

Investigating the similarity between the behavior of the order parameter derivatives and that of cumulants of the probability density for the thermal deconfining phase transition at zero chemical potential

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Abstract. In this work, we study the temperature driven deconfining phase transition, at zero chemical potential, in a statistical way, by means of the probability distribution of finding the mixed hadronic gas-QGP phase system in a finite volume, in a specific state defined by a parameter lying between 0 and 1, whose mean value represents the order parameter of the studied system. We calculate mean values of other characteristic quantities, called response functions, which are mainly the second, third and fourth cumulants of the probability distribution representing the variance, skewness and kurtosis respectively, as well as the three first thermal derivatives of the order parameter. Integral expressions are obtained for all these quantities, and their evaluation is done numerically. The such obtained results are adequately translated into graphs, illustrating the variations of the response functions with temperature, for various volumes. By examining their behavior, we notice a striking similarity between the behavior of the order parameter derivative and that of the cumulant, at the same order. We investigate this similarity and try to deeply analyze it, to obtain some important features characterizing the occurring deconfining phase transition.

1. Introduction

The thermodynamics of hadronic matter at high temperature has received great attention in the last few years. In this work, we study the thermal Deconfinement Phase Transition (DPT) from a Hadronic Gas (HG) phase to a color-singlet Quark Gluon Plasma (QGP) phase, at vanishing chemical potential, in a finite volume. For this purpose, we calculate the partition function of the QGP with the color-singletness requirement, using the projection method [1, 2]. The obtained partition function is then used for the calculation of mean values of some response functions, namely the order parameter and its three first thermal derivatives, whose behavior with temperature and volume is studied. Also, we examine the first four cumulants of the probability distribution $p(h)$ so we may obtain several characteristics of the DPT and get more information on the finite volume transition point and determine the effective transition temperature.

2. Thermal behavior of the order parameter and its derivatives



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This section is devoted to the extraction of information about the thermal DPT at vanishing chemical potential $\mu = 0$ via a deep analysis of the behavior with temperature of some response functions, well characterizing the mixed hadronic gas (HG)-QGP system. In the framework of the phase coexistence model [3], a parameter h representing the hadronic volume fraction is defined such that $h = V_{HG}/V$ with $V_{HG} + V_{QGP} = V$ the finite volume of the mixed (HG-QGP) system. The mean value of a given physical quantity $X(T, V)$ is then calculated based on the following definition [3, 4],

$$\langle X(T, V) \rangle = \frac{\int_0^1 X(h, T, V) Z(h) dh}{\int_0^1 Z(h) dh} \quad (1)$$

with $Z(h)$ the partition function of the mixed system and $X(h, T, V)$ the physical quantity in the state h . Assuming non-interacting phases, the total partition function $Z(h)$ is given by:

$$Z(h) = Z_{QGP}(h) Z_{HG}(h) \quad (2)$$

where Z_{QGP} and Z_{HG} are the partition functions of the individual QGP and HG phases, respectively.

$Z_{QGP}(h)$ is calculated taking into account the color-singletness condition in the QGP phase, by using the projection method [1, 2]. For this, we use a density of states which contains the volume term only, in a first step, as in [4, 5].

The total partition function of the mixed system composed of a HG of massless pions and a QGP consisting of gluons, massless u and d quarks and their antiquarks, in the state h , at temperature T and zero chemical potential, is after calculation given by [4, 5]:

$$Z(h) = \frac{4}{9\pi^2} \exp\left(\frac{\pi^2}{30} VT^3\right) \int_{-\pi}^{+\pi} \int_{-\pi}^{+\pi} M(\varphi, \psi) \exp\left\{(1-h)VT^3 \left[g_{\mu=0}(\varphi, \psi) - \frac{\pi^2}{30} - \frac{B}{T^4}\right]\right\} d\varphi d\psi \quad (3)$$

with:

$$M(\varphi, \psi) = \left\{ \sin\left[\frac{1}{2}\left(\psi + \frac{\varphi}{2}\right)\right] \sin\left(\frac{\varphi}{2}\right) \sin\left[\frac{1}{2}\left(\psi - \frac{\varphi}{2}\right)\right] \right\}^2 \quad (4)$$

and:

$$g_{\mu=0}(\varphi, \psi) = \frac{\pi^2}{12} \left(\frac{21}{30} d_Q + \frac{16}{15} d_G \right) + \frac{\pi^2}{12} \frac{d_Q}{2} \sum_{q=r,g,b} \left[\left(\frac{\alpha_q^2}{\pi^2} - 1 \right)^2 - 1 \right] - \frac{\pi^2}{12} \frac{d_G}{2} \sum_{g=1}^4 \left[\frac{(\alpha_g - \pi)^2}{\pi^2} - 1 \right] \quad (5)$$

with: $d_Q = 2N_f$ and $d_G = 2$ the degeneracy factors of quarks and gluons respectively, N_f being the number of flavors, taken here as $N_f = 2$ (the two lightest u and d quarks), $B^{1/4} = 192 \text{ MeV}$ and $\alpha_q (q = r, g, b)$ and $\alpha_g (g = 1, \dots, 4)$ the angles given by:

$$\begin{cases} \alpha_r = \frac{\varphi}{2} + \frac{\psi}{3}, & \alpha_g = \frac{-\varphi}{2} + \frac{\psi}{3}, & \alpha_b = \frac{-2\psi}{3} \\ \alpha_1 = \alpha_r - \alpha_g, & \alpha_2 = \alpha_g - \alpha_b, & \alpha_3 = \alpha_b - \alpha_r, & \alpha_4 = 0 \end{cases} \quad (6)$$

We have then to calculate mean values of some relevant quantities analytically and the obtained integral expressions have to be calculated numerically.

The first quantity of interest for our study is the order parameter of the occurring deconfining phase transition, which is the mean value of the hadronic volume fraction $\langle h(T, V) \rangle$. It is calculated using equation (1):

$$\langle h(T, V) \rangle = \frac{\int_0^1 h Z(h) dh}{\int_0^1 Z(h) dh} \quad (7)$$

The second quantity of interest is the thermal susceptibility $\chi(T, V)$ defined as the first derivative of the order parameter with respect to temperature T :

$$\chi(T, V) = \langle h(T, V) \rangle' = \frac{\partial \langle h(T, V) \rangle}{\partial T} \quad (8)$$

It is also worth to illustrate two additional thermal response functions, which are the second $\langle h(T, V) \rangle'' = \frac{\partial^2 \langle h(T, V) \rangle}{\partial T^2}$ and third $\langle h(T, V) \rangle''' = \frac{\partial^3 \langle h(T, V) \rangle}{\partial T^3}$ derivatives of the order parameter, obtained by differentiating this latter with respect to temperature twice and three times, respectively.

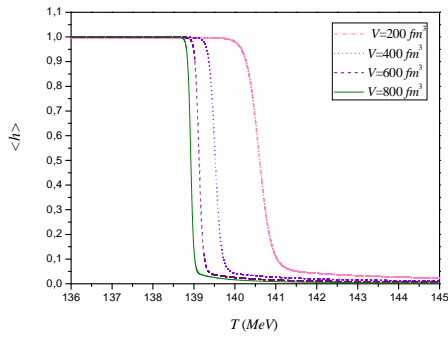


Figure 1. Plot of the order parameter versus temperature for different system volumes.

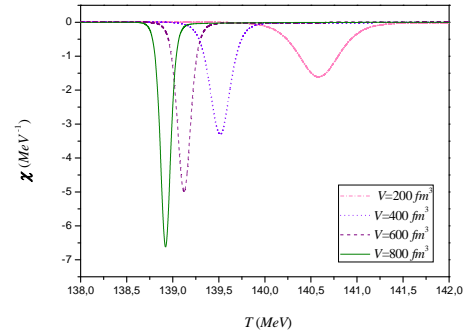


Figure 2. Plot of the susceptibility versus temperature for different system volumes.

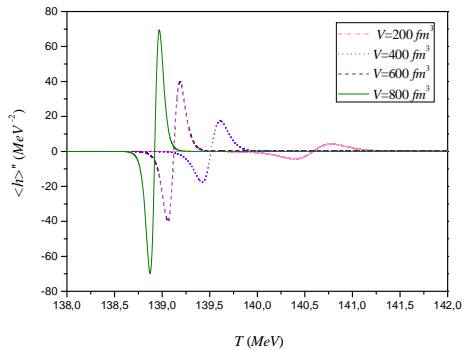


Figure 3. Plot of the second thermal derivative of the order parameter versus temperature for different system volumes.

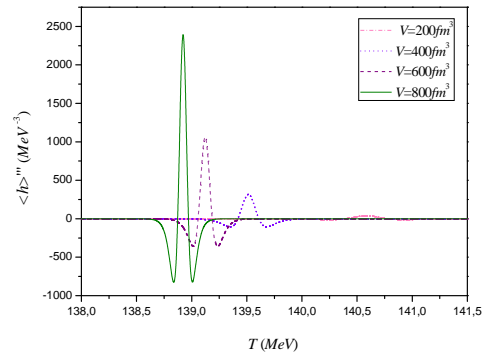


Figure 4. Plot of the third thermal derivative of the order parameter versus temperature for different system volumes.

The previous graphs illustrate the variations of the order parameter $\langle h(T,V) \rangle$ (Figure 1), its first derivative representing the susceptibility (Figure 2) and the second and third derivatives (Figure 3 and figure 4 respectively) with respect to temperature for different volumes of the system. It can clearly be seen that the order parameter exhibits a sharp discontinuity at large volumes, which appears at the level of the first derivative as a sharp peak. This occurs at the thermodynamic limit at a true transition temperature noted $T_c(\infty)$. In small volumes, the order parameter is rounded off, and the susceptibility peak is smeared out, acquiring a bigger width and a smaller height, smaller is the volume. The effective transition temperature in small volumes $T_c(V)$, shifted from the true one in the thermodynamic limit, can be defined at the value $\langle h(T,V) \rangle = 0.5$, meaning that both hadronic and QGP states contribute with equal probability to the total system. Similarly, it can be defined at the minimum of the susceptibility, at vanishing $\langle h(T,V) \rangle''$ and at the maximum of $\langle h(T,V) \rangle'''$. Also, the width $\delta T(V)$ of the transition region over which the transition is rounded off can be defined as the gap in temperature between the two extrema of the second derivative, and we can see that this gap decreases with increasing volume.

3. Probability Density Cumulants

In the following, we examine the first four cumulants of the probability density. The first order cumulant is the order parameter $\langle h(T,V) \rangle$ defined in equation (7).

The second order cumulant is the variance $\sigma^2(T,V)$ given by:

$$\sigma^2(T,V) = \langle (h - \langle h \rangle)^2 \rangle = \langle h^2 \rangle - \langle h \rangle^2 \quad (9)$$

and the third cumulant is the skewness times σ^3 , where the skewness measuring the asymmetry of the distribution is given by:

$$\Sigma(T,V) = \frac{\langle (h - \langle h \rangle)^3 \rangle}{\sigma^3} = \frac{\langle h^3 \rangle - 3\langle h \rangle \langle h^2 \rangle + 2\langle h \rangle^3}{(\langle h^2 \rangle - \langle h \rangle^2)^{3/2}} \quad (10)$$

The fourth cumulant is the kurtosis times σ^4 , where the kurtosis measuring the flatness of the distribution is given by:

$$K(T,V) = \frac{\langle (h - \langle h \rangle)^4 \rangle}{\sigma^4} - 3 = \frac{\langle h^4 \rangle - 4\langle h \rangle \langle h^3 \rangle + 6\langle h \rangle^2 \langle h^2 \rangle - 3\langle h \rangle^4}{(\langle h^2 \rangle - \langle h \rangle^2)^2} - 3 \quad (11)$$

The mean values $\langle h^n \rangle$ appearing in the expressions of the considered cumulants are calculated using the definition in equation (1) as follows:

$$\langle h^n(T,V) \rangle = \frac{\int_0^1 h^n Z(h) dh}{\int_0^1 Z(h) dh} \quad (12)$$

Figures 5 to 7 illustrate the variations of the variance, skewness and kurtosis respectively, and a clear similarity between the behavior of the first thermal derivative of the order parameter and the variance, and that of the second thermal derivative of the order parameter and the skewness, as well as that of the third thermal derivative of the order parameter and the kurtosis is noted. However, we notice that the derivatives are symmetric whereas the cumulants are not. For example, figure 8 which illustrates both the kurtosis and the third thermal derivative of the order parameter variations with temperature for $V=100 \text{ fm}^3$, shows that the two minima of the third derivative have the same height while kurtosis has a high first extrema and a short second one. A careful analysis shows that the effective transition

temperature can be defined at the maximum of the 3rd derivative and at the minimum of kurtosis, while the width of the rounded transition region may be the gap in temperature between the localizations of the two points at which the 3rd derivative vanishes, but cannot be extracted from kurtosis [5, 6].

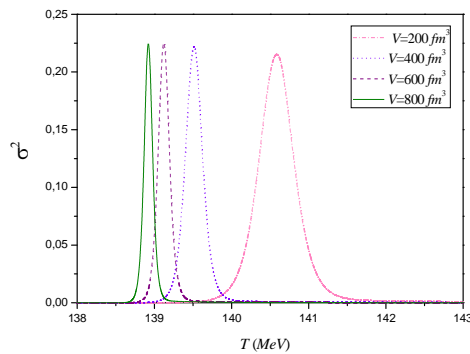


Figure 5. Plot of the variance versus temperature for different system volumes.

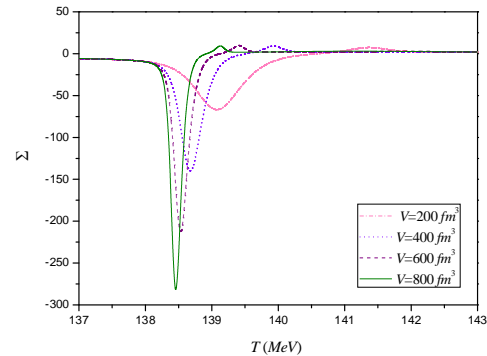


Figure 6. Plot of the skewness versus temperature for different system volumes.

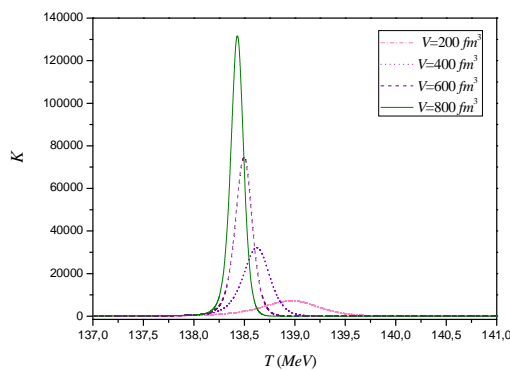


Figure 7. Plot of the kurtosis versus temperature for different system volumes.

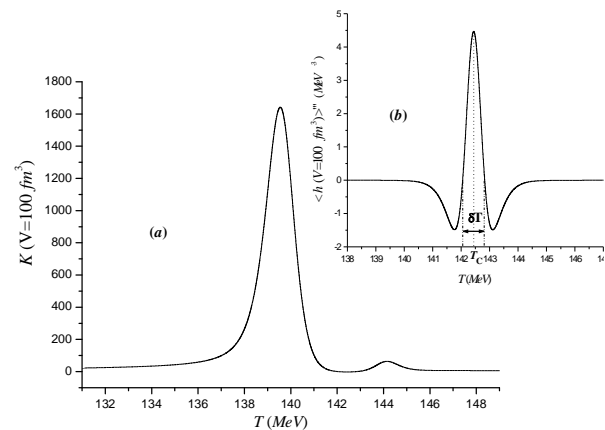


Figure 8. Variations with temperature of (a) kurtosis and (b) the third thermal derivative of the order parameter for $V=100 \text{ fm}^3$.

4. Conclusion

The present study gives some characteristics of the temperature driven deconfining phase transition in a finite volume, namely the effective transition temperature and the width of the rounded transition region in a finite volume. Besides, a similarity between the behavior of the first three derivatives of the order parameter and that of the second, third and fourth cumulants of the probability distribution is noted and is analyzed. A more complete work, where an explicit relationship between the variance and the susceptibility has been obtained can be found in [7]. More graphic illustrations are presented there and a deeper analysis is investigated.

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