MASS SINGULARITIES OF FEYNMAN AMPLITUDES

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Singularities of Feynman amplitudes associated specifically with vanishing internal masses appear frequently in various applications of field theory. The most well-known example is the infrared divergences in QED and non-Abelian gauge theories. In examining the large momentum behaviour of Feynman amplitudes, we also encounter with such singularities since large momentum limit with fixed masses may be translated to certain zero mass limit with fixed momenta. It is our ultimate purpose to construct a general method for treating these so-called mass singularities[1]. Here we give a brief account of the results obtained thus far. The details will be published elsewhere[2].

We illustrate the salient features of mass singularities choosing the box diagram G (fig.1) as an example. If we ignore spin, the corresponding Feynman amplitude F_G, which is regarded as a function of internal masses $\underline{m} = \{m_1,\ldots,m_4\}$ as well as external momenta $\underline{P} = \{P_1,\ldots,P_4\}$, is written using Feynman parameters $\underline{z} = \{z_1,\ldots,z_4\}$ as[3]

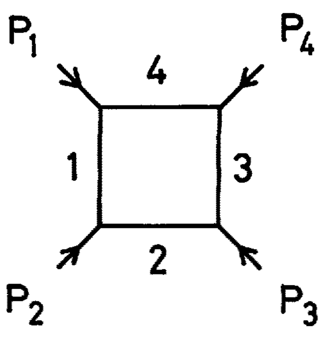

Fig.1. The box diagram G

$$F_G(\underline{m},\underline{P}) = \int \frac{\delta(1-\sum_{i=1}^{4} z_i)\ \prod_{j=1}^{4} dz_j}{[V(\underline{z},\underline{m},\underline{P})-i\varepsilon]^2}\ , \qquad (1)$$

where

$$V(\underline{z},\underline{m},\underline{P}) = \frac{1}{2}\sum_{i,j=1}^{4} v_{ij} z_i z_j\ , \qquad (2)$$

v_{ij} ($= v_{ji}$) being given by

$$((v_{ij})) = \begin{bmatrix} 2m_1^2 , & m_1^2+m_2^2-P_2^2 , & m_1^2+m_3^2-(P_2+P_3)^2 , & m_1^2+m_4^2-P_1^2 \\ & 2m_2^2 , & m_2^2+m_3^2-P_3^2 , & m_2^2+m_4^2-(P_3+P_4)^2 \\ & & 2m_3^2 , & m_3^2+m_4^2-P_4^2 \\ & & & 2m_4^2 \end{bmatrix} \quad (3)$$

The Feynman amplitude (1) develops a singularity when the singularity of the integrand given by $V(\underline{z},\underline{m},\underline{P})=0$ becomes unavoidable by any distortion of the hypersurface of integration $\sum_{i=1}^{4} z_i=1$. The corresponding singularity of F_G is classified into two types according to whether the subhypersurface of $\sum_{i=1}^{4} z_i=1$ along which V vanishes depends on (i) all, or (ii) only some of the Feynman parameters $z_1,\ldots,z_4$. We find that the former gives rise to ordinary threshold singularities. They are of no interest to us since it is not necessary for their appearance that any of the internal masses vanish. On the other hand, some of the internal masses inevitably vanish in the latter case (see below). In this sense, we call singularities of the latter type as mass singularities.

To analyze their structure, suppose, for instance, that $V(\underline{z},\underline{m},\underline{P})$ vanishes identically on the domain boundary D defined by $z_1=z_3=0, z_2+z_4=1$. We easily find from (2) and (3) that

$$V(\underline{z},\underline{m},\underline{P})\big|_{z_1=z_3=0} = m_2^2z_2^2 + \tfrac{1}{2}[m_2^2+m_4^2-(P_3+P_4)^2]z_2z_4 + m_4^2z_4^2 , \quad (4)$$

which agrees with the V function for the reduced diagram M shown in fig.2. We thus find two necessary conditions

$$(\alpha) \quad m_2 = m_4 = 0 , \qquad (\beta) \quad (P_3+P_4)^2 = (P_1+P_2)^2 = 0$$

for a mass singularity to occur. The first condition (α) justifies our definition of mass singularity. The second condition (β) says that the external momenta must be exceptional [4].

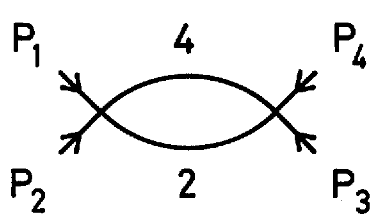

Fig.2. The reduced diagram M

The behaviour of $F_G(\underline{m},\underline{P})$ near the singularity depends strongly on how fast V vanishes at the boundary D. To examine this problem, consider the limit $\delta \to 0$ where z_1, $z_3 = O(\delta)$ and z_2, $z_4 = O(1)$, $z_2 + z_4 = 1 - O(\delta)$. The leading term of V in this limit is given by (4), which of course vanishes at the singularity. After substituting (α) and (β) into v_{ij}, we find that the $O(\delta)$ term is given by

$$(m_1^2 - P_2^2) z_1 z_2 + (m_1^2 - P_1^2) z_1 z_4 + (m_3^2 - P_3^2) z_3 z_2 + (m_3^2 - P_4^2) z_3 z_4 \quad . \qquad (5)$$

Since (5) does not necessarily vanish, V behaves as $O(\delta)$ in the general case. If $\underline{m}$ and $\underline{P}$ satisfy the further condition

$$(\gamma) \quad m_1^2 = P_1^2 = P_2^2 \qquad \text{and} \qquad m_3^2 = P_3^2 = P_4^2$$

in addition to (α) and (β), however, (5) vanishes identically and V behaves as $O(\delta^2)$, leading to an enhancement of mass singularity. Note that the three conditions (α), (β) and (γ) agree with those for the infrared singularity which appears in the forward Coulomb scattering amplitude in QED.

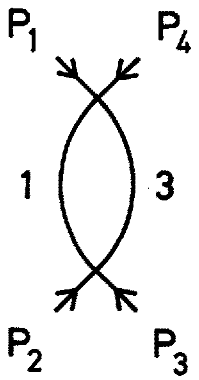

Fig.3. The reduced diagram $\tilde{M}$

The singularities we have examined above do not represent the most general ones associated with vanishing masses. In fact, z_1 and z_3 may make pinches instead of sticking to the boundary. If this happens, z_1 and z_3 have to satisfy

$$\partial V / \partial z_i = \sum_{j=1}^{4} v_{ij} z_j = 0 \quad (i=1,3) \ , \ (6)$$

for any values of z_2 and z_4. It follows that $v_{ij} = 0$ ($i=1,3$, $j=2,4$) and $v_{11} v_{33} - v_{13}^2 = 0$. The latter gives

$$(\delta) \quad (P_2 + P_3)^2 = (m_1 + m_3)^2 \ \text{and} \ m_1 z_1 - m_3 z_3 = 0$$

which are the threshold conditions for the reduced diagram $\tilde{M}$ shown in fig.3. Substituting these back into V and demanding it to vanish for arbitrary z_2 and z_4, we find that the four conditions (α), (β), (γ) and (δ) are all necessary. In a rough sense, this singularity may be interpreted as a product of the enhanced mass singularity at M (fig.2) and the threshold singularity of $\tilde{M}$ (fig.3).

All these considerations can be generalized in a straightforward way to arbitrary diagrams. In parallel with the simple example treated above, we find that mass singularities in general can be classified schematically as shown in fig.4 (in this figure, $\overline{M}'$ ($\overline{M}$) is a subdiagram (proper subdiagram) of $\overline{M}$ (G) and $G/\overline{M}$ etc. denotes the reduced diagram obtained from G by shrinking the lines of $\overline{M}$ to points, etc.).

mass singularity

$$\begin{cases} z_i = 0 \text{ for } i \in \overline{M}' \\ \partial V/\partial z_j = 0 \text{ for } j \in \overline{M} - \overline{M}' \\ V = 0 \end{cases}$$

<u>for any values of $z_k \in G - \overline{M}$.</u>

- occurs at some domain boundary $\overline{M}' = \overline{M}$
  - without enhancement $V = O(\delta)$ ...characterizes exceptional momenta
  - with enhancement $V = O(\delta^2)$ ...includes infrared singularity
- contains some pinches $\overline{M}' \subsetneq \overline{M}$ ......mass singularity† at $M = G/\overline{M}$ ⊗ threshold singularity of $\tilde{M} = H/(H-X)$ ($H = G/\overline{M}'$, $X = \overline{M}/\overline{M}'$)

(†with enhancement if X contains no loop.)

Fig.4. Classification of mass singularities.

Our future program is to establish general power counting rules for various types of mass singularities. Complications arising from ultraviolet divergences and their renormalization will be analyzed choosing $\lambda\phi^4$ theory as an example. Non-Abelian gauge theories will also be examined in connection with their infrared properties.

References

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