Aspects of
Branes in Supergravity

Paul Smyth
Theoretical Physics, Imperial College

A thesis presented for the degree of Doctor of Philosophy of the University of London and for the Diploma of Imperial College

October 2005

For my parents

Abstract

The aim of thesis is to study the aspects of branes in supergravity and string theory. We review the definition of energy in General Relativity, its extensions and Witten's proof of the positive energy theorem. We discuss the link between positive energy and classical supergravity, and then review the construction of p-brane solutions of supergravity, focusing on eleven-dimensional examples. We show that a certain class of braneworld models – five-dimensional Hořava-Witten domain walls – are stable by proving a generalised positive energy theorem. We also construct a set of simple constraints that can be used to check brane world models in various dimensions. We are particularly interested in understanding when such models can be realised as smooth solutions, i.e. without singular sources, and we show how previous no-go theorems can be evaded by considering more general geometry, such as a non-compact transverse space.

We also consider the constraints on supersymmetric D-branes preserving $\mathcal{N} = 1$ supersymmetry in compactifications of Type II string theory to four dimensions with general fluxes included. We show that these constraints can be understood in terms of generalised calibration conditions on the cycles wrapped by the branes in the internal manifold. We then show that these conditions can be written in an elegant geometrical language using pure spinors and generalised complex geometry, which is useful framework in which to study compactifications with flux. Using this language, we find that our constraints are the natural generalisation to manifolds with flux of the more familiar results on D-branes wrapping cycles on Calabi-Yau manifolds.

Preface

The work presented in this thesis was carried out between October 2001 and October 2005 in the Theoretical Physics Group at Imperial College London under the supervision of Prof. K. S. Stelle. This thesis has not been submitted for a degree or diploma at any other university.

The research in chapter 4 of the thesis was carried out in collaboration with Prof. K.S. Stelle and Mr J.-L. Lehners, and was reported in part in Class. Quant. Grav. **22** (2005) 2589. The research in chapter 5 was carried out in collaboration with Prof. K.S. Stelle and has not been reported elsewhere. The research in chapter 6 was carried out in collaboration Dr. L. Martucci and was reported in JHEP **0511** (2005) 048.

The author has also benefited from ideas developed in collaboration with Dr. J. Kalkkinen and Mr J.-L. Lehners.

The period between October 2001 and October 2004 was supported by a research studentship from the Particle Physics and Astronomy Research Council. The period between October 2004 and October 2005 was supported by a travel abroad studentship from the Leverhulme Trust, at the Institute for Theoretical Physics, K. U. Leuven under the supervision of Prof. A. Van Proeyen.

This thesis was typeset using LaTeX 2_ε.

Contents

Chapter 1

Introduction and Overview

The possibility of finding relevant four-dimensional physics from string theory was revitalised when it was realised that Ramond-Ramond fields were supported by extended objects known as Dp-branes[1], where 'D' refers to the Dirichlet boundary conditions of the open strings ending on p-dimensional spatial slice of the background spacetime [2]. An alternative to straightforward compactification became available as the extended objects naturally carry Ramond-Ramond gauge fields hence it was hoped that the standard model could be confined to four dimensions on them in a natural way. Initial models of intersecting D-branes already showed how one could find chiral fermions at intersections [3], with higher rank gauge groups being found on stacks of branes. This gives a simple way to reproduce aspects of the Standard Model and there has been a great deal of subsequent activity in this area [4].

While gauge fields are naturally localised on D-branes, gravity remains free to propagate in the ambient spacetime, commonly known as the *bulk*, which makes reproducing familiar four-dimensional General Relativity in intersecting brane models difficult. A complementary approach is to discard the initial goal of reproducing known particle physics and just try to find conventional four-dimensional gravity from a more exotic theory, which is usually some phenomenologically motivated toy model. The hope then is that this exotic theory can be successfully embedded in string theory or supergravity, in such a way as to complement the intersecting brane models, or more complicated compactifications [5, 6].

These exotic theories are usually minimal extensions of General Relativity to some arbitrary number of extra dimensions, which are not necessarily Planck size ($\sim$

[1] When it is not specifically useful to know the dimension of brane being discussed the label p will be dropped.

10^{-35}m). This is in contrast to the Kaluza-Klein supergravity programme [7], where the compact dimensions are near Planck size. In fact, these ideas were not recent [8, 9], but were brought to the fore by the work of Arkani-Hamed, Dimopoulos and Dvali [10–12], and of Randall and Sundrum [13, 14]. In [10] a proposal was made for a toy model with large extra dimensions ($\sim 10^{-4}$m) which was intriguing as it could be immediately tested by examining possible small distance modifications of the inverse-square gravitational force law in desktop experiments. Strict bounds now exist for such models [15–17]. One large extra dimension is immediately ruled out as it would cause modifications at the scale of galaxies, however models with two large sub-millimetre dimensions remain possible.

The large extra dimension models were initially proposed as a solution to the so-called hierarchy problem, which in its simplest form states that it is aesthetically un-pleasing to have large difference between two fundamental scales, the Planck scale ($\sim 10^{19}$ GeV) and the electroweak scale ($\sim 10^{3}$ GeV). The addition of n extra dimensions of length scale L causes the effective four-dimensional Planck scale that we appreciate to be much larger than the fundamental $4+n$:

$$M_4 = M_{4+n} L^{\frac{n}{2}} \ . \tag{1.1}$$

This appears to be an elegant solution to the problem. However, one quickly realises that all we have done is introduce a new hierarchy in length scales between our ultra-large extra dimensions and the n extra dimensions with $L \sim 10^{-4}$m.

The Randall-Sundrum (RS) models [13, 14] provide a different approach, consisting of five-dimensional anti-de Sitter space AdS_5 with gravity being localised on a negative tension 3-brane. Localisation is caused by suppression of the graviton wave-function away from the brane due to the appearance of an exponential *warp factor* multiplying the worldvolume metric. In their first model with two branes in AdS_5 (known as RS1), the hierarchy problem is solved by exactly the opposite effect – the positive exponential multiplies the fundamental Planck scale causing it to appear greatly increased in the effective four-dimensional theory. This model was superseded by RS2, which has only one brane, now with positive tension. Gravity can still be localised, however we no longer have a solution to the hierarchy problem [14].

The RS models appear at various stages in the work we shall describe later, so it will be useful to introduce them in a little more detail here. The metric on AdS_5

in Poincaré coordinates is[2]

$$ds^2 = e^{-2y/l}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 \ , \tag{1.2}$$

where $\mu, \nu = 0, \ldots, 3$. We note that $y \to -\infty$ is the Minkowski boundary and $y \to \infty$ is the AdS horizon. This metric solves the five-dimensional cosmological Einstein equations $G_{AB} = -\Lambda g_{AB}$, where $\Lambda = -\frac{6}{l^2}$. If we cut this space at $y = 0$ and glue onto it a second copy of the $y > 0$ region we find

$$ds^2 = e^{-2k|y|}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 \ , \tag{1.3}$$

which now has an obvious $\mathbb{Z}_2$-symmetry about $y = 0$. This spacetime also has the interpretation of AdS_5 with singular brane located at $y = 0$ with tension $\lambda = \sqrt{-6\Lambda/k\kappa_5^2}$, as in RS2 [14]. In order to construct the RS1 model, we enforce another $\mathbb{Z}_2$-symmetry at $y = kl$ with tension $-\lambda$. One finds that the graviton wave-function is localised here, along with the Planck scale being suppressed by a factor e^{-2k}. If we worry about the fact that the brane supporting the visible universe has negative tension, then we simply swap to the positive tension brane at the expense of solving the hierarchy problem. Requiring the Planck mass there to match the familiar Planck mass means we must resort to the scaling arguments of Arkani-Hamed et al's models.

Models of the form described above are generically known as *braneworlds*, and have attracted considerable attention from both the particle physics and cosmology communities in recent years (see [18] for a recent review). This is primarily owing to their seeming ability to predict anything of interest, conveniently at a scale that will be probed at the next generation of experiments. Observations from the string theory also suggest that warped RS braneworlds can easily arise from fundamental objects in the theory, D3-branes, which have $AdS_5 \times S^5$ geometry in the near horizon limit. An F-theory construction has shown this in some form [19, 20] and we shall present a detailed description of a more explicit supergravity realisation of the second RS model later.

[2] Here we follow the conventions of [18].

Overview

This thesis is divided into two parts, the first of which provides an introduction to some topics which will be useful for the work presented in the second.

We begin in chapter 2 by reviewing the definition of energy in General Relativity. We choose to concentrate on the Lagrangian pseudotensor approach of Abbott and Deser [21], which can applied to spacetimes with general asymptotics and correctly reproduces the results from the canonical Hamiltonian method of Arnowitt, Deser and Misner [22, 23] for the asymptotically flat case. We shall then discuss how this method is extended to non-trivial five-dimensional spacetimes, following Deser and Soldate [55]. Having introduced the concept of energy in General Relativity, we then discuss the proof of its positivity using the powerful spinor methods of Witten and Nester [61, 62]. This leads us to consider $\mathcal{N} = 1$ supergravity in four-dimensions, which provides a physically intuitive way to understand the positive energy proof.

In chapter 3 we review the main features of eleven-dimensional supergravity and its solitonic solutions, the M2 and M5-branes. We do this by considering a general gravity plus scalar plus form field Lagrangian. The solutions of this theory can describe the branes of the eleven-dimensional theory, and also those of both the Type IIA and IIB theories in ten dimension. We then describe the conserved charges, including the energy, of these extended objects and the relation to supersymmetry.

Part II of this thesis describes our original research. Chapters 4 and 5 are mainly devoted to the study of the consistency and stability of braneworlds. These models are generically inspired by domain wall solutions of five-dimensional gravity, however they have gravity localised on the four-dimensional worldvolume. In chapter 4 we are interested in Hořava-Witten spacetimes [94], which we define as singular $\mathbb{Z}_2$-symmetric solutions to a bosonic theory arising from the Kaluza-Klein reduction of supergravity in ten or eleven dimensions [97, 108]. In chapter 4 we discuss the 'breathing mode' reduction of type IIB supergravity on S^5. This leads to a five-dimensional theory with particularly interesting domain wall solutions, the singular versions of which resemble the Randall-Sundrum models [13, 14]. We proceed to define the generalised Abbott-Deser energy for these spacetimes, before discussing their stability. We present a simple argument based on studying the zero-modes associated with the domain wall's motion which shows that much of the previous work in this area has been overly restrictive. On identifying possibly dangerous modes that were overlooked before, we proceed to show that they are in fact safe by proving stability using the spinorial method introduced in part I. We also briefly consider asymmetric walls and conclude with a discussion of supersymmetry and the higher dimensional origin of $\mathbb{Z}_2$-symmetric domain walls.

In chapter 5 we discuss a set of simple rules with which one can easily test the consistency of phenomenologically motivated five-dimensional braneworld models [24] i.e. those which do not arise from a consistent truncation of a compactified supergravity theory. We present an extension of these rules to branes with more general transverse spaces and show the relation to well-known no-go theorems for smooth (non-singular) flat space and de Sitter compactifications [25–30]. We discuss examples in five and six dimensions, which illustrate positive features of these sum rules and show how previous no-go theorems can be circumvented. We also comment on the application to understanding smoothed p-brane solutions, where the flat transverse space is replaced by a Ricci-flat space. We present a generalised energy expression for such branes and comment on future directions for study.

Chapter 6 discusses the problem of finding supersymmetric compactifications of ten-dimensional string theory on generalised flux backgrounds with branes present. As this is somewhat removed from the other topics in the thesis, we shall give a brief review of supersymmetry and compactification, generalised flux backgrounds, and supersymmetric D-branes. We then describe the conditions on wrapped D-branes in generalised compactifications to four dimensions preserving the minimal amount of supersymmetry. We show that these conditions can be understood in terms of appropriately defined generalised calibrations for these backgrounds, and discuss the relation to generalised complex geometry.

We conclude with a review of our results and suggestions for future research. A collection of useful formulae and a description of our conventions are presented in the appendices.

Part I

Chapter 2

Energy in (Super)Gravity

2.1 Conserved Charges in General Relativity

Our approach to proving the stability of the braneworld models discussed later will be to first define the energy for these spacetimes and then then prove that this energy is positive. If we have a good definition of energy, i.e. one that is gauge invariant and conserved, then this will prove stability – if energy is positive and conserved then there is no decay. In order to study the stability in this way, we must have a good understanding of conserved charges in gravitational theories.

The concept, and problem, of charges in General Relativity has a long history (see, for instance, [31–34]). The definition of energy in particular is a delicate issue and remains an active area of research, having been reinvigorated by the AdS/CFT correspondence in string theory through a reassessment of charges in anti-de Sitter spacetime [35] and new classes of supergravity solutions [36, 37].

Local energy density is an ill-defined concept in General Relativity, as there is no useful definition of local gravitational energy density. If we follow the example of electromagnetism, we are led to the four-index Bel-Robinson tensor, constructed from the Weyl tensor in analogy with the energy-momentum tensor of the electromagnetic field strength. Unfortunately this does not have the units of energy density and so is therefore not very useful. What one can define is the total energy, or mass, of an isolated system.

To define an isolated system we would like to have some idea of a background, against which we can measure the fall-off of interesting quantities, such as curvature perturbations. Obviously this concept does not sit well with the general relativity in which there are no preferred frames. We can, however, use the asymptotic structure of a given spacetime. The simplest case is that of an *asymptotically flat* spacetime, which will we define as follows [32]. We will denote our spacetime metric by $g_{\mu\nu}$ ($\mu = 0, 1, 2, 3$), with local coordinates labelled by $x^\mu = x^0, x^i$ ($i = 1, 2, 3$). We will define the 'radial' coordinate $r = \sqrt{x^i \cdot x^i}$ and use $\eta_{\mu\nu}$ to denote the flat Minkowski metric. If, for any coordinate system x^μ, the metric components of the given spacetime behave like $g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}(1/r)$ as $r \to \infty$ along spacelike or null directions, then that spacetime is asymptotically flat. The appropriate background metric with which to compare the fall-off of interesting quantities is then simply the Minkowski metric $\eta_{\mu\nu}$.

Following standard Noether's theorem philosophy, we may then attempt to construct charges associated with a set of symmetries. Unlike in field theory however, we must do this on-shell, i.e. for the symmetries of a solution. In General Relativity these symmetries are defined by a set of vector fields ξ satisfying Killing's equation:

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \ , \tag{2.1}$$

Assuming that one such Killing vector is timelike, i.e. generates time translations, we can define the Komar integral [38] for the total energy of a stationary, asymptotically flat spacetime. It will be useful to recall that a spacetime is said to be *stationary* if there exists a timelike Killing vector, and *static* if there exists a family of spacelike hypersurfaces (constant-time slices) to which this Killing vector is orthogonal [32]. The Komar energy is given by

$$E_{\text{Komar}} = -\frac{1}{8\pi} \int_S d\Sigma^{\mu\nu} \epsilon_{\mu\nu\rho\sigma} \nabla^\rho \xi^\sigma \ , \tag{2.2}$$

where S is an arbitrary two-sphere and we define the volume element by

$$d\Sigma_{\mu\nu} \equiv \frac{1}{2\sqrt{-\overline{g}}} \epsilon_{\mu\nu\rho\sigma} dx^\rho \wedge dx^\sigma \ . \tag{2.3}$$

Here $\overline{g} = -1$ is the determinant of the background Minkowski metric and $\varepsilon_{\mu\nu\rho\sigma}$ is the four-dimensional volume form. This is related to the two-dimensional volume form on S by $\epsilon_{\mu\nu\rho\sigma} = -6 n_{[\mu\nu} \epsilon_{\rho\sigma]}$[1], where $n_{\mu\nu}$ is the normal bi-vector defined by the orthogonal vectors $n_\mu, \ \xi_\nu$

$$n_{\mu\nu} = 2\xi_{[\mu} n_{\nu]} \ . \tag{2.4}$$

n_μ is a unit normal to S and ξ_ν is the unit norm Killing vector appearing in (2.1)

[1] Our conventions for differential forms are explained in the appendix.

[32]. We shall not prove here that this provides a satisfactory definition of total energy, but one can easily check it reproduces the desired result for Schwarzschild's solution, for example.

A seemingly different definition of energy for an asymptotically flat spacetime was provided by Arnowitt, Deser and Misner (ADM) [22]. They introduced the canonical Hamiltonian formulation of General Relativity, from which a value for the total energy is easily calculated. The ADM procedure is somewhat subtle and we shall postpone its discussion until later. For now, let us note their result

$$E_{\mathrm{ADM}} \;=\; \oint dS_i \left[\partial^j h^i{}_j - \partial^i h^j{}_j\right] \;, \tag{2.5}$$

where h_{ij} is the spatial component of the metric perturbation and dS_i is now the measure on an asymptotic spatial two-surface which is the boundary of a three-dimensional spatial slice (See equation (A.16) for the definition of the volume element on a submanifold). One can show that (2.1) and (2.5) agree for asymptotically flat spacetimes, however it transpires that the Komar expression is somewhat ambiguous and one often has to compare with some alternate definition. For instance, if we were to calculate the angular momentum of the Kerr black hole using a Komar integral we would find an answer that differs by a factor of two from what is believed to be the correct result [39].

In practice, taking the asymptotic limits associated with calculating total energy can be tricky and a more formal approach has been developed to provide rigorous results. This involves defining a spacetime to be asymptotically flat if it can be conformally transformed to another spacetime which obeys certain rigorous conditions of flatness. The Bondi energy is essentially an extension of the Komar energy (2.2) to this conformal completion of an asymptotically flat spacetime, associated to the generators of timelike translations of the extended symmetry group at null infinity, the Bondi-Metzner-Sachs (BMS) group [32]. The Bondi energy is in fact not conserved; there can be a flux of gravitational radiation at null infinity. One can show that under certain assumptions (vanishing of the Bondi news tensor), the Bondi energy and ADM energy are the same, after gravitational radiation is accounted for [40].

In the next section we will review the Abbott-Deser (AD) construction of conserved charges in cosmological Einstein theory [21], i.e. General Relativity with non-zero cosmological constant Λ. We will then present the extension of this construction to higher dimensions [55] in preparation for the discussion of charges for supergravity solutions in the next chapter. The final section describes Witten's proof of the positivity of energy in classical General Relativity and its relation to simple $\mathcal{N}=1$ supergravity.

A comprehensive review of many of the gravitational properties of string theory and supergravity, covering some of the topics discussed here, is given in the recent monograph [34].

2.2 Energy in Cosmological Einstein Theory

2.2.1 The Definition of Conserved Charges

The Abbott-Deser method allows the construction of conserved charges for spacetimes with general asymptotic structures. The key to this is to correctly identify the vacuum and its symmetries, which will generally not be that of flat spacetime. We follow the conventions of [21, 32] $\big(R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} \sim +\partial_\alpha \Gamma^{\alpha}_{\mu\nu}\ ,\ \mu,\nu = 0,1,2,3\big)$, with mostly plus signature $(-+++)$, our other conventions are explained in the appendix.

We shall concentrate on solutions to the cosmological Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0\ . \tag{2.6}$$

We define a vacuum or 'background' solution as some metric $\overline{g}_{\mu\nu}$ which solves (2.6), with examples being de Sitter ($\Lambda > 0$) or anti-de Sitter ($\Lambda < 0$) spacetime. Let us now consider small fluctuations around background solutions, dividing the metric as follows

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu}\ , \tag{2.7}$$

where $h_{\mu\nu}$ is a perturbation which vanishes at infinity. We will use $^{(n)}$ to denote a perturbation of order n, with all background quantities being barred e.g. $\overline{\nabla}^{\mu} = \nabla^{\mu(0)}$. From now on we raise and lower indices by $\overline{g}_{\mu\nu}$ and it is worthwhile to note that the inverse metric is given by

$$g^{\mu\nu} = \overline{g}^{\mu\nu} - h^{\mu\nu} + h^{\mu}{}_{\rho} h^{\rho\nu} + \cdots\ . \tag{2.8}$$

We now expand (2.6) keeping terms linear in $h_{\mu\nu}$ on the left-hand side and define the gravitational energy-momentum pseudotensor $\tau_{\mu\nu}$ as terms of quadratic and higher order in perturbations, which we then move to the right-hand side,

$$\begin{aligned} G^{(1)}_{\mu\nu} - \Lambda h_{\mu\nu} &= R^{(1)}_{\mu\nu} - \frac{1}{2}\overline{g}_{\mu\nu}\overline{g}^{\rho\sigma}R^{(1)}_{\rho\sigma} + \frac{1}{2}\overline{g}_{\mu\nu}h^{\rho\sigma}R^{(0)}_{\rho\sigma} - \frac{1}{2}h_{\mu\nu}\overline{g}^{\rho\sigma}R^{(0)}_{\rho\sigma} - \Lambda h_{\mu\nu} \\ &= \tau_{\mu\nu} - \Lambda h_{\mu\nu} \ , \end{aligned} \tag{2.9}$$

where we have used the fact that $\overline{g}_{\mu\nu}$ solves (2.6). Noting that the Bianchi identity holds to all orders, and in particular

$$\overline{\nabla}^{\mu}\left(G^{(1)}_{\mu\nu} - \Lambda h_{\mu\nu}\right) = 0 \ , \tag{2.10}$$

and using (2.6), we can easily show the pseudotensor is conserved with respect to the background covariant derivative:

$$\overline{\nabla}^{\mu}\tau_{\mu\nu} = 0 \ . \tag{2.11}$$

We could, if we wish, also include matter sources on the right-hand side without effecting this definition. Recall now that we want to define conserved charges associated to the symmetries of the background solutions and as such, we expect there to be a set of background Killing vectors $\overline{\xi}^{\mu}$ obeying

$$\overline{\nabla}_{\mu}\overline{\xi}_{\nu} + \overline{\nabla}_{\nu}\overline{\xi}_{\mu} = 0 \ . \tag{2.12}$$

We can now construct the vector density $\sqrt{-\overline{g}}\overline{\xi}^{\mu}\tau_{\mu\nu}$ which is ordinarily conserved,

$$\overline{\nabla}_{\nu}\left(\overline{\xi}_{\mu}\tau^{\mu\nu}\right) = \frac{1}{\sqrt{-\overline{g}}}\partial_{\nu}\left(\sqrt{-\overline{g}}\overline{\xi}_{\mu}\tau^{\mu\nu}\right) = 0 \ . \tag{2.13}$$

If the perturbation falls off sufficiently quickly, we can construct the following set of quantities, which we shall call the Noether, or Killing, charges:

$$Q^{\mu}(\overline{\xi}) = \int_{V} dV \tau^{\mu\nu}\overline{\xi}_{\nu} \ , \tag{2.14}$$

where $dV = \sqrt{-\overline{g}}d^3x$ is the volume element on the spatial slice V and we are using natural units, such that $8\pi G = 1$. Showing that $Q^{\mu}(\overline{\xi})$ are conserved is then straightforward

$$\begin{aligned}\frac{\partial}{\partial t}Q^{\mu}(\bar{\xi}) &= \int_V dV \partial_0\{\tau^{\mu\nu}\bar{\xi}_\nu\} \\ &= -\int_V dV \partial_i\{\tau^{\mu\nu}\bar{\xi}_\nu\} \\ &= 0 \ , \end{aligned} \tag{2.15}$$

where we have used (2.13) in the second line, leaving a total derivative, and the third follows by Stokes' theorem. If the Killing vector $\bar{\xi}_\nu$ is timelike, the quantity $Q^0(\bar{\xi})$ is the Killing energy i.e. it is a Noether charge associated to an asymptotic time translation symmetry. The appropriate fall-off conditions for perturbations will be discussed in greater detail later for explicit examples.

We can now proceed to show that the volume integral may be written as an asymptotic flux integral over a spatial 2-surface. We begin by noting that the pseudotensor $\tau^{\mu\nu}$ can be rewritten in terms of a superpotential[2]

$$\tau^{\mu\nu} = \overline{\nabla}_\rho \overline{\nabla}_\sigma K^{\mu\rho\nu\sigma} + X^{\mu\nu} \ . \tag{2.16}$$

The quantity $X^{\mu\nu}$ can be written in terms of superpotential $K^{\mu\rho\nu\sigma}$ which is defined as

$$K^{\mu\rho\nu\sigma} = \frac{1}{2}\left[\bar{g}^{\mu\sigma}H^{\nu\rho} + \bar{g}^{\nu\rho}H^{\mu\sigma} - \bar{g}^{\mu\nu}H^{\rho\sigma} - \bar{g}^{\rho\sigma}H^{\mu\nu}\right] \ , \tag{2.17}$$

where

$$H^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\bar{g}^{\mu\nu}h^{\rho}{}_{\rho} \ . \tag{2.18}$$

The superpotential has the same symmetries as the Riemann tensor,

$$K^{\mu\rho\nu\sigma} = K^{\nu\sigma\mu\rho} = -K^{\rho\mu\nu\sigma} = -K^{\mu\rho\sigma\nu} \ . \tag{2.19}$$

The obvious definition of $X^{\mu\nu}$ from the perturbative expansion of the Einstein equations is

$$X^{\mu\nu} = \frac{1}{2}\left[\overline{\nabla}_\rho \ , \ \overline{\nabla}^\nu\right] H^{\mu\rho} - \Lambda H^{\mu\nu} \ , \tag{2.20}$$

with $X^{\mu\nu}$ symmetric in its two indices. We can then use the formula relating the covariant derivative commutator to the curvature (A.3) and the background

[2]This 'superpotential' is not related to the supergravity superpotential used later. The name is given as its derivative is related to a quantity of interest, in this case the pseudotensor $\tau^{\mu\nu}$.

Einstein equations to give

$$X^{\mu\nu} = \frac{1}{2}\overline{R}^{\nu}{}_{\lambda\rho\sigma}K^{\mu\lambda\rho\sigma} \ . \tag{2.21}$$

Using the Killing vector identity $\overline{\xi}_{\mu;\nu\rho} = \overline{R}_{\mu\nu\rho}{}^{\sigma}\overline{\xi}_{\sigma}$, we see that the integrand of the Killing energy (2.14) can be written as

$$\tau^{\mu\nu}\overline{\xi}_{\nu} = \overline{\nabla}_{\rho}\left[\left(\overline{\nabla}_{\sigma}K^{\mu\rho\nu\sigma}\right)\overline{\xi}_{\nu} - K^{\mu\sigma\nu\rho}\overline{\nabla}_{\sigma}\overline{\xi}_{\nu}\right] + \left[K^{\mu\rho\nu\sigma}\overline{\nabla}_{\sigma}\overline{\nabla}_{\rho} + X^{\mu\nu}\right]\overline{\xi}_{\nu} \ . \tag{2.22}$$

A little manipulation then shows that the last term can be removed, leaving a total derivative. The anti-symmetry of the resulting expression then allows us to rewrite the Killing charges as flux integrals over a spatial 2-surface. These are the AD charges:

$$\begin{aligned} Q^{\mu}(\overline{\xi}) &= \int_V dV \ \tau^{\mu\nu}\overline{\xi}_{\nu} \ , \\ &= \oint dS_i \ \sqrt{-\overline{g}}\left[\overline{\nabla}_{\sigma}K^{\mu i\nu\sigma} - K^{\mu j\nu i}\overline{\nabla}_{j}\right]\overline{\xi}_{\nu} \ , \end{aligned} \tag{2.23}$$

where once again $i, j = 1, 2, 3$ label spatial directions and dS_i is the measure on the two surface which is the boundary of the spatial slice V (see equation (A.16)). In fact, one can manipulate this expression into a more transparent form [41]

$$\begin{aligned} Q^{\mu}(\overline{\xi}) = \oint dS_i\sqrt{-\overline{g}} \ \Big[& \overline{\xi}_{\nu}\overline{\nabla}^{\mu}h^{i\nu} - \overline{\xi}_{\nu}\overline{\nabla}^{i}h^{\mu\nu} + \overline{\xi}^{\mu}\overline{\nabla}^{i}h - \overline{\xi}^{i}\overline{\nabla}^{\mu}h + h^{\mu\nu}\overline{\nabla}^{i}\overline{\xi}_{\nu} \\ & -h^{i\nu}\overline{\nabla}^{\mu}\overline{\xi}_{\nu} + \overline{\xi}^{i}\overline{\nabla}_{\nu}h^{\mu\nu} - \overline{\xi}^{\mu}\overline{\nabla}_{\nu}h^{i\nu} + h\overline{\nabla}^{\mu}\overline{\xi}^{i}\Big] \ . \end{aligned} \tag{2.24}$$

2.2.2 Some examples

As a consistency test of the Abbott-Deser (AD) energy (2.24), we can try to reproduce the standard result for asymptotically flat spacetime [22, 23]. In this case, the perturbative expansion of the metric is

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ , \tag{2.25}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and one can show that in order for the AD energy to be well defined the perturbations must fall off asymptotically like $h_{\mu\nu} \sim O(1/r)$ as $r \to \infty$, where once again $r = \sqrt{x^i \cdot x^i}$. The exact behaviour of the various components of $h_{\mu\nu}$ can be determined by studying the perturbed field equations, or the constraints in the Hamiltonian formalism [22].

Setting $\Lambda = 0$ and using the simple time-like Killing vector in flat spacetime $\bar{\xi}^\mu = (-1, 0, 0, 0)$, we find that the general AD expression reduces to the well known ADM energy formula

$$E_{(\mathrm{AD}\ \Lambda=0)} \;=\; E_{\mathrm{ADM}} \;=\; \oint dS_i \left[\partial^j h^i{}_j - \partial^i h^j{}_j\right] \;. \tag{2.26}$$

We can use this result to calculate the mass of Schwarzschild's solution. It is convenient to write the metric in isotropic coordinates,

$$ds^2 \;=\; -\left(\frac{1 - M/2\rho}{1 + M/2\rho}\right)^2 dt^2 + \left(1 + \frac{M}{2\rho}\right)^4 d\overline{X}^2 \;, \tag{2.27}$$

where $d\overline{X}^2$ is the line element on the 3-space in spherical coordinates, and M is a constant. The 'radius' ρ is related to the regular Schwarzschild radius by $r = \rho(1 + M/2\rho)^2$. A straightforward application of (2.26) then gives $E_{\mathrm{ADM}} = M$, identifying M as the mass, as expected.

Let us now apply the simplified form of the AD energy (2.24) to the general case of non-vanishing cosmological constant ($\Lambda \neq 0$), following [41]. The metric for Schwarzschild-(anti)de Sitter in static coordinates is

$$\begin{aligned} ds^2 \;&=\; -\,H\,dt^2 + H^{-1}dr^2 + r^2 d\Omega_2^2 \;, \\ H \;&=\; 1 - \frac{M}{r} - \frac{\Lambda r^2}{3} \;. \end{aligned} \tag{2.28}$$

Concentrating on the anti-de Sitter case (AdS), the background metric ($M = 0$) has a simple timelike Killing vector $\bar{\xi}^\mu = (-1, 0, 0, 0)$ that is globally defined. The energy integral is then easily evaluated to give

$$Q^0(\bar{\xi}) \;=\; E(r) \;=\; M\frac{\left(1 - \frac{\Lambda r^2}{3}\right)}{\left(1 - \frac{M}{r} - \frac{\Lambda r^2}{3}\right)} \;, \tag{2.29}$$

so that the asymptotic energy of Schwarzschild-AdS is $E(r \to \infty) = M$. One subtlety in this definition is that anti-de Sitter spacetime does not possess a global Cauchy surface as spatial infinity is timelike [42]. This means that there could be a flux of gravitational radiation, causing our energy defined above to be not conserved, like the Bondi energy in asymptotically flat spacetimes. To resolve this problem, one must fix appropriate boundary conditions to ensure there is no inflow of radiation [21].

The de Sitter (dS) background is somewhat more subtle. Looking at the norm of the Killing vector

$$\bar{g}_{\mu\nu}\bar{\xi}^{\mu}\bar{\xi}^{\nu} = -\left(1 - \frac{\Lambda r^2}{3}\right) , \tag{2.30}$$

we see that it is only timelike for $r < r_H$, $r_H = \sqrt{3/\Lambda}$, i.e. inside the de Sitter horizon. If we naively apply the energy expression (2.29), we find $E(r \to r_H) = 0$. Abbott and Deser argued that in order to sensibly calculate the energy we must consider the case where the Schwarzschild radius r_S is much smaller than the de Sitter horizon, allowing us to perform the integral in the region $r_S << r << r_H$. This once again leads to the expected result $E_{\mathrm{SdS}} = M$.

2.2.3 Problems and Other Approaches

The Abbott-Deser (AD) approach provides us with an intuitive way to define charges in the cosmological Einstein theory. In the flat spacetime limit it correctly reproduces the result of Arnowitt, Deser and Misner (ADM), which was originally derived using very different methods. It also gave the expected result for the mass of the Schwarzschild-(anti)de Sitter black holes. Unfortunately, both de Sitter and anti-de Sitter cases are problematic. Recall that in proving the conservation of the AD charges we used the Stokes theorem to convert the expression into an asymptotic boundary integral, presuming that there were no internal boundaries. The de Sitter horizon forced us to evaluate our energy integral at finite radius, and as a consequence the Killing energy is no longer conserved[3]. The AD approach necessarily involves fixing a background around which to study perturbations (2.7). This split was long known to cause problems in the definition of angular-momentum, even in asymptotically flat spacetimes. In this case the problem arises as the asymptotic symmetry group is not Poincaré, as implied by assuming the background metric $\bar{g}_{\mu\nu}$ is that of Minkowski spacetime, but is the infinite dimensional BMS group [32, 48].

In the case of anti-de Sitter spacetime, the asymptotic symmetry is correctly identified as the anti-de Sitter group $SO(3, 2)$, however the subtleties arise due to the lack of a global Cauchy surface and also in taking the asymptotic limit [46, 47]. Quite surprisingly, there are numerous approaches to the definition of charge in anti-de Sitter. The original refinement of Ashtekar and Magnon [46] generalised Penrose's treatment of conformal infinity in Minkowski spacetime, leading to later developments in charge definition by Wald and Zoupas [49]. Another approach, developed by Katz et al [50, 51], uses Noether current techniques and a superpotential, similar to the AD method we have described. Yet another approach uses

[3] This has studied by Shiromizu et al [43–45], who resolved the issue for certain restricted cases.

counter-term subtraction [52, 53], inspired by the AdS/CFT correspondence. All of these are based on using symmetries of the Lagrangian in various guises. A quite different tact was advocated by Henneaux and Teitelboim [54], who studied the Hamiltonian theory, following in the vein of the original work of ADM.

The state-of-art in this area has been recently reviewed by Ishibashi et al [35], where the Henneaux-Teitelboim, Ashtekar-Magnon-Das and counterterm subtraction methods, and an extension of the Wald-Zoupas method, were all found to agree. While this provides evidence for the equivalence of Lagrangian and Hamiltonian definitions, the relation to the original AD approach and that of Katz et al remains unclear. In particular, these methods both employ a superpotential which is related to a pseudotensor defined in the bulk. The other methods listed above are all based on boundary techniques, where the quantities being studied are genuine tensors, and at present it is unclear how to relate the two[4].

2.3 Conserved Charges in Higher Dimensions

The work that we shall describe in chapters 4 and 5 of this thesis is concerned with the energy and stability of extended objects in supergravity. Before we turn to this, it will be useful to understand the simpler case of pure gravity in five dimensional, Kaluza-Klein theory. In this section we will review the work of Deser and Soldate [55], who generalised the AD charges to five-dimensional spacetimes with one compactified direction. This introduces some techniques which will prove crucial in defining the energy of the domain wall solutions we shall study later.

2.3.1 Definitions

In this section Greek indices run over all five dimensions ($\mu, \nu = 0, \ldots, 4$), while upper case Latin indices run over $M, N = 1, 2, 3, 4$ and lower case Latin indices run over $m, n = 1, 2, 3$. We are interested in defining the energy of solutions to the five-dimensional Einstein equations, and we choose to follow the approach of Deser and Soldate, who generalised the Abbott-Deser pseudotensor definition of conserved charges to five dimensions. We begin by splitting the five-dimensional metric into the background and perturbation pieces which are taken to vanish asymptotically $g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu}$. We then define an energy-momentum pseudotensor $\tau_{\mu\nu}$ in terms of the perturbations in the same way as described previously (section (2.2)). We can

[4]We thank A. Ishibashi for a discussion on this point.

then define the usual Noether, or Killing, charges associated with the symmetries of the background

$$Q^\mu(\bar{\xi}) = \int_V dV\ \tau^{\mu\nu}\bar{\xi}_\nu\ , \tag{2.31}$$

where $\bar{\xi}_\nu$ is a Killing vector and now V is a four-dimensional submanifold of the spacetime.

Before proceeding to evaluate this integral in terms of perturbations $h_{\mu\nu}$, it is worthwhile to consider what type of solutions we would like to study. In particular, it is only meaningful to compare energies among solutions which have the same asymptotic structure i.e. the same background metric $\bar{g}_{\mu\nu}$. With our later example of the Kaluza-Klein in mind, we shall choose to limit to the case of background solutions with topology $M_{1,3} \times S^1$, where $x^4 \in 0, 2\pi R$. Any solutions to the five-dimensional Einstein equations that do not have this topology therefore fall outside of our discussion.

One further assumption is that once again there is a simple timelike Killing vector $\bar{\xi}^\mu = \delta^\mu{}_0$, and that there are no internal boundaries. We can then evaluate the Noether charge associated with $\bar{\xi}$, the Deser-Soldate (DS) energy:

$$\begin{aligned} E &= \int dx^4 \int d^3x\, \tau^{00} \\ &= \int_0^{2\pi R} dx^4 \int d^2S_i\ \left(\partial_j h^{ij} - \partial^i h^j{}_j - \partial^i h^4{}_4\right)\ . \end{aligned} \tag{2.32}$$

Note that any terms with x^4-derivatives are periodic in x^4 and so vanish under the integral over S^1. We now define the radial coordinate over the remaining three spatial directions $\rho = \sqrt{x^i \cdot x^i}$. Looking at the DS energy we then notice that the integral is only well defined if $h_{MN} \sim 1/r$, as opposed to $1/r^2$ as one might have expected for a five-dimensional theory. If one wishes, it is possible to show that this fall-off property is correct in a more concrete manner by carefully considering the field equation for the perturbations, or equivalently the integrand of the four-volume integral, with the appropriate Kaluza-Klein ansatz [55].

Consider now a class of compactifications of the form $M_{1,3} \times S^1$, with energies $E(h, \bar{g})$ given by (2.32). One way to find a possible preferred vacuum among this class would be to look for a minimum energy configuration amongst $E(h, \bar{g})$. To do this it is useful to fix the S^1 radius to be R in all compactifications, such that geometric differences are entirely encoded in $\bar{g}_{44}$. This allows the DS energy to be written as

$$E(h,\overline{g}) = \int_0^{2\pi R} dx^4 \int d^2S_i \sqrt{\overline{g}_{44}} \left(\partial_j h^{ij} - \partial^i h^j{}_j - \overline{g}^{44}\partial^i h_{44}\right) . \qquad (2.33)$$

Deser and Soldate then note it is not possible to compare energies amongst different compactifications as it would require $h_{44} \to constant$, whereas we know that this method of defining charges relies on studying perturbations that vanish asymptotically, i.e. $h_{\mu\nu} \to 0 \; \forall \; \mu, \nu$. This is an obvious limitation of the Deser-Soldate approach – we can not compare energies amongst spacetimes with different asymptotic behaviour. Of course, one can still compare energy among backgrounds with the same asymptotics but differing by small amounts of matter $T^{\mu\nu}_{mat}$.

2.3.2 An example - the Kaluza-Klein Monopole

When we constructed the Killing energy in section (2.2), we split the metric into background plus perturbations (2.7) and stated that the background contribution should satisfy the Einstein equations. The example that we shall now discuss, the five-dimensional Kaluza-Klein monopole [56, 57], is interesting as upon making this split the background metric is no longer a solution of the field equations. The monopole metric was found by noticing that a solution to the five-dimensional vacuum Einstein equations can be constructed by taking the direct product of $\mathbb{R}$ $(-dt^2)$ with any four-dimensional gravitational instanton (i.e. a solution to the four dimensional Euclidean vacuum Einstein equations) [56]. In this way one can construct the Kaluza-Klein monopole from the Taub-NUT instanton [58]. The metric is

$$\begin{aligned} ds^2 &= -dt^2 + H^{-1}\left(dx^4 + A_\phi \, d\phi^2\right) + H\left(dr^2 + r^2 d\Omega_2^2\right) \\ H &= \left(1 + \frac{4m}{r}\right) \qquad\qquad A_\phi = 4m(1 - \cos\theta) \, , \end{aligned} \qquad (2.34)$$

where θ is the azimuthal angle in the spherical metric $d\Omega_2^2$ and m is an as yet unfixed parameter. This metric is ultrastatic, and therefore possess the simple timelike Killing vector $\overline{\xi}^\mu = \delta^\mu{}_0$, and also has an additional rotational symmetry in ϕ. A spacetime is ultrastatic if $g_{00} = constant$ and it is static. Note that for the solution to be non-singular, the parameter m appearing in A_ϕ must be fixed by the radius R of the S^1 direction x^4 and so there is only one unique Kaluza-Klein monopole for this choice of topology $M_{1,3} \times S^1$. The asymptotic limit is given by $H \to 1$, however the Kaluza-Klein vector component A_ϕ does not vanish and thus one sees that asymptotic metric $\overline{g}_{\mu\nu}$ does not satisfy the background Einstein equations $G_{\mu\nu}(\overline{g}) = 0$ [57]. Crucially though, the 0-components of the Ricci tensor vanish

$$R_{0\mu}(\overline{g}) = 0 \ . \tag{2.35}$$

Let us now turn to the definition of energy for this background. In order to construct a Killing energy for the Kaluza-Klein monopole we need the exact form of the pseudotensor appearing in the integral (2.31), and it will be useful for us to reconsider its definition. Let us write the Einstein equations in the following form

$$\begin{aligned} \mathcal{G}^{\mu\nu} &\equiv \sqrt{-g}\, G^{\mu\nu} = \mathbb{X}^{\mu\nu\rho\sigma} R_{\rho\sigma} \\ &\equiv \sqrt{-g}\left(g^{\mu\rho}g^{\nu\sigma} - \frac{1}{2}g^{\mu\nu}g^{\rho\sigma}\right) R_{\rho\sigma} = T^{\mu\nu} \ , \end{aligned} \tag{2.36}$$

where we now include contributions given by the matter energy-momentum tensor $T_{\mu\nu}$ and for convenience we have defined the contravariant combination of metric factors $\mathbb{X}^{\mu\nu\rho\sigma}$. We then make the standard split of the metric into background and perturbation parts (2.7) and expand the above expression to give

$$\mathcal{G}^{\mu\nu} \equiv \overline{\mathcal{G}}^{\mu\nu} + \delta\mathbb{X}^{\mu\nu\rho\sigma} R_{\rho\sigma} + \overline{\mathbb{X}}^{\mu\nu\rho\sigma}\delta R_{\rho\sigma} + \mathcal{O}(h^2, \ldots) = \delta T^{\mu\nu} \ , \tag{2.37}$$

where

$$\begin{aligned} \delta R_{\rho\sigma} &\equiv \overline{\nabla}_\lambda \delta\Gamma^\lambda_{\rho\sigma} - \overline{\nabla}_\rho \delta\Gamma^\lambda_{\lambda\sigma} \\ &\equiv -\frac{1}{2}\left(\overline{\nabla}^\lambda\overline{\nabla}_\lambda h_{\rho\sigma} - \overline{\nabla}_\lambda\overline{\nabla}_\rho h^\lambda_\sigma - \overline{\nabla}_\lambda\overline{\nabla}_\sigma h^\lambda_\rho + \overline{\nabla}_\rho\overline{\nabla}_\sigma h\right) \ . \end{aligned} \tag{2.38}$$

Following our construction of conserved charges in section (2.2), we would now expect to expand the term $\delta\mathbb{X}^{\mu\nu\rho\sigma}$ keeping only terms linear in h in our definition of the energy-momentum pseudotensor $\tau^{\mu\nu}$, and moving terms of $\mathcal{O}(h^2)$ and higher into the total source on the right-hand of the perturbed field equation. However, Deser and Soldate note that this would not lead to a conserved quantity (as can be easily checked), thus we must move $\delta\mathbb{X}^{\mu\nu\rho\sigma}$, and the background term $\mathcal{G}^{\mu\nu}$, onto the right-hand side of (2.37) to define the new total source $T^{\mu\nu}_{\mathrm{T}}$:

$$\begin{aligned} \mathcal{G}^{\mu\nu}_{\mathrm{L}} = \sqrt{-\overline{g}}\tau^{\mu\nu} \equiv -\frac{1}{2}\sqrt{-\overline{g}}\Big[\, &\overline{\nabla}^\lambda\overline{\nabla}_\lambda h^{\mu\nu} - \overline{\nabla}^\lambda\overline{\nabla}^\mu h^\nu{}_\lambda - \overline{\nabla}^\lambda\overline{\nabla}^\nu h^\mu{}_\lambda + \overline{\nabla}^\mu\overline{\nabla}^\nu h \\ &-\overline{g}^{\mu\nu}\left(\overline{\nabla}^\lambda\overline{\nabla}_\lambda h - \overline{\nabla}_\rho\overline{\nabla}^\lambda h^{\rho\lambda}\right)\Big] = T^{\mu\nu}_{\mathrm{T}} \ . \end{aligned} \tag{2.39}$$

Returning to the question of conservation, we take the covariant divergence of this expression and find that it is still not zero,

$$\overline{\nabla}_\nu \mathcal{G}_{\rm L}^{\mu\nu} = \overline{R}^\mu{}_\sigma \overline{\nabla}_\nu \left(h^{\nu\sigma} - \frac{1}{2}\overline{g}^{\nu\sigma} h \right) + \frac{1}{2}\left(\overline{\nabla}_\rho \overline{R}^\mu{}_\sigma + \overline{\nabla}_\sigma \overline{R}^\mu{}_\rho - \overline{\nabla}^\mu \overline{R}_{\rho\sigma} \right) h^{\rho\sigma} \ . \tag{2.40}$$

The way around this problem is to use (2.35) and the simple form of the timelike Killing vector $\overline{\xi}^\mu = \delta^\mu{}_0$, along with the fact that $\overline{\nabla}_0 = 0$ in this background to show that

$$\overline{\nabla}_\nu \left(\sqrt{-\overline{g}}\,\overline{\xi}_\mu \mathcal{G}_{\rm L}^{\mu\nu} \right) = \partial_\nu \left(\sqrt{-\overline{g}}\,\overline{\xi}_\mu \mathcal{G}_{\rm L}^{\mu\nu} \right) = -\partial_\nu \mathcal{G}_{\rm L}^{0\nu} = 0 \ . \tag{2.41}$$

We may now define the conserved Killing energy (2.33) for the Kaluza-Klein monopole background

$$E_{\rm KKM} \ = \ \int dx^4 \int d^2 S_i \sqrt{-\overline{g}} \left(\overline{\nabla}_K h^{iK} - \overline{\nabla}^i h^j{}_j - \overline{\nabla}^i h^4{}_4 \right) \ , \tag{2.42}$$

where again $K, L = 1,2,3,4$ run over all spatial indices, whereas $i, j = 1,2,3$. To evaluate this expression explicitly it is most convenient to use Cartesian coordinates, such that the integrand can be rewritten as[5]

$$\sqrt{-\overline{g}} \left(\overline{\nabla}_K h^{iK} - \overline{\nabla}^i h^j{}_j - \overline{\nabla}^i h^4{}_4 \right) = h^{ik}{}_k - \overline{g}^{i4}{}_{,k} h^k{}_4 - \frac{1}{2}\overline{g}_{KL}{}^{,i} h^{KL} - h^K{}_K{}^{,i} \ . \tag{2.43}$$

The energy of the Kaluza-Klein monopole (2.34) is then found to be,

$$E_{\rm KKM} \ = \ \int d^3\overline{r} \nabla^2 \left(2H + H^{-1} \right) \ = \ m. \tag{2.44}$$

In [57], the authors calculated this five-dimensional energy using various methods and showed they all agreed if the Killing vector $\overline{\xi}$ is covariantly constant (i.e. if the Komar energy vanishes). Moreover, they carried out the dimensional reduction on the S^1 direction and proved that the resulting four-dimensional theory produces a canonical energy-momentum pseudotensor. It is interesting to note that the compactified monopole solution to the resulting four-dimensional Einstein-Maxwell-(dilatonic) scalar theory is singular, whereas the original five-dimensional solution was not. This is indicator of the later result of Gibbons et al, which showed that dilatonic singularities are artifacts of the dimensional reduction procedure [59].

[5] See appendix B of [55] for other relations that are useful in this calculation.

2.4 The Positive Energy Theorem

After the concept of global energy was introduced in General Relativity, the obvious next step was to show that it is positive for all reasonable solutions of the field equations. The problem was posed by Arnowitt, Deser and Misner [22], however a formal version of the statement took a considerable time to appear. Schoen and Yau [60] provided a proof that, although rigorous, involved mathematically complex analysis and lacked some physical intuition. This was superseded by Witten's proof using spinors [61], and its later refinements [62], which were inspired by work on the positivity of the supergravity Hamiltonian [63, 64]. An interesting perspective on the period between the rigorous proof and the later physical version is provided by [65].

We shall now review the proof of positive energy for asymptotically flat spacetimes in classical General Relativity, which will be of use for understanding the work we describe later in this thesis. We will begin by proving the positivity of an initially abstract quantity defined in terms of spinors, and then provide some more physical understanding by considering how it arises in simple $\mathcal{N} = 1$ supergravity, the supersymmetric extension of General Relativity.

2.4.1 Spinorial Stability Analysis

It will be useful to briefly review our conventions for spinors here, following [66, 67] (see also Appendix A). We use curved spacetime γ-matrices in a real representation obeying $\{\gamma_\alpha, \gamma_\beta\} = 2g_{\alpha\beta}$, which may be constructed from the usual γ-matrices with flat spacetime indices using the vierbein $e^{\underline{\alpha}}_{\alpha}$ ($g_{\alpha\beta} = e^{\underline{\alpha}}_{\alpha} e^{\underline{\beta}}_{\beta} \eta_{\underline{\alpha\beta}}$). We shall use $\underline{\alpha}, \underline{\beta}$ to label flat indices and all indices run over four dimensions. We define $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$, $\{\gamma^5, \gamma^\alpha\} = 0$, with the Lorentz generators defined as $\sigma_{\alpha\beta} = \frac{1}{4}[\gamma_\alpha, \gamma_\beta]$. The covariant derivative on a spinor Ψ is $\nabla_\mu \psi = \partial_\mu \psi + \frac{1}{2}\omega_\mu{}^{\alpha\beta}\sigma_{\alpha\beta}\,\psi$ and we define $\overline{\psi} = \psi^\dagger \gamma^0$.

We begin by defining the Witten-Nester (WN) four-momentum [61, 62]

$$P_\lambda v^\lambda = -\frac{1}{2}\int_{\partial V} dS_{\mu\nu} E^{\mu\nu} \ , \tag{2.45}$$

where the integral is taken over the boundary of the spatial volume element V and once again we set $8\pi G = 1$. The Nester 2-form $E^{\mu\nu}$ is defined as

$$E^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}\left(\overline{\psi}\,\gamma_5\gamma_\rho\nabla_\sigma\psi - \nabla_\sigma\overline{\psi}\,\gamma_5\gamma_\rho\psi\right) \ , \tag{2.46}$$

where ψ denotes here a commuting Dirac spinor function which tends to a constant asymptotic value ψ_∞.

$$\psi(x) \to \psi_\infty + O(\tfrac{1}{r}) \quad , \quad \text{as } r \to \infty \ . \tag{2.47}$$

Assuming there are no internal boundaries or horizons, we can use Gauss' law to rewrite (2.45) as

$$P_\lambda v^\lambda = -\int_V d\Sigma_\nu \nabla_\mu E^{\mu\nu} \ . \tag{2.48}$$

Using the following useful formula,

$$\begin{gathered} 2G^\sigma{}_\lambda = 2R^\sigma{}_\lambda - \delta^\sigma{}_\lambda R = \tfrac{1}{2}\epsilon^{\sigma\alpha\delta\beta} R^{\mu\nu}{}_{\alpha\beta}\,\epsilon_{\delta\mu\nu\lambda} \\ [\nabla_\mu, \nabla_\nu]\,\psi = \tfrac{1}{2}R^{\alpha\beta}{}_{\mu\nu}\,\sigma_{\alpha\beta}\psi \quad , \quad \{\gamma_\mu, \sigma_{\alpha\beta}\} = -\epsilon_{\mu\alpha\beta\lambda}\,\gamma^\lambda\gamma_5 \\ \epsilon^{\mu\nu\rho\sigma}\epsilon_{\rho\alpha\beta\lambda} = -(3!)\,\delta^{[\mu}_{[\alpha}\delta^\nu_\beta\delta^{\sigma]}_{\lambda]} \ , \end{gathered}$$

we can calculate the divergence of the Nester tensor to find

$$\nabla_\mu E^{\mu\nu} = -\overline{\psi}\gamma^\lambda\psi\, G^\mu{}_\lambda + 2\nabla_\mu\overline{\psi}\,\{\gamma^\nu, \sigma^{\rho\mu}\}\nabla_\rho\psi \ . \tag{2.49}$$

Using Einstein's equations, and defining the vector $u^\lambda = \overline{\psi}\gamma^\lambda\psi$, we can now rewrite our expression for the energy (2.48) as

$$\begin{aligned} E_{\text{WN}} &= -\int_V d\Sigma_\nu \nabla_\mu E^{\mu\nu} \\ &= \int_V d\Sigma_\nu\, T^\nu_\lambda u^\lambda - 4\int_V d\Sigma_\nu \nabla_\mu\overline{\psi}\,\{\gamma^\nu, \sigma^{\rho\mu}\}\nabla_\rho\psi \end{aligned} \tag{2.50}$$

From our earlier discussion of the ADM energy, we know that we should consider the measure of this integral as being over a spacelike hypersurface. Choosing $\nu = 0$, we then have the 3-volume element on a constant time hypersurface $d\Sigma_0 = dV$. We shall use $m, n = 1, 2, 3$ to label coordinates on the spatial hypersurfaces under this foliation. Concentrating on the second term in (2.50), we find it can be rewritten as

$$\begin{aligned} \int dV\, \nabla_m\overline{\psi}\,\{\gamma^0, \sigma^{nm}\}\nabla_n\psi &= 2\int dV\, \nabla_m\psi^\dagger\,\sigma^{nm}\nabla_n\psi \ , \\ &= -\int dV\, \nabla^m\psi^\dagger\,\nabla_m\psi + \int dV\, \nabla_m\psi^\dagger\,\gamma^m\gamma^n\nabla_n\psi \ , \end{aligned}$$

having used $\overline{\psi} = \psi^\dagger\gamma^0$, $(\gamma^0)^2 = -1$ and $2\sigma^{mn} = \gamma^m\gamma^n - \delta^{mn}$. The WN energy is then written as

$$E_{\rm WN} = P_0 v^0 = \int dV\, T^0_\lambda u^\lambda + \int dV\, \nabla^m \psi^\dagger\, \nabla_m \psi - \int dV\, \nabla_m \psi^\dagger\, \gamma^m \gamma^n \nabla_n \psi \ , \quad (2.51)$$

Let us now consider the positivity of this expression. The first term is positive if u^λ is a future-pointing non-spacelike vector field, and $T^{\mu\nu}$ satisfies the dominant energy condition, i.e.

$$T_{\mu\nu}\, u^\mu v^\nu \geq 0 \ , \quad (2.52)$$

for u^μ, v^ν non-spacelike. The second term is manifestly positive, as it is a square, however the sign of the third term is undetermined. The way around this problem is simply to remove the troublesome term by imposing the 'Witten condition'

$$\gamma^n \nabla_n \psi = 0 \ . \quad (2.53)$$

It has been shown [68, 69] that solutions to this equation, which is essentially the spatial Dirac equation, always exists on spatial hypersurfaces with the appropriate boundary conditions defined by (2.47). With this in hand, one can show that the WN energy is positive

$$E_{\rm WN} = \int dV\, T^0_\lambda u^\lambda + \int dV\, \nabla_m \psi^\dagger\, \nabla_n \psi \ \geq 0 \ , \quad (2.54)$$

Furthermore, we can see that the inequality is saturated when $T^0_\lambda = 0$ and $\nabla_n \psi = 0$. As we initially considered arbitrary hypersurfaces Σ, we can promote the second condition to $\nabla_\nu \psi = 0$ i.e. on all possible hypersurfaces. Also, as we considered an arbitrary commuting spinor parameter ψ, we in fact have a basis of such spinors that are covariantly conserved. Using this in conjunction with the integrability condition for covariant derivatives on spinors, one finds that $E_{\rm WN} = 0$ holds if and only if the spacetime is flat ($R_{\mu\nu\alpha\beta} = 0$), proving Minkowski spacetime is the minimal energy solution to Einstein's field equations. One can also make contact with the more physical definition of energy by perturbatively expanding the spin connections and vierbeins appearing in the Witten-Nester energy (2.54). Keeping only first order terms and using that it is always possible to form a Killing vector from a gamma matrix and two Killing spinors, one can show that (2.54) reduces to the Noether energy, or the equivalently the ADM energy, that we discussed earlier.

The assumption of no horizons rules out arguably the most interesting solutions of the field equations, black holes. The extension of the Witten proof in this case was carried out by Gibbons et al [70], and required a subtle analysis of the boundary conditions on the spinor parameter. The apparent horizon of the black hole forms an inner boundary H on the initial hypersurface Σ. Showing that the energy is positive then requires proving the existence of spinors ψ behaving as in the original proof outside the region H on Σ, with some appropriate boundary conditions on H. The obvious choice is $\psi|_H = 0$, however one can easily show that the Witten condition (2.53) then implies that ψ must vanish everywhere. The key is to identify an appropriate projection $\gamma^1\gamma^0\psi = \psi$ on H, which halves the degrees of freedom. One can then show that energy is positive and independent of data inside the region H [70].

Gibbons et al also considered black-holes with Reissner-Nordstrom electric and magnetic charges Q and B, respectively. The proof proceeds exactly as before, only one must modify the covariant derivative to include a field strength term. Eventually one finds the following inequality

$$\psi^\dagger \left(M - i\gamma^0(Q - \gamma^5 B)\right)\psi > 0, \tag{2.55}$$

which must hold for all ψ, thus the mass is bounded from below by the charges

$$M \geq \left((Q^2 + B^2)^{\frac{1}{2}}\right) . \tag{2.56}$$

This mass bound for Reissner-Nordstrom solutions is interesting as it resembles the bounds one finds for the mass of solitons in supersymmetric field theories i.e. supersymmetric version of the Bogomol'nyi bound on the mass of monopoles and dyons. Witten and Olive showed that this bound can be linked to the topological terms, the central charges, that are necessarily appear in supersymmetric theories with solitons [71]. Their arguments were extended to $\mathcal{N} = 2$ supergravity by Gibbons and Hull who first derived the bound (2.56) for the solitons of this theory [72]. This implies a strong connection between supersymmetry and positive energy, which we shall consider further in the next section.

The spinorial techniques described above provide an elegant and relatively simple way to prove that energy in classical General Relativity is positive. It is clear that the crux of this proof lies in the ability to impose the Witten condition, thus removing the negative term in the expression for energy. The existence proof for solutions to (2.53) is complex, however we can gain some physical intuition by considering the relation between classical supergravity and general relativity.

2.4.2 Simple Supergravity and Positive Energy

D=4, $\mathcal{N}=1$ Supergravity

Witten's original proof of positive energy was inspired by results in quantum supergravity, but in fact it was later shown that the same result can be derived from the classical theory, and it is this approach that we shall now discuss. We will begin by introducing the Deser-Zumino version of simple $\mathcal{N}=1$ supergravity [73][6]. The field content of this theory is the vierbein e^a_μ (gravity) plus the spin-$\frac{3}{2}$ Rarita-Schwinger field, which is a Majorana fermion. In the first order formalism described here, the connection ω is also an independent variable, which must be varied to produce its own equation of motion. The Lagrangian for simple supergravity is [73]

$$\mathcal{L} = \frac{1}{2} e\, R - \frac{i}{2} e\, \epsilon^{\lambda\mu\nu\rho} \overline{\psi}_\lambda \gamma_5 \gamma_\mu \nabla_\nu \psi_\rho \ , \tag{2.57}$$

where $e = \det e^\mu{}_a$ and $R = e^\mu{}_a e^\nu{}_b R_{\mu\nu}{}^{ab}$. The action of the covariant derivative on spinors is as defined in the section (2.4.1) (which is different from that used in [73]), although ω now includes torsion induced by the gravitino. The equations of motion for ψ_ρ, $\omega_{\mu ab}$ and $e^a{}_\mu$ are

$$R^\lambda \equiv \epsilon^{\lambda\mu\nu\rho} \left(\gamma_\mu \nabla_\nu \psi_\rho - \tfrac{1}{4} \gamma_\tau C_{\mu\nu}{}^\tau \psi_\rho \right) = 0 \ , \tag{2.58}$$

$$C_{\mu\nu}{}^\tau = \frac{i}{2} \overline{\psi}_\mu \gamma^\tau \psi_\nu \ , \tag{2.59}$$

$$G^{\tau\mu} = \frac{i}{2} \epsilon^{\lambda\mu\nu\rho} \overline{\psi}_\lambda \gamma_5 \gamma^\tau \nabla_\mu \psi_\rho \ . \tag{2.60}$$

One also find non-trivial boundary terms, that we shall discuss later. Note that the Einstein tensor is now non-symmetric due to the torsion, defined by

$$C_{\mu\nu}{}^{\underline{\mu}} = \nabla_\mu e^{\underline{\mu}}{}_\nu - \nabla_\nu e^{\underline{\mu}}{}_\mu \ . \tag{2.61}$$

Setting $\psi = 0$ reproduces the vacuum Einstein equations and one can easily check that this is a consistent truncation of the field content. To reproduce the second order results of Ferrara et al [74], one solves (2.59) for the connection $\omega_{\mu\underline{\alpha\beta}}$ to give

$$\omega_{\mu\underline{\alpha\beta}} = \omega_{\mu\underline{\alpha\beta}}(e) + \tfrac{i}{4} \left(\overline{\psi}_{\underline{\alpha}} \gamma_\mu \psi_{\underline{\beta}} + \overline{\psi}_\mu \gamma_{\underline{\alpha}} \psi_{\underline{\beta}} - \overline{\psi}_\mu \gamma_{\underline{\beta}} \psi_{\underline{\alpha}} \right) , \tag{2.62}$$

$$\omega_\mu^{\underline{\alpha\beta}}(e) = \tfrac{1}{2} e^{\underline{\rho}}{}_\mu \left(\Omega^{\underline{\alpha\beta}}{}_{\underline{\rho}} - \Omega^{\underline{\beta}}{}_{\underline{\rho}}{}^{\underline{\alpha}} - \Omega_{\underline{\rho}}{}^{\underline{\alpha\beta}} \right) \ , \tag{2.63}$$

[6]For a review of this, along with the superspace approach of Ferrara et al [74], see [75].

where $\omega_\mu^{\underline{\alpha\beta}}(e)$ is the torsion-free spin connection and $\Omega^{\underline{\alpha\beta}}{}_{\underline{\rho}}$ are the objects of anholonomity, defined by

$$\Omega_{\underline{\alpha\beta}}{}^{\underline{\rho}} = 2e_{\underline{\alpha}}{}^{\mu} e_{\underline{\beta}}{}^{\nu} \partial_{[\mu} e^{\underline{\rho}}{}_{\nu]} \ . \tag{2.64}$$

The action for (2.57) is invariant, up to total derivative, under local supersymmetry transformations with respect to an infinitesimal anti-commuting parameter ϵ

$$\delta_\epsilon e^{\underline{\mu}}{}_{\mu} = i\bar{\epsilon}\gamma^{\underline{\mu}}\psi_\mu \ , \tag{2.65}$$

$$\delta_\epsilon \psi_\mu = 2\nabla_\mu \epsilon \ , \tag{2.66}$$

$$\delta_\epsilon \omega_\mu{}^{\underline{\alpha\beta}} = B_\mu{}^{\underline{\alpha\beta}} - \tfrac{1}{2} e^{\underline{\beta}}{}_{\mu} B_{\underline{\rho}}{}^{\underline{\alpha\rho}} + \tfrac{1}{2} e^{\underline{\alpha}}{}_{\mu} B_{\underline{\rho}}{}^{\underline{\beta\rho}} \ , \tag{2.67}$$

where

$$B_{\underline{\alpha}}{}^{\lambda\mu} = i\bar{\epsilon}\gamma_5\gamma_{\underline{\alpha}}\nabla_\nu\psi_\rho\epsilon^{\lambda\mu\nu\rho} \ . \tag{2.68}$$

The product of two supersymmetry transformations affects a spacetime diffeomorphism, as can be seen by looking at the commutator

$$[\delta_\epsilon, \delta_{\epsilon'}] = \delta_G(K^\mu) + \delta_L(K^\mu \omega_{\mu\,\underline{\alpha\beta}}) + \delta_{-K^\mu\psi_\mu} \ , \tag{2.69}$$

$$K^\mu = 2i\bar{\epsilon}'\gamma^\mu\epsilon \ , \tag{2.70}$$

where the δ terms on the right-hand side are coordinate transformations, local frame rotations and supersymmetry transformations respectively. The supersymmetry transformations form a representation of the supersymmetry algebra, the super-Poincaré algebra, which closes on-shell.

The total derivative term that appears upon varying the action is given by

$$\delta_\epsilon S = \int d^4x \nabla_\mu \theta^\mu_\epsilon \ , \tag{2.71}$$

where $\theta^\mu_\epsilon = i\,\epsilon^{\mu\nu\rho\lambda}\,\overline{\psi}_\nu\gamma_5\gamma_\rho\nabla_\lambda\epsilon$. Using Noether's theorem we can derive the corresponding conserved currents J^μ and supercharges Q

$$J^\mu_\epsilon = \frac{\delta S}{\delta \nabla_\mu \Phi}\delta_\epsilon\Phi - \theta^\mu_\epsilon \epsilon^{\mu\nu\rho\lambda} = -2i\,\epsilon^{\mu\nu\rho\lambda}\,\overline{\psi}_\nu\gamma_5\gamma_\rho\nabla_\lambda\epsilon + \cdots \ , \tag{2.72}$$

$$Q\cdot\epsilon = \int_\Sigma d\Sigma_\mu J^\mu_\epsilon \ , \tag{2.73}$$

where Φ denote all fields in the theory and the ellipsis indicates torsion terms which vanish on-shell. Supersymmetry transformations are generated by the supercharges Q through $\delta_\epsilon = \epsilon^\alpha Q_\alpha$, and so the Q's also form a representation of the supersymmetry algebra

$$\{Q, Q\} = C\gamma_\alpha P^\alpha \ , \tag{2.74}$$

where C is the charge conjugation matrix in the chosen representation and P^α is the four-momentum.

The Witten condition from Supergravity

In order to thoroughly understand the positivity of gravitational energy one should formulate the Hamiltonian version of classical supergravity [76]. We shall not do this here, but instead reproduce only the key features required in order to understand the physical significance of the Witten condition, following the arguments of Horowitz and Strominger [77] (See also [78]).

In the Hamiltonian approach, transformations are generated by Dirac brackets with respect to the appropriate charge. The supersymmetry transformations (2.65)-(2.67) are generated by the Dirac bracket with the supercharge Q, and similarly, time translations are generated by Hamiltonian H. A fundamental property of supersymmetric theories is that time translations are also generated by the square of supersymmetry transformations, so that roughly $H \sim Q^2$. This relation was first noted in the quantum supergravity by Deser and Teitelboim [63], and later extended to the classical theory. In performing the Hamiltonian analysis, one must make an appropriate gauge choice for the gravitino,

$$\psi_0 = 0 \tag{2.75}$$

$$\gamma^m \psi_m = 0 \tag{2.76}$$

which holds on the spatial hypersurface Σ that the Hamiltonian will be defined on. On identifying ψ_m as the correct degrees of freedom for the gravitino, one can then argue that if the Witten condition (2.76) were not imposed, the supersymmetry transformation $\delta_\epsilon \psi$ would transform positive energy components of gravitino into positive and negative (unphysical) energy, components of the gravity sector [78]. Local supersymmetry combined with the Witten condition (2.76) then implies that the parameter ϵ is asymptotically covariantly constant

$$\gamma^m \nabla_m \varepsilon = 0 \ , \tag{2.77}$$

This equation has four non-zero solutions ϵ_N which determine four supercharges Q_N $(N = 1, \ldots, 4)$. Taking the Dirac bracket of two supercharges and comparing the result with the explicit form given above (2.69), one finds

$$\overline{\epsilon}^0_M \gamma^\mu \epsilon^0_N P_\mu = 2 \int d^3\Sigma \left(\nabla^m \epsilon_N\right)^\dagger \nabla_m \epsilon_M \ , \tag{2.78}$$

where ϵ^0_N is the asymptotic value of the $N-$th supersymmetry parameter. Recall that Witten's argument used *commuting* spinors, so in order to reproduce this we must factor out the anticommuting component ϵ,

$$\epsilon_N(x) = \theta_N \alpha_N(x) \ , \tag{2.79}$$

where θ_N form a basis of the Grassmann algebra. One then integrates out over the anticommuting variables and uses that $\overline{\alpha}^0_N \gamma_v \alpha^0_M$ should be timelike and future-directed in order for P_0 to be a sensible expression for energy. The resulting expression is

$$E = 2 \int d^3\Sigma \left(\nabla^m \alpha\right)^\dagger \nabla_m \alpha, \tag{2.80}$$

which is exactly Witten's energy expression for vacuum solutions to the field equations.

Chapter 3

Supergravity and p-Branes

In this chapter we will provide a short introduction to p-brane solutions in supergravity. This is a vast subject unto itself and is the topic of many good review articles; we shall mainly follow [79].

We begin by reviewing the main elements of eleven-dimensional supergravity, its equations of motion and supersymmetry algebra. This is the highest dimension in which one can formulate a theory of supergravity, and is the simplest of the supergravity theories related to string theory (See [80, 81] and references therein). Our main interest will be the extended objects which are solitonic solutions of these theories, generically known as *p-branes*, where p labels the number of spatial dimensions on the brane's worldvolume. For instance, a domain wall in four dimensions would be a 2-brane. We will study a general p-brane ansatz to the common sector of supergravity theories, i.e. D-dimensional gravity coupled to a scalar and a $n-1$-form gauge potential. We will see that two basic p-brane solutions exist; the extremal branes, which can be supported electrically or magnetically by the gauge field. As examples, we briefly describe the 2-brane and 5-brane solutions in eleven dimensions. We then discuss the definition of energy, charges and the supersymmetry of these branes and describe a more general class of solutions known as black branes. This will allow us to understand the special role that the extremal branes play.

3.1 11D Supergravity and its solutions

3.1.1 Action, Symmetries and Field Equations

In this chapter we will adopt the conventions common in the literature, which differs from that used earlier. In this section capital Latin indices M, N will run over eleven dimensions, with underlined indices once again being used for flat, tangent space indices. We define the eleven-dimensional vielbein $e^{\underline{M}}_{M}$ using $g_{MN} = e^{\underline{M}}_{M} e^{\underline{N}}_{N} \eta_{\underline{MN}}$, $\eta_{\underline{MN}}$ is the eleven-dimensional Minkowski metric.

The action of eleven-dimensional supergravity is [82]

$$
\begin{aligned}
S = \int d^{11}x \, \sqrt{-g} \, \Big[R &+ \frac{1}{192} \left(\overline{\Psi}_M \Gamma^{MNPQRS} \Psi_N + 12 \overline{\Psi}^P \Gamma^{QR} \Psi^S \right) \left(F_{PQRS} + \tilde{F}_{PQRS} \right) \\
&- \frac{1}{48} F_{MNPQ} F^{MNPQ} - \frac{\mathrm{i}}{2} \overline{\Psi}_M \Gamma^{MNP} D_N \left(\frac{1}{2} [\omega + \tilde{\omega}] \right) \Psi_P \Big] \\
&+ \frac{2}{(12)^4} \int d^{11}x \epsilon^{KLMNPQRSTUV} F_{KLMN} F_{PQRS} A_{TUV} \ , \qquad (3.1)
\end{aligned}
$$

where $F_{MNPQ} = 4\partial_{[M} A_{NPQ]}$, A_{NPQ} is a 3-form anti-symmetric tensor gauge potential and Ψ are spin 3/2 fermions satisfying the Majorana condition $\overline{\Psi} = \Psi^T C^{-1}$, with the charge conjugation matrix C being defined by $C^{-1}\Gamma_{\underline{A}} C = -\Gamma^T_{\underline{A}}$. The covariant derivative is then defined as $D(\omega)_M = \partial_M - \frac{1}{4}\omega_M^{\underline{AB}} \Gamma_{\underline{AB}}$ and $\tilde{\omega}_{M\underline{AB}} = \omega_{M\underline{AB}} + \frac{i}{4} \overline{\Psi}_N \Gamma_{M\underline{AB}}{}^{NP} \Psi_P$, where ω is the sum of spin connection and contorsion tensor. ϵ is the eleven-dimensional anti-symmetric tensor, and the final piece of (3.1) is a topological, Chern-Simons term.

Following the literature [7], in this chapter we will choose the eleven-dimensional gamma matrices forming a pure imaginary representation of the Clifford algebra

$$
\{\Gamma_{\underline{A}}, \Gamma_{\underline{B}}\} = -2\eta_{\underline{AB}} \ . \qquad (3.2)
$$

The action (3.1) is invariant under general coordinate transformations, SO(1, 10) local Lorentz transformations, Abelian gauge transformations and $\mathcal{N} = 1$ supersymmetry with infinitesimal anti-commuting parameter ε, the latter defined by

$$
\delta_\varepsilon e^{\underline{A}}_{M} = -\mathrm{i}\overline{\varepsilon}\Gamma^{\underline{A}}\Psi_M \ , \qquad (3.3)
$$

$$
\delta_\varepsilon \Psi = \tilde{D}_M \varepsilon \equiv D(\tilde{\omega})_M \varepsilon - \frac{\mathrm{i}}{144} \left(\Gamma_M{}^{NPQR} - 8\Gamma^{NPQ}\delta^R_M \right) \tilde{F}_{NPQR} \, \varepsilon \ , \qquad (3.4)
$$

$$
\delta_\varepsilon A_{MNP} = \frac{3}{2} \overline{\varepsilon} \Gamma_{[MN} \Psi_{P]} \ . \qquad (3.5)
$$

$\tilde{F}$ is the supercovariantised field strength defined by $\tilde{F}_{NPQR} = F_{NPQR} - 3\overline{\Psi}_{[N}\Gamma_{PQ}\Psi_{R]}$. The supersymmetry transformations and explicit form of the action are fixed by requiring that the appropriate equations of motion are reproduced. In particular, we expect to have the following supercovariant field equation for Ψ_M,

$$\Gamma^{MNP}\tilde{D}_N\Psi_P = 0 \ . \tag{3.6}$$

Comparing this with $\delta_\Psi S$ and requiring any extraneous terms to vanish fixes the coefficient of the Chern-Simons term in (3.1). Specifically, requiring the vanishing of all terms of the form $\bar{\varepsilon}\Psi F^2$ fixes the product of the Chern-Simons and δA coefficients, and then the δA coefficient is completely fixed by considering the terms $\bar{\varepsilon}\partial\Psi F$ and $\bar{\varepsilon}\Psi\partial F$.

The supersymmetry transformations (3.3)–(3.5) form a representation of the eleven-dimensional supersymmetry algebra,

$$\{Q,Q\} = C\left(\Gamma^{\underline{A}}P_{\underline{A}} + \Gamma^{\underline{AB}}U_{\underline{AB}} + \Gamma^{\underline{ABCDE}}V_{\underline{ABCDE}}\right) \ , \tag{3.7}$$

which is an extended version of the super-Poincaré algebra, where $U_{\underline{AB}}$ and $V_{\underline{ABCDE}}$ are 2-form and 5-form charges respectively, whose significance will become apparent when we discuss solutions to this theory. For the moment, let us note that we will be interested in solitonic solutions preserving some fraction of supersymmetry. These can be found by consistently truncating to the bosonic sector and solving the field equations there, which are given by

$$R_{MN}(\tilde{\omega}) - \frac{1}{2}g_{MN}R(\tilde{\omega}) = \frac{1}{3}\tilde{F}_{MPQR}\tilde{F}_N^{PQR} - \frac{1}{24}g_{MN}\tilde{F}_{PQRS}\tilde{F}^{PQRS} \ , \tag{3.8}$$

$$D(\tilde{\omega})_M\tilde{F}^{MNPQ} = -\frac{1}{576}\epsilon^{NPQRSTUVWXY}\tilde{F}_{RSTU}\tilde{F}_{VWXY} \ , \tag{3.9}$$

plus the Bianchi identity

$$\partial_{[M}\tilde{F}_{NPQR]} = 0 \ . \tag{3.10}$$

3.1.2 p-brane Solutions

Having introduced eleven-dimensional supergravity in the previous section, we will now discuss its solitonic solutions. We are going to do this by finding the general p-brane solution to a D-dimensional theory of gravity, a scalar field ϕ and one $(n-1)$-form potential $A_{[n-1]}$ $(n \neq D/2)$ with field strength $F_{[n]} = A_{[n-1]}$ [79]. The action for this theory is

$$S_D = \int d^D x \sqrt{-g} \left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2n!} e^{a\phi} F_{[n]}^2 \right) \ , \tag{3.11}$$

where now $M, N = 0, \ldots D-1$ and the topological term has been dropped out as we consider only one gauge field. The equations of motion derived from this action are

$$\begin{aligned} R_{MN} &= \frac{1}{2(n-1)!} e^{a\phi} \left(F_{MP_2 \ldots P_n} F_N{}^{P_2 \ldots P_n} - \frac{n-1}{n(D-2)} F^2 g_{MN} \right) \\ &\quad + \frac{1}{2} \partial_M \phi \partial_N \phi \end{aligned} \tag{3.12}$$

$$\nabla_{M_1} \left(e^{a\phi} F^{M_1 \ldots M_n} \right) = 0 \ , \tag{3.13}$$

$$\Box \phi = \frac{a}{2n!} e^{a\phi} F^2 \ . \tag{3.14}$$

In order to describe eleven-dimensional supergravity we set $a = 0 = \phi$, which one can easily check is a consistent truncation by looking at the field equations. We will now make an ansatz for solutions preserving (Poincaré)$_d \times SO(D-d)$ symmetry, which is appropriate for p-branes. We split the spacetime indices into $x^M = x^\mu, y^m$, where x^μ $(\mu, \nu = 0, \ldots, p)$ are coordinates on the $d = p+1$-dimensional Poincaré invariant space, the *worldvolume*, and y^m $(m, n = d, \ldots D-1)$ are coordinates on the isotropic transverse space. The metric for a solution preserving these symmetries is

$$ds^2 = e^{2A(r)} g_{\mu\nu} dx^\mu dx^\nu + e^{2B(r)} \delta_{mn} dy^m dy^n, \tag{3.15}$$

where δ_{mn} is the flat metric on the transverse space, and we have defined the radial coordinate $r = \sqrt{y^m \cdot y^m}$. The ansatz for the scalar field is simple $\phi = \phi(r)$, however the gauge field requires some more thought. In analogy with electrodynamics, we expect the $A_{[n-1]}$ potential to support an extended object with $p = (n-2)$ spatial dimensions, carrying the corresponding electric charge. In this case, our ansatz for the gauge potential is

$$A_{\mu_1 \ldots \mu_{n-1}} = \epsilon_{\mu_1 \ldots \mu_{n-1}} e^{C(r)} \ , \tag{3.16}$$

with field strength

$$F_{m\mu_1 \ldots \mu_{n-1}} = \epsilon_{\mu_1 \ldots \mu_{n-1}} \partial_m e^{C(r)} \ , \tag{3.17}$$

Alternatively, we could have an object supported *magnetically* by the gauge potential corresponding to the Hodge dual of original field strength $F_{[n]}$. This dual $(D-n)$-form field strength would support a $(D-n-2)$-brane, and although the dual action for this field is not straightforward to construct, the ansatz for the field strength supporting this magnetic brane is simple:

$$F_{m_1 \dots m_{\tilde{d}+1}} = \lambda \epsilon_{m_1 \dots m_{\tilde{d}} l} \frac{y^l}{r^{\tilde{d}+2}} \ , \tag{3.18}$$

where λ is an integration constant that we associate to the magnetic charge and we have defined $\tilde{d} = D - d - 2$. We must then show that this ansatz solves the equations of motion, fixing the form of the functions $A(r)$, $B(r)$ and $C(r)$. The task is greatly simplified if we state that we want our solution to obey

$$dA' + \tilde{d}B' = 0 \ , \tag{3.19}$$

where $' = \partial/\partial_r$, which one can show is implied by preservation of some fraction of supersymmetry. Introducing the definition

$$\Delta = a^2 - \frac{2d\tilde{d}}{(D-2)} \ , \tag{3.20}$$

and making some further educated guesses as to the form of the solution, one can re-express the ϕ field equation as a pure Laplace equation in the transverse space:

$$\nabla^2 e^{\frac{c\Delta}{2a}\phi} = 0 \ , \tag{3.21}$$

where $c = \pm 1$, depending on whether we look for an electric/magnetic solution respectively. This is easily solved to give,

$$e^{\frac{c\Delta}{2a}\phi} \equiv H(r) = 1 + \frac{k}{r^{\tilde{d}}} \ , \tag{3.22}$$

where k is a constant, and we have fixed $\phi_{|r \to \infty}$. A little more manipulation of the field equations then allows one to deduce the form of the function appearing in the field strength ansatz (3.17) for the electric brane

$$e^{C(r)} = \frac{2}{\sqrt{\Delta}} H(r)^{-1} \ , \tag{3.23}$$

and fix the charge of the magnetic brane

$$\lambda = \frac{2\tilde{d}}{\sqrt{\Delta}} k \ . \tag{3.24}$$

Combining the above results, we present the complete metric for the p-brane solutions of theory defined by (3.11):

$$ds^2 = H(r)^{\frac{-4\tilde{d}}{\Delta(D-2)}} dx^\mu dx^\nu \eta_{\mu\nu} + H(r)^{\frac{4d}{\Delta(D-2)}} dy^m dy^n \delta_{mn} \; , \tag{3.25}$$

$$e^\phi = H^{\frac{2a}{c\Delta}} \quad , \quad H(r) = 1 + \frac{k}{r^{\tilde{d}}} \; , \tag{3.26}$$

with the corresponding choice of field strengths,

$$F_{m\mu_1\ldots\mu_{n-1}} = \epsilon_{\mu_1\ldots\mu_{n-1}} \partial_m \left(H^{-1}\right) \qquad \text{(electric)} \; , \tag{3.27}$$

$$F_{m_1\ldots m_n} = -\epsilon_{m_1\ldots m_n l} \partial_l H \qquad \text{(magnetic)} \; . \tag{3.28}$$

3.1.3 M2 and M5 branes

Let's look at the soliton solutions of eleven-dimensional supergravity. As mentioned above, we can consistently truncate the action (3.11) by setting $a = 0 = \phi$, which means that we then fix $\Delta = 4$. The electric M2 brane solution is [83, 84]

$$\begin{aligned} ds^2_{\text{M2}} &= \left(1 + \frac{k}{r^6}\right)^{-\frac{2}{3}} dx^\mu dx^\nu \eta_{\mu\nu} + \left(1 + \frac{k}{r^6}\right)^{\frac{1}{3}} dy^m dy^n \delta_{mn} \\ A_{\mu\nu\lambda} &= \epsilon_{\mu\nu\lambda} \left(1 + \frac{k}{r^6}\right)^{-1} \quad , \qquad \mu, \nu = 0, \ldots, 2 \; . \end{aligned} \tag{3.29}$$

The magnetic M5 brane solution is [85]

$$\begin{aligned} ds^2_{\text{M5}} &= \left(1 + \frac{k}{r^3}\right)^{-\frac{1}{3}} dx^\mu dx^\nu \eta_{\mu\nu} + \left(1 + \frac{k}{r^3}\right)^{\frac{2}{3}} dy^m dy^n \delta_{mn} \\ F_{mnpq} &= 3k \, \epsilon_{mnpqr} \frac{y^r}{r^5} \; , \qquad \mu, \nu = 0, \ldots, 5 \; . \end{aligned} \tag{3.30}$$

Both are asymptotically flat by construction, but various coordinate transformations display other interesting features of these solutions. Transforming $r = (\tilde{r}^{\tilde{d}} - k)^{\frac{1}{6}}$, we find Schwarzschild-like coordinates. For the M2-brane one then finds a degenerate horizon at $\tilde{r}^6 = k$, where light-cones do not flip over (unlike the horizon in Schwarzschild geometry), and a timelike singularity at $\tilde{r} = 0$. So we see that M2 is more like the Reissner-Nordstrom black hole of General Relativity rather than the Schwarzschild black hole, as one would expect for a charged solution. For the M5-brane, one also finds a degenerate horizon at $\tilde{r}^3 = k$, only now there is no singularity at $\tilde{r} = 0$. We can see this by considering the interpolating coordinates, defined by $\tilde{r} = k^{\frac{1}{\tilde{d}}}(1 - R^d)^{-\frac{1}{\tilde{d}}}$. In this frame the M5 is symmetric under $R \to -R$, allowing a maximal analytic extension to a smooth spacetime, exactly like that used for Reissner-Nordstrom solution.

In fact the interpolating coordinates have a further use. The limit $R \to 1$ sends both solutions to flat space. On the other hand, the near horizon $R \to 0$ limit, takes M2 to $AdS_4 \times S^7$ and M5 to $AdS_7 \times S^4$, both of which, along with the flat space limits, are maximally supersymmetric solutions of the eleven-dimensional supergravity. So we see that the M-brane solutions *interpolate* between the maximally supersymmetric vacuum of the theory [59], and so we expect them to have some interesting supersymmetry properties, which we will now discuss.

3.2 Charges and Supersymmetry

3.2.1 Energy for p-Branes

Defining mass or energy for p-branes requires a little thought. The extended nature of these objects means that any integrated quantity will diverge due to the infinite world-volume i.e. we no longer have some localised, point-like source as in the case of a four-dimensional black hole. Instead, we will evaluate the energy density, with the integral taken over the boundary of transverse space $\partial\mathcal{M}_{\mathrm{T}}$ with the appropriate background metric being a transverse asymptotically flat spacetime [79, 86]. To remove the problem of the divergent worldvolume integral we can impose that the worldvolume directions are periodic, with p-volume V_p. We can then define the average of some quantity A in the obvious way [86]:

$$< A >= \frac{1}{V_p}\int d^p x \; A. \tag{3.31}$$

The p-brane energy density, first given in [79], is an extension of the Deser-Soldate energy for five-dimensional spacetimes (2.32) that was presented in section (2.3.1). Understanding that we have an appropriately normalised integral, as in (3.31), we can define the p-brane energy density by

$$E_{\mathrm{p-brane}} = \int_{\partial\mathcal{M}_{\mathrm{T}}} d^{D-d-1}\Sigma^m(\partial^n h_{mn} - \partial_m h^a_a) \;, \tag{3.32}$$

where Ω_{D-d-1} is the volume of the transverse $\mathcal{S}^{D-d-1}$ unit sphere, $i, j = 1, \ldots, d-1$ and early alphabet lower case indices run over all spatial directions $a, b = i, j \ldots m, n \ldots D-1 = 1, \ldots D-1$. For compactness we do not write the explicit p-volume integral as in (3.31). The integral (3.32) is easily evaluated for the general ansatz presented in the previous section (3.25), for which we find the following expressions for metric perturbations

$$h_{mn} = \frac{4kd}{\Delta(D-2)r^{\tilde{d}}}\delta_{mn} \quad , \quad h_{ij} = -\frac{4k\tilde{d}}{\Delta(D-2)r^{\tilde{d}}}\delta_{ij} \ , \tag{3.33}$$

$$\Rightarrow \quad h^b{}_b = \frac{8k(d+\frac{1}{2}\tilde{d})}{\Delta(D-2)r^{\tilde{d}}} \ , \tag{3.34}$$

where we have used $\partial^n r = y^n/r$. Using that $d^{D-d-1}\Sigma^m = r^{\tilde{d}} y^m d\Omega^{D-d-1}$, we then find

$$E_{\text{p-brane}} = \frac{4k\tilde{d}\Omega_{D-d-1}}{\Delta} \ , \tag{3.35}$$

where Ω_{D-d-1} is the volume of the unit $(D-d-1)$-sphere. To put this into some perspective, we will now consider a more general class of solutions to the equations of motion derived from (3.11): the black brane metrics.

3.2.2 Black Branes

The black brane metric, written in generalised Schwarzschild coordinates, is [87–89]

$$\begin{aligned}
ds^2 &= -e^{2u}dt^2 + e^{2a}dx^i dx^j \delta_{ij} + e^{2v}d\tilde{r}^2 + e^{2B}\tilde{r}^2 d\Omega^2_{D-d-1} \\
e^{2v} &= \frac{K_-^{\left(\frac{2a^2}{\Delta\tilde{d}}-1\right)}}{K_+} \quad , \quad e^{2A} = K_-^{\frac{4\tilde{d}}{\Delta(D-2)}} \quad , \quad e^{2u} = \frac{K_+}{K_-^{\left(1-\frac{4\tilde{d}}{\Delta(D-2)}\right)}} \\
e^{2B} &= K_-^{\frac{2a^2}{\Delta\tilde{d}}} \quad , \quad e^{\frac{c\Delta}{2a}\phi} = K_-^{-1} \quad , \quad K_\pm = 1 - \left(\frac{r_\pm}{\tilde{r}}\right)^{\tilde{d}} \ ,
\end{aligned} \tag{3.36}$$

and the general field strength parameter is now given by

$$\lambda = \frac{2\tilde{d}}{\sqrt{\Delta}(r_+ r_-)^{\tilde{d}/2}} \ . \tag{3.37}$$

This solution to the field equations (3.12)-(3.14) represents a two-parameter family that generalises the electric and magnetic branes discussed in the previous section. This class of solutions have an outer event horizon at $r = r_+$, which, like that in the Schwarzschild solution, is non-singular. There is also an inner horizon at $r = r_-$, which coincides with a proper curvature singularity. In the extremal limit $r_+ = r_-$, we recover the supersymmetric solutions of the previous section.

The parameterisation used above is useful as the DS energy density takes a simple, if somewhat lengthy form [89, 90],

$$E_{\text{B.B.}} = \left[\tilde{r}^{\tilde{d}+1}(d-1)(e^{2A})' + \tilde{r}^{\tilde{d}+1}(\tilde{d}-1)(e^{2B})' - \tilde{r}^{\tilde{d}}(\tilde{d}+1)\big(e^{2v} - e^{2B}\big)\right]\Big|_{r\to\infty}, \tag{3.38}$$

where $' = \partial_{\tilde{r}}$. Comparing with the result for the extremal branes (3.35), one sees that

$$E_{\text{B.B.}} > \frac{4k\tilde{d}\Omega_{D-d-1}}{\Delta} = E_{\text{p-brane}}\ . \tag{3.39}$$

Rewriting this expression using $k = \sqrt{\Delta}\lambda/(2\tilde{d})$, we find that the extremal brane ansatz (3.25) saturates the inequality

$$E \geq \frac{2\lambda\Omega_{D-d-1}}{\sqrt{\Delta}}\ . \tag{3.40}$$

Recall that the parameter λ determines the charge of the solution we were considering (3.37), and so we see that the charge acts as lower bound on the energy, which is saturated for extremal branes (c.f. the mass bound for the Reissner-Nordstrom black hole (2.56)). This type of inequality is similar to the Bogomol'nyi bound found on the energy of monopoles in Yang-Mills-Higgs theories, with the extremal limit being equivalent to BPS condition i.e. the extremal branes (3.25) solutions are the BPS states of our theory. To understand this point better, we need to reconsider the supersymmetry of these solutions and the definition of their electric/magnetic charges.

3.2.3 Supersymmetry of p-Branes

Let's begin by considering the M2 brane (3.29) and the definition of its electric charge. As this solution is singular, we expect it to produce a δ-function contribution in the field equation for $A_{[3]}$. In analogy with a point particle source in regular electromagnetism, we introduce the Nambu-Goto source action for the 2-brane [79],

$$I_{\text{M2}} = Q_e \int_{W_3} d^3\xi \left[-\sqrt{\gamma} + \frac{1}{3!}\epsilon^{\mu\nu\rho}\partial_\mu x^M \partial_\nu x^N \partial_\rho x^R A_{MNR}\right]\ , \tag{3.41}$$

where $\gamma_{\mu\nu} = \partial_\mu x^M \partial_\nu x^N g_{MN}$ ($\partial_\mu = \partial/\partial\xi^\mu$) is the pullback of the eleven-dimensional spacetime metric onto the three-dimensional worldvolume W_3 of the 2-brane, which has coordinates $\xi^\mu(\mu = 0, 1, 2)$. This produces a δ-function current in the $A_{[3]}$ field equation which, for brevity, we now write in form notation (See appendix for conventions),

$$d\left(*F_{[4]}+\tfrac{1}{2}A_{[3]}\wedge F_{[4]}\right)=*J_{[3]}\ , \tag{3.42}$$

where

$$J^{MNR}=Q_e\int_{W_3}d^3\xi\delta^3\left(x-x(\xi)\right)dx^M\wedge dx^N\wedge dx^R\ . \tag{3.43}$$

This leads to a generalised Page charge

$$\begin{aligned}U&=\int_{\partial M_8}\left(*F_{[4]}+\tfrac{1}{2}A_{[3]}\wedge F_{[4]}\right)\\&=\int_{M_8}*J_{[4]}=\frac{1}{3!}\int d^8\Sigma_{MN}\,J^{0MN}\ ,\end{aligned} \tag{3.44}$$

which then identifies $U=Q_e$ as the 2-brane electric charge. We can evaluate this integral explicitly for the M2-brane (3.29), for which the $A_{[3]}\wedge F_{[4]}$ term vanishes. Choosing M_8 to coincide with the M2 transverse space, we can use the explicit form of $F_{[4]}$ to find

$$Q_e=\int_{\partial M_8}d^7\Sigma_m F^m{}_{012}=\lambda\Omega_7\ . \tag{3.45}$$

Looking again at the ADM-mass formula for p-branes (3.35), with $k=\sqrt{\Delta}\lambda/(2\tilde{d})$, and using that $\Delta=4$ for the eleven-dimensional theory, we find by direct comparison that the Bogomol'nyi bound (3.40) is indeed saturated for the M2-brane

$$E_{\mathrm{M2}}=Q_e=\lambda\Omega_7\ . \tag{3.46}$$

One finds a similar result for the M5-brane, where now the magnetic charge defined by the Bianchi identity (3.10) is given by

$$V=\int_{\partial M_5}F_{[4]}. \tag{3.47}$$

Choosing the M5 transverse space to coincide with M_5 and using the ansatz for the field strength (3.30), we find

$$V=\int_{\partial M_5}d^7\Sigma_m\epsilon^m{}_{npqr}F^{npqr}=\lambda\Omega_4\ . \tag{3.48}$$

Once again one can easily see that this solution saturates the Bogomol'nyi bound (3.40). These arguments certainly imply that the M2 and M5-brane solutions preserve some supersymmetry, however one can go further. The M2 and M5 charges $U=Q_e$ and V are in fact the magnitudes of the 2-form U_{AB} and 5-form V_{ABCDE} charges appearing in the eleven-dimensional supersymmetry algebra (3.7) [91, 92].

For instance, the 2-form charge may be defined by

$$U^{AB} = Q_e \int_{S_2} dx^A \wedge dx^B \ , \tag{3.49}$$

where the integral is over the spatial 2-cycle S_2 of the M2 worldvolume. This expression will be used to re-express the anti-symmetric tensor charge U^{AB} appearing in the supersymmetry algebra in terms on the charge Q_e.

Let us now say that the M2 spatial directions S_2 coincide with x^1, x^2, then (ignoring the 5-form charge) we find the supersymmetry algebra can be written as [79]

$$\{Q, Q\} = C\left(\Gamma^0 P_0 + \Gamma^{12} U_{12}\right) \ . \tag{3.50}$$

From our discussion of energy bounds and charges above we can now use that $P_0 = E_{\mathrm{M2}} = Q_e$ and $C = \Gamma^0$, to find

$$\{Q, Q\} = 2E_{\mathrm{M2}}\, P_{012} \qquad , \qquad P_{012} = \tfrac{1}{2}\left(\mathbb{1} + \Gamma^{012}\right) \ . \tag{3.51}$$

Using that $(\Gamma^{012})^2 = \mathbb{1}$, where $\mathbb{1}$ is the unit matrix, one sees P^{012} is a projection operator with trace $\mathrm{tr} P^{012} = \frac{1}{2} \cdot 32$. This implies that half of the eigenvalues of P^{012} are zero. Any supersymmetry transformations preserved by the M2-brane solution must in turn satisfy the following relation

$$\{Q, Q\}\,\epsilon = 2E_{\mathrm{M2}}\, P_{012}\, \varepsilon = 0 \qquad \Rightarrow \qquad \tfrac{1}{2}\left(\mathbb{1} + \Gamma^{012}\right)\,\varepsilon = 0 \ , \tag{3.52}$$

where now we see that since P^{012} has half zero eigenvalues, the M2 brane solution preserves half of the supersymmetries of the background. We could have also studied the background supersymmetry transformations (3.3)-(3.5) directly. For bosonic solutions this means looking for background 'Killing spinors' i.e. spinors which solve

$$\delta_\varepsilon \Psi = \tilde{D}_M \varepsilon = 0 \ . \tag{3.53}$$

One can show that this results in the same projection condition in terms of P^{012} as one finds by considering just the supersymmetry algebra, again implying that half of the original 32 supersymmetries are preserved by the M2-brane. A similar procedure shows the M5-brane is also half supersymmetric.

Having seen that the extremal branes of eleven-dimensional supergravity preserve half the supersymmetry, we can step back and reconsider equation (3.50). We know that the left-hand side of this expression must be positive definite and so if we don't impose the extremal relation $P_0 = Q_e$, a little manipulation gives the following bound

$$P_0 = E_{\mathrm{M2}} \geq |Q_e| \ , \tag{3.54}$$

where we have used (3.49). This has the same form as the Bogomol'nyi bound we found by a direct construction of Noether charges for branes (3.40), and which also arises when one considers the positive energy theorem for charged black holes, as we noted earlier (2.56). In fact, (3.54) is nothing more than the extension of the supersymmetric bound of Gibbons and Hull [72] to solitons which are no longer point-like.

To recap, we have seen that supersymmetric (extremal) branes saturate Bogomol'nyi bounds linking mass and charge i.e. they are minimal energy solutions. We could show this by considering the charges directly or, more fundamentally, by showing that super-algebra itself implied that energy was minimised for solutions preserving some fraction of the background supersymmetry:

$$\begin{aligned} &\mathrm{E} \geq |Q_e| \\ &\mathrm{E} = |Q_e| \Leftrightarrow \tilde{D}_M \varepsilon = 0 \ . \end{aligned} \tag{3.55}$$

A spinorial proof of positive energy for p-brane spacetimes of this form has also been given [59, 93], extending the black hole version we discussed earlier.

Part II

Chapter 4

The Stability of Hořava-Witten Spacetimes

4.1 Introduction

Hořava-Witten (HW) theory is an interesting alternative to standard compactifications for generating models of four-dimensional physics [94, 95]. The theory links 11-dimensional supergravity on the orbifold $S^1/\mathbb{Z}_2$ with strongly coupled heterotic $E_8 \times E_8$ string theory, and upon further compactification to four dimensions on a Calabi-Yau can provide phenomenologically interesting models. One distinctive feature of such models is the prediction of an intermediate, five-dimensional, energy regime when particles previously bound to the four-dimensional boundaries M_4 can probe into the fifth (bulk) dimension [96, 97]. The full eleven-dimensional picture isn't recovered until energies reach the string scale.

This rigorous string construction inspired many particle physics and cosmology models suggesting intermediate scales that could be easily detected at the next generation of experiments. Most notable were the large extra dimensions model of Arkani-Hamed et al [10–12] and the Randall-Sundrum warped compactification models [13, 14], with many more following quickly (see [98, 99] for recent reviews). Many of such models are constructed with string theory as an inspiration, however their mathematical consistency is often not so clear.

Let us clarify our terminology here. Models which include some number of extra dimensions are now generically known as brane worlds, whether they have any relation to the branes of string theory or not. We will be interested only in Hořava-Witten spacetimes, which for our purposes we define as $\mathbb{Z}_2$-symmetric domain wall solutions, such as the ten-dimensional orbifold places in the original HW theory. A distinguishing feature of such spacetimes is that they have topology $M_4 \times \mathcal{I}$, where $\mathcal{I}$ is an interval. In the case where the interval is the orbifold $\mathcal{I} = S^1/\mathbb{Z}_2$, we have two possible interpretations of the spacetime: the 'upstairs' picture with a $\mathbb{Z}_2$ identified circle; or the 'downstairs' picture where we have the interval with boundary branes. For consistency, these boundary branes, or orbifold planes, must have equal and opposite tension, as one can easily see from their singular contributions to the field equations (see (5.2.1) and [24]) and so these models have negative tension objects present from the outset.

The appearance of negative tension branes seems troublesome. If we were to describe the low-energy excitations of these domain walls by some field theory, the negative tension brane would give rise to a scalar field Goldstone mode with a wrong sign kinetic term: the classic sign of an instability in the theory. However, it is often the case that these HW spacetimes are supersymmetric by construction, and as we discussed in part 1, supersymmetric objects have positive energy. There seems to be a flagrant contradiction, but in fact it's what the theory tells us. Consider the ten-dimensional orbifold planes in the original HW model. The supersymmetry algebra of eleven-dimensional supergravity contained topological extensions that are linked with the M2 and M5-brane solutions, and in section (3.2.3) we saw that it was the spatial charge components that were linked with these solutions e.g. U_{12} in the M2-brane case. It turns out that if one were to consider the time component of the 2-form charge, the corresponding extended solution is a 9-brane – the HW orbifold plane [100, 101]. In [94] , Hořava-Witten showed that the $\mathbb{Z}_2$ projection defining the orbifold planes commutes with half of the supersymmetry transformations, and that the condition of equal and opposite tension for the two planes was required for consistency of the theory. Hence one finds that the spacetime $M_{10} \times S^1/\mathbb{Z}_2$ is supersymmetric and thus should be stable, as supersymmetric states minimise energy, and therefore have no decay channels. On the other hand, the negative tension brane could give rise to "ballooning" modes, i.e. one would expect that it could lose energy by expanding, thus becoming unstable[1]. In this chapter we aim to resolve this issue by providing a comprehensive analysis of the energy and the fluctuations of HW braneworld solutions.

[1] Concerns to this effect were raised by Brandon Carter at Stephen Hawking's 60$^{\text{th}}$ birthday conference [102]

4.2 Domain Wall Perturbations

We will begin with a general consideration of the possible fluctuations of domain wall spacetimes. Much like our analysis of energy in chapter 2, let us identify a five-dimensional background solution around which we will consider perturbations:

$$ds_5^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{-8A(y)}dy^2 \ , \tag{4.1}$$

$$\phi \sim \ln H(y) \quad , \quad H = k|y| + c \ , \tag{4.2}$$

where ϕ is the scalar field supporting the branes and c, k are constants. We note that H is a linear harmonic function as expected for a domain wall, and that the metric data $A(y)$ is a function of H whose precise form we shall give later. The indices $\mu = 0, 1, 2, 3$ run over the worldvolume of the domain wall, with y labelling the transverse, bulk direction. Capital Latin indices M, N run over all directions. Choosing to parameterise the interval $\mathcal{I}$ as $S^1/\mathbb{Z}_2$, the $\mathbb{Z}_2$ identification appears as a symmetric kink in the harmonic function at the location of the branes. For clarity we note that to complete the HW picture, we should consider the case with two branes located at $y = y_i$, $(i = 1, 2)$, where the brane at $y = 0$ will have negative tension $H(y)'|_{y=0} < 0$ and we can choose the positive tension brane to be at $y = \pi$. For completeness, we note that when two branes are present the harmonic function is simply written as

$$H = k\left(|y| + |y - \pi|\right) + c \ , \tag{4.3}$$

and that the functional dependence of $A(y)$ on H remains unchanged. To simplify the formula we will often write the harmonic function with only one brane.

There are two types of motion that these branes can perform. The first, which we shall call the 'centre-of-mass' mode, corresponds to the branes keeping fixed separation but moving in the interval. The second mode, commonly known as the 'radion', corresponds to relative motion between the branes and we choose to discuss it first.

Labelling the radion mode by $r(x^\rho)$, we note that the motion can be understood by the following perturbation of the background metric

$$ds_5^2 = e^{r(x^\rho)+2A(y)}g^{(4)}_{\mu\nu}dx^\mu dx^\nu + e^{-2r(x^\rho)-8A(y)}dy^2 \ , \tag{4.4}$$

$$\phi \sim \ln H(y) + r(x^\rho) \ . \tag{4.5}$$

One approach to understanding the physics of this fluctuation is to consider the effective field theory on the domain wall worldvolume. This brane world dimensional reduction differs from the usual Kaluza-Klein procedure as the 'compactified' direction need not be near Planck size or even be compact. The low energy states, including gravity, are localised on the worldvolume by virtue of the metric warp factor $e^{2A(y)}$, which also ensures the that massive modes which appear in the mode expansion in the fifth direction, the Kaluza-Klein tower, are exponentially suppressed [13, 14].

Carrying out this reduction for our domain wall, it turns out that it is possible to truncate the effective theory to just four-dimensional gravity coupled to the scalar field describing the radion mode [104, 105], the equations of motion then being

$$R^{(4)}_{\mu\nu} = c'\partial_\mu r \partial_\nu r \tag{4.6}$$

$$\Box^{(4)} r = 0 \tag{4.7}$$

where c' is a positive constant. An appropriate field redefinition brings this into canonical form, and we then conclude that we should not expect any instability from it. It has been shown that if the background ansatz is a solution to a supersymmetric theory, then the effective field content in four dimensions can be completed to a Wess-Zumino multiplet [104].

Let us try the same procedure with the centre-of-mass mode. If we shift the brane positions by some amount s, then physically nothing as happened. This is just a diffeomorphism with the parameter s being a modulus. However, if we now allow the modulus to have dependence on the worldvolume coordinates $s(x^\rho)$ then it has the interpretation of a Goldstone mode, associated with the brane breaking translational invariance in the bulk spacetime [103]. The $\mathbb{Z}_2$ symmetry in our background solution then gets promoted to a local $\mathbb{Z}_2$ symmetry. This allows for relative shifts of different points on the brane, with $\mathbb{Z}_2$ symmetry then acting point by point. The centre-of-mass motion is described by the following ansatz

$$ds_5^2 = e^{2A(y-s(x^\rho))} g^{(4)}_{\mu\nu} dx^\mu dx^\nu + e^{-8A(y-s(x^\rho))} dy^2 \ , \tag{4.8}$$

$$H = k|y - s(x^\rho)| + c \ , \tag{4.9}$$

This mode therefore describes a sort of shear or warping of the HW end branes, which one can envisage as the twisting of a spring. In many studies of the cosmology of braneworld models this mode is not discussed, although it appears that it could allow the dangerous motion of the negative tension brane. The common assumption that is made is that the $\mathbb{Z}_2$ symmetry projects out this mode (see for example [106]), forcing the branes to remain in fixed positions. However, as we have seen above, this would only be the case for a global ($s(x^\rho) = s$) projection, which is a somewhat arbitrary truncation of the theory. Applying the braneworld reduction techniques to the centre-of-mass mode, one finds higher order couplings to five-dimensional modes on the right-hand side of the effective equations of motion. This tells us that we should consider the full five-dimensional theory plus boundaries in order to correctly understand the centre-of-mass mode's behaviour.

We shall finish this section with some comments. The original HW solution did not possess any difficulties with centre-of-mass mode as it simply does not appear. The bulk spacetime in this case is flat, and so shifting the brane positions has no effect. We can also view this from the perspective of consistent truncations of field content. Working in the upstairs picture, the radius R of the $\mathbb{Z}_2$-identified S^1 is related to the string coupling constant λ (i.e. the dilaton) of the ten-dimensional $E_8 \times E_8$ string theory by $R = \lambda^2/3$ [94]. This is the only scalar degree of freedom in the ten-dimensional theory, as the $\mathcal{N} = 1$ super-Maxwell multiplets contain no scalar fields. However, the HW solution found after compactifying the eleven-dimensional theory on a Calabi-Yau is a domain wall in five-dimensional curved space and the Goldstone mode reappears [97].

One interesting point that arises from the brief Goldstone mode analysis above is that both the radion (4.4,4.5) and centre-of-mass (4.8,4.9) motions can be identified with fluctuations of the bulk geometry, despite the presence of singular branes. For $\mathbb{Z}_2$-symmetric domain walls the Israel matching conditions become boundary conditions on the bulk fields [107]. This allows us to describe the domain wall dynamics entirely in terms of the bulk fields in the surrounding spacetime region, which will prove crucial when we come to consider the energy and stability of these fluctuations.

The model that we choose to focus on for studying the stability problem is not the compactified version of the original HW theory, but rather a five-dimensional model arising from a 'breathing mode' reduction of Type IIB supergravity. This is particularly interesting as it gives rise to $\mathbb{Z}_2$-symmetric singular domain wall solutions that display the gravity-trapping feature of the Randall-Sundrum model, and we review its properties in the next section. The work described in sections (4.2) and (4.3) of this chapter appeared in [1].

4.3 Supersymmetric Domain Walls

4.3.1 Randall-Sundrum solution

In order to study the stability of $\mathbb{Z}_2$-symmetric HW domain walls, it is useful to choose a particular model with certain generic features. The two Randall-Sundrum (RS) models [13, 14] are of phenomenological interest as they admit four-dimensional gravity at low-energies, but provide the intriguing possibility of experimental detection of the extra (fifth) dimensions at an intermediate (say, a few TeV) energy scales. The simplest RS model, which we previously called RS2, is a singular domain wall in five-dimensional anti-de Sitter space (AdS_5) [2]. In attempting to embed this into the fundamental framework of string theory, it's natural to look to a theory admitting AdS_5 as a vacuum. A good choice is ten-dimensional Type IIB string theory, and in fact it will suffice to concentrate on its supergravity limit. A simple application of the p-brane solution techniques described in chapter 3 shows that one solution of Type IIB supergravity is a self-dual 3-brane solution; the D3-brane. This solutions is similar in structure to the M5-brane of eleven-dimensional supergravity (3.30); for instance, it admits a analytic continuation to a completely smooth spacetime [59]. Writing the metric solution in interpolating coordinates, one finds that the near-horizon geometry of the D3-brane is $AdS_5 \times S^5$, suggesting that Type IIB compactified on S^5 is the good guess for a theory in which to embed the RS model.

We will now review the RS solution constructed in [109, 110]. The five-dimensional theory is derived from the S^5 dimensional reduction of Type IIB supergravity, where the volume modulus of the S^5 is promoted to the dynamical 'breathing mode'. The Type IIB field equations for gravity and the five-form $F_{[5]}$, and the Bianchi identity are most conveniently written as [108]

$$R_{\hat{A}\hat{B}} = \frac{1}{96} F_{\hat{A}\hat{C}\hat{D}\hat{E}\hat{G}} F_{\hat{B}}{}^{\hat{C}\hat{D}\hat{E}\hat{G}} \ , \tag{4.10}$$

$$F_{[5]} = *F_{[5]} \ , \tag{4.11}$$

$$dF_{[5]} = 0 = d * F_{[5]} \ , \tag{4.12}$$

and for our purposes it will suffice to consider only this sector of the theory. The hatted capital indices run over all ten dimensions $\hat{A}, \hat{B} = 0, \ldots, 9$. The ansatz for

[2] In this chapter we are not directly interested in the studying the hierarchy problem, which was only addressed in the first Randall-Sundrum model, and so we shall not differentiate between the two.

the Kaluza-Klein S^5 reduction is

$$ds_{10}^2 = e^{2\alpha\phi} ds_5^2 + e^{-\frac{6\alpha\phi}{5}} ds^2(S^5) \ , \quad (4.13)$$

$$F_{[5]} = 4m\, e^{8\alpha\phi} \epsilon_{[5]} + 4m\, \epsilon_{[5]}(S^5) \ , \quad (4.14)$$

where the ϵ's are the volume forms on the five-spaces, and we will now take unhatted capital Latin indices to run over the five dimensions in the line element ds_5^2, $M, N = 0, \ldots, 4$. The resulting five-dimensional equations of motion are

$$R_{MN} = \frac{1}{2}\partial_M\phi\partial_N\phi + \frac{8}{3}m^2\, e^{8\alpha\phi} g_{MN} - \frac{1}{3}R_5\, e^{\frac{16\alpha}{5}\phi} g_{MN} \ , \quad (4.15)$$

$$\Box\phi = 64\alpha m^2\, e^{8\alpha\phi} - \frac{16}{5}\alpha R_5\, e^{\frac{16\alpha}{5}\phi} \ , \quad (4.16)$$

where the constant m is Type IIB 5-form flux, R_5 is the scalar curvature of the S^5 and $\alpha = \frac{\sqrt{15}}{12}$. We stress again that ϕ is not the ten-dimensional dilaton, but the breathing mode scalar representing the volume of the 5-sphere compactification. A Lagrangian which one can vary to produce these equations of motion is [108]

$$\mathcal{L}_5 = \mathcal{L}_{\text{E.H.}} + \mathcal{L}_\phi = \sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right] \ , \quad (4.17)$$

where

$$V(\phi) = 8m^2 e^{8\alpha\phi} - R_5 e^{\frac{16\alpha}{5}\phi} \ , \quad (4.18)$$

That this gravity plus scalar theory is a consistent truncation of the dimensional reduction of Type IIB is interesting because the scalar is massive. Traditional Kaluza-Klein philosophy would say that if we include one massive scalar, we are forced to include a whole tower of massive states [7]. The breathing mode scalar ϕ evades this as it lies in a singlet of the SO(6) symmetry group of the S^5. To make contact with the more familiar Kaluza-Klein compactifications we note that the usual AdS_5 Freund-Rubin vacuum solution corresponds to the case where

$$\frac{\partial V}{\partial\phi} = 0 \qquad \Leftrightarrow \qquad e^{\frac{24\alpha\phi}{5}} = \frac{R_5}{20m^2} = \text{constant} \ . \quad (4.19)$$

As a consistent truncation of the compactified theory, we could expect the Lagrangian (4.17) to have a supersymmetric completion. The specific details of the reduction and truncation from the full $\mathcal{N} = 8$ theory are complicated and remain unclear; but for now, let us just note that it is possible to rewrite the potential for this theory in terms of a superpotential [110]

$$V(\phi) = W_{,\phi}^2 - \frac{2}{3}W^2 \ , \tag{4.20}$$

$$W(y,\phi) = \sqrt{2}(2me^{4\alpha\phi} - 5\sqrt{\frac{R_5}{20}}e^{\frac{8}{5}\alpha\phi}) \ , \tag{4.21}$$

which fits perfectly with the general expression for potentials in $\mathcal{N} = 2$ supergravity [111]. We shall return to the question of supersymmetry later.

The domain wall solution to (4.17), which we take to be our basic example of a HW domain wall, is given by

$$ds_5^2 = (b_1 H^{2/7} + b_2 H^{5/7})^{1/2}\eta_{\mu\nu}dx^\mu dx^\nu + (b_1 H^{2/7} + b_2 H^{5/7})^{-2}dy^2 \ , \tag{4.22}$$

$$\phi = -\frac{\sqrt{15}}{7}\ln(H) \quad , \quad H = k|y| + c \ , \tag{4.23}$$

with $b_1 = \pm\frac{28m}{3k}$, $b_2 = \pm\frac{14}{15k}\sqrt{5R_5}$, and $\mu, \nu = 0, \ldots 3$ running over the domain wall worldvolume. Here k denotes the tension, and the second $\mathbb{Z}_2$-symmetric brane of opposite tension is placed at $y = \pi$ such that the topology of the full spacetime is $\mathbb{R}_4 \times S^1/\mathbb{Z}_2$. Recall that the RS model was a gravity-trapping slice of AdS_5. If we want to have a limit where pure AdS_5 is reached then we must choose $b_2 > 0$ and $b_1 < 0$ [105, 110]. Also, in order for the metric to be real, we require

$$H(y)^{\frac{3}{7}} > \mid \frac{b_1}{b_2} \mid \ , \tag{4.24}$$

which can be satisfied if the constant c is chosen appropriately. At this point one could ask why we must have a singular domain wall, not just a smooth solution to the field equations. The answer comes from the fact that supergravity domain walls are well known to only have anti-de Sitter asymptotics on one side [112], so in order to have a gravity trapping domain wall we must introduce the modulus in the harmonic function (4.23).

4.3.2 Brane Actions and Supersymmetry

The appearance of the $|y|$-term in the domain wall metric (4.23) means that we get singular terms in the Einstein tensor and in the scalar field equation [113]:

$$G_{\mu\nu} = \frac{3k}{14}(2b_1 H^{-\frac{5}{7}} + 5b_2 H^{-\frac{2}{7}})(g_{55})^{-\frac{1}{2}} g_{\mu\nu}[\delta(y) - \delta(\pi - y)] + \text{Reg} \tag{4.25}$$

$$G_{yy} = 0 + \text{Reg} \tag{4.26}$$

$$\Box\phi = -\frac{2\sqrt{15}k}{7}(b_1 H^{-\frac{5}{7}} + b_2 H^{-\frac{2}{7}})(g_{55})^{-\frac{1}{2}}[\delta(y) - \delta(\pi - y)] + \text{Reg}\ , \tag{4.27}$$

where we use Reg to denote the regular non-singular terms solving the bulk field equations (i.e. for $y \neq 0$). One notices that singular terms appear only in 4 out of the 5 diagonal components of the Einstein tensor, and that there are two contributions corresponding to b_1 and b_2. This suggests that we couple two 3-brane source terms to our action. Rather than use the Nambu-Goto action as introduced for the M2 brane in chapter 3, it will be convenient for the moment to use the equivalent Howe-Tucker action [114], given by[3]

$$\begin{aligned} S_{3-\text{brane}} \;=\; -T\int_{M_4} d^4\xi \Big[&\frac{1}{2}\sqrt{-\gamma}\gamma^{\mu\nu}\partial_\mu X^M \partial_\nu X^N g_{MN}(X) f(\phi(X)) - \sqrt{-\gamma} \\ &+ \frac{1}{4!}\epsilon^{\mu\nu\rho\tau}\partial_\mu X^M \partial_\nu X^N \partial_\rho X^P \partial_\tau X^Q A_{MNPQ}(X)\Big] \end{aligned} \tag{4.28}$$

Here T denotes the tension, ξ^μ denote the worldvolume coordinates, $X^M(\xi)$ are embedding functions and $\partial_\nu \equiv \partial/\partial\xi^\nu$, $\gamma_{\mu\nu}(\xi)$ is the worldvolume metric on the brane and the function $f(\phi(X))$ is as yet unspecified. The topological Wess-Zumino term for the 4-form $A_{[4]}(X)$ is required for consistency and represents the charge of the brane. Without this it would not be possible to satisfy the 'brane-wave' equation i.e. the equation of motion resulting from $\delta S/\delta X$.

Let us pause our discussion of brane actions to consider where $A_{[4]}$ appears from. The five-dimensional theory (4.17) arises from the dimensional reduction of the gravity plus five-form $F_{[5]}$ sector of Type IIB, which is itself a consistent truncation of the full ten-dimensional theory. The reduction of this field strength's kinetic term is trivial, and the resulting five-form in five dimensions has no continuous degrees of freedom. However the flux parameter m, which appears in the reduction ansatz, is linked to the potential for the breathing mode scalar ϕ and so the $F_{[5]}$ does have some role to play [108]. Fields strengths of this form have become known as

[3]Note that we choose $\mu, \nu, ...$ indices to denote worldvolume directions $0, 1, 2, 3$ in anticipation of the fact that we will choose the comoving gauge later.

"theory-of-almost-nothing" fields. It now seems natural to identify $F_{[5]} = dA_{[4]}(X)$, providing a bulk kinetic term for the field we introduced for consistency of the brane. However, if we dualise the kinetic term for $F_{[5]}$ in the dimensionally reduced action we would find only half of the potential $V(\phi)$ in the Lagrangian. The second term in the potential $V(\phi)$ arises from the scalar curvature R_5 of the S^5 in the compactification. For our purposes it will be useful to introduce a second "theory-of-almost-nothing" field strength $\tilde{F}_{[5]} = d\tilde{A}_{[4]}$ in five dimensions, dual to this term of the potential. Recall that there are two singular sources in the field equations (4.25)-(4.27), so we now choose one brane to couple to $A_{[4]}$ and the other to $\tilde{A}_{[4]}$. In order to completely determine these couplings, we need to calculate the junction conditions for our domain wall. Before we do so, note that as $A_{[4]}$ descends from the Type IIB four-form in ten dimensions, the brane coupling to this field will have a natural interpretation as being the dimensional reduction of the self-dual D3-brane there. We shall leave the interpretation of the second component for the moment.

Israel Junction Conditions

Let us now explicitly calculate the Israel junction conditions coming from our total five-dimensional action $S_5 = S_{\text{E.H.}} + S_\phi + S_{3-\text{brane}}$ given by (4.17) and (4.28). We will work in the comoving gauge $X^\mu = \xi^\mu$, $X^5 = (0, \pi)$, which means that there are no conditions associated with the 55 and $\mu 5$ components of the Einstein equations as $T^{55}_{brane} = 0 = T^{\mu 5}_{brane}$. The singular contribution to the Einstein tensor is given by

$$G_{\mu\nu} = -\frac{T}{\sqrt{g_{55}}}\delta(y) g_{\mu\nu} f^2(\phi) + \text{Reg} \ , \tag{4.29}$$

which we can trace-reverse to get

$$R_{\mu\nu} = \frac{T}{3\sqrt{g_{55}}}\delta(y) g_{\mu\nu} f^2(\phi) + \text{Reg} \ . \tag{4.30}$$

As the non-trivial contributions come from the $T^{\mu\nu}_{brane}$, and we can derive the associated junction conditions by integrating the $\mu\nu$ components of the Einstein equations across the brane hypersurface, where the Reg terms do not contribute [23, 115]:

$$\int_{-\epsilon}^{+\epsilon} dy\, G_{\mu\nu} = \int_{-\epsilon}^{+\epsilon} dy\, T_{\mu\nu} \ . \tag{4.31}$$

On taking the limit $\epsilon \to 0$ we find

$$[K_{\mu\nu} - K g_{\mu\nu}]^+_- = t_{\mu\nu} \ , \tag{4.32}$$

where $t_{\mu\nu}$ is the integrated stress-energy tensor defined by the $\epsilon \to 0$ limit of the right-hand side of (4.31), $K_{\mu\nu}$ is the extrinsic curvature of the brane defined by $K_{\mu\nu} = \partial_\mu X^M \partial_\nu X^N \nabla_M n_N$ and n_N is the normal to the brane. We can now use several tricks to simplify this expression. In comoving frame $\partial_\nu X^N = \delta^N_\nu$ and for a domain wall we can always go to the Gaussian normal coordinate system locally, where $g_{yy} = 1$, which means that $K = \nabla^a n_a = \frac{1}{2} g^{\mu\nu} g_{\mu\nu,y}$. Using this in conjunction with our $\mathbb{Z}_2$ condition, we can evaluate (4.31) in trace-reversed form to find

$$g_{\mu\nu,y}\big|_{y=0} = -\frac{T}{3}\sqrt{g_{55}}\, g_{\mu\nu} f^2(\phi)\big|_{y=0} \ , \tag{4.33}$$

where the $\mathbb{Z}_2$ condition means that the total value of K is related to the value of the fields on the brane, rather than its difference. A similar junction condition can also be derived for the scalar field:

$$\phi_{,y}\big|_{y=0} = 2T\sqrt{g_{55}}\, f \frac{\partial f}{\partial \phi}\big|_{y=0} \ . \tag{4.34}$$

In fact, a quicker way to this result is to realise that the only δ-function contribution to the Einstein tensor comes from the term $\frac{1}{2} g_{\mu\nu,yy}$. One can easily see that integrating this term alone across the brane hypersurface would give the same result (4.33).

The Total Action

Using our previous results, we can now write down the complete action, where now we have four brane sources in total. The sources coupled to $A_{[4]}$ will be taken to be at X^M_i $(i = 1, 2)$, with those coupled to $\tilde{A}_{[4]}$ located at $\tilde{X}^M_i$. The total action is then given by

$$
\begin{aligned}
S_5 = & \int d^5x\sqrt{-g}\,[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2\cdot 5!}e^{-8\alpha\phi}F_{[5]}^2 + \frac{1}{4\cdot 4!}e^{-\frac{16}{5}\alpha\phi}\tilde{F}_{[5]}^2] \\
& -4m\sum_{i=1}^{2} s_i \int d^5x \int d^4\xi\delta^5(x - X_{(i)})[\sqrt{-\gamma}\gamma^{\mu\nu}\partial_\mu X_i^M \partial_\nu X_i^N g_{MN}e^{2\alpha\phi} - 2\sqrt{-\gamma} \\
& +\frac{2}{4!}\epsilon^{\mu\nu\rho\sigma}\partial_\mu X^M\partial_\nu X^N\partial_\rho X^P\partial_\sigma X^Q A_{MNPQ}] \\
& +\sqrt{5R_5}\sum_{i=1}^{2} s_i \int d^5x \int d^4\xi\delta^5(x - \tilde{X})[\sqrt{-\gamma}\gamma^{\mu\nu}\partial_\mu \tilde{X}^M \partial_\nu \tilde{X}^N g_{MN}e^{\frac{4}{5}\alpha\phi} - 2\sqrt{-\gamma} \\
& +\frac{2}{4!}\epsilon^{\mu\nu\rho\sigma}\partial_\mu \tilde{X}^M\partial_\nu \tilde{X}^N\partial_\rho \tilde{X}^P\partial_\sigma \tilde{X}^Q \tilde{A}_{MNPQ}],
\end{aligned}
\tag{4.35}
$$

where $s_1 = 1$, $s_2 = -1$ give the opposing charges of the (left, right) branes of each type. For clarity, we recall that $M, N = \mu, y = 0, \ldots 3, 4$. In the following, it will suffice to take the two brane types on each side of the interval to be coincident, i.e. $X_i^M = \tilde{X}_i^M$. Before doing so, it's useful to note that the five-form equations of motion are given by

$$
\nabla_y(e^{-8\alpha\phi}F_{[5]}^{y\mu\nu\rho\sigma}) = 8m[\delta(y) - \delta(y-\pi)]\frac{1}{\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma}, \tag{4.36}
$$

$$
\nabla_y(-\frac{5}{2}e^{-\frac{16}{5}\alpha\phi}\tilde{F}_{[5]}^{y\mu\nu\rho\sigma}) = 2\sqrt{5R_5}[\delta(y) - \delta(y-\pi)]\frac{1}{\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma}\ , \tag{4.37}
$$

which have the following solutions

$$
F_{MNPQT} = 4me^{8\alpha\phi}\theta(y)\sqrt{-g}\,\epsilon_{MNPQT}\ , \tag{4.38}
$$

$$
\tilde{F}_{MNPQT} = -\frac{2}{5}\sqrt{5R_5}e^{\frac{16}{5}\alpha\phi}\theta(y)\sqrt{-g}\,\epsilon_{MNPQT}\ , \tag{4.39}
$$

where

$$
\theta(y) = \begin{cases} +1 & \text{for } 0 \le y < \pi \\ -1 & \text{for } -\pi \le y < 0 \end{cases} \tag{4.40}
$$

and we impose the upstairs-picture identification $y \sim y + 2\pi$.

Once again, we have used the brane worldvolume reparameterisation freedoms and $D = 5$ general coordinate invariance to choose a comoving gauge. One might assume that this would not be possible for the two brane system, however Gregory et al [117] have shown that in this case the comoving gauge does not over-fix the coordinate and reparameterisation gauge freedoms. Note that we are not fixing the physical positions of the branes, just the choice of coordinates by which we label the two branes. They remain free to move in the five-dimensional spacetime, with their motion now being encoded in bulk supergravity fields. A simple way to see that the branes are not over constrained by the choice of comoving gauge is to note that the brane-wave equation is not trivially satisfied.

Supersymmetry in Singular Spaces

Five-dimensional theories with singular branes have formulated in a supersymmetric way by Bergshoeff, Kallosh and Van Proeyen (BKVP) [116]. There are several key steps involved in ensuring that such a theory is well-defined. First, one must identify the behaviour of the various fields under the $\mathbb{Z}_2$ projection. Just as for the domain wall solution presented in the previous section, one finds that it is necessary to introduce a five-form "theory-of-almost-nothing" field, which flips sign across the location of the brane, exactly as was found in (4.38,4.39). The last step is to identify the scalar function in the brane action $f(\phi)$ with the superpotential. An inspection of the action (4.35) shows that the scalar function, there written as two components, is indeed the superpotential that was tentatively identified earlier (4.21).

Let us make the correspondence to the BKVP [116] formalism explicit. In order to simplify the formula, we'll consider the theory containing only two brane sources. The action is given by

$$
\begin{aligned}
S = & \int d^5x\sqrt{-g}\left(R-\frac{1}{2}(\partial\phi)^2 - V(\phi,x)\right) \\
& + \frac{1}{6}\int d^5x\ \epsilon^{MNPQR}A_{MNPQ}\,\partial_R m(x) \\
& - 8m\int d^5x\sum_{i=1}^{2} s_i\int d^4\sigma\delta^5(x-X_i)\left(\sqrt{-\gamma}\,\tilde{W}(\phi(x)) + \frac{1}{4!}\epsilon^{\mu\nu\rho\sigma}A_{\mu\nu\rho\sigma}(x)\right) \quad (4.41)
\end{aligned}
$$

where $\gamma_{\mu\nu} = \partial_\mu x^M \partial_\nu x^N g_{MN}$ ($\partial_\mu = \partial/\partial\xi^\mu$) is now understood as the pullback of the five-dimensional spacetime metric onto the worldvolume, x-denotes bulk five-dimensional coordinates and ξ denotes worldvolume coordinates. $s_{1,2} = \pm 1$ as before and now

$$V(\phi,x) = 8m(x)^2\left(\tilde{W}(\phi)^2_{,\phi} - \frac{2}{3}\tilde{W}(\phi)^2\right) , \quad (4.42)$$

$$\tilde{W}(\phi(x)) = \frac{W}{2\sqrt{2}m} = e^{4\alpha\phi(x)} - \frac{5}{2}\sqrt{\frac{R_5}{20\ m^2}}e^{\frac{8}{5}\alpha\phi(x)} . \quad (4.43)$$

Written in comoving gauge, the important equations of motion become:

$$\partial_y m(y) = 2\ m\ \delta(y) \tag{4.44}$$

$$F_{MNPQR} \equiv 5\partial_{[M}A_{NPQR]} \ = \ \sqrt{-g}\,\frac{V(x,\phi)}{2\ m(x)}\epsilon_{MNPQR}\ . \tag{4.45}$$

4.4 Proving Stability for HW Spacetimes

Having defined the theory we wish to study, let us turn to the question of stability. Our analysis of the Goldstone modes showed that in order to understand the dynamics of the system fully we should consider the full five-dimensional theory, not the effective theory on the domain wall. We shall do this by proving a positive energy theorem for this type of background, before looking at the more intuitive Hamiltonian approach. Before doing so, we should properly define the energy for these backgrounds. We shall begin by considering the smooth version of the background, before discussing the contribution of the singularities later.

4.4.1 Energy Definition for Domain Walls

Following our discussion of conserved charges in chapter 2, we know that the definition of energy is a subtle business in General Relativity. In order to define energy for our domain wall we first note that the background metric has a simple timelike Killing vector $\overline{\xi}^M$, so we can apply the Abbott-Deser technique defined previously. However, as we have a five-dimensional solution with a non-trivial scalar field we shall have to use some other tricks too, as in the Kaluza-Klein monopole example of section (2.3.2).

We begin in the usual way by making the background/perturbation split, now in both the metric and scalar field

$$g_{MN} \ = \ \overline{g}_{MN} + h_{MN}\ , \tag{4.46}$$

$$\phi \ = \ \phi^{(0)} + \phi^{(1)}\ , \tag{4.47}$$

where, once again, the superscript denotes the order of perturbation and we use $\overline{g}_{MN} = g^{(0)}_{MN}$, $h_{MN} = g^{(1)}_{MN}$. We then expand the Einstein equations as in (2.9), where now we must also include the stress-energy tensor perturbations

$$\tau_{MN} \equiv T^{(2+\text{higher})}_{MN} - G^{(2+\text{higher})}_{MN} \tag{4.48}$$

$$= R^{(1)}_{MN} - \frac{1}{2}g^{(0)}_{MN}g^{(0)RS}R^{(1)}_{RS} + \frac{1}{2}g^{(0)}_{MN}h^{RS}R^{(0)}_{RS}$$
$$-\frac{1}{2}h_{MN}g^{(0)RS}R^{(0)}_{RS} - T^{(1)}_{MN} \ , \tag{4.49}$$

where we have the usual contributions due to gravitational energy and we can explicitly calculate $T^{(1)}_{MN}$ to find

$$T^{(1)}_{MN} = \frac{1}{2}\phi^{(0)}_{,M}\phi^{(1)}_{,N} + \frac{1}{2}\phi^{(0)}_{,N}\phi^{(1)}_{,M} - \frac{1}{4}h_{MN}\phi^{(0),L}\phi^{(0)}_{,L} + \frac{1}{4}g^{(0)}_{MN}h^{RL}\phi^{(0)}_{,R}\phi^{(0)}_{,L} \tag{4.50}$$
$$-\frac{1}{2}g^{(0)}_{MN}\phi^{(0),L}\phi^{(1)}_{,L} - \frac{1}{2}g_{MN}\frac{\partial V(\phi,x)}{\partial\phi}\phi^{(1)} - g_{MN}\frac{V(\phi,x)}{m(x)}m(x)^{(1)} \ .$$

While the Bianchi identity holds to all orders

$$\left[\nabla_M G^{MN}\right]^{(n)} = 0, \tag{4.51}$$

one finds that the perturbed Einstein tensor is not divergence free with respect to the background in general, and will contribute to the covariant divergence of τ_{MN} :

$$\overline{\nabla}^M G^{(1)}_{MN} = h^{MR}R^{(0)}_{MN;R} - \frac{1}{2}h^{MR}g^{(0)}_{MN}g^{(0)LS}R^{(0)}_{LS;R} + g^{(0)MR}(\Gamma^{(1)L}_{RM}\Gamma^{(0)}_{LN} + \Gamma^{(1)L}_{RN}\Gamma^{(0)}_{LM}) \ , \tag{4.52}$$

where we have defined

$$R^{(1)}_{MN} = \frac{1}{2}g^{(0)RS}(h_{SM;NR} + h_{SN;MR} - h_{MN;RS} - h_{RS;MN}) \ , \tag{4.53}$$

$$\Gamma^{(1)R}_{MN} = \frac{1}{2}g^{(0)RS}(h_{SM;N} + h_{SN;M} - h_{MN;S}) \ . \tag{4.54}$$

We then find that τ_{MN} is not background covariantly conserved, but satisfies

$$\overline{\nabla}_M\tau^{MN} \propto \phi^{(0),N} \times [\text{linearised } \phi \text{ field equation}] \ , \tag{4.55}$$

where the [linearised ϕ field equation] is the perturbed version of the ϕ field equation (4.104),

$$\frac{1}{2}\Box^{(0)}\phi^{(1)} - \frac{1}{2}h^{RL}\phi^{(0)}_{;RL} - \frac{1}{2}h^{RL}{}_{;L}\phi^{(0)}_{,R} + \frac{1}{4}h^{R}{}_{R;}{}^{L}\phi^{(0)}_{,L} - \frac{1}{2}\frac{\partial^2 V(\phi^{(0)},x)}{\partial\phi^{(0)2}}\phi^{(1)} = 0 \ . \tag{4.56}$$

One cannot now impose this linearised matter field equation as it would be in conflict with the fact that we are also imposing the full gravity plus matter field equations. However, as in the Kaluza-Klein monopole example, we are saved by a property of our solution (4.23). As our background satisfies

$$\phi^{(0),M}\overline{\xi}_M = 0 \ , \tag{4.57}$$

we have that

$$(\overline{\nabla}_M \tau^{MN})\overline{\xi}_N = 0 \ , \tag{4.58}$$

Using the background Killing's equation

$$\overline{\nabla}_M \overline{\xi}_N + \overline{\nabla}_N \overline{\xi}_M = 0 \ , \tag{4.59}$$

we can construct the ordinarily conserved vector density

$$\overline{\nabla}_M(\sqrt{-\overline{g}}\tau^{MN}\overline{\xi}_N) = \partial_M(\sqrt{-\overline{g}}\tau^{MN}\overline{\xi}_N) = 0 \ . \tag{4.60}$$

We can then define the AD energy for the domain wall as

$$E_{\rm DW} = \int_V dV \, \sqrt{-\overline{g}} \, \tau^{0M}\overline{\xi}_M \ , \tag{4.61}$$

where dV is a 4-spatial volume element. It is straightforward to show that this is ordinarily conserved, and thus provides a good candidate for energy:

$$\frac{\partial}{\partial t}E_{\rm DW} = -\int dV \partial_i[\sqrt{-\overline{g}}(G^{(1)Mi} - T^{(1)Mi})\overline{\xi}_0] \tag{4.62}$$

$$= -[\sqrt{-\overline{g}}(G^{(1)05} - T^{(1)05})\overline{\xi}_0]_{y=0}^{y=\pi} \tag{4.63}$$

$$= 0 \ , \tag{4.64}$$

where i, j are spatial indices. The last line follows because $G^{(1)05}$ and $T^{(1)05}$ are continuous and odd under the $\mathbb{Z}_2$ symmetry and so vanish at the location of the branes. The fact that they are continuous can be explained by observing that the brane energy-momentum tensor is given by

$$T^{05}_{brane} \propto \gamma^{\mu\nu}\partial_\mu X^0 \partial_\nu X^5 \ , \tag{4.65}$$

as can seen from (4.25) and (4.26) in section (4.3.2), and so in the comoving gauge we have

$$T^{05}_{brane} = 0. \tag{4.66}$$

This holds at every order in perturbation theory, and shows that there are no singular contributions to the 05-component of the Einstein equations. Moreover this shows that the bulk energy is conserved without any contribution from the brane variables.

As discussed in the introduction, the brane dynamics can be entirely understood in terms of bulk fields – the radion and centre-of-mass Goldstone modes. This was stated more formally in terms of the junction conditions (4.32) which, for $\mathbb{Z}_2$-symmetric domain walls, explicitly become boundary conditions on bulk fields (4.33). Hence, if we can prove that the bulk energy is positive and conserved, then we have shown the stability of this class of $\mathbb{Z}_2$-symmetric domain walls, which have metrics of the form (4.23) and a superpotential relation as in (4.21).

Before we continue the proof of positivity , let us note that it can be manipulated into a total derivative, allowing us to write a surface form for the energy [1, 41]

$$E_{\rm DW} = \frac{1}{2}\int_{\partial V} d\Sigma_i(\overline{\xi}_N h^{iN;0} - \overline{\xi}_N h^{0N;i} + \overline{\xi}^0 h^{,i} - \overline{\xi}^i h^{,0} + h^{0N}\overline{\xi}_{N;}{}^{i} - h^{iN}\overline{\xi}_{N;}{}^{0} + \overline{\xi}^i h^{0N}{}_{;N} - \overline{\xi}^0 h^{iN}{}_{;N} + h\overline{\xi}^{i;0} + \overline{\xi}^0 \phi^{(0),i}\phi^{(1)} - \overline{\xi}^i \phi^{(0),0}\phi^{(1)}) \ , \quad (4.67)$$

where the semicolons denote background covariant differentiation.

4.4.2 Positive Energy from Spinors

Our analysis tells us that for HW spacetimes it suffices to consider just the bulk fields alone. As such, if we are able to show that the energy is positive at a given time, it will remain so due to the bulk field equations alone, with no contribution from the boundary X^μ variables. In this section we will prove a positive energy theorem for the HW background (4.23) by using the spinorial methods introduced in section (2.4). We saw there that this proof was more intuitive when we consider how it arises in supergravity. It is therefore useful to note that we can propose the form of the supersymmetry transformations for fermions for the supersymmetric extension of the Lagrangian (4.17) [110, 119][4]

$$\delta\psi_M \equiv \mathcal{D}_M\epsilon = [\nabla_M - \frac{1}{6\sqrt{2}}\Gamma_M W(y,\phi)]\epsilon + \text{higher order in fermions} \quad (4.68)$$

$$\delta\lambda = (\frac{1}{2}\Gamma^M\nabla_M\phi + \frac{1}{\sqrt{2}}W_{,\phi})\epsilon + \text{higher order in fermions}\ , \quad (4.69)$$

[4]In this chapter we are return to using a real representation for the gamma matrices. We have listed some useful formula in Appendix A for convenience.

where W is the putative superpotential we introduced previously (4.21). From now on we shall denote background quantities by $^{(0)}$, with the overline notation being used for the Majorana conjugate on spinors.

The proof of positivity proceeds in generally the same manner as that for asymptotically flat spaces [61, 62], but extended to theories with scalar potentials [64, 120–122]. While we have tentatively identified supersymmetry transformations for the gravitino ψ and dilatino λ, Boucher [120] has shown that supersymmetry is not actually required for proof of positive energy, but it acts as a guide to identify quantities such as (4.68), (4.69) which will prove useful.

We begin by defining the Witten-Nester energy integral

$$E_{\rm WN} = \int_{\partial V} *E \ , \tag{4.70}$$

where the integral is taken over the boundary of the spatial volume element V, and where $*E$ is the Hodge dual of the Nester 2-form $E = \frac{1}{2}E_{MN}dx^M dx^N$, defined by

$$E^{MN} = \bar{\eta}\,\Gamma^{MNP}\mathcal{D}_P\eta - \overline{\mathcal{D}_P\eta}\,\Gamma^{MNP}\eta \ , \tag{4.71}$$

where we now use η to denote the commuting spinor function that asymptotically tends to a background Killing spinor, i.e. it satisfies

$$\begin{aligned} \mathcal{D}^{(0)}_M\eta &= 0 \qquad (4.72) \\ \frac{1}{2}\Gamma^M\overline{\nabla}_M\phi^{(0)}\eta + \frac{1}{\sqrt{2}}W^{(0)}_{,\phi}\eta &= 0 \ , \qquad (4.73) \end{aligned}$$

as $r \to \infty$, where r is the appropriately defined radial coordinate. The anti-commuting supersymmetry parameter appearing in the fermion transformations (4.68),(4.69) is given by η times an anticommuting constant.

We can rewrite the energy expression as a surface integral in usual way, where now the spatial bulk volume V should be thought of as extending up to an infinitesimal distance away from the brane hypersurfaces:

$$\begin{aligned} E_{\rm WN} &= \int_V d\Sigma_M\sqrt{-g}\,\nabla_N E^{MN} \qquad (4.74) \\ &= \int_V d\Sigma_M\sqrt{-g}\,[\overline{\mathcal{D}_N\eta}\Gamma^{MNP}\mathcal{D}_P\eta + \bar{\eta}\Gamma^{MNP}\mathcal{D}_N\mathcal{D}_P\eta + \text{h.c.}] \ . \qquad (4.75) \end{aligned}$$

We then choose to foliate our spacetime in terms of spatial slices at constant times and impose the Witten condition on the initial hypersurface

$$\Gamma^k \mathcal{D}_k \eta = 0 \ . \tag{4.76}$$

It is now straightforward to express the Witten-Nester energy in terms of $\tilde{\delta}\psi_i$, $\tilde{\delta}\lambda$, where $\tilde{\delta}$ has the same action as the supersymmetry transformations, with the anticommuting spinor parameter ϵ being replaced by the commuting η. A little manipulation leads to

$$E_{\rm WN} = 2\int_V dV\sqrt{-g}\,\left[(\tilde{\delta}\psi_i)^\dagger\tilde{\delta}\psi_i + \frac{1}{2}(\tilde{\delta}\lambda)^\dagger\tilde{\delta}\lambda\right] \geq 0 \ . \tag{4.77}$$

This expression is well defined if we impose the usual fall-off constraints on metric perturbations, and also that the superpotential behaves as

$$W(\phi) \underset{\phi\to\phi^{(0)}}{\sim} \frac{1}{2l} + O(\phi^2) \ , \tag{4.78}$$

where l is the AdS scale for our solution [122], and one can check this is true for (4.21) [110]. By expanding (4.77) in fluctuations about the background we can show that it correctly reproduces the surface integral form of AD energy given in the previous section (4.67). Using the following expression for the vielbein and spin connection

$$e^P{}_a = e^{(0)P}{}_a + \frac{1}{2}h^P{}_a \ , \tag{4.79}$$

$$\omega^{(1)}_{Pab} = \frac{1}{2}(h_{Pa;b} - h_{Pb;a}) \ , \tag{4.80}$$

one finds the Killing spinor equation can expanded to give

$$\mathcal{D}_P\eta = \frac{1}{4}\omega^{(1)}_{Pab}\Gamma^{ab}\eta - \frac{1}{6\sqrt{2}}\Gamma_P W^{(0)}_{,\phi}\phi^{(1)}\eta - \frac{1}{12\sqrt{2}}h_{Pa}\Gamma^a W^{(0)}\eta + \ldots \tag{4.81}$$

This allows us to write (4.70) as

$$\begin{aligned} E_{\rm WN} = \int_{\partial V} \quad & [\frac{1}{4}\bar{\eta}\Gamma^M\eta(h^{;N} - h_P{}^{N;P}) - \frac{1}{4}\bar{\eta}\Gamma^N\eta(h^{;M} - h_P{}^{M;P}) \\ & + \frac{1}{4}\bar{\eta}\Gamma^P\eta(h_P{}^{N;M} - h_P{}^{M;N}) \\ & - \frac{1}{12\sqrt{2}}\bar{\eta}(\Gamma^{MN}h + \Gamma^{NP}h_P{}^M + \Gamma^{PM}h_P{}^N)W(\phi)\eta \\ & - \frac{1}{2\sqrt{2}}\bar{\eta}\Gamma^{MN}W_{,\phi}\phi^{(1)}\eta]d\Sigma_{MN} + h.c. \ . \end{aligned} \tag{4.82}$$

We now relate Killing spinors to Killing vectors by

$$\xi^{(0)M} = \bar{\eta}\Gamma^M \eta \ , \tag{4.83}$$

and one can then easily see that (4.82) agrees with the AD energy defined previously (4.67). Thus, we have proved the AD energy for HW spacetimes is positive, and as it was also shown to be conserved, this implies that the spacetime is stable despite the presence of the negative tension brane at $y = 0$. It's interesting to note that the analogous problem of four-dimensional negative mass Schwarzschild has also recently shown to be stable, subject to linearised perturbations of finite total energy [123].

Our analysis has shown that one can prove this completely in terms of bulk modes, independent of all brane sources. A key point here was that the branes were $\mathbb{Z}_2$-symmetric, allowing the construction of an AD energy that is conserved. It would interesting to see what happens if we relax this assumption, and we shall return to this point later.

As with the asymptotically flat case discussed in the introduction, the WN method provides an elegant proof of positive energy for the HW spacetimes. In section (2.4.2) we saw that supergravity provided some physical reason for the Witten condition: it arose from a physical gauge choice for the gravitino (2.76), (2.77). We see now how supersymmetry provides further insight into positivity, as we were able to identify the appropriate quantities which allowed the WN energy to be written as a sum of squares. In the case of a supersymmetric theory, it is the sum of squares of the supersymmetry transformations [121].

4.4.3 Positivity at Quadratic Order

An alternative way to study stability and positive energy is to study the Hamiltonian for the given theory and then consider its perturbation. This is not as rigorous as the spinorial proof as one has to deal with gauge invariance in the theory, however it is somewhat more physically intuitive and was the common way to tackle this problem before the development of Witten's proof [65]. We will now construct the Hamiltonian version of our theory (4.17) and show that it is manifestly positive at quadratic order in a particular gauge. One would imagine that any potential instability would already manifest itself at this order, hence positivity provides a strong sign that our theory is stable at all orders.

We will follow the canonical ADM approach [21, 22, 124], making an explicit $(1+4)$

decomposition of the metric, and choosing spacetime to be foliated along constant time slices Σ. For clarity we will briefly review the main features of this process (see [32] for further details).

We will denote five-dimensional bulk coordinates by X^M and coordinates on Σ by x^i. The projector onto Σ is defined as $X_i^M = \partial X^M/\partial x^i$ so that the spatial metric on Σ is given by $g_{ij} = X_i^M X_j^N g_{MN}$, and we define the normal to Σ by $n_M X_i^M$, normalised such that $g_{MN}n^M n^N = -1$. The spacetime metric can then be written as

$$ds^2 = (N_i N^i - N^2)dt^2 + 2N_i dx^i dt + g_{ij}dx^i dx^j \ , \tag{4.84}$$

where $N \equiv -n_M \dot{X}^M$ is the lapse function and $N^i \equiv X^i_M \dot{X}^M$ is the shift function. Indices must now be raised and lowered by the appropriate metric, so for instance $X^i_M = g_{MN} g^{ij} X^N_j$. A dot on top of a quantity denotes a time derivative, while $_|$ denotes covariant differentiation with respect to the 4-dimensional metric g_{ij}. The embedding of the 4-dimensional hypersurface in the 5-dimensional bulk spacetime is characterised by the extrinsic curvature K_{ij}, which for this specific metric decomposition is given by

$$K_{ij} = \frac{1}{2N}(-\dot{g}_{ij} + N_{i|j} + N_{j|i}) \ . \tag{4.85}$$

It should be noted that this is not the same extrinsic curvature that appeared previously in the Israel junction condition (4.32). The "momentum" conjugate to the metric is defined as

$$\pi^{ij} \equiv \frac{\delta\mathcal{L}}{\delta\dot{g}_{ij}} = -g_\Sigma^{\frac{1}{2}}(K^{ij} - g^{ij}K) \ , \tag{4.86}$$

and the momentum P conjugate to the scalar field ϕ is

$$P \equiv \frac{\delta\mathcal{L}}{\delta\dot{\phi}} = \frac{g_\Sigma^{\frac{1}{2}}}{N}(\dot{\phi} - N^i\phi_{|i}) \ . \tag{4.87}$$

The determinant of the metric on Σ is denoted by $g_\Sigma^{\frac{1}{2}}$. In terms of the canonical variables, we can rewrite the action as [124]

$$S = \int dt d^4x \left(\pi^{ij}\dot{g}_{ij} + P\dot{\phi} - N_i H^i - N g^{-\frac{1}{2}} H\right) \ . \tag{4.88}$$

We see that N and N_i act as Lagrange multipliers enforcing the constraints

$$
\begin{aligned}
H &= \pi^{ij}\pi_{ij} - \frac{1}{3}\pi^2 - g(^{(4D)}R - V(\phi)) + \frac{1}{2}P^2 + \frac{1}{2}gg^{ij}\phi_{|i}\phi_{|j} = 0 \quad (4.89)\\
H^i &= -2\pi^{ij}{}_{|j} + \phi^{|i}P = 0\ . \quad (4.90)
\end{aligned}
$$

At background order, these constraints are automatically satisfied by the solution (4.23), where the background Einstein equations are

$$
{}^{(4D)}R = \frac{1}{2}g^{(0)ij}\phi^{(0)}_{|i}\phi^{(0)}_{|j} + V(\phi)\ , \quad (4.91)
$$

and we have that

$$
0 = P^{(0)} = \pi^{(0)ij} = N^{(0)i} = \dot{\phi}^{(0)}\ . \quad (4.92)
$$

We then impose the constraints at linear order:

$$
\begin{aligned}
&-{}^{(4D)}R^{(1)} + \frac{\partial V}{\partial \phi}\phi^{(1)} - \frac{1}{2}h^{ij}\phi^{(0)}_{|i}\phi^{(0)}_{|j} + g^{(0)ij}\phi^{(0)}_{|i}\phi^{(1)}_{|j} = 0 \quad (4.93)\\
&2\pi^{(1)ij}{}_{|j} = \phi^{(0)|i}P^{(1)}\ , \quad (4.94)
\end{aligned}
$$

where

$$
{}^{(4D)}R^{(1)} = h^{ij}{}_{|ji} - h^{i}{}_{i|j}{}^{j} - h^{ij(4D)}R^{(0)}_{ij}\ . \quad (4.95)
$$

Written in the form above (4.88), we can easily read off the Hamiltonian

$$
\mathcal{H} = \int d^4x\,[NH + N_iH^i]\ . \quad (4.96)
$$

We know that this procedure should give the ADM energy at first order in perturbations [22, 65], so in order to study positivity we should look at second order expression. To second order in perturbations, subject to the constraints imposed at linear order, we find the Hamiltonian is given by

$$
\begin{aligned}
\mathcal{H}^{(2)} = \int d^4x \Bigg(& N^{(0)}g^{(0)-\frac{1}{2}}\left[\pi^{(1)ij}\pi^{(1)}_{ij} - \frac{1}{3}\pi^{(1)2} + \frac{1}{2}P^{(1)2}\right]\\
& + N^{(0)}g^{(0)\frac{1}{2}}\Bigg[\frac{1}{4}h^{ij|k}h_{ij|k} + \frac{1}{4}h^{i}{}_{i|}{}^{j}h^{k}{}_{k|j} - \frac{1}{2}h^{ij}{}_{|j}h_{ik|}{}^{k}\\
& - \frac{1}{12}h^{ij}h_{ij}(V + \frac{1}{2}g^{(0)kl}\phi^{(0)}_{,k}\phi^{(0)}_{,l})\Bigg]\\
& + N^{(0)}g^{(0)\frac{1}{2}}\left[\frac{1}{2}\frac{\partial^2 V}{\partial\phi^2}\phi^{(1)2} + \frac{1}{2}(\phi^{(1)|i} - h^{ij}\phi^{(0)}_{|j})(\phi^{(1)}_{|i} - h_i{}^{k}\phi^{(0)}_{|k})\right]\Bigg)\ . \quad (4.97)
\end{aligned}
$$

One can easily check that this is conserved, subject to the linearised constraints. In order to prove positivity of this expression we remove the terms of negative or undetermined sign. Much like the spinorial version of the proof, we will employ the standard technique of choosing a convenient gauge [65]. A simple counting argument tells us we can make five gauge choices, which we choose to be

$$h^{ij}{}_{|j} = 0, \tag{4.98}$$

$$P^{(1)2} = \frac{2}{3}\pi^{(1)2} + g^{(0)} \mid \frac{\partial^2 V}{\partial\phi^2} \mid \phi^{(1)2} \ . \tag{4.99}$$

We then find that the Hamiltonian reduces to a sum of positive definite terms plus a term of undetermined sign, which one can show is positive if the following inequality holds

$$V + \frac{1}{2} g^{(0)ij} \phi^{(0)}_{|i} \phi^{(0)}_{|j} \leq 0 \ . \tag{4.100}$$

For our background solution (4.23), this can be written as the following condition on the metric data

$$\frac{3k^2}{196H}(16b_1^2 H^{-\frac{3}{7}} + 20b_1 b_2 - 5b_2^2 H^{\frac{3}{7}}) \leq 0 \ . \tag{4.101}$$

One can check that this is always satisfied if the metric is real, i.e. if (4.24) holds. This means that the Hamiltonian is manifestly positive at second order in perturbations, implying that energy is positive and that our HW domain wall is stable.

4.5 Asymmetric Domain Walls

In studying the energy of the HW domain wall we found that the $\mathbb{Z}_2$ projection played a crucial role. Recall that it is exactly this projection that allowed us to show that the energy was conserved by virtue of the bulk field equations alone, i.e. without the source contributions from the branes. A natural question to ask is what happens when we relax this assumption: is the energy defined as before still conserved? Consider the action with sources given in (4.41). For the purposes of this discussion, the important field equations are

$$
\begin{aligned}
G^{MN} &= T^{MN}_{bulk} + T^{MN}_{brane} && (4.102)\\
&= \frac{1}{2}\phi^{,M}\phi^{,N} - \frac{1}{4}g^{MN}\phi^{,R}\phi_{,R} - \frac{1}{2}g^{MN}V(\phi,x)\\
&\quad + \frac{T}{2\sqrt{-g}}\int d^4\sigma\delta^5(x-X)\sqrt{-\gamma}\gamma^{\mu\nu}X^M_{,\mu}X^N_{,\nu}W(\phi)\ , && (4.103)\\
\Box\phi &= \frac{\partial V(\phi,x)}{\partial\phi} - \frac{T}{\sqrt{-g}}\int d^4\sigma\delta^5(x-X)\sqrt{-\gamma}\gamma^{\mu\nu}X^M_{,\mu}X^N_{,\nu}g_{MN}\frac{\partial W(\phi)}{\partial\phi}\ , && (4.104)\\
0 &= \partial_\mu(\sqrt{-\gamma}\gamma^{\mu\nu}X^N_{,\nu}W(\phi)) + \sqrt{-\gamma}\gamma^{\mu\nu}X^M_{,\mu}X^R_{,\nu}\Gamma^N_{MR}W(\phi) && (4.105)\\
&\quad -\frac{1}{2}\sqrt{-\gamma}\gamma^{\mu\nu}X^M_{,\mu}X^R_{,\nu}g_{MR}\frac{\partial W}{\partial\phi}\phi^{,N} - \frac{1}{4!}\epsilon^{\mu\nu\rho\sigma}\partial_\mu X^M\partial_\nu X^R\partial_\rho X^S\partial_\sigma X^T F^N{}_{MRST}\ .
\end{aligned}
$$

The final expression is the brane-wave equation, which arises from varying the action with respect to the embedding functions X. One can easily check that the energy-momentum tensor defined by (4.103) is covariantly conserved, despite the presence of the brane source terms. In defining the bulk energy it was important that the vector density constructed from the linearised energy-momentum pseudotensor was covariantly conserved (4.60). We shall try to use the same construction for the asymmetric wall. The split into background and perturbations proceeds as before, where now the pseudotensor is defined as

$$
\begin{aligned}
\tau^{MN} &= \tau^{MN}_{bulk} + T^{(1)\ MN}_{brane}\\
&= \tau^{MN}_{bulk} + \frac{T}{2\sqrt{-g}}\int d^4\xi\delta^5(x-X)\delta^{(1)}\left[\sqrt{-\gamma}\gamma^{\mu\nu}X^M_{,\mu}X^N_{,\nu}W(\phi)\right]\\
&\quad -\frac{1}{2}T^{(0)\ MN}_{brane} && (4.106)
\end{aligned}
$$

where τ^{MN}_{bulk} is the bulk energy-momentum pseudotensor defined by (4.49) and the last term comes from perturbing the bulk metric determinant factor in the brane energy momentum tensor. We use $\delta^{(1)}[\ldots]$ to denote the first order perturbation of the term in square brackets, whose explicit form is not needed for this calculation. The source terms also modify the linearised Bianchi identity, which can expanded to give

$$
\begin{aligned}
\overline{\nabla}_M G^{(1)\ MN} &= -\frac{1}{2}h^N{}_{M;R}\partial^M\phi\partial^R\phi + \frac{1}{4}h_M^{\ N;M}(\partial\phi)^2 + \frac{1}{2}h_M^{\ N;M}V(\phi,x)\\
&\quad +\frac{1}{4}h_{MR}{}^{;N}\partial^M\phi\partial^R\phi - \frac{1}{4}h^R{}_{R;N}\partial^N\phi\partial^M\phi\\
&\quad -\Gamma^{(1)\ N}_{MR}T^{(0)\ MR}_{brane} - \frac{1}{2}h^R{}_{R;M}T^{(0)\ MN}_{brane} && (4.107)
\end{aligned}
$$

Taking the covariant derivative of (4.106) we now find

$$\overline{\nabla}_M \tau^{MN} \propto \phi^{(0),N} \times \text{ [linearised } \phi \text{ field equation]} + \text{[linearised brane wave equation]} \tag{4.108}$$

Now, constructing the vector density from τ_{MN} as before (4.60) we find that the first term once again vanishes by virtue of $\phi^{(0),M}\overline{\xi}_M = 0$, however the last term remains. We then find that

$$\overline{\nabla}_M(\sqrt{-\overline{g}}\tau^{MN}\overline{\xi}_N) = \sqrt{-\overline{g}}\,\text{[linearised brane wave equation]}^N\,\overline{\xi}_N \neq 0\ . \tag{4.109}$$

and the term in brackets is the linearised version of (4.105),

$$\delta^{(1)}\left[\partial_\mu(\sqrt{-\gamma}\gamma^{\mu\nu}X^N_{,\nu}W(\phi))\right] + \Gamma^{(1)\ N}_{MR}W(\phi)\sqrt{-\gamma}\gamma^{\mu\nu}X^M_{,\mu}X^R_{,\nu} - \frac{\partial^2 W}{\partial\phi^{(0)2}}\sqrt{-\gamma}\phi^{(1)}\partial^N\phi^{(0)}$$
$$+\delta^{(1)}\left[W(\phi)\sqrt{-\gamma}\gamma^{\mu\nu}X^M_{,\mu}X^R_{,\nu}\right]\Gamma^{(0)\ N}_{MR} - \frac{\partial W}{\partial\phi^{(0)}}\sqrt{-\gamma}h^{MN}\partial_M\phi^{(0)} - \frac{\partial W}{\partial\phi^{(0)}}\sqrt{-\gamma}\partial^N\phi^{(1)}$$
$$+\frac{1}{2}\frac{\partial W}{\partial\phi^{(0)}}\sqrt{-\gamma}h^M_M\partial^N\phi^{(0)} + \frac{1}{4!}\delta^{(1)}\left[\epsilon^{\mu\nu\rho\sigma}\partial_\mu X^M\partial_\nu X^R\partial_\rho X^S\partial_\sigma X^T F^N{}_{MRST}\right] = 0 \tag{4.110}$$

This expression can be simplified by using the background equation of motion (4.45), however it remains as an $O(1)$ perturbed equation of motion. Following our earlier discussion, it would be inconsistent to invoke a first order equation of motion to show conservation at first order. As this term remains, it appears that the charge defined using the AD formalism is no longer conserved. It is interesting to see that it appears as a perturbed equation of motion, as is the case for the bulk fields. However, the expression is not multiplied by an overall factor of a background field which would have allowed the same Killing vector trick to work.

It is not clear whether this observation is physically interesting or an artifact of the definitions and calculations used here. However, it would appear that the asymmetric domain walls are considerably more complicated the $\mathbb{Z}_2$-symmetric HW domain walls we have considered above, and one can certainly not draw any conclusion about their stability from the analysis presented. Initial studies of asymmetric brane worlds seem to suggest that even linear perturbation analysis breaks down, and so much remains to be understood about these models [125, 126].

4.6 Comments on Breathing Mode Reductions

Before concluding this chapter, we shall make some comments on the breathing mode reduction [108] used to find a five-dimensional theory that has a HW domain wall solution. As we stated previously, this five-dimensional theory is a consistent truncation of the dimensional reduction of Type IIB supergravity. As such, we should expect to be able to lift the domain wall solution up to ten dimensions and identify it with some known solution. In [110], the authors initially identified the lift with a stack of positive and negative tension D-branes. A deeper examination of the singularity structure proved that in fact there was a greater contribution than one could expect from the appropriate amount of D-branes, suggesting the presence of some other objects [113].

We would like work out whether our theory (4.17) has a supersymmetric extension. As a first step we can ask if it can be realised as the bosonic subsector of a well defined five-dimensional supergravity. We have already mentioned that the bosonic theory with singular source terms included fits the known prescription for supersymmetry in singular spaces [116], however this did not include any fermions. In fact this question has already been answered for smooth domain walls by Celi et al [127]. They studied the conditions under which domain wall solutions to so-called fake supergravities can be realised as solutions to $\mathcal{N} = 2$ gauged supergravity in five dimensions. A fake supergravity is a purely bosonic theory where one identifies a putative superpotential from which one can derive the scalar potential, as we did in (4.21). One then finds constraints on the scalar field ϕ supporting the domain wall. For instance, if the domain wall is curved then ϕ must lie in a vector multiplet; whereas for a flat domain wall, ϕ must lie in a hypermultiplet [127].

Of course our five-dimensional theory arises from the S^5 compactification of Type IIB and as such, we expect it to have $\mathcal{N} = 8$ supersymmetry in five dimensions, as spherical reductions preserve all supersymmetry [128]. The details of the full nonlinear version of this reduction remain unclear. Nevertheless, we can choose to focus on just an $\mathcal{N} = 2$ sector and write down our theory there. For the breathing mode reduction described above, this has been done by Liu and Sati [119].

Given this argument for believing that our five-dimensional is indeed supersymmetric, we can now ask what the lift of our domain wall solution is. As we are mainly interested in the supersymmetry of this solution, we won't concern ourselves with stacks of branes, but just try to work out what the type of branes we have in ten dimensions. In the next section we will review the general ansatz for breathing mode reductions and domain wall solutions given in [108]. We then present a simple method to determine the dimensions of the branes from which a domain wall solution has descended, focusing on ten and eleven-dimensional examples.

4.6.1 Domain Walls from Breathing Mode Reductions

Following [108][5], we shall consider a D-dimensional theory of gravity coupled to a n-form field strength

$$e^{-1}\mathcal{L} = \hat{R} - \frac{1}{2n!}\hat{F}_n^2 \ . \tag{4.111}$$

Hatted quantities will always be D-dimensional. The reduction ansatz for the D-dimensional metric is

$$d\hat{s}^2 = e^{2\alpha\phi}\, ds_x^2 + e^{2\beta\phi}\, ds_y^2 \ , \tag{4.112}$$

$$\beta = -\frac{\alpha(d_x-2)}{d_y} \ , \tag{4.113}$$

where $D = d_x + d_y$, α and β are constants, and at this point we only assume that the compactifying space is Einstein. In order for the breathing mode scalar field ϕ to have a canonical kinetic term in d_x dimensions, we require that

$$\alpha^2 = \frac{d_y}{2(d_x-2)(d_x+d_y-2)} \ , \tag{4.114}$$

although we will just write α, rather than its numerical value. The generic ansatz for the reduction of the n-form is

$$\hat{F}_n(x,y) = F(x) \ , \tag{4.115}$$

which will be modified in the special cases of $n = d_x$ and $n = d_y$. For the latter we make the ansatz,

$$\hat{F}_n(x,y) = F(x) + m\epsilon_{d_y} \ , \tag{4.116}$$

where ϵ_{d_y} is the volume form on the compactifying space. In the case $n = d_x$ we can dualise the resulting kinetic term for the field strength to give a cosmological term, which is a precursor of the "theory-of-almost-nothing" field discussed above. The reduction ansatz is then most easily written in this dual form, with

$$F_n = me^{2(n-1)\alpha\phi}\epsilon_{d_x} \ . \tag{4.117}$$

[5]In particular, see appendices A and B of [108]

In the case where $d_x = n = d_y$, we simply take the ansatz to be the sum of these two special cases, as in the example of the self-dual field strength in Type IIB (4.14). Upon applying this ansatz, one finds that the resulting d_x-dimensional field equations can be derived from the following action

$$e^{-1}\,\mathcal{L}_x = R - \frac{1}{2}(\partial\varphi)^2 + e^{2(\alpha-\beta)\varphi}\,R_y - \frac{1}{2}\zeta\,m^2\,e^{2(d_x-1)\alpha\varphi}\ . \tag{4.118}$$

Our HW domain wall (4.23) was a solution to a theory of the form [108]

$$e^{-1}\,\mathcal{L} = R - \frac{1}{2}(\partial\varphi)^2 - V(\varphi)\ , \tag{4.119}$$

with the potential $V(\varphi)$ given by

$$V(\varphi) = \frac{1}{2}g_1^2\,e^{a_1\varphi} - \frac{1}{2}g_2^2\,e^{a_2\varphi}\ . \tag{4.120}$$

Comparing this to the general form of the dimensionally reduced Lagrangian (4.118), we find the following relations between the parameters of (4.120) and the Kaluza-Klein parameters:

$$\begin{aligned} a_1 &= 2(d_x-1)\alpha\,, \\ a_2 &= \frac{2(d_y+d_x-2)\alpha}{d_y} \end{aligned} \tag{4.121}$$

The general HW domain wall solution to this theory is given by ($\mu,\nu = 0,\ldots,d_x-1$)

$$ds^2 = e^{2A}\,dx^\mu\,dx_\mu + e^{2B}\,dy^2 \tag{4.122}$$

$$e^{-\frac{1}{2}(a_1+a_2)\varphi} = H = c + k|y|\ , \tag{4.123}$$

$$e^{4A} = e^{-B} = \tilde{b}_1\,H^{\frac{a_2}{a_2+a_1}} + \tilde{b}_2\,H^{\frac{a_1}{a_2+a_1}}.\ , \tag{4.124}$$

where $\tilde{b}_i = b_i\,(a_1+a_2)/(2k)$ and b_i are defined by

$$b_i^2 = \frac{(D-2)a_i^2\,g_i^2}{(D-2)a_i^2 - 2(D-1)}\ . \tag{4.125}$$

One can easily check that for the appropriate choice of $a_1,\ a_2$ this gives the domain wall solution (4.23).

4.6.2 Singular Sources for HW Domain Walls

Let us now consider the source terms supporting this singular solution. In [110, 113] Duff et al identified these source terms by studying the singular contributions to the stress-energy tensor. As we already know that we have a $d_x - 1$-brane in d_x dimensions (i.e. a domain wall) we shall not take this approach, but rather study the singular term in the field equation for the breathing mode scalar ϕ. Following our discussion of brane charges in chapter 3, we expect the brane to contribute a charge term,

$$\Box\varphi = V_{,\varphi} + Q_{brane}, \tag{4.126}$$

We can fix the specific form of Q_{brane} by matching to the singular terms of the left-hand side using the scalar field junction conditions, just as we did for the stress-energy tensor components in section 4.3.2. Using (4.123) and focusing now only on the terms $H_{,yy}$ that we know produce δ-functions we find

$$\Box\varphi = \text{Reg} + \frac{4k}{(a_1 + a_2)} \frac{\delta(y)}{\sqrt{g_{55}}} \left(\tilde{b}_1 \, e^{\frac{a_1}{2}\phi} + \tilde{b}_2 \, e^{\frac{a_2}{2}\phi} \right) \tag{4.127}$$

As was noted earlier for the components of the stress-energy tensor, even for one brane in d_x dimensions there are two singular contributions implying that the domain wall descends from two branes in the D-dimensional theory. We can now compare the right-hand side of (4.127) to the standard brane source terms. In $D = d_x + d_y$, the simple Nambu-Goto action is given by[6]

$$S_{brane} = -T \int d^D\hat{x} \, \delta(\hat{x} - \hat{X}) \int d^p\hat{\xi} \, \sqrt{-\hat{\gamma}}. \tag{4.128}$$

where $\hat{\gamma}$ is the determinant of the pull-back of the D-dimensional metric to the worldvolume, and the other hatted quantities are the D-dimensional versions of those appearing in (4.41). After dimensional reduction to a domain wall in d_x this becomes

$$S_{source} = -T \int d^{d_x}x \, \delta(y) \int d^p\xi \, \sqrt{-\gamma} e^{(d_x-1)\alpha\varphi + n_y\beta\varphi}, \tag{4.129}$$

where n_y is the number of brane directions that lay in the transverse space $\hat{ds}^2_y$ in D dimensions. Note that as we have chosen to consider a reduction to a domain wall in the comoving gauge, the delta function has reduced to $\delta(y)$, where y is the direction transverse to the $d_x - 1$-dimensional worldvolume.

By comparing the conformal factors of φ in (4.127) and (4.129) and using the rela-

[6]We shall set all worldvolume fields to zero.

tions in (4.121) we can determine the dimensions of the branes prior to dimensional reduction. By branes here, we specifically mean objects contributing singular terms to the field equation. We can now see that there are two branes with the following dimensions:

$$brane_1: \qquad n_y = 0 \Rightarrow d_1 = d_x - 1 \ , \tag{4.130}$$

$$brane_2: \qquad n_y = d_y - 1 \Rightarrow d_2 = d_x - 1 + d_y - 1 = D - 2 \ . \tag{4.131}$$

This assumes that both branes appear separately as domain walls in d_x, i.e. we do not allow for lower dimensional defects, such as strings, on the domain worldvolume.

Let us now return to the five-dimensional domain wall we were interested in, given by (4.23). For the appropriate choice of values above, we see that this argument suggests our domain wall descended from a 3-brane and 7-brane in Type IIB. Of course, the 3-brane was expected as one of the components of the dimensional reduction was the self-dual five form field strength supporting this solution. In [108] the domain wall metric was lifted back to ten dimensions and shown to be equivalent to that of the D3-brane after a coordinate transformation. After a careful consideration of the tension contributions from the D3-branes, it was later realised that this was not the whole story [110, 113]. The second component in the singular terms was then seen to come from the $\mathbb{Z}_2$ action on ten-dimensional flat space, and it was suggested that this may have some relation to smeared 7-branes [19, 20, 113]. However, we know that we truncated to the gravity plus five-form sector of Type IIB whereas D7-branes (and D-instantons) are supported by the axion-dilaton sector. An alternative suggestion is that the smeared 7-brane is a purely gravitational solution of the type identified in [129]. We still need to produce the appropriate topological coupling found in five dimensions in order to solve the brane-wave equation, however as this gravitational brane is not charged under any field we must resort to another coupling. One possibility is the topological coupling to the $\hat{A}$ genus, however this is easily seen to be zero for this background [130, 131]. The correct identification of this extra source term in ten dimensions and of the correct ansatz for dimensional reduction of fermions in the $\mathbb{Z}_2$-background remains work in progress.

Before we conclude, we note that the same procedure can be applied to reductions of eleven-dimensional supergravity on S^4 and S^7. The resulting domain wall solutions can be lifted back to eleven dimensions to find the $M2$ and $M5$ branes respectively. It is interesting to note that the second singular component in each case lifts to an 8-brane, however no such solution is known to exist there. We see then that the $\mathbb{Z}_2$-symmetric domain walls supported by breathing mode scalars are quite strange. One would have expected that solutions derived from eleven-dimensional supergravity to lift back to the known supersymmetric solutions there –the HW orbifold planes– however we see that is not the case.

4.7 Conclusions

We have shown that a class of domain walls in five dimensions, known as HW domain walls, are stable despite the presence of a negative tension brane. The key to the stability proof was that we were able to write the energy as a sum of squares of putative supersymmetry transformations, which had been identified previously in initial studies of the supersymmetry of the bulk solution. Although the precise form of supersymmetry for these solutions remains unclear, we saw that the background solution is as we expect for a BPS solution: it acts as a minimum energy state so that the energy of perturbations is bounded from below.

A crucial point was that we were considering $\mathbb{Z}_2$-symmetric domain walls. This allowed us to construct an energy entirely in terms of bulk quantities and we saw explicitly how the $\mathbb{Z}_2$ projection forces the Israel junction conditions to become boundary conditions on the bulk fields. The constrained nature of this spacetime then meant that the proof of positive energy was completely unaffected by the negative tension brane at one end of the interval. As soon as the $\mathbb{Z}_2$ condition is dropped, we are no longer able to construct the conserved charges in the usual way. In this case the topology changes to $R_{1,3} \times S^1$, and these perturbations are not included in our analysis.

Returning to the question of supersymmetry we considered the higher dimensional origin of our domain wall. We saw that one component could be understood as a BPS solution of the supergravity theory: the D3-brane in the case of the five-dimensional domain wall. The origin of the second component remains unclear. Regardless of the issue of supersymmetry, our analysis of $\mathbb{Z}_2$-symmetric domain walls has answered a key question that was often overlooked in brane world models.

Chapter 5

Consistency Conditions for Brane Worlds

5.1 Introduction

The resurgence of interest in extra dimensions in particle physics and cosmology spurred by the work of Randall and Sundrum, and Arkani-Hamed et al led to a huge growth in the number of new models of physics being proposed. Almost every one claimed to have verifiable signals in accelerators or other experiments. While the focus was initially on models with one extra warped dimension, it soon became apparent that these simple ideas could be extended to six-dimensional models, and higher, without requiring a firm footing in string theory or supergravity. This slew of phenomenologically motivated brane world models, with varying assumptions and approximations, prompted Gibbons, Kallosh and Linde to define a set of simple rules aimed at checking the consistency of five-dimensional models case by case [24]. In this chapter we shall review their work, before presenting a generalised version of their consistency checks, applicable to higher dimensional models and other cases. We apply our generalised sum rules to two models of particular value, and show how they offer a more robust test of consistency than some other methods. This leads us to consider the case of supergravity p-branes with more general geometries, for which we propose a generalised form of the ADM energy.

5.2 Brane World Sum Rules

In their original paper [24] Gibbons, Kallosh and Linde (GKL) used components of Einstein's equations to find simple constraints on brane world models. Their aim was to derive a set of rules for such models that are relatively model independent. They chose to concentrate on five-dimensional scenarios, similar to the Randall-Sundrum models [13, 14]. We shall reproduce their arguments here, before presenting some generalisations in the next section.

Consider D-dimensional geometries given by the following warped product metric:

$$ds^2 = W^2(y)\, g_{\mu\nu}(x)\, dx^\mu dx^\nu + g_{mn}(y)\, dy^m dy^n \ , \tag{5.1}$$

where x^μ are coordinates on the $p+1 \equiv d$ dimensional brane and y^m are coordinates in the $D - d \equiv \tilde{d} + 2$ dimensional transverse space. We assume Poincaré symmetry on the brane however unlike the standard treatment of p-brane solutions in supergravity [79], we assume no symmetry in the transverse space.

It will be most convenient to use the trace-reversed D-dimensional Einstein's equations, which are given by

$$^{(D)}R_{AB} = 8\pi G_D \left[T_{AB} - \frac{1}{D-2} g_{AB} T^C_C \right] \ , \tag{5.2}$$

and we can decompose the D-dimensional Ricci tensor components as follows,

$$^{(D)}R_{\mu\nu} = \ ^{(d)}R_{\mu\nu} - \frac{g_{\mu\nu}}{d} \frac{1}{W^{d-2}(y)} \nabla^2 (W^d(y)) \tag{5.3}$$

$$^{(D)}R_{mn} = \ ^{(\tilde{d}+2)}R_{mn} - \frac{d}{W(y)} \nabla_m \nabla_n W(y) \ , \tag{5.4}$$

where ∇^2 is the transverse space Laplacian defined by the covariant derivative with respect to g_{mn}. We will use the traced version of Ricci tensor components and in order to keep the expression as compact as possible we will suppress the functional dependence of W(y):

$$\begin{aligned} ^{(D)}R^\mu_\mu &= \ ^{(d)}RW^{-2} - \frac{1}{W^d} \left(\frac{1}{\sqrt{g}} \partial_m (\sqrt{g}) \partial^m (W^d) + \tilde{\nabla}^2 (W^d) \right) \\ &= \ ^{(d)}RW^{-2} - \frac{1}{W} \left(d\tilde{\nabla}^2 W + \frac{d(d-1)}{W} (\partial_m W)^2 + \frac{d}{\sqrt{g}} \partial_m (\sqrt{g}) \partial^m W \right) \\ ^{(D)}R^m_m &= \ ^{(\tilde{d}+2)}R - \frac{d}{W} \left(\tilde{\nabla}^2 W + \frac{1}{\sqrt{g}} \partial_m (\sqrt{g}) \partial^m W \right) . \end{aligned} \tag{5.5}$$

Here $g = \det(g_{\mu\nu})$, $\tilde{\nabla}^2 = \partial_m \partial^m$ and we have made use of the identity

$$\nabla^2 W = \frac{1}{\sqrt{g}} \partial_m (\sqrt{g}\; g^{mn} \partial_n W) = \frac{1}{\sqrt{g}} \partial_m \sqrt{g} \partial^m W + \tilde{\nabla}^2 W \; . \tag{5.6}$$

GKL chose to restrict themselves to studying singular domain wall (i.e. 3-brane) solutions to some five-dimensional theory of gravity plus matter. The action for this class of theories is

$$S = S_{\text{E.H.}} + S_{\text{mat}} - \sum_{\alpha} \int d^{p+1}x \sqrt{-\det g_{\mu\nu}}\; \lambda_\alpha(\Phi) \; , \tag{5.7}$$

where $S_{\text{E.H.}}$ is the canonical Einstein-Hilbert term, S_{mat} gives some five-dimensional matter content and the explicit term is a collection of sources for the singular branes located at positions $y = y_i$ with tensions λ_i.

GKL were interested in cases where the internal manifold, now just y, is closed, i.e. compact without boundary. If we multiply the Ricci tensor components by appropriate powers of the warp factor W and integrate over y, some terms will drop out as total derivatives. This will allow us to find further constraints on the theory that are otherwise not obvious.

Following GKL, we take the combination of Ricci tensor components

$$^{(D)}R^\mu_\mu \times (1-n) W^n +^{(D)} R^\mu_\mu \times (n-4) W^n \; . \tag{5.8}$$

Denoting ∂_y by $'$ and substituting (5.5), we then find the following expression

$$\frac{(W^n)''}{n} = \frac{(1-n)W^n}{12} \left[^{(5)}R^\mu_\mu -^{(4)} R W^{-2} \right] + \frac{(n-4)W^n}{12} \; ^{(5)}R^m_m \; . \tag{5.9}$$

If we now assume that the warp factor can be written as an exponential $W(y) = e^{A(y)}$, as in the RS models [13, 14] for example, then the left-hand side of this expression becomes a total derivative, which will vanish when integrated over a closed manifold.

Using the trace reversed Einstein's equations (5.2) we then have

$$\left(A' e^{nA} \right)' = \frac{2\pi G_5}{3} W^n \left(T_\mu{}^\mu + (2n-4) T_y{}^y \right) - \frac{1-n}{12} \; W^{(n-2)} \; ^{(4)}R \; . \tag{5.10}$$

This is the five-dimensional 'sum rule' for a general brane world model. Assuming that our internal direction is closed, we can integrate, dropping this total derivative, to find a constraint

$$\oint e^{nA}\left(T_\mu{}^\mu + (2n-4)T_5{}^5\right) = \frac{1-n}{8\pi G_5} R_g \oint e^{(n-2)A} \ . \tag{5.11}$$

One immediately sees that interesting information can be found regardless of the exact form of T_{MN} derived from the variation of $S_{\rm mat}$. For instance, choosing $n = 1$ we find a constraint on the components of T_{MN},

$$\oint e^{A}\left(T_\mu{}^\mu - 2T_5{}^5\right) = 0 \ . \tag{5.12}$$

GKL note that this constraint was proposed in [132] as a condition for the vanishing of the four-dimensional cosmological constant, i.e. for flat branes. However from this simple expression we see that it is in fact independent of the curvature ${}^{(4)}R$. Also, even if the internal manifold is not closed, it is still interesting to see that some combinations of stress-energy components give a total derivative.

5.2.1 Applications of the Sum Rules

Let us now fix the form of $S_{\rm mat}$. GKL choose a five-dimensional scalar field theory with potential

$$S_{mat} = -\int d^5x\sqrt{-G}\left(\frac{1}{2}g_{XY}\partial_N\Phi^X\partial^N\Phi^Y + V(\Phi)\right) \ , \tag{5.13}$$

where $4M^3 = (8\pi G_5)^{-1}$, and g_{XY} is the metric on the scalar moduli space with X, Y labelling the fields. For the domain wall we are interested in the fields Φ will only have dependence on the internal direction y and the spacetime scalar product with respect to g_{AB} in the kinetic term will be denoted by $\Phi' \cdot \Phi'$. Including source terms, the stress-energy components of use are

$$T^\mu_\mu = -4\left(\frac{1}{2}\Phi'\cdot\Phi' + V(\Phi) + \sum_i \lambda_i(\Phi)\delta(y-y_i)\right) , \tag{5.14}$$

$$T^5_5 = \frac{1}{2}\Phi'\cdot\Phi' - V(\Phi) \ . \tag{5.15}$$

The sum rules of interest to this theory are given by $n = 0, 1, 4$:

n=0

$$\oint\left(\Phi'\cdot\Phi' + \sum_i \lambda_i(\Phi)\delta(y-y_i) + M^3W^{-2}(y)\ R\right) = 0 \ . \tag{5.16}$$

n=1

$$\oint W(y)\left(3\Phi'\cdot\Phi' + 2V(\Phi) + 4\sum_i \lambda_i(\Phi)\delta(y-y_i)\right) = 0\ . \tag{5.17}$$

n=4

$$\oint W^4(y)\left(2V(\Phi) + \sum_i \lambda_i(\Phi)\delta(y-y_i) - 3M^3W^2(y)\ R\right) = 0\ . \tag{5.18}$$

The first constraint is particularly interesting because the potential term, which places a key role in most discussions of compactifications and brane world scenarios, drops out. For instance, if we wish to model the current period of cosmic expansion [133] with a de Sitter solution, we fix ${}^{(4)}R > 0$. Looking at the $n = 0$ constraint, we then find

$$\oint \left(\Phi'\cdot\Phi' + \sum_i \lambda_i(\Phi)\delta(y-y_i)\right) < 0\ . \tag{5.19}$$

Let us consider the first term in this expression. This is required to be positive definite for all normal (i.e. non-tachyonic) matter. In certain gauged supergravities one finds non-positive definite terms arising, however such scalars are compensator fields arising from gauging a conformal symmetry, and as such are not in the physical sector of the theory [134].

Knowing that the kinetic term must be positive definite, we then require that the dominant brane contribution must be negative: i.e. at least one negative tension brane must be present. Similarly, if we wish to model four-dimensional Minkowski (flat) space-time ${}^{(4)}R = 0$ we must also have negative tension branes present. It is interesting to note that if we consider the case of smooth domain walls, that is without any delta function support, we are led to impossible constraints for four dimensional Minkowski or de Sitter:

$$\oint \Phi'\cdot\Phi' \ = \ 0 \qquad (R=0) \tag{5.20}$$

$$\oint \Phi'\cdot\Phi' \ < \ 0 \qquad (R>0)\ . \tag{5.21}$$

This is a simple proof of the no-go theorem for smooth Randall-Sundrum solutions supported by scalar fields. Reinstating the delta function supports for two flat branes and setting all scalar fields to zero, we quickly find the original Randall-Sundrum tension matching condition from (5.16)

$$\lambda_1 = -\lambda_2\ , \tag{5.22}$$

so the branes must have equal and opposite tension. If we now fix the form of the warp factor to be $e^{2A} = e^{-2k|y|}$ and fix the scalar potential such that it acts as a cosmological constant Λ, we find the second Randall-Sundrum condition from the $n = 1$ constraint (5.17):

$$\Lambda = k\lambda_1 \ . \tag{5.23}$$

The GKL constraints on four-dimensional models arising from compactifications or brane world scenarios agree with earlier work [25–30], but are considerably simpler in their derivation. They are particularly useful in that they are derived independently of any supersymmetry in the theory, and therefore offer a simple alternative to other approaches, for instance the renormalisation group flow methods developed by Freedman et al [26].

5.3 Generalised Sum Rules

Following the construction of the five dimensional sum rules discussed in the previous section, we shall now present a generalisation to arbitrary dimensions. We begin by making an ansatz for a useful linear combination of Ricci tensor components, analogous to (5.8), for arbitrary worldvolume dimension d:

$$^{(D)}R^\mu_\mu \times (1-n)W^n +^{(D)} R^m_m \times (n-d)W^n \ . \tag{5.24}$$

Evaluating this using (5.5) leads to a rather unpleasant expression, however by using the identities

$$\frac{\tilde{\nabla}^2(W^n)}{n} = W^{n-1}\tilde{\nabla}^2 W + (n-1)W^{n-2}(\partial_m W)^2 \ , \tag{5.25}$$

$$W^{n-1}\partial^m W = \frac{\partial^m(W^n)}{n}, \tag{5.26}$$

we find it can be brought into the following compact form

$$\begin{aligned}(1-n)W^n \left[^{(D)}R^\mu_\mu -^{(d)} RW^{-2}\right] + (n-d)W^n \left[^{(D)}R^m_m +^{(\tilde{d}+2)} R\right] \\ = d(d-1)\left(\frac{\tilde{\nabla}^2(W^n)}{n} + \frac{\partial_m(\sqrt{g})}{n\sqrt{g}}\partial^m(W^n)\right).\end{aligned} \tag{5.27}$$

This is further simplified by noting that the right-hand side is nothing but the covariant Laplacian (5.6). Hence we find the following convenient form

$$\frac{\nabla^2(W^n)}{n} = \frac{(1-n)W^n}{d(d-1)}\left[{}^{(D)}R^\mu_\mu - {}^{(d)}RW^{-2}\right] + \frac{(n-d)W^n}{d(d-1)}\left[{}^{(D)}R^m_m + {}^{(\tilde{d}+2)}R\right]. \quad (5.28)$$

One can easily check that this reduces to the previous expression of GKL (5.9) when $D = 5$ and $d = 4$. We can now use the trace-reversed Einstein's equations again to get

$$\begin{aligned}\frac{\nabla^2(W^n)}{n} &= \frac{W^n}{d(d-1)} 8\pi G_D T^\mu_\mu \left[1 - \frac{(\tilde{d}+1)(2n-d)}{d+\tilde{d}}\right] - W^{n-2}\frac{1-n}{d(d-1)}\,{}^{(d)}R \\ &+ \frac{W^n}{d(d-1)} 8\pi G_D T^m_m \left[\frac{(2n-d)(d-1)}{d+\tilde{d}}\right] - W^n \frac{n-d}{d(d-1)}\,{}^{(\tilde{d}+2)}R, \quad (5.29)\end{aligned}$$

where ${}^{(d)}R$ and ${}^{(\tilde{d}+2)}R$ are the worldvolume and internal space Ricci scalars respectively. We shall refer to (5.29) as the generalised sum rules.

At this point it is worthwhile commenting on the overlap between our work and that of other authors. After completing this work it was realised that an equivalent form of (5.29) had been found previously by Leblond, Myers and Winters [135]. We shall not repeat their analysis here, but comment on differences with our work. Having derived the general form of the sum rules, the authors of [135] chose to focus on a specific six dimensional example, the *AdS Soliton* [136]. Being a codimension two object, the integral of the transverse space curvature gave the Euler characteristic for this space. While this is an example of some interest (in the AdS/CFT correspondence for instance) it is worthwhile noting that it is singular, i.e. the delta function sources remain. Indeed, Leblond et al produce an illuminating discussion of delta function sources in curved spaces. They also choose to focus on compact transverse spaces, allowing the removal of the total derivative term as before. We shall not stipulate such a condition in our analysis.

The most common reference on constraints on compactifications in supergravity is the Maldacena-Nunez no-go theorem [30]. Simply put, this states that it is not possible to produce de Sitter or Randall-Sundrum type compactifications of supergravity. We shall now briefly show how one can derive this theorem from the generalised sum rules presented above. In fact, we shall see how this approach has some shortcomings when compared with the GKL constraints.

The metric ansatz of Maldacena-Nunez [30] is

$$ds^2 = W^2(y)\Big(g_{\mu\nu}(x)\, dx^\mu dx^\nu \ + \ g_{mn}(y)\, dy^m dy^n \Big) \,. \tag{5.30}$$

In order to compare with the GKL-inspired generalised sum rules, we must conformally rescale our original ansatz (5.1) $g_{mn} \to W^2(y) g_{mn}$. The worldvolume Ricci tensor component (5.3) then becomes

$${}^{(D)}R_{\mu\nu} = \ {}^{(d)}R_{\mu\nu} - g_{\mu\nu}\left(\frac{\nabla^2 W(y)}{W(y)} + (D-3)\frac{(\nabla W(y))^2}{W(y)^2} \right) \,, \tag{5.31}$$

where we have made use of the fact that under this transformation the Laplacian scales as

$$\nabla^2 W(y) \to \frac{\nabla^2 W(y)}{W^2(y)} + \tilde{d}\, \frac{W^3(y)}{(\nabla W)^2} \,. \tag{5.32}$$

Tracing over $g_{\mu\nu}$, we can we rewrite this expression in the form found in [30]:

$$\frac{1}{(D-2)W(y)^{(D-2)}}\nabla^2 W(y)^{D-2} = \ R + W(y)^2 \left(-T^\mu_\mu + \frac{d}{D-2}(T^\mu_\mu + T^m_m) \right) . \tag{5.33}$$

We can now see that any no-go theorem derived from this expression will correspond to one sum rule i.e. the one choice of $n = d$ in (5.29). This initially appears to be trivial. However, it corresponds to disregarding contributions from certain stress-energy components. For example, the $n = 0$ rule in five dimensions (5.16) does not contain a potential term and can provide a valuable constraint upon other components of the theory. It is useful to remember at this stage that we are dealing with Einstein's equation, and therefore must insist upon the consistency of all possible linear combinations of its components. In the next section we will explicitly see how the extended sum rules can show constraints on the theory that would be hidden if we relied only on the Maldacena-Nunez no-go theorem.

5.4 Applications of the Generalised Sum Rules

We shall now concentrate on two examples which can display the power of the generalised sum rules. We will begin by discussing the most straightforward extension of the original GKL results to include a non-compact internal manifold. The example we choose is a smooth domain wall solution to $\mathcal{N} = 2$ five-dimensional gauged supergravity with general matter content [137, 138]. In order to understand if the sum rules are of use in higher dimensions, our second example will focus on a six-dimensional analogue of Melvin's magnetic solution [140–142], where the spacetime is of the form $M_{1,3} \times M_2$. By allowing the internal manifolds to be noncompact, we shall be able to study the behaviour of the warp factor in both examples. For the domain wall this means we will be able to understand if the sign of the tension plays any role in allowing a smooth solution.

5.4.1 Domain Walls Don't Need Singular Sources

In their work on RG flow equations in five-dimensional gauged supergravity with simple matter content [27], Kallosh and Linde showed that it was not possible to have a smooth domain wall interpolating between two different infra-red critical points. Phrased differently, this says that there are no domain wall solutions to these theories in which the warp factor smoothly interpolates between two minima, as in the Randall-Sundrum model. Behrndt and Dall'Agata found an explicit solution circumventing this constraint by coupling five-dimensional supergravity to more general forms of matter [137, 138]. In particular, they found that the introduction of hypermultiplets plays a crucial role. The extra scalars then present modified the isometry structure of the coset manifold, which in turn changed the critical point structure, which then allowed the identification of a smooth domain wall solution [137]. It is reasonable to assume that the more complicated field content of this theory will allow the no-go theorem to be avoided, however the sum rules show that the geometry also plays a crucial role[1].

For our purposes it will suffice to consider just the bosonic sector of the general matter coupled Lagrangian, given by [2]:

$$\mathcal{L}_{\text{bosonic}} \;=\; R - \frac{1}{4}F_{AB}F^{AB} - \frac{1}{2}g_{XY}\mathcal{D}_A q^{X}\mathcal{D}^A q^{Y} - g^2\,\mathcal{V}(y,q).$$

For clarity we note that the covariant derivative on the scalar manifold is defined as

$$\mathcal{D}_B q^{X} = \partial_B q^{X} + g\,A_B k^{X}(q)\,, \tag{5.34}$$

where $k^{X}(q)$ is the gauged isometry's Killing vector and A_B is a vector field. The

[1] Further work on the domain wall structure of such supergravity theories has followed [139].

[2] Our choice of signature is different to that used in [137] and we have dropped the topological term.

scalar potential $\mathcal{V}$ is defined by

$$\mathcal{V} = -4P^r P^r + \frac{3}{4} k^X k^Y g_{XY}(q) . \tag{5.35}$$

where P^r is the prepotential and g_{XY} is the metric on the scalar manifold. The stress-energy tensor derived from the Lagrangian (5.34) is

$$T_{AB} = F_{AC} F_B{}^C + \frac{\delta_{Ay}\delta_{By}}{2} q' \cdot q' - g_{AB} \left(\frac{1}{4} F_{CD} F^{CD} + \frac{1}{2} q' \cdot q' + g^2 \mathcal{V} \right) , \tag{5.36}$$

where we have used $F_{yA} = 0$ and also that the scalar field supporting the domain wall will only have dependence on the transverse direction, so $\partial_A q = \partial_y q = q'$. The exact form of the metric solution for the domain wall will not be required.

To apply the sum rule arguments of the previous section, it will be useful to note the following combination of T_{AB} components

$$T^\mu_{\ \mu} = F_{\mu\nu} F^{\mu\nu} - 4 \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} q' \cdot q' + g^2 \mathcal{V} \right) \tag{5.37}$$

$$T^y_{\ y} = \frac{1}{2} q' \cdot q' - g^2 \mathcal{V} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} . \tag{5.38}$$

As we expect a domain wall solution to this theory to be BPS we can set ${}^{(4)}R = 0$ and the general sum rule (5.10) becomes

$$\left(A(y)' e^{nA} \right)' = \frac{2\pi G_5}{3} \, e^{nA} \left((n-4) q' \cdot q' - \frac{n}{4} g^2 \mathcal{V} - \frac{(n-2)}{2} F_{\mu\nu} F^{\mu\nu} \right) . \tag{5.39}$$

We see that this expression has the same form as the GKL sum rule, as expected – the $n = 0$ and $n = 4$ conditions have no contributions from the scalar potential and kinetic terms respectively. However, we no longer assume that our space is compact so the integral of the left-hand side does not vanish.

Let us first consider the $n = 0$ condition. Following standard no-go theorem techniques [30] and using that the field strengths satisfy $F^2 < 0$, we find the sum rule has a simple form

$$A'' = -4 q' \cdot q' - |F^2| \qquad \Rightarrow \; A'' < 0 . \tag{5.40}$$

Integrating this expression we find $A(y)'|_{-\infty}^{+\infty} < 0$, which for one symmetric domain wall means $A(y)' < 0$. Recall that for a domain wall space-time, Israel junction conditions imply that the tension is given by the change in extrinsic curvature across the wall. Using Gaussian normal coordinates and the $\mathbb{Z}_2$-symmetry then means the tension is entirely determined by the warp factor change across the domain wall.

An inspection of the junction conditions shows this means the domain wall has positive tension. We can now use this in conjunction with the other sum rules to try to constrain the scalar potential. Looking at the $n = d = 4$ rule we find

$$-|A''| + 4(A')^2 = -g^2\mathcal{V} + |F^2| \qquad \Rightarrow \; 4(A')^2 + g^2\mathcal{V} > 0 \; . \tag{5.41}$$

One sees that this inequality is obviously satisfied if $\mathcal{V} > 0$. However, it can also be satisfied for $\mathcal{V} < 0$ if $|\mathcal{V}| < 4(A')^2$. It is now worth recalling the Strong Energy Condition (SEC), which states that

$$\left(T_{AB} - \frac{1}{D-2} g_{AB} T^C_C \right) u^A u^B \geq 0 \; , \tag{5.42}$$

for non-spacelike vectors u. This can rewritten in terms of the trace-reversed stress-energy tensor (5.2), and in a local frame amounts to stating that all matter is attractive [25]. As stated in the original no-go theorem [25], the vector field A will obey the SEC and so any violation must be generated by the scalar field. For example, a positive scalar potential, such as a positive mass term ($\mathcal{V}(\phi) = \frac{1}{2}m^2\phi^2$) for a minimally coupled scalar ϕ field, will violate the SEC.

The Maldacena-Nunez no-go theorem assumes that the SEC holds for all fields, and in particular it states that any scalar potentials must be negative. Behrndt and Dall'Agata suggested that they were able to circumvent this no-go theorem by having a scalar potential that could become positive at some point [137], violating the SEC. However with a careful treatment of the brane world constraints for this theory we have seen that this is not necessary. It is possible that a smooth domain wall solution exists with $\mathcal{V} < 0$. Our result shows an extra requirement - that the domain wall must have positive tension, which, as we have seen previously in (5.19), is not obvious. For instance, for singular domain wall spacetimes with Minkowski or de Sitter worldvolumes, we specifically need negative tension branes. Had we employed the Maldacena-Nunez no-go theorem without the assuming the SEC holds, we would not have found a constraint independent of the scalar potential and the behaviour of the warp factor would have been unclear. In fact by setting $F_{CD} = 0$, one can show that it is exactly the non-compact nature of the transverse space that allows for the smooth domain wall solution, without requiring the scalar potential to violate the SEC.

5.4.2 Higher Dimensions and Melvin's Solution

Having shown how the sum rules give constraints upon the geometry as well as the field content of domain wall solutions, it is now natural to ask if this can be extended to higher codimension spacetimes. In [135] Leblond et al have studied singular branes in six dimensions with compact internal spaces. They concluded that negative source terms were not required if one has codimension two branes. In fact, we have already shown that it is possible to remove source terms in codimension one by having a transverse space with boundary.

We shall now use the sum rules to study smooth 3-brane solutions in six-dimensional Einstein-Maxwell theory. Scalar fields will not play a role as we are no longer considering domain wall solutions and we know the vector field obeys the SEC, hence it will be the geometry of the two-dimensional transverse space and the warp factor that are important.

The action for this theory is

$$S = \int d^6X\sqrt{-g}\left\{\frac{1}{2\kappa^2}\left(\hat{R} - \frac{\Lambda}{2}\right) - \frac{1}{4}F_{AB}F^{AB}\right\}. \tag{5.43}$$

Solutions to the equations of motion derived from this action have been constructed analytically by Wiltshire and collaborators [141, 142], and are given by

$$ds^2 = W^2(r)\, g_{\mu\nu}(x)\, dx^\mu dx^\nu + d\Omega^2_{(2)}\ , \tag{5.44}$$

$$\mathbf{F} = \frac{B}{\kappa r^4}\, dr \wedge d\varphi\ , \tag{5.45}$$

where r, φ are radial and angular coordinates, respectively, on the transverse 2-space $\mathcal{M}$ with metric $d\Omega^2_{(2)} = g_{mn}dy^m dy^n$ and κ is the six dimensional Newton's constant. While not essential for our considerations, the worldvolume is assumed to be Einstein:

$$R_{\mu\nu} = \lambda g_{\mu\nu}\ , \tag{5.46}$$

with λ now being the Gaussian curvature. The transverse space metric is defined by a function of Λ, λ, B and r, the zeros of which determine the different geometries. The only restrictions placed on the transverse space geometry is that it is geodesically complete and we are free to consider spaces with boundary. In [141] it was shown that if one were to consider this codimension two model with a two dimensional worldvolume, Melvin's solution [140] arises as one particular choice of coefficients.

The evaluation of the sum rule constraints is considerably simplified by the fact that this model considers a monopole configuration for the gauge field i.e. the only non-trivial component is F_{mn}, which is spacelike. Without loss of generality we will take $W(r) > 0$.

The appropriate contractions of the stress-energy tensor derived from (5.43) are

$$T^{\mu}_{\mu} = \frac{1}{4\pi} F_{mn} F^{mn} - \frac{\Lambda}{2\pi G} \tag{5.47}$$

$$T^{m}_{m} = \frac{1}{8\pi} F_{mn} F^{mn} - \frac{\Lambda}{4\pi G} \ , \tag{5.48}$$

which we can insert into the generalised sum rule (5.29) with $D = 6$ and $d = 4$ to find

$$\begin{aligned} \frac{\nabla^2 W^n(r)}{n} &= \frac{1}{6} W^n(r) \pi G \left[(5n - 14) F_{mn} F^{mn} + \left(\frac{n+2}{2} \right) \Lambda \right] \\ &+ \left(\frac{n-1}{12} \right) W^{n-2}(r) \ R \ + \ \left(\frac{4-n}{12} \right) W^n(r) \tilde{R} \ . \end{aligned} \tag{5.49}$$

Written in this form, we again see that particular choices of n will give constraints between different components of the theory. We shall now consider various cases in turn, and the simplest to begin with is a flat brane with vanishing bulk cosmological constant.

Flat Brane $R = 0$ and $\Lambda = 0$

In this case we see that choosing $n = 4$ in (5.49) leaves a relation between only the warp factor and F^2, which is positive definite as we know that the field strength is spacelike. Multiplying (5.49) by $W^4(r)$ and integrating by parts we find

$$\int_M \nabla_m (W^4(r) \ \nabla^m W^4(r)) \ - \ \int_M \ (\nabla W^4(r))^2 \ > \ 0 \ , \tag{5.50}$$

from which we find that

$$\int_M \ \nabla_m (W^4(r) \ \nabla^m W^4(r)) > 0 \Rightarrow \int_{\partial M} \ W^7(r) \ n_m \nabla^m W(r)) > 0 \ , \tag{5.51}$$

where n_m is the unit normal at the boundary (∂M) of the transverse space, which we choose to be positive. As we have shown that $\nabla W(r) > 0$, we can use this in the $n = 14/5$ sum rule, in which the field strength term drops out, to show that $\tilde{R} > 0$, i.e. the transverse space must have positive curvature.

Flat Brane $R = 0$ and $\Lambda \neq 0$

Allowing the bulk cosmological constant to be non-zero introduces some extra freedom into the sum rules. In particular, from the $n = 4$ sum rule we can now derive two cases of interest:

$$a)\ \nabla W(r) < 0 \quad \text{if} \quad 0 < GF^2 < \Lambda/2$$
$$b)\ \nabla W(r) > 0 \quad \text{if} \quad GF^2 > \Lambda/2$$

From $b)$ we can once again show that $\tilde{R} > 0$ by using the $n = -2$ sum rule, however little more can be found from $a)$.

Flat Brane $R = 0$ and Compact Transverse Space

It is interesting to study the possibility of smooth solutions of this sort with a compact transverse space. Leblond et al [135] have shown that negative tension source terms are not required for flat branes with codimension two, unlike the domain walls (codimension one) discussed by GKL. We shall now attempt to understand whether such solutions can be extended to smooth branes.

As $\partial M = 0$, integration of the generalised sum rules (5.29) now leads to the vanishing of the left-hand side, leaving constraints independent of the behaviour of the warp factor. It will be useful to list the sum rules of interest to understand how each component is constrained in turn:

n=4

$$\int_M W^4(r)\ G\ F^2 - \int_M W^4(r)\frac{\Lambda}{2} \ = \ 0 \qquad \rightarrow \qquad \Lambda \ > \ 0. \tag{5.52}$$

n=1

$$\int_M 6W(r)\ G\ F^2 \ = \ \int_M W(r)(\tilde{R} - \Lambda) \qquad \rightarrow \qquad \tilde{R} \ > \ 0. \tag{5.53}$$

n=14/5

$$\int_M 6W(r)^{14/5}\ \tilde{R} - \int_M 16\ W(r)^{14/5}\ \Lambda = 0 \qquad \rightarrow \qquad \tilde{R} \ = \ 4\ \Lambda \tag{5.54}$$

So we see that unlike the five dimensional Lagrangian considered in the previous sections, this model does allow a consistent smooth brane solution with compact transverse space. The only requirements being that both the bulk cosmological constant and the transverse space curvature are positive.

This model has provided us with some useful insights into the construction of smooth brane solutions. It was particularly important that we had a monopole compactification of the $U(1)$ vector field, which meant that we only dealt with positive definite field strength terms. Unfortunately as soon as we consider more general scenarios we lose all ability to constrain the various terms. Indeed for the case of curved branes we find that although there are still values of n that lead to simpler sum rules, we cannot fix the signs of the worldvolume and internal curvatures, even if we set $\Lambda = 0$. By fixing the values of each parameter in turn, it is possible to show agreement with the analytic curved worldvolume solutions in [142], but this is of little value.

Before concluding, we present a summary of our results in the following table,

D	$\partial\mathcal{M}$	${}^{(4)}R$	$\tilde{R}$	$\mathcal{V}(\phi)/\Lambda$	$\partial W(r)$
5	$\neq 0$	0	0	$\mathcal{V}(\phi) > 0$	> 0
6	$\neq 0$	0	> 0	$\Lambda = 0$	> 0
6	$\neq 0$	0	> 0	$\Lambda > 2GF^2 > 0$	> 0
6	$\neq 0$	0	?	$2GF^2 > \Lambda$	< 0
6	$= 0$	0	$\tilde{R} = 4\Lambda$	> 0	?

Table 5.1: Constraints on smooth 3-branes in five and six dimensions.

5.5 Comments on Generalised p-Branes

In chapter 3 we described a method to construct the one charge p-brane solutions to various supergravity theories. Focusing on the solutions to eleven-dimensional supergravity, we saw how the M2-brane had a timelike singularity which one could introduce a delta-function source to support. The M5 brane was completely non-singular and required no source terms. In ten dimensions one similarly finds that the self-dual 3-brane solution is completely non-singular, with all other branes having timelike or conical singularities. The common feature of all these branes is that their transverse spaces are asymptotically flat, however, this is nothing more than a simplifying assumption to aid in finding solutions. One can consider the more general case, where it transpires that the spaces need only be Ricci flat, rather than Riemann flat, in order to solve the equations of motion [79, 143–146]. Such solutions are known as generalised p-branes. One can then ask whether the brane solutions have their singularities resolved, or smoothed out, by the new geometry, in much the same way as we found that smooth domain wall solutions exist if the transverse space is not compact.

Explicit examples of generalised p-branes have been constructed by many authors. We shall choose to concentrate on a those given by Pope et al [147], which have the following form of metric

$$ds^2 = H(r)^{\frac{-\tilde{d}}{D-2}}\eta_{\mu\nu}(x)\,dx^\mu dx^\nu + H(r)^{\frac{d}{D-2}}g_{mn}(y)\,dy^m dy^n \tag{5.55}$$

$$H(r) = 1 + \frac{k}{r^{\tilde{d}}}\ , \tag{5.56}$$

where x^μ are coordinates on the $p+1 \equiv d$ dimensional brane and y^m are coordinates in the $D-p-1 = q$ dimensional Ricci-flat transverse space with metric g_{mn}. It will be more convenient to consider the following general metric ansatz for a p-brane with a q-dimensional transverse space (c.f. (3.15))

$$d\hat{s}^2 = e^{2\alpha\varphi}d\tilde{s}_q^2 + e^{2\beta\varphi}ds_{p+1}^2\ , \tag{5.57}$$

where $d\tilde{s}_q^2 = \tilde{g}_{mn}(y)dy^m dy^n$ and $ds_{p+1}^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$ are generic metrics on the $q = D-p-1$ and $p+1$ dimensional spaces respectively, and $\varphi = \varphi(y)$. To distinguish between the different parts of the geometry we will denote D-dimensional quantities with hats and q-dimensional quantities with tildes. Defining $\xi \equiv (q-2)\alpha + (p+1)\beta$, we then note the Ricci tensor decomposition is given by [3]

$$\hat{R}_{\mu\nu} = R_{\mu\nu} - \beta g_{\mu\nu}\ e^{2(\beta-\alpha)\varphi}\left(\nabla^2\varphi + \xi(\partial\varphi)^2\right)\ , \tag{5.58}$$

$$\begin{aligned}\hat{R}_{mn} &= \tilde{R}_{mn} - \xi\left(\nabla_m\partial_n\varphi + \alpha\tilde{g}_{mn}(\partial\varphi)^2\right) - \alpha\tilde{g}_{mn}\nabla^2\varphi \\ &\quad + \left(\alpha\xi + (\alpha\beta - \beta^2)(p+1)\right)\partial_m\varphi\partial_n\varphi\ ,\end{aligned} \tag{5.59}$$

where $R_{\mu\nu}$ and $\tilde{R}_{mn}$ are the Ricci tensor components on the $p+1$ and q-dimensional manifolds respectively. Tracing these we find

$$\begin{aligned}\hat{R} &= e^{-2\alpha\varphi}R + e^{-2\beta\varphi}\tilde{R} \\ &\quad - e^{-2\alpha\varphi}\left(e^{-\xi\varphi}\nabla^2 e^{\xi\varphi} + (\alpha(q-1) + \beta(p+1))\nabla^2\varphi + \left[q\alpha^2 + (p+1)\beta^2\right](\partial\varphi)^2\right).\end{aligned} \tag{5.60}$$

One can easily check the sum rule Ricci tensor components (5.3) and (5.4) are

[3] This choice of parameterisation was suggested by J. Kalkkinen, whom we thank for discussions on this point.

reproduced with the following choice:

$$\alpha = 0 \ , \quad \beta = 1, \quad \varphi(y) = \ln W(y) \ . \tag{5.61}$$

Pope et al constructed various p-brane solutions with general Ricci flat transverse spaces, all with metrics of the above form with $\xi = 0$. The key point is that these spaces admit suitable harmonic forms leading to modified Bianchi identities, and can smooth out any singularities in the full geometry if they are normalisable [147]. For example, consider the ten-dimensional Heterotic theory, whose bosonic field equations can be derived from the following Lagrangian,

$$\mathcal{L}_{\rm het} = \hat{R}\,\hat{*}1 - \frac{1}{2}\hat{*}d\phi\wedge d\phi - \frac{1}{2}e^{-\phi}\hat{*}F_{(3)}\wedge F_{(3)} - \frac{1}{2}e^{-\frac{1}{2}\phi}\hat{*}F_{(2)}\wedge F_{(2)} \ . \tag{5.62}$$

The field strengths are defined as

$$F_{(3)} = dA_{(2)} + \frac{1}{2}A_{(1)}\wedge A_{(1)} \ , \tag{5.63}$$

$$F_{(2)} = dA_{(1)} \ , \tag{5.64}$$

with the Bianchi the identity for $F_{(3)}$

$$dF_{(3)} = \frac{1}{2}F_{(2)}\wedge F_{(2)}. \tag{5.65}$$

The ansatz for the resolved 5-brane solution to this theory is,

$$d\hat{s}_{10}^2 = H^{-1/4}\,dx^\mu\,dx^\mu\,\eta_{\mu\nu} + H^{3/4}ds_4^2\,,$$
$$e^{-\phi}\,\hat{*}F_{(3)} = d^6x\wedge dH^{-1}\,, \qquad \phi = \frac{1}{2}\log H\,, \qquad F_{(2)} = m\,L_{(2)}\,, \tag{5.66}$$

where $L_{(2)}$ is a normalisable two-form on the transverse Ricci-flat four-manifold, which can be chosen to be self-dual or anti-self-dual. We could now study the sum rules using the Ricci tensor components given above, however as we have the exact solutions at our disposal we can begin by looking directly at the equations of motion. On inserting the ansatz, we find the equations of motion reduce to

$$\hat{\nabla}^2 H(y) = -\frac{1}{4}m^2\,L_{(2)}^2\,. \tag{5.67}$$

An inspection of the gravitino and dilatino supersymmetry transformations show the Ricci-flat four-manifold must also be Kähler with an appropriate orientation if supersymmetry is to be preserved by this more general background. Two examples of appropriate four-manifolds are the Eguchi-Hanson and Taub-NUT gravitational instantons, both of which can be described by the Gibbons-Hawking metric [58, 147, 148]. In fact, for all generalised p-branes supported by some Ricci-flat transverse one finds that the field equations reduced to the form (5.67) with an appropriate n-form replacing $L_{(2)}$. For normalisable forms $L_{(n)}$ one finds the solutions become completely smooth.

Unfortunately, when we apply the sum rules to such brane spacetimes we learn little new. The more general geometry supporting the n-form $L_{(n)}$ allows the brane singularity to be resolved. A straightforward integration of (5.67) tells us that $\partial_y H(y)|_\infty < 0$, fixing the overall sign of the non-constant term in $H(y)$. The significance of this is entirely dependent upon the particular example under question and no general constraints can be found. However, it is clear that the modified geometry of such generalised branes will effect the definition of conserved charges, to which we now turn.

5.5.1 Energy for Generalised p-Branes

While the sum rules offer little insight into generalised branes, it will be interesting to consider what can be said about their conserved charges. As such, we shall now present a formal definition of ADM-like energy for the generalised branes with metrics of the form (5.55). Following Deser et al [21, 41, 55], we assume the existence of a simple timelike vector and then define the Killing energy for a p-brane as

$$\mathcal{E} = \frac{1}{2\kappa^2} \int_{\partial \mathcal{M}_{\mathrm{T}}} d^{D-d-1} \Sigma^m (\overline{D}^n h_{mn} - \overline{D}_m h^a_a) \ , \tag{5.68}$$

where $\overline{D}$ is the covariant derivative with respect to the general background metric and the h's are asymptotic fluctuations about this background. The integral is taken over the surface defined in the transverse space to the brane. Once again, lower case Latin indices run over all spatial directions $a, b = i, j \ldots m, n$, where i, j run over spatial worldvolume directions. One can see that this is the covariantised version of the usual p-brane energy integral (3.32) and agrees with the general Deser-Tekin expression (2.24). One can easily show that this expression also gives the correct expression for the Kaluza-Klein monopole that was discussed in chapter 2, agreeing with Deser-Soldate result (2.44).

Let us now calculate the energy for the 5-brane examples, where we choose the transverse four-manifold to be the Eguchi-Hanson instanton. The metric on the four-manifold is given by

$$ds_4^2 = W(r)^{-1}dr^2 + \frac{1}{4}r^2 W(r)\left(d\psi + \cos\theta d\phi\right)^2 + \frac{1}{4}r^2 d\Omega_2^2 \ ,$$
$$W(r) = 1 - \frac{a^4}{r^4} \ , \tag{5.69}$$

where $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$, ψ has period 2π and the radial coordinate r lies in $a \leq r \leq \infty$. This space is asymptotically locally Euclidean and the condition on ψ means that the topology is $\mathbb{R} \times S^3/\mathbb{Z}_2$.

The appropriate normalisable 2-form on the space is defined by

$$L_{(2)} = r^{-3}dr \wedge (d\psi + \cos\theta d\phi) + \frac{1}{2}r^{-2}\sin\theta d\theta \wedge d\phi \ . \tag{5.70}$$

Assuming the "harmonic" function H only has dependence on r, the field equations reduce to

$$\left(r^3 W H'\right)' = -\frac{4m^2}{r^5} \ , \tag{5.71}$$

where $' = \partial/\partial r$. This can be solved to give

$$H(r) = 1 + \frac{m^2}{2a^4 r^2} \ , \tag{5.72}$$

after an appropriate choice of integration constants. For the purposes of calculating the Killing energy, we see that the metric is locally flat and so we can identify the perturbations just as in the original p-brane case (3.33)

$$h_{mn} = \frac{3m^2}{8a^4r^2}\delta_{mn} + O(\frac{1}{r^4}) \quad , \quad h_{ij} = -\frac{m^2}{8a^4r^2}\delta_{ij} \ . \tag{5.73}$$

Substituting into the energy integral we then find

$$\mathcal{E} = \frac{m^2}{a^4}\Omega_{S^3/\mathbb{Z}_2} \ , \tag{5.74}$$

where $\Omega_{S^3/\mathbb{Z}_2}$ is half the volume of the unit three-sphere. This result agrees with that for a p-brane with an asymptotically flat transverse space up to the half volume factor due the different asymptotic topology. Pope et al have shown that this solution preserves supersymmetry and so we expect that the Bogomol'nyi bound still holds with a suitable defined magnetic charge.

The Eguchi-Hanson example is useful as it is possible to calculate the energy explicitly. It would be of interest to consider cases that are not asymptotically transverse flat or locally flat. Unfortunately, such cases often lead to non-divergent integrals, as in the example of the Heterotic 5-brane on the Taub-NUT instanton [147]. It may be possible to circumvent this problem by using the methods of Barnich et al [149], whereby one calculates the energy integral over a finite path in solution space rather than asymptotically. The nature of these modified charges of generalised branes remains unclear and merits further investigation. In particular, it would be interesting to construct the full set of charges for such branes and check their "black brane" mechanics and Smarr relations [86].

5.6 Conclusions: What can Sum Rules tell us?

In their original work GKL [24] showed how simple manipulations of the Einstein equations could provide constraints on components of five-dimensional brane world models that would otherwise remain hidden without a detailed study of supersymmetry. We have provided a natural extension of these arguments to compactifications of higher dimensional theories and to more general internal manifolds. We showed how the evasion of previous no-go theorems for smooth Randall-Sundrum domain walls in five dimensions did not rely on the scalar field potential violating the Strong Energy Condition. We found that the potential could be negative if it was sufficiently small and if the transverse direction was non-compact. This means that the domain wall must have positive tension. This may initially seem like an obvious statement, however in the supergravity literature much is made of the necessity of negative tension objects in realisations of Randall-Sundrum models. In [150], Giddings et al showed that negative tension brane contributions were required if one wanted to find a RS type solution to compactified type IIB theory. Using the Maldacena-Nunez no-go theorem, they showed that a negative contribution was needed on the right-hand side of the traced Einstein's equations for consistency. They noted that this could be generated by negative tension D3 brane sources, or equivalently by D7 branes wrapped on appropriate four-cycles of the compactifying manifold. We have seen in this chapter that it is often more useful to extend the Maldacena-Nunez arguments to other constraints arising from the Einstein equations (the $n \neq d$ sum rules) and to consider non-compact transverse spaces. It would interesting to reconsider the constraints of [150] for the non-compact Calabi-Yau's [151], which are known to localise gravity, as the generalised sum rules could offer new insights into these models. We leave this for future work.

As a next step in testing the generalised sum rules we considered the Einstein-Maxwell theory in six dimensions. Although this had been used previously by other authors in generalised sum rules, we saw there was more interesting information to be found. From a simple analysis of the generalised sum rules we found that one can have a smooth 3-brane solution with compact transverse space, unlike the five-dimensional case. The requirements for this solution were that both the bulk cosmological constant Λ and the transverse space curvature ${}^{(2)}R$ must be positive. As originally discussed by Leblond et al [135], this latter constraint means that the Euler characteristic for the internal manifold must positive.

The Melvin-type solutions in six dimensions were interesting as they appeared as solitons supported magnetically by a vector potential (i.e. with no flux linking their worldvolume). This was reminiscent of the p-brane solutions of supergravity discussed in Chapter 3 and led us to consider generalised p-branes with non-flat transverse space geometry. Many examples of such branes appear in the literature, however we found that little could be learnt about their general features from the sum rule arguments. One common feature of solutions found by Pope et al was that the "harmonic" function should be asymptotically decreasing. We found that it was interesting to reconsider the Killing energy for such generalised branes, and provided the appropriate energy integral for this class of backgrounds (5.68). Considering the example of the Heterotic 5-brane with an Eguchi-Hanson transverse space, we saw how the modified geometry changed the result for the ADM energy. It would be extremely interesting to extend this to other conserved charges for such branes and consider even more general asymptotics. Also, we should consider the thermodynamics and stability of these general backgrounds to further understand how they relate to the familiar extremal branes. We leave this for future work.

Chapter 6

Branes on Generalised Backgrounds

6.1 Introduction

We shall now depart from studying the braneworld models of the previous two chapters and consider the formal question of finding the constraints placed on D-branes and geometry in generalised supersymmetric *flux* compactifications of Type II (i.e. $\mathcal{N} = 2, D = 10$) supersymmetric string theories to four dimensions. Such compactifications have received much attention recently, and may play an important role in the search for more realistic models derived from string theory. Their study has led to the development of significant links with Hitchin's work on generalised geometry - the study of manifolds deformed by fluxes - which has in turn provided more fundamental insights into the physics of these models (see [5] for a review).

Before discussing this work in detail, we shall give a brief overview of the ideas used in this chapter and our main results. One way to approach the problem of finding constraints on the geometry in supersymmetric compactifications is to the study the structure group on the compact six-dimensional manifold Y, i.e. the group of transformations of the tangent bundle T_Y, and the corresponding invariant tensors and spinors. We shall refer to such constraints as *supersymmetry conditions*. As we shall review in the next section, some amount of preserved supersymmetry implies that the structure group, which is SO(6) on an oriented six-dimensional Riemannian manifold, is reduced to some subgroup. We shall see that choosing to have the minimal amount of supersymmetry preserved in four dimensions, $\mathcal{N} = 2$, corresponds to the internal manifold Y having SU(3) structure. Compactifications with flux hope to reach more phenomenologically interesting models by reducing this to $\mathcal{N} = 1$ supersymmetry. It transpires that there is a natural extension of these ideas to flux compactifications, whereby one studies the structure group of the direct sum of the tangent and cotangent bundles $T_Y \oplus T_Y^\star$. We shall see how supersymmetric vacua can be described in terms of two O(6,6) *pure spinors* $\Psi^\pm$, which can also be understood as formal sums of forms $\Psi^\pm = \sum_k \Psi^\pm_{(k)}$, where k is even for Ψ^+ and odd for Ψ^- [157, 158].

This formalism introduces a natural relation between flux compactifications and generalised complex geometry [159–161]. The two pure spinors are associated to *generalised almost complex structures* whose (generalised) integrability corresponds, in turn, to non-closure of the pure spinors under the twisted derivative operator $d_H = d + H\wedge$, where H is a form-field on Y. In [158] it has been shown that the supersymmetry conditions provide the integrability of the almost complex structure associated to one pure spinor and that it defines a twisted generalised Calabi-Yau (CY) structure à la Hitchin [159] on the internal manifold. On the other hand, the second pure spinor is not integrable ($d_H\Psi \neq 0$) due to the presence of Ramond-Ramond (RR) field-strengths which act as an obstruction to integrability. For example, if we restrict ourselves to the case where the structure group is reduced to SU(3), the internal manifold will be either symplectic (IIA) or complex (IIB)[1]. In the more general SU(3) × SU(3) case, the manifold is a complex-symplectic hybrid, even if IIA and IIB continue to "prefer" symplectic and complex manifolds respectively [158]. Here we should understand that $\mathrm{SU}(3) \times \mathrm{SU}(3) \subset \mathrm{SU}(3,3) \subset$ O(6,6), where the structure is initially reduced to $\mathrm{SU}(3,3)$ by the existence of two invariant spinors, and then further reduced to $\mathrm{SU}(3)\times\mathrm{SU}(3)$ if the spinors are 'compatible' (in a sense we shall define later), or equivalently if a generalised metric structure exists on $T_Y \oplus T_Y^\star$ [161].

In the following sections we will see how it is possible to characterise the supersymmetric D-brane configurations completely in terms of the two pure spinors for a general class of $\mathcal{N} = 1$ backgrounds. We will mainly focus on the case of branes filling the flat four-dimensional space-time, however our results may be extended to other cases, where branes appear as defects in four dimensions. The resulting equations [see equations (6.78) and (6.79)] represent the generalisation to $\mathcal{N} = 1$ flux backgrounds of the conditions obtained in [162, 163] for branes wrapped on cycles of a Calabi-Yau threefold CY_3. This can be seen from the form these conditions take once we restrict to the $SU(3)$ case [see equations (6.93) and (6.94)],

[1] Here we use the standard notation of IIA and IIB for the ten-dimensional $\mathcal{N} = 2$ non-chiral and chiral theories respectively.

which can be considered as formally the closest to the CY case[2]. A *supersymmetric cycle* is defined to be any cycle which can be wrapped by a Dp-brane satisfying the supersymmetry conditions. We shall see how the supersymmetry conditions split into two parts involving the two pure spinors $\Psi^{\pm}$ and are completely symmetric under the exchange $\Psi^{+} \leftrightarrow \Psi^{-}$ as one goes from Type IIA backgrounds to Type IIB and vice-versa. This symmetry can be seen as a generalisation of the usual mirror symmetry between supersymmetric cycles on standard CY's.

The first supersymmetry condition for a space-time filling D-brane wrapping an n-cycle on the compact manifold Y can be written in the form

$$\Big\{P[(g^{mk}\imath_k + dx^m\wedge)\Psi]\wedge e^{\mathcal{F}}\Big\}_{(n)} = 0 \;, \tag{6.1}$$

where $\mathcal{F} = f + P[B]$ (f is the world-volume field-strength on the D-brane), $P[.]$ indicates the pullback on the worldvolume of the brane and Ψ is equal to Ψ^{-} in IIB and Ψ^{+} in IIA. On the left-hand side of this expression it should be understood that we only consider forms of rank equal to the dimension n of the wrapped cycle. For each case, IIA and IIB, these pure spinors are exactly the integrable ones and we will discuss how this condition means that supersymmetric cycles are generalised complex submanifolds with respect to the appropriate integrable generalised complex structure $\mathcal{J}$, as defined in [160]. We then understand that branes satisfying (6.1) wrap an appropriate generalisation of a complex submanifold in Type IIB and of coisotropic submanifolds in Type IIA, with this identification becoming precise in the $SU(3)$-structure case. This result is analogous to that found in [166], where D-branes on supersymmetric backgrounds with only nontrivial Neveu-Schwarz (NS) fields are considered (for previous work on branes in the context of generalised complex geometry see [167–171]).

The second supersymmetry condition is related to the stability of the D-brane and can be written as

$$\Big\{\mathrm{Im}\big(iP[\Psi]\big)\wedge e^{\mathcal{F}}\Big\}_{(n)} = 0 \;, \tag{6.2}$$

where now Ψ is equal to Ψ^{+} in IIB and Ψ^{-} in IIA (i.e. the non-integrable pure spinor in each case).

The two conditions (6.1) and (6.2) imply that for a suitable choice of orientation

[2]Equivalent conditions have recently been presented for D-branes on IIB $SU(3)$-structure backgrounds in [164], where several interesting applications to the warped Calabi-Yau subcase [165] are also discussed.

on the wrapped cycle, the D-brane configuration is supersymmetric. Note that as we are considering backgrounds with nontrivial RR fluxes turned on, reversing the orientation on the brane does not generally preserve supersymmetry.

The above two conditions can be rephrased in terms of a single condition which also encodes the necessary orientation requirement. For a D-brane wrapping an internal n-cycle, this is given by

$$\Big\{\mathrm{Re}\big(-iP[\Psi]\big)\wedge e^{\mathcal{F}}\Big\}_{(n)} = \frac{||\Psi||}{8}\sqrt{-\det(g+\mathcal{F})}d\sigma^1\wedge\dots d\sigma^n \ , \tag{6.3}$$

where again Ψ is equal to Ψ^+ in IIB and Ψ^- in IIA, and $||\Psi||^2 = \mathrm{Tr}(\Psi\Psi^\dagger)$. This condition will be identified as a calibration condition with respect to an appropriate *generalised calibration* $\omega = \sum_k \omega_{(k)}$, with $\omega_{(k)}$ being a k form, which is twisted closed by definition, i.e. $d_H\omega = 0$, and must fulfil a condition of minimisation of the D-brane energy density. More specifically, for any space-time filling D-brane wrapping any internal cycle Σ and with any worldvolume field strength $\mathcal{F}$ (such that $d\mathcal{F} = P_\Sigma[H]$), we must have

$$P_\Sigma[\omega]\wedge e^{\mathcal{F}} \leq \mathcal{E}(\Sigma,\mathcal{F}) \ , \tag{6.4}$$

where $\mathcal{E}$ represents the energy density [see equation (6.108)]. Once again, on the left-hand side we mean that only forms of rank equal to the dimension of the wrapped cycle are considered. An analogous definition of generalised calibration has recently been used in [166] for the case with only nontrivial NS fields, and our result represents an extension of that proposal in the presence of non-zero RR fluxes.

The remainder of this chapter is organised as follows. In (6.2) we shall review supersymmetric constraints on compactifications with and without flux, introducing the mathematical notation of generalised geometry used throughout the subsequent sections. In section (6.3) we introduce the basic conditions defining the general class of $\mathcal{N}=1$ backgrounds we are considering. In section (6.4) we derive the supersymmetry conditions for supersymmetric D-branes using κ-symmetry and express them in terms of the pure spinors $\Psi^\pm$ characterising our backgrounds. In sections (6.5), (6.6) and (6.7) we clarify the meaning of the conditions for the internal supersymmetric cycles, identifying them as generalised complex submanifolds calibrated with respect to the appropriate definition of generalised calibration. Some basic properties of the almost complex structure and (3,0)-form constructed from an internal spinor on a SU(3) structure manifold are presented in Appendix B.

6.2 Supersymmetry and Compactification

Let us begin by reviewing some basic concepts about the reduction of Type II string theory to four dimensions on manifolds without flux [153, 154]. We will focus mainly on the supersymmetry of such compactifications and study the problem at the level of the low-energy effective theory, namely supergravity. We shall not burden ourselves with the details of the reduction of the bosonic sectors of these theories which follow the standard Kaluza-Klein law [7]. Instead, we shall concentrate on the fermions and their supersymmetry transformations, which will be of most importance when we consider D-branes later. Focusing on the gravity sector, we recall that that the Rarita-Schwinger field, here called the gravitino ψ_M, has a supersymmetry transformation given by

$$\delta\psi_M = \nabla_M \varepsilon = 0 \ , \tag{6.5}$$

where ε is an infinitesimal anti-commuting parameter, ∇_M is the usual covariant derivative on spinors (A.2) and the indices run over $M, N = 0, \ldots, 9$.

Consider a ten-dimensional solution of the form $M = M_4 \times Y$, where M_4 is a maximally symmetric four-dimensional spacetime and Y is a compact Riemannian manifold. If we wish to have some fraction of supersymmetry preserved after compactifying our theory on Y, the supersymmetry transformation (6.5) tells us that we need one covariantly constant spinor for each unbroken supersymmetry. The standard integrability condition on spinors (A.23)

$$[\nabla_M, \nabla_N]\varepsilon = \frac{1}{4} R_{MNPQ}\Gamma^{PQ}\varepsilon = 0 \ , \tag{6.6}$$

then tells us that M_4 must be flat Minkowski spacetime [153]. Looking at (6.5) again, we see that ε must be independent of the coordinates on M_4,which means that the integrability condition (6.6) then implies that there must be a covariantly constant spinor on Y, known as a Killing spinor. This has two consequences. The first is the existence of a globally defined nowhere vanishing spinor on Y, implying that the structure group on Y is reduced. The second is that this spinor is covariantly constant, which means that M_4 is flat, as we have already seen, but also that Y is Ricci-flat[3]. However, this second condition also places further constraints on the compactifying manifold Y. In order to have a better understanding of these points, it will be worthwhile to review some details of the geometry of

[3] Y is Ricci-flat, rather than flat, as we make no assumption of maximal symmetry.

compactifications[4].

Recall that the vielbeins e_A^M define a local orthonormal frame on a manifold M, with the set of all such frames over M being known as the frame bundle. The frame bundle is the principle fibre bundle naturally associated the tangent bundle T_M over M [155]. The structure group can be understood as the group of transformations required to sew together the frames over the entire manifold. On an oriented Riemannian d-dimensional manifold M this group is $\mathrm{SO}(d)$. The manifold M is said to have reduced structure if this structure group is not $\mathrm{SO}(d)$, but one of its subgroups $G \subset \mathrm{SO}(d)$. If a manifold has a reduced structure group, one can prove that there exists a globally defined tensor which is covariantly constant with respect to the connection on the reduced structure bundle i.e. that there exists a tensor which is invariant under G-transformations [156]. Conversely, one can use the existence of a globally defined G-invariant tensor, or spinor, on M to prove that its structure group is reduced. It is worth noting that this is not unfamiliar. Had we not assumed that our manifold was Riemannian from the outset, the structure group would have been $\mathrm{GL}(d, \mathbb{R})$, which is the generic structure group on a frame bundle. By stating that the manifold possessed a metric which is covariantly constant with respect to the metric connection on the frame bundle, i.e. by stating that the manifold is Riemannian, we had reduced the structure group from $\mathrm{GL}(d, \mathbb{R})$ to $\mathrm{O}(d)$. This is then further reduced to $\mathrm{SO}(d)$ if the manifold is orientable. Furthermore, one can prove that there exists a unique, torsion-free metric connection - the familiar Levi-Civita connection - with the manifold then being defined to have $\mathrm{O}(d)$ holonomy. Generally, if the connection on a principle bundle over M is torsion-free, then the manifold is said to have G-holonomy, rather than G-structure.

We now appreciate that the existence of a covariantly constant spinor on Y implies that it has reduced G-structure. For the compactification we consider here, $G \subset \mathrm{SO}(6)$, where we have initially decomposed the ten-dimensional structure group as

$$\mathrm{SO}(1,9) \to \mathrm{SO}(1,3) \times \mathrm{SO}(6) \ , \tag{6.7}$$

but it is not yet clear what the subgroup G of $\mathrm{SO}(6)$ is. In order to determine this, we should provide some more details about the ten-dimensional theory in question.

The Type II theories have two real spinors, corresponding to the $\mathcal{N} = 2$ supersymmetry in ten dimensions. In Type IIA these spinors have opposite chirality and are denoted by $\varepsilon^{\pm}$, whereas in Type IIB they have same chirality which

[4] We refer the reader to [155, 156] for further definitions and proofs of the topics discussed here.

we can choose to be positive. Ten-dimensional spinors transform in the **16** representation of SO(6)[5], which splits into $(\mathbf{2},\mathbf{4}) + (\overline{\mathbf{2}},\overline{\mathbf{4}})$ under the decomposition $\mathrm{SO}(1,9) \to \mathrm{SO}(1,3) \times \mathrm{SO}(6)$. Concentrating now on Type IIA, we can write this decomposition as

$$\varepsilon^{\pm} = \xi^{\pm} \otimes \eta \ , \tag{6.8}$$

where $\xi^{\pm} \in \mathrm{SO}(1,3)$. Here we have chosen the minimal case of one internal spinor $\eta \in G \subset \mathrm{SO}(6)$, such that the two generic four-dimensional spinors $\xi^{\pm}$ give rise to the eight supercharges of $\mathcal{N} = 2$ supersymmetry in four dimensions. We know from the supersymmetry transformation (6.5) that η must be covariantly constant, and therefore it must lie in a singlet of the reduced structure group G. As the Lie algebra of SO(6) is isomorphic to SU(4), we can choose to look for subgroups of SU(4). An appropriate choice is SU(3), under which the **4** of SU(4) decomposes to $\mathbf{3} + \mathbf{1}$, thus allowing $\eta \in \mathbf{1}$. As stated above, one can prove that the existence of a covariantly conserved spinor implies that the structure group is reduced from SO(6) to SU(3), or a subgroup thereof. For example, if there were two independent covariantly constant spinors one would have SU(2) structure.

On further study of the implications of SU(3) holonomy [153], one finds that it is possible to construct a 2-form ϕ and $(3,0)$-form Ω using the spinor η. One can identify ϕ with the familiar fundamental form of complex geometry and from it construct the corresponding almost complex structure J, which is a map from from the tangent bundle onto itself, obeying $J^2 = -\mathbb{1}$. One can prove that J is integrable, thus providing a complex structure, and so one sees that the manifold is complex. The second form Ω can then be shown to be holomorphic with respect to the complex coordinates on the manifold, and the metric g can be shown to be Hermitian i.e. $g(u,v) = g(Ju, Jv)$ for all vector fields u, v on Y.

A complex manifold with a covariantly constant complex structure, as we have here by construction, is known as a Kähler manifold[6]. Thus for our six-dimensional internal manifold Y, the existence of a covariantly constant spinor η means that Y is a complex, Ricci-flat Kähler manifold i.e. Y is a Calabi-Yau manifold [153].

As an aside, we note that the statement that the manifold is Kähler holds if the reduced holonomy group is $\mathrm{U}(3) = \mathrm{SU}(3) \times \mathrm{U}(1)$. However, one can prove that the Ricci tensor of Kähler manifold defines the field strength of the U(1) part of the spin connection. As this vanishes for a Ricci-flat manifold, the holonomy group

[5]More concretely, the spinors lie in representations of Spin(1,9), the spin cover of ten-dimensional Lorentz group, which can be decomposed into $\mathrm{Spin}(1,3) \times \mathrm{Spin}(6)$, corresponding to (6.7).

[6]For equivalent definitions of a Kähler manifold see [155] and proposition 4.4.2 of [156].

U(3) is further reduced to SU(3). This can be rephrased formally in terms of the vanishing of the first Chern class of Y. For thorough treatment of these results on compactifications and the mathematics of reduced holonomy manifolds we refer the reader to [153, 156].

We have seen that by requiring our compactification to preserve the minimal amount of supersymmetry and to result in a maximally symmetric four-dimensional manifold M_4, the supersymmetry condition (6.5) forced M_4 to be flat Minkowski spacetime and the internal manifold Y to be Calabi-Yau. We understood this by realising that the existence of a covariantly constant spinor on Y meant that the holonomy group of Y was reduced from SO(6) to SU(3), which in turn implied that Y was a Ricci-flat Kähler manifold.

In this chapter we aim to consider D-branes on manifolds with reduced structure groups, rather than holonomy groups, i.e. manifolds with torsion [5, 152, 154, 157, 158]. In particular, we shall study the geometry of supersymmetric D-branes in a very general class of supergravity backgrounds preserving four-dimensional Poincaré invariance and $\mathcal{N} = 1$, rather than $\mathcal{N} = 2$, supersymmetry. Such backgrounds correspond to warped products of four-dimensional Minkowski space-time and an internal six-dimensional manifold M with general fluxes turned on, which we shall often refer to simply as generalised backgrounds. $\mathcal{N} = 1$ supersymmetry requires the existence of four supercharges, which correspond to four independent ten-dimensional Killing spinors, whose most general form can be written in terms of two internal six-dimensional Weyl spinors $\eta_+^{(1)}$ and $\eta_+^{(2)}$, analogous to (6.8). The flux on the internal manifold means that these spinors are no longer covariantly constant with respect to the Levi-Civita connection, and as such the corresponding forms constructed from the spinors are no longer closed. While not necessary for the discussion we present here, it is worth noting that the non-closure of the complex structure J and the holomorphic three-form Ω constructed from the spinors provides an elegant description of how far a G-structure manifold is from being a G-holonomy manifold in terms of intrinsic torsion classes [154, 156]. The analysis is involved, but one finds general constraints on Type II flux compactifications preserving $\mathcal{N} = 1$ supersymmetry; the internal manifold is constrained to be complex in IIB and symplectic (plus one complex case) in IIA (see [5], for instance, for further details).

The spinors $\eta_+^{(1)}$ and $\eta_+^{(2)}$ both define an SU(3) structure, each with a corresponding almost complex structure, which we by denote J_1 and J_2 respectively. If the spinors are parallel ($\eta_+^{(1)} \propto \eta_+^{(2)}$), then we have the case of SU(3) structure, analogous to torsion-free Calabi-Yau case described above. If the spinors are nowhere parallel, then then structure is further reduced to SU(2). However, one can also have the more general case of the two spinors becoming parallel only in some patches, which is complicated to describe.

There exists a unified language in which to treat these various cases, first suggested by Hitchin [159] and developed by other authors [160, 161], in which one considers the sum of the tangent and cotangent bundles over Y, $T_Y \oplus T_Y^\star$, and then demands that there is a SU(3) $\times$ SU(3) structure on the associated principle bundle over it. The familiar concepts of complex geometry can be extended to this bundle, defining *generalised complex geometry*. For instance, a generalised almost complex structure $\mathcal{J}$ is defined as a map of $T_Y \oplus T_Y^\star$ onto itself such that $\mathcal{J}^2 = -\mathbb{1}$, and obeys the Hermiticity condition $\mathcal{J}^t \delta \mathcal{J} = \delta$, where $\delta = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is the natural metric on $T_Y \oplus T_Y^\star$[7].

Spinors lie in representations of the Spin(6,6) cover group, although often one refers to the related representation of the Clifford algebra, denoted Clifford(6,6), which can be defined in terms of matrices λ^m, ρ_n obeying the following algebra [157, 158]

$$\{\lambda^m, \lambda^n\} = 0 \quad , \quad \{\lambda^m, \rho_n\} = \delta^m_n \quad , \quad \{\rho_m, \rho_n\} = 0 \; , \tag{6.9}$$

where δ^m_n is the $6+6$-dimensional metric on $T_Y \oplus T_Y^\star$, described above, and $m, n = 1, \ldots, 6$. One can also find a representation of this algebra in terms of forms using

$$\lambda^m = dx^m \wedge \quad , \quad \rho_n = \iota_n \; , \tag{6.10}$$

where $\iota_n \equiv \iota_{\partial_n} = dx^{a_1} \wedge \ldots \wedge dx^{a_p} = p\delta_n^{[a_1} dx^{a_2} \wedge \ldots \wedge dx^{a_p]}$ is the familiar contraction $\iota_n : \Lambda^p T^\star \to \Lambda^{p-1} T^\star$. A spinor is defined to be *pure* in six dimensions if there exist six linear combinations of λ^m, ρ_n which annihilate it. Defining an appropriate vacuum state, one can construct forms of all degrees from the complementary set of matrices. A generic Clifford(6,6) spinor can then be understood as a formal sum of forms, with positive and negative chirality spinors in Majorana-Weyl representation of Clifford(6,6) corresponding to sum of forms of even and odd degree respectively [160]. In fact, many of these results are extensions to $T_Y \oplus T_Y^\star$ of well-known work on spinors and forms on $T_Y^\star$, reviewed in [173].

An important element of this construction is the Clifford map, relating Clifford(6,6)

[7] See, for instance, [5] for further details with applications to supersymmetric compactifications.

spinors to Clifford(6) bispinors [157, 161]:

$$\chi \equiv \sum_k \frac{1}{k!} \chi^{(k)}_{m_1 \dots m_k} dx^{m_1} \wedge \dots \wedge dx^{m_k} \longleftrightarrow \not{\chi} \equiv \sum_k \frac{1}{k!} \chi^{(k)}_{m_1 \dots m_k} \hat{\gamma}^{m_1 \dots m_k} \ , \quad (6.11)$$

where $\hat{\gamma}$ are Clifford(6) gamma matrices satisfying the algebra $\{\hat{\gamma}^m, \hat{\gamma}^n\} = g^{mn}$ and g_{mn} is the metric on Y. For example, on a manifold with SU(3) structure we know there is a nowhere vanishing spinor, from which we can construct two nowhere vanishing bispinors in the following way

$$\not{\Psi}^+ = \eta_+ \otimes \eta_+^\dagger \quad , \quad \not{\Psi}^- = \eta_+ \otimes \eta_-^\dagger \ . \quad (6.12)$$

We can rewrite these in terms of the formal sums of forms

$$\eta_+ \otimes \eta_\pm^\dagger = \frac{1}{4} \sum_k \frac{1}{k!} \eta_\pm^\dagger \hat{\gamma}^{m_1 \dots m_k} \eta_+ \hat{\gamma}^{m_k \dots m_1} \ , \quad (6.13)$$

using the Fierz identities. One can act on the left and right of a bispinor with the six-dimensional gamma matrices

$$\overrightarrow{\gamma^m} = (\lambda^m + g^{mn} \iota_n) \quad , \quad \overleftarrow{\gamma^m} = \pm(\lambda^m - g^{mn} \iota_n) \ , \quad (6.14)$$

where we have dropped the $\hat{}$'s and the $\pm$ depends on the parity of the bispinor upon which $\overleftarrow{\gamma^m}$ acts. It also is useful to note the inverse relation

$$\lambda^m \chi \longleftrightarrow \frac{1}{2}(\overrightarrow{\gamma^m} \not{\chi} \pm \not{\chi} \overleftarrow{\gamma^m}) \ , \quad (6.15)$$

where once again the $\pm$ depends on the parity of the bispinor $\not{\chi}$. Using this, one can check that the SU(3) spinors (6.12) are pure, with each being annihilated by six Clifford(6,6) gamma matrices; three acting on the left and three on the right of the bispinor [158],

$$(\delta + iJ)_m^{\ n} \gamma_n \eta_+ \otimes \eta_\pm^\dagger = 0 \quad , \quad \eta_+ \otimes \eta_\pm^\dagger \gamma_n (\delta + iJ)_m^{\ n} = 0 \ , \quad (6.16)$$

where J is the almost complex structure on T_Y. Two pure spinors are said to be compatible if they share three annihilators in six dimensions, which in this case are provided by the three acting on the left. Furthermore, one can prove that there is a one-to-one correspondence between a pure spinor and a generalised complex structure associated $\mathcal{J}$, where the later are constructed from the familiar spinors defining the reduced structure in the usual way.

In the next section we will see how the supersymmetry conditions for flux compactifications with SU(3) × SU(3) can be written in a compact elegant form using this pure spinor formalism.

6.3 Basic results on $\mathcal{N} = 1$ vacua

We are interested in Type II warped backgrounds preserving four-dimensional Poincaré invariance and $\mathcal{N} = 1$ supersymmetry, with the most general fluxes and fields turned on. The ansatz for the ten dimensional metric g_{MN} is

$$ds^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + g_{mn}(y)dy^m dy^n \ , \tag{6.17}$$

where x^μ, $\mu = 0, \ldots, 3$ label the four-dimensional flat space, y^m, $m = 1, \ldots, 6$, the internal space, and $M, N = 0, \ldots, 9$. Let us introduce the modified RR field strengths

$$F_{(n+1)} = dC_{(n)} + H \wedge C_{(n-2)} \ , \tag{6.18}$$

where $dC_{(n)}$ are the standard RR field strengths[8]. In order to preserve four dimensional Poincaré invariance we can write

$$F_{(n)} = \hat{F}_{(n)} + Vol_{(4)} \wedge \tilde{F}_{(n-4)} \ . \tag{6.19}$$

The relation $F_{(n)} = (-)^{\frac{(n-1)(n-2)}{2}} \star_{10} F_{(10-n)}$ between the lower and higher rank field strengths translates into a relation of the form $\tilde{F}_{(n)} = (-)^{\frac{(n-1)(n-2)}{2}} \star_6 \hat{F}_{(6-n)}$ between their internal components. The ten dimensional gamma matrices $\Gamma_{\underline{M}}$ (underlined indices correspond to flat indices) can be chosen in a real representation and decomposed in the following way

$$\Gamma_{\underline{\mu}} = \gamma_{\underline{\mu}} \otimes \mathbb{1} \qquad , \qquad \Gamma_{\underline{m}} = \gamma_{(4)} \otimes \hat{\gamma}_{\underline{m}} \ , \tag{6.20}$$

where the four-dimensional gammas $\gamma_{\underline{\mu}}$ are real and the six-dimensional ones $\hat{\gamma}_{\underline{m}}$ are anti-symmetric and purely imaginary. The four- and six-dimensional chirality operators are given respectively by

$$\gamma_{(4)} = i\gamma^{\underline{0123}} \qquad , \qquad \hat{\gamma}_{(6)} = -i\hat{\gamma}^{\underline{123456}} \ , \tag{6.21}$$

[8] We will essentially follow the conventions of [157, 158], up to some differences consisting in a sign for H in Type IIB and the sign change $C_{(2n+1)} \to (-)^n C_{(2n+1)}$ in Type IIA.

so that the ten-dimensional chirality operator can be written as $\Gamma_{(10)} = \Gamma^{\underline{0\cdots 9}} = \gamma_{(4)} \otimes \hat{\gamma}_{(6)}$.

For Type IIA backgrounds the supersymmetry parameter is a ten-dimensional Majorana spinor ε that can be split into two Majorana-Weyl (MW) spinors of opposite chirality:

$$\varepsilon = \varepsilon_1 + \varepsilon_2 \quad , \quad \Gamma_{(10)}\varepsilon_1 = \varepsilon_1 \quad , \quad \Gamma_{(10)}\varepsilon_2 = -\varepsilon_2 \ . \tag{6.22}$$

Since we are interested only in four-dimensional $\mathcal{N} = 1$ backgrounds, they must have 4 independent Killing spinors that can be decomposed as

$$\begin{aligned} \varepsilon_1(y) &= \zeta_+ \otimes \eta_+^{(1)}(y) + \zeta_- \otimes \eta_-^{(1)}(y) \ , \\ \varepsilon_2(y) &= \zeta_+ \otimes \eta_-^{(2)}(y) + \zeta_- \otimes \eta_+^{(2)}(y) \ , \end{aligned} \tag{6.23}$$

where ζ_+ is a generic constant four-dimensional spinor of positive chirality, while the $\eta_+^{(a)}$ are two particular six-dimensional spinor fields of positive chirality that characterise the solution and

$$\zeta_- = (\zeta_+)^* \quad , \quad \eta_-^{(a)} = (\eta_+^{(a)})^* \ . \tag{6.24}$$

In Type IIB the two supersymmetry parameters $\varepsilon_{1,2}$ are MW real spinors of positive ten-dimensional chirality ($\Gamma_{(10)}\varepsilon_{1,2} = \varepsilon_{1,2}$). In this case

$$\varepsilon_a(y) = \zeta_+ \otimes \eta_+^{(a)}(y) + \zeta_- \otimes \eta_-^{(a)}(y) \ , \tag{6.25}$$

where again $\zeta_- = (\zeta_+)^*$ and $\eta_-^{(a)} = (\eta_+^{(a)})^*$. The existence of the internal spinors $\eta_+^{(1)}$ and $\eta_+^{(2)}$ associated to these $\mathcal{N} = 1$ backgrounds generally specifies an $SU(3) \times SU(3)$-structure on $T_Y \oplus T_Y^\star$, as we described previously.

As discussed in [158], in order to analyse the supersymmetry conditions for the background, it is convenient to use the bispinor formalism. Using the Clifford map (6.11), we can associate two pure spinors to our internal spinors $\eta_+^{(1)}$ and $\eta_+^{(2)}$

$$\not{\Psi}^+ = \eta_+^{(1)} \otimes \eta_+^{(2)\dagger} \quad , \quad \not{\Psi}^- = \eta_+^{(1)} \otimes \eta_-^{(2)\dagger} \tag{6.26}$$

corresponding to sums of forms of definite parity

$$\Psi^+ = \sum_{k\geq 0} \Psi^+_{(2k)} \quad , \quad \Psi^- = \sum_{k\geq 0} \Psi^-_{(2k+1)} \ , \tag{6.27}$$

Following [158], we also fix the norms of the two internal spinors to be

$$||\eta^{(1)}||^2 = |a|^2 \quad , \quad ||\eta^{(2)}||^2 = |b|^2 \,. \tag{6.28}$$

In [158] it was first shown how the Killing equations, i.e. the supersymmetry transformations, can be written in an elegant form in terms of the pure spinors $\Psi^{\pm}$ using the Clifford map. This calculation is rather involved, but crucial, and so we will take some time to review it in detail here.

Background supersymmetry conditions

Following [157, 158], we shall now show how the supersymmetry conditions for the ten-dimensional Type II theories with SU(3) $\times$ SU(3) structure backgrounds can be written as $d\Psi^{\pm}$, and in particular we adopt their conventions in the following calculation. The result can be translated back to our convention by using the rules specified in footnote 8 on page 110. We will use the democratic formalism of the Type II theories [174], in which the gravitino ψ_M and dilatino λ supersymmetry transformations can be written in the following way

$$\delta\psi_M = \nabla_M\varepsilon + \frac{1}{4}H_M\Gamma_{(10)}\varepsilon + \frac{1}{16}e^{\Phi}\sum_n \not{F}_{(2n)}\,\Gamma_M\mathcal{P}_n\,\varepsilon\,, \tag{6.29}$$

$$\delta\lambda = \left(\not{\partial}\Phi + \frac{1}{2}\not{H}\mathcal{P}\right)\varepsilon + \frac{1}{8}e^{\Phi}\sum_n(-1)^{2n}(5-2n)\not{F}_{(2n)}\,\mathcal{P}_n\,\varepsilon\,, \tag{6.30}$$

where the modified RR field strengths are now given by $F_{(2n)} = dC_{(2n-1)} - H\wedge C_{(2n-3)}$, with $n = 0,\ldots,5$ for IIA and $n = 1/2,\ldots,9/2$ for IIB, $H_M \equiv \frac{1}{2}H_{MNP}\Gamma^{NP}$ and, for instance, $\not{H} = \Gamma^{MNP}H_{MNP}$. Here the two Majorana-Weyl supersymmetry parameters have been arranged in a doublet $\varepsilon = (\varepsilon_1, \varepsilon_2)$. As this expression involves both field strengths and their duals we must impose the self-duality relation, which now takes the form

$$F_{(2n)} = (-)^{\mathrm{Int[n]}}\star_{10} F_{(10-2n)}\ , \tag{6.31}$$

where Int[n] denotes the integer part of n. For Type IIA we define $\mathcal{P} = \Gamma_{(10)}$ and $\mathcal{P}_n = \Gamma^n_{(10)}\sigma^1$, while for Type IIB we define $\mathcal{P} = -\sigma^3$, $\mathcal{P}_n = \sigma^1$ for $n+1/2$ even and $\mathcal{P}_n = i\sigma^2$ for $n+1/2$ odd, where σ^i , $i = 1,2,3$ are the usual Pauli matrices. For calculations it is convenient to consider the "modified dilatino" transformation, in which the RR terms vanish, defined by

$$\Gamma^M\delta\psi_M - \delta\lambda = \left(\not{\nabla} - \not{\partial}\Phi + \frac{1}{4}\not{H}\mathcal{P}\right)\varepsilon\,. \tag{6.32}$$

It shall be sufficient to concentrate on the calculation of $d\Psi$ in the Type IIA case, as the Type IIB case is completely analogous. The supersymmetry transformations (6.29), (6.32) are then given by

$$\delta\psi_M = \nabla_M\varepsilon + \frac{1}{4}H_M\Gamma_{(10)}\varepsilon + \frac{1}{16}e^{\Phi}\sum_n \not{F}_{(2n)}\,\Gamma_M\Gamma^n_{(10)}\sigma_1\,\varepsilon\,, \tag{6.33}$$

$$\Gamma^M\delta\psi_M - \delta\lambda = \left(\not{\nabla} - \not{\partial}\Phi + \frac{1}{4}\not{H}\Gamma_{(10)}\right)\varepsilon\,. \tag{6.34}$$

Using the four plus six dimensional decomposition of spinors, field strengths and gamma matrices described above, we may rewrite these transformations as conditions upon the internal spinors $\eta^{(a)}_{\pm}$. For the external component of the gravitino transformation $\delta\psi_\mu$ one then finds

$$\frac{1}{2}\not{\partial}A\eta^{(1)}_{+} + \frac{e^{\Phi}}{16}\left(\hat{\not{F}}_{A1} + i\tilde{\not{F}}_{A1}\right)\eta^{(2)}_{-} = 0\,, \tag{6.35}$$

$$\frac{1}{2}\not{\partial}A\eta^{(2)}_{-} + \frac{e^{\Phi}}{16}\left(\hat{\not{F}}_{A2} + i\tilde{\not{F}}_{A2}\right)\eta^{(1)}_{+} = 0\,, \tag{6.36}$$

where $F_{A1} = F_{(0)} - F_{(2)} + F_{(4)} - F_{(6)}$ and $F_{A2} = F_{(0)} + F_{(2)} + F_{(4)} + F_{(6)}$. The corresponding decomposition on the internal component $\delta\psi_m$ gives

$$\nabla_m\eta^{(1)}_{+} + \frac{1}{4}H_m\eta^{(1)}_{+} + \frac{e^{\Phi}}{16}\left(\hat{\not{F}}_{A1} - i\tilde{\not{F}}_{A1}\right)\hat{\gamma}_m\eta^{(2)}_{-} = 0\,, \tag{6.37}$$

$$\nabla_m\eta^{(2)}_{-} - \frac{1}{4}H_m\eta^{(2)}_{-} + \frac{e^{\Phi}}{16}\left(\hat{\not{F}}_{A2} - i\tilde{\not{F}}_{A2}\right)\hat{\gamma}_m\eta^{(1)}_{+} = 0\,, \tag{6.38}$$

and for the modified dilatino transformation we find

$$\left(\not{\nabla} + \frac{1}{4}\not{H} + 2\not{\partial}A - \not{\partial}\Phi\right)\eta^{(1)}_{+} = 0\,, \tag{6.39}$$

$$\left(\not{\nabla} - \frac{1}{4}\not{H} + 2\not{\partial}A - \not{\partial}\Phi\right)\eta^{(2)}_{+} = 0\,. \tag{6.40}$$

Consider the exterior derivative of the Clifford(6,6) spinors $\Psi^{\pm}$ in terms of Clifford(6) bispinors, given by

$$d\Psi^{\pm} = dx^m\wedge\nabla_m\Psi^{\pm} = dx^m\wedge\left[(\nabla_m\eta^{(1)}_{+})\otimes\eta^{(2)\dagger}_{\pm} + \eta^{(1)}_{+}\otimes(\nabla_m\eta^{(2)}_{\pm})^{\dagger}\right]\,. \tag{6.41}$$

We shall concentrate on $d\Psi^{-}$, with the aim of deriving the first supersymmetry constraint for Type IIA. The second condition follows in a similar manner. Using the definition of Clifford(6,6) spinor representations given in [157, 158], we can rewrite (6.41) as

$$2d\Psi^- = \nabla\!\!\!/ \eta_+^{(1)} \otimes \eta_-^{(2)\dagger} - \eta_+^{(1)} \otimes (\nabla\!\!\!/ \eta_-^{(2)})^\dagger + \hat{\gamma}^m \eta_+^{(1)} \otimes (\nabla_m \eta_-^{(2)})^\dagger - \nabla_m \eta_+^{(1)} \otimes \eta_-^{(2)\dagger} \hat{\gamma}^m \; . \tag{6.42}$$

One can now use the decomposition of internal gravitino and modified dilatino supersymmetry transformations to evaluate the right-hand side of this expression. We form the following bispinors from the external gravitino transformation

$$\frac{1}{2} \partial\!\!\!/ A \eta_-^{(1)} \otimes \eta_+^{(2)\dagger} - \frac{e^\Phi}{16} \left(\hat{F\!\!\!/}_{A1} - i \tilde{F\!\!\!/}_{A1} \right) \eta_+^{(2)} \otimes \eta_+^{(2)\dagger} = 0 \, , \tag{6.43}$$

$$\frac{1}{2} \eta_-^{(1)} \otimes \eta_+^{(2)\dagger} \partial\!\!\!/ A - \frac{e^\Phi}{16} \eta_-^{(1)} \otimes \eta_-^{(1)\dagger} \left(\hat{F\!\!\!/}_{A1} + i \tilde{F\!\!\!/}_{A1} \right) = 0 \, . \tag{6.44}$$

These two quantities can be added together with (6.42) to give,

$$\begin{aligned} 2d\Psi^- = & \left(\partial\!\!\!/ \Phi - \frac{1}{4} H\!\!\!/ - 2\partial\!\!\!/ A \right) \eta_+^{(1)} \otimes \eta_-^{(2)\dagger} - \eta_+^{(1)} \otimes \eta_-^{(2)\dagger} \left(\partial\!\!\!/ \Phi - \frac{1}{4} H\!\!\!/ - 2\partial\!\!\!/ A \right) \\ & - \frac{1}{4} \hat{\gamma}^m \eta_+^{(1)} \otimes \eta_-^{(2)\dagger} H_m + \frac{1}{4} H_m \eta_+^{(1)} \otimes \eta_-^{(2)\dagger} \hat{\gamma}^m - \partial\!\!\!/ A \eta_-^{(1)} \otimes \eta_+^{(2)\dagger} + \eta_-^{(1)} \otimes \eta_+^{(2)\dagger} \partial\!\!\!/ A \\ & - \frac{e^\Phi}{16} \left(\hat{\gamma}^m \eta_+^{(1)} \otimes \eta_+^{(1)\dagger} \hat{\gamma}_m + 2\eta_-^{(1)} \otimes \eta_-^{(1)\dagger} \right) (\hat{F\!\!\!/}_{A1} + i \tilde{F\!\!\!/}_{A1}) \\ & + \frac{e^\Phi}{16} (\hat{F\!\!\!/}_{A1} - i \tilde{F\!\!\!/}_{A1}) \left(\hat{\gamma}^m \eta_-^{(2)} \otimes \eta_-^{(2)\dagger} \hat{\gamma}_m + 2\eta_+^{(2)} \otimes \eta_+^{(2)\dagger} \right) \; . \end{aligned} \tag{6.45}$$

Making some manipulations, using the fact that $\hat{\gamma}_{(6)} \hat{F\!\!\!/}_{A1} = -i \tilde{F\!\!\!/}_{A1}$ and going back to our preferred notation, the Clifford map (6.11) allows us to write this equation in the form applicable to both Type IIA and IIB. The complete set of equations for $\Psi^\pm$ resulting from this procedure are[9]

$$\begin{aligned} e^{-2A+\Phi}(d + H\wedge)\left[e^{2A-\Phi}\Psi_1\right] &= dA \wedge \bar{\Psi}_1 + \frac{e^\Phi}{16}\left[(|a|^2 - |b|^2)\hat{F} + i(|a|^2 + |b|^2)\tilde{F}\right] \; , \\ (d + H\wedge)\left[e^{2A-\Phi}\Psi_2\right] &= 0 \; , \end{aligned} \tag{6.46}$$

where for Type IIA we have

$$\Psi_1 = \Psi^- \quad , \quad \Psi_2 = \Psi^+ \quad \text{and} \quad F = F_A = F_{(0)} + F_{(2)} + F_{(4)} + F_{(6)} \; , \tag{6.47}$$

[9] Note that, taking into account the different conventions, the first of (6.46) has some sign differences with equations (3.2) and (3.3) of [158]. We thank the authors of [158] for private communications confirming the sign mistakes appearing in equations (3.2) and (3.3) in the original version of their paper.

while for Type IIB

$$\Psi_1 = \Psi^+ \quad , \quad \Psi_2 = \Psi^- \quad \text{and} \quad F = F_B = F_{(1)} + F_{(3)} + F_{(5)} \ . \qquad (6.48)$$

The second condition means that the generalised almost complex structure associated to Ψ_2 is integrable, while in the first the RR fields represent an obstruction to the integrability of the generalised almost complex structure associated to Ψ_1. Using the gravitino Killing equations one can furthermore show that $\mathcal{N} = 1$ supersymmetry imposes the following constraint

$$d|a|^2 = |b|^2 dA \quad , \quad d|b|^2 = |a|^2 dA \ . \qquad (6.49)$$

As discussed in [158], it can be proven that equations (6.46) and (6.49) are completely equivalent to the full set of supersymmetric Killing equations and hence can be considered as necessary and sufficient conditions to have a supersymmetric background. Furthermore, one has to bear in mind that these equations only make sense if not all of the RR field strengths are vanishing, and that in order to have a complete supergravity solution one has to supplement these conditions with the Bianchi identities and the equations of motion for the fluxes [175].

The supersymmetry conditions (6.46) and (6.49) are identical in form for Type IIA and IIB and the two cases are exactly related by the exchange

$$\Psi^+ \leftrightarrow \Psi^- \quad \text{and} \quad F_A \leftrightarrow F_B \ . \qquad (6.50)$$

This relation can be seen as a generalised mirror symmetry for Type II backgrounds with $SU(3) \times SU(3)$ structure and, as we will see, the conditions for having supersymmetric branes respect this symmetry, providing further evidence that it is a fundamental feature. For further discussions on generalised mirror symmetry, see e.g. [152, 176–180].

It will be useful at this point to make some contact with the common terminology in the literature. A manifold is called generalised Calabi-Yau if there exists on it a closed pure spinor $d\Psi = 0$. The corresponding generalised almost complex structure is then integrable. Similarly, the existence of a twisted closed pure spinor, with closed H-field, $d_H\Psi = (d + H\wedge)\Psi = 0$ defines a "twisted" generalised Calabi-Yau. While it is not necessary for our purposes, we note that the integrability of the corresponding almost complex structure can be phrased in terms of an appropriate Courant bracket on $T_Y \oplus T_Y^\star$, which is a "twisted" version of the usual Lie bracket

[160].

Let us finally remember that these backgrounds contain as subcases the $SU(3)$ and $SU(2)$ structure backgrounds. In the $SU(3)$ case we have to require that the two $\eta_+^{(a)}$ are linearly dependent, i.e. $\eta_+^{(1)} = a\eta_+$ and $\eta_+^{(2)} = b\eta_+$ for a given six-dimensional spinor field η_+, with $\eta_+^\dagger \eta_+ = 1$. On the other hand we have $SU(2)$-structure when $\eta_+^{(1)}$ and $\eta_+^{(2)}$ are never parallel. We refer the reader to the detailed discussion of these cases given in [158].

6.4 Supersymmetric D-branes on $\mathcal{N} = 1$ vacua

Let us now turn to the main question of this chapter, namely: what are the constraints on supersymmetric D-branes in the general class of backgrounds we have described in the previous section? In chapter 3 we saw how preserved supersymmetry of the M2-brane enforced a projection condition upon the supersymmetry parameter ε (3.52), which we derived by using the conserved charges for the M2-brane and the eleven-dimensional supersymmetry algebra. In this section we shall take a complementary approach in which the fraction of preserved supersymmetry of a brane is determined from the worldvolume perspective. For clarity, let us now review our conventions for the Dp-action and symmetries, before proceeding to analyse the supersymmetry conditions for branes on our generalised backgrounds.

In general, a Dp-brane configuration of a Type II theory is defined by the embedding $\xi^\alpha \mapsto X^M = (x^\mu(\xi), y^m(\xi))$, $\alpha = 0, \ldots, p$ of the Dp-brane worldvolume with coordinates ξ^α into the ten-dimensional spacetime with coordinates X^M, and by the worldvolume two-form field strength $f_{(2)} = dA_{(1)}$, where $A_{(1)}$ is the gauge field living on the brane. We write the bosonic part of standard Dp-brane in standard Dirac-Born-Infield plus Chern-Simons form [181–183]

$$S_{Dp}^{(B)} = -\tau_{Dp} \int d^{p+1}\xi e^{-\Phi} \sqrt{-\det(g + \mathcal{F})} + \tau_{Dp} \int \sum_n P[C_{(n)}] e^{\mathcal{F}} \ , \tag{6.51}$$

where τ_{Dp}^{-1} is the brane tension. Here $\mathcal{F}_{\alpha\beta} = P[B]_{\alpha\beta} + f_{\alpha\beta}$ and $g_{\alpha\beta} = P[G]_{\alpha\beta}$, $P[B]_{\alpha\beta}$ are the pull-backs of the background spacetime metric G_{mn} and Neveu-Schwarz two-form gauge potential B_{MN}, respectively, on the worldvolume.

The total Dp-brane action may be written in superspace formalism, however this is somewhat complicated. In order to understand the physical couplings, it will suffice for our purposes to write the fermionic terms up to quadratic order, following

[184, 185]:

$$S_{Dp}^{(F)} = \frac{\tau_{Dp}}{2} \int d^{p+1}\xi e^{-\Phi} \sqrt{-\det(g+\mathcal{F})}\, \bar{\theta}(1-\Gamma_{Dp}) \big[(\tilde{M}^{-1})^{\alpha\beta}\Gamma_\beta D_\alpha - \Delta\big]\theta \ . \quad (6.52)$$

where the pull-back of a gamma matrix is defined by $\Gamma_\alpha = \Gamma_{\underline{M}} e^{\underline{M}}_M \partial_\alpha X^M$ and underlined indices run over flat tangent space directions. It is convenient to once again use a double spinor convention for both Type IIB and Type IIA , where in the IIA case the two spinors of opposite chirality are organised in a two component vector. We have also introduced

$$\tilde{M}_{\alpha\beta} = g_{\alpha\beta} + \tilde{\Gamma}_{(10)}\mathcal{F}_{\alpha\beta} \,, \quad (6.53)$$

where

$$\text{IIA}: \quad \tilde{\Gamma}_{(10)} = \Gamma_{(10)} \qquad , \qquad \text{IIB}: \quad \tilde{\Gamma}_{(10)} = \Gamma_{(10)} \otimes \sigma_3 \ . \quad (6.54)$$

The other operators appearing in (6.52) are given defined the pullbacks of the gravitino (6.30) and dilatino (6.30) supersymmetry transformations:

$$\delta_\varepsilon \psi_m = D_m \varepsilon \qquad , \qquad \delta_\varepsilon \lambda = \Delta\varepsilon \ . \quad (6.55)$$

The complete bosonic plus fermionic action is invariant under worldvolume diffeomorphisms and global supersymmetry, which takes the following form on the worldvolume

$$\delta_\varepsilon \theta = \varepsilon \ , \quad (6.56)$$

$$\delta_\varepsilon y^m = -\frac{1}{2}\bar{\theta}\Gamma^m \varepsilon \ , \quad (6.57)$$

$$\delta_\varepsilon A_\alpha = \frac{1}{2}\bar{\theta}\tilde{\Gamma}_{(10)}\Gamma_\alpha \varepsilon - \frac{1}{2} B_{\alpha m}\bar{\theta}\Gamma^m \varepsilon \ . \quad (6.58)$$

It also possess an additional local fermionic symmetry, κ-symmetry, which up to quadratic order in fermions is given by

$$\delta_\kappa \bar{\theta} = \bar{\kappa}(\mathbb{1} + \Gamma_{Dp}) \ , \quad (6.59)$$

$$\delta_\kappa y^m = -\frac{1}{2}\delta_\kappa \bar{\theta}\Gamma^m \theta \ , \quad (6.60)$$

$$\delta_\kappa A_\alpha = \frac{1}{2}\delta_\kappa \bar{\theta}\tilde{\Gamma}_{(10)}\Gamma_\alpha \theta - \frac{1}{2} B_{\alpha m}\delta_\kappa \bar{\theta}\Gamma^m \theta \ , \quad (6.61)$$

with transformation parameter κ. In fact, for all brane solutions one finds such an additional symmetry of the actions [181, 182], taking the following generic form on a spacetime spinor θ

$$\delta_\kappa \theta = (\mathbb{1} + \Gamma)\kappa \ , \tag{6.62}$$

where Γ is a Hermitian, traceless operator, with $\Gamma^2 = \mathbb{1}$, and therefore eigenvalues split equally between ± 1. Thus $\frac{1}{2}(\mathbb{1} + \Gamma)$ is a projection operator with the same properties as that found from the spacetime supersymmetry algebra (3.52) in chapter 3 i.e. half its eigenvalues are zero.

Using the κ-symmetry transformation (6.59), one can show that a Dp-brane preserves a given supersymmetry ε of the background if it satisfies the condition [183][10]

$$\delta_\kappa \theta + \delta_\varepsilon \theta = 0 \quad \Rightarrow \quad \bar{\varepsilon}\Gamma_{Dp} = \bar{\varepsilon} \ , \tag{6.63}$$

where Γ_{Dp} is the same worldvolume chiral operator entering the κ-symmetry transformations (6.59)[183]. Using the explicit form of the κ-operators in our notation[11], we find this condition reduces to

$$\hat{\Gamma}_{Dp}\varepsilon_2 = \varepsilon_1 \ , \tag{6.64}$$

where

$$\hat{\Gamma}_{Dp} = \frac{1}{\sqrt{-\det(P[G] + \mathcal{F})}} \sum_{2l+s=p+1} \frac{\epsilon^{\alpha_1 \ldots \alpha_{2l}\beta_1 \ldots \beta_s}}{l! s! 2^l} \mathcal{F}_{\alpha_1\alpha_2} \cdots \mathcal{F}_{\alpha_{2l-1}\alpha_{2l}} \Gamma_{\beta_1 \ldots \beta_s} \ , \tag{6.65}$$

and $\hat{\Gamma}^{-1}_{Dp}(\mathcal{F}) = (-)^{\mathrm{Int}[\frac{p+3}{2}]}\hat{\Gamma}_{Dp}(-\mathcal{F})$.

Let us now consider the implications of the Dp-brane supersymmetry condition (6.64) in our general class of backgrounds. We begin by restricting our attention to Dp-branes filling the time plus q flat directions (with no worldvolume flux in these directions), and wrapping an internal $(p-q)$-cycle in the compact manifold Y. Using (6.20) we can decompose the above operators into four- and six-dimensional components as follows

$$\hat{\Gamma}_{Dp} \quad = \quad \gamma_{\underline{0 \ldots q}} \gamma_{(4)}^{p-q} \otimes \hat{\gamma}'_{(p-q)} \ , \tag{6.66}$$

[10]See [162] for an earlier discussion of the M2-brane supersymmetry on CY_3.

[11]Here we use the κ-symmetry operators constructed from T-duality in [184], which are identical to those given in [182] up to some different overall signs. Their explicit form in double spinor notation in both IIA and IIB can be found in [185].

where

$$\hat{\gamma}'_{(r)} = \frac{1}{\sqrt{\det(P[g]+\mathcal{F})}} \sum_{2l+s=r} \frac{\epsilon^{\alpha_1\ldots\alpha_{2l}\beta_1\ldots\beta_s}}{l!s!2^l} \mathcal{F}_{\alpha_1\alpha_2}\cdots\mathcal{F}_{\alpha_{2l-1}\alpha_{2l}}\hat{\gamma}_{\beta_1\ldots\beta_s} \; , \tag{6.67}$$

is a unitary operator acting on the internal spinors.

By considering general Dp-branes in both Type IIA/IIB backgrounds and using (6.23), (6.65) and (6.66), it is possible to see that the supersymmetry condition (6.63) can be split into the four-dimensional condition

$$\gamma_{\underline{0\ldots q}}\zeta_+ = \alpha^{-1}\zeta_{(-)^{q+1}} \; , \tag{6.68}$$

and the internal six-dimensional one

$$\hat{\gamma}'_{(p-q)}\eta^{(2)}_{(-)^{p+1}} = \alpha\eta^{(1)}_{(-)^{q+1}} \; . \tag{6.69}$$

By consistency with the complex conjugate of these expressions and the fact that $\gamma^2_{\underline{0\ldots q}} = -(-)^{\frac{q(q+1)}{2}}$, it can be seen that the case $q=0$, i.e. the case where we have an effective four-dimensional particle, can never be supersymmetric, while for $q=1,2,3$ one has the condition that $\alpha = e^{i\varphi}$, i.e. α is a pure phase. More explicitly $\varphi = 0$ or π for $q=1$ (effective string), φ is arbitrary for $q=2$ (domain-wall) and $\varphi = -\pi/2$ for $q=3$ (space-time filling branes). From the unitarity of the operator $\hat{\gamma}'_{(r)}$, it also follows that we must have the following constraints on the internal spinors

$$||\eta^{(1)}||^2 = ||\eta^{(2)}||^2 \; , \tag{6.70}$$

and from (6.49) we then see that once the condition (6.70) is fulfilled at one point for our backgrounds, it is automatically valid at all points.

For the purposes of this work we are interested in spacetime filling branes and hence from this point on we shall consider only these cases. Supersymmetry conditions for the other cases listed above are easily found by reinstating φ-dependence in the appropriate way. In the case of four-dimensional space-time filling branes, the four-dimensional condition is automatically satisfied once we set $\varphi = -\pi/2$, leaving the following internal conditions

$$\begin{cases} i\hat{\gamma}'_{(2k)}\eta^{(2)}_+ = \eta^{(1)}_+ & , \quad \text{in IIB} \; , \\ i\hat{\gamma}'_{(2k+1)}\eta^{(2)}_+ = \eta^{(1)}_- & , \quad \text{in IIA} \; . \end{cases} \tag{6.71}$$

We would now like to write the supersymmetry conditions (6.71) in terms of the geometrical objects Ψ^+ and Ψ^- introduced in section 6.3. In order to do this, it is useful to decompose the spinorial quantities entering (6.71) in the basis defined by

$$\eta_+^{(1)} \quad , \quad \eta_-^{(1)} \quad , \quad \hat{\gamma}_m \eta_+^{(1)} \quad \text{and} \quad \hat{\gamma}_m \eta_-^{(1)} \ . \tag{6.72}$$

By decomposing the supersymmetry conditions (6.71) in this basis, one obtains a set of equations written in a more geometric fashion in terms of the pull-back to the worldvolume of Ψ^+ and Ψ^-. Explicitly, for even $2k$-cycles we have the conditions

$$\begin{aligned} \left\{P[\Psi^+] \wedge e^{\mathcal{F}}\right\}_{(2k)} &= \frac{i|a|^2}{8} \sqrt{\det(P[g]+\mathcal{F})} d\sigma^1 \wedge \ldots \wedge d\sigma^{2k} \ , \\ \left\{P[dx^m \wedge \Psi^- + g^{mn} \imath_n \Psi^-] \wedge e^{\mathcal{F}}\right\}_{(2k)} &= 0 \ , \end{aligned} \tag{6.73}$$

while for odd $(2k+1)$-cycles we have

$$\begin{aligned} \left\{P[\Psi^-] \wedge e^{\mathcal{F}}\right\}_{(2k+1)} &= \frac{i|a|^2}{8} \sqrt{\det(P[g]+\mathcal{F})} d\sigma^1 \wedge \ldots \wedge d\sigma^{2k+1} \ , \\ \left\{P[dx^m \wedge \Psi^+ + g^{mn} \imath_n \Psi^+] \wedge e^{\mathcal{F}}\right\}_{(2k+1)} &= 0 \ . \end{aligned} \tag{6.74}$$

Note that these equations are identical if we interchange

$$\Psi^+ \leftrightarrow \Psi^- \ . \tag{6.75}$$

They then respect the generalised mirror symmetry (6.50) that relates the Type IIA and IIB $\mathcal{N}=1$ supersymmetric backgrounds we are considering.

In the following section we will discuss the geometrical interpretation of the supersymmetry conditions (6.73) and (6.74). As a preliminary step, it is useful to observe that they are not independent. Indeed, we obtained these conditions by expanding (6.71) in the basis (6.72). Using the unitarity of $\hat{\gamma}'(\mathcal{F})$, it is easy to see that the first equations of (6.73) and (6.74) imply the second ones. Vice-versa, the second conditions determine the first up to an overall arbitrary (in general, point dependent) phase. Moreover, once again using the unitarity of $\hat{\gamma}'(\mathcal{F})$, the first conditions can be furthermore restricted in such a way that we can characterise the supersymmetry cycles in the following way:

$$\begin{aligned} \left\{\mathrm{Im}\left(iP[\Psi^+]\right) \wedge e^{\mathcal{F}}\right\}_{(2k)} &= 0 \ , \\ \left\{P[dx^m \wedge \Psi^- + g^{mn} \imath_n \Psi^-] \wedge e^{\mathcal{F}}\right\}_{(2k)} &= 0 \ , \end{aligned} \tag{6.76}$$

for even $2k$-cycles, while

$$\begin{aligned} \Big\{\mathrm{Im}\big(iP[\Psi^-]\big)\wedge e^{\mathcal{F}}\Big\}_{(2k+1)} &= 0\ , \\ \Big\{P[dx^m\wedge\Psi^+ + g^{mn}\imath_n\Psi^+]\wedge e^{\mathcal{F}}\Big\}_{(2k+1)} &= 0\ , \end{aligned} \tag{6.77}$$

for odd $(2k+1)$-cycles.

Note that these conditions do not strictly speaking imply that the wrapped brane is supersymmetric but in general it is supersymmetric for one choice of orientation. If the RR fields were turned off, the orientation would be arbitrary because a change of orientation would amount to considering an anti D-brane instead of a D-brane or vice-versa, and these feel the background fields in the same way. However, we are considering the case with nontrivial RR fields. D-branes and anti D-branes then react to the background in a different way and the orientation cannot be ignored, meaning that the conditions given in (6.76) and(6.77) are in fact necessary and sufficient only for the brane to admit at least an orientation making it supersymmetric.

The above conditions can be substituted by the following single condition that encodes also the necessary orientation requirement:

$$\Big\{\mathrm{Re}\big(-iP[\Psi^+]\big)\wedge e^{\mathcal{F}}\Big\}_{(2k)} = \frac{|a|^2}{8}\sqrt{\det(P[g]+\mathcal{F})}d\sigma^1\wedge\ldots\wedge d\sigma^{2k}, \tag{6.78}$$

for even $2k$-cycles, while for odd $(2k+1)$-cycles

$$\Big\{\mathrm{Re}\big(-iP[\Psi^-]\big)\wedge e^{\mathcal{F}}\Big\}_{(2k+1)} = \frac{|a|^2}{8}\sqrt{\det(P[g]+\mathcal{F})}d\sigma^1\wedge\ldots\wedge d\sigma^{2k+1}. \tag{6.79}$$

Note that since we are assuming that the internal spinors have the same norm, in the above expressions we can write $|a|^2$ in terms of any of the two pure spinors as follows

$$|a|^4 = ||\Psi||^2 = \mathrm{Tr}(\not\Psi\not\Psi^\dagger) = 8\sum_k|\Psi_{(k)}|^2\ . \tag{6.80}$$

We will see in section 6.7 that we can interpret the equations (6.78) and (6.79), and then also (6.76) and (6.77) plus an appropriate choice of the orientation, as generalised calibration conditions.

6.5 The geometry of the supersymmetric D-branes

We shall now discuss the geometrical meaning of the second conditions of (6.76) and (6.77). As we will see, supersymmetric branes wrapping even cycles in Type IIB and odd cycles in Type IIA must correspond to a correctly generalised definition of holomorphic and coisotropic branes respectively. For the cases we are interested in we can adapt the discussion presented in [166, 168] for backgrounds with only nontrivial NS fields.

Let us first recall that, for the general r-cycle, the second conditions of (6.76) and (6.77) come from the requirement that $\hat{\gamma}'_{(r)}(\mathcal{F})\eta_+^{(2)}$ must be parallel to $\eta_{(-)^r}^{(1)}$. It is then possible to see [166] that this condition is equivalent to

$$J_1|_\Sigma = (-)^r R J_2 R^{-1}|_\Sigma \ , \tag{6.81}$$

where J_1 and J_2 are the almost complex structures associated to the six-dimensional spinors $\eta_+^{(1)}$ and $\eta_+^{(2)}$ respectively (see Appendix B for the explicit construction in the SU(3) structure case). $J|_\Sigma$ denotes the restriction of a complex structure to the D-brane worldvolume wrapping a cycle Σ of the internal manifold Y. The action of the rotation matrix R on $T_M|_\Sigma = T_\Sigma \oplus \mathcal{N}_\Sigma$ is defined as follows. If $p_{||}$ and $p_\perp$ are the projectors on the tangent and normal bundle of the brane respectively, then R acts as a reflection in the normal directions ($Rp_\perp = p_\perp R = -p_\perp$) while the action of R along T_Σ is defined by

$$p_{||}^T(g-\mathcal{F})p_{||} = p_{||}(g+\mathcal{F})p_{||}R \ , \tag{6.82}$$

where $\mathcal{F}$ is now naturally thought of as a section of $\Lambda^2 T_M^\star|_\Sigma$ such that $p_\perp^T\mathcal{F} = \mathcal{F}p_\perp = 0$. The pure spinors Ψ^+ and Ψ^- are associated to generalised almost complex structures $\mathcal{J}_+$ and $\mathcal{J}_-$ on $T_M \oplus T_M^\star$. One can prove that these can be written in terms of J_1 and J_2 as follows [160, 166, 168]:

$$\mathcal{J}_\pm = \frac{1}{2}\begin{pmatrix} J_1 \mp J_2 & (J_1 \pm J_2)g^{-1} \\ g(J_1 \pm J_2) & g(J_1 \mp J_2)g^{-1} \end{pmatrix} \ . \tag{6.83}$$

One can then see that (6.81) is equivalent to the following condition for $\mathcal{J}_\pm$ restricted on $T_M \oplus T_M^\star|_\Sigma$

$$\mathcal{J}_{(-)^{r+1}} = \mathcal{R}^{-1}\mathcal{J}_{(-)^{r+1}}\mathcal{R} \ , \tag{6.84}$$

where $\mathcal{R}$ acts in the following way on $T_M \oplus T^\star_M|_\Sigma$

$$\mathcal{R} = \frac{1}{2}\begin{pmatrix} r & 0 \\ \mathcal{F}r + r^T\mathcal{F} & -r^T \end{pmatrix} , \tag{6.85}$$

with $r = p_{||} - p_\perp$.

The D-brane worldvolume wrapping the internal cycle Σ, fully specified by the couple $(\Sigma, \mathcal{F})$ where $\mathcal{F}$ is such that $d\mathcal{F} = P_\Sigma[H]$, can be seen as a generalised submanifold as defined by Gualtieri in [160]. Gualtieri also defines a *generalised tangent bundle* $\tau^{\mathcal{F}}_\Sigma$ associated to the brane. The key point is that the elements $X \in T_M \oplus T^\star_M|_\Sigma$ belonging to $\tau^{\mathcal{F}}_\Sigma$ can be characterised by the condition [168]

$$\mathcal{R}X = X \ . \tag{6.86}$$

The subsequent step is to remember that, given an (integrable) generalised complex structure $\mathcal{J}$ on M, Gualtieri defines a *generalised complex submanifold* as a generalised submanifold $(\Sigma, \mathcal{F})$ with generalised tangent bundle $\tau^{\mathcal{F}}_\Sigma$ stable under $\mathcal{J}$ i.e. it is mapped to itself under that action of $\mathcal{J}$.

From (6.84) and (6.86) we arrive at the conclusion that the second conditions in (6.76) and (6.77) are each equivalent to the following requirement:

Supersymmetric D-branes wrapping even-cycles in Type IIB and odd-cycles in Type IIA must be generalised complex submanifolds with respect to the (integrable) generalised complex structures $\mathcal{J}_-$ and $\mathcal{J}_+$ respectively.

These generalised complex submanifolds can be seen as the most natural generalisation of complex (holomorphic) cycles with $\mathcal{F}$ of kind $(1,1)$ in Type IIB and of coisotropic cycles in Type IIA [160, 186].

6.6 D-branes on $SU(3)$-structure manifolds

In this section we pause the discussion of $SU(3) \times SU(3)$ structure manifolds to comment on the $SU(3)$ structure subcase. We recall that this is obtained when we can write $\eta^{(1)}_+ = a\eta_+$ and $\eta^{(2)}_+ = b\eta_+$ ($\eta^\dagger_+\eta_+ = 1$), remembering that in order to have supersymmetric branes we have to fulfil the necessary condition $|a| = |b|$. In this case, the pure spinors $\Psi^\pm$ can be defined in terms of the almost complex structure

J and the $(3,0)$-form Ω associated to η_+ as explained in appendix B,

$$J_{mn} = -\frac{i}{|a|^2}\eta_+^\dagger\hat{\gamma}_{mn}\eta_+ \quad , \quad \Omega_{mnp} = -\frac{i}{a^2}\eta_-^\dagger\hat{\gamma}_{mnp}\eta_+ \ . \tag{6.87}$$

Using the Fierz decomposition it is possible to show that

$$\eta_\pm \otimes \eta_\pm^\dagger = \frac{1}{8}e^{\mp iJ}\!\!\!/ \quad , \quad \eta_+ \otimes \eta_-^\dagger = -\frac{i}{8}\Omega\!\!\!/ \quad . \tag{6.88}$$

We immediately see that in the $SU(3)$-structure case Ψ^+ and Ψ^- reduce to

$$\Psi^+ = \frac{a\bar{b}}{8}e^{-iJ} \quad , \quad \Psi^- = -\frac{iab}{8}\Omega \quad . \tag{6.89}$$

Since we must require that $|a| = |b|$, we can pose

$$\frac{a}{b} \equiv e^{i\phi} \quad , \quad \frac{a}{b^*} \equiv e^{i\tau} \ , \tag{6.90}$$

and the supersymmetry conditions for the wrapped branes now read

$$\begin{aligned} &\Big\{\mathrm{Im}\left(ie^{i\phi}P[e^{-iJ}]\right) \wedge e^{\mathcal{F}}\Big\}_{(2k)} = 0 \ , \\ &\Big\{P[dx^m \wedge \Omega + g^{mn}\imath_n\Omega] \wedge e^{\mathcal{F}}\Big\}_{(2k)} = 0 \ , \end{aligned} \tag{6.91}$$

for even $2k$-cycles, and

$$\begin{aligned} &\Big\{\mathrm{Im}\left(e^{i\tau}P[\Omega]\right) \wedge e^{\mathcal{F}}\Big\}_{(2k+1)} = 0 \ , \\ &\Big\{P[dx^m \wedge e^{iJ} + g^{mn}\imath_n e^{iJ}] \wedge e^{\mathcal{F}}\Big\}_{(2k+1)} = 0 \ , \end{aligned} \tag{6.92}$$

for odd $(2k+1)$-cycles. Again, these conditions really imply that it is possible to choose an orientation on the D-brane in order for it to be supersymmetric and generally reversing the orientation does not preserve supersymmetry. As in the general case, they can be substituted by the following equivalent conditions which also provide the necessary requirement on the orientation

$$\Big\{\mathrm{Re}\big(-ie^{i\phi}P[e^{-iJ}]\big) \wedge e^{\mathcal{F}}\Big\}_{(2k)} = \sqrt{\det(P[g]+\mathcal{F})}d\sigma^1 \wedge \ldots \wedge d\sigma^{2k} \ , \tag{6.93}$$

for even $2k$-cycles, and

$$\Big\{\mathrm{Re}\big(-e^{i\tau}P[\Omega]\big) \wedge e^{\mathcal{F}}\Big\}_{(2k+1)} = \sqrt{\det(P[g]+\mathcal{F})}d\sigma^1 \wedge \ldots \wedge d\sigma^{2k+1} \ , \tag{6.94}$$

for odd $(2k+1)$-cycles. Note that in the $SU(3)$-case Type IIB and IIA backgrounds have complex and symplectic internal manifolds respectively. The above conditions have the same form as those derived in [163] for branes with nontrivial worldvolume fluxes on spaces with no fluxes, and can be seen as their natural generalisation (see also the discussion in [164] for the Type IIB case). In particular, from the discussion of the previous section, the second conditions in (6.91) and (6.92) now require that supersymmetric branes are complex branes with $(1,1)$ field strength $\mathcal{F}$ in Type IIB and coisotropic branes of the kind discussed in [186] in Type IIA (see section 7.2 of [160]). Also, the above conditions are obviously exchanged by the generalised mirror symmetry, that in this case takes the form

$$e^{i\phi}e^{-iJ} \leftrightarrow -ie^{i\tau}\Omega \ . \tag{6.95}$$

6.7 Generalised calibrations for $\mathcal{N}=1$ vacua

We shall now proceed to discuss the meaning of the supersymmetry conditions in the general $SU(3)\times SU(3)$ case. We will see how the conditions in the form (6.78) and (6.79) can be interpreted as generalised calibration conditions. Then the first of each pair of conditions (6.76) and (6.77) encodes the necessary requirement related to the stability of the supersymmetric D-brane that must be added to the geometrical characterisation given in section 6.5.

Calibrations were originally introduced in [187] as a means to construct volume minimising submanifolds of Riemannian manifolds. They are constructed using closed forms and therefore have close ties with manifolds of reduced holonomy which naturally possess such forms, as we discussed in section (6.2). A nice review of these points is given in [156], which we shall follow to introduce the basic definition of a calibrated submanifold.

Let (M,g) be an oriented Riemannian manifold possessing a closed p-form ω. An oriented tangent p-plane V on M is a p-dimensional vector subspace T_xM of a tangent space to M at a point $x\in M$. The restriction of g to V, denoted $g|_V$, can be combined with the orientation on V to give a natural volume form vol_V on V.

The p-form ω is said to be a calibrating form, or calibration, if for every oriented tangent p-plane V on M one has that $\omega|_V \leq vol_V$. A given oriented submanifold Σ is defined to be a calibrated submanifold if $\omega|_\Sigma = vol_{T_x\Sigma}$ for all $x\in\Sigma$. One can easily show that a calibrated submanifold is volume-minimising within its homology class, however we shall postpone this proof until we discuss the more general case in which we are interested.

The relation of calibrations to supersymmetric brane solutions was developed in a series of papers [162, 188–191]. By studying the supersymmetry conditions derived from κ-symmetry arguments, it was shown that supersymmetric instantonic branes wrap volume-minimising submanifolds [162], with such submanifolds then being called supersymmetric cycles. The relation between such supersymmetric cycles in string theory models and the mathematical theory of calibrations was put on a firmer footing in subsequent works [188–191].

A more intuitive picture is provided by recalling the Bogomol'nyi bound for a supersymmetric object, such as the M2 brane bound (3.54) discussed in chapter 3. In a background with all fields but the metric set to zero, the energy density of a brane with static worldvolume is given entirely in terms of the Nambu-Goto piece of the Dirac-Born-Infield action. Considering the supersymmetry algebra with a central extension, as is appropriate for a general (non-dyonic) p-brane solution, and contracting with background Killing spinors, the familiar Bogomol'nyi bound for a supersymmetric brane can be understood as a calibration bound, with the calibration form being constructed as a spinor bilinear of the central charge term in the algebra [173, 192, 193, 195]. This bound is then easily seen to imply that supersymmetric branes in this class of backgrounds are volume minimising. Once again, we shall provide further details of these points when we discuss the case we are particularly interested in.

Let us first of all introduce the appropriate definition of generalised calibration for the general class of $\mathcal{N} = 1$ manifolds we are considering, starting from the supersymmetry conditions for four-dimensional space-time filling branes derived in the previous sections. We will see how it is possible to naturally introduce a generalised calibration that minimises the energy density and with respect to which supersymmetric cycles are calibrated. The notion of generalised calibration was first introduced in [192, 193] to describe supersymmetric branes on backgrounds with fluxes, and studied in several subsequent papers (see for example [194, 195]). The idea is that the calibration should minimise the brane energy density, which does not necessarily coincide with the volume wrapped by the brane. It has been shown in [166] how, in the case of pure NS supersymmetric backgrounds, it is possible to introduce another notion of generalised calibration which naturally takes into account the role of the worldvolume field strength f. We will now see how an analogous definition of generalised calibration can also be used for general $\mathcal{N} = 1$ backgrounds with nontrivial RR fluxes.

We define a *generalised calibration* as a sum of forms of different degree $\omega = \sum_k \omega_{(k)}$ such that $d_H\omega = (d + H\wedge)\omega = 0$ and

$$P_\Sigma[\omega] \wedge e^{\mathcal{F}} \leq \mathcal{E}(\Sigma, \mathcal{F}) \ , \tag{6.96}$$

for any D-brane $(\Sigma, \mathcal{F})$ characterised by the wrapped cycle Σ and the worldvolume field strength $\mathcal{F}$ and with energy density $\mathcal{E}$ [12]. In (6.96) and all other expressions in this section involving sums of forms of different degree on the cycle wrapped by the brane, we understand that only forms of rank equal to the dimension of the cycle are selected. Furthermore, the inequalities between these forms refer to the associated scalar components in the one-dimensional base given by the standard (oriented) volume form.

A D-brane $(\Sigma, \mathcal{F})$ is then *calibrated in a generalised sense* by $\omega = \sum_k \omega_{(k)}$, if it satisfies the condition

$$P_\Sigma[\omega] \wedge e^{\mathcal{F}} = \mathcal{E}(\Sigma, \mathcal{F}) \ . \tag{6.97}$$

Since the generalised calibration ω is d_H-closed, one can immediately prove that the saturation of the calibration bound is a minimal energy condition. Let $E(\Sigma, \mathcal{F})$ be the four-dimensional energy density of a calibrated wrapped D-brane $(\Sigma, \mathcal{F})$. Consider a continuous deformation to a different brane configuration $(\Sigma', \mathcal{F}')$ such that we can take a chain $\mathcal{B}$ and a field-strength $\hat{\mathcal{F}}$ on it (with $d\hat{\mathcal{F}} = P_{\mathcal{B}}[H]$), such that $\partial\mathcal{B} = \Sigma - \Sigma'$ and the restriction of $\hat{\mathcal{F}}$ to Σ and Σ' gives $\mathcal{F}$ and $\mathcal{F}'$ respectively. We then have

$$E(\Sigma, \mathcal{F}) = \int \mathcal{E}(\Sigma, \mathcal{F}) = \int_\Sigma P[\omega] \wedge e^{\mathcal{F}} \tag{6.98}$$

$$= \int_{\mathcal{B}} P[d_H\omega] \wedge e^{\hat{\mathcal{F}}} + \int_{\Sigma'} P[\omega] \wedge e^{\mathcal{F}'} \tag{6.99}$$

$$= \int_{\Sigma'} P[\omega] \wedge e^{\mathcal{F}'} \leq \int \mathcal{E}(\Sigma', \mathcal{F}') = E(\Sigma', \mathcal{F}') \ . \tag{6.100}$$

A calibration condition can then be seen as a stability condition for a D-brane under continuous deformations, i.e. a D-brane wrapping a supersymmetric cycle is the lowest energy object in its homology class.

We will now see how the supersymmetry conditions in (6.78) and (6.79) can be

[12] This definition is completely equivalent to the definition used in [166] where a generalised calibration $\tilde{\omega}$ is closed, i.e. $d\tilde{\omega} = 0$, and satisfies the relation $P_\Sigma[\omega] \wedge e^f \leq \mathcal{E}(\Sigma, \mathcal{F})$. The two generalised calibrations are obviously related by $\tilde{\omega} = \omega \wedge e^B$. We prefer our choice as it involves the worldvolume field-strength f only through the gauge invariant combination $\mathcal{F}$.

rephrased as generalised calibration conditions. In order to prove this, we have to construct the generalised calibration appropriate to our case. Let us start by recalling that we are restricting to the case in which $\eta^{(1)}$ and $\eta^{(2)}$ have the same norm. Then the standard Cauchy-Schwarz inequality

$$||i\hat{\gamma}'_{(r)}(\mathcal{F})\eta^{(2)}_+ + \eta^{(1)}_{(-)^r}|| \leq ||i\hat{\gamma}'_{(r)}(\mathcal{F})\eta^{(2)}_+|| + ||\eta^{(1)}_{(-)^r}|| \ , \tag{6.101}$$

implies that we have the following completely general inequalities

$$\mathrm{Re}\left[i\eta^{(1)\dagger}_+\hat{\gamma}'_{(2k)}(\mathcal{F})\eta^{(2)}_+\right] \leq |a|^2 \quad , \quad \mathrm{Re}\left[i\eta^{(1)\dagger}_-\hat{\gamma}'_{(2k+1)}(\mathcal{F})\eta^{(2)}_+\right] \leq |a|^2 \ , \tag{6.102}$$

which, remembering the supersymmetry conditions (6.71), are clearly saturated when we are considering supersymmetric cycles. Using expression (6.67) for $\hat{\gamma}'_{(r)}$ it is not difficult to see that from these relations we obtain the conditions

$$\Big\{\mathrm{Re}\big(-iP[\Psi^+]\big)\wedge e^{\mathcal{F}}\Big\}_{(2k)} \leq \frac{|a|^2}{8}\sqrt{\det(P[g]+\mathcal{F})}d\sigma^1\wedge\ldots\wedge d\sigma^{2k} \ , \tag{6.103}$$

$$\Big\{\mathrm{Re}\big(-iP[\Psi^-]\big)\wedge e^{\mathcal{F}}\Big\}_{(2k+1)} \leq \frac{|a|^2}{8}\sqrt{\det(P[g]+\mathcal{F})}d\sigma^1\wedge\ldots\wedge d\sigma^{2k+1} \ . \tag{6.104}$$

Once we impose that the D-branes must wrap generalised complex submanifolds in M, one sees that requiring these inequalities in (6.103) and (6.104) to be saturated is equivalent to requiring that the D-branes we are considering satisfy the supersymmetry conditions (6.78) and (6.79).

We would now like to use these inequalities to construct a generalised calibration for this space-time filling branes. Given the RR field-strength ansatz specified in (6.19), we can analogously decompose the RR potentials in the following way

$$C_{(n)} = \hat{C}_{(n)} + dx^0\wedge\ldots\wedge dx^3\wedge e^{4A}\tilde{C}_{(n-4)} \ , \tag{6.105}$$

and then express the internal RR field strengths in terms of the internal RR potentials

$$\hat{F}_{(k+1)} = d\hat{C}_{(k)} + H\wedge\hat{C}_{(k-2)} \ , \tag{6.106}$$

$$\tilde{F}_{(k+1)} = d\tilde{C}_{(k)} + H\wedge\tilde{C}_{(k-2)} + 4dA\wedge\tilde{C}_{(k)} \ . \tag{6.107}$$

Our space-time filling branes couple only to the "tilded" RR fields. Since we are considering static configurations, we can extract from the Dirac-Born-Infield plus Chern-Simons action the following effective energy density for a space-time filling brane wrapping an internal n-cycle [192]

$$\mathcal{E} = e^{4A}\Big\{e^{-\Phi}\sqrt{\det(P[g]+\mathcal{F})}d\sigma^1\wedge\ldots\wedge d\sigma^n - \Big(\sum_k P[\tilde{C}_{(k)}]\wedge e^{\mathcal{F}}\Big)_{(n)}\Big\}\ , \tag{6.108}$$

where for simplicity we have omitted the overall factor given by the D-brane tension. We can now write the inequalities (6.103) and (6.104) in terms of a lower bound on the energy density

$$P[\omega]\wedge e^{\mathcal{F}} \leq \mathcal{E}\ , \tag{6.109}$$

where we have used the sum of forms of different degrees $\omega = \sum_k \omega_{(k)}$ given by

$$\omega_{IIA} = e^{4A}\Big[\mathrm{Re}\Big(\frac{-8i}{|a|^2}e^{-\Phi}\Psi^-\Big) - \sum_k \tilde{C}_{(2k+1)}\Big]\ , \tag{6.110}$$

$$\omega_{IIB} = e^{4A}\Big[\mathrm{Re}\Big(\frac{-8i}{|a|^2}e^{-\Phi}\Psi^+\Big) - \sum_k \tilde{C}_{(2k)}\Big]\ . \tag{6.111}$$

Note that in the left-hand side of (6.109) one can completely factorise the contributions of the background quantities through the pullback on the cycle of ω and B, and the contribution from the worldvolume field-strength f.

It is clear from (6.109) that the ω's defined in (6.110) and (6.111) represent a good candidate for generalised calibrations as described at the beginning of this section. To prove that this is indeed the case, it remains to show that the ω's in (6.110) and (6.111) are d_H-closed. In order to do this, it will be enough to use the equations (6.46) and (6.49), which characterise our $\mathcal{N}=1$ backgrounds.

Let us impose the vanishing of the d_H-differential of the ω's defined in (6.110) and (6.111). This gives the following condition to have properly defined calibrations

$$d_H\omega_{IIA} = 0 \Leftrightarrow \big[d + (H+4dA)\wedge\big]\Big[\frac{1}{|a|^2}e^{-\Phi}\mathrm{Re}\Big(i\Psi^-\Big)\Big] = -\frac{1}{8}\sum_{k=0,1,2,3}\tilde{F}_{(2k)}, \tag{6.112}$$

$$d_H\omega_{IIB} = 0 \Leftrightarrow \big[d + (H+4dA)\wedge\big]\Big[\frac{1}{|a|^2}e^{-\Phi}\mathrm{Re}\Big(i\Psi^+\Big)\Big] = -\frac{1}{8}\sum_{k=0,1,2}\tilde{F}_{(2k+1)}. \tag{6.113}$$

One immediately sees that the background supersymmetry conditions (6.46) and (6.49) imply that the above requirements are indeed satisfied. This concludes our proof that our $\mathcal{N}=1$ backgrounds are generalised complex manifolds with generalised calibrations defined in (6.110) and (6.111), such that supersymmetric four-dimensional spacetime filling branes wrap generalised complex submanifolds that are also generalised calibrated.

One can get more intuition on the structure of the above generalised calibrations by considering the $SU(3)$-structure subcase. The generalised calibrations then take the form

$$\omega_{IIA} = e^{4A}\Big[\text{Re}\Big(- e^{i\tau}e^{-\Phi}\Omega\Big) - \sum_k \tilde{C}_{(2k+1)}\Big] \ , \tag{6.114}$$

$$\omega_{IIB} = e^{4A}\Big[\text{Re}\Big(- ie^{i\phi}e^{-\Phi}e^{-iJ}\Big) - \sum_k \tilde{C}_{(2k)}\Big] \ . \tag{6.115}$$

We then explicitly see how these calibrations generalise the usual calibrations in Calabi-Yau spaces through crucial modifications introduced by the nontrivial dilaton, warp-factor and fluxes.

Note also that the generalised calibrations (6.110) and (6.111) are naturally related by the mirror symmetry (6.50) if we exchange $\sum_k \tilde{C}_{(2k)}$ and $\sum_k \tilde{C}_{(2k+1)}$. These sums can be seen as H-twisted potentials of the sums of internal field strengths $\tilde{F}_A$ and $\tilde{F}_B$ as defined in (6.47) and (6.48). If we think in terms of untwisted quantities we then get a mirror symmetry for the potentials of the form

$$\sum_k \tilde{C}_{(2k)} \wedge e^B \leftrightarrow \sum_k \tilde{C}_{(2k+1)} \wedge e^B \ , \tag{6.116}$$

which clearly recalls the form of the transformation rules of the RR-potentials under T-duality.

Let us observe that the generalised calibration ω defined above is a sum of forms which are not generally globally defined, since they are not invariant under the RR gauge transformations. Indeed, consider the gauge transformation

$$\sum_n \delta\tilde{C}_{(n)} = e^{-4A} d_H\lambda \ , \tag{6.117}$$

preserving the decomposition (6.105), where λ is a sum of even (odd) forms for Type IIA (IIB). Then ω transforms as $\omega \to \omega - d_H\lambda$, since it is related to the D-brane energy density which naturally depends on the RR gauge potentials. As an alternative, we could also introduce an equivalent globally defined generalised calibration $\hat{\omega} = \sum_n \hat{\omega}_{(n)}$ which is more in the spirit of that adopted in [192, 193]. First, in our class of backgrounds, $\hat{\omega}$ is no longer d_H closed, but must satisfy the condition

$$d_H\hat{\omega} = e^{4A}\sum_k \tilde{F}_{(k)} \ . \tag{6.118}$$

Secondly, the energy density minimisation condition (6.109) is replaced by the condition

$$P_\Sigma[\hat{\omega}] \wedge e^{\mathcal{F}} \leq e^{4A-\Phi}\sqrt{\det(P[g]+\mathcal{F})}d\sigma^1 \wedge \ldots d\sigma^n \ , \tag{6.119}$$

for any D-brane $(\Sigma, \mathcal{F})$ wrapping an internal n-dimensional cycle. It is clear from our previous discussion that such an alternative generalised calibration is given by

$$\hat{\omega} = -\frac{8e^{4A-\Phi}}{|a|^2}\mathrm{Re}(i\Psi) \ , \tag{6.120}$$

where $\Psi = \Psi^+$ for Type IIB and $\Psi = \Psi^-$ for Type IIA. We obviously have that $\omega = \hat{\omega} - e^{4A}\sum_n \tilde{C}_{(n)}$ and the alternative generalised calibration $\hat{\omega}$ can be essentially identified with the imaginary part of the non-integrable pure spinor characterising the $\mathcal{N} = 1$ background considered.

As we are assuming $|a| = |b|$, the condition (6.118) is equivalent to the imaginary part of the first background supersymmetry condition of (6.46), thus giving a physical interpretation for it. It is interesting to note that an analogous conclusion can be reached for the remaining equations in (6.46). Indeed, we have seen in section (6.4) how we could also consider supersymmetric branes filling only two or three flat space-time directions, giving rise to an effective string or domain wall respectively with appropriately chosen phases α in (6.68). One can then repeat the arguments of this section for these cases, with the generalised calibrations now given by

$$\omega^{(string)} = \frac{8e^{2A-\Phi}}{|a|^2}\mathrm{Re}(\Psi_1) \quad , \quad \omega^{(DW)} = \frac{8e^{3A-\Phi}}{|a|^2}\mathrm{Re}(e^{i\varphi}\Psi_2) \ , \tag{6.121}$$

where $\Psi_1 = \Psi^+$ (Ψ^-) and $\Psi_2 = \Psi^-$ (Ψ^+) for Type IIB (IIA), and φ is an arbitrary (constant) phase. The generalised calibrations $\omega^{(string)}$ and $\omega^{(DW)}$ now satisfy the condition (6.119) with e^{4A} substituted by e^{2A} and e^{3A} respectively. Furthermore, they must now be d_H-closed, since the coupling to the background RR-fields vanishes for these configurations . It is easy to see that the condition $d_H\omega^{(string)} = 0$ is equivalent to the real part of the first of (6.46) (with $|a| = |b|$), while $d_H\omega^{(DW)} = 0$ for any φ is equivalent to the second of (6.46) . We then see how, in the subcase where the two internal spinors have the same norm, the background supersymmetry conditions (6.46) have a physical interpretation as conditions for the existence of generalised calibrations for the allowed supersymmetric D-brane configurations. This correspondence between background supersymmetry conditions and generalised calibrations has been extensively discussed in [194] and we see here how it works perfectly in the cases we have considered.

6.8 Conclusions

In this chapter we have studied the conditions for having supersymmetric D-branes in Type II backgrounds with general NS and RR fields preserving four-dimensional Poincaré invariance and $\mathcal{N} = 1$ supersymmetry, focusing on D-branes filling the four flat directions. It transpired that the supersymmetry conditions for D-branes obtained from κ-symmetry arguments can be elegantly expressed in terms of the two pure spinors that define the $SU(3)\times SU(3)$-structure on the internal six-dimensional manifold. We have shown that the supersymmetry conditions give two important pieces of information about the supersymmetric D-branes, involving the two pure spinors separately. These conditions were related to the geometry and the stability of the branes, just as in previous cases in the absence of fluxes.

Firstly, the we found that the D-brane must wrap a generalised complex submanifold defined with respect to the integrable generalised complex structure of the internal manifold. This can be introduced thanks to the integrability of one of the two pure spinors coming from the requirement of $\mathcal{N} = 1$ supersymmetry. The $SU(3)$ structure subcase provides a clear example where this condition means that the brane must wrap a holomorphic cycle with $(1, 1)$ field strength $\mathcal{F}$ in Type IIB and a coisotropic cycle of the kind discussed in [160, 186] in Type IIA. In the more general $SU(3) \times SU(3)$ case the equivalent Type IIA/IIB identifications become slightly mixed.

Secondly, we found that on the wrapped internal n-cycle one must furthermore impose a condition of the form $\{\mathrm{Im}(P[i\Psi]) \wedge \mathcal{F}\}_{(n)} = 0$, where Ψ is the non-integrable pure spinor. This condition is related to the stability of the D-brane. Note that it is the non-integrable pure spinor that now plays the relevant role and the fact that it should be connected to some dynamical information for the D-branes can be linked to the role of the nontrivial RR-fields as obstructions to the integrability of the pure spinor. A supersymmetric D-brane configuration must then satisfy the above two conditions, plus an appropriate choice of its orientation which is in general not arbitrary due to presence of nontrivial background RR fields.

The above requirements that characterise supersymmetric D-branes are equivalent to the condition that the D-brane must be calibrated in a generalised sense with respect to an appropriate definition of generalised calibration. This encodes a requirement of minimisation of the energy density of the brane, rather than volume-minimisation, and involves the non-integrable pure spinor. The non-integrability of this pure spinor is due to the non-vanishing RR-fields, which also couple to D-branes and so must enter the associated generalised calibrations. One then sees that the non-integrability of the pure spinor is exactly what is needed to compensate for the presence of the RR terms in the generalised calibration in order for the calibration to be well defined. This strict relation between the non-integrable pure spinor and a generalised calibration can be made even more explicit by using the equivalent alternative definition of generalised calibration given in (6.118) and (6.119). Furthermore, as we discussed at the end of section 6.7, by considering D-branes filling only two or three flat directions, the conditions for the existence of well defined calibrations associated to supersymmetric D-branes are completely equivalent to the background supersymmetry conditions (6.46), thus giving a clear physical interpretation for them.

To conclude, it is intriguing to see how the two pure spinors can be fruitfully used in the description of the geometrical and stability features of supersymmetric D-branes. In both Type IIA and IIB, a supersymmetric D-brane must wrap a generalised complex submanifold with respect to the integrable pure spinor and be calibrated in a generalised sense with respect to the non-integrable pure spinor. Also, we have seen how all the results discussed in this chapter confirm the interpretation of the symmetry (6.50) relating Type IIA and IIB backgrounds as a generalised mirror symmetry, exchanging also odd and even dimensional supersymmetric cycles and the corresponding generalised calibrations. These results may hide some deeper insight into the understanding of string theory on general backgrounds with fluxes and its relation to generalised geometry.

Chapter 7

Conclusions and Future Directions

In this thesis we have discussed several aspects of branes in supergravity, ranging from issues of consistency in phenomenologically motivated five-dimensional braneworld models, to the rigorous constraints placed on D-branes in general flux compactifications. We shall now conclude with a review of our main results and some suggestions for future work.

In chapter 4 we studied the problem of the stability of Hořava-Witten spacetimes, which we identified as generic domain solutions of the form $M_4 \times \mathcal{I}$, where $\mathcal{I}$ is an interval. We were particularly interested in the case where the interval $\mathcal{I}$ could be understood as an orbifold $S^1/\mathbb{Z}_2$, as in the much studied Randall-Sundrum models. A key feature required for the consistency of such models is the appearance of negative tension branes and we showed that, despite receiving much attention, it was not clear whether such spacetimes were in fact stable.

To tackle the issue of stability, we chose to concentrate on a class of five-dimensional models of gravity coupled to a scalar field with a double exponential potential. This potential could be written in terms of a superpotential using standard tools from supergravity, allowing us to define a consistent action for both the bulk fields and the brane sources supporting our $\mathbb{Z}_2$-symmetric domain wall solution. A careful treatment of the definition of energy for this class of spacetimes proved that the $\mathbb{Z}_2$-symmetry was crucial here. In particular, the Israel junction conditions for our domain walls simplified significantly, reducing to boundary conditions for the bulk fields ((4.33), (4.34)). This meant that if we were able to prove the stability of the bulk theory alone, it would be sufficient to show that the entire bulk plus brane system was stable.

The proof of stability followed using the standard spinorial techniques of the positive energy theorem in general relativity and classical supergravity. A key point was the identification of the superpotential for our bulk theory, which allowed us to assume the form of the supersymmetry transformations for the gravitino and dilatino [110]. Regardless of whether this supersymmetric completion could be realised, this identification meant we could rewrite our spinorial energy expression in terms of a sum of squares of exactly these supersymmetry transformations (4.77), thus proving the positivity of energy. We were also able to show that the spinorial energy expression agreed perturbatively with the conserved energy which we had constructed previously using the Abbott-Deser pseudotensor technique. The spinorial proof of positive energy then implied that this class of $\mathbb{Z}_2$-symmetric spacetimes was stable. Thus we have seen how the background domain wall solution behaves as one would expect for a supersymmetric solution, acting as a ground state which bounds the energy of perturbations from below.

While our proof of positive energy was successful, it also raised several questions. An obvious next step was to ask what happens when we relax the $\mathbb{Z}_2$-symmetry. An analysis of this question showed that we quickly run into difficulties. The $\mathbb{Z}_2$-symmetry was crucial in allowing us to prove positive energy at the level of the background solution. In fact, when considering non-symmetric domain walls we do not have a definition of energy that is conserved by virtue of the background equations of motion alone. In this case, the Israel junction conditions no longer simplify and one must solve the equations order by order in perturbations. This would suggest that, as it stands, our approach is not valid for studying perturbations of these more general spacetimes, and hence we are unable to draw any conclusions about their stability. One possible area for development would be to consider singular braneworld models with higher codimension. Using our methods, it should be possible to prove the stability of the symmetric models without resorting to a perturbative analysis.

A second question of considerable interest relates to the supersymmetry of our five-dimensional theory and its singular domain wall solutions. We know that the bulk theory is derived from a consistent truncation of an S^5 dimensional reduction of Type IIB supergravity in ten dimensions, and thus should lie in a subsector of a theory with $\mathcal{N} = 8$ supersymmetry. The complete nonlinear ansatz for this reduction is not known, however the appropriate $\mathcal{N} = 2$ subsector has been constructed [119], and one can show that the supersymmetry transformations that we putatively identified for the gravitino and dilatino agree with the reduction of the Type IIB fermionic terms in this subsector.

With this in hand, one is naturally led to lift the singular domain wall solution back to ten dimensions to understand its relation to the known supersymmetric p-brane solutions [108, 110, 113]. One finds that there is a coupling in the brane source which is indicative of a D3-brane in ten dimensions, as one would expect due to the five-form field strength being non-zero in the original reduction for the ansatz for the bulk fields. However there is also a second contribution in the brane source term [113]. A simple counting argument for the conformal factors in the source term shows that this object should have an eight-dimensional worldvolume, suggesting a D7-brane. The utility of D7-branes in constructing compactifications to warped five-dimensional models has been suggested before [19, 20], however in the breathing mode reduction we considered, the axion and dilaton fields, which source the D7-brane, were not present. As this is a consistent truncation of the field equations in ten dimensions one should not expect it to cause any difficulty, therefore we are left with the problem of identifying what creates the second term in the brane source. One may attempt to identify this eight-dimensional object, previously called the 'turtle' [113], with one of the exotic gravitational solutions found in [129], however this does not correctly reproduce the couplings we see in the lifted brane source term.

Alternatively, one can search for a ten-dimensional projection operator that reduces to the $\mathbb{Z}_2$ action on the domain wall after compactification, analogous to the original Hořava-Witten scenario in eleven-dimensional supergravity. In that case, the boundary brane projection operators commuted with half the supersymmetry transformations, implying that the solution preserved half the supersymmetry. It had been proposed in [119] that such a projection for the Type IIB case necessarily includes an orientation flip on the S^5 directions, and therefore in the five-form flux parameter, thus causing a disparity in the supersymmetry transformation at the position of the domain wall. Work in this direction in ongoing, but initial attempts suggest that it is not possible to construct a projection operator in ten dimensions that commutes with the supersymmetry transformations[1]. As such, the supersymmetry and ten-dimensional origin of the class of singular domain wall solutions we have studied remains unclear.

We also noted that the argument used to determine the dimensions of the branes from which a singular domain wall descends in a breathing mode reduction could be extended to other parent theories. For instance, one can apply this simple technique to domain walls which are solutions to the bosonic theories arising from S^4 and S^7

[1] This work is in progress with J. Kalkkinen, J. L. Lehners and K. S. Stelle, whom we thank for useful discussions on these points.

compactifications of eleven-dimensional supergravity. In doing so, one finds that the domain walls lift back to $M2$ and $M5$-brane sources respectively, however once again there is a second contribution in each case. This corresponds to an object with a nine-dimensional worldvolume, an 8-brane, which is not a known supersymmetric solution of eleven-dimensional supergravity. The interpretation of this result, and the original question of determining what supersymmetry, if any, is preserved for this class of singular domain walls, is work in progress.

In chapter 5 we considered the generalisation of braneworld sum rules [24], which provide a straightforward scheme to test the consistency of five-dimensional braneworld models of gravity. By making use of convenient combinations of components of Einstein's equations, one can determine constraints on the various components of a given model. We extended these rules to incorporate more general spacetimes (5.29), including non-compact internal spaces, and we were able to provide further insights into models that had been studied previously. For instance, we were able to reconsider gravity-trapping domain wall models in five dimensions, such as the second (one brane) Randall-Sundrum model. Using the generalised sum rules, we were able to show that if the internal space is non-compact, the Strong Energy principle need not be violated for these solutions to exist (5.41), which, to our knowledge, had not been appreciated before.

Unfortunately, the generalised sum rules offered little insight into supergravity p-brane solutions with more general, Ricci-flat transverse space geometries [147]. However, our investigation of these solutions led us to propose a generalised version of the ADM energy for such branes. Our expression agreed with previous formulae in the flat transverse space limit, and we were able to evaluate this energy explicitly for the case of a Heterotic 5-brane on an Eguchi-Hanson instanton. As the topology of the transverse space is $\mathbb{R} \times S^3/\mathbb{Z}_2$ in this example, we found the result to be half the energy of a regular p-brane, as expected. It would be interesting to apply our improved ADM energy integral to more general p-brane spacetimes and to construct the complete set of charges for them. This should offer further insights into their supersymmetry and "black brane" mechanics [86, 147], and also their relation to the more familiar p-brane solutions, such as those reviewed in chapter 3. We leave this for future work.

Chapter 6 was dedicated to the constraints on supersymmetric D-branes in general flux compactifications of Type II string theories. We were particularly interested in compactifications to four-dimensional Minkowski space preserving $\mathcal{N} = 1$ supersymmetry, rather than the more familiar $\mathcal{N} = 2$ supersymmetry of Calabi-

Yau compactifications. Such scenarios are realised by turning on general Neveu-Schwarz and Ramond-Ramond fields, and including a warp factor multiplying the four-dimensional metric component [5]. Recent developments in this area have shown how such compactifications have an elegant description in terms of reduced SU(3) × SU(3) structure on the formal sum of the tangent and cotangent bundles $T_Y \oplus T_Y^\star$ [157–161]. We reviewed this prescription and the main definitions of the associated generalised geometry, along with the definition of pure Clifford(6,6) spinors on this bundle. This proved useful in allowing us to rewrite the background conditions for preserved $\mathcal{N} = 1$ supersymmetry in a very compact manner (6.46). We then considered D-branes filling the four flat spacetime directions and wrapping cycles in the internal manifold in these backgrounds. Using standard κ-symmetry techniques we determined the conditions for preserved supersymmetry from the D-brane worldvolume perspective, and showed that these too could be rewritten in an elegant form in terms of the two pure spinors associated to the SU(3) × SU(3) structure on $T_Y \oplus T_Y^\star$ ((6.78), (6.79)).

The first condition implied that the D-brane must wrap a generalised complex submanifold of the internal manifold and was given by the integrable pure spinor, i.e. the pure spinor which was twisted closed $d_H\Psi = 0$. This provides a generalisation of the well-known results on branes wrapping cycles of Calabi-Yau manifolds [162, 163], where now our condition includes the effects of both Neveu-Schwarz and Ramond-Ramond fields. The second condition was related to the stability of the D-brane and was given in terms of the non-integrable pure spinor ($d_H\Psi \neq 0$), with the Ramond-Ramond fields forming the obstruction to this spinor being twisted closed.

We were able to incorporate both supersymmetry conditions, plus the necessary requirement of choice of orientation, into a single expression involving the non-integrable pure spinor in each case. This was shown to be equivalent to requiring that a D-brane should be *generalised calibrated* with respect to an appropriate definition of calibration on the generalised background. This statement was then shown to imply that supersymmetric D-branes on these backgrounds are energy density minimising within their homology class. This is nothing more than the familiar Bogomol'nyi bound for supersymmetric solitons.

That this calibration condition is given in terms of the non-integrable pure spinor is understood by considering the energy density of a D-brane with static worldvolume (6.108), which includes couplings to Ramond-Ramond fields. It is exactly the Ramond-Ramond fields that act as an obstruction to integrability for one of the pure spinors in Type IIA and IIB, and so we see that it is natural that calibration condition should be phrased in terms of this pure spinor in each case. In fact, it is precisely the non-integrability of the pure spinor that is needed to account for the Ramond-Ramond terms in generalised calibration ((6.110),(6.111)) in order to make the calibration well defined (6.96). The final step in ensuring that we had a good definition for our calibration forms was to prove that they are closed with respect to the twisted derivative operator. This followed directly from the background supersymmetry conditions written in terms of the pure spinors ((6.112),(6.113)).

One appealing feature of our formulation of the supersymmetry conditions for D-branes on generalised backgrounds is the symmetry of the equations under the exchange of the pure spinors $\Psi^+ \leftrightarrow \Psi^-$ and Ramond-Ramond fields $F_A \leftrightarrow F_B$ (6.50). This symmetry has been proposed in the literature as a generalised form of the usual mirror symmetry on Calabi-Yau manifolds [152, 154, 176–180], interchanging Type IIA and IIB backgrounds. We have seen that supersymmetric D-branes also respect this symmetry, which now exchanges odd and even dimensional supersymmetric cycles in the generalised backgrounds and the corresponding generalised calibrations. This provides further strength to the argument that generalised mirror symmetry is a fundamental property of Type II string theories compactified on manifolds with flux.

The are several interesting directions for the future development of this work. In [196, 197] a proposal was made relating calibrations on Calabi-Yau manifolds to the superpotential of the low energy effective theories generated by compactifying M-theory or Type IIA, with branes wrapping internal cycles. It would be interesting to extend our analysis of D-branes in general compactifications to instantonic branes. This would allow us to compare the superpotentials defined from our calibrations with those found by dimensional reduction of the gravitino supersymmetry transformations of Type II theories in [152]. It would be interesting to consider whether these superpotential corrections have an effect on moduli stabilisation in the low energy theory. Also, the form of the potentials generated will have implications for cosmological scenarios, such as those of Kachru et al [198, 199]. To understand these points we should construct explicit examples of instantonic D3-branes and determine whether they possessed the correct number of fermionic zero-modes to produce superpotential corrections [200]. Initial investigations have appeared in the literature [201–203], however these issues deserve to be reassessed for D-branes on more general $SU(3) \times SU(3)$ backgrounds preserving $\mathcal{N} = 1$ supersymmetry in four dimensions, as described by our results.

Acknowledgements

I thank all the friends I have made in the physics community over the last 5 years, and my friends from elsewhere who have abided me during that time.

At various stages I have enjoyed the hospitality of CERN, the Newton Institute and the Institute for Theoretical Physics at K.U. Leuven. I gratefully acknowledge the support of PPARC and the Leverhulme Trust.

I would like to thank the Theoretical Physics group at Imperial College, in particular Rachel Bean, Aidan Burch, Pedro Gandra, Dan Hook and Marko Ivin. Over the last 3 years I have worked with Jean-Luc Lehners, to whom I extend my sincere thanks as a friend and collaborator. Special thanks go to Pete Dodd for a careful reading of this manuscript and for being a great friend.

I especially thank Stephon Alexander, Tim Evans, Hugh Jones, Joao Magueijo and Ray Rivers for support at various stages. I thank Brandon Carter, Gary Gibbons, Jonathan Halliwell, Akihiro Ishibashi, Chris Pope, Paul Townsend, Arkady Tseytlin, Antoine Van Proeyen and in particular Jussi Kalkkinen and Mohab Abou-Zeid for useful comments and discussions, and Luca Martucci for initiating our fruitful collaboration.

I thank my friends and colleagues in Leuven, and especially Caroline Vandenplas for her patience and love.

Throughout my time as research student I have enjoyed the unwavering support, guidance and patience of Kelly Stelle, whom I thank for countless conversations that have shaped and deepened my understanding of theoretical physics.

Finally, I extend my deepest thanks to my parents, Geoffrey and Elizabeth Smyth. Their moral support, encouragement, love and patience is endless, and I cannot thank them enough. I dedicate this thesis to them.

Appendix A

Conventions

We will mainly follow Wald [32] for metric, curvature and stress-energy tensor conventions. The D-dimensional metric will be mostly plus throughout $(-+++ \ldots +)$, and for compactness we shall use lower case latin indices $a, b = 0, \ldots, D-1$. The vielbein are defined by $g_{ab} = e^{\underline{a}}{}_{a} e^{\underline{b}}{}_{b} \eta_{\underline{ab}}$, where underlined indices run over flat tangent space directions. Covariant derivatives are defined by

$$\nabla_a e^{\underline{a}}_b = \partial_a e^{\underline{a}}_b + \omega_a{}^{\underline{ab}} e_{b\underline{b}} - \Gamma^c_{ab} e^{\underline{a}}_c = 0 \ , \tag{A.1}$$

$$\nabla_a \psi = \partial_a \psi + \tfrac{1}{2} \omega_a{}^{\underline{bc}} \sigma_{\underline{bc}} \psi \ . \tag{A.2}$$

The Riemann tensor is defined by

$$[\nabla_a, \nabla_c] V_b = R_{acb}{}^d V_d \ , \tag{A.3}$$

for an arbitrary vector field V_b. We choose $R_{acb}{}^d = +\partial_c \Gamma^d_{ab} - \cdots$, such that the Ricci tensor is

$$R_{ab} = \partial_c \Gamma^c_{ab} - \partial_a \Gamma^c_{cb} + \Gamma^c_{ab} \Gamma^d_{cd} - \Gamma^d_{ac} \Gamma^c_{bd} \ . \tag{A.4}$$

The Einstein-Hilbert term in the action is defined in natural units with a plus sign $S_{\rm EH} = +\int \sqrt{-g} R$, where g denotes the metric determinant, R the Ricci scalar and we use $8\pi G_4 = 1$ in four dimensions. The stress-energy tensor is then defined as minus the variation of the matter action:

$$T_{ab} = -\frac{1}{\sqrt{-g}} \frac{\delta S_{\rm matter}}{\delta g^{ab}} \ . \tag{A.5}$$

We use weight one for (anti)symmeterisation

$$A_{[ab]} = \tfrac{1}{2}\left(A_{ab} - A_{ba}\right) \qquad , \qquad A_{(ab)} = \tfrac{1}{2}\left(A_{ab} + A_{ba}\right) \tag{A.6}$$

We use the standard notation for differential forms, the wedge product and the exterior derivative on a D-dimensional manifold (See [204])

$$\alpha = \frac{1}{p!}\alpha_{a_1\ldots a_p} dx^{a_1} \wedge \ldots \wedge dx^{a_p} \quad , \quad \alpha \in \wedge^p \ , \tag{A.7}$$

$$\alpha \wedge \beta = (-1)^{pq}\beta \wedge \alpha \quad , \quad \alpha \in \wedge^p \ , \beta \in \wedge^q \ , \tag{A.8}$$

$$d\alpha \equiv \frac{1}{p!}\partial_{[b}\alpha_{a_1\ldots a_p]} dx^b \wedge dx^{a_1} \wedge \ldots \wedge dx^{a_p} \ . \tag{A.9}$$

The epsilon symbol is defined by

$$\varepsilon_{a_1\ldots a_D} \equiv (+1, -1, 0) \ , \tag{A.10}$$

for (odd, even, no) permutations of the order of the indices. We can also define the symbol with upper indices by

$$\varepsilon^{a_1\ldots a_D} \equiv (-1)^t \varepsilon_{a_1\ldots a_D} \ , \tag{A.11}$$

where t is the number of timelike coordinates. We define the epsilon tensors by

$$\epsilon_{a_1\ldots a_D} = \sqrt{g}\varepsilon_{a_1\ldots a_D} \qquad , \qquad \epsilon^{a_1\ldots a_D} = \frac{1}{\sqrt{g}}\varepsilon^{a_1\ldots a_D} \ , \tag{A.12}$$

and note the following useful identity

$$\epsilon_{a_1\ldots a_r, a_{r+1}\ldots a_p}\epsilon^{a_1\ldots a_r, b_{r+1}\ldots b_p} = (-1)^t r!(n-r)!\delta^{b_{r+1}\ldots b_p}_{a_{r+1}\ldots a_p} \ , \tag{A.13}$$

We define the Hodge dual by

$$*(dx^{a_1} \wedge \ldots \wedge dx^{a_p}) \equiv \frac{1}{(D-p)!}\epsilon_{b_1\ldots b_{D-p}}{}^{a_1\ldots a_p} dx^{b_1} \wedge \ldots \wedge dx^{b_{D-p}} \ . \tag{A.14}$$

Taking $p = 0$ in this formula we find

$$*1 = \epsilon = \frac{1}{n!}\epsilon_{b_1\ldots b_D} dx^{b_1} \wedge \ldots \wedge dx^{b_D} = \sqrt{|g|}dx^1 \wedge \ldots \wedge dx^D. \tag{A.15}$$

which gives a natural definition of the volume element $\sqrt{|g|}d^n x$ for a D-dimensional manifold. The volume element on a (D-n)-dimensional submanifold is then defined by

$$d^{D-n}\Sigma_{b_1\dots b_n} \equiv \frac{1}{(D-n)!\sqrt{|g|}}\epsilon_{b_1\dots b_n a_1\dots a_{D-n}} dx^{a_1} \wedge \dots \wedge dx^{a_{D-n}} \ . \tag{A.16}$$

For a more discussion of integration and submanifolds were refer the reader to [32].

For spinors and gamma matrices we follow the conventions of [67] (another useful reference is [34]). The D-dimensional Gamma matrices satisfy $\{\Gamma_a, \Gamma_b\} = 2g_{ab}$. Using the anti-symmeterisation defined above we note

$$\Gamma^{a_1\dots a_n} \equiv \Gamma^{[a_1}\Gamma^{[a_2}\cdots\Gamma^{a_n]} \tag{A.17}$$

$$\Gamma^a\Gamma^b = \Gamma^{ab} + \eta^{ab} \ , \tag{A.18}$$

$$\Gamma^a\Gamma^b\Gamma^c = \Gamma^{abc} + \eta^{ab}\Gamma^c - \eta^{ac}\Gamma^b + \eta^{bc}\Gamma^a \ , \tag{A.19}$$

$$\Gamma^{abc}\Gamma_d = \Gamma^{abc}{}_d + 3\Gamma^{[ab}\delta^{c]}{}_d. \tag{A.20}$$

It is worthwhile to note the following contractions

$$\Gamma^{abc}\Gamma_c = (D-2)\Gamma^{ab} \qquad , \qquad \Gamma^{ca}\Gamma_c = -(D-1)\Gamma^a \ . \tag{A.21}$$

Repeated use of these identities leads to the formula

$$\Gamma^{abc}\Gamma^{de}R_{bcde} = 4\Gamma^b G_b{}^a \ , \tag{A.22}$$

where now we have converted to gamma matrices with curved space indices using the vielbein. Using the integrability condition

$$[\nabla_c, \nabla_d]\,\eta = \frac{1}{4}R_{cdab}\Gamma^{ab}\eta \ , \tag{A.23}$$

for a spinor η, one finds the following combination of the above equations

$$\overline{\eta}\Gamma^{abc}\nabla_b\nabla_c\eta = \frac{1}{2}T^{ab}\overline{\eta}\Gamma_b\eta \ , \tag{A.24}$$

where T^{ab} is the energy-momentum tensor, $\overline{\eta} = \eta^\dagger C$ and C is the charge conjugation matrix.

A useful expression for gamma matrices acting on scalar fields is

$$\Gamma^b\Gamma^a\Gamma^c\phi_{,b}\phi_{,c} = 2\phi^{,a}\Gamma^c\phi_{,c} - \Gamma^a\phi^{,c}\phi_{,c} \ . \tag{A.25}$$

Appendix B

Basic definitions for $SU(3)$ structure manifolds

In this section we will review some basic facts about an $SU(3)$-structure manifold M, that is characterised by the existence of a globally defined spinor η_+, such that $||\eta||^2 = |a|^2$ (for a nice review on this subject see for example [154]). This spinor allows one to introduce an associated almost complex structures with respect to which the six-dimensional metric g_{mn} is Hermitian, where now $m, n = 1, \ldots, 6$. For our purposes the most useful choice is given by

$$J_{mn} = -\frac{i}{|a|^2}\eta_+^\dagger \hat{\gamma}_{mn}\eta_+ \ . \tag{B.1}$$

Using the Fierz identities, it is possible to show that

$$J_m{}^p J_p{}^n = -\delta_m^n \quad , \quad J_m{}^p J_n{}^q g_{pq} = g_{mn} \ , \tag{B.2}$$

the second of which is the Hermiticity condition. This almost complex structure allows one to introduce the projector on holomorphic indices

$$\mathcal{P}_m{}^n = \frac{1}{2}(\delta_m^n - iJ_m{}^n) \ , \tag{B.3}$$

and the associated anti-holomorphic projector $\bar{\mathcal{P}}_m{}^n = (\mathcal{P}_m{}^n)^*$. One can then split r-forms in (p, q)-forms, with $p + q = r$, in the standard way.

The following relations hold

$$\eta_+^\dagger \hat{\gamma}^m \hat{\gamma}_n \eta_+ = 2|a|^2 \bar{\mathcal{P}}^m{}_n \quad , \quad \eta_-^\dagger \hat{\gamma}^m \hat{\gamma}_n \eta_- = 2|a|^2 \mathcal{P}^m{}_n \ . \tag{B.4}$$

Then $\hat{\gamma}_m\eta_+ = \mathcal{P}_m{}^n\hat{\gamma}_n\eta_+$ (it is of the kind $(1,0)$ in the index m), and the base (6.72) is indeed eight dimensional. The general six-dimensional Dirac spinor χ can then be decomposed as

$$\chi = \lambda_1\eta_+ + \lambda_2\eta_- + \xi_1^m\hat{\gamma}_m\eta_+ + \xi_2^m\hat{\gamma}_m\eta_- \ , \tag{B.5}$$

where ξ_1^m is a $(1,0)$-vector ($\mathcal{P}_m{}^n\xi_1^m = \xi_1^n$) and ξ_2^m is a $(0,1)$-vector ($\bar{\mathcal{P}}_m{}^n\xi_1^m = \xi_1^n$). Then,

$$\begin{aligned} \lambda_1 &= \frac{1}{|a|^2}\eta_+^\dagger\chi \quad , \quad \lambda_2 = \frac{1}{|a|^2}\eta_-^\dagger\chi \quad , \\ \xi_1^m &= \frac{1}{2|a|^2}\eta_+^\dagger\hat{\gamma}^m\chi \quad , \quad \xi_2^m = \frac{1}{2|a|^2}\eta_-^\dagger\hat{\gamma}^m\chi \ . \end{aligned} \tag{B.6}$$

Analogously to the CY_3 case, we can also introduce a $(3,0)$ form Ω defined by

$$\Omega_{mnp} = -\frac{i}{a^2}\eta_-^\dagger\hat{\gamma}_{mnp}\eta_+ \ . \tag{B.7}$$

By applying Fierz identities it is possible to see that

$$\frac{1}{3!}J\wedge J\wedge J = \frac{i}{8}\Omega\wedge\bar{\Omega} \quad , \quad J\wedge\Omega = 0 \ , \tag{B.8}$$

as for Calabi-Yau manifolds. The existence of a globally defined non-degenerate (real) J and a globally defined non-degenerate (complex) Ω satisfying the conditions (B.8) actually characterises $SU(3)$-structure manifolds. In our case we are considering the more general case of internal manifolds M with $SU(3)\times SU(3)$-structure group for $T_M\oplus T_M^\star$. This contains as subcases the $SU(3)$-structure manifolds case and the even more restricted manifolds with $SU(2)$-structure, that contain two different independent $SU(3)$ structures and requires the vanishing of the Euler characteristic χ of M.

Bibliography

[1] J. L. Lehners, P. Smyth and K. S. Stelle, "Stability of Hořava-Witten spacetimes", Class. Quant. Grav. **22** (2005) 2589 [hep-th/0501212].

[2] J. Polchinski, "Dirichlet-Branes and Ramond-Ramond Charges", Phys. Rev. Lett. **75** (1995) 4724 [hep-th/9510017].

[3] M. Berkooz, M. R. Douglas and R. G. Leigh, "Branes intersecting at angles", Nucl. Phys. B **480** (1996) 265 [hep-th/9606139].

[4] R. Blumenhagen, M. Cvetic, P. Langacker and G. Shiu, "Toward realistic intersecting D-brane models", [hep-th/0502005].

[5] M. Grana, "Flux compactifications in string theory: A comprehensive review", [hep-th/0509003].

[6] L. Martucci and P. Smyth, "Supersymmetric D-branes and calibrations on general N = 1 backgrounds", [hep-th/0507099].

[7] M. J. Duff, B. E. W. Nilsson and C. N. Pope, "Kaluza-Klein Supergravity", Phys. Rept. **130** (1986) 1.

[8] K. Akama, "An Early Proposal Of 'Brane World' ", Lect. Notes Phys. **176** (1982) 267 [hep-th/0001113].

[9] V. A. Rubakov and M. E. Shaposhnikov, "Do We Live Inside A Domain Wall?", Phys. Lett. B **125** (1983) 136.

[10] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, "The hierarchy problem and new dimensions at a millimeter", Phys. Lett. B **429** (1998) 263 [hep-ph/9803315].

[11] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, "New dimensions at a millimeter to a Fermi and superstrings at a TeV", Phys. Lett. B **436** (1998) 257 [hep-ph/9804398].

[12] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, "Phenomenology, astrophysics and cosmology of theories with sub-millimeter dimensions and TeV scale quantum gravity", Phys. Rev. D **59** (1999) 086004 [`hep-ph/9807344`].

[13] L. Randall and R. Sundrum, "A large mass hierarchy from a small extra dimension", *Phys. Rev. Lett.* **83** (1999) 3370, [`hep-th/9905221;`].

[14] L. Randall and R. Sundrum, "An alternative to compactification", *Phys. Rev. Lett.* **83** (1999) 4690, [`hep-th/9906064`].

[15] S. J. Smullin, A. A. Geraci, D. M. Weld, J. Chiaverini, S. Holmes and A. Kapitulnik, "New constraints on Yukawa-type deviations from Newtonian gravity at 20-microns", [`hep-ph/0508204`].

[16] J. Chiaverini, S. J. Smullin, A. A. Geraci, D. M. Weld and A. Kapitulnik, "New experimental constraints on non-Newtonian forces below 100-microns", Phys. Rev. Lett. **90** (2003) 151101 [`hep-ph/0209325`].

[17] S. Eidelman *et al.* [Particle Data Group Collaboration], "Review of particle physics", Phys. Lett. B **592** (2004) 1.

[18] R. Durrer, "Braneworlds", AIP Conf. Proc. **782** (2005) 202 [`hep-th/0507006`].

[19] H. L. Verlinde, "Holography and compactification", Nucl. Phys. B **580** (2000) 264 [`hep-th/9906182`].

[20] C. S. Chan, P. L. Paul and H. L. Verlinde, "A note on warped string compactification", Nucl. Phys. B **581** (2000) 156 [`hep-th/0003236`].

[21] L.F. Abbott and S. Deser, "Stability of Gravity with a Cosmological Constant", *Nucl.Phys.* **B195** (1982) 76.

[22] R. Arnowitt, S. Deser and C.W. Misner, "The Dynamics of General Relativity", in *Gravitation: An Introduction to Current Research,* 1962, ed. L. Witten, Wiley, [`gr-qc/0405109`].

[23] C.W. Misner, K.S. Thorne and J.A. Wheeler, "Gravitation", 1973, ed. W.H. Freeman & Co.

[24] G. W. Gibbons, R. Kallosh and A. D. Linde, "Brane world sum rules", JHEP **0101** (2001) 022 [`hep-th/0011225`].

[25] G. W. Gibbons, "Aspects of Supergravity Theories", in *Supersymmetry, Supergravity and Related Topics*, eds. F. del Aguila, J. A. de Azcaragga and L. E. Ibanez, World Scientific (1985).

[26] D. Z. Freedman, S. S. Gubser, K. Pilch and N. P. Warner, "Renormalization group flows from holography supersymmetry and a c-theorem", Adv. Theor. Math. Phys. **3** (1999) 363 [`hep-th/9904017`].

[27] R. Kallosh and A. Linde, "Supersymmetry and the brane world", JHEP **0002**, 005 (2000) [`hep-th/0001071`].

[28] K. Behrndt and M. Cvetic, "Anti-de Sitter vacua of gauged supergravities with 8 supercharges", Phys. Rev. **D61**, 101901 (2000) [`hep-th/0001159`].

[29] G. W. Gibbons and N. D. Lambert, "Domain walls and solitons in odd dimensions", Phys. Lett. **B488**, 90 (2000) [`hep-th/0003197`].

[30] J. Maldacena and C. Nunez, "Supergravity description of field theories on curved manifolds and a no go theorem", Int. J. Mod. Phys. A **16** (2001) 822 [`hep-th/0007018`].

[31] S. Deser, "Energy In Gravitational Theories: Definition, Positivity Theorems And Stability", Print-83-0258 (BRANDEIS), Invited lecture given at 11th Texas Symposium on Relativistic Astrophysics, Austin, Tex., Dec 13-17, 1982.

[32] R. M. Wald, "General Relativity", 1984, ed. The University of Chicago Press.

[33] J. M. Nester, "General pseudotensors and quasilocal quantities", Class. Quantum Grav. **21**, S261-S280 (2004).

[34] T. Ortin, "Gravity and strings", 2004, ed. Cambridge University Press.

[35] S. Hollands, A. Ishibashi and D. Marolf, "Comparison between various notions of conserved charges in asymptotically AdS-spacetimes", Class. Quant. Grav. **22** (2005) 2881 [`hep-th/0503045`].

[36] M. Cvetic, G. W. Gibbons, H. Lu and C. N. Pope, "Rotating black holes in gauged supergravities: Thermodynamics, supersymmetric limits, topological solitons and time machines", [`hep-th/0504080`].

[37] I. Papadimitriou and K. Skenderis, "Thermodynamics of asymptotically locally AdS spacetimes", [`hep-th/0505190`].

[38] A. Komar, "Covariant Conservation Laws In General Relativity", Phys. Rev. **113** (1959) 934.

[39] J. Katz, "A note on Komar's anomalous factor", Class. Quant. Grav. **2** (1985) 423.

[40] A. Ashtekar and A. Magnon-Ashtekar "Energy-Momentum in General Relativity", Phys. Rev. Lett. **43** (1979) 181.

[41] S. Deser and B. Tekin, "Energy in Generic Higher Curvature Gravity Theories", *Phys. Rev.* **D67** (2003) 084009, `hep-th/0212292.`

[42] S. W Hawking and G. F. R. Ellis, "The large scale structure of space-time", 1973, ed. Cambridge University Press.

[43] T. Shiromizu, "Positivity of gravitational mass in asymptotically de Sitter space-times", Phys. Rev. D **49** (1994) 5026.

[44] S. A. Hayward, T. Shiromizu and K. i. Nakao, "A Cosmological constant limits the size of black holes", Phys. Rev. D **49** (1994) 5080 [`gr-qc/9309004`].

[45] K. Nakao, T. Shiromizu and K. Maeda, "Gravitational mass in asymptotically de Sitter space-times", Class. Quant. Grav. **11** (1994) 2059.

[46] A. Ashtekar and A. Magnon, "Asymptotically anti-de Sitter space-times", Class. Quant. Grav. **1** (1984) L39.

[47] A. Ashtekar and S. Das, "Asymptotically anti-de Sitter space-times: Conserved quantities", Class. Quant. Grav. **17** (2000) L17 [`hep-th/9911230`].

[48] J. Winicour, "Angular Momentum in General Relativity". In *General Relativity And Gravitation. 100-Years After The Birth Albert Einstein. Vol. 1* 1980, ed. A. Held, Plenum (New York).

[49] R. M. Wald and A. Zoupas, "A General Definition of "Conserved Quantities" in General Relativity and Other Theories of Gravity", Phys. Rev. D **61** (2000) 084027 [`gr-qc/9911095`].

[50] J. Katz, J. Bicak and D. Lynden-Bell, "Relativistic conservation laws and integral constraints for large cosmological perturbations", Phys. Rev. D **55** (1997) 5957 [`gr-qc/0504041`].

[51] N. Deruelle and J. Katz, "On the mass of a Kerr-anti-de Sitter spacetime in D dimensions", Class. Quant. Grav. **22** (2005) 421 [`gr-qc/0410135`].

[52] M. Henningson and K. Skenderis, "The holographic Weyl anomaly", JHEP **9807** (1998) 023 [`hep-th/9806087`].

[53] V. Balasubramanian and P. Kraus, "A stress tensor for anti-de Sitter gravity", Commun. Math. Phys. **208** (1999) 413 [`hep-th/9902121`].

[54] M. Henneaux and C. Teitelboim, "Asymptotically Anti-De Sitter Spaces", Commun. Math. Phys. **98** (1985) 391.

[55] S. Deser and M. Soldate, "Gravitational Energy in Spaces with Compactified Dimensions", *Nucl. Phys.* **B311** (1989) 739.

[56] R. D. Sorkin, "Kaluza-Klein Monopole", Phys. Rev. Lett. **51** (1983) 87.

[57] L. Bombelli, R. K. Koul, G. Kunstatter, J. H. Lee and R. D. Sorkin, "On Energy In Five-Dimensional Gravity And The Mass Of The Kaluza-Klein Monopole", Nucl. Phys. B **289** (1987) 735.

[58] S. W. Hawking, "Gravitational Instantons", Phys. Lett. A **60** (1977) 81.

[59] G.W. Gibbons, G.T. Horowitz and P.K. Townsend, "Higher dimensional resolution of dilatonic black hole singularities", Class. Quantum Grav. **12**, 297 (1995), [`hep-th/9410073`].

[60] R. Schoen and S. T. Yau, "On The Proof Of The Positive Mass Conjecture In General Relativity", Commun. Math. Phys. **65** (1979) 45; **79** (1981) 231.

[61] E. Witten, "A New Proof of the Positive Energy Theorem", Commun. Math. Phys. **80** (1981) 381.

[62] J.M. Nester, "A New Gravitational Energy Expression with a Simple Positivity Proof", Phys. Lett. **83A** (1981) 241.

[63] S. Deser and C. Teitelboim, "Supergravity Has Positive Energy", Phys. Rev. Lett. **39** (1977) 249.

[64] S. Deser,"Positive Classical Gravitational Energy From Classical Supergravity", Phys. Rev. D **27** (1983) 2805.

[65] D. Brill and P. S. Jang, "The Positive Mass Conjecture". In *General Relativity And Gravitation. 100-Years After The Birth Albert Einstein. Vol. 1* 1980, ed. A. Held, Plenum (New York).

[66] K. S. Stelle, "Gravity and Supergravity" unpublished notes.

[67] A. Van Proeyen, "Tools for supersymmetry", Annals of the University of Craiova, Physics AUC **9** 1 (1999) 1 [hep-th/9910030].

[68] T. Parker and C. H. Taubes, "On Witten's Proof Of The Positive Energy Theorem", Commun. Math. Phys. **84** (1982) 223.

[69] O. Reula, "Existence Theorem For Solutions Of Witten's Equation And Non-negativity Of Total Mass", J. Math. Phys. **23** (1982) 810.

[70] G. W. Gibbons, S. W. Hawking, G. T. Horowitz and M. J. Perry, "Positive Mass Theorems For Black Holes",Commun. Math. Phys. **88**, 295 (1983).

[71] E. Witten and D. I. Olive, "Supersymmetry Algebras That Include Topological Charges", Phys. Lett. B **78** (1978) 97.

[72] G. W. Gibbons and C. M. Hull, "A Bogomolny Bound For General Relativity And Solitons In N=2 Supergravity", Phys. Lett. B **109** (1982) 190.

[73] S. Deser and B. Zumino, "Consistent Supergravity", Phys. Lett. B **62** (1976) 335.

[74] D. Z. Freedman, P. van Nieuwenhuizen and S. Ferrara, "Progress Toward A Theory Of Supergravity", Phys. Rev. D **13** (1976) 3214.

[75] P. Van Nieuwenhuizen, "Supergravity", Phys. Rept. **68** (1981) 189.

[76] S. Deser, J. H. Kay and K. S. Stelle, "Hamiltonian Formulation Of Supergravity", Phys. Rev. D **16** (1977) 2448.

[77] G. T. Horowitz and A. Strominger, "On Witten's Expression For Gravitational Energy", Phys. Rev. D **27** (1983) 2793.

[78] C. M. Hull, "The Positivity of Gravitational Energy and Global Supersymmetry", Commun. Math. Phys. **90** (1983) 545.

[79] K. S. Stelle, "BPS branes in supergravity", [hep-th/9803116].

[80] P. C. West,"Supergravity, brane dynamics and string duality", [hep-th/9811101].

[81] M. R. de Traubenberg, "Clifford algebras in physics", [hep-th/0506011].

[82] E. Cremmer, B. Julia and J. Scherk, "Supergravity Theory In 11 Dimensions", Phys. Lett. B **76** (1978) 409.

[83] E. Bergshoeff, E. Sezgin and P. K. Townsend, "Supermembranes And Eleven-Dimensional Supergravity", Phys. Lett. B **189** (1987) 75.

[84] M. J. Duff and K. S. Stelle, "Multi-Membrane Solutions Of D = 11 Supergravity", Phys. Lett. B **253** (1991) 113.

[85] R. Gueven, "Black p-brane solutions of D = 11 supergravity theory", Phys. Lett. B **276** (1992) 49.

[86] P. K. Townsend and M. Zamaklar, "The first law of black brane mechanics", Class. Quant. Grav. **18** (2001) 5269 [hep-th/0107228].

[87] G. T. Horowitz and A. Strominger, "Black strings and P-branes", Nucl. Phys. B **360** (1991) 197.

[88] M. J. Duff and J. X. Lu, "Black and super p-branes in diverse dimensions", Nucl. Phys. B **416** (1994) 301 [hep-th/9306052].

[89] M. J. Duff, H. Lu and C. N. Pope, "The black branes of M-theory", Phys. Lett. B **382** (1996) 73 [hep-th/9604052].

[90] J. X. Lu, "ADM masses for black strings and p-branes", Phys. Lett. B **313** (1993) 29 [hep-th/9304159].

[91] J. A. de Azcarraga, J. P. Gauntlett, J. M. Izquierdo and P. K. Townsend, "Topological Extensions Of The Supersymmetry Algebra For Extended Objects", Phys. Rev. Lett. **63** (1989) 2443.

[92] P. K. Townsend, "M-theory from its superalgebra", [hep-th/9712004].

[93] A. Dabholkar, G. W. Gibbons, J. A. Harvey and F. Ruiz Ruiz, "Superstrings And Solitons", Nucl. Phys. B **340** (1990) 33.

[94] P. Hořava and E. Witten, "Heterotic and Type I String Dynamics from Eleven-Dimensions", *Nucl. Phys.* **B460** (1996) 506, `hep-th/9510209`; "Eleven-Dimensional Supergravity on a Manifold with Boundary", *Nucl. Phys.* **B475** (1996) 94, [`hep-th/9603142`].

[95] B. A. Ovrut, "Lectures on heterotic M-theory", [`hep-th/0201032`].

[96] T. Banks and M. Dine, "Couplings and Scales in Strongly Coupled Heterotic String Theory", *Nucl. Phys.* **B479** (1996) 173, [`hep-th/9605136`].

[97] A. Lukas, B.A. Ovrut, K.S. Stelle and D. Waldram, "The Universe as a Domain Wall", *Phys. Rev.* **D59** (1999) 086001, `hep-th/9803235`; "Heterotic M-Theory in Five Dimensions", *Nucl. Phys.* **B552** (1999) 246, [`hep-th/9806051`].

[98] P. Brax, C. van de Bruck and A.-C. Davis, "Brane World Cosmology", *Rept. Prog. Phys.* **67** (2004) 2183, [`hep-th/0404011`].

[99] R. Sundrum, "To the fifth dimension and back. (TASI 2004)", [`hep-th/0508134`].

[100] C. M. Hull, "Gravitational duality, branes and charges", Nucl. Phys. B **509** (1998) 216 [`hep-th/9705162`].

[101] E. Bergshoeff, J. Gomis and P. K. Townsend, "M-brane intersections from worldvolume superalgebras", Phys. Lett. B **421** (1998) 109 [`hep-th/9711043`].

[102] B. Carter, "Energy dominance and the Hawking-Ellis vacuum conservation theorem", In *The future of theoretical physics and cosmology* 2002, ed. A. Held, Cambridge University Press,[`gr-qc/0205010`].

[103] T. Adawi, M. Cederwall, U. Gran, B.E.W. Nilsson and B. Razaznejad, "Goldstone Tensor Modes", *JHEP* **9902** (1999) 001, [`hep-th/9811145`].

[104] J.-L. Lehners and K.S. Stelle, "D=5 M-Theory Radion Supermultiplet Dynamics", *Nucl. Phys.* **B661** (2003) 273, [`hep/th-0210228`].

[105] J.T. Liu, H. Lü and C.N. Pope, "The Radion Mode in Consistent Brane World Reductions", [`hep-th/0212037`].

[106] C.P. Burgess, F. Quevedo, S.-J. Rey, G. Tasimato and I. Zavala, "Cosmological Spacetimes from Negative Tension Brane Backgrounds", *JHEP* **0210** (2002) 028, [`hep-th/0207104`].

[107] I. Moss, "Boundary terms for supergravity and heterotic M-theory", [hep-th/0403106].

[108] M. Bremer, M.J. Duff, H. Lü, C.N. Pope and K.S. Stelle, "Instanton Cosmology and Domain Walls from M Theory and String Theory", *Nucl. Phys.* **B543** (1999) 321, [hep-th/9807051].

[109] M. Cvetič, H. Lü and C.N. Pope, "Domain walls and massive gauged supergravity potentials", *Class. Quant. Grav.* **17** (2000) 4867, [hep-th/0001002].

[110] M.J. Duff, J.T. Liu and K.S. Stelle, "A supersymmetric Type IIB Randall–Sundrum realization", *J. Math. Phys.* **42** (2001) 3027, [hep-th/0007120].

[111] S. Cecotti, L. Girardello and M. Porrati, "Constraints On Partial Superhiggs", Nucl. Phys. B **268** (1986) 295.

[112] M. Cvetic and H. H. Soleng, "Supergravity domain walls", Phys. Rept. **282** (1997) 159 [hep-th/9604090].

[113] M. Cvetič, M.J. Duff, J.T. Liu, H. Lü, C.N. Pope and K.S. Stelle, "Randall–Sundrum brane tensions", *Nucl. Phys.* **B 605** (2000) 141, [hep-th/0011167].

[114] P. S. Howe and R. W. Tucker, "A Locally Supersymmetric And Reparametrization Invariant Action For A Spinning Membrane", J. Phys. A **10** (1977) L155.

[115] H. A. Chamblin and H. S. Reall, "Dynamic dilatonic domain walls", Nucl. Phys. B **562** (1999) 133 [hep-th/9903225].

[116] E. Bergshoeff, R. Kallosh and A. Van Proeyen, "Supersymmetry in Singular Spaces", *JHEP* **0010** (2000) 033, [hep-th/0007044].

[117] C. Charmousis, R. Gregory and V.A. Rubakov, "Wave function of the radion in a brane world", *Phys. Rev.* **D62** (2000) 067505, [hep-th/991216].

[118] S. Weinberg, "Gravitation and Cosmology", 1972, ed. John Wiley & Sons.

[119] J.T. Liu and H. Sati, "Breathing Mode Compactifications and Supersymmetry of the Brane-World", *Nucl. Phys.* **B605** (2001) 116, [hep-th/0009184].

[120] W. Boucher, "Positive Energy Without Supersymmetry", *Nucl. Phys.* **B242** (1984) 282.

[121] P.K. Townsend, “Positive Energy and the Scalar Potential in Higher Dimensional (Super)Gravity Theories”, *Phys. Lett.* **B148** (1984) 55.

[122] D.Z. Freedman, C. Núñez, M. Schnabl and K. Skenderis, “Fake Supergravity and Domain Wall Stability”, [hep-th/0312055].

[123] G. W. Gibbons, S. A. Hartnoll and A. Ishibashi, “On the stability of naked singularities”, [hep-th/0409307].

[124] J.J. Halliwell, “Global Spacetime Symmetries in the Functional Schrödinger Picture”, *Phys. Rev.* **D43** (1991) 2590.

[125] R. A. Battye, B. Carter, A. Mennim and J. P. Uzan, “Einstein equations for an asymmetric brane-world”, Phys. Rev. D **64** (2001) 124007 [hep-th/0105091].

[126] A. Padilla, “Infra-red modification of gravity from asymmetric branes”, Class. Quant. Grav. **22** (2005) 1087 [hep-th/0410033].

[127] A. Celi, A. Ceresole, G. Dall’Agata, A. Van Proeyen and M. Zagermann, “On the fakeness of fake supergravity”, Phys. Rev. D **71** (2005) 045009 [hep-th/0410126].

[128] M. Cvetic, H. Lu and C. N. Pope, “Consistent Kaluza-Klein sphere reductions”, Phys. Rev. D **62** (2000) 064028 [hep-th/0003286].

[129] E. Bergshoeff, U. Gran and D. Roest, “Type IIB seven-brane solutions from nine-dimensional domain walls”, Class. Quant. Grav. **19** (2002) 4207 [hep-th/0203202].

[130] M. B. Green, J. A. Harvey and G. W. Moore, “I-brane inflow and anomalous couplings on D-branes”, Class. Quant. Grav. **14** (1997) 47 [hep-th/9605033].

[131] J. Kalkkinen, Private Communication.

[132] U. Ellwanger, “Constraints on a brane-world from the vanishing of the cosmological constant”, Phys. Lett. **B473**, 233 (2000) [hep-th/9909103].

[133] A. G. Riess *et al.* [Supernova Search Team Collaboration], “The Farthest Known Supernova: Support for an Accelerating Universe and a Glimpse of the Epoch of Deceleration”, Astrophys. J. **560** (2001) 49 [astro-ph/0104455].

[134] E. Bergshoeff, S. Cucu, T. de Wit, J. Gheerardyn, S. Vandoren and A. Van Proeyen, "N = 2 supergravity in five dimensions revisited", Class. Quant. Grav. **21**, 3015 (2004) [hep-th/0403045].

[135] F. Leblond, R. C. Myers and D. J. Winters, "Consistency conditions for brane worlds in arbitrary dimensions", JHEP **0107** (2001) 031 [hep-th/0106140].

[136] G. T. Horowitz and R. C. Myers, "The AdS/CFT correspondence and a new positive energy conjecture for general relativity", Phys. Rev. D **59** (1999) 026005 [hep-th/9808079].

[137] K. Behrndt and G. Dall'Agata, "Vacuua of $N = 2$ gauged supergravity derived from non-homogeneous quaternionic spaces", [hep-th/0112136].

[138] A. Ceresole and G. Dall'Agata, "General matter coupled N = 2, D = 5 gauged supergravity", *Nucl. Phys.* **B585** (2000) 143–170 [hep-th/0004111].

[139] See, for instance, L. Anguelova and C. I. Lazaroiu, "Domain walls of N = 2 supergravity in five dimensions from hypermultiplet moduli spaces", JHEP **0209** (2002) 053 [hep-th/0208154].

[140] M. A. Melvin, "Pure Magnetic And Electric Geons", Phys. Lett. **8** (1964) 65.

[141] G. W. Gibbons and D. L. Wiltshire, "Space-Time As A Membrane In Higher Dimensions", Nucl. Phys. B **287** (1987) 717 [hep-th/0109093].

[142] J. Louko and D. W. Wiltshire, "Brane Worlds with Bolts", [hep-th/0109099].

[143] M. J. Duff, H. Lu, C. N. Pope and E. Sezgin, "Supermembranes with fewer supersymmetries", Phys. Lett. B **371** (1996) 206 [hep-th/9511162].

[144] K. Becker and M. Becker, "M-Theory on Eight-Manifolds", Nucl. Phys. B **477** (1996) 155 [hep-th/9605053].

[145] J. P. Gauntlett, G. W. Gibbons, G. Papadopoulos and P. K. Townsend, "Hyper-Kaehler manifolds and multiply intersecting branes", Nucl. Phys. B **500** (1997) 133 [hep-th/9702202].

[146] D. Brecher and M. J. Perry, "Ricci-flat branes", Nucl. Phys. B **566** (2000) 151 [hep-th/9908018].

[147] M. Cvetic, H. Lu and C. N. Pope, "Brane resolution through transgression", Nucl. Phys. B **600** (2001) 103 [hep-th/0011023].

[148] G. W. Gibbons and S. W. Hawking, "Gravitational Multi - Instantons", Phys. Lett. B **78** (1978) 430.

[149] G. Barnich and G. Compere, "Generalized Smarr relation for Kerr AdS black holes from improved surface integrals", Phys. Rev. D **71** (2005) 044016 [gr-qc/0412029].

[150] S. B. Giddings, S. Kachru and J. Polchinski, "Hierarchies from fluxes in string compactifications", Phys. Rev. D **66** (2002) 106006 [hep-th/0105097].

[151] I. Antoniadis, R. Minasian and P. Vanhove, "Non-compact Calabi-Yau manifolds and localized gravity", Nucl. Phys. B **648** (2003) 69 [hep-th/0209030].

[152] M. Grana, J. Louis and D. Waldram, "Hitchin functionals in N = 2 supergravity", [hep-th/0505264].

[153] M. B. Green, J. H. Schwarz and E. Witten, "Superstring Theory. Vol. 2: Loop Amplitudes, Anomalies And Phenomenology", 1987, ed. Cambridge University Press.

[154] S. Gurrieri, J. Louis, A. Micu and D. Waldram, "Mirror symmetry in generalised Calabi-Yau compactifications", Nucl. Phys. B **654** (2003) 61 [hep-th/0211102].

[155] M. Nakahara, "Geometry, Topology and Physics", 2003 (2nd. edition), ed. Institute of Physics.

[156] D. D. Joyce, "Compact Manifolds with Special Holonomy", 2000, ed. Oxford University Press.

[157] M. Grana, R. Minasian, M. Petrini and A. Tomasiello, "Supersymmetric backgrounds from generalised Calabi-Yau manifolds", JHEP **0408** (2004) 046 [hep-th/0406137].

[158] M. Grana, R. Minasian, M. Petrini and A. Tomasiello, "Generalised structures of N = 1 vacua", JHEP **0511** (2005) 020 [hep-th/0505212].

[159] N. Hitchin, "Generalised Calabi-Yau manifolds", Quart. J. Math. Oxford Ser. **54** (2003) 281 [math.dg/0209099].

[160] M. Gualtieri, "Generalised complex geometry", Oxford University PhD Thesis, [math.dg/0401221].

[161] C. Jeschek and F. Witt, "Generalised G(2)-structures and type IIB superstrings", JHEP **0503** (2005) 053 [hep-th/0412280];

F. Witt, "Generalised G_2-manifolds", [math.dg/0411642];

F. Witt, "Special metric structures and closed forms", [math.dg/0502443].

[162] K. Becker, M. Becker and A. Strominger, "Five-branes, membranes and nonperturbative string theory", Nucl. Phys. B **456** (1995) 130 [hep-th/9507158].

[163] M. Marino, R. Minasian, G. W. Moore and A. Strominger, "Nonlinear instantons from supersymmetric p-branes", JHEP **0001** (2000) 005 [hep-th/9911206].

[164] J. Gomis, F. Marchesano and D. Mateos, "An open string landscape", JHEP **0511** (2005) 021 [hep-th/0506179].

[165] M. Grana and J. Polchinski, "Supersymmetric three-form flux perturbations on AdS(5)", Phys. Rev. D **63** (2001) 026001 [hep-th/0009211].

[166] P. Koerber, "Stable D-branes, calibrations and generalised Calabi-Yau geometry", JHEP **0508** (2005) 099 [hep-th/0506154].

[167] A. Kapustin, "Topological strings on noncommutative manifolds", Int. J. Geom. Meth. Mod. Phys. **1** (2004) 49 [hep-th/0310057].

[168] M. Zabzine, "Geometry of D-branes for general N = (2,2) sigma models", Lett. Math. Phys. **70** (2004) 211 [hep-th/0405240].

[169] R. Zucchini,"Generalized complex geometry, generalized branes and the Hitchin sigma model", JHEP **0503** (2005) 022 [hep-th/0501062].

[170] A. Kapustin and Y. Li, "Open string BRST cohomology for generalized complex branes", [hep-th/0501071].

[171] A. Kapustin, "A-branes and noncommutative geometry", [hep-th/0502212].

[172] M. R. Douglas, "Lectures on D-branes on Calabi-Yau manifolds", Prepared for ICTP Spring School on Superstrings and Related Matters, Trieste, Italy, 2-10 Apr 2001.

[173] F. R. Harvey, "Spinors and Calibrations", 9 of Perspectives in Mathematics (1990) , ed. Academic Press, Boston.

J .Dadok and F. R. Harvey, "Calibrations and Spinors", Acta. Math. **170** (1993) 83.

[174] E. Bergshoeff, R. Kallosh, T. Ortin, D. Roest and A. Van Proeyen, "New formulations of D = 10 supersymmetry and D8 - O8 domain walls", Class. Quant. Grav. **18** (2001) 3359 [hep-th/0103233].

[175] D. Lust and D. Tsimpis, "Supersymmetric AdS(4) compactifications of IIA supergravity", JHEP **0502** (2005) 027 [hep-th/0412250].

[176] S. Fidanza, R. Minasian and A. Tomasiello, "Mirror symmetric SU(3)-structure manifolds with NS fluxes", Commun. Math. Phys. **254** (2005) 401 [hep-th/0311122].

[177] O. Ben-Bassat, "Mirror symmetry and generalized complex manifolds", [math.ag/0405303].

[178] C. Jeschek, "Generalized Calabi-Yau structures and mirror symmetry", [hep-th/0406046];

S. Chiantese, F. Gmeiner and C. Jeschek, "Mirror symmetry for topological sigma models with generalized Kaehler geometry", [hep-th/0408169].

[179] P. Grange and R. Minasian, "Modified pure spinors and mirror symmetry", [hep-th/0412086].

[180] A. Tomasiello, "Topological mirror symmetry with fluxes", JHEP **0506** (2005) 067 [hep-th/0502148].

[181] M. Cederwall, A. von Gussich, B. E. W. Nilsson, P. Sundell and A. Westerberg, "The Dirichlet super-p-branes in ten-dimensional type IIA and IIB supergravity", Nucl. Phys. B **490** (1997) 179 [hep-th/9611159].

[182] E. Bergshoeff and P. K. Townsend, "Super D-branes", Nucl. Phys. B **490** (1997) 145 [hep-th/9611173].

[183] E. Bergshoeff, R. Kallosh, T. Ortin and G. Papadopoulos, "kappa-symmetry, supersymmetry and intersecting branes", Nucl. Phys. B **502** (1997) 149 [hep-th/9705040].

[184] D. Marolf, L. Martucci and P. J. Silva, "Fermions, T-duality and effective actions for D-branes in bosonic backgrounds", JHEP **0304** (2003) 051 [hep-th/0303209];

D. Marolf, L. Martucci and P. J. Silva, “Actions and fermionic symmetries for D-branes in bosonic backgrounds”, JHEP **0307** (2003) 019 [hep-th/0306066].

[185] L. Martucci, J. Rosseel, D. Van den Bleeken and A. Van Proeyen, “Dirac actions for D-branes on backgrounds with fluxes”, Class. Quant. Grav. **22** (2005) 2745 [hep-th/0504041].

[186] A. Kapustin and D. Orlov, “Remarks on A-branes, mirror symmetry, and the Fukaya category”, J. Geom. Phys. **48** (2003) [hep-th/0109098].

[187] F. R. Harvey and H. B. Lawson, “Calibrated Geometries”, Acta. Math. **148** (1982) 47.

[188] M. Bershadsky, C. Vafa and V. Sadov, “D-Branes and Topological Field Theories,” Nucl. Phys. B **463** (1996) 420 [hep-th/9511222].

[189] K. Becker, M. Becker, D. R. Morrison, H. Ooguri, Y. Oz and Z. Yin, “Supersymmetric cycles in exceptional holonomy manifolds and Calabi-Yau 4-folds”, Nucl. Phys. B **480** (1996) 225 [hep-th/9608116].

[190] G. W. Gibbons and G. Papadopoulos, “Calibrations and intersecting branes”, Commun. Math. Phys. **202** (1999) 593 [hep-th/9803163].

[191] J. P. Gauntlett, N. D. Lambert and P. C. West, “Branes and calibrated geometries,” Commun. Math. Phys. **202** (1999) 571 [hep-th/9803216].

[192] J. Gutowski and G. Papadopoulos, “AdS calibrations”, Phys. Lett. B **462** (1999) 81 [hep-th/9902034].

[193] J. Gutowski, G. Papadopoulos and P. K. Townsend, “Supersymmetry and generalised calibrations”, Phys. Rev. D **60** (1999) 106006 [hep-th/9905156].

[194] J. P. Gauntlett, N. w. Kim, D. Martelli and D. Waldram, “Fivebranes wrapped on SLAG three-cycles and related geometry”, JHEP **0111** (2001) 018 [hep-th/0110034];

J. P. Gauntlett, D. Martelli, S. Pakis and D. Waldram, “G-structures and wrapped NS5-branes”, Commun. Math. Phys. **247** (2004) 421 [hep-th/0205050];

J. P. Gauntlett, D. Martelli and D. Waldram, “Superstrings with intrinsic torsion”, Phys. Rev. D **69** (2004) 086002 [hep-th/0302158];

D. Martelli and J. Sparks, "G-structures, fluxes and calibrations in M-theory", Phys. Rev. D **68** (2003) 085014 [`hep-th/0306225`].

[195] J. F. G. Cascales and A. M. Uranga, "Branes on generalised calibrated submanifolds", JHEP **0411** (2004) 083 [`hep-th/0407132`].

[196] S. Gukov, C. Vafa and E. Witten, "CFT's from Calabi-Yau four-folds," Nucl. Phys. B **584** (2000) 69 [Erratum-ibid. B **608** (2001) 477] [`hep-th/9906070`].

[197] S. Gukov, "Solitons, superpotentials and calibrations," Nucl. Phys. B **574** (2000) 169 [`hep-th/9911011`].

[198] S. Kachru, R. Kallosh, A. Linde and S. P. Trivedi, "De Sitter vacua in string theory," Phys. Rev. D **68** (2003) 046005 [`hep-th/0301240`].

[199] S. Kachru, R. Kallosh, A. Linde, J. Maldacena, L. McAllister and S. P. Trivedi, "Towards inflation in string theory," JCAP **0310** (2003) 013 [`hep-th/0308055`].

[200] E. Witten, "Non-Perturbative Superpotentials In String Theory," Nucl. Phys. B **474** (1996) 343 [`hep-th/9604030`].

[201] P. K. Tripathy and S. P. Trivedi, "D3 brane action and fermion zero modes in presence of background flux," JHEP **0506**, 066 (2005) [`hep-th/0503072`].

[202] N. Saulina, "Topological constraints on stabilized flux vacua," Nucl. Phys. B **720**, 203 (2005) [`hep-th/0503125`].

[203] R. Kallosh, A. K. Kashani-Poor and A. Tomasiello, "Counting fermionic zero modes on M5 with fluxes," JHEP **0506**, 069 (2005) [`hep-th/0503138`];

E. Bergshoeff, R. Kallosh, A. K. Kashani-Poor, D. Sorokin and A. Tomasiello, "An index for the Dirac operator on D3 branes with background fluxes," [`hep-th/0507069`].

[204] C. N. Pope, "Useful Formula in Differential Geometry", unpublished notes.