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Frequentist Limit for $D^0 \to \mu^+\mu^-$

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Abstract

We present a new Frequentist calculation of the limits for the $D^0 \to \mu^+\mu^-$ analysis with $360\,\mathrm{pb}^{-1}$. Code is developed for a multi-channel Frequentist calculation, with a Feldman-Cousins ordering principal, and integration of nuisance parameters in the method of Cousins and Highland. The result agrees with the initial Frequentist calculation done in this analysis, but the sensitivity is in more reasonable agreement with the result. This confirms that the Frequentist limit is about half the Bayesian limit found for this analysis, and reasons why are explored.

1 History of $D^0 \to \mu^+\mu^-$ Results

Recall the reason for performing this check:

In the original analysis [?], we estimated the sensitivity as the most probable outcome of the experiment, yielding sensitivities of 7.3×10^{-7} and 9.1×10^{-7} for the 90% and 95% confidence level upper limits, respectively for the expected backgrounds. For 3 observed events in the CMU-CMU channel, 0 in CMU-CMX, and 1 in CMX-CMX, we derived upper limits on the branching fraction of 2.3×10^{-7} and 3.0×10^{-7} at the 90% and 95% confidence level, respectively.

The significant difference between the sensitivity and the limits, a factor of 3, raised some suspicions about the validity of the calculation, and prompted a recalculation of the sensitivity and the limits using a Bayesian approach.

We used the `genlimit` code written by Joel Heinrich [?]. This code allows for multiple measurement channels with nuisance parameters. The calculation yielded the result that was blessed on 28 February 2008, namely, a sensitivity of $6.5(7.8) \times 10^{-7}$ at the 90% (95%) credibility level, and a limit of $\mathcal{B}(D^0 \to \mu^+\mu^-) < 4.3(5.3) \times 10^{-7}$ at the 90% (95%) credibility level.

That the Bayesian analysis gave a somewhat smaller sensitivity but limits almost a factor of 2 larger than the original frequentist analysis raised some concern for an error. One might presume the error occurred in the original frequentist calculation (limits a factor of 3 times smaller than the sensitivities), but the complexity of the calculation make this impossible to prove. If the frequentist calculation were correct, then the substantially stronger limit makes this a very appealing result. Therefore, it was decided to try once more, with a new, independent frequentist calculation.

Note that Bayesian and Frequentist limits result from different calculations and have different interpretations. As physicists, we tend to take a 95% limit at face value, but there are differences. Often, despite the differences, the results are remarkably similar, reinforcing the tendency to blur the differences when quoting a result. More will be said about the difference between Bayesian and Frequentist limits at the end of this note.

2 New Frequentist Limit Calculation

The probability for observing n_i events, $i = 1$ to N is given by

$$P(n_i) = \prod_{i=1}^{N} \frac{(e_i s + b_i)^{n_i}}{(n_1)!} e^{-(e_i s + b_i)} \tag{1}$$

where the e_i are the efficiencies in each measurement, and b_i are the expected number of background events in each measurement. This calculation is performed as a Feldman-Cousins procedure for two measurement channels ($N = 2$) for the same signal, s, with each channel having a different efficiency for detecting the signal, e_i, and background, b_i.

We couldn't find existing code to do a multiple channel Feldman-Cousins calculation, so we wrote code in `Python`. Several checks of the code were performed.

First, we reproduced the single channel results of Feldman and Cousins [?], as discussed in Appendix **??**. Next, we verify that a two channel calculation with identical backgrounds and efficiencies in both channels produces results that are symmetric on interchange of the number of observed events in each channel. That is, the limit for (n_1, n_2) is the same as the limit for (n_2, n_1). Finally, we calculate sensitivities and outcome probabilities and verify that these are also as expected.

The calculation time grows exponentially with the number of channels, and the 2 channel calculation takes of order 2 hours. Since both previous calculations found that the CMX-CMX channel added little sensitivity to the result, we stopped with just the CMU-CMU and CMU-CMX channels.

The sensitivity for background values around our nominal values are displayed in Table **??**. No significant non-monotonic behavior is seen, so we quote the results for the nominal background expectations of 5.2 and 6.0×10^{-7} at the 90% and 95% confidence level, respectively.

Table 1: Calculation of sensitivity at 3 levels of confidence. Sensitivities are taken as the median upper limit that would be obtained when the signal is zero. As a check on the influence of the background estimate, the sensitivities are calculated for a range of background values centered around the estimated backgrounds. All sensitivities are in units of 10^{-7}.

b-cl		68.27 %	90.00 %	95.00 %
4.5 +/- 1.3	2.3 +/- 1.0	3.02	5.12	5.96
4.7 +/- 1.3	2.3 +/- 1.0	3.08	5.14	5.95
4.9 +/- 1.3	2.3 +/- 1.0	2.88	5.08	5.88
5.1 +/- 1.3	2.3 +/- 1.0	2.94	5.12	5.95
5.3 +/- 1.3	2.3 +/- 1.0	3.23	5.43	6.24
4.5 +/- 1.3	2.5 +/- 1.0	2.93	5.01	5.83
4.7 +/- 1.3	2.5 +/- 1.0	2.98	5.05	5.88
4.9 +/- 1.3	2.5 +/- 1.0	3.12	5.22	6.05
5.1 +/- 1.3	2.5 +/- 1.0	3.10	5.29	6.05
5.3 +/- 1.3	2.5 +/- 1.0	3.20	5.36	6.19
4.5 +/- 1.3	2.7 +/- 1.0	2.96	5.04	5.83
4.7 +/- 1.3	2.7 +/- 1.0	3.10	5.16	5.94
4.9 +/- 1.3	2.7 +/- 1.0	3.08	5.19	6.04
5.1 +/- 1.3	2.7 +/- 1.0	3.07	5.21	6.04
5.3 +/- 1.3	2.7 +/- 1.0	3.27	5.38	6.19
4.5 +/- 1.3	2.9 +/- 1.0	2.99	5.09	5.92
4.7 +/- 1.3	2.9 +/- 1.0	3.12	5.25	6.08
4.9 +/- 1.3	2.9 +/- 1.0	3.20	5.42	6.17
5.1 +/- 1.3	2.9 +/- 1.0	3.19	5.42	6.21
5.3 +/- 1.3	2.9 +/- 1.0	3.25	5.39	6.16
4.5 +/- 1.3	3.1 +/- 1.0	3.06	5.21	6.00
4.7 +/- 1.3	3.1 +/- 1.0	3.17	5.33	6.11
4.9 +/- 1.3	3.1 +/- 1.0	3.12	5.26	6.07
5.1 +/- 1.3	3.1 +/- 1.0	3.06	5.23	6.03
5.3 +/- 1.3	3.1 +/- 1.0	3.19	5.28	6.12

The limits are calculated for 3 observed CMU-CMU events and 0 CMU-CMX events. The 90% limits for backgrounds spaced around the nominal values are shown in Table **??**, and the similar 95% limits in Table **??**. Some non-monotonic behavior is visible, but is generally small and almost washes out when one rounds the results to two significant figures, therefore we quote the 90 and 95% upper limits of 2.1×10^{-7} and 3.0×10^{-7}.

Table 2: 90.00 % confidence limits for $(n_1, n_2) = (3, 0)$. All limits are in units of 10^{-7}.

b_2-b_1	4.5 ± 1.3	4.7 ± 1.3	4.9 ± 1.3	5.1 ± 1.3	5.3 ± 1.3
2.3 ± 1.0	0.00,2.19	0.00,2.19	0.00,2.13	0.00,1.92	0.00,1.80
2.5 ± 1.0	0.00,2.04	0.00,2.27	0.00,1.95	0.00,1.93	0.00,1.83
2.7 ± 1.0	0.00,2.19	0.00,2.14	0.00,2.08	0.00,1.92	0.00,1.84
2.9 ± 1.0	0.00,2.16	0.00,1.98	0.00,2.00	0.00,1.84	0.00,1.78
3.1 ± 1.0	0.00,3.08	0.00,2.07	0.00,2.01	0.00,1.92	0.00,1.87

Table 3: 95.00 % confidence limits for $(n_1, n_2) = (3, 0)$. All limits are in units of 10^{-7}.

b_2-b_1	4.5 +/- 1.3	4.7 +/- 1.3	4.9 +/- 1.3	5.1 +/- 1.3	5.3 +/- 1.3
2.3 +/- 1.0	0.00,3.02	0.00,3.05	0.00,3.00	0.00,2.80	0.00,2.65
2.5 +/- 1.0	0.00,2.95	0.00,3.20	0.00,2.87	0.00,2.83	0.00,2.73
2.7 +/- 1.0	0.00,3.02	0.00,3.02	0.00,3.00	0.00,2.79	0.00,2.58
2.9 +/- 1.0	0.00,3.07	0.00,2.77	0.00,2.91	0.00,2.71	0.00,2.65
3.1 +/- 1.0	0.00,4.03	0.00,2.97	0.00,2.82	0.00,2.79	0.00,2.76

3 Comparison with Initial and Bayesian Results

The limit results from this calculation (multi-channel frequentist), and from the initial frequentist calculation of Ivan, and the Bayesian calculation, are shown in Table **??**. The 90 and 95% confidence level limits agree for the two frequentist based calculations. The Bayesian credibility level limits are about a factor of two larger. The sensitivities don't agree for the frequentist calculations. Different criteria were used to determine the sensitivities – value with the highest probability given zero signal in the initial Frequentist calculation, and median of the distribution of possible limits given zero signal in the multi-channel Frequentist calculation. It is basically impossible to reconstruct the code used in the initial Frequentist calculation, so this difference may be a moot point. The sensitivities for the Bayesian and multi-channel Frequentist calculation were calculated in a consistent manner as the median of the distribution of possible limits in the case of zero signal.

Table 4: Summary of the limit calculations. All limits are in units of 10^{-7}.

	Initial Frequentist	Bayesian	Multi-Channel Frequentist
Sensitivity 90%	7.3	6.5	5.2
Sensitivity 95%	9.1	7.8	6.0
Upper limit 90%	2.3	4.3	2.1
Upper limit 95%	3.0	5.3	3.0

3.1 Discussion

The obvious question is which result to quote, or phrased differently, why are the frequentist limits about a factor of two different from the Bayesian limit?

The Bayesian limit is larger than either Frequentist limit because of the way that downward fluctuations are handled, especially when the observed number is zero. In the Bayesian calculation, an observation of fewer events than the estimated background causes the background to be modified. For instance, if 3 events are observed when 4.9 are expected from the background estimate, then the Bayesian calculation allows for the background to be 0, 1, 2, or 3 events, only. The result is that the Bayesian limits don't decrease as much as the Frequentist limits do when there is a downward fluctuation.

An important special case occurs when the observed events is 0, and the predicted background is nonzero. In this case, the Bayesian limit becomes the same value (2.3 events at 90% credibility level) irrespective of the number of predicted background events. The Frequentist limit behaves rather differently, and decreases with increasing background prediction.

In this analysis, all the channels show a downward fluctuation: 3 observed when 4.9 ± 2.3 background events are expected in the CMU-CMU channel; 0 observed when 2.7 ± 1.0 background events are expected in the CMU-CMX channel; and 1 observed when XX background events are expected in the CMX-CMX channel. In the Bayesian calculation, the limit on the number of signal events is about the same in the CMU-CMU and CMU-CMX, but since the effective number of normalization mode events is more than twice as large in the CMU-CMU channel as the CMU-CMX channel, the resulting limit is determined primarily by the data for the CMU-CMU channel alone.

In the Frequentist calculation, the limit on the number of signal events in the CMU-CMX channel is about half that for the CMU-CMU channel, such that both channels contribute roughly equally to the resulting limit. The effect is almost as if there were twice as many CMU-CMU events, and the resulting limit is about a factor of two smaller.

4 Bayesian Versus Frequentist

The Bayesian approach begins with the relation

$$P(n_t|n_o) = \frac{P(n_o|n_t)P(n_t)}{P(n_o)}$$

where n_o is the observation, n_t is the true value of some quantity, $P(a)$ represents the probability of a, and the notation $P(a|b)$ means the probability of a given b.

5 Conclusion

A multi-channel Frequentist limit calculation has been performed, and yields limits consistent with the initial Frequentist calculation. These limits are about a factor of two smaller than the Bayesian limits, and the mechanism that produces the smaller outcome is understood. The sensitivities are not well understood, but since these aren't integral to the result – they serve as a check that the observed outcome has a reasonable probability – the differences can be ignored.

A Check Against Feldman-Cousins Published Tables

As a check of the validity of the code, we reproduce the tables published by Feldman and Cousins [**?**]. These are to be compared to the tables in the paper. The agreement is nearly exact when the observed number is large (compared to expected background).

Table 5: 90.00 % confidence limits

n_0-b	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	5.0
0	0.00, 2.43	0.00, 1.93	0.00, 1.60	0.00, 1.33	0.00, 1.08	0.00, 1.17	0.00, 0.95	0.00, 1.05	0.00, 0.85	0.00, 0.77
1	0.11, 4.35	0.00, 3.85	0.00, 3.35	0.00, 2.90	0.00, 2.52	0.00, 2.18	0.00, 1.87	0.00, 1.59	0.00, 1.33	0.00, 1.19
2	0.53, 5.91	0.03, 5.41	0.00, 4.91	0.00, 4.41	0.00, 3.91	0.00, 3.44	0.00, 3.03	0.00, 2.66	0.00, 2.32	0.00, 1.72
3	1.10, 7.41	0.60, 6.92	0.10, 6.41	0.00, 5.92	0.00, 5.41	0.00, 4.92	0.00, 4.41	0.00, 3.95	0.00, 3.52	0.00, 2.78
4	1.47, 8.59	1.16, 8.09	0.74, 7.59	0.24, 7.09	0.00, 6.59	0.00, 6.09	0.00, 5.59	0.00, 5.09	0.00, 4.59	0.00, 3.59
5	1.84, 9.98	1.53, 9.48	1.25, 8.98	0.93, 8.48	0.43, 7.98	0.00, 7.48	0.00, 6.98	0.00, 6.48	0.00, 5.98	0.00, 4.98
6	2.20,11.46	1.90,10.96	1.61,10.46	1.33, 9.96	1.08, 9.46	0.65, 8.96	0.15, 8.46	0.00, 7.96	0.00, 7.46	0.00, 6.46
7	3.56,12.53	3.06,12.03	2.56,11.52	2.08,11.02	1.59,10.52	1.18,10.02	0.89, 9.52	0.39, 9.02	0.00, 8.52	0.00, 7.53
8	3.96,13.99	3.46,13.49	2.96,12.99	2.51,12.49	2.14,11.99	1.81,11.49	1.51,10.99	1.06,10.49	0.66, 9.99	0.00, 8.99
9	4.36,15.29	3.85,14.79	3.35,14.29	2.91,13.79	2.53,13.29	2.18,12.79	1.88,12.29	1.59,11.79	1.33,11.29	0.43,10.29
10	5.50,16.50	5.00,16.00	4.50,15.50	4.00,15.00	3.50,14.50	3.04,14.00	2.63,13.50	2.27,13.00	1.94,12.50	1.19,11.50
11	5.91,17.80	5.41,17.30	4.91,16.80	4.41,16.30	3.91,15.81	3.45,15.31	3.04,14.81	2.67,14.31	2.33,13.81	1.73,12.81
12	7.00,19.00	6.50,18.50	6.00,18.00	5.50,17.50	5.00,17.00	4.50,16.50	4.00,16.00	3.54,15.50	3.12,15.00	2.38,14.00
13	7.42,20.05	6.92,19.55	6.42,19.05	5.92,18.55	5.42,18.05	4.92,17.55	4.42,17.05	3.95,16.55	3.53,16.05	2.78,15.05
14	8.50,21.50	8.00,21.00	7.50,20.50	7.00,20.00	6.50,19.50	6.00,19.00	5.50,18.50	5.00,18.00	4.50,17.50	3.59,16.50
15	9.48,22.51	8.98,22.01	8.48,21.51	7.98,21.01	7.48,20.51	6.98,20.01	6.48,19.51	5.98,19.01	5.48,18.51	4.48,17.51
16	9.99,23.99	9.49,23.49	8.99,22.99	8.49,22.49	7.99,21.99	7.49,21.49	6.99,20.99	6.49,20.49	5.99,19.99	4.99,18.99
17	11.04,25.01	10.54,24.51	10.04,24.01	9.54,23.51	9.04,23.01	8.54,22.51	8.04,22.01	7.54,21.51	7.04,21.01	6.04,20.01
18	11.47,26.15	10.97,25.65	10.47,25.15	9.97,24.65	9.47,24.15	8.97,23.65	8.47,23.15	7.96,22.65	7.46,22.15	6.46,21.15
19	12.51,27.42	12.01,26.95	11.51,26.48	11.01,26.00	10.51,25.50	10.01,25.00	9.51,24.50	9.01,24.00	8.51,23.50	7.51,22.50
20	13.55,28.32	13.05,27.88	12.55,27.43	12.05,26.97	11.55,26.50	11.05,26.01	10.55,25.51	10.05,25.01	9.55,24.51	8.55,23.51

One thing that was learned is that there is a *roughness* to the frequentist limits such that the the upper limit is sometimes not monotonic with the background. This is particularly true when the observed number is smaller than expected background. Feldman and Cousins do some smoothing to their results that is not easily reproduced. Several simple algorithms were tried, but none were found to be effective. Therefore, we accept the roughness as part of the procedure, and scan our result for background values around the default values to see if there is any significant effect on the result.

References

[1] E. Berry, I.K. Furić, R.F. Harr, and Y.K. Kim, "Search for Flavor-Changing Neutral Current $D^0 \to \mu^+\mu^-$ Decays Using $360\,\mathrm{pb}^{-1}$ of CDF Run II Data", CDF note 8791.

[2] J. Heinrich, "Bayesian limit software: multi-channel with correlated backgrounds and efficiencies", CDF note 7587.

[3] G. Feldman and R. Cousins, "A Unified approach to the classical statistical analysis of small signals", Phys. Rev. **D**57, 3873 (1998).

[4] R.D. Cousins, "Why isn't every physicist a Bayesian?", Am. J. Phys. **63**, 5 (1995).

Table 6: 90.00 % confidence limits

n_0-b	6.0	7.0	8.0	9.0	10.0	11.0	12.0	13.0	14.0	15.0
0	0.00, 0.70	0.00, 0.91	0.00, 0.84	0.00, 0.79	0.00, 0.74	0.00, 0.70	0.00, 0.89	0.00, 0.85	0.00, 0.81	0.00, 0.78
1	0.00, 1.08	0.00, 0.99	0.00, 0.92	0.00, 0.85	0.00, 0.80	0.00, 1.00	0.00, 0.95	0.00, 0.90	0.00, 0.86	0.00, 0.82
2	0.00, 1.56	0.00, 1.10	0.00, 1.01	0.00, 0.93	0.00, 1.14	0.00, 1.07	0.00, 1.01	0.00, 0.95	0.00, 0.90	0.00, 0.86
3	0.00, 2.14	0.00, 1.59	0.00, 1.44	0.00, 1.33	0.00, 1.24	0.00, 1.15	0.00, 1.08	0.00, 1.02	0.00, 0.96	0.00, 0.91
4	0.00, 2.82	0.00, 2.56	0.00, 1.97	0.00, 1.81	0.00, 1.35	0.00, 1.25	0.00, 1.16	0.00, 1.09	0.00, 1.02	0.00, 0.97
5	0.00, 4.06	0.00, 3.27	0.00, 2.59	0.00, 2.37	0.00, 1.83	0.00, 1.69	0.00, 1.57	0.00, 1.47	0.00, 1.39	0.00, 1.31
6	0.00, 5.46	0.00, 4.54	0.00, 3.73	0.00, 3.02	0.00, 2.39	0.00, 2.20	0.00, 1.70	0.00, 1.59	0.00, 1.48	0.00, 1.39
7	0.00, 6.53	0.00, 5.53	0.00, 4.58	0.00, 3.76	0.00, 3.05	0.00, 2.80	0.00, 2.23	0.00, 2.06	0.00, 1.59	0.00, 1.49
8	0.00, 7.99	0.00, 6.99	0.00, 5.99	0.00, 5.04	0.00, 4.21	0.00, 3.48	0.00, 2.82	0.00, 2.62	0.00, 2.08	0.00, 1.94
9	0.00, 9.29	0.00, 8.29	0.00, 7.29	0.00, 6.29	0.00, 5.29	0.00, 4.29	0.00, 3.92	0.00, 3.24	0.00, 2.63	0.00, 2.09
10	0.22,10.50	0.00, 9.50	0.00, 8.50	0.00, 7.50	0.00, 6.50	0.00, 5.55	0.00, 4.71	0.00, 3.95	0.00, 3.26	0.00, 2.64
11	1.01,11.81	0.02,10.81	0.00, 9.81	0.00, 8.81	0.00, 7.81	0.00, 6.81	0.00, 5.81	0.00, 4.81	0.00, 4.38	0.00, 3.69
12	1.57,13.00	0.83,12.00	0.00,11.00	0.00,10.00	0.00, 9.00	0.00, 8.00	0.00, 7.00	0.00, 6.04	0.00, 5.19	0.00, 4.41
13	2.14,14.05	1.50,13.05	0.65,12.05	0.00,11.05	0.00,10.05	0.00, 9.05	0.00, 8.05	0.00, 7.04	0.00, 6.07	0.00, 5.21
14	2.83,15.50	2.13,14.50	1.39,13.50	0.47,12.50	0.00,11.50	0.00,10.50	0.00, 9.50	0.00, 8.50	0.00, 7.50	0.00, 6.54
15	3.48,16.51	2.56,15.51	1.98,14.51	1.26,13.51	0.30,12.51	0.00,11.51	0.00,10.51	0.00, 9.51	0.00, 8.51	0.00, 7.52
16	4.07,17.99	3.28,16.99	2.59,15.99	1.82,14.99	1.13,13.99	0.14,12.99	0.00,11.99	0.00,10.99	0.00, 9.99	0.00, 8.99
17	5.04,19.01	4.11,18.01	3.32,17.01	2.37,16.01	1.81,15.01	0.98,14.01	0.00,13.01	0.00,12.01	0.00,11.01	0.00,10.01
18	5.46,20.15	4.54,19.15	3.73,18.15	3.02,17.15	2.39,16.15	1.70,15.15	0.82,14.15	0.00,13.15	0.00,12.15	0.00,11.15
19	6.51,21.50	5.51,20.50	4.58,19.50	3.77,18.50	3.05,17.50	2.21,16.50	1.58,15.50	0.67,14.50	0.00,13.50	0.00,12.50
20	7.55,22.51	6.55,21.51	5.55,20.51	4.55,19.51	3.55,18.51	2.81,17.51	2.23,16.51	1.48,15.51	0.53,14.51	0.00,13.51

Table 7: 95.00 % confidence limits

n_0-b	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	5.0
0	0.00, 3.08	0.00, 2.62	0.00, 2.33	0.00, 2.04	0.00, 1.77	0.00, 1.52	0.00, 1.62	0.00, 1.38	0.00, 1.48	0.00, 1.37
1	0.05, 5.13	0.00, 4.63	0.00, 4.13	0.00, 3.69	0.00, 3.30	0.00, 2.95	0.00, 2.62	0.00, 2.32	0.00, 2.04	0.00, 1.87
2	0.36, 6.71	0.00, 6.21	0.00, 5.71	0.00, 5.21	0.00, 4.71	0.00, 4.25	0.00, 3.83	0.00, 3.45	0.00, 3.10	0.00, 2.47
3	0.82, 8.25	0.32, 7.75	0.00, 7.25	0.00, 6.75	0.00, 6.25	0.00, 5.75	0.00, 5.25	0.00, 4.77	0.00, 4.34	0.00, 3.57
4	1.37, 9.75	0.87, 9.25	0.37, 8.75	0.00, 8.25	0.00, 7.75	0.00, 7.25	0.00, 6.75	0.00, 6.25	0.00, 5.75	0.00, 4.83
5	1.84,11.25	1.47,10.75	0.97,10.25	0.47, 9.75	0.00, 9.25	0.00, 8.75	0.00, 8.25	0.00, 7.75	0.00, 7.25	0.00, 6.25
6	2.20,12.75	1.90,12.25	1.61,11.75	1.11,11.25	0.61,10.75	0.11,10.25	0.00, 9.75	0.00, 9.25	0.00, 8.75	0.00, 7.75
7	2.58,13.81	2.27,13.31	1.97,12.81	1.69,12.31	1.29,11.81	0.79,11.31	0.28,10.81	0.00,10.31	0.00, 9.81	0.00, 8.81
8	2.94,15.29	2.63,14.79	2.33,14.29	2.04,13.79	1.78,13.29	1.48,12.79	0.98,12.29	0.48,11.79	0.00,11.29	0.00,10.29
9	4.36,16.76	3.85,16.26	3.35,15.76	2.91,15.26	2.46,14.76	1.96,14.26	1.62,13.76	1.19,13.26	0.70,12.76	0.00,11.76
10	4.75,17.81	4.25,17.31	3.75,16.81	3.30,16.31	2.92,15.81	2.56,15.31	2.25,14.81	1.82,14.31	1.42,13.81	0.43,12.81
11	5.14,19.28	4.64,18.78	4.14,18.28	3.69,17.78	3.30,17.28	2.95,16.78	2.62,16.28	2.33,15.79	2.04,15.29	1.16,14.29
12	6.32,20.33	5.82,19.83	5.32,19.33	4.82,18.83	4.32,18.33	3.85,17.83	3.43,17.33	3.06,16.83	2.69,16.33	1.88,15.33
13	6.72,21.80	6.22,21.30	5.72,20.80	5.22,20.30	4.72,19.80	4.25,19.30	3.83,18.80	3.46,18.30	3.10,17.80	2.47,16.80
14	7.83,22.94	7.33,22.44	6.83,21.94	6.33,21.44	5.83,20.94	5.33,20.44	4.83,19.94	4.37,19.44	3.94,18.94	3.10,17.94
15	8.25,24.30	7.75,23.80	7.25,23.30	6.75,22.80	6.25,22.30	5.75,21.80	5.25,21.30	4.78,20.80	4.35,20.30	3.58,19.30
16	9.34,25.39	8.84,24.89	8.34,24.39	7.84,23.89	7.34,23.39	6.84,22.89	6.34,22.39	5.84,21.89	5.34,21.39	4.42,20.39
17	9.76,26.65	9.26,26.19	8.76,25.73	8.26,25.26	7.76,24.78	7.26,24.30	6.76,23.80	6.26,23.30	5.76,22.80	4.84,21.80
18	10.83,27.73	10.33,27.23	9.83,26.73	9.33,26.23	8.83,25.76	8.33,25.29	7.83,24.81	7.33,24.33	6.83,23.83	5.83,22.83
19	11.25,28.66	10.75,28.26	10.25,27.86	9.75,27.44	9.25,27.03	8.75,26.59	8.25,26.15	7.75,25.69	7.25,25.23	6.25,24.28
20	12.33,29.65	11.83,29.30	11.33,28.92	10.83,28.51	10.33,28.01	9.83,27.51	9.33,27.05	8.83,26.61	8.33,26.17	7.33,25.26

Table 8: 95.00 % confidence limits

n_0-b	6.0	7.0	8.0	9.0	10.0	11.0	12.0	13.0	14.0	15.0
0	0.00, 1.27	0.00, 1.49	0.00, 1.40	0.00, 1.33	0.00, 1.26	0.00, 1.47	0.00, 1.40	0.00, 1.34	0.00, 1.29	0.00, 1.24
1	0.00, 1.73	0.00, 1.61	0.00, 1.51	0.00, 1.42	0.00, 1.63	0.00, 1.55	0.00, 1.48	0.00, 1.41	0.00, 1.35	0.00, 1.56
2	0.00, 2.27	0.00, 2.10	0.00, 1.97	0.00, 1.85	0.00, 1.74	0.00, 1.65	0.00, 1.57	0.00, 1.49	0.00, 1.42	0.00, 1.63
3	0.00, 2.90	0.00, 2.68	0.00, 2.14	0.00, 2.00	0.00, 1.87	0.00, 1.76	0.00, 1.67	0.00, 1.58	0.00, 1.79	0.00, 1.72
4	0.00, 4.04	0.00, 3.35	0.00, 2.72	0.00, 2.53	0.00, 2.37	0.00, 2.23	0.00, 2.10	0.00, 1.99	0.00, 1.90	0.00, 1.81
5	0.00, 5.33	0.00, 4.51	0.00, 3.79	0.00, 3.14	0.00, 2.93	0.00, 2.39	0.00, 2.25	0.00, 2.12	0.00, 2.01	0.00, 1.91
6	0.00, 6.75	0.00, 5.81	0.00, 4.98	0.00, 4.24	0.00, 3.57	0.00, 2.96	0.00, 2.78	0.00, 2.62	0.00, 2.48	0.00, 2.35
7	0.00, 7.81	0.00, 6.81	0.00, 5.87	0.00, 5.03	0.00, 4.28	0.00, 4.00	0.00, 3.37	0.00, 2.80	0.00, 2.63	0.00, 2.49
8	0.00, 9.29	0.00, 8.29	0.00, 7.29	0.00, 6.34	0.00, 5.49	0.00, 4.73	0.00, 4.03	0.00, 3.78	0.00, 3.19	0.00, 3.01
9	0.00,10.76	0.00, 9.76	0.00, 8.76	0.00, 7.77	0.00, 6.82	0.00, 5.96	0.00, 5.17	0.00, 4.46	0.00, 3.81	0.00, 3.59
10	0.00,11.81	0.00,10.81	0.00, 9.81	0.00, 8.81	0.00, 7.81	0.00, 6.86	0.00, 6.00	0.00, 5.20	0.00, 4.49	0.00, 4.23
11	0.17,13.29	0.00,12.29	0.00,11.29	0.00,10.29	0.00, 9.29	0.00, 8.29	0.00, 7.33	0.00, 6.46	0.00, 5.66	0.00, 4.92
12	0.92,14.33	0.00,13.33	0.00,12.33	0.00,11.33	0.00,10.33	0.00, 9.33	0.00, 8.33	0.00, 7.37	0.00, 6.49	0.00, 5.69
13	1.68,15.80	0.69,14.80	0.00,13.80	0.00,12.80	0.00,11.80	0.00,10.80	0.00, 9.80	0.00, 8.80	0.00, 7.84	0.00, 6.96
14	2.28,16.94	1.46,15.93	0.46,14.93	0.00,13.93	0.00,12.93	0.00,11.93	0.00,10.93	0.00, 9.93	0.00, 8.93	0.00, 7.94
15	2.91,18.30	2.11,17.30	1.25,16.30	0.25,15.30	0.00,14.30	0.00,13.30	0.00,12.30	0.00,11.30	0.00,10.30	0.00, 9.30
16	3.60,19.39	2.69,18.39	1.98,17.39	1.04,16.39	0.04,15.39	0.00,14.39	0.00,13.39	0.00,12.39	0.00,11.39	0.00,10.39
17	4.04,20.80	3.35,19.80	2.63,18.80	1.83,17.80	0.83,16.80	0.00,15.80	0.00,14.80	0.00,13.80	0.00,12.80	0.00,11.80
18	4.91,21.83	4.11,20.83	3.18,19.83	2.53,18.83	1.63,17.83	0.63,16.83	0.00,15.83	0.00,14.83	0.00,13.83	0.00,12.83
19	5.33,23.30	4.52,22.30	3.79,21.30	3.14,20.30	2.37,19.30	1.44,18.30	0.44,17.30	0.00,16.30	0.00,15.30	0.00,14.30
20	6.33,24.31	5.39,23.33	4.57,22.33	3.82,21.33	2.94,20.33	2.23,19.33	1.25,18.33	0.25,17.33	0.00,16.33	0.00,15.33