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On area and entropy of a black hole

A. Alekseev [a,b], A.P. Polychronakos [c], M. Smedbäck [a]

[a] *Institute for Theoretical Physics, Uppsala University, Box 803, S-75108 Uppsala, Sweden*
[b] *Section of Mathematics, University of Geneva, c.p. 240, 1211 Geneva 24, Switzerland*
[c] *Physics Department, City College of the CUNY, New York, NY 10031, USA*

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Abstract

We consider a model of a black hole consisting of a number of elementary components. Examples of such models occur in loop quantum gravity and in M-theory. We show that treating the elementary components as completely distinguishable leads to the area law for the black hole entropy. Contrary to previous results, we show that no Bose condensation occurs and that the horizon area has big local fluctuations. We discuss a regularization which leads to the equidistant area spectrum in loop quantum gravity.

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In the framework of quantum gravity black holes are treated as quantum objects. As such, they are characterized by quantum numbers: mass, electric charge, angular momentum, etc. For Schwarzschild black holes (neutral, nonrotating) the only quantum number which is left is the mass M. It is related to the area A of the black hole horizon by formula, $A = 16\pi G^2M^2/c^4$, where G is the gravitational constant. Important questions in black hole physics are what the spectrum of A looks like and what the degeneracies of states are for a given value of A. These will determine the quantum and statistical mechanical properties of black holes [1].

In the absence of a definitive quantum gravity theory, the answers to these questions depend on the model of a black hole. In several approaches one assumes that a black hole consists of elementary components contributing additively to its area. For instance, in the loop quantum gravity approach (see, e.g., [2,3]) these are Wilson lines of the Ashtekar's connection A_μ^a. In the M-theory approach [4] these are D_0-branes. Each of the elementary components can be in a number of states. An elementary component in the state n gives a contribution A_n to the total area, and has a degeneracy $g(n)$. The eigenvalues of the area operator then acquire the form $A = \sum_{n=1}^{\infty} N_n A_n$, where N_n is the number of elementary components in the state n. The multiplicity of states with area A is $\Omega(A)$ and its asymptotics when A is macroscopically large defines the entropy of the black hole, $S(A) = k \ln \Omega(A)$.

E-mail addresses: alekseev@math.unige.ch (A. Alekseev), alexios@sci.ccny.cuny.edu (A.P. Polychronakos), mikael@teorfys.uu.se (M. Smedbäck).

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The spectrum of the area A and the behaviour of the entropy $S(A)$ depend on the spectrum of the elementary area A_n, the elementary multiplicities $g(n)$ and the degree of distinguishability of the components. We will discuss the form of A_n in the framework of loop quantum gravity [2,3]. The classical result in this field is that the index n can be associated with a spin j of an $SU(2)$ representation. Then, j takes integer and half-integer values and according to [5] $A_j = \sqrt{j(j+1)}$ in some units. We discuss a regularization which gives instead

$$A_j = j + \frac{1}{2}. \tag{1}$$

In particular, this implies that the spectrum of the area operator is equidistant. Such a situation was first considered by Bekenstein, based on the fact that the black hole area behaves as an adiabatic quantum invariant [1]. This was later further analyzed by Bekenstein and Mukhanov [6], who showed that such a spectrum implies certain specific properties of the black hole radiation: there exists an energy quantum $\hbar\omega_0$, energy can be radiated only in integer multiples of this energy quantum and the black hole radiation spectrum becomes discrete. An equidistant area spectrum has been proposed by several other authors, although not in the loop quantum gravity approach [7].

Furthermore, we will consider the issue of black hole entropy. It will be seen that the *area law* $S \propto A$ is to a large extent generic. A more subtle issue is what the statistically preferable state of the system is. The analysis of [10] in the case of loop quantum gravity and of [11] in the case of black holes composed of D_0-branes showed that the most probable configuration has all the elementary components in the same multiplet. This can be viewed as Bose condensation. We show instead that for completely distinguishable elementary components one obtains a Gibbs distribution for the N_n's with many of them nonvanishing in the most probable configuration. Our result implies that the area operator has large fluctuations, $\Delta A \sim \sqrt{A}$, whereas in the case of Bose condensation the fluctuation of area would have been strongly suppressed.

We turn first to the issue of the area spectrum in loop quantum gravity. One of the virtues of Ashtekar's formulation is that instead of the metric $g_{\mu\nu}$ one deals with the gauge field A_μ^a [2,3]. In the Euclidean gravity the gauge group is $SU(2)$ and the isotopic index a takes values $1, 2, 3$ corresponding to three Pauli matrices. In the Hamiltonian formulation one uses the spatial components A_i^a as generalized coordinates and the components of the electric field, $E_i = \partial_0 A_i - \partial_i A_0 - [A_0, A_i]$ as conjugate momenta. It is convenient to view the gauge field as a 1-form $A = A_i\,dx^i$, and the conjugate momenta E_i as components of a 2-form, $E = \frac{1}{2}\varepsilon_{ijk} E_i\,dx^j\,dx^k$.

The physical Hilbert space is obtained by imposing the Gauss law constraint, the diffeomorphism constraint and the Hamiltonian constraint (the Wheeler–DeWitt equation). While constructing the full space of wave functionals which annihilate all those constraints proves to be a difficult task [2], one believes that the spectrum of the area operator is captured by studying the 'kinematical Hilbert space' containing the Wilson lines [12],

$$W_\Gamma^j(A) = \operatorname{Tr} P \exp\left(\int_\Gamma A^a T_j^a\right), \tag{2}$$

where Γ is a closed contour, j is a positive integer or half-integer and T_j^a are generators of the Lie algebra $su(2)$ in the representation with spin j.

In terms of the Ashtekar's variables, the area operator corresponding to the 2-dimensional surface Σ acquires the form [5],

$$\mathcal{A}_\Sigma = \int_\Sigma d^2x \sqrt{\operatorname{Tr} E^2}, \tag{3}$$

where the integrand is the natural density which can be integrated over the 2-dimensional surface Σ. The Wilson lines W_Γ^j are eigenstates of the operators $\mathcal{A}_\Sigma$, at least when all the intersections of Γ and Σ are transversal. Moreover, each intersection gives a contribution A_j which depends only on the spin j of the Wilson line. It is our next task to compute the numbers A_j.

Canonical quantization suggests that in the A-representation the conjugate momentum E_i^a act as derivatives, $E_a^i(x) = -i\delta/\delta A_a^i(x)$. Ignoring the singularity arising from the coincident arguments in the expression $\sqrt{E_i^a(x)E_i^a(x)}$, one easily obtains

$$\mathcal{A}_\Sigma W_\Gamma(A) = \sum_p \operatorname{Tr} \sqrt{T_j^a T_j^a}\, P \exp\left(\int_p A_i^a T_j^a\,dx_i\right),$$

where p are the intersection points of Γ and Σ, and $\int_p$ stands for the integration over Γ with the starting point at p. The expression $T_j^a T_j^a$ is proportional to the unit matrix with coefficient the quadratic Casimir $c_j = j(j+1)$ of $SU(2)$. This observation led [5] to the conclusion that $A_j = \sqrt{j(j+1)}$. We shall argue that the spectrum of the area operator depends on the choices in the definition of the path integral and discuss the particular choice giving rise to formula (1).

The area operator $(E_i^a(x)E_i^a(x))$ contains a product of two fields $E_a^i(x)$ at the same point and potentially needs a regularization. The answer $c_j = j(j+1)$ is certainly correct to the leading order in j which might be used as the parameter in the semi-classical expansion. However, there might be quantum corrections to this formula similar to the shift by $\hbar\omega/2$ in the energy spectrum of the harmonic oscillator $E_n = \hbar\omega(n+\frac{1}{2})$. We shall argue that a similar shift by $\frac{1}{4}$ in the Casimir arises, giving rise to the spectrum $A_j = j + \frac{1}{2}$.

Instead of using the A-representation it will be convenient to write the wave functional $W_\Gamma(A)$ in the E-representation by means of the functional Fourier transform,

$$\widetilde{W}_\Gamma(E) = \int \mathcal{D}A\, e^{i\int \mathrm{Tr}(EA)} W_\Gamma(A). \tag{4}$$

This expression can be simplified using the geometric quantization formula for the Wilson line [13,14],

$$W_\Gamma(A) = \int \mathcal{D}g\, e^{i\int_\Gamma \mathrm{Tr}(\tau g^{-1}\partial_s g\, ds + A(g\tau g^{-1}))}, \tag{5}$$

where the auxiliary field g is a group valued function on the contour Γ, τ is a constant diagonal matrix, and s is a parameter along Γ. The right-hand side of (5) is well-defined only when the eigenvalues of p are integers or half-integers [14]. Putting together the functional Fourier transform (4) and the geometric quantization (5) yields,

$$\widetilde{W}_\Gamma(E) = \int dg\, \delta\big(E - g\tau g^{-1}\delta_\Gamma\big), \tag{6}$$

where δ_Γ stands for the δ-function supported on Γ. Eq. (6) suggests that E vanishes outside Γ. On the contour, we obtain $\sqrt{\mathrm{Tr}\,E^2} = \sqrt{\mathrm{Tr}\,\tau^2}\delta_\Gamma$. Hence, the contribution to the area operator coming from the transversal intersection of Γ and Σ is given by $A_j = \sqrt{\mathrm{Tr}\,\tau^2}$. The relation between τ and j depends on the regularization of the functional integral over g. Using the regularization of [13] one obtains $A_j = j + \frac{1}{2}$ (the regularization of [14] yields $A_j = j$).

Choosing the right regularization is a delicate issue. There is at least one example of a gauge theory where the spectrum of the operator $\mathrm{Tr}\,E^2$ can be determined unambiguously. This is 2-dimensional Yang–Mills theory as considered by Witten [15]. There, the eigenvalues of the operator $\mathrm{Tr}\,E^2$ can be related to the volumes of the moduli spaces of flat connections on Riemann surfaces which can be computed independently without use of field theory (see, e.g., [16]). In this case, the correct regularization indeed happens to be $c_j = (j+\frac{1}{2})^2$ [15].

The one-dimensional path integral which occurs in formula (5) is a particular case of a Poisson σ-model [8] (with target the space R^3 with Poisson bracket $\{x_i, x_j\} = \varepsilon_{ijk}x_k$). A Poisson σ-model is a two-dimensional topological field theory which admits a one-dimensional formulation of the type (5). In this formalism, one possible regularization of the operators on the one-dimensional contour (like the operator $\sqrt{\mathrm{Tr}\,E^2}$) is to define them as limits of the bulk operators when the distance to the boundary $\varepsilon \to 0$ [9]. Such a regularization maps the center of the Poisson algebra to the center of the algebra of observables preserving multiplication (see, [9, Section 4]). In our case, this requirement again fixes $A_j = j + \frac{1}{2}$.

In conclusion, we collected evidence in favour of the formula (1) for the spectrum A_j of the elementary objects. If correct, it implies a Bekenstein–Mukhanov type discrete spectrum of the black hole area, and a discrete spectrum of the black hole radiation.

We now turn to the black hole thermodynamics. We consider a quantum black hole as consisting of a large number of identical elementary components. As stated, examples of such elementary components are the Wilson lines in loop quantum gravity, or, equivalently, the spin variables in spin lattice gravity, and D_0-branes in M-theory. Different viewpoints on the degree of distinguishability of these components will be discussed, and the entropy will be calculated in each case.

To facilitate the counting of states, it is convenient to view the set of elementary components as the constituents of a grand canonical system, with a temperature-like parameter β dual to the area controlling the thermodynamics. Standard statistical mechanical arguments determine the entropy S of the black hole to be given by

$$S = k(A\beta + \ln Z), \tag{7}$$

where k is Boltzmann's constant, A is the conserved black hole area and Z is the area partition function

$$Z = \sum_{A} \Omega(A) e^{-\beta A} = \sum_{A,N} \Omega(A,N) e^{-\beta A}. \tag{8}$$

Here $\Omega(A)$ is the multiplicity of states of area A, and $\Omega(A,N)$ is the multiplicity of state of area A and N elementary components. The summation is over all possible areas and number of components.

To proceed, we will consider the elementary components as identical but *completely distinguishable* independent quantum systems. Then the partition function becomes

$$Z = \sum_{N} Z_1^N = \frac{1}{1 - Z_1}, \tag{9}$$

where $Z_1 = \sum_A \Omega(A,1) e^{-\beta A}$ is the partition function of a single component. The above expression for Z implies that β can never be less than the *Hagedorn temperature* parameter β_0, fixed by the relation $Z_1(\beta_0) = 1$.

From Eq. (8) the area is related to the partition function by formula,

$$A = -\frac{d \ln Z}{d\beta} = -\frac{1}{1 - Z_1} \cdot \frac{dZ_1}{d\beta}.$$

Restricting our considerations to macroscopic areas $A \gg 1$ only, this relation implies that $1 - Z_1 \sim A^{-1}$. Thus $\beta \to \beta_0$ and $\ln Z$ grows only as $\ln A$. The dominant contribution to the entropy will therefore be given by $S = k\beta_0 A$. The entropy is always proportional to the area. This is a completely general result, valid for any system consisting of distinguishable components.

To fix the proportionality constant, we restrict our considerations to the case of the area spectrum given by Eq. (1), i.e., $A_n = n$, $n = 1, 2, 3, \ldots$, and the degeneracy function $\Omega(A,1) = \Omega(n,1) = g(n) = n$. Note that the states enumerated by the spin quantum number j, taking integer or half-integer values, are now enumerated by $n = 2j + 1$, a positive integer. We obtain $Z_1 = (2\sinh\frac{\beta}{2})^{-2}$. The partition function diverges at the Hagedorn parameter $\beta = \beta_0 = \ln\frac{3+\sqrt{5}}{2}$ and the area becomes macroscopic as $\beta \to \beta_0$. In that limit the entropy becomes

$$S = \frac{kA}{4\pi\gamma l_p^2} \ln\left(\frac{3+\sqrt{5}}{2}\right), \tag{10}$$

where we have acknowledged that the area is measured in units of $4\pi\gamma l_p^2$, where l_p is the Planck length and γ is the Immirzi parameter [17]. The above result can also be derived by direct calculation of the multiplicity of states at fixed area A. The grand canonical derivation presented above, however, is significantly simpler.

Eq. (10) represents our final result for the entropy of a black hole. The validity of this derivation depends crucially on two important and independent claims: that Eq. (1) gives the true area spectrum and that the area constituents, i.e., the edges in the spin lattice gravity approach, are completely distinguishable. Justification for the former claim has already been given. Let us now turn to the latter.

The elementary components of a black hole could a priori be either *identical*, *partially distinguishable* or *completely distinguishable*. Completely distinguishable components would mean that:

(1) There is a difference between assigning two given spin values j_1 and j_2 to two different edges according to (edge 1, edge 2) $= (j_1, j_2)$ and (edge 1, edge 2) $= (j_2, j_1)$;
(2) Even if $j_1 = j_2$, there is a difference between assigning two different spin states m_1 and m_2 to the two different edges.

In calculating the entropy of Eq. (10), the edges were considered to be *completely distinguishable*, i.e., both claims (1) and (2) are imposed. Since the positions of different edges are determined by the *spin network*, which is formed by the way in which the edges are connected to each other and to the outside world, this viewpoint appears to be the one best describing the physical model, and as such, the one we should adopt (see also [18,19]).

In papers [10,11] a different view is adopted: the edges are *partially distinguishable*. Indeed, the multiplicity formula $\Omega(\{N_n\}) = \prod_n g(n)^{N_n}$ applies if claim (1) above is *not* adopted while (2) still applies.

Evaluating (8) with this counting of states we obtain

$$Z = \sum_{\{N_n\}} \Omega(\{N_n\}) e^{-\beta A} = \prod_n \frac{1}{1 - g(n) e^{-\beta A(n)}}. \quad (11)$$

If no edge has an exponentially increasing density of states (in which case it would be itself a macroscopic black hole) the only poles of the above expression are at $e^{\beta A(n)} = g(n)$. Calling $\bar{\beta}_0$ the lowest of these values of β, occurring for some n_0, we deduce that for macroscopic areas the model will exhibit Bose condensation at the spin $j_0 = (n_0 - 1)/2$ and the entropy will be $S = \bar{\beta}_0 A$. For our area function (1) and the degeneracy function $g(n) = n$ this gives an entropy

$$S = \frac{kA\gamma_0}{4l_p^2} \quad \text{with } \gamma_0 = \frac{\ln 3}{3\pi}.$$

Bose condensation occurs at spin $j = 1$. For Ashtekar's choice of area function the result is instead $\gamma_0 = \frac{\ln 2}{\pi\sqrt{3}}$, and Bose condensation occurs at $j = \frac{1}{2}$.

Finally, if the edges are *identical* we adopt neither claim (1) nor (2) and have the equivalent of a Bose–Einstein gas. Then, it turns out that the entropy is related to the area by $S \propto A^t$, where the exponent satisfies $t < 1$ if no edge has an exponential density of states. For the choices of area spectrum (1) and degeneracy function $g(n) = n$, the exponent acquires the value $t = 2/3$. No Bose condensation occurs.

In conclusion, the area law $S = \beta_0 A$ and the appearance of a Hagedorn parameter are the main results. These are extremely generic, requiring only *some* distinguishability of the elementary components. Since the Immirzi parameter is not fixed by the quantum theory [17], the different values of β_0 are of somewhat secondary importance. There is, however, a crucial physical difference between our result and the result of [10,11]. While the latter suggest that Bose condensation occurs and area fluctuations are strongly suppressed, in our fully distinguishable case no Bose condensation occurs and the area within any solid angle of the black hole will exhibit fluctuations of order $\sqrt{A}$.

It is important to consider whether the appearance of such fluctuations is reasonable and whether it has observable consequences. A fluctuation $\Delta A_\Omega \sim \sqrt{A_\Omega}$ for the horizon area within a solid angle Ω implies a local fluctuation in the radius R of the black hole of order $\Delta R \sim 1$ in Planck units, as well as the fact that such fluctuations in parts of the horizon separated by more that a few Planck lengths are statistically independent. These are certainly reasonable and expected in any theory of quantum gravity. This also implies that any observational implication of such fluctuations would involve probing physics at the Planck scale. Such effects are clearly yet to be detected.

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