_{\mu\nu}(\psi^{(-\frac{1}{2})})\big)\Big]\,, \qquad (3.23)
\end{aligned}$$

where $\mathcal{T}_{\mu\nu}=T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T$. Using the lowest-order field equation, eq. (3.17), we are left with only terms that will vanish in the limit of $\alpha\to\infty$. The slowest vanishing term goes as $\alpha^{-\frac{1}{2}}$ such that $p_1=\min\{-\frac{1}{2},0,\frac{1}{2}\}=-\frac{1}{2}$. Taking the limit $\alpha\to\infty$ provides the second-order metric field equation

$$R^{(-\frac{1}{2})}_{\mu\nu}+\psi^{(-\frac{1}{2})}R^{(0)}_{\mu\nu}=8\pi G\phi_0^{-1}\mathcal{T}^{(-\frac{1}{2})}_{\mu\nu}+\frac{1}{2}\Box^{(0)}\psi^{(-\frac{1}{2})}g^{(0)}_{\mu\nu}+\nabla^{(0)}_\mu\partial_\nu\psi^{(-\frac{1}{2})}+\frac{1}{8\phi_0}M^{(3)(0)}_{\mu\nu}[\psi^{(-\frac{1}{2})}]\,. \qquad (3.24)$$

Preforming a PN expansion of (3.24) recovers the metric corrections arising from the modifications to gravity. Using that $\psi^{(-\frac{1}{2})}=\psi^{(-\frac{1}{2},1)}+\mathcal{O}_{PN}(>1)$, the metric $g^{(0)}_{\mu\nu}=\eta_{\mu\nu}+h^{(0,2)}_{\mu\nu}+\mathcal{O}_{PN}(>2)$ and $g^{(-\frac{1}{2})}_{\mu\nu}=h^{(-\frac{1}{2},l)}_{\mu\nu}+\mathcal{O}_{PN}(>l)$, the field equation to lowest order is

$$R^{(-\frac{1}{2},l)}_{\mu\nu}=\frac{1}{2}\overline{\nabla}^2\psi^{(-\frac{1}{2},1)}\eta_{\mu\nu}+\partial_\mu\partial_\nu\psi^{(-\frac{1}{2},1)}\,. \qquad (3.25)$$

Examining the 00 component of the metric, the equation of motion is

$$-\frac{1}{2}\overline{\nabla}^2h^{(-\frac{1}{2},l)}_{00}=-\frac{1}{2}\overline{\nabla}^2\psi^{(-\frac{1}{2},1)}\,. \qquad (3.26)$$

As the right-hand side is $\mathcal{O}_{PN}(1)$, so must the left-hand side and thus $l=1$.

As in the case of GR, $h^{(-\frac{1}{2},1)}_{mn}$ should be diagonal, hence the choice of gauge in eq. (3.20). Collecting the lowest-order terms, eq. (3.25) becomes

$$-\frac{1}{2}\overline{\nabla}^2h^{(-\frac{1}{2},1)}_{mn}=\frac{1}{2}\overline{\nabla}^2\psi^{(-\frac{1}{2},1)}\delta_{mn}\,. \qquad (3.27)$$

Note that eqs. (3.26) and (3.27) are equivalent to the second part of eqs. (3.44) and (3.49) in ref. [15], and so we recover the same results to $\mathcal{O}_{PN}(2)$ as with the employment of dual Lagrangians.

3.3.2 Second-order expansion for $0j$ and the third-order for 00

To find the higher-order PN corrections to the metric, we will need the higher-order PN correction to the scalar field. The scalar field perturbation $\psi^{(-\frac{1}{2})}$ is a solution to the field equation (3.18),

$$\frac{1}{4}\mathcal{S}^{(2)(0)}\left[\psi^{(-\frac{1}{2})}\right] = 8\pi G T^{(0)}\,, \tag{3.28}$$

and $\psi^{(-\frac{1}{2},1)}$ satisfies the equation of motion (3.22). The next order PN expansion of the trace of the stress-energy tensor is of $\mathcal{O}_{PN}(4)$. Writing down all terms in eq. (3.18) which are $\mathcal{O}_{PN}(4)$ or linear in $\psi^{(-\frac{1}{2},k)}$, the field equation (3.18) becomes

$$\begin{aligned}&\left(\overline{\nabla}^2\psi^{(-\frac{1}{2},1)}\right)^2 h_i^{i(0,2)} - \left(\overline{\nabla}_i\overline{\nabla}_j\psi^{(-\frac{1}{2},1)}\right)^2 h_k^{k(0,2)} - \frac{1}{4}\overline{\nabla}^2 h_{ij}^{(0,2)}\overline{\nabla}_i\psi^{(-\frac{1}{2},1)}\overline{\nabla}_j\psi^{(-\frac{1}{2},1)}\\ &+\left(\overline{\nabla}^2\psi^{(-\frac{1}{2},1)}\right)\left(\overline{\nabla}^2\psi^{(-\frac{1}{2},k)}\right) - \left(\overline{\nabla}_m\overline{\nabla}_n\psi^{(-\frac{1}{2},1)}\right)\left(\overline{\nabla}_m\overline{\nabla}_n\psi^{(-\frac{1}{2},k)}\right) = 16\pi G\rho\left(3\frac{p}{\rho} - \Pi\right)\,.\end{aligned} \tag{3.29}$$

One can see that in order to balance the PN orders in this equation, we must have that $\mathcal{O}_{PN}(\psi^{(-\frac{1}{2},k)}) = \mathcal{O}_{PN}(3)$.

With this relation, we are able to find the second-order PN correction to the metric for the $0j$ components and the third order of 00. We can then use these results to map the modified gravity theory into the PPN formalism (see section 1.7). Using the linear terms from the expansion of the $0j$ components of the Ricci tensor, eq. (1.35), and the two gauge conditions (3.20) and (3.21), it is easily shown that

$$R_{0j} \approx -\frac{1}{2}\left(\overline{\nabla}^2 h_{0j} + \frac{1}{2}h_{00,0j} - 2\psi_{,0j}\right) \tag{3.30}$$

to lowest order. The lowest PN order for the $0j$ component in equation (3.24) is $\mathcal{O}_{PN}(2)$ after inserting the expansion of the Ricci tensor, the metric and scalar field. The resulting field equation is given by

$$\overline{\nabla}^2 h_{0j}^{(-\frac{1}{2},2)} + \frac{1}{2}h_{00,0j}^{(-\frac{1}{2},1)} = 0\,. \tag{3.31}$$

This is equivalent to the PN expansion of the $0j$ component in GR, eq. (1.41), but without the matter source.

Applying the gauge conditions with third-order corrections to $R_{00}^{(-\frac{1}{2})}$, eq. (1.36), one finds that

$$R_{00}^{(-\frac{1}{2},3)} = -\frac{1}{2}\Bigg[\overline{\nabla}^2 h_{00}^{(-\frac{1}{2},3)} - 2\psi_{,00}^{(-\frac{1}{2},1)} - \psi_{,j}^{(-\frac{1}{2},1)} h_{00,j}^{(0,2)} + 2h_{00,j}^{(-\frac{1}{2},1)} h_{00,j}^{(0,2)} - h_{jk}^{(0,2)} h_{00,jk}^{(-\frac{1}{2},1)} - h_{jk}^{(-\frac{1}{2},1)} h_{00,jk}^{(0,2)}\Bigg] . \quad (3.32)$$

Thus the field equation (3.24) to $\mathcal{O}_{PN}(3)$ is

$$\begin{aligned}\overline{\nabla}^2 h_{00}^{(-\frac{1}{2},3)} &= \overline{\nabla}^2 \psi^{(-\frac{1}{2},3)} - \psi_{,00}^{(-\frac{1}{2},1)} - h_{00}^{(0,2)} \overline{\nabla}^2 \psi^{(-\frac{1}{2},1)} \\ &+ \psi_{,j}^{(-\frac{1}{2},1)} h_{00,j}^{(0,2)} - 2h_{00,j}^{(-\frac{1}{2},1)} h_{00,j}^{(0,2)} + h_{jk}^{(-\frac{1}{2},1)} h_{00,jk}^{(0,2)} + h_{jk}^{(0,2)} h_{00,jk}^{(-\frac{1}{2},1)} \\ &- \psi^{(-\frac{1}{2},1)} \overline{\nabla}^2 h_{00}^{(0,2)} - \frac{1}{4\phi_0} M_{00}^{(3)(0,0)} \left[\psi^{(-\frac{1}{2},1)}\right] + 8\pi G \phi_0^{-1} \rho h_{00}^{(-\frac{1}{2},1)} \\ &- \frac{1}{2}\left(2\partial_j h_{ij}^{(0,2)} - \partial_i h_{jj}^{(0,2)} - \partial_i h_{00}^{(0,2)}\right) \psi_{,i}^{(-\frac{1}{2},1)} , \end{aligned} \quad (3.33)$$

where the last line arises from the contribution of the Christoffel symbols.

Note that the equations for the metric components, eqs. (3.26), (3.27), (3.31), and (3.33) (with exception of the term proportional to $M_{00}^{(3)(0,0)}$), are all equivalent to the corresponding field equations found for Brans-Dicke theory with no matter content [142]. The differences arise from different scalar field equations, eqs. (3.22) and (3.29). That there is no matter content in eq. (3.31) results in the gauge condition (3.21) being trivially satisfied.

3.3.3 Metric solution

The metric components can be solved in terms of the scalar field and source terms derived from the matter content, such as the mass density or Newtonian potential. It will prove convenient to use the notation

$$\left(\overline{\nabla}^2\right)^{-1}\left[f(\overline{x}, t)\right] \equiv -\frac{1}{4\pi} \int \frac{f(\overline{x}', t)}{|\overline{x} - \overline{x}'|} d^3x' \quad (3.34)$$

for a function $f(\overline{x}, t)$.

From the equations of motion for $h_{00}^{(-\frac{1}{2},1)}$, $h_{mn}^{(-\frac{1}{2},1)}$ and $h_{0j}^{(-\frac{1}{2},2)}$, i.e., eqs. (3.26), (3.27) and (3.31), respectively, it follows that

$$h_{00}^{(-\frac{1}{2},1)} = \psi^{(-\frac{1}{2},1)}\,, \tag{3.35}$$

$$h_{mn}^{(-\frac{1}{2},1)} = -\psi^{(-\frac{1}{2},1)}\delta_{mn}\,, \tag{3.36}$$

$$\begin{aligned} h_{0j}^{(-\frac{1}{2},2)} &= \big(\overline{\nabla}^2\big)^{-1}\big[-\frac{1}{2}h_{00,0j}^{(-\frac{1}{2},1)}\big] \\ &= \frac{1}{2}\big(\mathcal{V}_j^{(-\frac{1}{2},2)} + 3\mathcal{W}_j^{(-\frac{1}{2},2)} + \mathcal{O}_{PN}(>2)\big)\,. \end{aligned} \tag{3.37}$$

In the last line, we have used the identity (1.42), and in analogy to the PPN potentials V_i and W_i in eqs. (1.43) and (1.44), we have defined

$$\mathcal{V}_i^{(j,k+1)} \equiv -\frac{1}{4\pi}\int \frac{\psi'^{(j,k)}v'_j}{|\overline{x}-\overline{x}'|^3}d^3x'\,, \tag{3.38}$$

$$\mathcal{W}_i^{(j,k+1)} \equiv -\frac{1}{4\pi}\int \frac{\psi'^{(j,k)}[\overline{v}'\cdot(\overline{x}-\overline{x}')](x-x')_j}{|\overline{x}-\overline{x}'|^5}d^3x'\,, \tag{3.39}$$

where $\overline{v} = \partial_t\overline{x}$.

Finally, from the equation of motion for $h_{00}^{(-\frac{1}{2},3)}$, eq. (3.33), we have

$$\begin{aligned} h_{00}^{(-\frac{1}{2},3)} &= \psi^{(\frac{1}{2},3)} + \tilde{\Phi}_1^{(-\frac{1}{2},3)} - 3\mathcal{A}_\psi^{(-\frac{1}{2},3)} \\ &\quad - \mathcal{B}_\psi^{(-\frac{1}{2},3)} + 6G\phi_0^{-1}\tilde{\Phi}_2^{(-\frac{1}{2},3)} - \Phi_{cubic}^{(-\frac{1}{2},3)} \end{aligned} \tag{3.40}$$

$$= \Phi_{BD}^{(-\frac{1}{2},3)} - \Phi_{cubic}^{(-\frac{1}{2},3)}\,. \tag{3.41}$$

We have absorbed all terms arising from the Brans-Dicke-like part of the action into Φ_{BD} and in analogy to the PPN potentials, we have defined

$$\begin{aligned} &\tilde{\Phi}_1^{(p,q+2)} \equiv -\frac{1}{4\pi}\int \frac{\psi'^{(p,q)}v'^2}{|x-x'|^3}d^3x'\,, \quad \mathcal{A}_\psi^{(p,q+2)} \equiv -\frac{1}{4\pi}\int \frac{\psi'^{(p,q)}[\overline{v}'\cdot(\overline{x}-\overline{x}')]^2}{|\overline{x}-\overline{x}'|^5}d^3x'\,, \\ &\tilde{\Phi}_2^{(p,q+2)} \equiv \int \frac{\rho'\psi'^{(p,q)}}{|x-x'|}d^3x'\,, \qquad \mathcal{B}_\psi^{(p,q+2)} \equiv -\frac{1}{4\pi}\int \frac{\psi'^{(p,q)}[\overline{a}'\cdot(\overline{x}-\overline{x}')]}{|\overline{x}-\overline{x}'|^3}d^3x'\,, \\ &\Phi_{cubic}^{(-\frac{1}{2},3)} \equiv (\overline{\nabla}^2)^{-1}\left[\frac{1}{4\phi_0}|\overline{\nabla}\psi^{(-\frac{1}{2},1)}|^2\overline{\nabla}'^2\psi^{(-\frac{1}{2},1)}\right]\,, \end{aligned} \tag{3.42}$$

where $\overline{a} = \partial_t\overline{v}$.

In solving the metric components we have kept the scalar field explicit, rather than implementing its solution in terms of the matter source. This complicates our results in comparison to expansions given for other theories such as Brans-

Dicke gravity [191], where the solution to the scalar field equation is used and the metric components are solved for in terms of the matter content. However, leaving the scalar field explicit allows for a generic solution in terms of potentials derived from it, which will prove useful when examining the chameleon model in section 3.4. One may worry that the potentials $\mathcal{A}$ and $\mathcal{B}$ arise from the Brans-Dicke part of the action but have no analogue in its PN expansion. This is as $\psi^{(-\frac{1}{2},1)}_{,00}$ would be removed through the use of the Brans-Dicke scalar field equation.

Interestingly, the solution for $h^{(-\frac{1}{2},1)}_{00}$ in eq. (3.35) identifies this perturbation directly with the perturbation in the scalar field of the same PN order. Recall that the force that a test particle feels in the low-energy static limit is associated with the derivative of the 00 component of the metric. Hence, we recognise that the inclusion of the α-correction gives rise to a fifth force interaction between the test particle and the scalar field.

3.3.4 Mapping to the parameterised post-Newtonian formalism

Next, we are left with putting all of the parts together to map the modifications to the PPN framework. This can be done either through the introduction of new potentials or through adaptation of the PPN parameters. Considering the corrections to the metric as new potentials with corresponding parameters is in the spirit of the PPN formalism, which was built through adding new potentials as they were discovered [190]. These potentials then carry their own parameter that can be constrained by experiments. We present such a metric at the end of this section.

However, we also present an adaptation of the PPN formalism that captures the spatial dependence of screening effects by promoting the PPN parameters into functions of the coordinates. This constitutes changing the summation order from *summing over α then over PN* to the *summing over PN then over α* by combining the PN summation from the GR field equations (3.17) with that coming from the α-corrections in eq. (3.24). The results of this section will be summarised in table 3.1.

In order to match the expression to the form of the PPN metric in eq. (1.73a), the 00 component of the metric should go as $\approx -1 + 2GU$. Hence to PN leading order, we get that

$$
\begin{aligned}
g_{00} &= -1 + 2G\phi_0^{-1}U + \alpha^{-\frac{1}{2}}\psi^{(-\frac{1}{2},1)} \\
&= -1 + 2G_{\mathit{eff}}U\,,
\end{aligned}
\tag{3.43}
$$

where we have defined the *effective gravitational coupling*

$$
\begin{aligned}
G_{\mathit{eff}} &\equiv G^{(0)} + G^{(-\frac{1}{2})} \\
&= G\phi_0^{-1} + \alpha^{-\frac{1}{2}}\frac{\psi^{(-\frac{1}{2},1)}}{2U}\,.
\end{aligned}
\tag{3.44}
$$

Note that the correction to G_{eff} is controlled by the parameter α.

The ij component of the metric must be $\approx \delta_{ij}(1 + 2G_{\mathit{eff}}\gamma U)$. From this we can find the PPN parameter γ as

$$
\begin{aligned}
g_{ij} &= \delta_{ij}(1 + 2G\phi_0^{-1}U - \alpha^{-\frac{1}{2}}\psi^{(-\frac{1}{2},1)}) \\
&= \delta_{ij}\left(1 + 2G_{\mathit{eff}}U\frac{2G^{(0)}U - \alpha^{-\frac{1}{2}}\psi^{(-\frac{1}{2},1)}}{2G^{(0)}U + \alpha^{-\frac{1}{2}}\psi^{(-\frac{1}{2},1)}}\right) \\
&\equiv \delta_{ij}\left(1 + 2G_{\mathit{eff}}U\gamma\right)\,,
\end{aligned}
\tag{3.45}
$$

where γ becomes a function of space, as expected. We have

$$
\begin{aligned}
\gamma =& \gamma^{(0)} + \gamma^{(-\frac{1}{2})} \\
=& 1 - \frac{2\alpha^{-\frac{1}{2}}\psi^{(-\frac{1}{2},1)}}{2G^{(0)}U + \alpha^{-\frac{1}{2}}\psi^{(-\frac{1}{2},1)}}\,.
\end{aligned}
\tag{3.46}
$$

and when taking the limit of $\alpha \to \infty$, we recover $\gamma = 1$.

For the $0i$ components of the metric, the PPN expansion (1.73b) introduces several parameters: ξ which is responsible for preferred location effects, ζ_1 which is responsible for a breakdown in the conservation of momentum, α_1 and α_2 which are responsible for preferred frame effects. As this theory comes from a Lagrangian we have no reason to expect a breakdown of conservation laws, and as a scalar-tensor theory with a constant background field we expect no preferred frame effects (see ref. [86] for a discussion of how preferred frame effects can appear in a cosmological context). However, we would expect preferred location effects from our expansion. This is as the approximation $\alpha^q \ll \psi$ is only valid for a sufficiently screened (or unscreened) region, and so the expansion favours these locations. This manifests as the gravitational coupling varying through space, which is responsible for preferred location effects. Hence, we will use the parameter ξ to put the metric into PPN form.

The PPN metric for the $0i$ components is given by

$$g_{0i} = -\frac{1}{2}(4\gamma + 3 - 2\xi)V_i G_{\mathit{eff}} - \frac{1}{2}(1 + 2\xi)W_i G_{\mathit{eff}}, \tag{3.47}$$

where in section 3.3.3 we have found the metric solution

$$g_{0i} = -\frac{7}{2}V_i G^{(0)} - \frac{1}{2}W_i G^{(0)} + \alpha^{-\frac{1}{2}}\big(\frac{1}{2}\mathcal{V}_i^{(-\frac{1}{2},3)} + \frac{3}{2}\mathcal{W}_i^{(-\frac{1}{2},3)}\big). \tag{3.48}$$

For clarity, we will drop the order notation on $\mathcal{V}$ and $\mathcal{W}$.

The method used for finding the parameters γ and G_{eff} in both the 00 and ij components will not work for ξ. The reason for this is that ξ will need to solve two different equations coming from the prefactors of the two vector potentials. Furthermore, Ψ_{0i} may not lie in the span of V_i and W_i and so will have a component that cannot be absorbed into their prefactors even without this constraint.

We therefore propose here that the parameter ξ should be promoted to a matrix such that $\xi_{ij} = \xi^{(0)}\delta_{ij} + \alpha^{-\frac{1}{2}}\xi_{ij}^{(-\frac{1}{2})}$. The reasoning behind this is that the PPN parameters are meant to indicate how much a PPN potential is transformed from one theory to another. For scalar potentials, the most general linear transformation is scalar multiplication. However, for vector potentials, we must consider a matrix acting upon the potential. Moreover, as we focus on screened models, the parameters of this matrix would themselves be functions of position, as for γ. This changes eq. (3.47) to

$$\begin{aligned} g_{0i} &= -\frac{1}{2}\big((4\gamma+3)\delta_{ij} - 2\alpha^{-\frac{1}{2}}\xi_{ij}^{(-\frac{1}{2})}\big)V_j G_{\mathit{eff}} - \frac{1}{2}\big(\delta_{ij} + 2\alpha^{-\frac{1}{2}}\xi_{ij}^{(-\frac{1}{2})}\big)W_j G_{\mathit{eff}} \\ &= -\frac{7}{2}V_i G^{(0)} - \frac{1}{2}W_i G^{(0)} + \alpha^{-\frac{1}{2}}\frac{1}{2}(\mathcal{V}_i + 3\mathcal{W}_i), \end{aligned} \tag{3.49}$$

where we have used that $\xi^{(0)} = 0$. We now wish to separate this into two different matrix equations for $\xi_{ij}V_j$ and $\xi_{ij}W_j$. There is a redundancy as there is no unique way to doing this, but the analogous form of the potentials $\mathcal{V}_i$ and $\mathcal{W}_j$ to V_i and

W_j suggests

$$G^{(0)}\xi_{ij}^{(-\frac{1}{2})}V_j = \frac{1}{2}G^{(0)}V_i\big(4\gamma^{(-\frac{1}{2})} + 7G^{(-\frac{1}{2})}/G^{(0)}\big) + \frac{1}{2}\mathcal{V}_i \equiv G^{(0)}V_{\mathit{eff}\ i}\,, \tag{3.50}$$

$$G^{(0)}\xi_{ij}^{(-\frac{1}{2})}W_j = -\frac{3}{2}\mathcal{W}_i - \frac{1}{2}W_iG^{(-\frac{1}{2})} \equiv G^{(0)}W_{\mathit{eff}\ i}\,, \tag{3.51}$$

where we have defined the effective potentials $W_{\mathit{eff}\ i}$ and $V_{\mathit{eff}\ i}$. The equations (3.50) and (3.51) are not sufficient for specifying the matrix $\xi_{ij}^{(-\frac{1}{2})}$ and another three constants are required. These extra constraints need not be physical as they are a remnant of the mathematical construction we have chosen. As such, any constraints chosen which allow for a solution will be equivalent as they only remove redundant degrees of freedom. We impose here that the diagonal elements $\xi_{ii}^{(-\frac{1}{2})}$ vanish as this leads to an elegant solution. We can thus solve for $\xi_{ij}^{(-\frac{1}{2})}$ to find

$$\begin{aligned}\xi_{ij}^{(-\frac{1}{2})} &= \frac{(\epsilon_{jkl}W_kV_l)_{\mathit{eff}}}{\epsilon_{ikl}V_kW_l} \quad \text{if } i \neq j,\\ &= 0 \qquad\qquad\qquad \text{if } i = j\,,\end{aligned} \tag{3.52}$$

where we define $(\epsilon_{jkl}W_kV_l)_{\mathit{eff}}$ as $(\epsilon_{1kl}W_kV_l)_{\mathit{eff}} = W_{eff\ 2}V_3 - V_{eff\ 2}W_3$, with analogous definitions for $(\epsilon_{2kl}W_kV_l)_{\mathit{eff}}$ and $(\epsilon_{3kl}W_kV_l)_{\mathit{eff}}$. Note that this solution does not depend on the form of $V_{\mathit{eff}\ i}$ or $W_{\mathit{eff}\ i}$, which is specific to the theory and only requires that the denominators do not vanish. We have only used the PPN parameter ξ, however, we suspect that the other parameters associated with the vector potentials (α_1, α_2 and ζ_1) would also have to be promoted to matrices to incorporate more general gravity theories.

Finally, we have the fourth-order PN corrections to the 00 component of the metric. The new parameter that enters into the PPN metric at this level is β which describes the non-linearity of gravity as it is the coupling strength of U^2. The 00 component of the metric in the PPN formalism takes the form

$$\begin{aligned}g_{00} \approx& -1 + 2G_{\mathit{eff}}U - 2\beta G_{\mathit{eff}}^2U^2 + (2\gamma + 2 - 2\xi)G_{\mathit{eff}}\Phi_1\\ &+ 2(3 + \gamma - 2\beta + 1 + \xi)G_{\mathit{eff}}^2\Phi_2 + 2G_{\mathit{eff}}\Phi_3\\ &- 2\xi G_{\mathit{eff}}^2\Phi_W + 2(\gamma - 2\xi)G_{\mathit{eff}}\Phi_4 + 2\xi G_{\mathit{eff}}\mathcal{A}\,,\end{aligned} \tag{3.53}$$

[or (1.73a)]. Since ξ now appears beside a scalar, we cannot directly use the matrix ξ_{ij} that we have previously defined. Rather, we need to use a scalar object made from ξ_{ij} such that it reduces to the scalar PPN parameter ξ. That we do not find the ξ parameter in our expansion despite the existence of ξ_{ij} is not unexpected. This is as the object to identify ξ in eq. (3.53) is $\frac{1}{3}\mathrm{tr}(\xi_{ij})$. The trace will ensure that the correct order in α is conserved as the determinant will raise the order by a factor of three. Since $\xi^{(0)}$ vanishes, as in GR, and $\xi_{ij}^{(-\frac{1}{2})}$ has vanishing trace by definition, $\frac{1}{3}\mathrm{tr}(\xi_{ij})$ does not appear in eq. (3.53). Thus we have that

$$\begin{aligned} g_{00} \approx &-1 + 2G_{\mathit{eff}}U - 2G^{(0)2}U^2 + 4G^{(0)}\Phi_1 + 4G^{(0)2}\Phi_2 \\ &+ 2G^{(0)}\Phi_3 + 6G^{(0)}\Phi_4 + \alpha^{-\frac{1}{2}}\left(\Phi_{BD}^{(-\frac{1}{2},3)} - \Phi_{cubic}^{(-\frac{1}{2},3)} \right). \end{aligned} \tag{3.54}$$

From this we can extract β such that

$$\beta = 1 + \alpha^{-\frac{1}{2}}\left(\beta_{BD}^{(-\frac{1}{2})} + \beta_{Cubic}^{(-\frac{1}{2})}\right), \tag{3.55}$$

$$\beta_{BD}^{(-\frac{1}{2})} = \frac{-\Phi_{BD}^{(-\frac{1}{2},3)} + G^{(-\frac{1}{2})}\dfrac{\delta h_{00}^{(0,4)}}{\delta G^{(0)}} + \gamma^{(-\frac{1}{2})}\dfrac{\delta h_{00}^{(0,4)}}{\delta \gamma^{(0)}}}{2G^{(0)2}U^2 + 4G^{(0)2}\Phi_2}, \tag{3.56}$$

$$\beta_{Cubic}^{(-\frac{1}{2})} = \frac{\Phi_{cubic}^{(-\frac{1}{2},3)}}{2G^{(0)2}U^2 + 4G^{(0)2}\Phi_2}, \tag{3.57}$$

where we have defined β_{BD} as the terms that arise from the Brans-Dicke-like part of the action, and β_{cubic} as the term from the additional contributions of the cubic galileon model responsible for screening. The reason for this separation will become apparent in the next section.

Rather than adapting the PPN parameters into functions in order to capture the spatial dependence of screening, we can also consider a direct parameterisation of the new potentials. We parameterise all potentials found from $\psi^{(-\frac{1}{2},1)}$ with σ_1, those from $\psi^{(-\frac{1}{2},3)}$ with σ_2 and $\Phi_{cubic}^{(-\frac{1}{2},3)}$ with σ_{cubic}. The metric is then parameterised as

$$\begin{aligned} g_{00} =& g_{00}^{PPN} + \sigma_1(\psi^{(-\frac{1}{2},1)} + \tilde{\Phi}_1^{(-\frac{1}{2},3)} - 3\mathcal{A}_\psi^{(-\frac{1}{2},3)} - \mathcal{B}_\psi^{(-\frac{1}{2},3)} + 6G\phi_0^{-1}\tilde{\Phi}_2^{(-\frac{1}{2},3)}) \\ &+ \sigma_2\psi^{(-\frac{1}{2},3)} + \sigma_{cubic}\Phi_{cubic}^{(-\frac{1}{2},3)}, \end{aligned} \tag{3.58}$$

$$g_{0i} = g_{0i}^{PPN} + \sigma_1(\frac{1}{2}\mathcal{V}_i + \frac{3}{2}\mathcal{W}_i), \tag{3.59}$$

$$g_{ij} = g_{ij}^{PPN} - \sigma_1\psi^{(-\frac{1}{2},1)} \tag{3.60}$$

where $g_{\mu\nu}^{PPN}$ are given in eq. (1.73a) through (1.73c). In this case of the cubic galileon, the value for all these parameters coincide as $\alpha^{-\frac{1}{2}}$.

3.4 Post-Newtonian expansion of a chameleon screened model

The chameleon screening mechanism was discovered in ref. [96], where it was found that the fifth force caused by the coupling of a scalar field to matter near massive bodies can be suppressed with respect to the Newtonian force exerted by the same object. The mechanism relies on the scalar field taking the minimum value of an effective potential, a function of the ambient mass density and the self-interaction of the field. The mechanism is a non-linear effect, with a linearisation leading to large deviations from GR within regions that would otherwise be screened [44, 84]. This causes an environmentally dependent effective mass such that in regions of high density, the effective mass grows and the force becomes Yukawa suppressed. In contrast, in regions of low density, for instance, the large-scale structure, this mass is small, and the fifth force introduces deviations from GR. Chameleon screening has been of particular interest in $f(R)$ gravity theories [37, 44, 52, 84], which are equivalent to a Brans-Dicke model with $\omega = 0$ and a scalar field potential [167] (also see, e.g., ref. [113] for a review of observational constraints). Note that the chameleon model in ref. [96] is presented with an action in the Einstein frame whereas we will work entirely in the Jordan frame. For convenience, we adopt a simple power-law potential in this frame that exhibits chameleon screening.

More specifically, we consider the Brans-Dicke-type chameleon model with the action

$$S_{cham} = \frac{M_p^2}{2} \int d^4x \sqrt{-g} \left[\phi R + \frac{2\omega}{\phi} X - \alpha(\phi - \phi_{min})^n \right] + S_m[g] \,, \tag{3.61}$$

where $0 < n < 1$ (see ref. [116] for a discussion of this model). The equations of motion for the model are given by

$$\begin{aligned} (3 + 2\omega)\Box\phi &= M_p^{-2} T - \alpha(\phi - \phi_{min})^{n-1}(2(\phi - \phi_{min}) - n) \,, & (3.62)\\ \phi R_{\mu\nu} &= M_p^{-2}[T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T] + \frac{\omega}{\phi} \nabla_\mu\phi\nabla_\nu\phi + \frac{1}{2}\Box\phi g_{\mu\nu} + \nabla_\mu\nabla_\nu\phi + \frac{1}{2} g_{\mu\nu}\alpha(\phi - \phi_{min})^n \,. & (3.63) \end{aligned}$$

The limit that corresponds to screening is $\alpha \to 0$ with $q = \frac{1}{1-n}$ and $\phi_0 = \phi_{min}$, and the screened field equations are found to be (see chapter 2 for more details on the derivation)

$$\phi_0 R^{(0)}_{\mu\nu} = 8\pi G \left[T^{(0)}_{\mu\nu} - \frac{1}{2} T^{(0)} g^{(0)}_{\mu\nu} \right] , \tag{3.64}$$

$$\phi_0 \psi^{(\frac{1}{1-n})} = \left(-\frac{M_p^{-2} T^{(0)}}{n} \right)^{\frac{1}{n-1}} . \tag{3.65}$$

The PN expansion for the metric $g^{(0)}_{\mu\nu}$ at zeroth α order is again that of GR, as expected for a screened limit, and the leading-order α-correction of the scalar field is

$$\phi_0 \psi^{(\frac{1}{1-n},\frac{2}{n-1})} = \left(\frac{M_p^{-2} \rho}{n} \right)^{\frac{1}{n-1}} . \tag{3.66}$$

The field equation for the first metric α-correction is given by

$$R^{(\frac{1}{1-n})}_{\mu\nu} + \psi^{(\frac{1}{1-n})} R^{(0)}_{\mu\nu} = 8\pi G \mathcal{T}^{(\frac{1}{1-n})}_{\mu\nu} + \frac{1}{2} \Box^{(0)} \psi^{(\frac{1}{1-n})} g^{(0)}_{\mu\nu} + \nabla^{(0)}_{\mu} \partial_{\nu} \psi^{(\frac{1}{1-n})} - \frac{1}{2} g^{(0)}_{\mu\nu} \phi_0^{n-1} \psi^{(\frac{1}{1-n})\, n} . \tag{3.67}$$

From a comparison to eq. (3.24), one can see that the derivation of the PN expansion to $\mathcal{O}_{PN}(2)$ is analogous to the calculation for the cubic galileon model. Performing the PN expansion on the 00, ij and $0j$ components of the field equation yields

$$-\frac{1}{2} \nabla^2 h_{00}^{(\frac{1}{1-n},\frac{2}{n-1})} = -\frac{1}{2} \nabla^2 \psi^{(\frac{1}{1-n},\frac{2}{n-1})} , \tag{3.68}$$

$$-\frac{1}{2} \overline{\nabla}^2 h_{mn}^{(\frac{1}{1-n},\frac{2}{n-1})} = \frac{1}{2} \overline{\nabla}^2 \psi^{(\frac{1}{1-n},\frac{2}{n-1})} \delta_{mn} , \tag{3.69}$$

$$0 = -\frac{1}{2} (\overline{\nabla}^2 h_{0j}^{(\frac{1}{1-n},\frac{n+1}{n-1})} + \frac{1}{2} h_{00,0j}^{(\frac{1}{1-n},\frac{2}{n-1})}) , \tag{3.70}$$

respectively, where we have used the gauge conditions (3.20) and (3.21). These equations are equivalent to eqs. (3.26), (3.27) and (3.31) but with a different leading-order scalar field solution. The reason for recovering the same results up to the scalar field equation is that for both the chameleon and cubic galileon model, we have a Brans-Dicke-like action amended by a non-linear screening term that vanishes in the PN expansion; the difference arises only in the scalar field profile capturing the screening mechanism.

We now consider the next-order PN correction to the scalar field equation (3.65).

This is easily found to be

$$\psi^{(\frac{1}{1-n},\frac{2n}{n-1})} = \left(\frac{M_p^{-2}\rho}{n\,\phi_0^n}\right)^{\frac{1}{n-1}} \frac{\Pi - 3p/\rho}{n-1}\,. \tag{3.71}$$

Again using the expansion for R_{00} up to quadratic terms in the metric and employing the gauge conditions (1.32) and (1.33), we find that

$$\begin{aligned}
\overline{\nabla}^2 h_{00}^{\left(\frac{1}{1-n},\frac{2n}{n-1}\right)} &= \overline{\nabla}^2 \psi^{\left(\frac{1}{1-n},\frac{2n}{n-1}\right)} - \psi_{,00}^{\left(\frac{1}{1-n},\frac{2}{n-1}\right)} - h_{00}^{(0,2)}\,\overline{\nabla}^2 \psi^{\left(\frac{1}{1-n},\frac{2}{n-1}\right)} \\
&+ \psi_{,j}^{\left(\frac{1}{1-n},\frac{2}{n-1}\right)}\, h_{00,j}^{(0,2)} - 2h_{00,j}^{\left(\frac{1}{1-n},\frac{2}{n-1}\right)}\, h_{00,j}^{(0,2)} + h_{jk}^{\left(\frac{1}{1-n},\frac{2}{n-1}\right)}\, h_{00,jk}^{(0,2)} \\
&+ h_{jk}^{(0,2)}\, h_{00,jk}^{\left(\frac{1}{1-n},\frac{2}{n-1}\right)} - \psi^{\left(\frac{1}{1-n},\frac{2}{n-1}\right)}\,\overline{\nabla}^2 h_{00}^{(0,2)} - \phi_0^{n-1}\,\psi^{\left(\frac{1}{1-n},\frac{2}{n-1}\right)n} \\
&- \frac{1}{2}\left(2\partial_j h_{ij}^{(0,2)} - \partial_i h_{jj}^{(0,2)} - \partial_i h_{00}^{(0,2)}\right)\psi_{,i}^{\left(\frac{1}{1-n},\frac{2}{n-1}\right)}\,.
\end{aligned} \tag{3.72}$$

As the functional form of these equations is identical to the results found for the cubic galileon, the PPN parameters γ and ξ can be defined equivalently. The parameter β differs as the screening term enters into the equation through the expansion in h_{00} at $\mathcal{O}_{PPN}(4)$. As such, β changes in eq. (3.55) by replacing β_{Cubic} with

$$\beta_{Cham} = \frac{-\Phi_{Cham}^{\left(\frac{1}{1-n},\frac{2n}{n-1}\right)}}{2G^{(0)2}U^2 + 4G^{(0)2}\Phi_2}\,, \tag{3.73}$$

$$\Phi_{Cham}^{\left(\frac{1}{1-n},\frac{2n}{n-1}\right)} = (\overline{\nabla}^2)^{-1}[\phi_0^{n-1}\psi^{\left(\frac{1}{1-n},\frac{2}{n-1}\right)n}]\,. \tag{3.74}$$

Alternatively, the direct parameterisation of the new potentials leads to an analogous result to that in eqs. (3.58)-(3.60). The parameterisation differs by the scalar field satisfying a different field equation, and so the σ_i parameters must be replaced with τ_1, τ_2 and τ_3. The potential $\Phi_{cubic}^{(-\frac{1}{2},3)}$ in eq. (3.58) is also replaced with $\Phi_{cham}^{(\frac{1}{1-n},\frac{2n}{n-1})}$.

3.5 Application to Shapiro time delay and measurements of γ

As the presence of gravitational fields distorts space-time away from flat space, light trajectories are not straight lines in the Newtonian sense, but rather are perturbed and the time taken to travel between two points is delayed. The constant PPN parameter γ was constrained by the Cassini mission to within 10^{-5} of the GR value ($\gamma = 1$) by measuring the time delay caused by a radio echo passing the Sun [24]. This tight constraint has proven to be among the most useful tools for constraining gravitational modifications in the Solar System.

If a light ray is emitted in a weak gravitational field at a point $\overline{x} = \overline{x}_e$ at a time $t = t_e$ in a direction $\hat{n}$, the light signal will follow the path

$$\overline{x}(t) = \overline{x}_e - \hat{n}(t - t_e) + \overline{x}_p(t)\,, \tag{3.75}$$

where $\overline{x}_p(t)$ is the perturbation away from a straight line. The Shapiro time delay of this can be found from the component of the perturbation in the direction of $\hat{n}$, $x_{p\|}$. To leading order, this component of the perturbation satisfies the equation

$$\frac{dx_{p\|}}{dt} = -\frac{1}{2}(h_{00} + h_{ii}) \tag{3.76}$$

$$= -(1+\gamma)G_{\mathit{eff}}U\,, \tag{3.77}$$

where there is no summation over the i index.

Importantly, it is this combination that is measured by the Cassini mission and it deviates from the standard equations for these components in the traditional PPN. This is because we allow both G_{eff} and γ to be functions of position. Recall that to leading order in α, the value of $G^{(0)}$ and $\gamma^{(0)}$ within in the Solar System are the ones of GR and so eq. (3.76) makes the same prediction as GR to this order.

Using the values for G_{eff} and γ, see table 3.1, one finds

$$\begin{aligned} G_{\mathit{eff}}U(1+\gamma) =& G\phi_0^{-1}U + \frac{1}{2}\alpha^q\psi^{(q,p)} + G\phi_0^{-1}U - \frac{1}{2}\alpha^q\psi^{(q,p)} \\ =& 2G\phi_0^{-1}U, \end{aligned} \tag{3.78}$$

where $q = -\frac{1}{2}$ and $p = 1$ for the cubic galileon model and $q = \frac{1}{1-n}$ and $p = \frac{2}{n-1}$ for the chameleon model. The combination hence contains no α-corrections and so both the cubic galileon and chameleon models cause no deviations in the second-order α perturbations of the metric. The only deviation from the standard result is the factor of ϕ_0^{-1}. However, the combination $G\phi_0^{-1}$ corresponds to the gravitational constant measured in a screened regime to leading order, eq. (3.17), and the result can thus be considered equivalent to that of the bare gravitational constant in GR.

The cancellation of modifications to the time delay also reoccurs at the next order α perturbation in the cubic galileon model. This is in agreement with the known result for Dvali-Gabadadze-Porrati (DGP) gravity [64] that no deviation in lensing occurs between DGP and GR [124]. Hence, we expect that the cancellation also applies to all higher orders in our method. Note that ref. [15] also found no change in the time delay for the cubic galileon at $\mathcal{O}_{PPN}(2)$. Similarly, it has been shown that for $f(R)$ gravity ($\omega = 0$) there is no deviation in the path of light rays from GR to $\mathcal{O}_{PN}(2)$ [23, 45] (cf. [52, 68, 84]), which is also consistent with our results.

Importantly, for the theories we have considered, the combination

$$(h_{ii} - h_{00}) = G_{\text{eff}} U(1 - \gamma) = 2\alpha^q \psi^{(q,p)} \tag{3.79}$$

is not constrained by time delay experiments. Hence, the often quoted Cassini bound [24] (applicable to standard PPN),

$$|\gamma - 1| < 2.3 \times 10^{-5}, \tag{3.80}$$

cannot straightforwardly be employed to constrain the value of the scalar field in the Solar System for screened models. This result further suggests that one may also want to be cautious in the interpretation of other Solar-System tests in the context of screening mechanisms.

It should be noted, however, that local constraints can be inferred on $f(R)$ gravity [84] and other chameleon models [116] from the requirement that the scalar field can settle from its unscreened value in the environment of the Milky Way to its screened value in the Solar-System region or within the region of the measured Milky Way rotation curve. The requirement of residing in a screened regime is also applicable here.

3.6 Conclusions

We have demonstrated how an efficient and systematic PPN expansion can be performed in the screened regimes of different modified gravity models with fundamentally different screening mechanisms. For this purpose, we extend the scaling method developed in chapter 2 to a higher-order expansion that can be applied in the screened or unscreened limits of the field equations of a given

Parameter	Interpretation	GR	Screened MG
G_{eff}	Gravitational strength.	G	$G\phi_0^{-1} + \alpha^q \frac{\psi^{(q,p)}}{2U}$
γ	Amount of curvature caused by mass.	1	$1 - \frac{2\alpha^q \psi^{(q,p)}}{2G\phi_0^{-1}U + \alpha^q \psi^{(q,p)}}$
β	Amount of non-linearity in superposition.	1	$1 + \alpha^q \left(\beta_{BD}^{(q)} + \beta_{Scr}^{(q)}\right)$
ξ_{ij}	Preferred location effects.	0	$\alpha^q \frac{(\epsilon_{jkl} W_k V_l)_{eff}}{\epsilon_{ikl} V_k W_l}$

Table 3.1 *The PPN parameters for the screened regions of the cubic galileon and chameleon models with their physical interpretations. Contrary to their usual definitions, they are promoted to functions and the models additionally introduce a varying gravitational coupling* $G_{\rm eff}$*. The scalar field perturbations* ψ *are solutions to the leading-order scalar field equations* (3.22) *for the cubic galileon model, where* $q = -\frac{1}{2}$, $p = 1$, *and* (3.66) *for the chameleon model, where* $q = \frac{1}{1-n}$, $p = \frac{2}{n-1}$. *Furthermore,* α *is the coupling to the screening terms in the action, used as the scaling parameter,* V_i *and* W_j *are the usual PPN vector potentials, and* $(\epsilon_{jkl} W_k V_l)_{\rm eff}$ *is defined in eq.* (3.52). *Finally,* $\beta_{\rm BD}$ *is a complicated function given by eq.* (3.56) *which is shared between the cubic galileon and chameleon models and* $\beta_{\rm Scr}$ *is either* $\beta_{\rm Cubic}$ *or* $\beta_{\rm Cham}$*, which are specified for each theory in eqs.* (3.57) *and* (3.73).

theory. This can then be combined with a low-energy, static expansion of the metric and scalar fields in these regimes. In particular, the procedure finds the PPN parameters for screened modified gravity models along with an effective gravitational coupling, which are generalised to time and space dependent functions. Moreover, we propose that the PPN parameter ξ, characterising preferred location effects, should be promoted to a matrix that rotates and stretches the PPN vector potentials. This is so that the expansion of the screened models can be placed into to the PPN framework without introducing new, theory-dependent potentials. Our method results in the PPN parameters being expressed as a series where the size of successive terms is controlled by the coupling constant associated with the screening term in the action of the modified gravity theory. Alternatively, we also present a direct parameterisation of the new potentials found in the expansion of the two screened theories in the spirit of the original PPN formalism.

We apply our method to calculate the PPN parameters for a cubic galileon and chameleon model and summarise our results in table 3.1. As expected due to screening, we find that these theories recover GR at leading order. We then compute the first-order correction to these parameters as functions of the scalar field. The functional forms for these corrections to the PPN parameters are equivalent among the two gravity models, which is attributed to the Brans-Dicke-type action adopted in both cases. The corrections themselves, however, differ due to the scalar field solving different equations of motion. As a further application of our method, we re-examine measurements of the Shapiro time delay. We find that the constraint traditionally quoted for the PPN parameter γ cannot directly be applied to modifications of gravity with screening effects. In particular, this is attributed to the generalised effective gravitational strength and γ combining to cancel deviations in observables from the predictions of GR. We explicitly show this for the cubic galileon and chameleon model, finding the time delay in screened regions predicted by both theories to be identical to that of GR. As a result, bounds on γ found from time delay experiments in the Solar System such as the Cassini mission [24] do not constrain these theories directly. That is, there are no deviations from GR as long as the Solar System can be considered screened. Hence, future tests of γ that depend on the time delay of light (see, e.g., ref. [137] for a review) will not trivially constrain these theories, regardless of accuracy, and more subtle considerations need to be made.

The natural extension of this work will be the application of our method to the Horndeski action [83], in particular, the subspace of models that allow for a screened Einstein limit as described in chapter 2. This seems feasible since in our method the equations of motion for the higher-order corrections become linear with the non-linearities restricted to the leading order and the background terms of each correction. Thus, the corrections to the metric and scalar field remain functions of known parameters and the leading-order solutions, which also contain the field profile required for screening. A caveat of our approach so far, however, is that in finding the PPN parameters, we have restricted to time independent background fields, which excludes screening effects relying on an evolving cosmological background (see, e.g., ref. [161]). That our method recovers the same results for the cubic galileon to $\mathcal{O}_{PPN}(2)$ as that found in ref. [15] is worthy of note. This is as the calculation therein uses Lagrange multipliers which one would not assume to be related to our method. Moreover, the order of expansions is reversed between the two calculations and so even should they both converge to the same result, it is interesting that they seem to recover the same result order by order. This correspondence warrants further study. Finally, the expansion developed in this chapter can also be applied to the calculation of gravitational waves in modified gravity theories [105, 192]. This in turn would allow for tests using both binary pulsar system [186] and direct gravitational wave detection [1], complimenting existing work [36]. It would furthermore be of interest to use our method to examine the power emitted by the quartic galileon, which does not converge, as shown in ref. [57], to test the convergence and consistency of the result with a different expansion prescription. We leave these analyses for future work.

Chapter 4

The effect of screening on the gravitational waveform

4.1 Introduction

The detection of gravitational waves has been a massive undertaking, resulting in the LIGO detector reporting their first detection in early 2016, [3], with which we are entering the era of gravitational wave astronomy, allowing a new window into the universe. So far, every gravitational wave detected has been from the merger of compact, stellar mass binary objects, either black holes or neutron stars [8]. The gravitational radiation emitted during the merger of more massive black holes will also be detectable by the eLISA mission, [11], detecting lower frequency gravitational waves. Thus, a wealth of new data about the universe will soon be available and new tests of gravity are available for the first time.

Upon detecting a signal, templates of gravitational waveforms are used to identify gravitational wave signals with the data using matched filtering processes. The use of waveforms predicted by general relativity allows for tests of astrophysics such as finding the merger rate [4], the mass function of black holes [99], or, when combined with electromagnetic signals, using these sources as standard candles [82].

However, the detection of gravitational waves also allows for unique tests of the theory of gravity. The waveform predicted by general relativity is assumed in the LIGO analysis, but modified gravity theories will in principle predict different waveforms. Thus, if one has waveforms from multiple gravity theories, one can perform model selection tests comparing the different theories. This would compliment existing tests of gravity, which test local solar system scales, [191], relativistic systems, [186], and in cosmology [151].

In order to boost the signal to noise in the match filtering process, a highly accurate waveform is required. An expansion of the gravitational waveform emitted during the inspiral of a compact binary can be found using post-Newtonian theory. This involves both finding the post-Newtonian (PN) dynamics of the binary system and of the post-Newtonian expansion of the metric. The post-Newtonian expansion is an expansion in orders of v/c, where a term of order $(v/c)^{2n}$ is said to be of nPN order. This convention is in contrast to how the post-Newtonian order is counted when considering the expansion for solar system tests, where this would be of 2nPN order. This work focuses on the calculation of the post-Newtonian expansion of the metric.

It should be noted that the total waveform, inspiral, merger and ringdown, cannot all be found analytically. The ringdown can be found using black-hole perturbation theory and the close limit approximation, [108]. However, the merger is inherently non-linear and requires numerical relativity simulations to find, [18, 155]. The numerical solutions can be combined with the inspiral signal in order to find the whole waveform, allowing for tests of gravity in the relativistic, strong-field regime.

The calculation of gravitational waveforms can be performed via multiple methods, such as that of Will and Wiseman, [192], the post-Minkowskian method of Blanchet, Damour and Iyer, [29, 30] as well as more modern derivations, [72, 76, 90]. These were developed independently but found to produce the same results confirming the validity of the results. This work will follow the method of Will and Wiseman, with [192] referred to as WW in the body of the text. This method was also chosen by Lang in [105] to calculate the gravitational waveform to second post-Newtonian order for Brans-Dicke theory, a prototypical scalar-tensor theory. We will refer to [105] to as Lang in the body of the test.

We calculate the gravitational waveform for a phenomenological model of screened gravity. This calculation is performed to 2PN beyond the leading order quadrupole term, as was done in WW and in Lang. It should be noted, however, that in general relativity, the expansion has been performed to 3PN, [29]. We take interest in screened gravity theories as screening mechanisms are often invoked

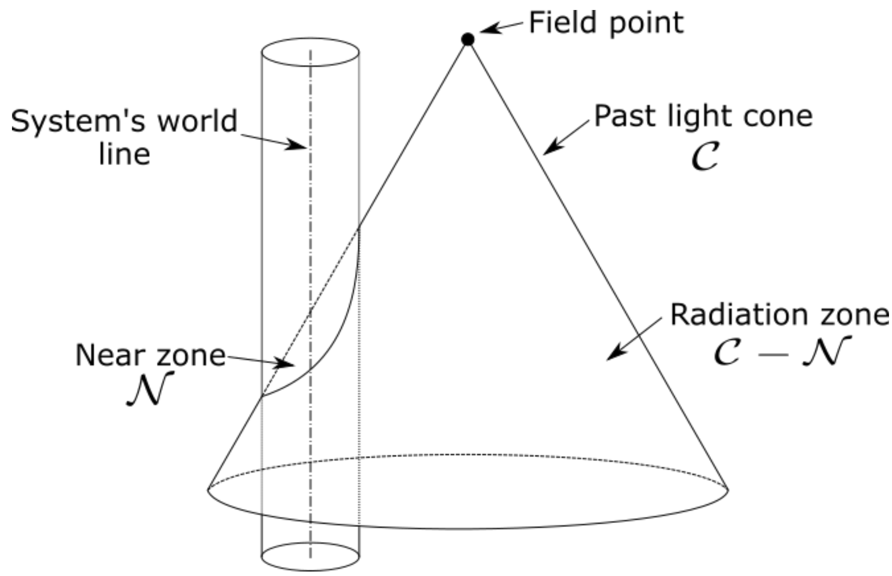

Figure 4.1 *The past light cone $\mathcal{C}$ of the field point of a detector. The near zone, $\mathcal{N}$, is the intersection of the past light cone and the world tube of the system's world line with radius $\mathcal{R}$. The radiation zone, $\mathcal{C} - \mathcal{N}$, is the remainder of the light cone. We assume that the metric field equations take on their screened form within $\mathcal{N}$, while they take their unscreened form in $\mathcal{C} - \mathcal{N}$.*

to pass solar system tests, and so we aim to provide another avenue to constrain these theories. The particular models we are interested in are those which at leading order mimic general relativity in screened regions and Brans-Dicke theory in unscreened regions. We can use the scaling method outlined in chapter 2 to justify taking the formal limit of the screened theory, allowing for the exact use of general relativity and Brans-Dicke theory.

The field equations are cast into their relaxed form; flat space wave equations sourced by non-linear self-interactions and the stress energy tensor. This allows for direct integration of the field equations using retarded Green's function. The evaluation of these integrals forms the bulk of this chapter. The past light cone $\mathcal{C}$ of the field point is split into two subsets. The near zone $\mathcal{N}$ describes the intersection of the past light cone with a world tube of a neighbourhood of the gravitating system, where the system has radius $\mathcal{R}$. The remainder of the past light cone, $\mathcal{C} - \mathcal{N}$, is called the radiation zone. The radiation zone is treated as containing no matter component, but will still contribute to the integral because of the gravitational field's non-linear self-interactions. The field point where we evaluate the gravitational fields can lie in either the near zone or the radiation zone, and the retarded integral has to be performed over the near zone and radiation zone in both cases. This leads to four distinct integrals that need to be performed and each is approached differently. We will let radius of the near zone be equal to the screening radius of our theory, such that we can treat gravity in the region $\mathcal{N}$ as screened and $\mathcal{C} - \mathcal{M}$ as unscreened. This geometry of the past light cone is shown in figure 4.1.

In section 4.2, we discuss the matter distribution and the field equations for our model in both their usual and relaxed forms. Section 4.3 finds the metric across the past light cone which will act as a source for the gravitational wave to 2PN beyond the quadrupole, which we calculate in section 4.4. The modified waveform is found to include all terms present in the general relativity waveform, with two new terms arising from the modification to gravity on large scales.

We specialise to the case where the system is a black hole binary in section 4.5. The gravitational waveform is then written in terms of the properties of the binary through use of the binary's equations of motion. The waveform is used with idealised data, made to reproduce the results of GW150914 when a GR template is used, [3] in order to show potential constraints on these modifications to gravity.

4.2 The modified field equations

We begin with a discussion of the matter distribution. In particular, the matter source is modeled as a collection of point particles with the stress energy tensor

$$T^{\alpha\beta} = \sum_A m_A \sqrt{-g} \frac{u_A^\alpha u_A^\beta}{U_A^0} \delta^3(\overline{x} - \overline{x}_A(t)) \,, \tag{4.1}$$

where u_A^α is the 4-velocity of the Ath body, m_A the gravitational mass including the gravitational binding energy of the Ath body, g the determinant of the metric. As the mass distribution is modelled as a series of point particles, they do not experience tidal effects. We define the total mass of the system as the sum of each point mass,

$$m = \sum_A m_A \,. \tag{4.2}$$

Motivated by the field equations for various screened gravity theories, see chapter

2, we consider the following tensor and scalar field equations,

$$\phi G_{\mu\nu} = 8\pi T_{\mu\nu} + \frac{\omega(\phi)}{\phi}(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda}) + (\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi) + A_{\mu\nu}(g_{\alpha\beta}, \phi), \tag{4.3}$$

$$\Box\phi = \frac{1}{3+2\omega(\phi)}(8\pi T - 16\pi\frac{\partial T}{\partial\phi} - \frac{d\omega}{d\phi}\phi_{,\lambda}\phi^{,\lambda}) + B(g_{\mu\nu}, \phi)\,. \tag{4.4}$$

Here, $g_{\mu\nu}$ is the metric, ϕ the scalar field, $G_{\mu\nu}$ is the Einstein tensor, $T_{\mu\nu}$ the stress energy tensor, T its trace, $\omega(\phi)$ the Brans-Dicke coupling function, $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ and commas denote partial derivatives. The additional two functions $A(g_{\mu\nu}, \phi)$ and $B(g_{\mu\nu}, \phi)$ are terms responsible for the screening effects in our model, for example, these could be the higher order derivatives of Galileon theories or scalar potentials of chameleon theories.

The modifications generated by A and B are to be chosen such that the screened limit is described by GR and the unscreened limit, Brans-Dicke theory. The functions can be chosen to satisfy this requirement through the use of the scaling method outlined in chapter 2. However, it should be noted that some screening mechanisms are also dependent on the symmetries of the matter distribution as well as its density, [33], but a dynamic black-hole binary causes screening for all terms investigated therein.

In line with the scaling method, we make a separation,

$$\phi = \phi_0(1 + \alpha^q\psi)\,. \tag{4.5}$$

where ϕ_0 is the background value of the scalar field, ψ the dynamic part of the scalar field, q a real number and α a scale in the theory appearing within A and B. Upon taking the limits α to infinity or zero, the screening terms A and B cause the metric field equations, (4.3), to take the forms:

$$G_{\mu\nu} = \frac{8\pi}{\phi_0}T_{\mu\nu}\,, \tag{4.6}$$

in the screened limit while in the unscreened limit, it becomes

$$\phi G_{\mu\nu} = 8\pi T_{\mu\nu} + \frac{\omega(\phi)}{\phi}(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda}) + (\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi)\,. \tag{4.7}$$

Note that A vanishes in both limits. The dynamic part of the scalar field ψ does not appear in the metric field equation (4.6) as it is the screened limit, where as (4.7) is the metric field equation for Brans-Dicke theory and describes the unscreened limit.

To calculate the gravitational waveform, we make use of the field equations in their relaxed form. This is achieved through a series of field redefinitions.

As the scalar field couples non-minimally to the metric, the derivation is different than in GR, hence we follow [131]. The scalar field is redefined as

$$\varphi = \frac{\phi}{\phi_0} = (1 + \alpha^q \psi) \,, \tag{4.8}$$

where we now choose the value of ϕ_0 to be the cosmological value of the scalar field. We assume that this is static for the purpose of this calculation as the time scales over which the background value varies is much larger than the orbital time scales of the system. That the gravitational wave may be produced at cosmological distances may have an effect on this assumption, but we will not consider it further.

The following redefinition of the metric is used,

$$h^{\mu\nu} = \eta^{\mu\nu} - \varphi\sqrt{-g}g^{\mu\nu} \,, \tag{4.9}$$

where g is the determinant of $g_{\mu\nu}$. The gravitational field is then identified with $h^{\mu\nu}$ as it describes the deviation from flat space, [134]. Note that we do not yet assume that $h^{\mu\nu}$ is small.

The harmonic gauge is imposed on the gravitational field $h^{\mu\nu}$,

$$h^{\mu\nu}{}_{,\nu} = 0 \,. \tag{4.10}$$

The reason for the somewhat strange field redefinition (4.9) is such that the metric field equation (4.3) can be re-written as

$$\Box_\eta h^{\mu\nu} = -16\pi\tau^{\mu\nu} - 2(-g)A^{\mu\nu}\frac{\phi}{\phi_0} \,, \tag{4.11}$$

where

$$\tau^{\mu\nu} \equiv (-g)\frac{\varphi}{\phi_0}T^{\mu\nu} + \frac{1}{16\pi}(\Lambda^{\mu\nu} + \Lambda_s^{\mu\nu}) \,. \tag{4.12}$$

One can think of Λ and Λ_s as the stress-energy held in the gravitational wave and scalar field respectively. These in turn are given by

$$\Lambda^{\mu\nu} \equiv 16\pi(-g)t^{\mu\nu}_{LL} + h^{\mu\alpha}{}_{,\beta}h^{\nu\beta}{}_{,\alpha} - h^{\alpha\beta}h^{\mu\nu}{}_{,\alpha\beta}\,, \tag{4.13}$$

where

$$\begin{aligned}(-g)t^{\mu\nu}_{LL} \equiv \frac{1}{16\pi}\big[&g_{\lambda\alpha}g^{\beta\rho}h^{\mu\lambda}{}_{,\beta}h^{\nu\alpha}{}_{,\rho} + \frac{1}{2}g_{\lambda\alpha}g^{\mu\nu}h^{\lambda\beta}{}_{,\rho}h^{\rho\alpha}{}_{,\beta} - 2g_{\alpha\beta}g^{\lambda(\mu}h^{\nu)\beta}{}_{,\rho}h^{\rho\alpha}{}_{,\lambda} \\ &+ \frac{1}{8}(2g^{\mu\lambda}g^{\nu\alpha} - g^{\mu\nu}g^{\lambda\alpha})(2g_{\beta\rho}g_{\sigma\tau} - g_{\rho\sigma}g_{\beta\tau})h^{\beta\tau}{}_{,\lambda}h^{\rho\sigma}{}_{,\alpha}\big]\,,\end{aligned} \tag{4.14}$$

which is the Landau-Lifshitz tensor, and

$$\Lambda^{\mu\nu}_s \equiv \frac{3+2\omega}{\varphi^2}\varphi_{,\alpha}\varphi_{,\beta}\left((\eta^{\mu\alpha} - h^{\mu\alpha})(\eta^{\nu\beta} - h^{\nu\beta}) - \frac{1}{2}(\eta^{\mu\nu} - h^{\mu\nu})(\eta^{\alpha\beta} - h^{\alpha\beta})\right)\,. \tag{4.15}$$

The scalar field satisfies the equation

$$\Box_\eta\varphi = -8\pi\tau_s + B \tag{4.16}$$

where the source of the scalar field equation is given by

$$\begin{aligned}\tau_s = &-\frac{1}{3+2\omega}\sqrt{-g}\frac{\varphi}{\phi_0}\left(T - 2\varphi\frac{\partial T}{\partial\psi}\right) - \frac{1}{8\pi}h^{\alpha\beta}\varphi_{,\alpha\beta} \\ &+ \frac{1}{16\pi}\frac{d}{d\varphi}\left[\ln\left(\frac{3+2\omega}{\varphi^2}\right)\right]\varphi_{,\alpha}\varphi_{,\beta}(\eta^{\alpha\beta} - h^{\alpha\beta})\,.\end{aligned} \tag{4.17}$$

When taking the screened limit, one finds that finds $\varphi \to 1$. Thus, the relaxed field equations reduce to

$$\Box_\eta h^{\mu\nu} = -16\pi(-g)\frac{\psi}{\phi_0}T^{\mu\nu} - \Lambda^{\mu\nu}\,. \tag{4.18}$$

and

$$0 = \frac{8\pi\sqrt{-g}T}{\phi_0(3+2\omega)} + B\,. \tag{4.19}$$

In the unscreened limit, the field equations take the form

$$\Box_\eta h^{\mu\nu} = 16\pi g\frac{\psi}{\phi_0}T^{\mu\nu} - (\Lambda^{\mu\nu} + \Lambda^{\mu\nu}_s)\,, \tag{4.20}$$

and

$$\Box_\eta \varphi = -8\pi\tau_s \tag{4.21}$$

Note that in both limits, we have that the effective stress-energy tensor $\tau^{\mu\nu}$ is conserved, $\tau^{\mu\nu}_{\mu} = 0$.

As the field equations take the form of wave equations, one can express them as retarded integrals,

$$h^{\mu\nu}(x,t) = 4\int_{\mathcal{C}} \frac{\tau^{\mu\nu}(t-|\overline{x}-\overline{x}'|,\overline{x}')}{|\overline{x}-\overline{x}'|}d^3x\,, \tag{4.22}$$

$$\varphi(x,t) = 2\int_{\mathcal{C}} \frac{\tau_s(t-|\overline{x}-\overline{x}'|,\overline{x}')}{|\overline{x}-\overline{x}'|}d^3x\,, \tag{4.23}$$

where $\mathcal{C}$ is the past light cone of the field point x. However, we can split the integration domain into parts of the past light cone which are screened, $\mathcal{N}$, and the parts that are unscreened, $\mathcal{C}-\mathcal{N}$. Importantly, the form that both $\tau^{\mu\nu}$ and τ_s take are different in these two regimes, This leads to four possibilities:

- The field point is in a screened regime and we are integrating over the screened regime (section 4.3.1).
- The field point is in a screened regime and we are integrating over the unscreened regime.
- The field point is in a unscreened regime and we are integrating over the screened regime, section 4.3.2 and 4.4.1.
- The field point is in a unscreened regime and we are integrating over the unscreened regime, (4.3.2 and 4.4.2).

Each possibility will require a different method, while the second does not need to be found in order to find the gravitational wave to order $(v/c)^4$ ([192]).

Finding these contributions will make the bulk of this chapter.

4.3 Finding the metric to 2PN

As we aim to find the gravitational waveform to 2PN beyond the quadrupole approximation (4PN), and so need the metric across the past light cone to 2PN.

This seems counter-intuitive as one would expect the fields would have to be known to 3PN. However, that the source satisfies $\partial_\mu \tau^{\mu\nu}$, τ^{ij} can be replaced with two time derivatives of $\tau^{00}x^i x^j$, raising the PN order in the near zone because of the slow motion limit, $v \ll c$. Thus we only need to find the metric in the near zone to 2PN, the aim of section 4.3.1.

The metric in the radiation zone also needs to be found to 2PN, this is as higher order terms would not contribute to the 2PN waveform when evaluated far from the source and so we find the metric in the radiation zone in section 4.3.2. To this order, contributions from the integration over the near zone and the radiation zone both contribute.

4.3.1 Metric in the near-zone

The gravitational field in the near zone is needed in order to calculate the gravitational waveform to 2PN beyond the quadrupole approximation. The contributions made to the waveform by the metric in the near zone are two-fold. It acts as a source for higher order contributions within the "gravitational stress-energy", $\Lambda^{\mu\nu} \in \tau^{\mu\nu}$, (4.13). It also contributes indirectly by sourcing the metric in the radiation zone, which in turn modifies the gravitational wave. This can be understood as the light cone being modified by the gravitational field, changing the propagation of the gravitational wave.

For a many body system, we define the characteristic size of the system $\mathcal{S}$ as the $\mathcal{S} = \max\{r_{AB}\}$. The radius $\mathcal{R}$ of near zone is typically defined to be the characteristic wavelength of the gravitational radiation about the centre of mass of the system. As the wavelength of the radiation is much larger than the system, $\mathcal{R} \gg \mathcal{S}$, the retardation of the fields is negligible and time derivatives can be treated as a higher order term justifying the use of the static approximation. This is the same as saying that the metric can be found in terms of instantaneous potentials. We will denote the constant time hyper-surface at time $u = t - R$ as $\mathcal{M}$. We will set the boundary size, $\mathcal{R}$, to the screening radius of the system. This is so that all field points within the radius will experience GR as the effective gravity theory.

Because the metric field equation in the screened region is the same as GR, the near zone metric is the same as that from GR. Below the metric in the near zone is given, for the details of the calculation see section III of WW. The calculation is very similar to the post-Newtonian expansions performed in the previous chapter, section 1.4.3 and section 1.6.1, and so we will not discuss it further.

The gravitational field components are

$$h^{00} =4\left(\phi_0{-}1U + \frac{1}{2}\phi_0^{-1}\partial_t^2 X - \phi_0^{-2}P + 2\phi_0^{-2}U^2\right)\,, \tag{4.24a}$$

$$h^{0i} =4\phi_0{-}1U_i\,, \tag{4.24b}$$

$$h^{ij} =4\phi_0{-}2P_{ij}\,, \tag{4.24c}$$

and for the metric

$$g^{00} = -(1 + 2\phi_0{-}1U + \phi_0{-}1\partial_t^2 X + 2\phi_0{-}2U^2) + \mathcal{O}(\epsilon^{5/2})\,, \tag{4.25a}$$

$$g^{0i} = -4\phi_0{-}1U_i + \mathcal{O}(\epsilon^{5/2})\,, \tag{4.25b}$$

$$g^{ij} = (1 - 2\phi_0{-}1U - \phi_0{-}1\partial_t^2 X)\delta^{ij} + \mathcal{O}(\epsilon^{5/2})\,, \tag{4.25c}$$

from which the determinant of the metric is found to be

$$-g = 1 + 4\phi_0^{-1}(U + \frac{1}{2}\delta_t^2 X) - 8\phi_0^{-2}(P - U^2) + \mathcal{O}(\epsilon^{5/2})\,. \tag{4.26}$$

As we have found the metric in the Newtonian screened regime, we can ascribe Newton's constant to the inverse of the background value of the scalar field,

$$G = \frac{1}{\phi_0}. \tag{4.27}$$

This is so that Poisson's equation is recovered correctly. Contrast this with $G = \frac{4+2\omega_0}{\phi_0(3+2\omega)}$ in Brans-Dicke theory, [34].

The instantaneous potentials used are defined as

$$U(u,\overline{x}) = \int_{\mathcal{M}} \frac{d^3x}{|\overline{x} - \overline{x}'|}(T^{00} + T^{ii})(u,\overline{x}')\,, \tag{4.28a}$$

$$X(u,\overline{x}) = \int_{\mathcal{M}} d^3x|\overline{x} - \overline{x}'|(T^{00} + T^{ii})(u,\overline{x}')\,, \tag{4.28b}$$

$$U_i(u,\overline{x}) = \int_{\mathcal{M}} \frac{d^3x}{|\overline{x} - \overline{x}'|}T^{0i}(u,\overline{x}')\,, \tag{4.28c}$$

$$P_{ij}(u,\overline{x}) = \int_{\mathcal{M}} \frac{d^3x}{|\overline{x} - \overline{x}'|}\left[\phi_0 T^{ij} + \frac{1}{4\pi}\left(U_{,i}U_{,j} - \frac{1}{2}\delta_{ij}U_{,k}U_{,k}\right)\right](u,\overline{x}')\,, \tag{4.28d}$$

With the metric in the near zone found to $\mathcal{O}(\epsilon^{5/2})$, the stress-energy tensor in the near zone can also be found to $\mathcal{O}(\epsilon^{5/2})$. This is needed when finding the radiation zone metric. The stress-energy tensor is found to be

$$\begin{aligned}
T^{00} =& \sum_A m_A \big[1 - \phi_0^{-1} U + \frac{\phi_0^{-1}}{2}\partial_t^2 X + \frac{1}{2} v_A^2 + \frac{\phi_0^{-2}}{2} U^2 + \frac{3\phi_0^{-1}}{2} U v_A^2 \\
&+ 4\phi_0^{-2} P + \frac{3}{8} v_A^4 - 4\phi_0^{-1} U_i v_A^i + \mathcal{O}(\epsilon^{5/2})\big] \delta^3(\overline{x} - \overline{x}_A(t)) \,, \qquad (4.29a)\\
T^{ij} =& \sum_A m_A v_A^i v_A^j \big[1 - \phi_0^{-1} U + \frac{1}{2} v_A^2 + \mathcal{O}(\epsilon^2)\big] \delta^3(\overline{x} - \overline{x}_A(t)) \,, \qquad (4.29b)\\
T^{0j} =& \sum_A m_A v_A^i \big[1 - \phi_0^{-1} U - \frac{\phi_0^{-1}}{2}\partial_t^2 X + \frac{1}{2} v_A^2 + \mathcal{O}(\epsilon^{5/2})\big] \delta^3(\overline{x} - \overline{x}_A(t)) \qquad (4.29c)
\end{aligned}$$

We will only need the source $\tau^{\mu\nu}$ for the calculations performed in section 4.3.2 to the orders $\mathcal{O}(\rho\epsilon)$ for τ^{00}, $\mathcal{O}(\rho\epsilon)$ for τ^{ij} and $\mathcal{O}(\rho\epsilon^{3/2})$ for τ^{0i}. With the stress-energy tensor defined above, we are left needing to show $\Lambda^{\mu\nu}$ to the required orders.

The non-compact part of the effective stress energy tensor, eq.(4.13), with the near zone metric (eq. (4.24) and (4.25)) is found to be

$$\begin{aligned}
\Lambda^{00} =& 14(U + 1/2\partial_t^2 X)^2_{,k} + 16[-U\ddot{U} + U_{,k}\dot{U}_{,k} - 2U_k\dot{U}_{,k} + 5/8\dot{U}^2 \\
&+ 1/2 U_{m,k}(U_{m,k} + 3U_{k,m}) + 2P_{,k}U_{,k} - P_{kl}U_{,kl} \\
&- 7/2 U U_{,k}^2] + \mathcal{O}\left(\rho\epsilon^3\right) \,, \qquad (4.30a)\\
\Lambda^{0i} =& 16[(U + \partial_t^2 X)_{,k}(U_{k,i} - U_{i,k}) + \frac{3}{4}\dot{U}(U + \partial_t^2 X)_{,i}] + \mathcal{O}\left(\rho\epsilon^{5/2}\right) \,, \qquad (4.30b)\\
\Lambda^{\text{ij}} =& 4[(U + \partial_t^2 X)_{,i}(U + \partial_t^2 X)_{,j} - \frac{1}{2}\delta_{ij}(U + \partial_t^2 X)_{,k}(U + \partial_t^2 X)_{,k}] \\
&+ 16[2U_{,(i}\dot{U}_{j)} - U_{k,i}U_{k,j} - U_{i,k}U_{j,k} + 2 - U_{k,(i}U_{j),k} \\
&- \delta_{ij}(3/2\dot{U}^2 + U_{,k}\dot{U}_k - U_{m,k}U_{[m,k]}] + \mathcal{O}\left(\rho\epsilon^{5/2}\right) \,. \qquad (4.30c)
\end{aligned}$$

We do not need to find Λ_s in this region as it vanishes due to the screening effect.

In section 4.3.2, we will also need τ_s evaluated in the near zone. Restricting τ_s to the near zone amounts to the limit $\varphi \to 1$, which substantially simplifies (4.17) to

$$\begin{aligned}
\tau_s =& -\frac{\sqrt{-g}\, GT}{(3+2\omega)} \\
=& \frac{G}{(3+2\omega)} \sum_A m_A \left(1 - \frac{1}{2} m_A v_A^2 + m_A U\right) \delta^3(\overline{x} - \overline{x}_A(t)) \,. \qquad (4.31)
\end{aligned}$$

4.3.2 Metric in the radiation-zone

At leading order, there is no contribution to the gravitational waveform due to the radiation zone. This is as the leading order gravitational waves are sourced from the compact matter component which lies solely in the near zone. However as we calculate the gravitational wave to higher orders, the non-compact stress energy components, Λ^{ij} and Λ_s^{ij} in eq.(4.12), contribute to the waveform the scattering of gravitational waves of the background. As such, we need to compute the metric in the radiation zone as sourced by the near zone, which we will denote $h^{\mu\nu}_{\mathcal{N}}$ in line with [105, 192].

One should expect that the results of this section mimic those of general relativity as to find $h^{\mu\nu}_{\mathcal{N}}$, we calculate the retarded integral

$$h^{\mu\nu}_{\mathcal{N}}(t', x') = 4 \int_{\mathcal{N}'} \frac{\tau^{\mu\nu}(t' - |\overline{x}' - \overline{x}''|, \overline{x}'')}{|\overline{x}' - \overline{x}''|} d^3x' \,. \tag{4.32}$$

The domain of the integral, $\mathcal{N}'$, is not the same as the near zone $\mathcal{N}$ in the detectors past light cone. This is as we are finding the metric at an arbitrary point in the radiation zone with the system at a distance R' from the field point, in the direction $\hat{N}'i = x'^i/R'$. The metric at the detector will be found by taking $R' = R$, as we will see in section 4.4.2.

$\tau^{\mu\nu}$ takes the same form as in general relativity and hence the metric solution will be the same as in GR.

Further, we can guess from no-hair theorems such as [79] that the metric should mimic GR, but see [143, 168, 168] for black holes in general scalar tensor theories. However, we are considering a field point far from the source during the inspiral in the Brans-Dicke limit, so the system should look almost static and spherical, and the assumptions of no hair theorem approximately applies. Importantly, this will not mean that our final answer will be subject to such theorems and produce a waveform identical to GR as the presence of gravitational waves explicitly breaks the static assumption.

Again we will leave out the details of the calculation and restrict this section to an outline of how it is performed.

The retarded integral (4.32) is evaluated over the near zone, and so the retarded integral can be expressed as the sum of moments. This expansion is valid as the distance to the field point, R, is much greater than the characteristic size of the system, $\mathcal{S}$ leading to an expansion in R'^{-1}. As such,

$$h^{\mu\nu}(t,\overline{x}) = 4\sum_{q=0}^{\infty}\frac{(-1)^q}{q!}\left(\frac{1}{R'}M^{\mu\nu k_1\ldots k_q}\right)_{,k_1\ldots k_q}, \tag{4.33}$$

where

$$M^{\mu\nu k_1\ldots k_q} = \int_{\mathcal{M}'}\tau^{\mu\nu}x'^{k_1}\ldots x'^{k_q}d^3x'. \tag{4.34}$$

We evaluate these moments at across a constant time hyper-surface $\mathcal{M}'$ at the retarded time $\tau = t - R'$. The integral is still over the near zone and so the effective stress-energy tenser takes its GR form and Λ_s does not contribute.

To 2PN, only the compact part of $\tau^{\mu\nu}$ contributes, where higher order compact contributions arise through inserting the instantaneous potentials into $\sqrt{-g}$. This renders these integrals relatively simple because of the δ functions in the stress-energy tensor.

Many of the higher order moments can be written in terms of the lower moments together with the momentum current moments

$$\mathcal{J}^{iQ} = \epsilon^{iab}\int_{\mathcal{M}}\tau^{0b}x^{aQ}d^3x\,, \tag{4.35}$$

where ϵ^{iab} is the Levi-Civita symbol.

After some calculation, the metric in the radiation zone is found to be (WW eq.5.5, cf. Lang eq.4.33),

$$h^{00}_{\mathcal{N}} = 4\frac{\tilde{m}}{\phi_0 R'} + 7\left(\frac{m}{\phi_0 R'}\right)^2 + 2\left(\frac{M^{ij}}{R'}\right)_{,ij} - \frac{2}{3}\left(\frac{M^{ijk}}{R'}\right)_{,ijk}, \tag{4.36a}$$

$$h^{0i}_{\mathcal{N}} = -2\left(\frac{\dot{M}^{ij} - \epsilon^{ija}\mathcal{J}^a}{R'}\right)_{,j} + \frac{2}{3}\left(\frac{\dot{M}^{ijk} - 2\epsilon^{ika}\mathcal{J}^{aj}}{R'}\right)_{,jk}, \tag{4.36b}$$

$$h^{ij}_{\mathcal{N}} = \left(\frac{m}{\phi_0 R'}\right)^2\hat{N}'^{ij} + 2\frac{\ddot{M}^{ij}}{R'} - \frac{2}{3}\left(\frac{\ddot{M}^{ijk} - 4\epsilon^{(i|ka}\dot{\mathcal{J}}^{a|j)}}{R'}\right)_{,k}, \tag{4.36c}$$

where all moments are evaluated on the constant $u' - r'$ hyper-surface $\mathcal{M}'$, $\tilde{m} = \sum_A m_A + \frac{1}{2}m_A v_A^2 - \frac{1}{2}m_A\phi_0^{-1}U_A$ and U_A is the total gravitational potential at the point x_A ignoring the infinite self energy contribution of body A.

We are left to find the scalar field in the radiation zone, $\varphi_{\mathcal{N}}$, to 2PN. The integral

(4.23) evaluated over the near zone can also be expanded into the sum of moments,

$$\varphi_{\mathcal{N}}(t,\overline{x}) = 2\sum_{q=0}^{\infty}\frac{(-1)^q}{q!}\left(\frac{1}{R}M_s^{\mu\nu k_1\ldots k_q}\right)_{,k_1\ldots k_q}, \tag{4.37}$$

where

$$M_s^{k_1\ldots k_q} = \int_{\mathcal{M}}\tau_s x'^{k_1}\ldots x'^{k_q}d^3x', \tag{4.38a}$$

where we have τ_s from eq.(4.31), the calculation is otherwise equivalent to that of the metric expansion. The $\mathcal{O}(\rho\epsilon^{3/2})$ moments are found to be

$$M_s = \frac{1}{\phi_0(3+2\omega)}\sum_A m_A(1-\frac{1}{2}v_A^2+\phi_0^{-1}U_A)\,, \tag{4.38b}$$

$$M_s^i = \frac{1}{\phi_0(3+2\omega)}\sum_A m_A x_A^i(1-\frac{1}{2}v_A^2+\phi_0^{-1}U_A)\,, \tag{4.38c}$$

$$M_s^{ij} = \frac{1}{\phi_0(3+2\omega)}\sum_A m_A x_A^{ij}\,, \tag{4.38d}$$

$$M_s^{ijk} = \frac{1}{\phi_0(3+2\omega)}\sum_A m_A x_A^{ijk}\,. \tag{4.38e}$$

Note that these differ from the results of Lang eq.6.11 due to the scalar source $\Lambda_s^{\mu\nu}$ being different due to the presence of screening. Because of this difference, when evaluated in the center of mass frame corrected to 2PN, the scalar dipole moment vanishes, unlike in the Brans-Dicke case. We will see this has a major effect as many terms present in Brans-Dicke theory will no longer contribute.

The scalar field in the radiation zone is found by inserting these moments into (4.37),

$$\varphi_{\mathcal{N}} = 2\frac{M_s}{r'} - 2\left(\frac{M_s^i}{r'}\right)_{,i} + \left(\frac{M_s^{ij}}{r'}\right)_{,ij} - \frac{1}{3}\left(\frac{M_s^{ijk}}{r'}\right)_{,ijk}. \tag{4.39}$$

Hence we have found the metric and scalar field in the radiation zone which is sourced by the near zone to $\mathcal{O}(\rho\epsilon^{3/2})$.

4.4 Finding the waveform

With the metric found in both the near and radiation zones, we are free to calculate the gravitational waveform. The calculation of the waveform contributions from both the near and radiation zones differs just as when finding the metric. As such we will examine both separately.

Recall that the observer of the gravitational wave resides in a screened region. This means that only the metric wave needs to be found as the scalar field does not interact with the detector. Hence we only need to solve (4.22) to 2PN beyond the quadrupole approximation.

Moreover, the calculation occurs for a field point where $R \gg \mathcal{R} > \mathcal{S}$. As a result, we may discard terms that fall off faster than $1/R$. We will refer to this region of spacetime as the far zone to distinguish it from the radiation zone.

In section 4.4.1, we outline how the near zone contributes to the gravitational wave. No further calculation is needed as the source in the near zone is equivalent to that of general relativity, and hence we can use the results of WW. In section 4.4.2, we discuss the contribution from the radiation zone. As the metric found from Λ^{ij} again takes a form equivalent to GR, we will defer to WW for the bulk of the calculation. However, unlike in GR, the wave is also sourced by the Λ_s leading to new terms.

4.4.1 Waveform contribution from the near-zone

The calculation of the gravitational waveform to 2PN beyond quadrupole at the detector is effectively just a continuation of the calculation performed in section 4.3.2. The main difference is that now we need only to keep terms in the metric expansion that fall off as R^{-1} because terms that fall off faster are entirely negligible when evaluated at the detector in the far zone ($\mathcal{S} < \mathcal{R} \ll R$). Further, as we are evaluating the waveform and not whole the metric, we must remember that only the spatial parts of the metric solution are needed and will undergo a transverse and traceless projection. This implies that terms proportional to either δ_{ij} or the position vector of the detector, $\hat{N}^i = x^i/R$, will not survive the projection. The calculation of the gravitational waveform due to the near zone is analogous to WW, so again we defer to [192] for the details and only give a brief overview.

The moments in the expansion (4.33) can in turn be expanded in R^{-1}, and using that we are calculating the metric in the far zone only the leading order is required. The expansion can then be expressed in terms of "Epstein-Wagoner" moments by using that the effective stress energy tensor is conserved, eq.(4.21). The simplifies eq.(4.33) to

$$h^{ij}_{\mathcal{N}} = \frac{2}{R}\frac{d^2}{dt^2}\sum_{m=0}^{\infty}\hat{N}_{k_i}\ldots\hat{N}_{k_m}I^{ijk_1\ldots k_m}_{EW}\,, \tag{4.40}$$

where

$$I^{ij}_{EW} = \int_{\mathcal{M}}\tau^{00}x^ix^jd^3x + I^{ij}_{EW(surf)}\,, \tag{4.41a}$$

$$I^{ijk}_{EW} = \int_{\mathcal{M}}(2\tau^{0(i}x^{j)}x^k - \tau^{0k}x^ix^j)d^3x + I^{ijk}_{EW(surf)}, \tag{4.41b}$$

$$I^{ijk_1\ldots k_m}_{EW} = \frac{2}{m!}\frac{d^{m-2}}{dt^{m-2}}\int_{\mathcal{M}}\tau^{ij}x^{k_1}\ldots x^{k_m}d^3x \quad (m\geq 2)\,, \tag{4.41c}$$

and

$$I^{ij}_{EW(surf)} = \oint_{\delta\mathcal{M}}(4\tau^{l(i}x^{j)} - (\tau^{kl}x^ix^j)_{,k})\mathcal{R}^2\hat{n}^ld^2\Omega\,, \tag{4.42a}$$

$$I^{ijk}_{EW(surf)} = \oint_{\delta\mathcal{M}}(2\tau^{l(i}x^{j)}x^k - \tau^{kl}x^ix^j)\mathcal{R}^2\hat{n}^ld^2\Omega\,, \tag{4.42b}$$

where $\hat{n}$ is a radially outward unit vector.

The explicit calculation of these moments is a large undertaking, making up the bulk of both WW and Lang, [105, 192]. Due to the similarity of the calculation performed in WW and here, we will not replicate it.

In brief, τ^{ij} is again written out in terms of the instantaneous potentials using the results (4.29) and (4.30). The potentials are then be solved using the matter distribution (4.1). Once found, the Epstein-Wagoner moments are inserted in the expansion (4.40)

The solutions for the Epstein-Wagoner moments are given in WW 6.6. However, in the calculation of the gravitational waveform in GR, terms that depend on $\mathcal{R}$ from the near zone contribution and the radiation zone contribution cancel out.

There is no guarantee of this for our model so we highlight the terms

$$
\begin{aligned}
I_{EW}^{ij} \supset & -\frac{14}{5} m\mathcal{R}\ddot{M}^{ij} \\
= & -\frac{28}{5} m\mu\mathcal{R}\big(v^{ij} - x^{(i}a^{j)}\big)\,, && (4.43a) \\
I_{EW}^{ijkl} \supset & -\frac{8}{35} m\mathcal{R}\ddot{M}^{ij}\delta^{kl} \\
= & -\frac{8}{35} m\mu\mathcal{R}\big(v^{ij} - x^{(i}a^{j)}\big)\delta^{kl}\,, && (4.43b) \\
I_{EW}^{ijklmn} \supset & -\frac{2}{315} m\mathcal{R}\ddot{M}^{ij}\delta^{(kl}\delta^{mn)} \\
= & -\frac{2}{315} m\mu\mathcal{R}\big(v^{ij} - x^{(i}a^{j)}\big)\delta^{(kl}\delta^{mn)}\,. && (4.43c) \\
& && (4.43d)
\end{aligned}
$$

4.4.2 Waveform contribution from the radiation zone

When finding the metric in the radiation zone in section 4.3.2, we found that the metric sourced by the near zone remains equivalent to that in general relativity. Deviations in the metric occur due to self-interaction in the radiation zone. That the gravity theory in the radiation region is Brans-Dicke theory only begins to have an effect on the metric in this region at $\mathcal{O}(\epsilon^2)$. Hence we would only expect these terms to contribute at order $\mathcal{O}(\epsilon^{1.5})$ beyond the quadrupole approximation.

Λ in the radiation zone up to $\mathcal{O}(\epsilon^3)$ is found to be

$$\Lambda^{ij} \supset -h_{(1)}^{00}\ddot{h}_{(2)}^{ij} + \frac{1}{4}h_{(1),(i}^{00}h_{(2),j)}^{00} + \frac{1}{2}h_{(1),(i}^{00}h_{(2),j)}^{kk} + 2h_{(1)}^{00,(i}\dot{h}_{(2)}^{j)0}\,, \tag{4.44}$$

where the subscript denotes the PN order used.

As such, we may take the results of WW eq. 5.8 for the metric components h^{ij} to order $\mathcal{O}(\epsilon^2)$ beyond the quadrapole order,

$$h^{ij} \supset \frac{4m}{R}\int_0^\infty ds\partial_t^4 M^{ij}(u-s)\left[\ln\left(\frac{s}{2R+s}\right) + \frac{11}{12}\right]. \tag{4.45}$$

This is the first tail term, and is of order $\mathcal{O}(\epsilon^{3/2})$ beyond the quadrapole, so only one more iteration is required.

The components of the gravitational stress-energy tensor that we need to consider are

$$\Lambda^{ij} \supset -h^{00}_{(1)}\ddot{h}^{ij}_{(1.5)} + \frac{1}{2}h^{00}_{(1),i}h^{00}_{(1.5),j} + \frac{1}{2}h^{00}_{(1),(i}h^{kk}_{(1.5),j)}. \tag{4.46}$$

For the first term, the source is proportional to δ_{ij} and so is not transverse-traceless. In the second and third, monopole-monopole couplings will go as $R^{-1}{}_{,(i}R^{-1}{}_{,j)}$, which when the derivatives are evaluated will go as $\hat{n}^{ij}$ which is not transverse-traceless. Hence, there only terms from monopole-quadrapole, mono-pole quadrapole and monopole-current quadrapole are relevant in Λ^{ij}.

The metric components due to the near zone again mimic those of GR leading to the result

$$\begin{aligned} h^{ij} \supset & \frac{4m}{3R}\hat{n}^k \int_0^\infty ds \partial_t^5 M^{ijk}(u-s)\left[\ln\left(\frac{s}{2R+s}\right) + \frac{97}{60}\right] \\ & - \frac{16m}{3R}\epsilon^{(i|ka}\hat{n}^k \int_0^\infty ds\partial_t^4 J^{a|j)}(u-s)\left[\ln\left(\frac{s}{2R+s}\right) + \frac{7}{6}\right] \\ & + \frac{1912}{315}\frac{m}{R}\partial_t^4 M^{ij}(u)\mathcal{R}\,, \end{aligned} \tag{4.47}$$

which is the remainder of WW eq.5.8.

We see that the contribution from the boundary, the term linear in $\mathcal{R}$, exactly cancels the contributions from (4.43). This is somewhat disappointing, as a contribution from the boundary would be a function of $\mathcal{R}$ and so allow for constraints to be placed upon the screening radius. It is possible that a contribution will arise at higher orders as the Brans-Dicke modification begins to propagate through the calculations.

The final contributions to be found come from Λ_s^{ij}, (4.15). The calculation is similar to that of Lang section VI.C, however, we have differing prefactors. In section 4.3.2, we found the spatial part of the metric perturbation $\tilde{h}^{ij}$ went as R^{-2} to leading order (1PN beyond the quadrapole). Hence, the leading order does not contribute to the gravitational waveform in the far zone.

For the next order, we expand Λ_s to 0.5PN beyond the quadrapole to find

$$\tau^{ij} \supset \frac{\pi}{16}(3+2\omega)\varphi^{,(i}_{Mono}\varphi^{,j)}_{Di}\,, \tag{4.48}$$

where φ_{Mono} is the monopole term and φ_{Di} the leading order dipole term in the scalar solution (4.39). However, the dipole vanishes in centre of mass coordinates. Also, this would not leave any transverse-traceless terms after taking the projection. Hence there are no new 0.5PN contributions.

At 1PN beyond the monopole, we find

$$\tau^{ij} \supset \frac{\pi}{16}(3+2\omega)\big(\varphi^{,(i}_{Mono}\varphi^{,j)}_{Quad} + \varphi^{,(i}_{Mono}\varphi^{,j)}_{Mono}\big)\,, \tag{4.49}$$

where φ_{Quad} is the quadrapole term, and the cross term monopoles terms are evaluated at 1PN order.

The expansion in terms of the moments of the scalar field is analogous to Lang eq.6.29, giving additional contribution to the gravitational wave,

$$h^{ij} \supset \frac{4m}{\phi_0 R}\bigg(- \frac{1}{12}\partial^3_\tau M^{ij}_s\bigg)\,. \tag{4.50}$$

This term enters at 1.5PN beyond the quadrapole and is a new contribution to the gravitational wave arising from the scalar quadrapole, similar to how we found the metric quadrapole in eq.(4.44). Interestingly, in Brans-Dicke theory there is a new tail term coming from dipole-dipole interactions appears which does not appear in our screened theory, reduce the effect of gravitational memory relative to Brans-Dicke theory.

Finally, to find the contribution to the metric due to Λ_s at 2PN beyond quadrupole, we need only consider monopole-octupole terms as all other contributions at this order are not transverse-traceless,

$$\tau^{ij} \supset \frac{\pi}{16}(3+2\omega)\varphi^{,(i}_{Mono}\varphi^{,j)}_{Oct}\,, \tag{4.51}$$

where φ_{Oct} is the octupole term.

Calculating the relevant contributions, we are left with the final piece of the waveform,

$$h^{ij} \supset \frac{4m}{\phi_0 R}\left(-\frac{1}{60}\partial^4_\tau M^{ija}_s \hat{N}^a\right)\,, \tag{4.52}$$

which enters at 2PN beyond the quadrapole. Again, we lose the additional tail terms found in Brans-Dicke theory due to the vanishing scalar dipole moment.

Thus the waveform is the GR result together with the contributions from $\Lambda^{\mu\nu}_S$, eq.(4.50) and eq.(4.52).

4.5 Waveform from a late binary inspiral

The metric h^{ij} found in the previous section is valid for any compact mass distribution. However, the sources of gravitational waves detected by LIGO are the late inspiral of binary objects. We will restrict this section to the case of a black hole - black hole binary so that we may neglect complications that arise when one of the bodies is a neutron star, see [29, 104, 170] for such examples.

In particular, the binary system will have been emitting gravitational waves through its evolution, which to leading order are sourced by the matter quadrupole. This has the effect of circularising the orbit of the binary system as the gravitational waves extract angular momentum and hence eccentricity, [29]. As a result we can treat the system as being quasi-circular, the binaries move in an approximately circular orbit but for the continued loss of angular momentum. We stress that the orbit is not truly circular as the continued emission of gravitational waves extracts energy and the system will fall inwards. However, the rate of change in the radius is too small to affect our analysis. The bigger effect is the change in angular phase which increases dramatically and will cause divergences at the time of collision.

Calculating the change in the binary dynamics due to the two additional contributions to the gravitational wave, (4.50) and (4.52), as well as the scalar field radiation, not calculated in this chapter, is beyond the scope of this work. The calculation is done for Brans-Dicke theory in [106]. We will instead assume the dynamics of the binary are those of general relativity. This seems like a reasonable assumption as the binary lies in a screened region and the gravitational wave only differs at high post-Newtonian order, and only with two of many terms.

We will work in center of mass coordinates correct to second post-Newtonian order as described in Appendix, C. In practice, we will only need the leading order definition in the two new contributions. We define the usual two body variables: total mass $m = m_1 + m_2$, reduced mass $\mu = m_1 m_2/m$, symmetric mass ratio $\eta = \mu/m$, relative position $\overline{x}^i = \overline{x}^i_1 - \overline{x}_2$, relative separation $r = |\overline{x}|$, unit vector $\hat{n}^i = x^i/r$, relative velocity $v^i = \dot{x}^i$, angular momentum $\overline{L} = \mu\overline{x} \times \overline{v}$, unit normal to the orbital plain $\overline{\lambda} = \hat{L} \times \hat{n}$ and angular frequency $\omega = |\overline{L}|/(\mu r^2)$. We also introduce the parameter $x = (\phi_0^{-1} m\omega)^{2/3}$ to allow for easy comparison to the results presented in [31]. The post-Newtonian equations of motion from the expansion of GR to second post-Newtonian order are presented in appendix

C, eq.(C.3a). The post-Newtonian orbital phase can easily be found, (C.5) from which We will use the leading term $\omega^2 \approx m/r^3$ in order to express the binary separation in terms of the angular frequency.

With these quantities, we are now equipped to find the new contributions to the gravitational wave in terms of the Newtonian parameters. We begin with the $\mathcal{O}(1.5)$ term (4.50),

$$\begin{aligned} h^{ij} \supset& \frac{4m}{\phi_0 R}\bigg(- \frac{1}{12}\frac{1}{\phi_0(3+2\omega)}\sum_A m_A \partial_\tau^3 x_A^{ij}\bigg) \\ =& -\frac{1}{12}\frac{4m\mu}{R}\frac{1}{\phi_0^2(3+2\omega)}\partial_\tau^3\big(x^{ij} + m f^{(1)ij} + \mathcal{O}(\epsilon^3)\big) \\ =& -\frac{1}{12}\frac{4m\mu}{R}\frac{1}{\phi_0^2(3+2\omega)}\big(2\dot{a}^{(i}x^{j)} + 6a^{(i}v^{j)}\big)\,. \end{aligned} \tag{4.53}$$

While we have shown the $\mathcal{O}(2)$ correction to the quadrapole in the second line, it is not needed and is left to illustrate that change to center of mass coordinates, (C.1), can introduce new post-Newtonian effects.

We now assume that the orbit is circular due to the gravitational wave emission which amounts to $\dot{r} = 0$, simplifying the calculation substantially. Expressing the result in terms of the Newtonian parameters using eq.(C.7) and (C.8) with $G = \phi_0^{-1}$ inserted, the waveform contribution is

$$h^{ij} \supset \frac{8\mu}{3R\phi_0(3+2\omega)}(x^{5/2} + (-3+\eta)x^{7/2})\hat{n}^{(i}\lambda^{j)}\,. \tag{4.54}$$

The final term is too high an order, entering at 2.5PN beyond the quadrapole and will not be used in the analysis. Again, it is presented only for illustrative purposes.

Repeating the same analysis for the new 2PN term, (4.52), we find

$$\begin{aligned} h^{ij} \supset& \frac{4m}{\phi_0 R}\left(-\frac{1}{60}\partial_\tau^4 M_s^{ija}\hat{N}^a\right) \\ =& \frac{1}{15}\frac{\mu\delta m}{R}\frac{1}{\phi_0^2(3+2\omega)}\partial_\tau^4 x^{ija}\hat{N}^a \\ =& x^3\frac{m\eta}{\phi_0 R}\frac{\delta m}{5m(3+2\omega)}(7\hat{n}^{ij}\hat{n}\cdot\hat{N} - 20\lambda^{ij}\hat{n}\cdot\hat{N} - 20\lambda^{(i}\hat{n}^{j)}\lambda\cdot\hat{N}) \end{aligned} \tag{4.55}$$

Hence to find the gravitational wave contribution, we must take the transverse-traceless projection of this metric. However, it is more useful to decompose the metric into plus and cross polarization states directly, which will have the benefit of also making use of only the waveform transverse-traceless part. To do this, the coordinate system at the detector is needed.

In order to express the waveform as observed at the detector, we use the vector $\hat{N}$, the radial vector from the centre of mass of the system to the observer defined above. The inclination angle i of the orbital plane is measured as the angle between $\overline{L}$ and $\hat{N}$. The orbital phase ϕ is the phase of body 1 measured from the ascending node of the orbital plane.

In finding the polarisations, we will use the identities WW eq.7.4, giving the final modified waveform contributions as

$$h^{ij} \supset \frac{m\eta x}{R}\frac{8x^{3/2}}{3\phi_0^2(3+2\omega)}\left(-\frac{1}{4}(1+\cos^2(i))\sin(2\phi)\right), \tag{4.56}$$

$$\begin{aligned} h^{ij} \supset & \frac{m\eta x}{R}\frac{\delta m x^2}{5m\phi_0^2(3+2\omega)}\left(\frac{7}{4}(\sin^2(i)+(1+\cos^2(i))\cos(2\phi))\sin(i)\sin(\phi)\right. \\ & - 5(\sin^2(i)-(1+\cos^2(i))\cos(2\phi))\sin(i)\sin(\phi) \\ & \left. + 5(1+\cos^2(i))\sin(2\phi)\sin(i)\cos(\phi)\right), \end{aligned} \tag{4.57}$$

for the plus polarisation and

$$h^{ij} \supset \frac{m\eta x}{R}\frac{8x^{3/2}}{3\phi_0^2(3+2\omega)}\left(\frac{1}{2}\cos(i)\cos(2\phi)\right), \tag{4.58}$$

$$\begin{aligned} h^{ij} \supset & \frac{m\eta x}{R}\frac{\delta m x^2}{5m\phi_0^2(3+2\omega)}\left(\frac{7}{2}\cos(i)\sin(2\phi)\sin(i)\sin(\phi)\right. \\ & \left. + 10\cos(i)\sin(2\phi)\sin(i)\sin(\phi) - 10\cos(i)\cos(2\phi)\sin(i)\cos(\phi)\right), \end{aligned} \tag{4.59}$$

for the cross.

Thus we finally have the additional terms to the gravitational wave in a screened theory which recovers Brans-Dicke theory on large scales.

4.5.1 Parameter Estimation

As the gravitational wave is modified, we wish to place bounds upon the additional terms. This would amount to constraining the value of $(3 + 2\omega)$. To do this, one would like to perform a full analysis of a LIGO detection using our model as a template for the inspiral. However, this lies beyond the scope of this work.

Rather, we will model idealized LIGO data of a binary black hole merger. For the data, we will use a GR signal with the measured parameters of the first detection, GW150914 [3]. In particular, we will use that the median masses of the two bodies are $36^{+5}_{-4}M_{\odot}$ and $29^{+4}_{-4}M_{\odot}$, where we quote the 90% credible intervals. The wave is detectable for approximately 0.2s before merger, LIGO samples at 16384 Hz (2^{14}hz), and the noise of the detector is not white, [111]. As a conservative estimate, our model of LIGO data model will have white, time-independent noise. The sampling rate we use is 300 Hz. We chose the rate as the LIGO detectors are most sensitive across the range of around 100-300 Hz. Moreover, the detected wave has a frequency in the 0.2 seconds prior to merger goes from around 35 Hz to 150 Hz across 8 cycles and so we capture the higher end as our Nyquist frequency.

To make our signal of the late inspiral, we find the gravitational waveform using the results of [31]. We may use these results as we have shown that the gravitational waveform in our phenomenological theory mimics that of GR, but with two extra terms, (4.50) and (4.52). This waveform is accurate up to second post-Newtonian order, but higher orders have been found for general relativity, [132, 133]. We only use the second order waveform as it is the same order our calculation is performed to.

We multiply the waveform by the luminosity distance and take an inclination angle $i = \pi/2$ so the binary is seen edge on. We also assume the detector is perpendicular to $\hat{N}$ so that we do not need to include the inclination of the detector relative to $\hat{N}$ in out analysis. On top of the waveform, we place Gaussian noise with standard deviation $\sigma = 2$. This value is chosen so that the errors on the masses of the binary are similar to those quoted for the original detection, [3].

With an idealised realisation of the data, we perform a consistency test through a maximum likelihood analysis. The analysis is performed using the MCMC python package emcee, [71], where we use 100 walkers, 4000 time-steps and treat the first 1000 steps as a burn-in period. For all variables, we take a uniform prior over the ranges $0 < M_1 < 100$, $0 < M_2 < M_1$ for the masses of the black holes, $0 \leq \phi_c < 2\pi$ for the phase at collision and $-10 \leq t_c < 10$ for the time of collision. The results are presented in figure 4.2. The results are quote the 90 percentile errors and do not vary much across different realisations of the data. The results are consistent with the GR signal the data is made from except for the time of collision t_c. This is as we can only model the gravitational wave during the inspiral and so when $t_c < 0$, data after the merger isn't used and hence unfavored.

Recall that the modified waveform introduces one new parameter, ω, arising from the gravity being Brans-Dicke like in the radiation zone. We use the modified waveform for the wave template in a maximum likelihood on the same data used in the consistency test in order to constrain ω. We use the same MCMC code and priors for the analogous parameters. However, ω can take the range of values -1.5 to ∞, and hence $3 + 2\omega$ spans 0 to ∞. We define $\tan(\pi\zeta/2) = 3 + 2\omega$, and place a uniform prior on ζ across the range of allowed values, 0 to 1, where $\zeta = 1$ recovers GR.

The effect of the modification to the MCMC parameter estimation is shown in figure 4.3. There is a substantial right-hand tail that does not go to zero in the posterior of ζ. This is to be expected as ζ increases, the modification shrinks and so contributes less to the gravitational wave. As a result, there will be little change in the posterior as ω increases, each value becomes almost equally likely. This means that the median value quoted is not very useful. However there is still a substantial peak in the posterior at a value of $\zeta = 0.1$ corresponding to a value of $\omega = -1.42$. The choice of prior on ζ is questionable, and as a result, the peak in the posterior may not be physical.

We stress that the constant ω is not the traditional Brans-Dicke parameter, but rather ω measures how kinetically heavy the scalar field is on large scales. As a result, we cannot compare our result to the value of $\omega > 40000$ placed by the Cassini mission, [24]. This is as the Cassini mission measures ω in the solar system, where we expect the theory to be screened. This being said, the standard result of $\omega \to \infty$ recovering GR still holds for our model in the radiation zone. As such, our analysis suggests that there is a rather big modification on large scales. It is interesting to look at constraints of Brans-Dicke theory that are placed on large scales, for example, [14, 27, 110, 144]. All of these place much larger constraints, around $\omega > 100$, implying that our model is incompatible with existing constraints on Brans-Dicke theory on large scales. It is worth noting that if we have insisted upon Bran-Dicke theory on large scales to derive our results, but one could add a potential which Yukawa suppresses the CMB constraints. The effect such a potential would have on our waveform is unclear, however. If one accepts our phenomenological model as an accurate description of leading order effects in screened gravity theories, this would appear to rule out screened theories which are Brans-Dicke like on large scales.

4.6 Conclusions

We have demonstrated that the scaling method developed in chapter 2 can be used to find gravitational waves in screened gravity theories. This is as the leading order metric and scalar field equations can be found in both the screened and unscreened regions. The method of calculation closely follows [105, 192], which includes a separation of the past light cone of the detector into a near zone, where the light cone intersects the world tube of the system, and the radiation zone, the rest of the past light cone. This introduces an arbitrary radius $\mathcal{R}$ which we set as the screening radius of the system. As a result, we can find the gravitational wave components due to the near zone and radiation zone using general relativity in the near zone and Brans-Dicke theory in the radiation zone corresponding to large scales. This has the effect of providing two new contributions to the gravitational waveform, (4.50) at 1.5PN beyond the quadrupole and (4.52) at 2PN. The size of these new terms is determined by $(3 + 2\omega)^{-1}$, where ω is the Brans-Dicke parameter on large scales. We stress this is not the same as the traditional Brans-Dicke parameter, so solar system constraints such as [24], do not apply. We have not been able to constrain the screening scale controlled by the parameter α. Constrains could be placed if terms arising from the boundary contributions from the screened near zone and unscreened far zone where found in the waveform. Should these terms not cancel each other at higher orders, then terms in the wave form proportional to $\mathcal{R}$ would survive and place a direct constraint on the screening radius.

We used idealised data, made to reproduce the results of [3] when a maximum likelihood analysis is performed. We assume that neither of the black holes in the binary has spin and that the inclination angle of the orbital plane to the line of sight is $\pi/2$ so that the system is seen edge on. The best-fit parameters for the modified gravitational waveform is also found from a maximum likelihood analysis. The posterior of parameter ω has a substantial tail, and so the median value and 90th percentile errors do not contain much information. However, there is still a peak in the posterior at $\omega = -1.42$. This lies in stark contrast to constraints placed on ω from cosmology, $\omega > 100$ [14, 27, 110, 144]. This would imply that gravitational waves generated in screened modified gravity theories cause sufficient contradictions as to make them unviable.

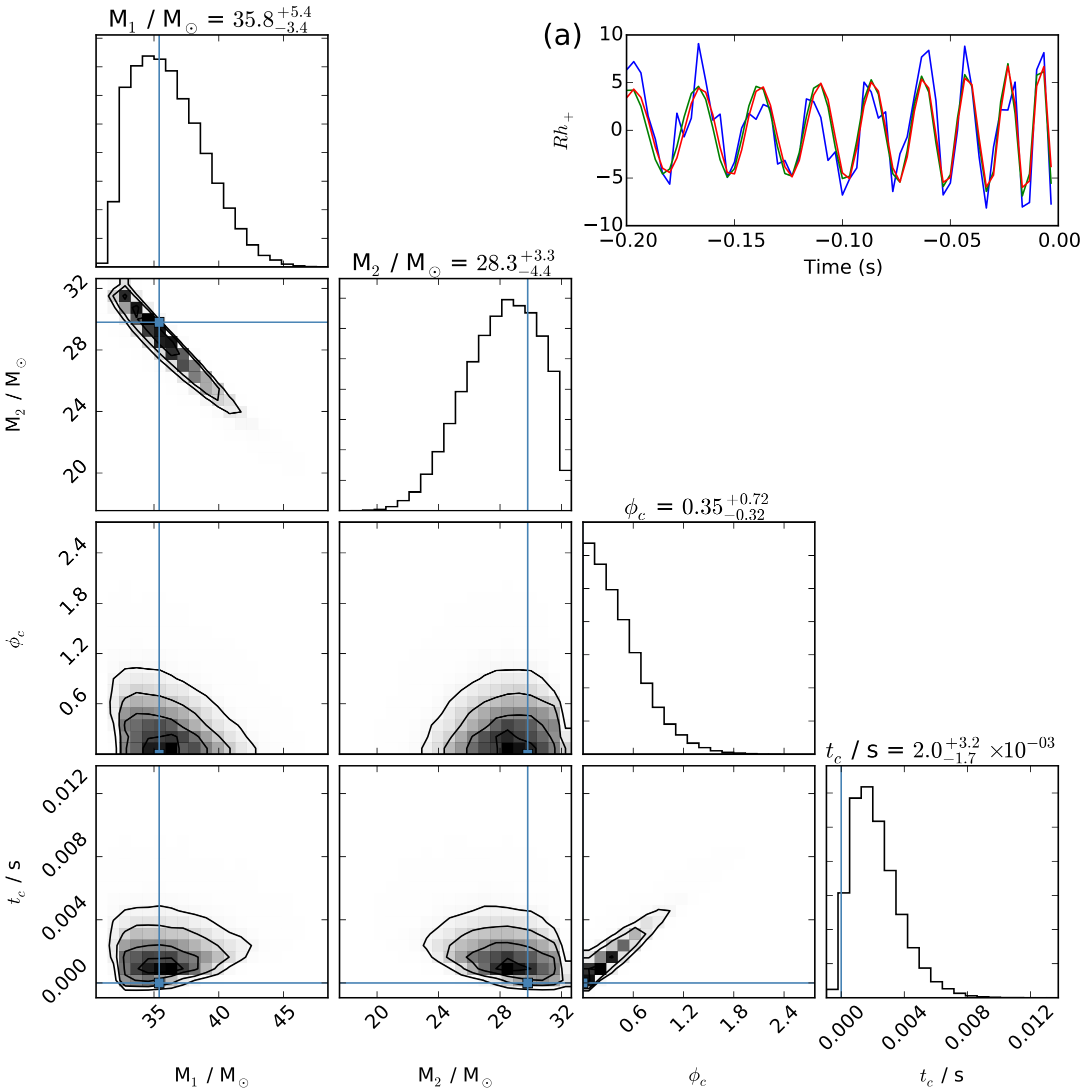

Figure 4.2 *Triangle diagram varying the mass of the primary, M_1, the mass of the secondary, M_2, the phase at time of collision, ϕ_c, and the time of collision, t_c. We present the median values and 90% credible intervals. The MCMC analysis was performed using the emcee python package, [71]. The priors used are constant for the ranges $0 < M_1/M_\odot < 100$, $0 < M_2/M_\odot < M1$, $0 < \phi_c < 2\pi$, $-10 < t_c/s < 10$. Inset (a): The signal with noise is presented in blue, the background signal in red and the GR gravitational wave best fit in green.*

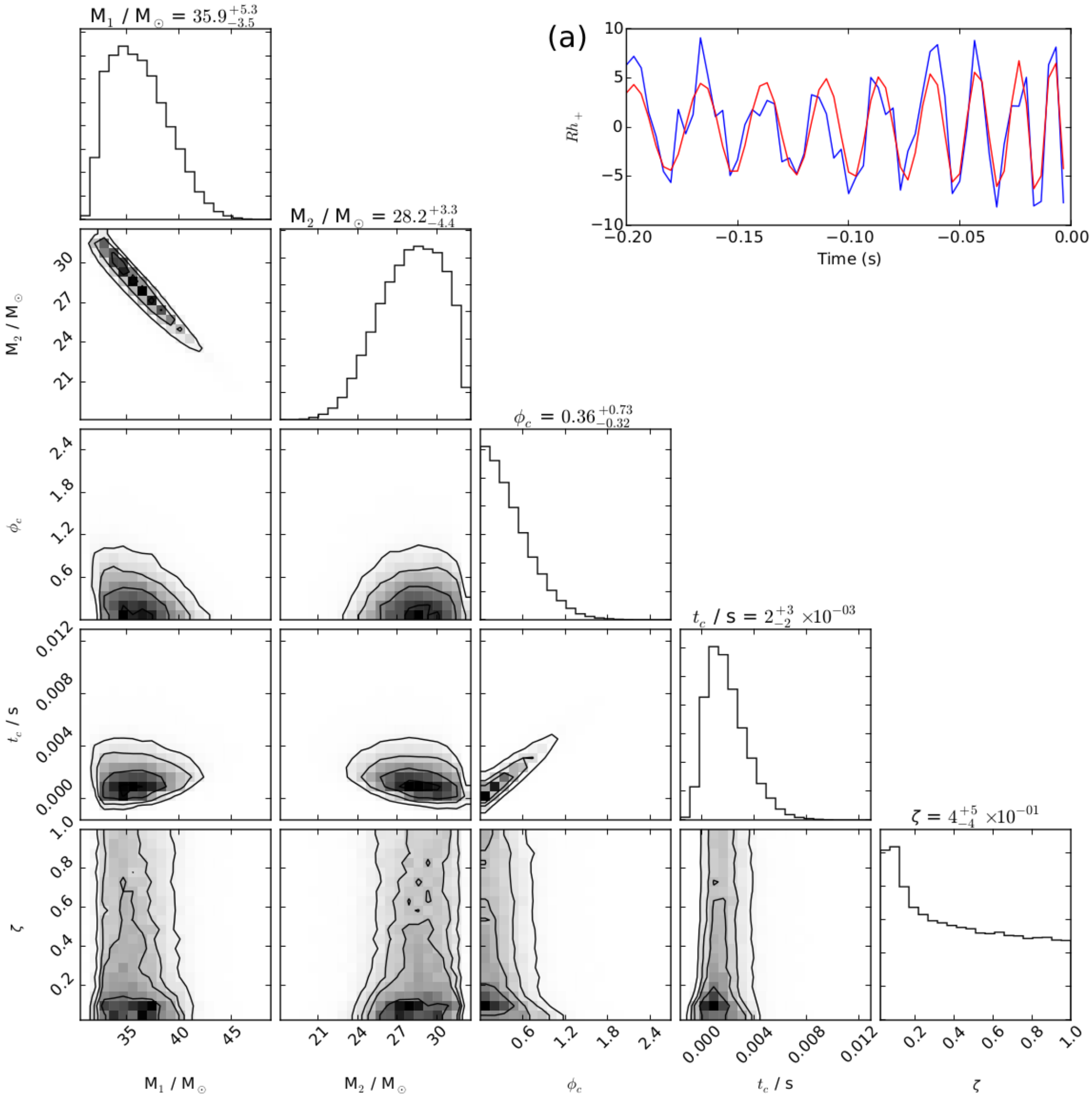

Figure 4.3 *Triangle diagram varying the mass of the primary, M_1, the mass of the secondary, M_2, the phase at time of collision, ϕ_c, the time of collision, t_c and the modification parameter, ζ. We present the median values and 90% credible intervals. The MCMC analysis was performed using the emcee python package, [71]. The priors used are uniform for the ranges $0 < M_1/M_\odot < 100$, $0 < M_2/M_\odot < M1$, $0 < \phi_c < 2\pi$, $-10 < t_c/s < 10$, $0 < \zeta < 1$. Inset (a): The signal with noise is presented in blue and the best fit signal in red.*

This being said, our analysis leaves much to be desired. Our analysis is performed on heavily idealised data. It would be beneficial to get a constraint from analysis of real data in order to ensure our result is not an artefact of the artificial data. Further, the prior used on ω (or more specifically ζ as defined in section 4.5.1) may not be a good choice. It would be worth changing the prior in order to see if the peak of the posterior is physical. Note that while we have assumed Brans-Dicke theory on large scales, the presence of a potential which introduces a mass scale would elevate the cosmological constraints. The effect this would have on the waveform is unknown, however, and is an interesting potential follow up to this work. We also assume that the entirety of the radiation zone is unscreened. Performing the calculation with a "patchy" radiation zone of both screened and unscreened regions would be interesting. Finally, in finding the waveform, we use the equations of motion for the binary in general relativity. Moreover, we also use that the energy emitted in radiation is the same as in general relativity. This is not unreasonable as we have shown that the metric radiation is the same as in GR up to two additional terms that enter at high post-Newtonian order. However, there will also be scalar radiation emitted which we have not calculated. This would have the effect of changing the orbital frequency and its derivative. This is important to include as LIGO is more sensitive to phase changes than the amplitude change we have calculated, [171].

The work of this chapter is still ongoing.

Chapter 5

Conclusions

Theories of relativistic gravity predate Einstein's general relativity, for example, the work of Poncaré, Minkowski and Sommerfeld [181]. But Einstein's theory proved not only to offer a revolutionary way of thinking about space-time but importantly makes accurate predictions about the universe, crowning it the successor to Newtonian gravity. These predictions cover laboratory scales with Cavendish experiments, solar system tests such as time delay experiments and highly accurate orbital calculation, to the physics of black holes, gravitational waves and cosmology.

However, the energy content of the universe leaves open questions about general relativity. Both dark matter and dark energy are needed in the field equations to accurately predict the CMB power spectra, the formation of structure and lensing maps to name a few. Moreover, dark matter is the standard explanation for the excess mass inferred from rotational velocities in galaxies, and, crucially, the mass distribution in the bullet cluster [47]. The study of type Ia supernovae also suggests that the universe is accelerating [148], implying the existence of a negative pressure fluid which the cosmological constant provides.

The study of modified gravity predates these discoveries, rather, a large driving force behind initial modifications were questions about the nature of general relativity and gravity. This is helped by the mathematical building blocks easily allowing for modifications, with early examples being Brans-Dicke theory and Kaluza-Klein gravity [34, 146]. These all depend upon the same mathematical principles as general relativity, such as coordinate invariance, while breaking parts of the equivalence principle.

With the wealth of modified gravity theories that can easily be constructed, a need to test in a theory-independent manner was needed. A body of work on this topic resulted in the parameterised post-Newtonian formalism, [172, 189, 190]. It was found that common potentials entered into the metric components across multiple gravity theories at the same order. Thus, if the effects of such potentials were to be found and measurements made, all one would have to do in order to place solar system constraints upon any gravity theory would be to find the post-Newtonian expansion. With the prefactors found, the corresponding parameter places constraints on the theory. This proved to be an effective method of testing gravity. Experiments showed that the parameters predicted by general relativity were consistent in all cases, while the parameters of modified gravity theories were constrained so as to either render the theory indistinguishable on all scales from general relativity, or contradictory to results from other scales, [191].

The invention of screening mechanisms revitalized the field as they allowed for models to decouple the predictions of gravity on solar system scales from the predictions on cosmological scales. Together with the discovery of late time acceleration and the existence of a non-zero but small cosmological constant in general relativity, this sparked hope that modified gravity models could be used to explain late time acceleration dynamically. While the study of modified gravity in cosmology, structure growth and late time acceleration is interesting, we have focused on mathematical aspects of modified gravity theories and application to low energy physics.

While the comparison of every modified gravity theory one could invent to observations is infeasible, so too is even writing them all down. Take for example scalar-tensor theories built from the Brans-Dicke action, eq.(1.56), where in principle, just wring down any potential can cause different phenomenology. Rather, it would be better to talk of classes of modified gravity theories, but even with just adding a potential, we could have chameleon models, symmetron models, scalar field masses or quartic self-interactions. As well as this, we can consider non-standard kinetic terms such as in k-moflage gravity or higher derivatives such as galileon models. This wealth of models only describes scalar-tensor theories, and one can easily write multiple-scalar-tensor theories, vector-tensor, scalar-vector-tensor theories, higher dimensional theories or with more care, super-gravity theories to name but a few. This thesis concerned itself with the study of single scalar-tensor theories as the simplest modification, outlining directions of work in more complex gravity models. But in order to ensure the work is generically applicable to any scalar-tensor theory, we began with a study of the Horndeski action. The Horndeski action is the most general local scalar-tensor theory with only second order derivatives in the equations of motion, and as such any statement we can make about it will carry to any scalar-tensor theory of Horndeski type.

In chapter 2, we began by examining screening effects in the Horndeski action. As the mathematical reason for many screening effects vary wildly, we do not approach this at the level of the field solutions. We, however, examine the capacity of the metric and scalar field equation to have a general relativity limit, by which we mean the metric field equation reduces to Einstein's field equations and the scalar field equation remains constant. This is important as many tests place constraints on the four free functions of the Horndeski action in different regimes. Thus the ability to easily check if a screening mechanism exists for a given choice of these four functions allows for easy discrimination between models viable on solar system scales.

In order to provide such a tool, we developed a new scaling method which consists of scaling field perturbations by the parameters of the theory raised to some exponent, for example

$$\phi = \phi_0(1 + \alpha^q \psi)\,, \tag{5.1}$$

for ϕ a scalar field, ϕ_0 the background value, ψ a perturbation and α a parameter of the theory. Taking the formal limits $\alpha \to \infty$ or $\alpha \to 0$ then provides the field equations for the metric and scalar field when different terms dominate. Should the metric field equation reduce to Einstein's equations, a screening mechanism may be in effect and the scalar field equation highlights the terms responsible for such an effect. At this point, we show how a simple argument applied to the scalar field equation can show whether this is the realisation of a screening effect or not. However, should the method not apply, further inspection of the scalar field equation is necessary. To show the efficiency of this method, it was applied to the cubic galileon and a chameleon model. We use a strong definition in this section about what constitutes a screening effect, in some limit general relativity is recovered. Often, the Newtonian and first few post-Newtonian terms are all that need to be recovered when screening effects are invoked.

While screening mechanisms are often cited as sufficient to pass solar system tests, even claimed above, this is only a leading order statement. In principle, no matter how strong a screening mechanism, the modifications should not vanish but rather just be sub-dominant. Having developed a unified method for determining the presence of screening mechanisms, we proceed in chapter 3 to expand upon this method so as to provide a perturbative series about the limits of the field equations. This is accomplished through the inclusion of further terms in the field expansion, all with their own prefactor controlling their size.

As a series of field equations is inherently more complex than one field equation, we prove several statements about the construction. As a consistency test, we demonstrate that the leading order field equation of the series is the same as that found using the method of chapter 2. A series expansion has no advantage over the complete field equations should they not be able to be solved iteratively. To this end, we demonstrate that our expansion does not require higher order solution to find the lower solutions. Finally, we show that the field equations for all orders beyond the first are linear, simplifying the method of calculation.

The scaling expansion was combined with a post-Newtonian expansion in order to find corrections to the metric resulting from the screened modifications to gravity. This was done for both the cubic galileon and chameleon models to order $(v/c)^4$, consistent with the parameterised post-Newtonian formalism. The expansion introduces several new potentials not present in the traditional formalism, and as such the formalism needs to be adapted in order to incorporate these new screening potentials. Moreover, we've shown that subtlety is needed when applying existing tests of gravity to the new potentials by examining the Shapiro time delay as measured by the Cassini mission, [24].

One problem with screening mechanisms, in so far as comparing to observations, is that they are intrinsically nonlinear. This nonlinearity can cause the field solution in a region of interest to depend heavily upon the neighbourhood of the system, [67]. This is unlike the low energy expansion of GR. For example, when calculating the orbit of planets in the solar system, that the solar system resides within the Galaxy is irrelevant. Moreover, the simulation of dynamics in such theories becomes much more complex as nonlinear physics is inherently harder to calculate. Our method absorbs the nonlinearities into the leading order scalar field equation, leaving just Einstein's equations in the screened limit and a series of linear equations for the perturbations around the screened limit. This could potentially be useful for simplifying dynamic simulations in screened regimes in a theory-independent way with potential use in galactic dynamics, [119].

While we have focused on solar system tests of the low-energy static limit, other astrophysical tests this limit. For example, comparisons between the rotation curves between presumably screened stars and unscreened gas in nearby dwarf galaxies, [179]. Stellar physics can also be modified leading to a wide range of effects such as for red giants having a screened core but unscreened envelope resulting in fifth force and hence being smaller and hotter, [42]. A more thorough analysis taking into account a change of gravitational strength throughout the star can be used on both Cepheid variables and the tip of the red giant branch in the same neighborhood to look for discrepancies in distance measurements, [88]. Further references can be found in [38], or for the Vainshtein mechanism, [103]. The methods we have developed would allow for one to find the perturbative effects of modifying gravity analytically, and hence the effects can be parameterised.

Finally, the new field of gravitational wave astronomy allows for novel tests of gravity. At the most basic level, just detecting gravitational waves is a test of general relativity. Parameterizations of the gravitational wave, in turn, allow for tests through finding best fit values and checking for consistency with the predictions of general relativity. The detection of electromagnetic counterparts also provides a very strong but simple test of gravity through the difference between the speed of light and that of gravitational waves being vanishingly small thanks to the cosmological distances probed. That these speeds do not differ drastically reduces the space of viable theories for late time acceleration, [122]. However, finding the predictions for gravitational waveforms emitted by binary mergers in modified gravities still needs to be calculated in order to highlight what deviations should actually be expected. Such calculations have been performed for Brans-Dicke theory [105], highlighting that new terms in the gravitational wave expansion are to be expected, not just changes in phase. Brans-Dicke theory is an interesting testbed for calculation, but it is not a physically relevant theory and so the calculation only serves to highlight potential deviations from general relativity.

We considered the generation of gravitational waves in screened models of scalar-tensor gravity in chapter 4. Because of the limiting procedure we have developed, we can justify taking both the screened and unscreened limit of these models, asserting that in the neighbourhood of a black hole binary, general relativity is recovered while beyond the screening radius, gravity is described by Brans-Dicke theory. That we can find such a model was shown in chapter 2 where we explicitly demonstrated that this is true for the cubic galileon model. A similar calculation to that of finding the conditions for a general relativity limit would also give conditions on the existence of Brans-Dicke limit on large scales, but this is left for future work. These two limits reduce the number of new calculations needed to find the gravitational waveform as the results from general relativity and Brans-Dicke theory can both be adapted to this phenomenological model.

The gravitational waveform was found to almost mimic that of general relativity up to order $(v/c)^4$ beyond the quadrupole approximation due to the bulk of the waveform being generated in the screened region. However, the gravitational wave also scatters off the gravitational background in the Brans-Dicke regime. As a result, two additional terms are found to contribute in our model. Both of these terms are also found in the Brans-Dicke expansion but with different prefactors. This demonstrates that any attempts to search for modifications in a parameterised manner needs to include new terms beyond those of only general relativity if screened gravity theories are to be tested. We use idealised LIGO data to perform a MCMC maximum likelihood analysis to find the effect such terms have upon parameters of our modified waveform. Unfortunately, this modification is hard to constrain as we find a wide posterior but with a clear best fit value. The best fit parameter would place a constraint on our unscreened Brans-Dicke parameter inconsistent with that placed in cosmology. This chapter highlights that a more thorough analysis may allow for screened modified gravity theories with the properties we consider to be ruled out.

Appendix A

Horndeski theories with an Einstein gravity limit

A.1 Horndeski field equations

The field equations for Horndeski gravity given in Ref. [98] are reproduced here for convenience. Varying the action (2.1) with respect to the metric and scalar field yields the equations of motion,

$$0 = \sum_{i=2}^{5} \mathcal{G}^{(i)}_{\mu\nu}, \tag{A.1}$$

$$0 = \sum_{i=2}^{5} (P^{(i)}_{\phi} - \nabla^{\mu} J^{(i)}_{\mu}), \tag{A.2}$$

respectively, which we rearrange to get the equations (2.4) and (2.6). Hereby, we have defined the rank-2 tensors

$$R^{(2)}_{\mu\nu} \equiv -\frac{1}{2} G_{2X} \nabla_{\mu}\phi \nabla_{\nu}\phi - \frac{1}{2} g_{\mu\nu}(G_{2X}X - G_2)\,, \tag{A.3}$$

$$R^{(3)}_{\mu\nu} \equiv G_{3X}\left\{\frac{1}{2}\Box\phi\nabla_{\mu}\phi\nabla_{\nu}\phi + \nabla_{(\mu}X\nabla_{\nu)}\phi + \frac{1}{2}g_{\mu\nu}X\Box\phi\right\} + G_{3\phi}\nabla_{\mu}\phi\nabla_{\nu}\phi\,, \tag{A.4}$$

$$R^{(4)}_{\mu\nu} \equiv G_{4X}\left\{ -\frac{1}{2}R\nabla_{\mu}\phi\nabla_{\nu}\phi - \Box\phi\nabla_{\mu}\nabla_{\nu}\phi + \nabla_{\lambda}\nabla_{\mu}\phi\nabla^{\lambda}\nabla_{\nu}\phi + 2R_{\lambda(\mu}\nabla_{\nu)}\phi\nabla^{\lambda}\phi \right.$$

$$
\begin{aligned}
&+ R_{\mu\alpha\nu\beta}\nabla^\alpha\phi\nabla^\beta\phi - \frac{1}{2}g_{\mu\nu}(RX + R_{\alpha\beta}\nabla^\alpha\phi\nabla^\beta\phi)\Big\} + G_{4\phi}\Big\{ - \nabla_\mu\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\Box\phi)\Big\} \\
&+ G_{4XX}\Big\{ - \frac{1}{2}[(\Box\phi)^2 - (\nabla_\alpha\nabla_\beta\phi)^2]\nabla_\mu\phi\nabla_\nu\phi + 2\nabla_\lambda X\nabla^\lambda\nabla_{(\mu}\phi\nabla_{\nu)}\phi \\
&- \nabla_\lambda X\nabla^\lambda\phi\nabla_\mu\nabla_\nu\phi - 2\nabla_{(\mu}X\nabla_{\nu)}\phi\Box\phi - \nabla^\alpha\phi\nabla_\alpha\nabla_\mu\phi\nabla^\beta\phi\nabla_\beta\nabla_\nu\phi \\
&- \frac{1}{2}g_{\mu\nu}(X[(\Box\phi)^2 - (\nabla_\alpha\nabla_\beta\phi)^2] + 2\nabla^\alpha X\nabla^\beta\phi\nabla_\alpha\nabla_\beta\phi - \nabla_\alpha X\nabla^\alpha\phi\Box\phi \\
&+ (\nabla_\alpha\nabla_\lambda\phi\nabla^\alpha\phi)^2)\Big\} + G_{4\phi\phi}\Big\{ - \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(-2X)\Big\} \\
&+ G_{4X\phi}\Big\{2\nabla_\lambda\phi\nabla^\lambda\nabla_{(\mu}\phi\nabla_{\nu)}\phi + 2X\nabla_\mu\nabla_\nu\phi - 2\nabla_\mu\phi\nabla_\nu\phi\Box\phi\Big\}, \qquad \text{(A.5)}
\end{aligned}
$$

$$
\begin{aligned}
R^{(5)}_{\mu\nu} \equiv & G_{5X}\Big\{R_{\alpha\beta}\nabla^\alpha\phi\nabla^\beta\nabla_{(\mu}\phi\nabla_{\nu)}\phi - R_{\alpha(\mu}\nabla_{\nu)}\phi\nabla^\alpha\phi\Box\phi \\
&- \frac{1}{2}R_{\alpha\beta}\nabla^\alpha\phi\nabla^\beta\phi\nabla_\mu\nabla_\nu\phi - \frac{1}{2}R_{\mu\alpha\nu\beta}\nabla^\alpha\phi\nabla^\beta\phi\Box\phi + R_{\alpha\lambda\beta(\mu}\nabla_{\nu)}\phi\nabla^\lambda\phi\nabla^\alpha\nabla^\beta\phi \\
&+ R_{\alpha\lambda\beta(\mu}\nabla_{\nu)}\nabla^\lambda\phi\nabla^\alpha\phi\nabla^\beta\phi + \nabla^\alpha X\nabla^\beta\phi R_{\alpha(\mu\nu)\beta} - \nabla_{(\mu}XG_{\nu)\lambda}\nabla^\lambda\phi \\
&- \nabla^\lambda XR_{\lambda(\mu}\nabla_{\nu)}\phi - \frac{1}{2}G_{\alpha\beta}\nabla^\alpha\nabla^\beta\phi\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}\Box\phi\nabla_\alpha\nabla_\mu\phi\nabla^\alpha\nabla_\nu\phi \\
&+ \frac{1}{2}(\Box\phi)^2\nabla_\mu\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\Big(R_{\alpha\beta}[\nabla^\alpha\phi\nabla_\lambda\phi\nabla^\beta\nabla^\lambda\phi - \nabla^\alpha\phi\nabla^\beta\phi\Box\phi \\
&+ \nabla^\alpha X\nabla^\beta\phi] + R_{\alpha\lambda\beta\rho}[\nabla^\rho\nabla^\lambda\phi\nabla^\alpha\phi\nabla^\beta\phi + \nabla^\alpha X\nabla^\rho\phi g^{\lambda\beta}] + G_{\alpha\beta}[\nabla^\alpha\nabla^\beta\phi X \\
&- \nabla^\alpha X\nabla^\beta\phi] - \frac{1}{2}\Box\phi\nabla_\alpha\nabla_\beta\phi\nabla^\alpha\nabla^\beta\phi - \frac{1}{6}\Box\phi(\nabla_\alpha\nabla_\beta\phi)^2 \\
&+ \frac{1}{3}(\nabla_\alpha\nabla_\beta\phi)^3\Big)\Big\} + G_{5\phi}\Big\{\frac{1}{2}\nabla_{(\mu}\nabla_{\nu)}\phi\Box\phi - \nabla_\lambda\nabla_{(\mu}\phi\nabla_{\nu)}\nabla^\lambda\phi + \frac{1}{2}\Box\phi\nabla_\mu\nabla_\nu\phi \\
&+ \nabla^\alpha\phi\nabla^\beta\phi R_{\alpha(\mu\nu)\beta} - \nabla_{(\mu}\phi G_{\nu)\lambda}\nabla^\lambda\phi - \nabla^\lambda\phi R_{\lambda(\mu}\nabla_{\nu)}\phi \\
&- \frac{1}{2}g_{\mu\nu}\Big(\nabla^\alpha\phi\nabla^\beta\phi R_{\alpha\rho\lambda\beta}g^{\rho\lambda} - \frac{1}{2}R^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi - G_{\alpha\beta}\nabla^\alpha\phi\nabla^\beta\phi\Big)\Big\} \\
&+ G_{5XX}\Big\{ - \frac{1}{2}\nabla_{(\mu}X\nabla^\alpha\phi\nabla_\alpha\nabla_{\nu)}\phi\Box\phi \\
&- \frac{1}{2}\nabla_\alpha X\nabla_\beta\phi\nabla^\alpha\nabla^\beta\phi\nabla_\mu\nabla_\nu\phi + \frac{1}{2}\nabla_{(\mu}X\nabla_{\nu)}\phi[(\Box\phi)^2 - (\nabla_\alpha\nabla_\beta\phi)^2] \\
&+ \nabla_\alpha X\nabla_\beta\phi\nabla^\alpha\nabla_{(\mu}\phi\nabla^\beta\nabla_{\nu)}\phi - \nabla_\beta X[\Box\phi\nabla^\beta\nabla_{(\mu}\phi - \nabla^\alpha\nabla^\beta\phi\nabla_\alpha\nabla_{(\mu}\phi]\nabla_{\nu)}\phi \\
&+ \frac{1}{2}\nabla^\alpha\phi\nabla_\alpha X[\Box\phi\nabla_\mu\nabla_\nu\phi - \nabla_\beta\nabla_\mu\phi\nabla^\beta\nabla_\nu\phi] + \frac{1}{12}[(\Box\phi)^3 - 3\Box\phi(\nabla_\alpha\nabla_\beta\phi)^2 \\
&+ 2(\nabla_\alpha\nabla_\beta\phi)^3]\nabla_\mu\phi\nabla_\nu\phi \\
&- \frac{1}{2}g_{\mu\nu}\Big(\nabla^\alpha\phi\nabla^\beta X[\nabla^\lambda\nabla_\beta\phi\nabla_\lambda\nabla_\alpha\phi + \nabla_\beta\nabla^\lambda\phi\nabla_\alpha\nabla_\lambda\phi - \frac{3}{2}\nabla_\alpha\nabla_\beta\phi\Box\phi - \frac{1}{2}\nabla_\beta\nabla_\alpha\phi\Box\phi \\
&+ \frac{1}{2}g_{\alpha\beta}[(\Box\phi)^2 - (\nabla_\rho\nabla_\lambda\phi)^2]] + \nabla^\alpha X\nabla^\beta X[\nabla_\alpha\nabla_\beta\phi - g_{\alpha\beta}\Box\phi] - \frac{1}{6}X[(\Box\phi)^3
\end{aligned}
$$

$$
\begin{aligned}
&- 3\Box\phi(\nabla_\alpha\nabla_\beta\phi)^2 + 2(\nabla_\alpha\nabla_\beta\phi)^3]\Big)\Big\} + G_{5\phi X}\Big\{ - \frac{1}{2}\nabla_{(\mu}\phi\nabla^\alpha\phi\nabla_\alpha\nabla_{\nu)}\phi\Box\phi \\
&+ \frac{1}{2}\nabla_{(\mu}X\nabla_{\nu)}\phi\Box\phi - \nabla_\lambda X\nabla_{(\mu}\nabla_{\nu)}\nabla^\lambda\phi + \frac{1}{2}[\nabla_\lambda X\nabla^\lambda\phi - \nabla_\alpha\phi\nabla_\beta\phi\nabla^\alpha\nabla^\beta\phi]\nabla_\mu\nabla_\nu\phi \\
&+ \frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi[(\Box\phi)^2 - (\nabla_\alpha\nabla_\beta\phi)^2] + \nabla_\alpha\phi\nabla_\beta\phi\nabla^\alpha\nabla_{(\mu}\phi\nabla^\beta\nabla_{\nu)}\phi - \nabla_\beta\phi[\Box\phi\nabla^\beta\nabla_{(\mu}\phi \\
&- \nabla^\alpha\nabla^\beta\phi\nabla_\alpha\nabla_{(\mu}\phi]\nabla_{\nu)}\phi - X[\Box\phi\nabla_\mu\nabla_\nu\phi - \nabla_\beta\nabla_\mu\phi\nabla^\beta\nabla_\nu\phi] \\
&- \frac{1}{2}g_{\mu\nu}\Big(\nabla^\alpha\phi\nabla^\beta\phi[-2\nabla_\alpha\nabla_\beta\phi\Box\phi + \frac{1}{2}g_{\alpha\beta}[(\Box\phi)^2 - (\nabla_\rho\nabla_\lambda\phi)^2]] \\
&+ \nabla^\alpha\phi\nabla^\beta X[-2g_{\alpha\beta}\Box\phi + \nabla_\alpha\nabla_\beta\phi]\Big)\Big\} \\
&+ G_{5\phi\phi}\Big\{\frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi\Box\phi - \nabla_\lambda\phi\nabla_{(\mu}\phi\nabla_{\nu)}\nabla^\lambda\phi - X\nabla_\mu\nabla_\nu\phi\Big\}\,. \qquad \text{(A.6)}
\end{aligned}
$$

To simplify the scalar field equation, we have defined the scalars

$$P_\phi^{(2)} \equiv G_{2\phi}\,, \tag{A.7}$$

$$P_\phi^{(3)} \equiv \nabla_\mu G_{3\phi}\nabla^\mu\phi\,, \tag{A.8}$$

$$P_\phi^{(4)} \equiv G_{4\phi}R + G_{4\phi X}[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2], \tag{A.9}$$

$$P_\phi^{(5)} \equiv -\nabla_\mu G_{5\phi}G^{\mu\nu}\nabla_\nu\phi - \frac{1}{6}G_{5\phi X}[(\Box\phi)^3 - 3\Box\phi(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3] \tag{A.10}$$

and the covariant four-vectors

$$J_\mu^{(2)} \equiv -\mathcal{L}_{2X}\nabla_\mu\phi\,, \tag{A.11}$$

$$J_\mu^{(3)} \equiv -\mathcal{L}_{3X}\nabla_\mu\phi + G_{3X}\nabla_\mu X + 2G_{3\phi}\nabla_\mu\phi\,, \tag{A.12}$$

$$
\begin{aligned}
J_\mu^{(4)} \equiv& -\mathcal{L}_{4X}\nabla_\mu\phi + 2G_{4X}R_{\mu\nu}\nabla^\nu\phi - 2G_{4XX}(\Box\phi\nabla_\mu X - \nabla^\nu X\nabla_\mu\nabla_\nu\phi) \\
&- 2G_{4\phi X}(\Box\phi\nabla_\mu\phi + \nabla_\mu X)\,, \qquad \text{(A.13)}
\end{aligned}
$$

$$
\begin{aligned}
J_\mu^{(5)} \equiv& -\mathcal{L}_{5X}\nabla_\mu\phi - 2G_{5\phi}G_{\mu\nu}\nabla^\nu\phi - G_{5X}[G_{\mu\nu}\nabla^\nu X + R_{\mu\nu}\Box\phi\nabla^\nu\phi - R_{\nu\lambda}\nabla^\nu\phi\nabla^\lambda\nabla_\mu\phi \\
&- R_{\alpha\mu\beta\nu}\nabla^\nu\phi\nabla^\alpha\nabla^\beta\phi] + G_{5XX}\Big\{\frac{1}{2}\nabla_\mu X[(\Box\phi)^2 - (\nabla_\alpha\nabla_\beta\phi)^2] \\
&- \nabla_\nu X(\Box\phi\nabla_\mu\nabla^\nu\phi - \nabla_\alpha\nabla_\mu\phi\nabla^\alpha\nabla^\nu\phi)\Big\} + G_{5\phi X}\Big\{\frac{1}{2}\nabla_\mu\phi[(\Box\phi)^2 - (\nabla_\alpha\nabla_\beta\phi)^2] \\
&+ \Box\phi\nabla_\mu X - \nabla^\nu X\nabla_\nu\nabla_\mu\phi\Big\}\,, \qquad \text{(A.14)}
\end{aligned}
$$

where we have used the different components of the Horndeski Lagrangian

$$
\begin{aligned}
\mathcal{L}_2 \equiv& G_2(\phi, X)\,, && \text{(A.15)}\\
\mathcal{L}_3 \equiv& -G_3(\phi, X)\Box\phi\,, && \text{(A.16)}\\
\mathcal{L}_4 \equiv& G_4(\phi, X)R + G_{4X}(\phi, X)[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2]\,, && \text{(A.17)}\\
\mathcal{L}_5 \equiv& G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{G_{5X}(\phi, X)}{6}[(\Box\phi)^3 - 3\Box\phi(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3]\,. && \text{(A.18)}
\end{aligned}
$$

A.2 Coordinate-dependent scaling method

We briefly present here another method that uses the limiting argument developed in section 2.3, but where α enters as a scaling of the coordinates instead. This method is most useful when applied to theories with derivative interactions such as galileon models.

For simplicity, let the scalar field be a function of one variable only $\phi = \phi(r)$ that scales as $\phi(\alpha^q r)$ with q a real number. Note, however, that one can easily generalise the following results to also include a dependence, for instance, on angular coordinates. We again consider a generic scalar field equation of the form

$$\alpha^s F_u(\phi, \partial_r\phi, \partial_r^2\phi) + \alpha^t F_v(\phi, \partial_r\phi, \partial_r^2\phi) = \rho\,, \tag{A.19}$$

for some functions F_u, F_v and real numbers u, v. After a redefinition of the coordinates, eq. (A.19) becomes

$$\alpha^s F_u(\phi, \alpha^q\partial_{\alpha^q r}\phi, \alpha^{2q}\partial^2_{\alpha^q r}\phi) + \alpha^t F_v(\phi, \alpha^q\partial_{\alpha^q r}\phi, \alpha^{2q}\partial^2_{\alpha^q r}\phi) = \rho\,. \tag{A.20}$$

We now put the condition on both F_u and F_v that α^q is factorized out with order their subscript, which yields the exponents $s + uq$ and $t + vq$ of α. The values of q must ensure a term independent of α on the left-hand side, so that

$$q \in \left\{-\frac{s}{u}, -\frac{t}{v}\right\} = Q\,. \tag{A.21}$$

In order to prevent divergences, we must adopt the most negative value in Q when $\alpha \to \infty$ and the most positive when $\alpha \to 0$, respectively. The resulting equations for ϕ, should they exist, describe the field in both limits.

For a specific example, consider a cubic galileon model with the scalar field equation (2.93) in approximately flat space. Assuming a cylindrical mass distribution $\rho = \rho(r)$ and field profile $\phi = \phi(r)$, the equation of motion becomes

$$\frac{\rho(r)}{M_p} = \partial_r^2\phi + \frac{\partial_r\phi}{r} + \frac{2\alpha\partial_r\phi\partial_r^2\phi}{r}\,. \tag{A.22}$$

To find the leading term in the limit of $\alpha \to \infty$, we insist that the coordinate dependence of the field goes as $\alpha^q r$ such that upon applying the chain rule we get

$$\frac{\rho(r)}{M_p} = \alpha^{2q}\partial^2_{\alpha^q r}\phi + \alpha^q\frac{\partial_{\alpha^q r}\phi}{r} + 2\alpha^{1+3q}\frac{\partial_{\alpha^q r}\phi\partial^2_{\alpha^q r}\phi}{r}\,, \tag{A.23}$$

from which we conclude that $q = -\frac{1}{3}$. Hence, we arrive at the simple scalar field equation

$$\frac{r\rho(r)}{M_p} = 2\partial_{\alpha^{-1/3}r}\phi\partial^2_{\alpha^{-1/3}r}\phi = \partial_{\alpha^{-1/3}r}(\partial_{\alpha^{-1/3}r}\phi)^2\,. \tag{A.24}$$

Inside of a cylindrical mass where $\rho(r) = \rho_0$, then this is trivially integrated to

$$\phi = \sqrt{\frac{\rho_0}{4M_p}}r^2\alpha^{3q/2} = \sqrt{\frac{\rho_0}{4M_p}}r^2\alpha^{-1/2}\,, \tag{A.25}$$

where we have insisted that $\phi \to 0$ as $r \to 0$. Notice that we again recover that there is an overall factor of $\alpha^{-1/2}$, which agrees with what we found with the method described in section 2.3.2. An advantage of this alternative scaling method, in contrast to the scaling method introduced in section 2.3, is that one can adopt a different scaling for each coordinate direction, and so encode the morphological dependence of the screening mechanism into the limiting procedure. A disadvantage, however, is that the method is computationally more involved, requiring the equations of motion to be written for a given coordinate choice. With that we also lose the benefits of the covariant method, making an analysis like the one performed in section 2.4 infeasible.

Appendix B

Properties of the expansion using the scaling parameter

In section 3.2, we have used that the extremal values in a set Q_j obtained from taking an α limit in the scalar field equation are functions of only $q_{i<j}$, where i and j denote the orders of the α-correction. Furthermore, we have used that the higher-order corrections $\psi^{(q_{i>1})}$ obey linear field equations with non-linear equations restrained to the leading order $\psi^{(q_1)}$. We will discuss these aspects in more detail in appendices B.1 and B.2, respectively.

B.1 Perturbative α-corrections

In the following, we aim to show that the value of q_1 in the expansion of ϕ to all orders, given by eq. (3.1), is the same as the value q in eq. (2.9), where only the term at leading order in α is considered. That is, q_1 is not affected by higher-order α-corrections. For this purpose, let Z denote all fields of a modified gravity theory other than the scalar field ϕ, for example, the metric, and consider a field equation of the form

$$F(Z, \phi, \partial\phi) = T/M_p^2\,, \tag{B.1}$$

where the source T does not depend on any of the fields. As in section 3.2, we will only consider the first derivative of the scalar field in the equations of motion. The argument is easily generalised to second or higher derivatives and so for simplicity, we will not consider them.

For ease of reference, we repeat the expansion (3.1),

$$\phi = \phi_0 \left(1 + \sum_{i=1}^{\infty} \alpha^{q_i} \psi^{(q_i)}\right) . \tag{B.2}$$

As we want the i^{th} term to be larger than the $(i+1)^{th}$, we need an ordering of the exponents q_i: for the $\alpha \to \infty$ limit, the ordering is $0 \geq q_1 > q_i > q_{i+1}$ and for $\alpha \to 0$, $0 \leq q_1 < q_i < q_{i+1}$ for integers $i > 1$. In the following, we shall specify to the limit $\alpha \to \infty$ so that $q_i > q_{i+1}$. An analogous discussion can, however, be made for the limit $\alpha \to 0$.

We will consider the perturbations $\psi^{(q_i)}$ to be independent fields, so that F is a function of many variables, $\{\alpha^{q_i}\psi^{(q_i)}\ \alpha^{q_i}\partial\psi^{(q_i)}\}$. We then Taylor expand F in all $\psi^{(q_i)}$ variables for $i > 1$,

$$F = F|_{\overline{\psi}=0} + \sum_{i>1} \alpha^{q_i} [(\delta_{\alpha^{q_i}\psi^{(q_i)}} F)|_{\overline{\psi}=0} \psi^{(q_i)} + (\delta_{\alpha^{q_i}\partial\psi^{(q_i)}} F)|_{\overline{\psi}=0} \partial\psi^{(q_i)}] + \dots , \tag{B.3}$$

where the ellipsis contain quadratic and higher order terms, and $\overline{\psi} = (\psi^{(q_2)}, \psi^{(q_3)}, ...)$. Importantly, F and its functional derivatives are still only functions of $\alpha^{q_1}\psi^{(q_1)}$ and Z due to the expansion, and $F|_{\overline{\psi}=0} = F(X, \psi^{(q_1)}, \partial\psi^{(q_1)})$.

When only considering the first-order correction in the expansion (B.2), one finds q_1 by examining the field equation corresponding to $F|_{\overline{\psi}=0}$, which is independent of all $\psi^{(i>2)}$ and hence $q_{i>2}$. Thus, we first wish to ensure that this result is replicated when including higher-order corrections.

Let Q_1 denote the set of all values that q_1 can take such that an exponent of α in F will be equal to zero. Then this set will contain elements found from $F|_{\overline{\psi}=0}$ which will be the same values found when considering the expansion (2.9). However, terms such as $(\delta_{\alpha^{q_i}\psi^{(q_i)}} F)|_{\overline{\psi}=0}\psi^{(q_i)}$ give rise to values for q_1 that are functions of q_i for $i > 1$. We shall show that the minimum value of the set Q_1, and hence the value that q_1 takes, is independent of any $q_{i>1}$ and hence q_1 coincides with the value found using eq. (2.9).

Consider a generic coefficient in the Taylor expansion, $((\prod \delta^{m_i}_{\alpha^{q_i}\psi^{(q_i)}} \delta^{n_i}_{\alpha^{q_i}\partial\psi^{(q_i)}})F)|_{\overline{\psi}=\overline{\partial\psi}=0}$ with finitely many non-vanishing natural numbers m_i and n_i. For such a term not to vanish through performing the functional derivatives on F and the evaluation at $\overline{\psi} = \overline{\partial\psi} = 0$, then in the expansion of F, it must be that

$$F \supset f(\psi^{(q_1)}, \partial\psi^{(q_1)}) \prod (\alpha^{q_i}\psi^{(q_i)})^{m_i} (\alpha^{q_i}\partial\psi^{(q_i)})^{n_i} \tag{B.4}$$

for some function f. But as we are using the expansion (B.2), there must also be a term

$$F \supset f(\psi^{(q_1)}, \partial\psi^{(q_1)}) \prod (\alpha^{q_1}\psi^{(q_1)})^{m_i} (\alpha^{q_1}\partial\psi^{(q_1)})^{n_i}. \tag{B.5}$$

The exponents of α found from both eq. (B.4) and eq. (B.5) are then $p+\sum_{i>1}(m_i+n_i)q_i$ and $p+\sum_{i>1}(m_i+n_i)q_1$, respectively, where the exponent p is obtained from f. The value of q_1 can be solved for implicitly by letting $p = p' + q_1$ as well as using the symmetric nature of eq. (B.4) and (B.5). Thus

$$Q_1 \supset \left\{-p' - \sum(m_i + n_i)q_1, -p' - \sum(m_i + n_i)q_i\right\}. \tag{B.6}$$

The value that q_1 must take is less than or equal to the minimum of these two combinations. Using the ordering that $0 \geq q_1 > q_i \ \forall i > 1$, it is clear that the smaller of the two terms is $-p' - \sum(m_i + n_i)q_1$. Hence for any generic term in the expansion such as in eq. (B.4), the value of q_1 which it predicts as a function of all q_i is bounded below by a value that is only a function of q_1 from a term such as eq. (B.5).

Thus we have found that when including the higher-order corrections in the expansion (3.1), q_1 does not become dependent on higher-order exponents $q_{i>1}$. Moreover, its value remains the same as that obtained from only considering the leading-order term, eq. (2.9). A similar argument can be made for $q_{i>1}$ being independent of $q_{j>i}$, as the key requirement is the ordering of the exponents.

B.2 Linearity of higher-order α-corrections

Finally, we aim to show that the field equations for the higher-order perturbations $\psi^{(i>1)}$ are linear for the generic field equation (B.1). When finding the field equation for the perturbation $\psi^{(q_1)}$, there are terms within $F|_{\overline{\psi}=0}$ that vanish when taking a limit of α as otherwise $\psi^{(q_1)}$ solves the full field equation. Moreover, only terms that vanish when taking a limit of α remain upon inserting the first two terms of the summation (B.2) into the generic field equation (B.1) and using that ψ^{q_1} solves a field equation which corresponds to the terms proportional to α^0. The remaining terms that are not functions of ψ^{q_2} are those that will source its field equation. The slowest vanishing source upon taking a limit is the source that ψ^{q_2} must balance.

Suppose that this source term takes the form $\alpha^t F_t(\psi^{(q_1)}, \partial\psi^{(q_1)})$ for a homogeneous function F_t of order t. The exponent q_2 needs to take a value such that a term in F that is a function of $\psi^{(q_2)}$ is of the same α order as the source term for it to balance the field equation order by order. In contrast, q_1 has to balance the non-vanishing matter source. From appendix B.1 we need only consider the terms in the Taylor expansion (B.3) that are functions of $\psi^{(1,2)}$ or $\partial\psi^{(1,2)}$,

$$F = F|_{\overline{\psi}=0} + \sum_{n,m}(\delta^n_{\alpha^{q_2}\psi^{(q_2)}}\delta^m_{\alpha^{q_2}\partial\psi^{(q_2)}}F)|_{\overline{\psi}=0}(\alpha^{q_2}\psi^{(q_2)})^n(\alpha^{q_2}\partial\psi^{(q_2)})^m . \tag{B.7}$$

Following a similar argument to that in appendix B.1, we will show that the value for q_2 predicted by a generic term is larger than what a term linear in $\psi^{(2)}$ or $\partial\psi^{(2)}$ would predict for q_2. For a generic coefficient $(\delta^n_{\alpha^{q_2}\psi^{(q_2)}}\delta^m_{\alpha^{q_2}\partial\psi^{(q_2)}}F)|_{\overline{\psi}=0}$ to not vanish, there must be a term in F of the form

$$F \supset f(\psi_1, \partial\psi_1)(\alpha^{q_2}\psi^{(q_2)})^n(\alpha^{q_2}\partial\psi^{(q_2)})^m . \tag{B.8}$$

As we are using the expansion (B.2), the existence of such a term implies that there be a term linear in $\psi^{(2)}$ and $\partial\psi^{(2)}$,

$$\begin{aligned} F \supset & f(\psi_1, \partial\psi_1)(\alpha^{q_1}\psi^{(q_1)})^{n-1}(\alpha^{q_1}\partial\psi^{(q_1)})^{m-1}\times \\ & \left[(\alpha^{q_1}\partial\psi^{(q_1)})(\alpha^{q_2}\psi^{(q_2)}) + (\alpha^{q_1}\psi^{(q_1)})(\alpha^{q_2}\partial\psi^{(q_2)})\right] . \end{aligned} \tag{B.9}$$

The exponents found from both eq. (B.8) and eq. (B.9) are then $p + (n + m)q_2$ and $p + (n + m - 1)q_1 + q_2$, respectively, where the exponent p is obtained from f. Again, let $p = p' + q_2$. Then we can find the values of q_2 that these terms contribute to Q_2 as

$$Q_2 \supset \{t - p' - (n + m)q_2, t - p' - (n + m - 1)q_1 - q_2\} . \tag{B.10}$$

Restricting to the limit $\alpha \to \infty$ and using the ordering $0 \geq q_1 > q_2$, we are left with the value for q_2 from eq. (B.8) being bounded below by the value from eq. (B.9). Hence, any term non-linear in $\psi^{(q_2)}$ or $\partial\psi^{(q_2)}$ provides a value for q_2 which is bounded below by a term which is linear in $\psi^{(q_2)}$ or $\partial\psi^{(q_2)}$, and hence any term that contributes to the field equation for $\psi^{(q_2)}$ will be linear in these quantities. The analogous conclusion holds in the $\alpha \to 0$ limit.

This argument again can be generalised to the i^{th} case through the use of the ordering. The result that the field equations for $\psi^{(q_2)}$ are linear may be intuitive as it comes from a Taylor expansion, but we have shown it explicitly here for rigour.

Appendix C

Calculating the gravitational waveform

In section 4.5, we calculate the gravitational waveform from a binary system with masses $m_{1,2}$ and positions $x^i_{1,2}$. This appendix is intended to fill the details overlooked in the body of the text

In performing this calculation we change coordinates to place the center of mass of the system at the origin. The is a trick commonly performed in classical physics where the mass of each body is only their rest mass. However we consider post-Newtonian corrections to the system, and so their respective masses are also changed by their potential and kinetic energies. Using the corrected mass changes the center of mass and so its definition changed accordingly.

The center of mass X^i is found to be

$$X^i = m^{-1}(m_1 x^i_1 + m_2 x^i_2) + f^{(1)} + \mathcal{O}(\epsilon^2)\,, \tag{C.1}$$

where $m = m_1 + m_2$. One may worry that as we are calculating the metric to $\mathcal{O}(\epsilon^4)$, we may need the center of mass up to $\mathcal{O}(\epsilon^2)$, but the only place it would the second correction enter is in the leading order quadrapole term, where it cancels out exactly.

We wish to describe the system in terms of the relative separation $x^i = x^i_1 - x^i_2$

and its derivatives. In the center of mass coordinate system, $X = 0$ and

$$x_1^i = (m_2/m)x^i - \delta_x + \mathcal{O}(\epsilon^2)\,, \tag{C.2a}$$
$$x_2^i = -\,(m_1/m)x^i - \delta_x + \mathcal{O}(\epsilon^2)\,. \tag{C.2b}$$

similarly, we define the standard two body variables $\mu = m_1 m_2/m$, $\eta = \mu/m$, $\delta m = m_1 - m_2$ and $r = |\overline{x}|$.

It will be useful to define the system in terms of Newtonian objects in the final expression. So we define the Newtonian angular momentum, $L^i = \mu(\overline{x} \times \overline{v})^i$, unit normal to the orbital plain $\lambda^i = (\overline{L} \times \hat{n})^i$ and angular velocity $\omega = |\overline{L}|/\mu r^2$.

When finding the time derivatives of the center of mass coordinates, we will make use of the equations of motion for the black hole binary. As the near zone is described by GR, the equations of motion can be taken readily from sources such as [29]. The accelerations due to the gravitational forces to first post-Newtonian order are found to be

$$a^i = a_N^i + a_1^i + \epsilon^{\in}\,, \tag{C.3a}$$
$$a_N^i = -\,\frac{m}{r^2}\hat{n}^i\,, \tag{C.3b}$$
$$a_1^i = -\,\frac{m}{r^2}(A_1\hat{n}^i + B_1\dot{r}v^i)\,, \tag{C.3c}$$
$$\tag{C.3d}$$

where for convenience we define

$$A_1 = -\,2(2+\eta)\frac{m}{r} + (1+3\eta)v^2 - \frac{3}{2}\eta\dot{r}^2, \tag{C.4a}$$
$$B_1 = -\,2(2-\eta)\,. \tag{C.4b}$$

The orbital phase can be rather easily found to be

$$\omega^2 = \frac{m}{r^3}\left(1 - \frac{m}{r}(3-\eta) + \left(\frac{m}{r}\right)\left(6 + \frac{41}{4}\eta + \eta^2\right)\right) \tag{C.5}$$

through the same analysis as in Newtonian mechanics but with the Newtonian force supplemented with the post-Newtonian forces of eq.(C.3a)

Derivatives of the equations of motion are also needed. In order to calculate $\dot{a}^j$

and $\ddot{a}^j$, we make use of the following identities:

$$\dot{\hat{n}}^i = \frac{v^i}{r} - \frac{\dot{r}\hat{n}^i}{r}\,, \tag{C.6a}$$

$$\ddot{r} = \ddot{r}_N + \ddot{r}_1\,, \tag{C.6b}$$

$$\ddot{r}_N = -\frac{m}{r^2} + \frac{v^2}{r} - \frac{\dot{r}^2}{r}\,, \tag{C.6c}$$

$$\ddot{r}_1 = -\frac{m}{r^2}(\alpha\frac{m}{r} + \beta v^2 + (\gamma+\sigma)\dot{r}^2), \tag{C.6d}$$

which gives $\dot{a}^j$to the required order as

$$\dot{a}^i = (\dot{a}_N)^i_N + (\dot{a}_1)^i_N\,, \tag{C.7a}$$

$$(\dot{a}_N)^i_N = \frac{2m\dot{r}}{r^3}\hat{n}^i - \frac{m}{r^3}v^i\,, \tag{C.7b}$$

$$\begin{aligned}(\dot{a}_1)^i_N = & -18\frac{m^2\dot{r}}{r^4}\hat{n}^i + 4\eta\frac{m^2}{r^4}v^i + (3+12\eta)\frac{m\dot{r}v^2}{r^3}\hat{n}^i - \frac{15}{2}\eta\frac{m\dot{r}^3}{r^3}\hat{n}^i \\ & -(12-\frac{15}{2}\eta)\frac{m\dot{r}^2}{r^3}v^i + (3-5\eta)\frac{mv^2}{r^3}v^i\end{aligned} \tag{C.7c}$$

and $\ddot{a}^j$ to the required order as

$$\ddot{a}^i = (\ddot{a}^i_N)_N + (\ddot{a}^i_N)_1 + (\ddot{a}^i_1)_N\,, \tag{C.8a}$$

$$(\ddot{a}^i_N)_N = -2\frac{m^2}{r^5}\hat{n}^i - 15\frac{m\dot{r}^2}{r^4}\hat{n}^i + 3\frac{mv^2}{r^4}\hat{n}^i + 6\frac{m\dot{r}}{r^4}v^i\,, \tag{C.8b}$$

$$\begin{aligned}(\ddot{a}^i_N)_1 = & (8+4\eta)\frac{m^3}{r^6}\hat{n}^i - (2+6\eta)\frac{m^2v^2}{r^5}\hat{n}^i + (12-3\eta)\frac{m^2\dot{r}^2}{r^5}\hat{n}^i \\ & -(4-2\eta)\frac{m^2\dot{r}}{r^5}v^i\,,\end{aligned} \tag{C.8c}$$

$$\begin{aligned}(\ddot{a}^i_1)_N = & -36\eta\frac{\dot{r}m^2}{r^5}v^i - (30-42\eta)\frac{m\dot{r}v^2}{r^4}v^i - (60-45\eta)\frac{m\dot{r}^3}{r^4}v^i \\ & +(126-57\eta)\frac{m^2\dot{r}^2}{r^5}\hat{n}^i - (15+165/2\eta)\frac{mv^2\dot{r}^2}{r^4}\hat{n}^i + 102/2\eta\frac{m\dot{r}^4}{r^4}\hat{n}^i \\ & +(22+2\eta)\frac{m^3}{r^6}\hat{n}^i + (8-10\eta)\frac{m^2v^2}{r^5}\hat{n}^i + (3+15\eta)\frac{mv^4}{r^4}\hat{n}^i\,.\end{aligned} \tag{C.8d}$$

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