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A Note on the Quantum Corrections to SU(2) and SU(3) Monopoles

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Abstract. We study monopole solutions of the quantum exact low-energy effective $N = 2$ super Yang-Mills theories of Seiberg and Witten. We review the known facts about the the motion in moduli space for gauge group SU(2) and derive, using the known effective action, some analoguous effects for gauge group SU(3).

Introduction

Supersymmetric theories with $\mathcal{N} = 2$ supersymmetry serve as a wonderful toy model, since due to supersymmetry they have a very restricted behavior (compared to $\mathcal{N} = 1$ or non-supersymmetric theories) but they have a running coupling and thus interesting quantum features (unlike $\mathcal{N} = 4$).

Duality is a kind of symmetry between electric and magnetic degrees of freedom, dating back to the idea of pointlike magnetic charges. The dual of an electron is a monopole—in usual theories a finite energy configurations with a nonzero magnetic charge. If duality holds, physics should be completely democratic in treating electrons and monopoles. So to each theory where the electron couples locally to the photon and the monopole is a finite energy configuration, there should be a dual theory with the roles of the electron and monopole reversed (i.e. in the dual theory the monopole couples locally to a dual photon and the electron is a field excitation).

Classical Monopoles

The classical supersymmetric Yang-Mills-Higgs theory with a general gauge group describes the behavior of a scalar field (Higgs field) and spinor fields coupled to a generalized electric and magnetic field (Yang-Mills field), both in the adjoint representation. We shall concentrate only on the bosonic terms, and set all spinors to zero. The action for the Yang-Mills field, the kinetic term for the Higgs field and the coupling of the Higgs field to the Yang-Mills field are as in the nonsupersymmetric case. The fact that the action is obtained from a $\mathcal{N} = 2$ supersymmetric action imposes a precise form of the potential for the Higgs field $V = [\phi, \bar{\phi}]^2$.

Monopoles are configurations with finite energy, so they must approach at infinity the vacuum with zero potential. The Hamiltonian of these theories can be always split in two parts: a positive definite term and a surface term (at spatial infinity).

Thus the energy has a lower bound given by the surface term. This is identified with the bound given in central charge of a $\mathcal{N} = 2$ superalgebra, which is $Z = n_m \tau a + n_e a$ (τ is the complex coupling) [*Witten, Olive*, 1978]. Using the surface integrals we define magnetic and electric quantum numbers $n_m a_D = \int d\vec{S}\vec{B}\phi_D$, $n_e a = \int d\vec{S}\vec{\Pi}\phi$, with $\vec{\Pi}$ the conjugate momentum of the potential $\vec{A}$ and a, a_D the limiting values of the scalar ϕ resp. the dual scalar $\phi_D = \tau\phi$. In order to have nonzero quantum numbers we must have nonzero values of the scalar field ϕ at infinity, thus breaking the symmetry from $SU(2)$ to $U(1)$. Note that the expectation value of ϕ is not gauge invariant, for a gauge invariant description we must use ϕ^2.

The bound is saturated if a first order equation (called BPS equation) [*Bogomol'nyi*, 1976, *Prasad, Sommerfield*, 1975] is satisfied $\vec{B} + i\vec{E} + \sqrt{2}e^{i\alpha}\vec{\nabla}\phi = 0$, where α is a phase related to the central charge phase $\alpha + \arg Z = \pi/2$. A solution to this equation is also a solution to the equations of motion. However, this equation is too complex to be solved analytically in the generic case.

The t'Hooft Polyakov Monopole

The BPS equation can be solved for the $SU(2)$ case under certain assumptions, giving the t'Hooft-Polyakov monopole [*t' Hooft*, 1974, *Polyakov*, 1974]. One imposes a so-called radial ansatz, ie. one requires the solution to be invariant under diagonal subgroup of the product of the rotational and gauge $SO(3)$'s combined with a parity transformation. This leaves us with only two functions which must be determine. The fields are expressed in terms of these functions as $\phi^A = K(r)\hat{r}^A$ and $A_i^A = A(r)\epsilon_{Aim}\hat{r}^m$, where $\hat{r}$ is a unit radial vector. The solution to the BPS equations is then $K = v/\coth[v(r+\delta)]$, $A = 1/r - v/\sinh[v(r+\delta)]$ with two constants. The integration constant v is determined by the vaccuum expectation value of ϕ, the integration constant δ is set to zero by requiring finiteness (this will be different in the quantum case). The magnetic quantum number of the t'Hooft-Polyakov monopole is $+1$. The electric field can be added in the form $A_0^A = b(r)\hat{r}^A$, which changes the scalar field to $\operatorname{Re} e^{i\alpha}K = v/\coth[v(r+\delta)]$. The magnetic and electric fields split in two components, one parallel to the direction of the Higgs field called abelian and the perpendicular component, which is called nonabelian. The long-range behavior is in the abelian field, which is $1/r^2$.

Quantum Effective Action

General $\mathcal{N} = 2$ supersymmetric theories describing the Yang-Mills and the Higgs field have their action determined by a single holomorphic function $\mathcal{F}$, called prepotential.

In their work [*Seiberg, Witten*, 1994] Seiberg and Witten found the full quantum effective action for the $SU(2)$ case. In their derivation they used the fact that the prepotential $\mathcal{F}$ is a holomorphic function and imposed duality. Furhermore they used the beta function of the classical theory to determine the one-loop corrections to the dual field and then considered monodromies. This is sufficient to determine the prepotential. Actually they gave the prepotential in an indirect form, not as a function of the scalar field. Instead they found the description of the theory in terms of a complex moduli u. The field and dual field (which is the derivative of the prepotential) are given by elliptic integrals of this moduli

$$\phi = \frac{4}{\pi q}\mathrm{E}(q) \qquad \phi_D = -i\frac{4}{\pi q}(\mathrm{E}(q\prime) - \mathrm{K}(q\prime)) \qquad \tau = i\frac{\mathrm{K}(q\prime)}{\mathrm{K}(q)} \tag{1}$$

with $q^2 = 2/(u+1)$, $q^2 + q\prime^2 = 1$ and τ the complex coupling $\partial\phi_D/\partial\phi$. The moduli space is the complex plane with two symmetric punctures at $u = \pm\Lambda$ and the metric $ds^2 = \operatorname{Im}\tau d\phi d\bar{\phi}$, Λ is a length scale dependent on the renormalization scheme used. The point $\phi^2 = 0$ where classically the full $SU(2)$ symmetry is present, has "split" into two punctures but itself has disappeared from moduli space. In the quantum case, these points are connected with a monopole or dyon becoming massless and a break down of the low energy effective description via $\mathcal{F}$.

Quantum Corrected SU(2) Monopoles

Quantum corrected monopoles, ie. monopoles in the full quantum theory, satisfy the same BPS equations as the classical monopoles. Another requirement arises—the imaginary part of the matrix of second derivatives of the prepotential $\frac{\partial^2\mathcal{F}}{\partial\phi^A\partial\phi^B}$ must be positive definite. This determines a region in moduli space, where the BPS equations need not hold. The border of this region is called the curve of marginal stability, since upon crossing certain states in the spectrum become marginally stable and decay [*Ferrari*, 1994, *Bilal*, 1994]. In terms of the fields it is given by the equation $\operatorname{Im}\phi_D/\phi = 0$.

If one assumes that the spatial dependence of the monopole arises only from the spatial dependence of the moduli, one can analyse the motion in moduli space.

The Bianchi identity and the Gauss law imply a second order differential equation which is satisfied both by the scalar and its dual

$$\operatorname{Re} e^{i\alpha}\nabla^2\phi = 0 \qquad \operatorname{Re} e^{i\alpha}\nabla^2\phi_D = 0.$$

This, together with the asymptotic behavior implies that

$$\mathrm{Re}\, e^{i\alpha}(n_m\phi_D + n_e\phi) = 0.$$

Thus we can define a local central charge $Z(r)$ which has the "algebraic" central charge as its limit, and which has a constant phase [*Chalmers et al.*, 1996]. These relations also imply that the magnetic field and the momentum are not independent $n_e\vec{B} - 4\pi n_m\vec{\Pi} = 0$. Combining this with the explicit form of the conjugate momentum and the electromagnetic fields in the radial ansatz gives an implicit relation for the undetermined function b. Inserting in the BPS equation one ends up with a differential equation for the moduli

$$\frac{du}{dr} = \frac{\pi}{2}\left(A^2 - 2\frac{A}{r}\right)\sqrt{u+1}\frac{e^{-i\alpha}}{\mathrm{K}(q)}\frac{i}{\mathrm{Im}\,\tau}\left(\frac{n_e}{n_m} + \bar{\tau}\right). \tag{2}$$

Since the right hand side is known only as a function of u and cannot be integrated easily, this equation must be solved numerically. The factor $(A^2 - 2A/r)$ is a known function and expresses a reparametrisation. The parameter δ enters in the differential equation only through this reparametrisation. Thus the shape of the solution does not depend on δ but the exact parametrisation does.

We can get rid of the term $(A^2 - 2A/r)$ by changing the parametrisation $t(r)$ such that $dt/dr = (A^2 - 2A/r)$. In terms of the central charge we find after this reparametrisation that

$$\frac{dZ}{dt} = \frac{ie^{-i\alpha}}{n_m\mathrm{Im}\,\tau}|n_m\tau + n_e|^2 \tag{3}$$

and since the reparametrisation factor is negative and the phase constant, the absolute value of the central charge decreases. In fact the reparametrisation factor can change its sign, for δ smaller than a certain critical value depending on the asymptotic $u(\infty)$. This corresponds to bouncing solutions, solutions that change their direction at a certain point, go back along the same curve towards infinity.

In fact one can write the equation for the absolute value of the central charge as $d|Z|/dt = -1/(\sqrt{2}n_m)\partial|Z| \cdot \partial|Z|$, which is an attractor equation as in [*Ferrara, Kallosh*, 1996, *Denef*, 1999,2000] and can be derived as the zero gravity limit in the attractor formalism.

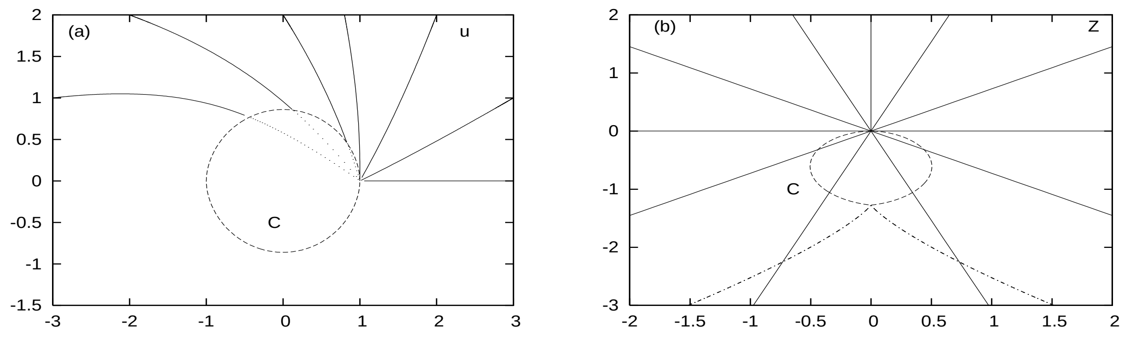

Figure 1. Solutions in moduli space (a) and in terms of the central charge (b). The dashed line C is the curve of marginal stability. The dash-dotted line in the Z-plane plot is a branch cut.

In fig. (1) some typical solutions are shown, together with the curve of marginal stability for the quantum numbers $n_m = 1$, $n_e = 0$. There are two types of behavior, depending on whether the curve of marginal stability is hit at $Z = 0$ (Z-poles) or not (X-poles).

Classical SU(3) Monopoles

The group $SU(3)$ is the next simplest case we can study. The Cartan subalgebra is two dimensional, whereas in $SU(2)$ it is one dimensional. We can choose the expectation value of the scalar field to lie in the Cartan subalgebra. The direction is given by a vector $\vec{h}$, which determines the symmetry breaking. If it is orthogonal to a root of the algebra, the symmetry breaking is nonabelian $SU(3) \to SU(2) \times U(1)$. Otherwise the symmetry breaking is maximal $SU(3) \to U(1) \times U(1)$. The magnetic charge $\vec{g}$ is now a dual root, $\vec{g} = \vec{\beta}/\beta^2$ [*Goddard et al.*, 1977]. We can expand it in terms of the dual simple roots, the coefficients (if the symmetry breaking is maximal) are topological invariants called magnetic numbers. For SU(2) there was no need to work explicitly with roots, dual roots etc. since there is only one root and only one direction in the Cartan subalgebra.

An easy way to find monopoles in $SU(3)$ is to use the $SU(2)$ t'Hooft-Polyakov monopoles [*Lee et al.*, 1996]. Each root has a $SU(2)$ subgroup associated with it, so we can use embed the t'Hooft-Polyakov monopole in this subgroup. However, since the asymptotic value of the field need not lie in this $SU(2)$, there will be an additional constant component of the field to compensate this. It can be chosen to lie in the Cartan subalgebra, orthogonal to the chosen $SU(2)$. So for each root we can rewrite a BPS solution (the functions A, b and K are as for $SU(2)$)

$$A_i = \sum_{s=1}^{3} \epsilon_{sim} \hat{r}^m A(r) t^s(\beta) \qquad A_0 = \sum_{s=1}^{3} b(r) \hat{r}^s t^s(\beta)$$

$$\phi = \sum_{i=1}^{3} K(r) \hat{r}^s t^s(\beta) + (\vec{h} - (\vec{h} \cdot \vec{\beta}) \vec{\beta}^*) \vec{H}, \tag{4}$$

where $t(\beta)$ are the generators of $SU(2)_\beta$ and $\vec{H}$ generate the Cartan subalgebra.

Since SU(3) has three roots, there are three distinct monopoles (for each $\vec{h}$) with magnetic numbers $(1,0)$, $(0,1)$ and a composite monopole with $(1,1)$. As in the SU(2) case, the vacuum expectation value $\vec{h}$ itself is not gauge invariant. For a gauge invariant description we should use the gauge invariant quantities $u = tr\phi^2$, $v = tr\phi^3$. In terms of these it is easy to see that the condition for nonmaximal symmetry breaking is $4u^3 - 27v^2 = 0$, where the gauge bosons of the unbroken $SU(2)$ have zero mass. Of course $u = v = 0$ corresponds to the full SU(3) symmetry.

SU(3) Effective Action

The quantum action for $SU(3)$ was studied in [*Argyres, Faraggi*, 1994, *Klemm et al.*, 1994]. The effective action depends now on two complex moduli u, v. The singular curve splits into two curves $4u^3 - (v \pm \Lambda^3) = 0$. As for SU(2) these are curves on which the mass of a dyon goes to zero. But, due to the higher dimension of moduli space, there are more singular points than that. The intersection of the two curves is an example, here a pair of dyons becomes massless. These dyons are mutually local, ie. they satisfy the condition $\vec{n_m}^{(1)} \vec{n_e}^{(2)} - \vec{n_e}^{(1)} \vec{n_m}^{(2)} \in Z$ and the low energy effective action can be described by a $U(1) \times U(1)$ theory. There are also points were mutually nonlocal particles become massless [*Argyres, Dougals*, 1995], the low energy description in terms of a local effectieve field theory of such a system is not known. Altogether, there are six different monopoles/dyons which become massless at certain points in moduli space, whereas there are only two in SU(2).

The scalar expectation value is written $< \phi >= \mathrm{diag}(\phi_1, \phi_2 - \phi_1, -\phi_2)$ and the duals defined by $\phi_{Di} = \partial \mathcal{F} / \partial \phi_i$. Then the explicit dependence on the moduli was found, but a closed form is possible only in certain coordinate patches of the moduli space. For SU(2) the fields ϕ, ϕ_D were elliptic integrals, which are special cases of hypergeometric functions, that have well known analytic continuations. Here, the fields are given in terms of Appell functions, ie. generalized hypergeometric functions of two variable. More precisely, in terms of F_4

$$F_4(a,b,c,c\prime;x,y)=\sum_{n,m\geq 0}\frac{(a)_{n+m}(b)_{n+m}}{(1)_n(1)_m(c)_n(c\prime)_m}x^n y^m$$

where $(a)_n$ is the Pochhammer symbol $(a)_n=\Gamma(a+n)/\Gamma(a)$. These functions converge only for $|\sqrt{x}|+|\sqrt{y}|<1$ and, unfortunately, not much seems to be known about their analytic continuations. The explicit formulas of ϕ_i and ϕ_{Di} are given in [*Klemm, Theisen*, 1995] (they are quite lengthy), together with the generalization of the coupling τ which, since it is the second derivative of $\mathcal{G}$ is now a 2×2 matrix, τ_{ij}. Only certain combinations of u, v and the scale Λ enter in the functions, so it is possible eg. to convert the Appell functions to hypergeometric functions for $u=0$, $v=0$ or to use the integral representation when on the singular curve.

Quantum Corrected SU(3) Monopoles

The central charge is a direct generalization of SU(2) $Z=n_{mi}\phi_{Di}+n_{ei}\phi_i$ and can be written also as a surface integral. By an analagous argument one can show that it has a constant phase. By a similar derivation as before we can find the differential equation for the central charge

$$\frac{dZ}{dr}=\frac{1}{2\sqrt{2}}ie^{-i\alpha}\left(A^2-2\frac{A}{r}\right)\frac{|\sum_{i,j}n_{mi}n_{ei}+n_{mi}\tau_{ij}n_{mj}|^2}{\sum_{ij}n_{mi}\mathrm{Im}(\tau_{ij})n_{mj}}. \tag{5}$$

So as before, the central charge has a constant phase and a decreasing absolut value. The reparametrisation is the same as for SU(2), including the parameter δ which can, for certain values, change the direction of the solution. The modulis satisfy the differential equations

$$\begin{aligned}\frac{du}{dr}&=\left(n_{m1}\frac{\partial u}{\partial\phi_1}+n_{m2}\frac{\partial u}{\partial\phi_2}\right)\frac{ie^{-i\alpha}}{2\sqrt{2}}\left(A^2-2\frac{A}{r}\right)\frac{\sum_{i,j}n_{mi}n_{ei}+n_{mi}\bar{\tau}_{ij}n_m mj}{\sum_{i,j}n_{mi}\mathrm{Im}\,\tau_{ij}n_{mj}}\\ \frac{dv}{dr}&=\left(n_{m1}\frac{\partial v}{\partial\phi_1}+n_{m2}\frac{\partial v}{\partial\phi_2}\right)\frac{ie^{-i\alpha}}{2\sqrt{2}}\left(A^2-2\frac{A}{r}\right)\frac{\sum_{i,j}n_{mi}n_{ei}+n_{mi}\bar{\tau}_{ij}n_m mj}{\sum_{i,j}n_{mi}\mathrm{Im}\,\tau_{ij}n_{mj}}\end{aligned} \tag{6}$$

As before these equations should be solved numerically. The equations are direct generalizations of the SU(2).

The curve of marginal stability generalizes to a curve in the (u,v) plane, which is given by three equations.

We shall point out some aspects, analoguous to $SU(2)$, for $v=0$. In this case all Appell functions reduce to hypergeometric functions. Furthermore the scalar fields and their duals are pairwise equal, ie. $\phi_1=\phi_2$, $\phi_{D1}=\phi_{D2}$. The conditions for the curve of marginal stability reduce to a single one, which is the same as for SU(2). It can be easily found, that there are three monopoles/dyons which become massless at $|u|=(27/4)^{1/3}\Lambda^2$, with quantum numbers $(n_{m1},n_{m2},n_{e1},n_{e2})$ resp. $(1,0,0,0)$, $(0,1,0,0)$, $(1,1,0,0)$. For the dyon solutions to equations (6) which lie in $v=0$. Thus the moduli motion of the dyon $(1,1,0,0)$ in the $v=0$ plane is analoguous to the motion of the monopole $(1,0)$ in SU(2). Since the curve of marginal stability is the same, the division into Z-poles and X-poles is exactly the same. The only difference lies in the different shape of the curve of marginal stability. As can be seen from fig. 2 for SU(3) this is not a closed curve, and has in fact three branches.

Conclusions

We have presented the most important well known features of quantum corrected SU(2) monopoles and their generalization to SU(3). We found differential equations for the actual space dependence of the moduli in SU(3), which will be studied more in the future, and showed some aspects for the restriction to $v=0$. Most features can be generalized directly from SU(2) to SU(3), although the generalization is not always straightforward but must be checked

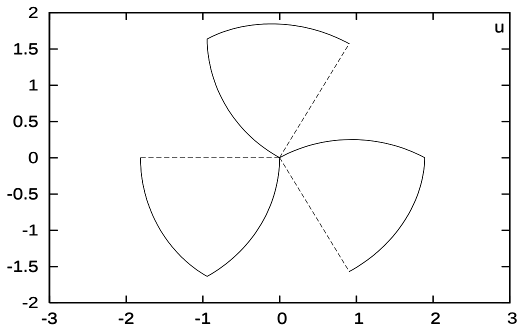

Figure 2. The curve of marginal stability and the branch cuts for SU(3) in $v = 0$ plane. The solid lines are the actual curve, the dashed lines are the branch cuts.

separately. Due to the higher dimension of the SU(3) subalgebra, new effects show up in SU(3). On the other hand the expressions are more complicated, (in fact it is even a problem to find a closed form) and are more difficult to handle numerically.

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