The gravitational field of a laser beam

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In this proceedings article, we review the results presented in [Fabienne Schneiter et al. 2018 Class. Quantum Grav. 35 195007] on the gravitational field of light in a laser beam, modeled as a solution to Maxwell's equations perturbatively expanded in the beam divergence. Using this approach, wave properties of light, such as diffraction, are taken into account that have been neglected in earlier studies. Interesting features of the gravitational field of laser beams become apparent like frame-dragging due to the intrinsic angular momentum of light and the deflection of parallel co-propagating test beams for short distances to the source beam.

Keywords: Gravitational properties of light, laser beams, linearized gravity.

1. Introduction

The gravitational field of a light beam has first been studied in Ref. 15 by Tolman, Ehrenfest and Podolski in 1931, who described the light beam as a one-dimensional "pencil of light". Later in Ref. 3 by Bonnor, a description for the gravitational field of a cylindrical beam of light of a finite radius was presented. Light was modeled as a continuous fluid moving at the speed of light. A central feature of the models of Ref. 15 and Ref. 3 is the lack of diffraction; the beams do not diverge. This corresponds to the short wavelength limit where all wavelike properties of light are neglected. Further studies to the gravitational field of light that share this feature include the investigation of two co-directed parallel cylindrical light beams of finite radius [2,10], spinning non-divergent light beams [9], non-divergent light beams in the framework of gravito-electrodynamics [5] and the gravitational field of a point like particle moving with the speed of light [1,17].

In contrast, the wavelike properties of light were taken into account in Ref. 16, where the gravitational field of a plane electromagnetic wave was investigated. An approach to take finite wavelengths into account for the case of a laser pulse was given in Refs. 11, 12, where, however, diffraction was neglected. Here, we describe the laser beam as a solution to Maxwell's equations. This is done perturbatively by an expansion in the beam divergence, which is considered to be small. The zeroth order of the expansion corresponds to the paraxial approximation and coincides with the result of Ref. 3. In the first order in the beam divergence, frame-dragging due to the internal angular momentum of circularly polarized beams occurs. In the forth order, a parallel co-propagating test beam of light is found to be deflected by the gravitational field of the laser beam.

2. The model

In the following, we will employ dimensionless coordinates by dividing the Cartesian coordinates corresponding to the lab reference by the beam waist w_0 as $\tau = ct/w_0$, $\xi = x/w_0$, $\chi = y/w_0$ and $\zeta = z/w_0$, where c is the speed of light. A laser beam in its simplest mode is accurately described by a Gaussian beam. The Gaussian beam is an almost monochromatic electromagnetic plane wave which has the property that its intensity distribution decays with a Gaussian factor with the distance to the beamline. It is obtained as a perturbative solution of Maxwell's equations, an expansion in the beam divergence θ, the opening angle of the beam, which is assumed to be small. When the beam divergence is small, the beam may be thought of as a bunch of almost, but not exactly, parallel propagating rays of light. The electromagnetic four-vector potential describing the Gaussian beam is obtained by a plane wave multiplied by an envelope function, which is assumed to be varying slowly in the direction of propagation, in agreement with the property that the divergence of the beam is small. Corresponding to these features, we make the ansatz for the four-vector potential $A_\alpha = \mathcal{A} v_\alpha(\xi, \chi, \theta\zeta) \exp(2i(\zeta - \tau)/\theta)$, were $\mathcal{A}$ is the amplitude, v_α the envelope function, and the exponential factor describes a plane wave propagating in ζ-direction with angular wave number $k = 2/\theta$, where w_0 is the beam waist at its focal point, a measure of the radius of the beam (see Fig. 1).

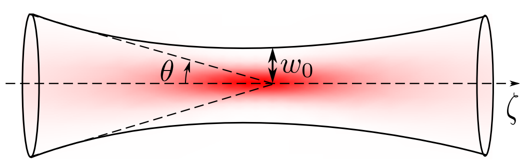

Fig. 1. Schematic illustration of the Gaussian beam, the beam waist w_0 and the beam divergence θ. More specifically, the figure illustrates the scalar envelope function v_0 of the vector potential of the Gaussian beam in a plane that contains the optical axis (represented by the dashed horizontal line). Due to the rotational symmetry of the envelope function around the optical axis, the vertical axis can be any direction transversal to the optical axis. The thick curved lines mark the distance $w(\zeta) = w_0\sqrt{1 + (\theta\zeta)^2}$ from the optical axis at which the absolute value of the envelope function reaches $1/e$ times its maximum.

In the following, the beam waist is kept constant. In this case, since the beam divergence is small, the angular wave number is large. As for any beam of radiation in flat spacetime, the four-vector potential satisfies the wave equations

$\left(\partial_\xi^2 + \partial_\chi^2 + \partial_\zeta^2 - \partial_\tau^2\right) A_\mu = 0$, which follow from Maxwell's equations when the Lorenz gauge is chosen. The wave equations for the four-vector potential reduce to a Helmholtz equation for the envelope function, $\left(\partial_\xi^2 + \partial_\chi^2 + \theta^2 \partial_{\theta\zeta}^2 + 4i\partial_{\theta\zeta}\right) v_\alpha = 0$. We assume that the envelope function is slowly varying in the direction of propagation, i.e. v_α is a function of $\theta\zeta$. Then, the Helmholtz equation can be solved iteratively by writing the envelope function as a power series in the divergence angle θ (see Ref. 4). For each term in the expansion of the envelope function, one obtains a Helmholtz equation with a source term. These source terms are proportional to terms of lower order in the expansion of the envelope function, and even and odd orders do not mix. We start from the zeroth order solution, which we assume to have a Gaussian profile in the focus plane, and derive the vector potential up to fourth order in the divergence angle.

To obtain the gravitational field of the laser beam, the energy momentum tensor $T_{\alpha\beta}$ has to be calculated from the field strength $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. We only consider vector potentials with field strength tensors that are eigenfunctions with eigenvalue $\lambda = \pm 1$ of the generator of the duality transformation of the electromagnetic field, which in our case is given by $F_{\alpha\beta} \mapsto -i\epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta}/2$, where $\epsilon_{\alpha\beta\gamma\delta}$ is the completely anti-symmetric tensor with $\epsilon_{0123} = -1$. In that case the rapidly oscillating contributions of the plane wave factor in the vector potential drop out and the energy momentum tensor assumes the simple form $T_{\alpha\beta} = c^2\epsilon_0 \,\mathrm{Re}\left(F_\alpha{}^\sigma F^*_{\beta\sigma} - \frac{1}{4}\eta_{\alpha\beta}F^{\delta\rho}F^*_{\delta\rho}\right)/2$. We interpret the eigenvalues λ of the generator duality transformation as the helicity of the beam since the standard notion of circular polarization is recovered in the zeroth order in θ.

3. The gravitational field

Since the energy of a laser beam is small, we may expect its gravitational field to be weak. The spacetime metric describing the gravitational field is thus assumed to consist of the metric for flat spacetime in the rescaled coordinates $\eta = w_0^2 \mathrm{diag}(-1,1,1,1)$ plus a small perturbation $h_{\alpha\beta}$, where small means $|h_{\alpha\beta}| \ll w_0^2$ for all α and β. Terms quadratic in the metric perturbation are neglected; this is the linearized theory of general relativity. When the Lorenz-gauge $\partial^\alpha h_{\alpha\beta} = \frac{1}{2}\partial_\beta h^\alpha{}_\alpha$ is chosen, Einstein's field equations reduce to wave equations for the metric perturbation[8]: $\left(\partial_\xi^2 + \partial_\chi^2 + \partial_\zeta^2 - \partial_\tau^2\right) h_{\alpha\beta} = -\kappa w_0^2 T_{\alpha\beta}$, where $\kappa = 16\pi G/c^4$ and G is Newton's constant. Solving the linearized Einstein equations for the energy momentum tensor of the laser beam with emitter and absorber at general positions can be quite cumbersome. Therefore, we consider two different limiting situations instead; the distance between emitter and absorber being very large in one case and very small in the case.

For a large distance between emitter and absorber, we can neglect the rapid change of the field strength at the emitter and the absorber. Then we can take into account that $T_{\alpha\beta}$ is changing slowly in ζ. In particular, we have $T^\lambda_{\alpha\beta} = T^\lambda_{\alpha\beta}(\xi, \chi, \theta\zeta)$.

Therefore, the metric perturbation can be expanded in orders of the beam divergence angle as $h_{\alpha\beta} = \sum_{n=0}^{\infty} \theta^n h_{\alpha\beta}^{(n)}$. The linearized Einstein equations lead to the differential equations

$$\left(\partial_\xi^2 + \partial_\chi^2\right) h_{\alpha\beta}^{\lambda(0)} = -w_0^2 \kappa\, t_{\alpha\beta}^{\lambda(0)} \,, \tag{1}$$

$$\left(\partial_\xi^2 + \partial_\chi^2\right) h_{\alpha\beta}^{\lambda(1)} = -w_0^2 \kappa\, t_{\alpha\beta}^{\lambda(1)} \,, \tag{2}$$

$$\left(\partial_\xi^2 + \partial_\chi^2\right) h_{\alpha\beta}^{\lambda(n)} = -w_0^2 \kappa\, t_{\alpha\beta}^{\lambda(n)} - \partial_{\theta\zeta}^2 h_{\alpha\beta}^{\lambda(n-2)} \,, \text{ for n} > 1 \,, \tag{3}$$

where the $t_{\alpha\beta}^{\lambda(n)}(\xi,\chi,\theta\zeta)$ are given by $T_{\alpha\beta}^{\lambda}(\xi,\chi,\theta\zeta) = \sum_{n=0}^{\infty} \theta^n t_{\alpha\beta}^{\lambda(n)}(\xi,\chi,\theta\zeta)$. The solutions $h_{\alpha\beta}^{\lambda(n)}$ of Eqs. (1), (2) and (3) can be found by direct calculation as we did in Ref. 13. These solutions have to be constructed such that the components of the Riemann curvature tensor vanish at infinite distance from the beamline. The Riemann curvature tensor governs the spread and the contraction of the trajectories of test particles. This means, if the Riemann tensor vanishes, parallel geodesics stay parallel and there is no physical effect as the only reference for a test particle in linearized gravity can be another test particle. As the energy distribution of the laser beam decays like a Gaussian function with the distance from the beamline, no gravitational effect should remain at infinite spatial distances from the beamline. Therefore, we have to ensure that the Riemann curvature tensor $R^{\mu}{}_{\rho\sigma\alpha}$ vanishes for $\rho \to \infty$. General solutions of Eqs. (1), (2) and (3) can be given in terms of the free space Green's function for the Poisson equation in two dimensions as

$$h_{\alpha\beta}^{\lambda(n)}(\xi,\chi,\theta\zeta) = \frac{1}{4\pi}\int_{-\infty}^{\infty} d\xi' d\chi' \log\left((\xi-\xi')^2 + (\chi-\chi')^2\right) Q_{\alpha\beta}^{\lambda(n)}(\xi',\chi',\theta\zeta') \,, \tag{4}$$

where $Q^{\lambda(n)}$ are the right hand sides of Eqs. (1), (2) and (3), respectively.

In the second situation, where we assume a short distance between emitter and absorber, the rapid change of the field strength at emitter and absorber cannot be neglected. Then, we solve the linearized Einstein equations by making use of the corresponding Green's function.

Iteratively solving the Poisson equations for the terms in the expansion of the metric perturbation up to a given order in the divergence angle, we obtain an expression for the spacetime metric which contains all information about the gravitational field of the laser beam to the given order. Knowing the spacetime metric allows us to study the motion of test particles in the gravitational field of the laser beam; their world lines γ^μ satisfy the geodesic equations which depend on the Christoffel symbols, and the distance between two initially parallel geodesics is governed by the geodesic deviation equation which is given in terms of the Riemann curvature tensor. Both Christoffel symbols and curvature tensor are expanded in orders of the divergence angle θ.

4. Conclusions

In zeroth order in θ, all non-vanishing components of the metric perturbation are equal up to a sign. Explicitly, we have $h_{\tau\tau}^{(0)} = h_{\zeta\zeta}^{(0)} = -h_{\zeta\tau}^{(0)} = -h_{\tau\zeta}^{(0)} =: I^{(0)}$. For

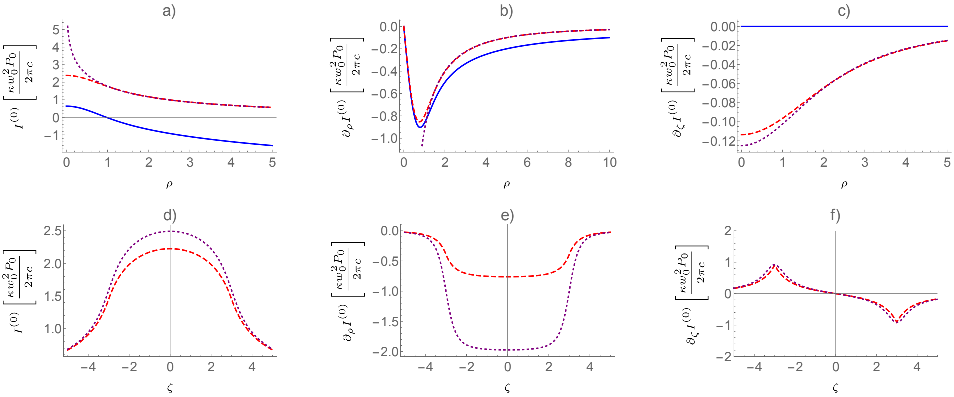

Fig. 2. These plots show the value of the leading order of the metric perturbation $I^{(0)}$ and its first derivatives for the Gaussian beam with infinite distance between emitter and absorber (plain, blue), the Gaussian beam with short distance between emitter and absorber (dashed, red), and the infinitely thin beam (dotted, purple). ρ is defined as $\sqrt{\xi^2+\chi^2}$. In the second and the third case, the distance between emitter and absorber is chosen to be $6w_0$. In the first row, the functions are plotted for $\zeta = 1$ and in the second row for $\rho = 1/2$. The second row does not contain plots for large distances between emitter and absorber as there is no dependence of $I^{(0)}$ on ζ in that case. We find that the values for $I^{(0)}$ and its first derivatives are usually larger for the infinitely thin beam than for the other two cases. This is due to the divergence at the beamline for the case of the infinitely thin beam. In the other two cases, the gravitational field is spread out as the sources are. In b), we see that the absolute value of the first ρ-derivative of $I^{(0)}$ reaches a maximum at a finite distance from the beamline. Note that $\partial_\rho I^{(0)}$ is proportional to the acceleration that a test particle experiences if it is initially at rest at a given distance ρ to the beamline. We see that the acceleration is always directed towards the beamline. It is larger in the case of an infinite distance between emitter and absorber than in the case of a finite distance, which we can attribute to the larger extension of the source (and thus the larger amount of energy) in the former than in the latter. In d), which shows plots for finite distance between emitter and absorber, we see that $\partial_\rho I^{(0)}$ still is the largest at the center between emitter and absorber and decays quickly once their positions at $\zeta = \pm 3$ are passed.

small values of the beam waist and for $\theta = 0$, which corresponds to the paraxial approximation in our case, our solution for the laser beam corresponds to the solution for the infinitely thin beam[15]. If we consider the laser beam to be infinitely long and assume $\theta = 0$, we recover the solution for an infinitely long cylinder[3]. In Fig. 2, the function $I^{(0)}$ and its derivatives are illustrated for the three cases of the infinitely long Gaussian beam, the Gaussian beam with short distance between emitter and absorber, and the infinitely thin beam.

In first order in the divergence angle, we find frame dragging due to spin angular momentum of the circular polarized laser beam. This is similar to the result of Ref. 14 for beams with intrinsic orbital angular momentum. In contrast to frame dragging induced by orbital angular moment, the effect that we find decays like a Gaussian with the distance from the beamline.

The statement of Ref. 15 by Tolman et al. that a non-divergent light beam does not gravitationally deflect a co-directed parallel light beam has been recovered

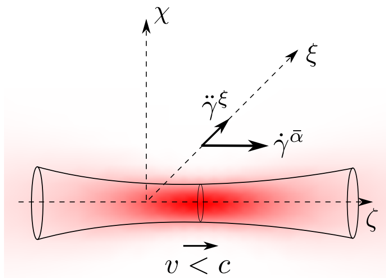

Fig. 3. Schematic illustration of the laser beam and the parallel co-propagating test ray of light: We look at the deflection of the test ray of light due to the gravitational field of the laser beam.

in different contexts: two co-directed parallel cylindrical light beams of finite radius[2,3,10], spinning non-divergent light beams[9], non-divergent light beams in the framework of gravito-electrodynamics[5] and parallel co-propagating light-like test particles in the gravitational field of a one-dimensional light pulse[11]. In fourth order in the divergence angle, we find a deflection of parallel co-propagating test beams. This shows that the result of Ref. 15 and Ref. 3 only holds up to the third order in the divergence angle. This could have been expected from the fact that the group velocity of light in a Gaussian beam along the beamline is not the speed of light[6,7]. However, the deflection of parallel co-propagating light beams by light in a focused laser beam decays like a Gaussian with the distance from the beamline. This means that the effect does not persist outside of the distribution of energy given by the laser beam.

In Ref. 13, we compare the result to the deflection that one obtains from a rod of matter boosted to a speed close to the speed of light. We conclude that focused light does not simply behave like massive matter moving with the reduced velocity identified in Refs. 11, 14. We argue that this difference is due to the divergence of the laser beam along the beamline which leads to additional non-zero components of the metric perturbation which do not appear in the case of the boosted rod. These additional contributions cancel the effect of the reduced propagation speed of light in the focused beam for large distances from the beamline.

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