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Quantum circuit for the direct measurement of the three-tangle of three-qubit states

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Understanding how to decompose quantum computations in the language of the shortest possible sequence of quantum gates is of interest to many researchers due to the importance of the experimental implementation of the desired quantum computations. We contribute to this research by providing a quantum circuit to directly measure the three-tangle of three-qubit quantum states. Direct measurement of outcome probabilities in the computational basis quantifies the three-tangle of the three-qubit quantum states.

Subject Index quantum circuit, Direct measurement, three-tangle

1. Introduction

Quantum entanglement has been regarded as a main nonlocal resource for quantum information processing [1]. It is widely used in quantum information science, including quantum key distribution [2], quantum computation [3], quantum metrology [4], and so on. Over recent decades the growing interest in producing entangled states in an experimental setup, such as atoms in cavities [5], entanglement among 14 trapped ions [6], five superconducting qubits [7], or trapped ions [8], has led to extensive research in this field. Naturally, there are two main problems: entanglement detection (determining whether a given quantum state is entangled or separable) and entanglement measurement (measuring the degree of entanglement between the qubits of a quantum system).

We start with the relevant definitions of multipartite entanglement. An N-partite pure state is called fully separable if it can be written as $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_N\rangle$. On the other hand, a mixed state is fully separable if it can be written as a convex combination of such fully separable pure states, $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$, where the coefficients p_k form a probability distribution, i.e. $p_k \geq 0$, $\sum_k p_k = 1$. One important method for entanglement detection is entanglement witness [10, 9]. Lu and coworkers [11] used machine learning methods to construct a new entanglement-separability classifier that had an advantage over the other methods in speed and accuracy. Also, by constructing neural networks, they can simultaneously encode convex sets of multiple entanglement witness inequalities [12].

Another open problem in quantum information is the quantification of the degree of entanglement for an arbitrary quantum system [13]. Concurrence as an entanglement measure to compute the entanglement of the formation of two-qubit pure and mixed states was introduced in Refs. [14–16]. Until now, many entanglement measures such as negativity, geometric measures, genuine multipartite entanglement, three-tangle, and a polynomial invariant of degree 2 have been defined to specify the degree of entanglement in a quantum system [17–19]. Also, there are several works for estimating entanglement using available quantum simulators [20–28].

A set of one- and two-qubit quantum gates can be used to perform *n*-qubit quantum computation [1, 29]. Recent advances in the experimental implementation of quantum computation have led to an interest in analyzing quantum circuits [30, 31]. Quantum circuits consist of wires as qubits as well as gates or quantum operators, and describe the quantum computation. Romero et al. proposed a general scheme to measure the concurrence of an arbitrary two-qubit pure state in atomic systems based on quantum gates that act on two available copies of the bipartite system followed by a global qubit readout [32]. We obtained a realistic protocol for directly measuring the polynomial invariant of degree 2 of an even *N*-qubit pure state [33].

In this paper we propose a protocol to measure the three-tangle of three-qubit pure states. The proposed method is based on the availability of four copies of the bipartite state and the direct measurement of the probability of occupying the collective state of all of the copies. We show that the three-tangle is equivalent to the probability of finding two specific configurations. Estimating the probabilities needs huge sampling. A theory for quantum metrology that makes high-precision measurements of given parameters with quantum systems is quantum parameter estimation, and a mathematical tool to estimate the quantum parameter is the quantum Cramér–Rao bound. In the quantum Cramér–Rao bound, two fundamental quantities to represent the limit of the precision of single- and multi-parameter estimations are, respectively, the quantum Fisher information and quantum Fisher information matrix [34]. But we do not use quantum parameter estimation in the three-qubit system. Instead, we implement a program related to this protocol in mathematics. In this program, we obtain all 2^{12} possible configurations, but only two of them are equivalent to three-tangle.

2. Method

For three-qubit systems, some ways of characterizing and quantifying entanglement have been presented [35–37]. Three-tangle is a necessary polynomial invariant for quantifying the entanglement of three-qubit states [38]. It is defined, for the general three-qubit state $\mid \phi\rangle = \sum_{i,j,k=0}^{1} b_{ijk} \mid ijk\rangle$ with $\sum_{i,j,k=0}^{1} \mid b_{ijk}|^2 = 1$ as [39]

$$\begin{aligned}\tau(|\phi\rangle) &= 2\left|\varepsilon_{i_1i_2}\varepsilon_{j_1j_2}\varepsilon_{k_1k_3}\varepsilon_{k_2k_4}\varepsilon_{i_3i_4}\varepsilon_{j_3j_4}b_{i_1j_1k_1}b_{i_2j_2k_2}b_{i_3j_3k_3}b_{i_4j_4k_4}\right| \\ &= 4\left|k_1 - 2k_2 + 4k_3\right|, \end{aligned} \tag{1}$$

where $\varepsilon_{01} = -\varepsilon_{10} = 1$, $\varepsilon_{00} = \varepsilon_{11} = 0$, the sum is over all the indices, and

$$\begin{aligned} k_1 &= b_{000}^2 b_{111}^2 + b_{001}^2 b_{110}^2 + b_{010}^2 b_{101}^2 + b_{100}^2 b_{011}^2, \\ k_2 &= b_{000}b_{111}b_{011}b_{100} + b_{000}b_{111}b_{101}b_{010} + b_{000}b_{111}b_{110}b_{001} + b_{011}b_{100}b_{101}b_{010} \\ &\quad + b_{011}b_{100}b_{110}b_{001} + b_{101}b_{010}b_{110}b_{001}, \\ k_3 &= b_{000}b_{110}b_{101}b_{011} + b_{111}b_{001}b_{010}b_{100}. \end{aligned}$$

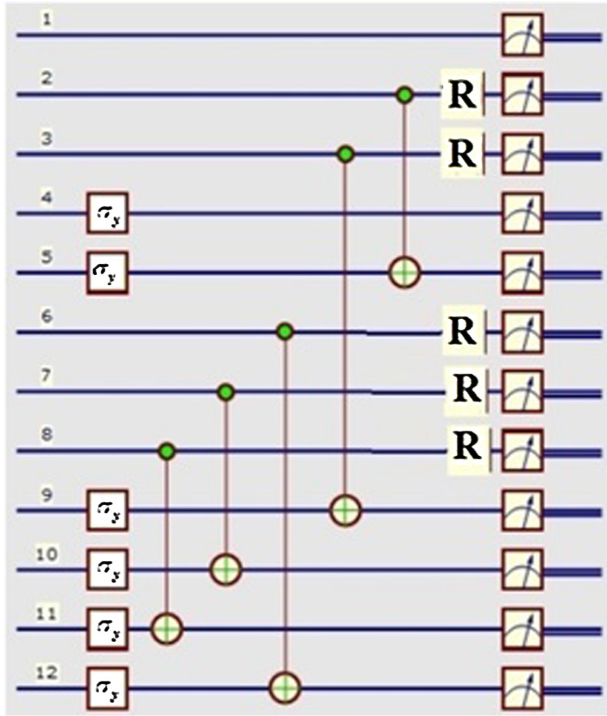

Fig. 1. Quantum circuit describing a direct measurement of the three-tangle of a three-qubit pure state, where four copies are available. Qubits 1–3 are the first copy, 4–6 are the second copy, 7–9 are third copy, and 10–12 are fourth copy of the three-qubit quantum state. It involves five controlled-NOT gates, as well as σ_y and rotation gates R, followed by the joint measurement of the 12 qubits.

In this paper we provide the protocol that implements Eq. (1) quantum mechanically. Our purpose is to give the protocol of implementation of Eq. (1); its circuit model is displayed in Fig. 1.

According to Eq. (1), four copies of the three-qubit system $|\,\phi\rangle$ are required, so that the indices of (i_1, j_1, k_1), (i_2, j_2, k_2), (i_3, j_3, k_3), and (i_4, j_4, k_4) are related to the first, second, third, and fourth copies of the three-qubit quantum system, respectively. For the sake of brevity, let us denote them by the numbers (1, 2, 3), (4, 5, 6), (7, 8, 9), and (10, 11, 12), respectively. The protocol can be thought of as the following steps:

1. Prepare four copies of the three-qubit state given by $|\,\phi\rangle$ as:

$$|\,\eta_0\rangle = |\,\phi\rangle \otimes |\,\phi\rangle \otimes |\,\phi\rangle \otimes |\,\phi\rangle.$$

2. The Pauli y gate is applied to the fourth, fifth, ninth, tenth, eleventh, and twelfth qubits:

$$|\,\eta_1\rangle = \sigma_{y_4} \otimes \sigma_{y_5} \otimes \sigma_{y_9} \otimes \sigma_{y_{10}} \otimes \sigma_{y_{11}} \otimes \sigma_{y_{12}}\,|\,\eta_0\rangle.$$

3. We need five controlled-NOT gates (CNOTs) so that for each, the control and target qubits are specified by the epsilon indices in Eq. (1). That is, $\varepsilon_{j_1(\equiv 2) j_2(\equiv 5)}$shows that one of the CNOTs acts on qubits 2 (the second qubit of the first copy) and 5 (the second qubit of the second copy) as control and target qubits, respectively, and so on for the next CNOTs. So, the CNOT gates are applied between the qubits as:

$$|\,\eta_2\rangle = C_{2,5}C_{3,9}C_{6,12}C_{7,10}C_{8,11}\,|\,\eta_1\rangle,$$

where C is a CNOT gate and the subscripts in the $C_{i,j}$ gate denote the control and the target gate, respectively.

4. Finally, the rotation gates R are applied to the second, third, sixth, seventh, and eighth qubits:

$$|\ \eta_3\rangle = R_2 R_3 R_6 R_7 R_8\ |\ \eta_2\rangle.$$

The unitary R gate rotates the state of the qubit as:

$$R\ |\ 0\rangle = \frac{|\ 0\rangle - |\ 1\rangle}{\sqrt{2}}, \qquad R\ |\ 1\rangle = \frac{|\ 0\rangle + |\ 1\rangle}{\sqrt{2}}.$$

Applying steps 1 to 4 for the quantum state $|\ \phi\rangle = \sum_{i,j,k=0}^{1} b_{ijk}\ |\ ijk\rangle$, in which none of the coefficients are zero, we obtain the following result:

$$|\ \eta_3\rangle = \frac{1}{4\sqrt{2}}(-k_1 + 2k_2 - 4k_3)\ |\ 000000000000\rangle + \frac{1}{4\sqrt{2}}(-k_1 + 2k_2 - 4k_3)\ |\ 100100000000\rangle + \cdots \tag{2}$$

Note that the $\cdots$ in the above relation indicates other sentences whose coefficients, unlike the first two sentences, are not related to the three-tangle, and we do not write them to avoid prolonging the result. Also, k_1, k_2, andk_3 are given at the bottom of Eq. (1). Comparing Eqs. (1) and (2), we obtain

$$\tau = 16\sqrt{2P_{000000000000}} \qquad \text{or} \qquad \tau = 16\sqrt{2P_{100100000000}}, \tag{3}$$

where $P_{000000000000}$ and $P_{100100000000}$ are the success probability of getting the state $|\ 000000000000\rangle$and $|\ 100100000000\rangle$, respectively (a detailed proof is given in the appendix).

For the entangled (GHZ) state $|\ \phi\rangle = \alpha\ |\ 000\rangle + \beta\ |\ 111\rangle$ (in which only two of the eight possible coefficients for the three-qubit quantum state are nonzero), the three-tangle is equal to $\tau(|\ \phi\rangle) = 4\alpha^2\beta^2$. Also, similar to the calculations of Eq. (2), for the GHZ state we obtain the following results:

$$\tau = 16\sqrt{2P_{\{0,1,2,3\}\{0,1\}\{0,2,4,6\}\{0\}}},$$
$$\tau = 16\sqrt{2P_{\{0,1,2,3\}\{0,1\}\{1,3,5,7\}\{1\}}},$$
$$\tau = 16\sqrt{2P_{\{4,5,6,7\}\{4,5\}\{0,2,4,6\}\{0\}}},$$
$$\tau = 16\sqrt{2P_{\{4,5,6,7\}\{4,5\}\{1,3,5,7\}\{1\}}},$$
$$\tau = 16\sqrt{2P_{\{0,1,2,3\}\{6,7\}\{0,2,4,6\}\{6\}}},$$
$$\tau = 16\sqrt{2P_{\{4,5,6,7\}\{2,3\}\{0,2,4,6\}\{6\}}},$$

in which we have converted the binary form to a decimal form for shorthand as

$$000 \rightarrow 0,$$
$$001 \rightarrow 1,$$
$$010 \rightarrow 2,$$
$$011 \rightarrow 3,$$
$$100 \rightarrow 4,$$
$$101 \rightarrow 5,$$
$$110 \rightarrow 6,$$
$$111 \rightarrow 7.$$

So. $\tau = 16\sqrt{2P_{\{0,1,2,3\}\{0,1\}\{1,3,5,7\}\{1\}}}$, in which $\{0, 1, 2, 3\}$, $\{0, 1\}$, $\{1, 3, 5, 7\}$, and $\{1\}$ are related to the first, second, third, and fourth copies of the three-qubit quantum state, respectively, means that:

$$\tau = 16\sqrt{2P_{0011}} \quad \text{or} \quad \tau = 16\sqrt{2P_{0031}} \quad \text{or} \quad \tau = 16\sqrt{2P_{0051}} \quad \text{or} \quad \tau = 16\sqrt{2P_{0071}} \quad \text{or}$$
$$\tau = 16\sqrt{2P_{0111}} \quad \text{or} \quad \tau = 16\sqrt{2P_{0131}} \quad \text{or} \quad \tau = 16\sqrt{2P_{0151}} \quad \text{or} \quad \tau = 16\sqrt{2P_{0171}} \quad \text{or}$$
$$\tau = 16\sqrt{2P_{1011}} \quad \text{or} \quad \tau = 16\sqrt{2P_{1031}} \quad \text{or} \quad \tau = 16\sqrt{2P_{1051}} \quad \text{or} \quad \tau = 16\sqrt{2P_{1071}} \quad \text{or}$$
$$\tau = 16\sqrt{2P_{1111}} \quad \text{or} \quad \tau = 16\sqrt{2P_{1131}} \quad \text{or} \quad \tau = 16\sqrt{2P_{1151}} \quad \text{or} \quad \tau = 16\sqrt{2P_{1171}} \quad \text{or}$$
$$\tau = 16\sqrt{2P_{2011}} \quad \text{or} \quad \tau = 16\sqrt{2P_{2031}} \quad \text{or} \quad \tau = 16\sqrt{2P_{2051}} \quad \text{or} \quad \tau = 16\sqrt{2P_{2071}} \quad \text{or}$$
$$\tau = 16\sqrt{2P_{2111}} \quad \text{or} \quad \tau = 16\sqrt{2P_{2131}} \quad \text{or} \quad \tau = 16\sqrt{2P_{2151}} \quad \text{or} \quad \tau = 16\sqrt{2P_{2171}} \quad \text{or}$$
$$\tau = 16\sqrt{2P_{3011}} \quad \text{or} \quad \tau = 16\sqrt{2P_{3031}} \quad \text{or} \quad \tau = 16\sqrt{2P_{3051}} \quad \text{or} \quad \tau = 16\sqrt{2P_{3071}} \quad \text{or}$$
$$\tau = 16\sqrt{2P_{3111}} \quad \text{or} \quad \tau = 16\sqrt{2P_{3131}} \quad \text{or} \quad \tau = 16\sqrt{2P_{3151}} \quad \text{or} \quad \tau = 16\sqrt{2P_{3171}}.$$

Similarly, for $\tau = 16\sqrt{2P_{\{4,5,6,7\}\{4,5\}\{0,2,4,6\}\{0\}}}$, $\tau = 16\sqrt{2P_{\{4,5,6,7\}\{4,5\}\{1,3,5,7\}\{1\}}}$, $\tau = 16\sqrt{2P_{\{0,1,2,3\}\{6,7\}\{0,2,4,6\}\{6\}}}$, and $\tau = 16\sqrt{2P_{\{4,5,6,7\}\{2,3\}\{0,2,4,6\}\{6\}}}$, according to the decimal form of each, the corresponding results are obtained.

Then the three-tangle can be obtained using one of the formulas shown above. It is interesting to note that for this case, which is a particular state of the X state, the genuine multipartite (GM) entanglement is equal to the square root of the three-tangle ($2 \mid \alpha\beta \mid$). Recall that if, in a system consisting of N qubits, each qubit is entangled with all of the other qubits and not only with some of them, we say that the system has GM entanglement. One of the measures to compute the GM entanglement is GM concurrence [40, 41], which, for a pure state $\mid \phi\rangle$, is defined as:

$$C_{\mathrm{GM}}(\mid \phi\rangle) =: \min_{\zeta\in\beta}\sqrt{2}\sqrt{1 - \mathrm{Tr}(\rho^2_{A_\zeta})},$$

where β denotes the set of all possible bipartitions $\{A_\zeta \mid B_\zeta\}$, and ρ_{A_ζ} is the reduced density matrix:$\rho_{A_\zeta} = \mathrm{Tr}_{B_\zeta}(\mid \phi\rangle\langle\phi \mid)$.

3. Conclusion

In conclusion, we have provided a circuit for the direct measurement of the three-tangle as an entanglement measure of a three-qubit pure quantum state, where four copies are available. For implementation, it needs five controlled-NOT gates, σ_2 unitaries, and other simple R qubit rotations. We hope that this proposal could be implemented using present technology. Also, analyzing various aspects of the quantum circuit, such as noise entering the problem gates or incomplete quantum state copies, are our next goals to be explored in future work.

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Appendix A.

We use a mathematical program to give Eq. (3). In this appendix we summarize the steps of the proof by stating a few sentences from each step.

The three-tangle is defined as the following form for the general three-qubit state $|\phi\rangle = \sum_{i,j,k=0}^{1} b_{ijk}|ijk\rangle$, where $\sum_{i,j,k=0}^{1} |b_{ijk}|^2 = 1$:

$$\begin{aligned} \tau(|\phi\rangle) &= 2\left|\varepsilon_{i_1i_2}\varepsilon_{j_1j_2}\varepsilon_{k_1k_3}\varepsilon_{k_2k_4}\varepsilon_{i_3i_4}\varepsilon_{j_3j_4}b_{i_1j_1k_1}b_{i_2j_2k_2}b_{i_3j_3k_3}b_{i_4j_4k_4}\right| \\ &= 4\left|k_1 - 2k_2 + 4k_3\right|, \end{aligned} \tag{A1}$$

where $\varepsilon_{01} = -\varepsilon_{10} = 1,\ \ \varepsilon_{00} = \varepsilon_{11} = 0$, the sum is over all the indices, and

$$\begin{aligned} k_1 &= b_{000}^2 b_{111}^2 + b_{001}^2 b_{110}^2 + b_{010}^2 b_{101}^2 + b_{100}^2 b_{011}^2, \\ k_2 &= b_{000}b_{111}b_{011}b_{100} + b_{000}b_{111}b_{101}b_{010} + b_{000}b_{111}b_{110}b_{001} + b_{011}b_{100}b_{101}b_{010} \\ &\quad + b_{011}b_{100}b_{110}b_{001} + b_{101}b_{010}b_{110}b_{001}, \\ k_3 &= b_{000}b_{110}b_{101}b_{011} + b_{111}b_{001}b_{010}b_{100}. \end{aligned}$$

The detailed proof steps of Eq. (3) are as follows:

1. Prepare four copies of the three-qubit state given by $|\phi\rangle$ as:

$$\begin{aligned} |\eta_0\rangle &= |\phi\rangle \otimes |\phi\rangle \otimes |\phi\rangle \otimes |\phi\rangle \\ &= \left(\sum_{i,j,k=0}^{1} b_{ijk}|ijk\rangle\right) \otimes \left(\sum_{i,j,k=0}^{1} b_{ijk}|ijk\rangle\right) \otimes \left(\sum_{i,j,k=0}^{1} b_{ijk}|ijk\rangle\right) \otimes \left(\sum_{i,j,k=0}^{1} b_{ijk}|ijk\rangle\right) \\ &= b_{000}^4|000000000000\rangle + b_{000}^3 b_{001}|000000000001\rangle + b_{000}^3 b_{010}|000000000010\rangle \\ &\quad + b_{000}^3 b_{011}|000000000011\rangle + b_{000}^3 b_{100}|000000000100\rangle + b_{000}^3 b_{101}|000000000101\rangle \\ &\quad + b_{000}^3 b_{110}|000000000110\rangle + b_{000}^3 b_{111}|000000000111\rangle + \cdots \end{aligned}$$

2. The Pauli y gate is applied to the fourth, fifth, ninth, tenth, eleventh, and twelfth qubits:

$$\begin{aligned} |\eta_1\rangle &= \sigma_{y_4} \otimes \sigma_{y_5} \otimes \sigma_{y_9} \otimes \sigma_{y_{10}} \otimes \sigma_{y_{11}} \otimes \sigma_{y_{12}}|\eta_0\rangle \\ &= -b_{000}^4|000110001111\rangle + b_{000}^3 b_{001}|000110001110\rangle + b_{000}^3 b_{010}|000110001101\rangle \\ &\quad -b_{000}^3 b_{011}|000110001100\rangle + b_{000}^3 b_{100}|000110001011\rangle - b_{000}^3 b_{101}|000110001010\rangle \\ &\quad +b_{000}^3 b_{110}|000110001001\rangle + b_{000}^3 b_{111}|000110001000\rangle + \cdots \end{aligned}$$

3. The CNOT gates are applied between the qubits as:

$$\begin{aligned}|\eta_2\rangle &= C_{2,5}C_{3,9}C_{6,12}C_{7,10}C_{8,11}|\eta_1\rangle\\ &= -b_{000}^4|000110001111\rangle + b_{000}^3 b_{001}|000110001110\rangle + b_{000}^3 b_{010}|000110001101\rangle\\ &\quad - b_{000}^3 b_{011}|000110001100\rangle + b_{000}^3 b_{0100}|000110001011\rangle - b_{000}^3 b_{101}|000110001010\rangle\\ &\quad - b_{000}^3 b_{110}|000110001001\rangle + b_{000}^3 b_{111}|000110001000\rangle + \cdots,\end{aligned}$$

where C is a CNOT gate and the subscripts in the $C_{i,j}$ gate denote the control and the target gate, respectively.

4. Finally, the rotation gates are applied to second, third, sixth, seventh, and eighth qubits:

$$|\eta_3\rangle = R_2R_3R_6R_7R_8|\eta_2\rangle.$$

The unitary gate R rotates the state of the qubit as:

$$R|0\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}, \qquad R|1\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

The result is as follows, only for the first sentence of the third step ($|000110001111\rangle$):

$$\begin{aligned}|\eta_3\rangle = -b_{000}^4(&|000110001111\rangle - |000110101111\rangle - |000111001111\rangle + |000111101111\rangle\\ &-|000110011111\rangle + |000110111111\rangle + |000111011111\rangle + |000111111111\rangle\\ &-|001110001111\rangle + |001110101111\rangle + |0011110011111\rangle - |001111101111\rangle\\ &-|001110011111\rangle + |001110111111\rangle + |001111011111\rangle + |001111111111\rangle\\ &-|010110001111\rangle + |010110101111\rangle + |010111001111\rangle - |010111101111\rangle\\ &+|010110011111\rangle - |010110111111\rangle - |010111011111\rangle - |010111111111\rangle\\ &+|0111110001111\rangle - |011110101111\rangle - |011111001111\rangle + |011111101111\rangle\\ &-|011110011111\rangle + |011110111111\rangle + |011111011111\rangle + |011111111111\rangle) + \cdots\end{aligned}$$

By factoring the sentences and summarizing them, the following expression is obtained:

$$\begin{aligned}|\eta_3\rangle = &\frac{1}{4\sqrt{2}}(-k_1 + 2k_2 - 4k_3)|000000000000\rangle\\ &+\frac{1}{4\sqrt{2}}(-k_1 + 2k_2 - 4k_3)|100100000000\rangle + \cdots \end{aligned} \tag{A2}$$

The coefficients of the first two terms in the above expression are the same as the three-tangle. It should be noted that the three-tangle does not appear in any of the coefficients of the other sentences. By comparing Eqs. (A.1) and (A.2), we get the following equation:

$$\tau = 16\sqrt{2P_{000000000000}} \qquad \text{or} \qquad \tau = 16\sqrt{2P_{100100000000}}, \tag{A3}$$

where $P_{000000000000}$and $P_{100100000000}$are the success probability of getting the state $|000000000000\rangle$and $|100100000000\rangle$, respectively.

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