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Differential Geometric Methods in Theoretical Physics

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Ixtapa-Zihuatanejo, México, September 20-24, 1993

Editors

Jaime Keller
Facultad de Estudios Superiores Cuautitlán
Universidad Nacional Autónoma de México

Zbigniew Oziewicz
University of Wroclaw, Poland

Universidad Nacional Autónoma de México

XXIIth International Conference on
DIFFERENTIAL GEOMETRIC METHODS IN THEORETICAL PHYSICS

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Ixtapa - Zihuatanejo, México, September 20-24, 1993

Conference Chairman: Jaime Keller

Centro de Investigaciones Teóricas
Facultad de Estudios Superiores Cuautitlán
Universidad Nacional Autónoma de México
Apartado 70-528, 04510 México, D.F., México
keller@redvax1.dgsca.unam.mx, Fax: 525 616 2010 and 525 872 0984

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FOREWORD

JAIME KELLER
Centro de Investigaciones Teóricas
Facultad de Estudios Superiores-Cuautitlán,
Universidad Nacional Autónoma de México

(October 1993)

The XXIIth International Conference on Differential Geometry Methods in Theoretical Physics was held in México during five days, September 20-24, 1993, in the most attractive tropical Pacific coast, 200 km from Acapulco, in Ixtapa-Zihuatanejo. This series of annual conferences outgrowth of the meetings in the Professor Konrad Bleuler's house and garden.

Konrad Bleuler, an outstanding Swiss theoretical physicist, was born in 1912 in HerzogenBuchsee, Switzerland. Died on January 1, 1992, in Bonn. He was educated at Eidgenössische Technische Hochschule in Zürich, influenced mostly by his teacher and friend Wolfgang Pauli and since 1959 affiliated later to the Institut für Theoretische Kernphysik, University of Bonn.

Konrad Bleuler was fascinated how abstract mathematical structures tie into empirical quantities and that the relations among physics and geometry guided developments of both. His examples of the mathematical works inspired by physics include the abstract structure of quantum mechanics, supermathematics, Yang-Baxter and braid equations, non-Abelian gauge theory and conformal field theories.

Konrad Bleuler organized the first conferences and since this inception in 1971 he was the permanent member of the International Advisory Committees and main force behind the organizing efforts. Bleuler's intention was to bring mathematicians and physicists together and he succeed in this fusion as for example M. F. Atiyah, R. J. Baxter, M. Jimbo, Vaughan F. R. Jones, Bertram Kostant, André Lichnerowicz, Yuri I. Manin, Krzysztof Maurin, Jean M. Souriau, Shlomo Sternberg, Julius Wess, Edward Witten, Chen Ning Yang, served during many years as active members of the International Advisory Committees. Another Bleuler's successful intention was to help contacts among East and West.

The first conference in this series was held in 1972. The Proceedings of the first DGM conferences, including XIVth conference in Salamanca in 1985, were published by Springer-Verlag in the series Lectures Notes in Mathematics. There were two exceptions, the conference in Aix-en-Provence in 1974 and Warsaw conference in 1976. The last one was published as the special volume of Reports on Mathematical Physics. Since 1986 the Proceedings were published either by Plenum Press and by Kluwer (1986, 1987 and 1989), three times by World Scientific (Chester 1988, New York 1991 and Tianjin 1992) and Rapallo conference in 1990 was published again by Springer-Verlag in the series Lecture Notes in Physics.

It was decided that starting 1993 these conferences will be organized every two years.

In honour of Professor Konrad Bleuler and in order to remember his contribution in the organization of this series of conferences, we proposed a *Bleuler Medal* as a bi-ennial award to a young scientist for an outstanding contribution to Geometrical Methods in Theoretical Physics. According to a selection made by the International Advisory Committee the distinction will be certified by a diploma signed by the Chairman of the conference and a silver medal containing a universal symbol for arts and science and the engraving *Bleuler Medal,* a *year* it is been awarded and a *name* of the victor.

The Bleuler Medal 1993', the first one, we delivered to Shahn Majid from the Department of Applied Mathematics and Theoretical Physics, University of Cambridge in United Kingdom. This silver medal contains the Aztec calendar from the year 1385.

The conference in Ixtapa attracted 62 participants from 14 countries: U.S.A. (20), Germany (8), Italia (8), México (7), Russia (6), Canada (4), France (2) and with the single representatives from Belgium, Finland, Japan, New Zealand, Poland and Serbia.

Organizer and Editors:

The Symposium was organized by Jaime Keller (Chairman), Mrs A. Irma Vigil de Aragón, Mrs Maria Esther Monroy Baldi, Adolfo Obaya Valdivia, Garret Sobczyk and with the help of the International Advisory Committee:

Lawrence C. Biedenharn (University of Texas at Austin), Sultan Catto (City University of New York), Alain Connes (Institut des Hautes Etudes Scientifiques, Paris), Frank Flaherty (Oregon State University), Jurg Fröhlich (Zürich), Mo Lin Ge (Tianjin, China), Vaughan F. R. Jones (New Zealand), Louis H. Kauffman (University of Illinois at Chicago), Werner Nahm (Universitát Bonn), Cupatitzio Ramirez (Universidad Autónoma de Puebla, México), Adolfo Sanchez Valenzuela (Centro de Investigacion en Matematicas, Guanajuato, México), Julius Wess (Max-Planck-Institüt für Physik und Astrophysik, München), Chen Ning Yang (Stony Brook New York) and Bruno Zumino (University of California and Lawrence Berkeley Laboratory).

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The organizer would like to thank the financial support of Facultad de Química, Consejo Nacional de Ciencia y Tecnología, México, Dirección General de Asuntos del Personal Académico, U.N.A.M. and Sistema Nacional de Investigadores. The Editors would like to thank Mrs. A. Irma Vigil de Aragón, Claudia Rosas, Jacqueline Rosas, Dr Mićo Đurđević and Q. José Luis Aguilera, for all help and assistance in the preparation of the Proceedings.

REMEMBERING KONRAD BLEULER

WERNER NAHM
Physikalisches Institut der Universität Bonn
nahm@pib1.physik.uni-bonn.de

(Received: September 5, 1994)

It was only during his last years that I got to know Konrad Bleuler more closely, thus the memories of his many friends will do him more justice. But one aspect comes first to the mind of everyone who knew him: Bleuler had an overwhelming, never wavering enthusiasm for physics. Even after his emeritation, he regularly came to seminars, full of curiosity for new fundamental developments. After all mathematically demanding talks, he emphatically congratulated the speaker. He impressed all the students, and respect for him increased every year.

In spite of his long stay in Germany, Bleuler's Swiss origins were unmistakable. He was born on September 23, 1912 in HerzogenBuchsee, canton Bern, into a family of economically successful mechanical engineers. When he started to study in 1931 in Zürich at the Eidgenössische Technische Hochschule, it still seemed obvious that he would become an engineer, too. Soon, however, he was seduced by the beauty of mathematical forms in nature. The decisive step for his turn towards physics was his study of Riemannian geometry and Einstein's theory of gravity.

In mathematics, Heinz Hopf had the strongest influence on Bleuler, in physics it was Wolfgang Pauli. For a while, he oscillated between the two subjects, and between Zürich and Geneva. After obtaining his physics diploma in 1936 he went to Geneva for two years, then back to Zürich, where in 1942 he got his PhD in mathematics under George Polya. He became assistant of Ernest Stückelberg in Geneva, later of Walter Heitler and Gregor Wentzel in Zürich, where he obtained a titulary professorship in 1945.

Bleuler's lifelong concern was the fundamental understanding of the atomic nucleus. He stressed the importance of experimental data, but was convinced that a good understanding only would result from an esthetically pleasing theoretical basis. Quantum electrodynamics already was phenomenologically satisfying, but Bleuler very much disliked with the lack of Lorentz symmetry of the calculations. In part he certainly was influenced by Stückelberg's work, but even more the insistence on symmetry and mathematical transparency was a part of his own nature. In 1950, Suraj Narayan Gupta published his famous paper on the quantum theory of the electromagnetic field. To some, it may have looked like a pure intellectual exercise, since Gupta only considered non-interacting photons, but Bleuler immediately saw the potential of this approach. He streamlined it and showed that it could be applied equally well to the interaction with charges. Thus the Gupta-Bleuler formalism was created, which later was developed into the great number of BRST type approaches to the physical states.

In 1957, Bleuler became ordinary professor at the university of Neuchâtel. He worked hard on the understanding of the nucleus, but already had his many interests outside of physics. In particular, he had contacts to several writers, e.g. rather close ones to Carl Zuckmayer, and to Friedrich Dürrenmatt as the most famous. Bleuler decided to introduce Pauli to the latter, which led to three nights of wine and conversation. Dürrenmatt was intrigued and somewhat intimidated, so he made sure to prove his superiority in drinking. At the 1962 conference on Chaumont close to Neuchâtel, Dürrenmatt was guest of honor and gave a speech. A much more important result, however, was his play "The Physicists" (by the way, one of my first few contacts with physics). Pauli became the model for one of the physicists, but the central male character got the name Beutler and was modeled on Bleuler himself. Wolfgang Pauli told me that Bleuler's way of expressing himself was very recognizable on stage.

In 1959, Bleuler got simultaneous professorship offers to Bonn an Freiburg. In the same year, he married Tinette Specogna. In Bonn he founded the Institute für Theoretische Kern Physik, which he directed even after his emeritation until 1983. For many years, his main interest was the replacement of the older phenomenological nucleon-nucleon potentials by ones based on meson exchange. The resulting Bonn potential proved to be rather satisfactory. Nevertheless, the advent of quantum chromodynamics immediately motivated Bleuler to cast aside this framework and to argue for models based directly on quarks. He was an organizer of three conferences on quarks and nuclei.

Though he did not want to abandon the understanding of the nucleus as his own task, he always encouraged young physicists to work on more fundamental questions. The conference series on Differential Geometrical Methods in Theoretical Physics was the most influential result of his persistent attempts to stimulate the interaction of mathematicians and physicists. Before the ADHM instanton paper, the importance of such interactions was far from universally accepted. Despite good preparations in discussions with Rolf Nevanlinna and others, it hardly could be foreseen, how far Bleuler's enthusiasm would carry, when he started the series in 1971 in an almost private setting next to his home in Bonn. Personal friendship, the green environment and the pleasant atmosphere created by his wife helped to establish a tradition. Later, conferences expanded and took place in France, Poland, Italy, Israel and in the United States. Important topics since the first years were geometric quantization, supersymmetry, and the geometry of gauge theories. The conferences were always attended by leading researchers in the field.

The series continued every year without interruption, with undiminishing involvement of Bleuler himself. For me, it is a pleasant memory how much he enjoyed the Lake Tahoe meeting in 1989 and deeply moving to remember how he prepared the Tianjin conference in 1992. In spite of his illness, he personally wrote the most important initial letters. Later, he had to observe the partial decay of his sense of space and time and suffered much from it. Still, his enthusiasm for physics was undiminished. Till the end, he longed to participate in the conference. The support of his family greatly helped him during his last months. He died on January 1, 1992, leaving his wife, two children and three grandchildren.

BLEULER's MEDAL WINNER: SHAHN MAJID

JAIME KELLER and ZBIGNIEW OZIEWICZ
Centro de Investigaciones Teóricas
Facultad de Estudios Superiores-Cuautitlán,
Universidad Nacional Autónoma de México

Shahn Majid is interested in mathematics closely tied to fundamental problems in theoretical physics: non-commutative geometry, quantum groups, integrable systems and Yang-Baxter equations. His aspiration has been the development of quantum geometry provided by quantum groups (Hopf algebras).

A main result of Majid's Ph.D. thesis (Harvard University 1988) was Hopf algebras obtained by his 'bicrossproduct' construction. These Hopf algebras arose as the algebras of observables of quantum particles on homogeneous spaces and were of self-dual type. These models have some features in common with black-holes and formed Majid's quantum-geometric approach to the unification of quantum mechanics and gravity. The resulting Hopf-von Neumann algebras have been pursued further by mathematicians working in the theory of operator algebras.

Drinfeld and Jimbo introduced quasitriangular Hopf algebras. Majid found that Drinfeld's quantum double Hopf algebra could be understood as an example of his 'double cross product' construction. These quasitriangular Hopf algebras are connected with knot-invariants. Majid studied the representations of a quantum group as a braided category and interpreted the q-dimension as the category-theoretic rank.

Majid proved generalized Tannaka-Krein reconstruction theorems for quantum groups and quasi-quantum groups. One application used quantum groups to connect properties of the Wess-Zumino-Witten model to number theory.

Majid introduced a Hopf algebra in a braided category, a braided group. This is like super-group but with the $\mathbb{Z}_2$-graded transposition replaced by noninvolutive braid statistics. Braided groups include the degenerate Sklyanin algebra and Manin's quantum plane. Braided groups lie on the interface between algebra and knot theory. To prove results about braided groups one must draw braid and knot diagrams or play with pieces of string. There are braided lines, planes, matrices, differential calculi, a theory of integration on such spaces, and Lie-algebras all developed by Majid in analogy with the supergeometry.

A theory of quantum group principal bundles and connections (gauge fields) on them, including the example of a q-monopole was introduced in a joint paper by Tomasz Brzezinski and Shahn Majid in 1993.

Shahn Majid was born on November 1, 1960, in India.

SHAHN MAJID: LIST OF PUBLICATIONS

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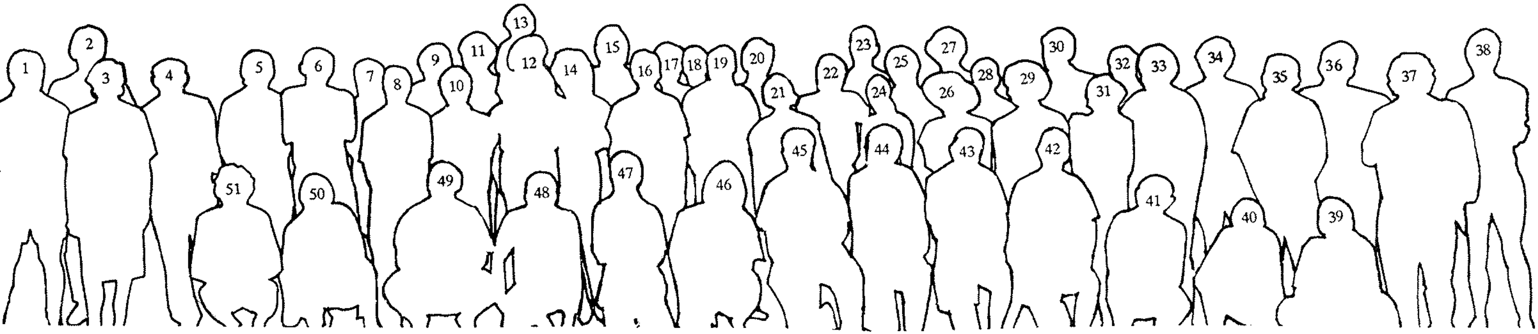

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CHAPTER I

QUANTUM GROUPS = HOPF ALGEBRAS AND BRAIDED GEOMETRY

RIBBON HOPF ALGEBRAS AND INVARIANTS OF 3-MANIFOLDS

LOUIS H. KAUFFMAN*
Department of Mathematics, Statistics and Computer Science (M/C 249)
University of Illinois at Chicago,
851 South Morgan Street, Chicago, Illinois 60607-7045.
u10451@uicvm.bitnet

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Abstract. This paper studies invariants of 3-manifolds derived from certain finite dimensional Hopf algebras. The invariants are based on right integrals for these Hopf algebras. It is shown that the resulting class of invariants is definitely distinct from the class of Witten-Reshetikhin-Turaev invariants.

Introduction

The purpose of this paper is to indicate a method of defining invariants of 3-manifolds intrinsically in terms of right integrals on certain Hopf algebras. We call such an invariant a *Hennings invariant* [5], as Hennings was the first person to point out that invariants could be defined in this way. The work reported in this paper appears more fully in joint work of the author and David Radford [10].

Hennings invariants were originally defined using oriented links. It is not necessary to use invariants that are dependent on link orientation to define 3-manifold invariants via surgery and Kirby calculus. For that reason the invariants discussed in this paper are formulated for unoriented links. This results in a simplification and conceptual clarification of the relationship of Hopf algebras and link invariants. The practical benefit is a simplified algorithmic structure for the calculation of reasoning about the invariants. *Further reference to invariants of 3- manifolds in this paper will, unless otherwise specified, be to this version of the Hennings invariant for unoriented links.*

We show in [10] that invariants defined in terms of right integrals, as considered in this paper, are definitely distinct from the invariants of Reshetikhin and Turaev. We show that our invariant is non-trivial for the quantum group $U_q(sl_2)$' when q is an fourth root of unity. The Reshetikhin Turaev invariant is trivial at this quantum group and root of unity. The non-triviality of our invariant is exhibited by showing that it distinguishes all the Lens spaces

* The author thanks the National Science Foundation for support of this research under grant number DMS-9205277.

$L(n,1)$ from one another. This proves that there is non-trivial topological information in the non-semisimplicity of $U_q(sl_2)$'.

The paper is organized as follows. Section 1 recalls Hopf algebras, quasi-triangular Hopf algebras and ribbon Hopf algebras. Section 2 discusses the conceptual setting of the invariant. This involves a summation over labellings of the link diagram by elements of the Hopf algebra. We work in a category that allows immersed diagrams so that the special grouplike element in the Hopf algebra and the ribbon element in the Hopf algebra both have diagrammatic interpretations. A trace function on the Hopf algebra that is invariant under the antipode is shown to yield a link invariant. In section 3 we show that traces of the kind discussed in section 2 are constructed from right integrals in many cases and that under suitable conditions these traces yield invariants of the 3-manifolds obtained by surgery on the links. Section 4 sketches the promised application to $U_q(sl_2)$'.

1. Algebra

Recall that a *Hopf algebra* A [20] is a bialgebra over a commutative ring k that has associative multiplication, coassociative comultiplication and is equipped with a counit, a unit and an antipode. The ring k is usually taken to be a field.

In order to be an algebra, A needs a multiplication $m : A \otimes A \to A$. The associative law for m is expressed by the equation $m(m \otimes 1) = m(1 \otimes m)$ where 1 denotes the identity map on A.

In order to be a bialgebra, an algebra needs a coproduct $\Delta : A \to A \otimes A$. The coproduct is a map of algebras, and is regarded as the dual of a multiplicative structure. Δ is coassociative. Coassociativity of Δ is expressed by the equation $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$ where 1 denotes the identity map on A.

The unit is a mapping from k to A taking 1 in k to 1 in A, and thereby defining an action of k on A. It will be convenient to just identify the units in k and in A, and to ignore the name of the map that gives the unit.

The counit is an algebra mapping from A to k denoted by $E : A \to k$. The following formulas for the counit dualize the structure inherent in the unit: $(E \otimes 1)\Delta = 1 = (1 \otimes E)\Delta$. Here the 1 denotes the identity map on A.

It is convenient to write formally $\Delta(x) = \Sigma\, x_{(1)} \otimes x_{(2)} \in A \otimes A$ to indicate the decomposition of the coproduct of x into a sum of first and second factors in the two-fold tensor product of A with itself. We shall further adopt the summation convention that $\Sigma\, x_{(1)} \otimes x_{(2)}$ can be abbreviated to just $x_{(1)} \otimes x_{(2)}$. Thus we shall $\Delta(x) = x_{(1)} \otimes x_{(2)}$.

The antipode is a mapping $s : A \to A$ satisfying equations $m(1 \otimes s)\Delta(x) = E(x)1$, and $m(s \otimes 1)\Delta(x) = E(x)1$ where 1 on the right hand side of these equations denotes the unit of k as identified with the unit of A. It is a consequence of this definition that $s(xy) = s(y)s(x)$ for all x and y in A.

A *quasitriangular Hopf algebra* A [3] is a Hopf algebra with an element $\rho \in A \otimes A$ satisfying the following equations:

1) $\rho\Delta = \Delta'\rho$ where Δ' is the composition of Δ with the map on $A \otimes A$ that switches the two factors.
2) $\rho_{12}\rho_{13} = (1 \otimes \Delta)\rho, \quad \rho_{13}\rho_{23} = (\Delta \otimes 1)\rho.$

These conditions imply that ρ has an inverse, and that $\rho^{-1} = (1 \otimes s^{-1})\rho = (s \otimes 1)\rho$.
It follows easily from the axioms of the quasitriangular Hopf algebra that p satisfies the Yang-Baxter equation

$$\rho_{12}\rho_{13}\rho_{23} = \rho_{23}\rho_{13}\rho_{12}.$$

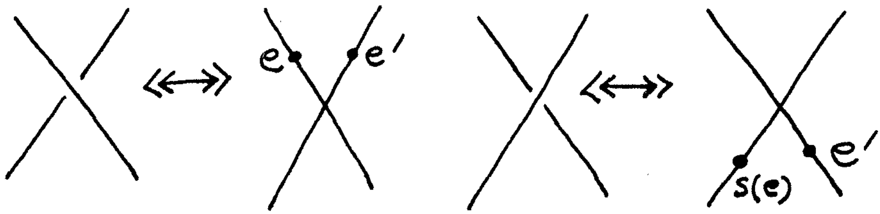
A less obvious fact about quasitriangular Hopf algebras is that there exists an element u such that u is invertible and $s^2(x) = uxu^{-1}$ for all x in A. In fact, we may take $u = \Sigma\, s(e')e$ where $\rho = \Sigma\, e \otimes e'$.
An element G in a Hopf algebra is said to be *grouplike* if $\Delta(G) = G \otimes G$ and $E(G) = 1$ (from which it follows that G is invertible and $s(G) = G^{-1}$).
A quasitriangular Hopf algebra is said to be a *ribbon Hopf algebra* [18],[9] if there exists a grouplike element G such that (with u as in the previous paragraph) $v = G^{-1}u$ is in the center of A and $s(u) = G^{-1}uG^{-1}$. We call G a *special* grouplike element of A.
Since $v = G^{-1}u$ is central, $vx = xv$ for all x in A. Therefore $G^{-1}ux = xG^{-1}u$, whence $s^2(x) = uxu^{-1} = GxG^{-1}$. Thus $s^2(x) = GxG^{-1}$ for all x in A. Similarly, $s(v) = s(G^{-1}u) = s(u)s(G^{-1}) = G^{-1}uG^{-1}G = G^{-1}u = v$. Thus the square of the antipode is represented by conjugation by the special grouplike element in a ribbon Hopf algebra, and the central element $v = G^{-1}u$ is invariant under the antipode.

2. Diagrammatic Geometry and the Trace

A function tr: $A \to k$ from the Hopf algebra to the base ring k is said to be a *trace* if

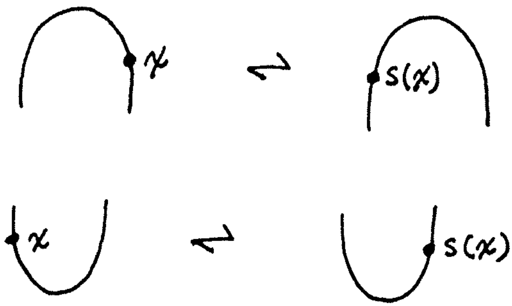
$$\begin{aligned} \mathrm{tr}(xy) &= \mathrm{tr}(yx) \quad \text{and} \\ \mathrm{tr}(s(x)) &= \mathrm{tr}(x) \end{aligned}$$

for all x and $y \in A$. In this section we describe how a trace function on a ribbon Hopf algebra yields and invariant, $TR(K)$, of regular isotopy of knots and links [6],[7].
The link diagram is arranged with respect to a vertical direction so that the crossings form the two types indicated below, and so that other than the crossings the only critical points of the height function are maxima and minima. Each crossing is decorated with elements of the Hopf algebra as shown below. Here $\rho = \Sigma\, e \otimes e'$ is the Yang-Baxter element in $A \otimes A$, and s denotes the antipode.



It is implicit in this formalism that there is a summation over all the pairs e, e' for each Yang-Baxter element.

Hopf algebra elements may be moved across maxima or minima at the expense of application of the antipode. That is, if a Hopf algebra element is moved across a maximum or minimum, then it is replaced by the application of the antipode to that element if the motion is anti-clockwise. If the motion is clockwise, then the inverse of the antipode is applied to the element. See the diagram below.



The link diagram is subject to deformations that generate regular isotopy [8]. Since the diagram is presented with respect to a choice of vertical direction (discriminating the maxima, minima and crossing types), regular isotopy is generated by a set of moves that include the cancellation of adjacent pairs of maxima and minima and the switching of an arc across a maximum or minimum. The full set of moves is shown in Figure 1. We have labelled these moves as

=. (cancellation of maxima and minima)

II. (cancellation of opposite crossings)

III. (braiding)

IV. (switching)

IV'. (twist of crossings)

IV' is equivalent to IV in the presence of the cancellation of maxima and minima. These moves generate regular isotopy for diagrams arranged with respect to a vertical direction.

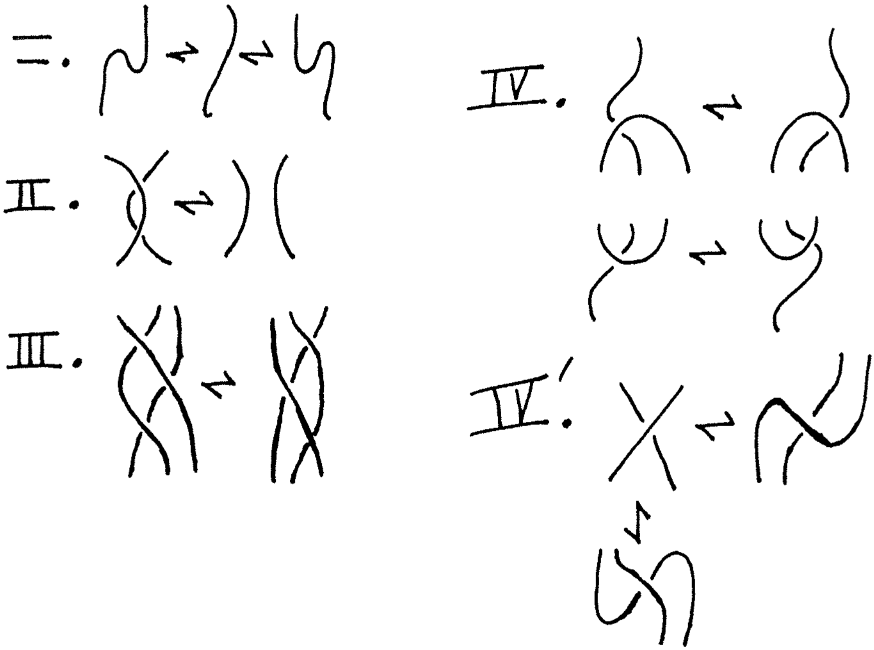
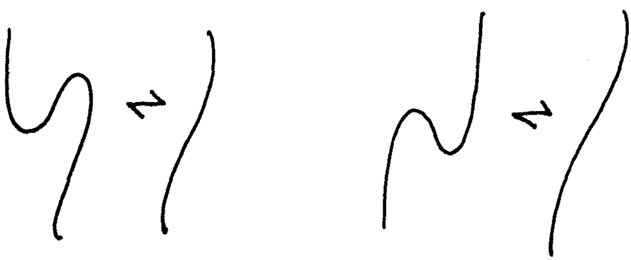
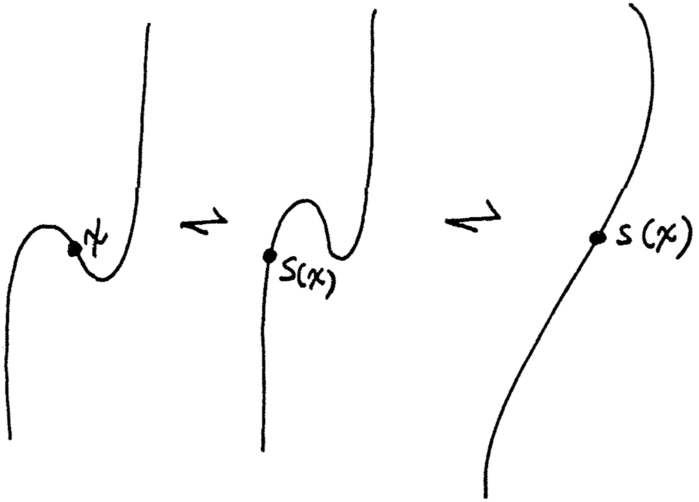


Figure 1

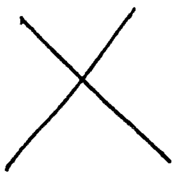
Remark. The symbol $\rightleftharpoons$ is used to denote the replacement of one figure by an equivalent figure. We shall sometimes use an equals sign (=) to perform the same purpose. The symbol $\twoheadrightarrow$ or $\twoheadleftarrow\!\!\twoheadrightarrow$ will be used to indicate a correspondence. For example, a link diagram corresponds to the diagram obtained from it by decoration with elements of the Hopf algebra.
An invariant of regular isotopy must remain unchanged by the moves shown in Figure 1. The simplest move is the cancellation of a pair consisting of a maximum and a minimum.



This pair cancellation gives a reformulation of the slide rule for the antipode: The antipode is accomplished by "composition with a maximum and a minimum".

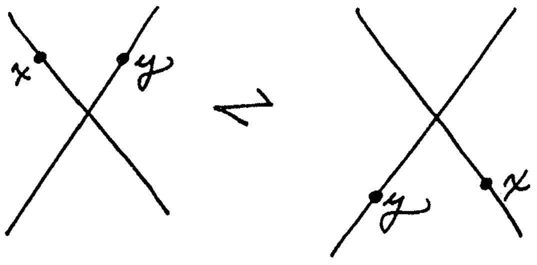


Note also that once the crossings of a link diagram have been labelled with elements of the Hopf algebra, the resulting diagram is depicted as a labelled immersion of a curve or curves in the plane. This is quite natural since the translation from algebraic braiding element to knot-theoretic braiding element is accomplished via the composition with a transposition, and the simplest diagrammatic representation of a transposition is the crossing of two arcs in the plane.

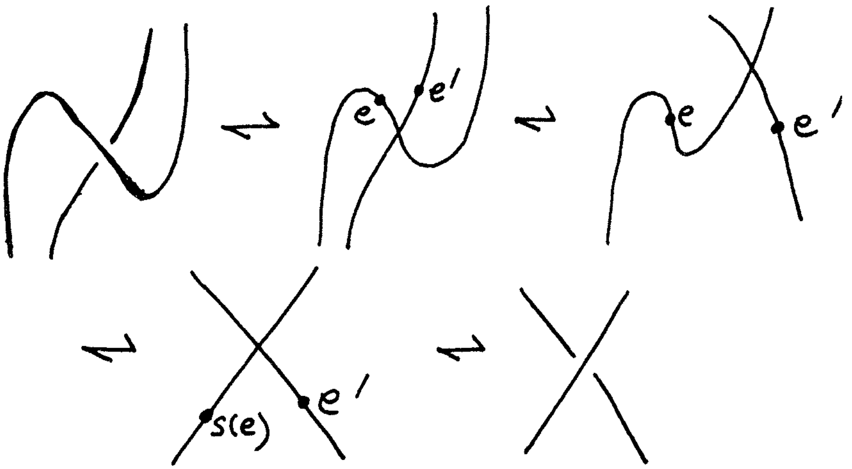


These immersions can be deformed up to regular homotopy that respects the given vertical direction. In other words, one can perform the projected forms of the moves of Figure 1. If algebra is present on the lines then the following extra move is added (sliding an external line past an algebra element).

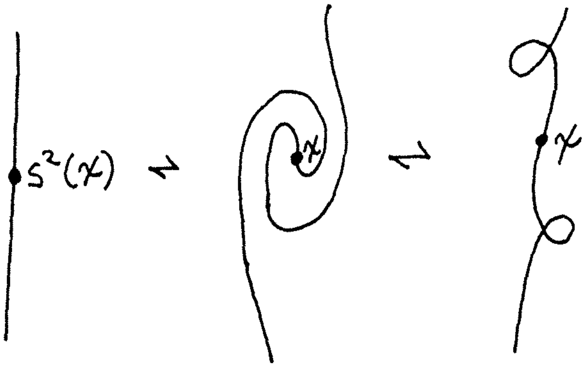
V. (*slide rule*)



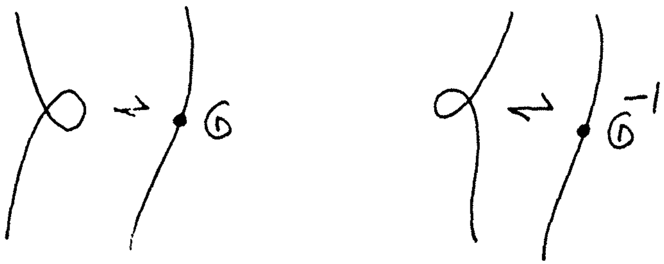
Since algebra elements are configured with respect to the vertical direction, we do not allow the cancellation of a maximum and a minimum that have an algebra element between them. This allows the representation of the antipode as described above.
It is now easy to check the twist relation (IV') for crossings:



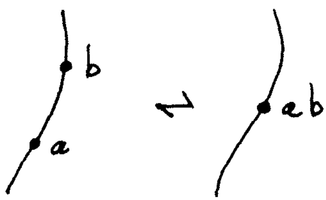
With these conventions, the square of the antipode is equivalently diagrammed as a "composition with two curls" as shown below:



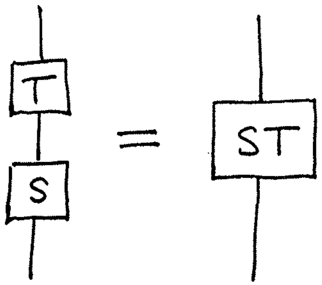
These curls are identified with the special grouplike elements G and G^{-1} in the Hopf algebra.



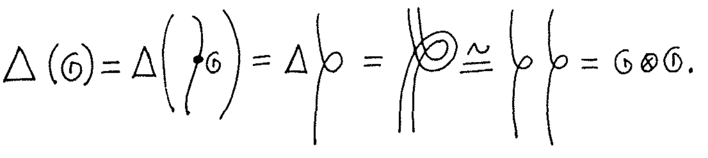
Thus the diagram for the square of the antipode represents directly the formula $s^2(x) = GxG^{-1}$.
Along a vertical line, algebra elements combine by multiplication.



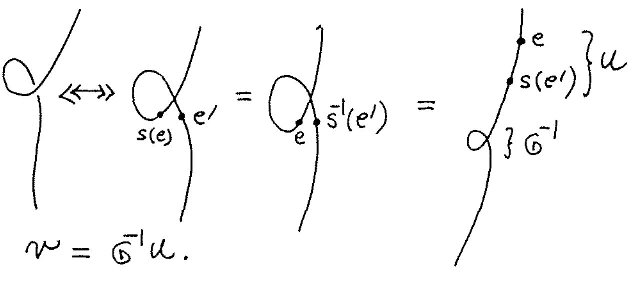
The product in the Hopf algebra corresponds to the multiplication of single strand tangles. A single strand tangle is a bit of link diagram with two free ends arranged with respect to the vertical so that one end is down and the other end is up. Tangles are multiplied by attaching the down end of one tangle to the top end of the other.



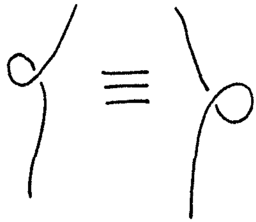
The coproduct $\Delta : A \to A \otimes A$ in the Hopf algebra corresponds to a mapping on tangles $\Delta : T^{(1)} \to T^{(2)}$ from single strand tangles to double strand tangles obtained by forming the parallel (two strand) cable of the given tangle. The tangles in question can be immersions.
For example, we see that the formula $\Delta(G) = G \otimes G$ corresponds to the regular isotopy shown below.



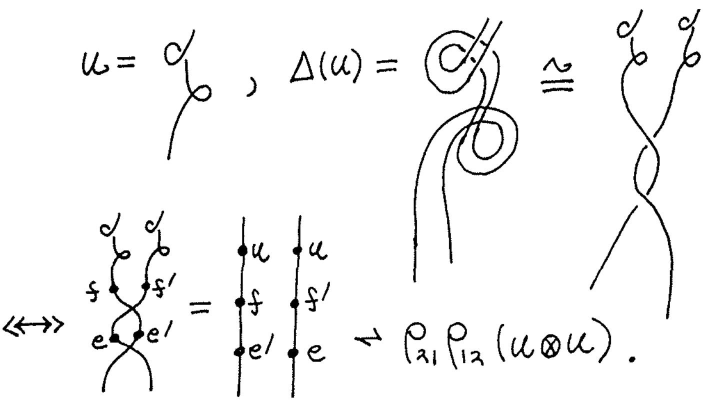
In this way knots on a line can be resolved into algebra elements. For example the twist shown below is equivalent to the ribbon element v. Note how the factorization of v into a product of G^{-1} and $u = \Sigma\, s(e')e$ is related to the slide convention for the antipode (in the diagrammatic calculation shown below we use the fact that $(s \otimes s)\rho = \rho$.)



Note that $s(v) = v$ corresponds to the identification shown below.



When this identification is added to regular isotopy, the twists catalog only the framing, and the equivalence relation on the link diagrams is equivalent to ambient isotopy of framed links.
Finally, returning to the diagrammatic coproduct we see the interpretation of the following formula of Drinfeld $\Delta(u) = \rho_{21}\rho_{12}(u \otimes u)$:



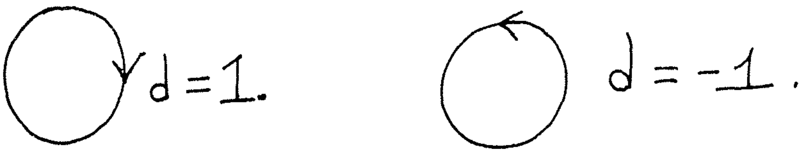
In general, if T is a single strand tangle, and $F(T)$ is the corresponding element in the Hopf algebra A that is determined by our correspondence, then $F(\Delta(T)) = \Delta(F(T))$ where the first Δ is the diagrammatic coproduct and the second Δ is the algebraic coproduct. This fact follows from the axioms for a quasi-triangular Hopf algebra in conjunction with our diagrammatic conventions.

Definition and Computation of $TR(K)$.

Suppose that tr: $A \to k$ is a trace function. That is, tr is a linear function satisfying
1. $\text{tr}(xy) = \text{tr}(yx)$ and
2. $\text{tr}(s(x)) = \text{tr}(x)$.

To define the trace $TR(K)$ for a knot diagram K, slide all of the algebra into one vertical portion of the diagram. Amalgamate this algebraic expression according to the rule for multiplying algebra elements on the diagram, as we have done above. Call this localized algebra element w. It is a sum of products, and can be formally represented as a product where it is understood that there is a sum over all pairs of the type e, e'.

Let d be the Whitney degree of the flat diagram for K that is obtained by traversing K upward from the vertical portion where the algebra has been concentrated. The Whitney degree is the total turn of the tangent vector to the curve as one traverses it in the given direction. For example:



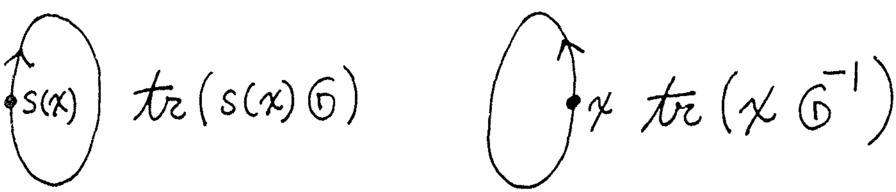
Define $TR(K)$ by the formula $TR(K)$ $=\mathbf{tr}(wG^d)$. Note that w is itself a summation over all the pairs x, x' corresponding to Yang-Baxter elements on the diagram. $TR(K)$ *defines a regular isotopy invariant of unoriented knots.* (The proof is primarily a matter of checking that $TR(K)$ is independent of the place where we concentrate the algebra. This reduces to checking the independence in the case where the concentration is moved around a maximum or a minimum. See example 2 below: and for a complete proof see Theorem 5.1 of [7].)

In order to define an invariant of unoriented links, concentrate the algebra for each component of the link, and define

$$TR(K) = \mathrm{tr}(w_1G^{d_1})\mathrm{tr}(w_2G^{d_2})\mathrm{tr}(w_3G^{d_3})\cdots\mathrm{tr}(w_nG^{d_n})$$

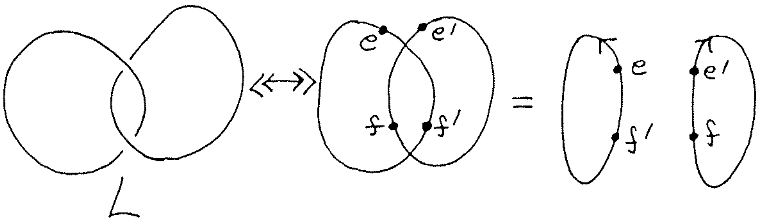
where the labels $1, 2, ..., n$ refer to the components of the link, and the implicit summation is the sum over all the pairs x, x' in these words. The elements $w_1, \ldots, w_n$ are the algebra concentrations for each link component, and the degrees $d_1, \ldots, d_n$ are the Whitney degrees of the components of the link.

Example. This example points out how the $TR(K)$ is invariant under algebra slides:



$$\mathrm{tr}(s(x)G) = \mathrm{tr}(s(s(x)G)) = \mathrm{tr}(G^{-1}s^2(x)) = \mathrm{tr}(G^{-1}GxG^{-1}) = \mathrm{tr}(xG^{-1}).$$

Example 3. Here is the form of calculation for a link.



$TR(L) = \Sigma\, \mathrm{tr}(f'eG^{-1})\mathrm{tr}(fe'G)$.

If $\rho = \sum_{i=1}^{n} x_i \otimes y_i$ then $TR(L) = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{tr}(y_j x_i G^{-1})\mathrm{tr}(x_j y_i G)$.
This is how the regular isotopy invariant of the link would look as a specific sum of traces of algebra elements.

III. Invariants of 3-manifolds

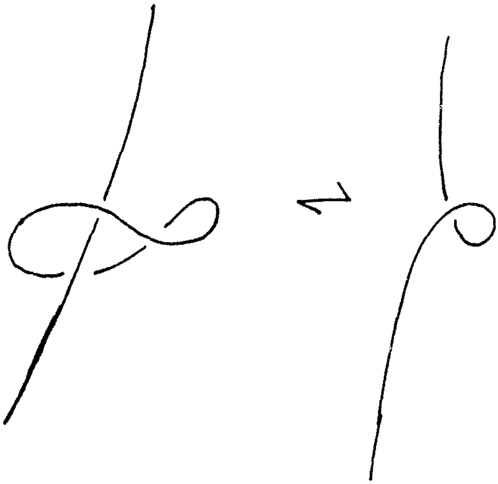
The structure we have built so far can be used to construct invariants of 3-manifolds presented in terms of surgery on framed links. We sketch here our technique that simplifies an approach to 3-manifold invariants of Mark Hennings [5].
Recall that an element λ of the dual algebra A^* is said to be a *right integral* if $\lambda(x)1 = m(\lambda \otimes 1)(\Delta(x))$ for all x in A. For a unimodular [12],[15] finite dimensional ribbon Hopf algebra A there is a right integral λ satisfying the following properties for all x and y in A:
0) λ is unique up to scalar multiplication when k is a field.
1) $\lambda(xy) = \lambda(s^2(y)x)$
2) $\lambda(gx) = \lambda(s(x))$ where $g = G^2$, G is the special grouplike element for the ribbon element $v = G^{-1}u$.

Given the existence of this λ, define a functional tr: $A \to k$ by the formula $\mathrm{tr}(x) = \lambda(Gx)$.

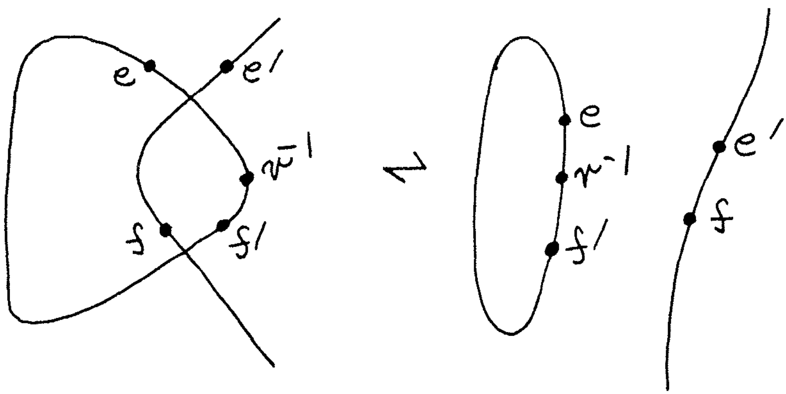
Theorem. With tr defined as above, then
1) $\mathrm{tr}(xy) = \mathrm{tr}(yx)$ for all x, y in A.
2) $\mathrm{tr}(s(x)) = \mathrm{tr}(x)$ for all x in A.
3) $[m(\mathrm{tr}\otimes 1)(\Delta(u^{-1}))]u = \lambda(v^{-1})v$ where $v = G^{-1}u$ is the ribbon element.

Proof. The proof is a direct consequence of the properties 1) and 2) of λ. Thus $\mathrm{tr}(xy) = \lambda(Gxy) = \lambda(s^2(y)Gx) = \lambda(GyG^{-1}Gx) = \lambda(Gyx) = \mathrm{tr}(yx)$, and $\mathrm{tr}(s(x)) = \lambda(Gs(x)) = \lambda(gG^{-1}s(x)) = \lambda(s(G^{-1}s(x))) = \lambda(s^2(x)s(G^{-1}))$ $= \lambda(s^2(x)G) = \lambda(GxG^{-1}G) = \lambda(Gx) = \mathrm{tr}(x)$. Finally, $[m(\mathrm{tr}\otimes 1)(\Delta(u^{-1}))]u = G^{-1}[m(\lambda \cdot G \otimes G)(\Delta(u^{-1}))]u = [m(\lambda \otimes 1)(\Delta(Gu^{-1}))]$ $G^{-1}u = \lambda(Gu^{-1})G^{-1}u = \lambda(v^{-1})v$. This completes the proof. //
The upshot of this Theorem is that for a unimodular finite dimensional Hopf algebra there is a natural trace defined via the existent right integral.

Remarkably, this trace is just designed by property 3) of the Theorem to behave well with respect to the Kirby move. The Kirby move is the basic transformation on framed links that leaves the corresponding 3-manifold obtained by framed surgery unchanged. See [11], [19]. This means that a suitably normalized version of this trace on framed links gives an invariant of 3-manifolds. Here is a sample Kirby move



The cable going through the loop can have any number of strands. The loop has one strand and the framing as indicated. The replacement on the right hand side puts a 360 degree twist in the cable with blackboard framing as shown above. Here we calculate the case of a single strand cable:



The diagram shows that the trace contribution is (with implicit summation on the repeated primed and unprimed pairs of Yang-Baxter elements)

$$\begin{aligned}&\mathrm{tr}(f'v^{-1}eG^{-1})fe' = \mathrm{tr}(f'ev^{-1}G^{-1})fe' = \mathrm{tr}(f'eu^{-1})fe'\\&= [m(\mathrm{tr}\otimes 1)(f'eu^{-1}\otimes fe'u^{-1})]u = [m(\mathrm{tr}\otimes 1)(\rho_{21}\rho_{12}(u^{-1}\otimes u^{-1}))]u\\&= [m(\mathrm{tr}\otimes 1)(\Delta(u^{-1}))]u = \lambda(v^{-1})v. \quad (\Delta(u^{-1}) = \rho_{21}\rho_{12}(u^{-1}\otimes u^{-1}))\end{aligned}$$

It follows from this calculation that the evaluation of the lefthand picture in the Kirby move is $\lambda(v^{-1})$ times the evaluation of the right hand picture. The corresponding result for an n-strand cable is obtained by applying the coproduct to the equation above, and using the functoriality of the coproduct with respect to tangles and tensor powers of the Hopf algebra.
Thus a proper normalization of $TR(K)$ gives an invariant of the 3- manifold obtained by framed surgery on K. More precisely, (assuming that $\lambda(v)$ and $\lambda(v^{-1})$ are non-zero) let

$$INV(K) = \{[\lambda(v)\lambda(v^{-1})]^{-c(K)/2}[\lambda(v)/\lambda(v^{-1})]^{-\sigma(K)/2}\}TR(K)$$

where $c(K)$ denotes the number of components of K, and $s(K)$ denotes the signature of the matrix of linking numbers of the components of K (with framing numbers on the diagonal), then $INV(K)$ is an invariant of the 3-manifold obtained by doing framed surgery on K in the blackboard framing. This is our reconstruction of Hennings invariant [5] in an intrinsically unoriented context.

IV. $U_q(sl_2)$'

The purpose of this section is to set up part of the general calculations for $U_q(sl_2)$', and to sketch the calculation of the special case of the evaluation of the right integral on powers of the ribbon element v in the case $n = 8$. This will give us the result that the invariant $INV(K)$ is distinct from the Witten- Reshetikhin-Turaev invariant at this root of unity. Complete details are found in [10].
Recall the algebraic structure of $U_q(sl_2)$'.
Let t be a primitive n-th root of unity, $q = t^2$, m =order(t^4). Assume $m \neq 1$ (that is $n \neq 1,2,4$).
The algebra has generators and relations as given below.

$$\begin{aligned}&ae = qea\\&af = q^{-1}fa\\&a^n = 1\\&e^m = 0 = f^m\\&[e,f] = ef - fe = (a^2 - a^{-2})/(q - q^{-1})\end{aligned}$$

The Yang-Baxter element is given by the formula below [11],[16].

$$R = \sum_{v=0}^{m-1} \sum_{i,u\in Z/nZ} [(t^{-uv-i(u-v)-v}(q-q^{-1})^v)/(n(v)_q!)]\, f^v a^i \otimes e^v a^{-u}.$$

The coproduct is described by the formulas

$$\begin{aligned}\Delta a &= a \otimes a \\ \Delta x &= x \otimes a^{-1} + a \otimes x, \quad x = e, f\end{aligned}$$

The counit is determined by the formulas

$$E(e) = E(f) = 0 \quad \text{and} \quad E(a) = 1.$$

It follows from the definition of the antipode s that for $x = e$ or f, $0 = E(x)1 = m(s \otimes 1)\Delta(x) = s(x)a^{-1} + s(a)x = s(x)a^{-1} + a^{-1}x$. ($s(a) = a^{-1}$ since$\Delta(a) = a \otimes a$.).
This means $s(x) = -a^{-1}xa$, whence

$$s(e) = -q^{-1}e \quad \text{and} \quad s(f) = -qf.$$

The special grouplike element is $G = a^{-2}$.
The special element u such that $s^2(x) = uxu^{-1}$ for all x, is given by the formula $u = \sum s(R^{(2)})R^{(1)}$. The next Lemma gives a specific formula for u.

Lemma 1. $u = \sum_{v=0}^{m-1} \sum_{i,j \in Z/nZ} [(t^{j(i-v)-i^2-3v}(q^{-1} - q)^v)/(n(v)_q!)]a^j e^v f^v.$

Proof. See [10].//

Change of Basis

We now make the following change of basis.
Replace e by $-(q - q^{-1})e$. Then

$$\begin{aligned}ae &= qea \\ af &= q^{-1}fa \\ a^n &= 1 \\ e^m &= 0 = f^m \\ [f, e] &= a^2a^{-2}\end{aligned}$$

Note that in this basis the formula for u becomes

$$u = \sum_{v=0}^{m-1} \sum_{i,j \in Z/nZ} [(t^{j(i-v)-i^2-3v})/(n(v)_q!)]a^j e^v f^v.$$

Right Integral

A right integral λ for $A = U_q(sl_2)$' is described as follows. Consider the linear basis for A given by the set $\{a^i e^j f^k | 0 \leq i < n, 0 \leq j, k < m\}$. Then $\lambda(w)$ for $w \in A$ is the coefficient of $a^{2(m-1)}e^{m-1}f^{m-1}$ in a writing of w in this basis. We can write $\lambda = \overline{a^{2(m-1)}e^{m-1}f^{m-1}}$ where the bar over the expression denotes the characteristic function of this element of the algebra A. That this formula gives the right integral can be verified by direct calculation [17].

Orthogonal Idempotents

Let $\wedge_i = (1/n) \sum_{j \in Z/nZ} t^{ij} a^j$. Then $\wedge_i \wedge_j = \wedge_i \delta_{ij}$ where δ_{ij} is the Kronecker delta and $1 = \wedge_0 + \wedge_1 + \ldots + \wedge_{n-1}$.
Thus $\{\wedge_0, \wedge_1, \ldots, \wedge_{n-1}\}$ form a set of orthogonal idempotents for the group algebra $k[G]$ where $G = (a) = Z/nZ$.

From the relation $\sum_{i \in Z/nZ} t^{ik} = \begin{cases} n \text{ if } k = 0 \\ 0 \text{ if } k \neq 0 \end{cases}$. for $k \in Z/nZ$, we have

Lemma 2. $a = \sum_{i \in Z/nZ} t^{-i} \wedge_i$.

Proof: See [10]. //

Hence $u = \sum_{v=0}^{m-1} \sum_{i \in Z/nZ} \left(\sum_{j \in Z/nZ} \left[\left(t^{-i^2-3v} \right) / ((v)_q!) \right] \left[t^{j(i-v)} a^j / n \right] \right) e^v f^v$

$$= \sum_{v=0}^{m-1} \left(\sum_{i \in Z/nZ} \left(t^{-i^2-3v} / (v)_q! \right) \wedge_{i-v} \right) e^v f^v$$

Lemma 3. $u = c \left(\sum_{v=0}^{m-1} \left[(t^{-3v-v^2}) / (v)_q! \right] a^{2v} e^v f^v \right)$ where $c = \sum_{i \in Z/nZ} t^{-i^2} \wedge_i$

Proof. See [10].//

The Special Case $n = 8$.

Let $n = 8$. Then $m = 2$, $q = \sqrt{-1}$ and the algebraic relations for $U_q(sl(2))'$ are

$$t^8 = 1, \; q = t^2$$
$$ae = qea$$
$$af = q^{-1} fa$$
$$a^8 = 1$$
$$e^2 = 0 = f^2$$
$$[f, e] = a^2 - a^{-2}.$$

Note that by the previous calculation,

$$u = c(1 + t^{-4} a^2 ef) = c(1 - a^2 ef)$$

with c given as in Lemma 3.

Recall that $\lambda = \overline{a^{2(m-1)} e^{m-1} f^{m-1}}$ is a right integral for $U_q(sl_2)'$. Thus, when $n = 8$, the right integral is $\lambda = \overline{a^2 ef}$.

Lemma 4. Let $X = -a^2 ef$. Then $u = c\,(1 + X)$ and
$X^2 = (a^4 - 1)X = -2 \left(\sum_{i \text{ odd}} \wedge_i \right) X$.

Proof. See [10].//

The special grouplike in this case is $G = a^{-2}$. Thus the ribbon element is $v = G^{-1}u = a^2u$. Thus $v = a^2c(1+X)$. To evaluate $\lambda(v^k)$, let $H = (a)$ be the cyclic group generated by a. Note that $v^k = c_0 + c_1X$, where $c_i \in k[H]$.

Lemma 5. Writing $c_1 = \sum_{i \in Z/8Z} \alpha_i \Lambda_i$, with $\alpha_i \in k$, then

$$\lambda(v^k) = (-1/8) = \sum_{i \in Z/8Z} \alpha_i.$$

Proof. See [10].//

Lemma 6. Let $n = 8$ and let λ be the right integral and v be the ribbon element for $U_q(sl(2))$' as described above.

Then $\lambda(v^k) = -k/2$.

Proof. See [10].//

Corollary. The value of the 3 manifold invariant $INV(L(k1))$ for $n = 8$ is given by the formula $INV(L(k,1)) = \sqrt{-1}k$ for $k \neq 0$.

Proof. The surgery datum for $L(k,1)$ is an unknotted loop with k curls. Hence the unnormalized invariant is given by the formula
$TR(v^kG^{-1}) = \lambda(Gv^kG^{-1}) = \lambda(v^kG^{-1}G) = \lambda(v^k) = -k/2$. The normalized invariant is given by the formula
$INV(L(k,l)) = [\lambda(v)\lambda(v^{-1})]^{-c(K)/2}[\lambda(v)/\lambda(v^{-1})]^{-\sigma(K)/2}TR(K)$.
Here $c(K) = 1$ and $\sigma(K) = 1$ if $k > 0$, $\sigma(K) = -1$ if $k < 0$ since the link has one component, and the linking matrix is (k). We know that $\lambda(v) = -1/2$ and $\lambda(v^{-1}) = 1/2$. Therefore
$INV(L(k,1)) = [(1/2)(-1/2)]^{-1/2}[(1/2)/(-1/2)]^{\pm 1}(-k/2)$
$= (-2^2)^{1/2}(-1)(-k/2) = (-1)^{1/2}k$.
This completes the proof.//

Remark. This finishes our verification that the invariant INV is definitely different from the WRT invariant in the case $n = 8$, where WRT is trivial. During the preparation of our paper [10] it came to our attention that similar results have been independently obtained by Tomotada Ohtsuki [14]. He finds that invariants defined for $U_q(sl_2)$' in a manner equivalent to ours necessarily vanish for 3-manifolds that are not rational homology spheres, and he performs calculations similar to ours for Lens spaces.

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ON THE CLASSICALITY OF BRODA'S SU(2) INVARIANTS OF 4-MANIFOLDS

LOUIS CRANE*
Department of Mathematics, Kansas State University
Manhattan, KS 66506-2602

LOUIS H. KAUFFMAN**
Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago, Chicago, IL 60680

and

DAVID N. YETTER
Department of Mathematics, Kansas State University
Manhattan, KS 66506-2602

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Abstract. Recent work of Roberts has shown that the surgical 4-manifold invariant of Broda [B1] and (up to an unspecified normalization factor) the state-sum 4-manifold invariant of Crane-Yetter [CY] are equivalent to the signature of the 4-manifold. Subsequently Broda [B2] defined another surgical invariant of 4-manifolds in which the 1- and 2-handles are treated differently. We use a refinement of Roberts' techniques developped in [CKY] to identify the normalization factor to show that the "improved" surgical invariant of Broda [B2] also depends only on the signature and Euler character.

As a starting point, let us first observe that the construction of Crane-Yetter [CY] does not really depend on the use of labels chosen from the irreps of $U_q(sl_2)$ at the principal r^th root of unity: the simple objects of any artinian semi-simple tortile category (cf. [S, Y]) in which all objects are self-dual and the fusion rules are multiplicity free will suffice. In particular, if we restrict to the integer spin (bosonic)[1] irreps, we obtain a construction of a different invariant of 4-manifolds.

In what follows, we use Temperley-Lieb recoupling theory (cf. [KL,L,R]). In particular, arcs are labelled with elements of $\{0, 1, ...r-2\}$ (twice the

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[1] This use of bosonic is a hideous abuse of language. Everything in sight has braid statistics. The "bosons" of this paper are the result of q-deforming honest bosons.

spin), $A = e^{2\pi i/4r}$, $q = A^2$, $\Delta(n) = (-1)^n \frac{q^{n+1}-q^{-n-1}}{q-q^{-1}}$, $\theta(a,b,c)$ denoted the evaluation of the theta-net with edge labelled a, b, and c, and $15-j$ denotes the evaluation of the Temperley-Lieb version of the Crane-Yetter quantum $15j$-symbol (with indices suppressed).

We then adopt the following further notational conventions:

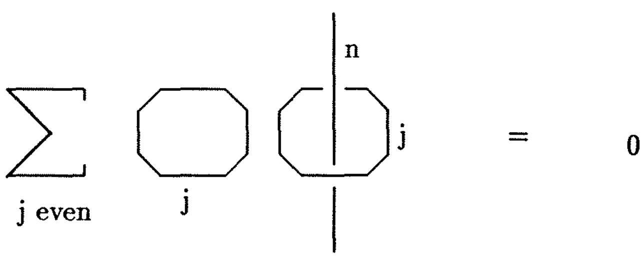
Arcs labelled ω denote the linear combination of arcs labelled $0, 1, \ldots, r-2$ in which the coefficient of i is $\Delta(i)$. Arcs labelled $\tilde{\omega}$ denote the linear combination of arcs labelled $0, 2, \ldots, 2\lfloor\frac{r-2}{2}\rfloor$ (even integers) in which the coefficient of i is $\Delta(i)$. N denotes the sum of the squares of the $\Delta(i)$'s, $\tilde{N}$ denotes the sum of the squares of the $\Delta(i)$'s for i even. Let κ be as in [KL,R], the evaluation of an ω labelled 1-framed unknot divided by the positive square root of N, and let $\tilde{\kappa}$ be the evaluation of an $\tilde{\omega}$ labelled 1-framed unknot divided by $\tilde{N}$.

If L is a framed link, then $\tilde{\omega}(L)$ denotes the evalutation of the link with all components labelled $\tilde{\omega}(L)$ If $\mathcal{L}$ is a set of 4-manifold surgery instructions (cf. Kirby [K]), that is a link L with a distinguished 0-framed unlink $\dot{L}$, then $\mathcal{B}^!(\mathcal{L})$ denotes the evaluation of the link L with all components of $\dot{L}$ (one-handle attachments) colored ω and all other components of L (two-handle attachments) colored $\tilde{\omega}$.

LEMMA 1. *$\tilde{\omega}(L)$ is invariant under handle-sliding. $\mathcal{B}^!\mathcal{L}$ is invariant under handle-sliding of 1- and 2-handles 1-handles and of 2-handles over 2-handles.*

proof: This follows immediately from handle-sliding over components labelled ω and the analysis given in Remark 17 §12.6 of Kauffman/Lins [KL] once it is observed that pairs of bosons only couple to produce bosons. □

LEMMA 2. **(The bosonic encirclement lemma)**



whenever n is even and non-zero.

proof: This follows from the same proof as the encirclement lemma of Lickorish [L] (cf. also Kauffman/Lins [KL]) with the "auxiliary loop" labelled 2 instead of 1. □

Let

$$CY_B(W) = \tilde{N}^{n_0-n_1} \sum_{\substack{\text{even labellings}\\ \lambda \text{ of faces and}\\ \text{tetrahedra}}} \prod_{\substack{\text{faces}\\ \sigma}} \Delta(\lambda(\sigma))$$

$$\prod_{\substack{\text{tetrahedra}\\ \tau}} \frac{\Delta(\lambda(\sigma))}{\theta(\lambda(\tau), \lambda(\tau_0), \lambda(\tau_2))\theta(\lambda(\tau), \lambda(\tau_1), \lambda(\tau_3))} \prod_{\text{4-simplexes}} 15-j$$

be the bosonic Crane-Yetter invariant.

Let $|L|$ (resp. $\nu(L)$, $\sigma(L)$) denote the number of components of a link L (resp. the nullity of the linking matrix of L, the signature of the linking matrix of L).

We can then define a purely bosonic version of Broda's original invariant by

$$Br_B(W) = \frac{\tilde{\omega}(L)}{\tilde{N}^{\frac{|L|+\nu(L)}{2}}}$$

where L is the underlying link of a surgery presentation of W; while a bosonic version of the Reshetikhin/Turaev [RT] 3-manifold invariant is given by

$$I_B(M) = \tilde{\kappa}^{-\sigma(L)}\tilde{N}^{\frac{|L|+1}{2}}\tilde{\omega}(L)$$

where L is a framed link giving surgery instructions for M.

Applying the two lemmas above in an analysis otherwise identical to that of given by Roberts [R] of the original Broda invariant [B1] shows that

PROPOSITION 3. $\qquad Br_B(W) = \tilde{\kappa}^{\sigma(W)}$

Similarly it follows from the bosonic encirclement lemma that

$$CY_B(W) = \tilde{N}^{n_0-n_1-n_3}\tilde{\omega}(L)$$

where n_d is the number of d-simplexes in a triangulation, and L is the link derived from a triangulation by putting a 0-framed unknot in each tetrahedron, and a loop around each 2-simplex (running mostly through 4-simplexes but linking each tetrahedron's unknot) after the manner of Roberts [R].

It then follows as in [CKY] that

PROPOSITION 4. $CY_B(W) = \tilde{\kappa}^{\sigma(W)}\tilde{N}^{\frac{\chi(W)}{2}}$ (*)

Now, Broda's new invariant is defined by

$$\mathcal{B}(W) = \frac{\mathcal{B}^!(L)}{\tilde{N}^{\nu(L)}N^{\frac{|L|-\nu(L)}{2}}}$$

For convenience we first analyse a slightly different normalization (for which the proof of invariance is effectively identical to that for B(W): let

$$\mathbf{B}(W) = \frac{\mathcal{B}^!(L)}{\tilde{N}^{|L-\dot{L}|-|\dot{L}|}N^{|\dot{L}|}}$$

Now, it follows from the original encirclement lemma of Lickorish [L] that

$$CY_B(W) = \tilde{N}^{n_0-n_1}N^{-n_3}\mathcal{B}^!(\mathcal{L}) \qquad (**)$$

where $\mathcal{L}$ is the surgery instructions given by assiociating the link L to the triangulation as above, and letting $\dot{L}$ be the unlink of loops in the tetrahedra.

Observe that $\mathbf{B}$ is multiplicative under connected sum, and that $\mathbf{B}(S^1 \times S^3) = \tilde{N}$ (an easy calculation). As shown in Roberts [R], $\mathcal{L}$ is a surgery presentation for $W\#(\overset{n_4-1}{\#} S^1 \times S^3)$.

From this and the fact that for $\mathcal{L}$, $|L - \dot{L}| = n_2$ and $|\dot{L}| = n_3$, we see that

$$\begin{aligned}\frac{\mathcal{B}^!(\mathcal{L})}{\tilde{N}^{n_2-n_3}N^{n_3}} &= \mathbf{B}(W\#(\overset{n_4-1}{\#} S^1 \times S^3)) \\ &= \mathbf{B}(W)\tilde{N}^{n_4-1}.\end{aligned}$$

Thus

$$\mathcal{B}^!(L) = \mathbf{B}(W)\tilde{N}^{n_2-n_3+n_4-1}N^{n_3}. \qquad (***)$$

It then follows from (*), (**) and (***) that

$$\mathbf{B}(W) = \tilde{\kappa}^{\sigma(W)}\tilde{N}^{\frac{\chi(W)}{2}-1}$$

To return to Broda's [B2] original normalization, note that

$$\mathcal{B}(W) = \mathbf{B}(W)(\tilde{N}N^{-\frac{1}{2}})^{|L-\dot{L}|-|\dot{L}|-\nu(L)}$$

From which we obtain

THEOREM 5. *If W is a connected closed oriented smooth 4-manifold, then*

$$\mathcal{B}(W) = \tilde{\kappa}^{\sigma(W)} \left(\frac{\tilde{N}}{N}\right)^{\frac{\chi(W)}{2}-1}$$

proof: It suffices to shown that if W is given by the surgery instruction $\mathcal{L}$, then

$$|L - \dot{L}| - |\dot{L}| - \nu(L) = \chi(W) - 2.$$

But this follows immediately from the observation that $\nu(L)$ is the number of 3-handles attached in completing the construction of W. □

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ON THE FAILURE OF THE LICKORISH ENCIRCLEMENT LEMMA FOR TEMPERLEY-LIEB RECOUPLING THEORY AT CERTAIN ROOTS OF UNITY

LOUIS CRANE*
Department of Mathematics, Kansas State University
Manhattan, KS 66506-2602

LOUIS H. KAUFFMAN**
Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago, Chicago, IL 60680

and

DAVID N. YETTER
Department of Mathematics, Kansas State University
Manhattan, KS 66506-2602

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Abstract. Lickorish [L] proved his encirclement lemma for Temperley-Lieb only in the case where the Kauffman bracket variable A is the principal $4r^{th}$ root of unity. The analogous statement does not hold for $A = ie^{\frac{2\pi i}{4r}}$ for r odd. As a consequence the interpretation given by the authors in [CKY] based on the work of Roberts [R] of the Crane-Yetter [CY] and Broda [B] invariants does not hold when the theories are constructed from this case of T-L theory, as is shown by the example of $S^2 \times S^2$.

The encirclement lemma of Lickorish [L] (cf. also [KL]) is used crucially in Roberts' elegant proof [R] that the Turaev-Viro invariant [TV] is the absolute square of the Reshetikhin-Turaev-Witten invariant [RT], and in the reduction of the 4-manifold invariants of Broda [B] and Crane-Yetter [CY] to classical homeomorphism invariants (cf. [B, CKY]).

In the following we present the representation theory of $U_q(sl_2)$ in terms of

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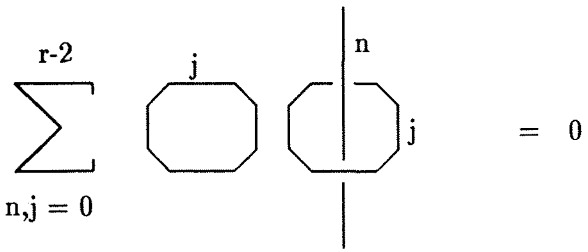
Temperley-Lieb recombination diagrams following Kauffman and Lins [KL]. In this version of the theory, the basic deformation variable is the Kauffman bracket variable A. We denote the "standard truncation" at a root of unity (the "small representations") by $Rep_!^A(U_q(sl_2))$, where A is the Kauffman bracket variable, a particular choice of $4r^{th}$ root of unity. In particular we are interested in 4^{th} roots of the principal r^{th} root.

The encirclement lemma can then be stated:

The Encirclement Lemma [L]

In $Rep_!^{e^{\frac{2\pi i}{4r}}}(U_q(sl_2))$



unless $n = 0$.

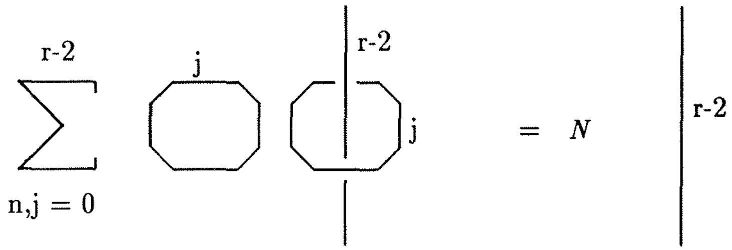
Analyses of the Temperley-Lieb recoupling theory have been carried out exclusively in the cases where A is the principal $4r^{th}$ root of unity (cf. [KL], [L], [B], [CKY]). A careful reading of [KL] shows that with the exception of the Handle-Sliding Lemma, and the Encirclement Lemma, the entire theory can be carried through with A some other root of unity, in particular for $A = ie^{\frac{2\pi i}{4r}}$ (which is a primitive $2r^{th}$ (resp. r^{th}) root of unity when $r = 1 \pmod 4$ (resp. $r = 3 \pmod 4$)). Note, however, that this A is nonetheless, a 4^{th} root of $e^{\frac{2\pi i}{r}}$.

The Handle-Sliding Lemma hold for $A = ie^{\frac{2\pi i}{4r}}$, as may be seen by considering Remark 17 of [KL] §12.6.

However, the Encirclement Lemma is false in the case $A = ie^{\frac{2\pi i}{4r}}$ for r odd. In particular

Proposition

In $Rep_!^{ie^{\frac{2\pi i}{4r}}}(U_q(sl_2))$ for r odd,



where N is the sum of the squares of the quantum dimensions, and in particular is non-zero.

proof: It suffices to show that the square of the braiding applied to $(r-2)$ and j (for any $0 \leq j \leq r-2$) is the identity, since this will imply that the map on the left-hand side is the identity map on $(r-2)$ multiplied by $N \neq 0$ (the sum of the squares of the quantum dimensions).

Now, in $Rep_!^A(U_q(sl_2))$, $(r-2) \otimes j$ is isomorphic to $r-2-j$. Thus the square of the braiding is a scalar multiple of the identity. Applying the formula for the braiding (cf. Kauffman/Lins [KL]), we find that the scalar is

$$\begin{aligned}
(-1)^{(r-2)+j-[(r-2)-j]}A^{(r-2)r+j(j+2)-(r-2-j)(r-j)} &= (-1)^{2j}A^{2rj} \\
&= (ie^{\frac{2\pi i}{4r}})^{2rj} \\
&= i^{2rj}e^{j\pi i} \\
&= (-1)^j(-1)^j \\
&= 1
\end{aligned}$$

So we are done. □

As a consequence the analogue of the result of the authors [CKY] (cf. also [R]) interpreting the Crane-Yetter invariant in the case where A is the principal $4r^{th}$ root of unity is without proof in the case of $A = ie^{\frac{2\pi i}{4r}}$ for r odd.

In fact there is no expression for $CY(W)$ of the form $\kappa^{\sigma(W)}N^{\frac{\chi(W)}{2}}$ in this case (for κ a phase, and N the sum of the squares of the quantum dimensions): direct calculation shows for $A = ie^{\frac{2\pi i}{20}}$ that $CY(S^2 \times S^2) = 2N^2$, while $CY(S^1 \times S^3) = 1$.

Similarly, Roberts [R] proof that the Turaev-Viro [TV] invariant is the absolute square of the Reshetikhin-Turaev-Witten [RT] invariant will not work in this case, since the reduction of Turaev-Viro to "chain mail" fails. We have not been able to see whether this delicacy arises in the proofs of

Turaev [T] or Walker [W] of the same result, nor to find examples which would show the result fails in this case.

Another consequence of the failure of the encirclement lemma in this case is the observation that the two $Rep_!^A(U_q(sl_2))$'s for the different values of A appearing above are inequivalent as *abstract braided monoidal categories.*

To be precise:

Definition The center $\mathcal{Z}(\mathcal{X})$ of a braided monoidal category $\mathcal{X}$ is the full subcategory of objects A satisfying

$$\sigma_{A,X}^{-1} = \sigma_{X,A}$$

for all objects X in $\mathcal{X}$.

Definition A braided monoidal equivalence between two braided monoidal categories is a monoidal equivalence which, moreover, satisfies

$$\begin{array}{ccc} F(A\otimes B) & \xrightarrow{F(\sigma)} & F(B\otimes A) \\ \Big\downarrow F^{\#} & & \Big\downarrow F^{\#} \\ F(A)\otimes F(B) & \xrightarrow[\sigma]{} & F(B)\otimes F(A) \end{array}$$

where σ denotes the braiding in the relevant category, and $(F, F^{\#}, F_0)$ is one of the functors in the equivalence.

We then have

Theorem The center of $Rep_!^{e^{\frac{2\pi i}{4r}}}(U_q(sl_2))$ is braided monoidally equivalent to $\mathbf{C}-v.s.,\otimes$, while the center of $Rep_!^{ie^{\frac{2\pi i}{4r}}}(U_q(sl_2))$ is braided monoidally equivalent to $\mathbf{Z}/2$-gr-$\mathbf{C}$-$v.s.$, $\otimes_{un-signed}$, (the "non-super" tensor product). Consequently the two categories are not braided monoidally equivalent.

proof: To calculate the center in each case, it suffices, by semisimplicity to determine which simple objects are in the center. In the first case the objects isomorphic to **0** are the only simple objects in the center; in the second, the objects isomorphic to 0 and 3 are the only simple object in the center. It is trivial to verify that the functors in a braided monoidal equivalence map the center to the center.□

By way of concluding speculations, it would be extremely interesting to determine whether there is a way of modifying the category $Rep_{|}^{ie^{\frac{2\pi i}{4r}}}(U_q(sl_2))$ in such a way that the center becomes the category of super-vector-spaces. Doing so could potentially lead to cancellation in the calculation for $S^2 \times S^2$ giving an invariant unstable under addition of 2-handles–a first requisite for a non-trivial invariant of differentiable structures.

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QUANTUM PRINCIPAL BUNDLES

MIĆO ĐURĐEVIĆ
Faculty of Physics, P.O.BOX 550
Studentski Trg 12, Beograd, Serbia, Yugoslavia

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Abstract. A noncommutative-geometric generalization of the theory of principal bundles is sketched. A differential calculus over corresponding quantum principal bundles is analyzed. The formalism of connections is presented. In particular, operators of covariant derivative and horizontal projection are described and analyzed. Quantum counterparts of the Bianchi identity and the Weil's homomorphism are found.

1. Introduction

The purpose of this letter is to present basic structural elements of a quantum theory of principal bundles, in which quantum groups play the role of structure groups, and quantum spaces the role of base manifolds. All considerations are performed within the conceptual scheme of non-commutative differential geometry [1, 2]. A detailed exposition of the theory is given in papers [3, 4].

The paper is organized as follows. Section II begins with a definition of quantum principal bundles. Then, questions related to differential calculus are discussed. Section III is devoted to the formalism of connections. In Section IV a generalization of the Weil's theory of characteristic classes is sketched. Finally, in Section V some examples of quantum principal bundles are considered, and some remarks are made.

Before passing to quantum principal bundles we shall fix the notation, and introduce relevant quantum group entities. Here, we shall deal with compact matrix quantum groups [9]. Let G be such a group. The algebra of 'polynomial functions' on G will be denoted by $\mathcal{A}$. The group structure on G is determined by the comultiplication $\phi\colon \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$, the counit $e\colon \mathcal{A} \rightarrow \mathbf{C}$, and the antipode $k\colon \mathcal{A} \rightarrow \mathcal{A}$. The result of the $(n-1)$-fold comultiplication of $a \in \mathcal{A}$ will be symbolically written as $a^{(1)} \otimes \ldots \otimes a^{(n)}$. We shall denote by $ad\colon \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ the adjoint action of G on itself. Explicitly, this map is given by $ad(a) = a^{(2)} \otimes k(a^{(1)})a^{(3)}$.

Let (Γ, d) be a first-order differential calculus [10] over G, and let

$$\Gamma^{\wedge} = \sum_{k\geq 0}^{\oplus} \Gamma^{\wedge k}$$

be the universal differential envelope ([3]-Appendix B) of (Γ, d) (with $\Gamma^{\wedge 0} = \mathcal{A}$ and $\Gamma^{\wedge 1} = \Gamma$). For each $k \geq 0$ let $p_k: \Gamma^{\wedge} \rightarrow \Gamma^{\wedge k}$ be the corresponding projection map. Further, let

$$\Gamma^{\otimes} = \sum_{k\geq 0}^{\oplus} \Gamma^{\otimes k}$$

be the tensor bundle algebra over Γ ($\Gamma^{\otimes k} = \Gamma \otimes_{\mathcal{A}} \ldots \otimes_{\mathcal{A}} \Gamma$ (k-times) and $\Gamma^{\otimes 0} = \mathcal{A}$). Let us assume that (Γ, d) is left-covariant. We shall denote by Γ_{inv} the space of left-invariant elements of Γ while $\mathcal{R} \subseteq \ker(e)$ will be the right $\mathcal{A}$-ideal which canonically, in the sense of [10] corresponds to (Γ, d). The map $\pi: \mathcal{A} \rightarrow \Gamma_{inv}$ given by

$$\pi(a) = k(a^{(1)})da^{(2)}$$

is surjective, and $\ker(\pi) = \mathbf{C}1 \oplus \mathcal{R}$. Because of this, there exists a natural isomorphism

$$\Gamma_{inv} = \ker(e)/\mathcal{R}.$$

The above isomorphism induces a right $\mathcal{A}$-module structure on Γ_{inv}, which will be denoted by $\circ$. Explicitly,

$$\pi(a) \circ b = \pi(ab),$$

for each $a \in \ker(e)$ and $b \in \mathcal{A}$. The tensor product of k copies of Γ_{inv} will be denoted by $\Gamma_{inv}^{\otimes k}$. The tensor algebra over Γ_{inv} will be denoted by $\Gamma_{inv}^{\otimes}$. It is naturally isomorphic to the space of left-invariant elements of $\Gamma^{\otimes}$. The differential subalgebra of left-invariant elements of $\Gamma^{\wedge}$ will be denoted by $\Gamma_{inv}^{\wedge}$. We have

$$\Gamma_{inv}^{\wedge} = \sum_{k\geq 0}^{\oplus} \Gamma_{inv}^{\wedge k}$$

where $\Gamma_{inv}^{\wedge k}$ consists of k-th order left-invariant elements. The following natural isomorphism holds

$$\Gamma_{inv}^{\wedge} = \Gamma_{inv}^{\otimes} / I_{inv}^{\wedge}.$$

Here $I_{inv}^{\wedge} \subseteq \Gamma_{inv}^{\otimes}$ is the ideal generated by elements of the form

$$q = \pi(a^{(1)}) \otimes \pi(a^{(2)})$$

where $a \in \mathcal{R}$. The right $\mathcal{A}$-module structure $\circ$ can be uniquely extended from Γ_{inv} to $\Gamma_{inv}^{\wedge,\otimes}$ such that

$$1 \circ a = e(a)1$$
$$(\vartheta\eta) \circ a = (\vartheta \circ a^{(1)})(\eta \circ a^{(2)})$$

for each $\vartheta, \eta \in \Gamma_{inv}^{\wedge,\otimes}$ and $a \in \mathcal{A}$.

Let us assume that (Γ, d) is bicovariant, and let $\varpi\colon \Gamma_{inv} \to \Gamma_{inv} \otimes \mathcal{A}$ be the adjoint action of G on Γ_{inv} (coinciding with the restriction of the right action of G on Γ_{inv}). We have

$$\varpi\pi = (\pi \otimes id)ad.$$

In the following, we shall denote by $\varpi^{\otimes}, \varpi^{\otimes k}, \varpi^{\wedge}, \varpi^{\wedge k}$ the adjoint actions of G on the corresponding spaces.

The map $\phi\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ admits the unique extension to the homomorphism $\widehat{\phi}\colon \Gamma^{\wedge} \to \Gamma^{\wedge} \widehat{\otimes} \Gamma^{\wedge}$ of (graded) differential algebras.

2. Quantum Principal Bundles and the Corresponding Differential Calculus

The aim of this section is to introduce quantum principal bundles, and to describe differential calculus over them.

Let M be a quantum space, represented by a (unital) *-algebra $\mathcal{V}$. The elements of $\mathcal{V}$ play the role of appropriate 'functions' on M.

DEFINITION 2.1. *A quantum principal G-bundle* over M is a triplet of the form $P = (\mathcal{B}, i, F)$ where $\mathcal{B}$ is a (unital) *-algebra, while $F\colon \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ and $i\colon \mathcal{V} \to \mathcal{B}$ are unital *-homomorphisms such that

(i) The following identities hold

$$(id \otimes e)F = id$$
$$(id \otimes \phi)F = (F \otimes id)F.$$

(ii) The map $i\colon \mathcal{V} \to \mathcal{B}$ is injective and

$$i(\mathcal{V}) = \{ b \in \mathcal{B} \mid F(b) = b \otimes 1 \},$$

for each $b \in \mathcal{B}$.

(iii) A linear map $X\colon \mathcal{B} \otimes \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ defined by

$$X(q \otimes b) = qF(b)$$

is surjective.

The map F plays the role of the dualized right action of G on P. Condition (i) justifies this interpretation. The map $i\colon \mathcal{V} \to \mathcal{B}$ can be interpreted as the dualized projection of P on M. Condition (ii) says that M can be identified with the corresponding 'orbit space' of P. Finally, condition (iii) is an effective quantum counterpart of the classical requirement that G acts freely on P.

Let $P = (\mathcal{B}, i, F)$ be a quantum principal G-bundle over M.

We are going to construct a graded differential algebra representing verticalized differential forms on P. Let us fix a bicovariant first-order differential *-calculus (Γ, d) over G. The *-involution naturally extends from Γ to $\Gamma^{\wedge,\otimes}$ (such that $(\vartheta\eta)^* = (-1)^{\partial\vartheta\partial\eta}\eta^*\vartheta^*$ for each $\vartheta, \eta \in \Gamma^{\wedge,\otimes}$). Algebras $\Gamma_{inv}^{\wedge,\otimes} \subseteq \Gamma^{\wedge,\otimes}$ are *-invariant.

Let us consider a (graded) vector space $\mathrm{ver}(P) = \mathcal{B} \otimes \Gamma_{inv}^{\wedge}$.

LEMMA 2.1. *The formulas*

$$(q \otimes \vartheta)(b \otimes \eta) = \sum_k qb_k \otimes (\vartheta \circ c_k)\eta$$

$$(b \otimes \eta)^* = \sum_k b_k^* \otimes (\eta^* \circ c_k^*)$$

$$d_v(b \otimes \eta) = b \otimes d\eta + \sum_k b_k \otimes \pi(c_k)\eta$$

where $F(b) = \sum_k b_k \otimes c_k$, *determine the structure of a graded differential *-algebra on* $\mathrm{ver}(P)$. *As a differential algebra,* $\mathrm{ver}(P)$ *is generated by* $\mathcal{B}$. □

We shall assume that a differential calculus over the bundle P is specified by a graded differential *-algebra $\Omega(P)$ such that

(diff1) The differential algebra $\Omega(P)$ is generated by $\mathcal{B} = \Omega^0(P)$.

(diff2) The map $F: \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ is extendable to a homomorphism

$$\widehat{F}: \Omega(P) \to \Omega(P) \widehat{\otimes} \Gamma^{\wedge}$$

of (graded) differential algebras.

The map $\widehat{F}$ is uniquely determined by the above conditions. We have

$$(\widehat{F} \otimes id)\widehat{F} = (id \otimes \widehat{\phi})\widehat{F}.$$

The formula

$$F^{\wedge} = (id \otimes p_0)\widehat{F}$$

defines the action $F^{\wedge}: \Omega(P) \to \Omega(P) \otimes \mathcal{A}$ of G on differential forms (extending the action F). The map $F^{\wedge}$ is a *-homomorphism and

$$(id \otimes e)F^{\wedge} = id$$

$$(F^{\wedge} \otimes id)F^{\wedge} = (id \otimes \phi)F^{\wedge}$$

$$F^{\wedge}d = (d \otimes id)F^{\wedge}.$$

Let us construct a quantum analog of the verticalising homomorphism. For each $w \in \Omega^k(P)$ the element

$$\pi_v(w) = (id \otimes \pi_{inv} p_k)\widehat{F}(w)$$

belongs to $ver^k(P)$. Here $\pi_{inv}:\Gamma^\wedge \to \Gamma^\wedge_{inv}$ is the canonical projection map. In other words, the above formula defines a linear grade-preserving map $\pi_v:\Omega(P) \to ver(P)$.

LEMMA 2.2. *The introduced map is an epimorphism of graded differential *- algebras.* □

Now, horizontal forms will be defined. Intuitively speaking, they can be characterized as forms possessing trivial differential properties along vertical fibers.

DEFINITION 2.2. The elements of the graded *-subalgebra

$$hor(P) = \widehat{F}^{-1}[\Omega(P) \otimes \mathcal{A}]$$

of $\Omega(P)$ are called *horizontal forms.*

The algebra $hor(P)$ is $F^\wedge$-invariant, in the sense that

$$F^\wedge(hor(P)) \subseteq hor(P) \otimes \mathcal{A}.$$

Horizontal forms w satisfying $F^\wedge(w) = w \otimes 1$ are interpretable as differential forms on M. They constitute a graded differential *-subalgebra $\Omega(M)$ of $\Omega(P)$, with $\Omega^0(M) = i(\mathcal{V})$.

3. The Formalism of Connections

Before introducing connections in the game, we shall define (pseudo)tensorial forms.

Let $\psi(P)$ be the space of linear maps $f:\Gamma_{inv} \to \Omega(P)$ satisfying

$$F^\wedge f = (f \otimes id)\varpi.$$

This space is naturally graded. The elements of $\psi^k(P)$ are imaginable as *pseudotensorial k-forms* on P, with values in the 'Lie algebra' of G. Further, $\psi(P)$ is closed with respect to compositions with $d:\Omega(P) \to \Omega(P)$. Let $\tau(P)$ be the graded subspace of $\psi(P)$ consisting of *tensorial forms* (pseudotensorial forms with values in $hor(P)$).

The formula

$$f^*(\vartheta) = f(\vartheta^*)^*$$

determines a *-involution on $\psi(P)$ (and $\tau(P)$).

DEFINITION 3.1. *A connection on P* (relative to $\Omega(P)$) is every first-order linear map $\omega:\Gamma_{inv} \to \Omega(P)$ such that

$$\widehat{F}\omega(\vartheta) = (\omega \otimes id)\varpi(\vartheta) + 1 \otimes \vartheta$$
$$\omega(\vartheta^*) = \omega(\vartheta)^*$$

for each $\vartheta \in \Gamma_{inv}$.

Connections can be equivalently defined as hermitian pseudotensorial 1-forms ω satisfying

$$\pi_v \omega(\vartheta) = 1 \otimes \vartheta$$

for each $\vartheta \in \Gamma_{inv}$.

THEOREM 3.1. *The bundle P admits at least one connection.* □

Let $con(P)$ be the set of all connections on P. This is a real affine subspace of $\psi^1(P)$. The corresponding vector space consists of hermitian tensorial 1-forms.

Let us fix a linear map $\delta: \Gamma_{inv} \to \Gamma_{inv}^{\otimes 2}$ with the following properties

(i) If $\delta(\vartheta) = \sum_k \vartheta_k^1 \otimes \vartheta_k^2$ then $d\vartheta = \sum_k \vartheta_k^1 \vartheta_k^2$ and $\delta(\vartheta^*) = -\sum_k \vartheta_k^{2*} \otimes \vartheta_k^{1*}$.

(ii) We have

$$\varpi^{\otimes 2}\delta = (\delta \otimes id)\varpi$$

(the right-covariance of δ).

For given linear maps $\varphi, \eta: \Gamma_{inv} \to \Omega(P)$ let us define new linear maps $<\varphi, \eta>, [\varphi, \eta]: \Gamma_{inv} \to \Omega(P)$ by

$$<\varphi, \eta> = m_\Omega(\varphi \otimes \eta)\delta$$
$$[\varphi, \eta] = m_\Omega(\varphi \otimes \eta)c^\top$$

where $c^\top = (id \otimes \pi)\varpi: \Gamma_{inv} \to \Gamma_{inv}^{\otimes 2}$ and $m_\Omega: \Omega(P) \otimes \Omega(P) \to \Omega(P)$ are the 'transposed commutator' [10] and the multiplication map.

If $\varphi \in \psi^i(P)$ and $\eta \in \psi^j(P)$ then $<\varphi, \eta>, [\varphi, \eta] \in \psi^{i+j}(P)$.

For each $\omega \in con(P)$ let us consider a map

$$R_\omega = d\omega - <\omega, \omega> .$$

LEMMA 3.2. *We have*

$$\widehat{F} R_\omega(\vartheta) = (R_\omega \otimes id)\varpi(\vartheta)$$
$$R_\omega(\vartheta^*) = R_\omega(\vartheta)^*,$$

For each $\vartheta \in \Gamma_{inv}$. In other words, R_ω is a tensorial hermitian 2-form. □

DEFINITION 3.2. The map R_ω is called *the curvature of* ω.

It is worth noticing that R_ω depends on the choice of δ. This dependence disappears if ω satisfies the following multiplicativity property.

DEFINITION 3.3. A connection ω is called *multiplicative* iff

$$\omega\pi(a^{(1)})\omega\pi(a^{(2)}) = 0$$

for each $a \in \mathcal{R}$.

If ω is multiplicative then it can be uniquely extended, by multiplicativity, to a unital (*-) homomorphism $\omega^\wedge\colon\Gamma_{inv}^\wedge \to \Omega(P)$. Another interesting class of connections consists of those having the following regularity property.

DEFINITION 3.4. A connection ω is called *regular* iff

$$\omega(\vartheta)\varphi = (-1)^{\partial\varphi}\sum_k \varphi_k\omega(\vartheta\circ c_k)$$

for each $\vartheta \in \Gamma_{inv}$ and $\varphi \in hor(P)$, where $F^\wedge(\varphi) = \sum_k \varphi_k \otimes c_k$.

Regular connections (if exist) form an affine subspace $\rho(P)$ of $con(P)$. The corresponding vector space consists of forms $f = f^* \in \tau^1(P)$ satisfying

$$f(\vartheta)\varphi = (-1)^{\partial\varphi}\sum_k \varphi_k f(\vartheta\circ c_k)$$

for each $\vartheta \in \Gamma_{inv}$ and $\varphi \in hor(P)$.

Let $\sigma\colon\Gamma_{inv}^{\otimes 2} \to \Gamma_{inv}^{\otimes 2}$ be the canonical flip-over operator [10]. Explicitly, this map is given by

$$\sigma(\eta\otimes\vartheta) = \sum_k \vartheta_k \otimes \eta\circ a_k$$

where $\sum_k \vartheta_k \otimes a_k = \varpi(\vartheta)$.

LEMMA 3.3. *If* $\omega \in \rho(P)$ *then*

$$m_\Omega(\omega\otimes\varphi) = (-1)^k m_\Omega(\varphi\otimes\omega)\sigma$$

for each $\varphi \in \tau^k(P)$. □

Now, we are going to introduce the operator of covariant derivative. This operator will be first defined on a restricted domain consisting of horizontal forms. After introducing the operator of horizontal projection, the domain of covariant derivative will be extended to the whole algebra $\Omega(P)$.

For each $\omega \in con(P)$ and $\varphi \in hor(P)$ let us define a new form

$$D_\omega(\varphi) = d\varphi - (-1)^{\partial\varphi}\sum_k \varphi_k\omega\pi(c_k),$$

where $F^\wedge(\varphi) = \sum_k \varphi_k \otimes c_k$.

The form $D_\omega(\varphi)$ is horizontal, too.

DEFINITION 3.5. A linear map $D_\omega\colon hor(P) \to hor(P)$ is called *the covariant derivative* associated to ω.

PROPOSITION 3.4. *(i) The map D_ω intertwines the action $(F^\wedge|hor(P))$ with itself.*

(ii) If ω is multiplicative then

$$D^2_\omega(\varphi) = -\sum_k \varphi_k R_\omega \pi(c_k),$$

for each $\varphi \in hor(P)$.

(iii) If ω is regular then

$$D_\omega(\varphi\psi) = D_\omega(\varphi)\psi + (-1)^{\partial\varphi}\varphi D_\omega(\psi)$$
$$D_\omega(\varphi^*) = D_\omega(\varphi)^*$$

for each $\varphi, \psi \in hor(P)$.

(iv) If $\varphi \in \Omega(M)$ then $D_\omega(\varphi) = d\varphi$. □

The space $\tau(P)$ is closed under taking compositions with D_ω. This fact enables us to define the action of the covariant derivative on tensorial forms.

LEMMA 3.5. *We have*

$$D_\omega(\varphi) = d\varphi - (-1)^{\partial\varphi}[\varphi,\omega]$$

for each $\varphi \in \tau(P)$. □

Let us consider a linear map $q_\omega: \psi(P) \to \psi(P)$ defined by

$$q_\omega(\varphi) = <\omega,\varphi> - (-1)^{\partial\varphi} <\varphi,\omega> - (-1)^{\partial\varphi}[\varphi,\omega].$$

We have then

$$q_\omega \tau(P) \subseteq \tau(P).$$

Moreover, if $\omega \in \rho(P)$ then $(q_\omega|\tau(P)) = 0$.

The following lemma gives the quantum counterpart for the classical Bianchi identity.

LEMMA 3.6. *We have*

$$(D_\omega - q_\omega)(R_\omega) = <\omega, <\omega,\omega>> - <<\omega,\omega>,\omega>$$

for each $\omega \in con(P)$. □

If the connection ω is multiplicative, then the right hand side of the above equality vanishes. On the other hand, if ω is regular then the second summand of the left hand side vanishes. It is worth noticing that regular connections are not necessarily multiplicative. However, there exists a common obstruction to multiplicativity for all regular connections, so that they are multiplicative, or not, at the same time.

In general, the lack of multiplicativity of the connection ω is measured by a map $r_\omega: \mathcal{R} \to \Omega(P)$ given by $r_\omega(a) = \omega\pi(a^{(1)})\omega\pi(a^{(2)})$.

LEMMA 3.7. *(i) The following identities hold*

$$r_\omega(k(a)^*) = -r_\omega(a)^*$$
$$\pi_v r_\omega(a) = 0$$
$$\widehat{F} r_\omega(a) = (r_\omega \otimes id)ad(a).$$

In particular, $r_\omega(a)$ is horizontal for each $a \in \mathcal{R}$.

(ii) The map $\omega \mapsto r_\omega$ is constant on cosets from the space $con(P)/\rho(P)$. If $\omega \in \rho(P)$ then

$$r_\omega(a)\varphi = \sum_k \varphi_k r_\omega(ac_k)$$

for each $a \in \mathcal{R}$ and $\varphi \in hor(P)$, where $F^\wedge(\varphi) = \sum_k \varphi_k \otimes c_k$. Further,

$$dr_\omega(a) = <\omega,\omega> \pi(a^{(1)})\omega\pi(a^{(2)}) - \omega\pi(a^{(1)}) <\omega,\omega> \pi(a^{(2)}). \square$$

Let us assume that P admits regular connections, and let $J(P)$ be the ideal in $\Omega(P)$ generated by the space $r_\omega(\mathcal{R})$, for some $\omega \in \rho(P)$. The previous lemma implies

$$J(P)^* = J(P)$$
$$\widehat{F} J(P) \subseteq J(P) \otimes \Gamma^\wedge$$
$$\pi_v J(P) = \{0\}$$
$$dJ(P) \subseteq J(P).$$

Consequently, it is possible to project the whole formalism on the factoralgebra $\Omega(P)/J(P)$. In the framework of this projected calculus regular connections become multiplicative.

The last topic in this section is the construction and the analysis of horizontal projection operators. Let us fix a splitting of the form

$$\Gamma_{inv}^{\otimes} = \Gamma_{inv}^{\wedge} \oplus I_{inv}^{\wedge}$$

in which $\Gamma_{inv}^{\wedge}$ is realized as a complement of the space $I_{inv}^{\wedge}$, with the help of a grade-preserving hermitian section $\iota: \Gamma_{inv}^{\wedge} \to \Gamma_{inv}^{\otimes}$, intertwining the adjoint actions. Further, let us assume that $\delta(\vartheta) = \iota d(\vartheta)$. Finally, let us consider a linear map $m_\omega: hor(P) \otimes \Gamma_{inv}^{\wedge} \to \Omega(P)$ given by

$$m_\omega(\varphi \otimes \vartheta) = \varphi\omega^\wedge(\vartheta).$$

Here, $\omega^\wedge = \omega^{\otimes}\iota$ and $\omega^{\otimes}: \Gamma_{inv}^{\otimes} \to \Omega(P)$ is the unital multiplicative extension of ω. This extends the previous definition of $\omega^\wedge$, formulated for multiplicative connections. In particular, if ω is multiplicative then the map m_ω is ι-independent.

THEOREM 3.8. *(i) The map m_ω is bijective. It intertwines the product of actions $(F^\wedge|hor(P))$ and $\varpi^\wedge$, with the action $F^\wedge$.*

*(ii) If ω is regular and if $J(P)=\{0\}$ then m_ω is an isomorphism of *-algebras. Here, it is assumed that $hor(P)\otimes\Gamma_{inv}^\wedge$ is endowed with a (graded) *-algebra structure specified by*

$$(\psi\otimes\eta)(\varphi\otimes\vartheta)=\sum_k(-1)^{\partial\varphi\partial\eta}\psi\varphi_k\otimes(\eta\circ c_k)\vartheta$$
$$(\varphi\otimes\vartheta)^*=\sum_k\varphi_k^*\otimes\vartheta^*\circ c_k^*,$$

where $F^\wedge(\varphi)=\sum_k\varphi_k\otimes c_k$. □

The 'horizontal projection' operator $h_\omega\colon\Omega(P)\to hor(P)$ can be now defined as follows

$$h_\omega=(id\otimes p^*)m_\omega^{-1},$$

where $p^*\colon\Gamma_{inv}^\wedge\to\mathbf{C}$ is the zero-component projection. Clearly, h_ω projects $\Omega(P)$ onto $hor(P)$.

With the help of h_ω, the domain of the covariant derivative can be extended to the whole algebra $\Omega(P)$. Indeed, the map $D_\omega\colon\Omega(P)\to hor(P)$ given by

$$D_\omega=h_\omega d$$

extends the previously defined covariant derivative.

PROPOSITION 3.9. *(i) The maps h_ω, D_ω intertwine the action $F^\wedge$.*

*(ii) If $\omega\in\rho(P)$ and if $J(P)=\{0\}$ then h_ω is a *-homomorphism, and*

$$D_\omega(wu)=D_\omega(w)h_\omega(u)+(-1)^{\partial w}h_\omega(w)D_\omega(u)$$
$$D_\omega(w^*)=D_\omega(w)^*$$

for each $w,u\in\Omega(P)$. □

Compositions of pseudotensorial forms with D_ω are tensorial. Hence, it is possible to define the covariant derivative $D_\omega\colon\psi(P)\to\tau(P)$. The following lemma gives an equivalent, more geometrical, description of the curvature.

LEMMA 3.10. *We have*

$$R_\omega=D_\omega(\omega)$$

for each $\omega\in con(P)$. □

4. Characteristic Classes

In this section we shall sketch a quantum generalization of the Weil's theory of characteristic classes. We shall assume that the bundle P admits regular connections, and that $J(P) = \{0\}$. For each $k \geq 0$ let $\mathcal{I}^k \subseteq \Gamma_{inv}^{\otimes k}$ be the subspace of *ad*-invariant elements, and let $\mathcal{I}$ be the direct sum of all these spaces. Clearly, $\mathcal{I}$ is a unital *-subalgebra of the tensor algebra $\Gamma_{inv}^{\otimes}$. Let $H(M)$ be the graded *-algebra of cohomology classes associated to $\Omega(M)$.

Let us consider a connection ω. There exists the unique unital homomorphism $R_\omega^{\otimes}\colon \Gamma_{inv}^{\otimes} \to \Omega(P)$ extending the curvature R_ω. The map $R_\omega^{\otimes}$ is *-preserving, and intertwines $\varpi^{\otimes}$ and $F^{\wedge}$.

PROPOSITION 4.1. *(i) If* $\vartheta \in \mathcal{I}^k$ *then* $R_\omega^{\otimes}(\vartheta) \in \Omega^{2k}(M)$.

(ii) If $\omega \in \rho(P)$ *then* $dR_\omega^{\otimes}(\vartheta) = 0$ *for each* $\vartheta \in \mathcal{I}$.

(iii) The cohomological class of $R_\omega^{\otimes}(\vartheta)$ *in* $\Omega(M)$ *is independent of the choice of a regular connection* ω, *for each* $\vartheta \in \mathcal{I}$.

(iv) The map $W\colon \mathcal{I} \to H(M)$ *given by* $W(\vartheta) = [R_\omega^{\otimes}(\vartheta)]$ *is a unital *-homomorphism.* □

The homomorphism W plays the role of the Weil's homomorphism in classical differential geometry [6]. In fact, in classical geometry the domain of the Weil's homomorphism is restricted on the algebra of *symmetric* invariant elements of the corresponding tensor algebra. However, besides simplifying the domain of W, such a restriction gives nothing new: the image of the Weil's homomorphism will be the same.

A similar situation holds in the noncommutative case. Let $\mathcal{S}$ be the *-algebra obtained from $\Gamma_{inv}^{\otimes}$ by factorising through the ideal $\mathcal{J}$ generated by $\mathrm{Im}(I - \sigma) \subseteq \Gamma_{inv}^{\otimes 2}$. The algebra $\mathcal{S}$ plays the role of polynoms over the 'Lie algebra' of G. The adjoint action $\varpi^{\otimes}$ is naturally projectable on $\mathcal{S}$. Let $\mathcal{I}_{sym} \subseteq \mathcal{S}$ be the subalgebra of elements invariant under the projected action (playing the role of invariant polynomials). Clearly, $\mathcal{I}_{sym} = \mathcal{I}/(\mathcal{I} \cap \mathcal{J})$.

LEMMA 4.2. *If* $\omega \in \rho(P)$ *then*

$$R_\omega^{\otimes}\sigma(\vartheta) = R_\omega^{\otimes}(\vartheta)$$

for each $\vartheta \in \Gamma_{inv}^{\otimes 2}$. □

The above statement implies that W and $R_\omega^{\otimes}$ are factorizable through the ideal $\mathcal{J}$. In this sense they naturally operate on $\mathcal{I}_{sym}$ and $\mathcal{S}$ respectively.

5. Examples and Remarks

(A) All quantum phenomena characteristic for the presented theory of quantum principal bundles already figure in a special version of this theory dealing with bundles over classical smooth manifolds. The theory of principal bundles of this kind is developed in [3].

The main structural result is that G-bundles P over a classical manifold M are in a natural correspondence with classical bundles P_{cl} over the same manifold, with the structure group G_{cl} consisting of *classical points* of G. More precisely, the elements of G_{cl} are *-characters $g\colon \mathcal{A} \to \mathbf{C}$. The product and the inverse in G_{cl} are given by

$$gg' = (g \otimes g')\phi, \qquad g^{-1} = gk,$$

while the counit $e\colon \mathcal{A} \to \mathbf{C}$ is the neutral element. The correspondence $P \leftrightarrow P_{cl}$ can be roughly described as follows. The bundle P_{cl} consists of classical points of P (*-characters of $\mathcal{B}$). Conversely, if P_{cl} is given then P can be recovered by applying an analog of the classical construction of extending structure groups.

In developing a differential calculus on such semiclassical bundles P it is natural to assume that all local trivializations of the bundle locally trivialize the calculus, too. This requirement, together with the specification of the calculus $\Gamma^\wedge$ over G, uniquely fixes the algebra $\Omega(P)$. However, the calculus (Γ, d) can not be chosen arbitrarily. It must satisfy specific consistency requirements, interpretable as compatibility properties with certain 'retrivialization maps' of the bundle. Such differential calculi are called 'admissible' in [3]. It turns out that a left-covariant calculus (Γ, d) is admissible iff $(X \otimes id)ad(\mathcal{R}) = \{0\}$, for each $X \in lie(G_{cl})$. Here, the Lie algebra of G_{cl} is understood as the space of hermitian functionals X on $\mathcal{A}$ satisfying $X(ab) = e(a)X(b) + e(b)X(a)$, for each $a, b \in \mathcal{A}$.

There exists the minimal admissible left-covariant calculus: it is based on the right-ideal $\mathcal{R} \subseteq \ker(e)$ consisting of elements killed by all operators $(X \otimes id)ad$. This calculus is also *-covariant and right-covariant. If G is an ordinary compact matrix group then the minimal admissible calculus coincides with the usual one (based on differential forms). However, small quantum deformations of the classical group structure may cause drastical changes at the level of the minimal admissible calculus. For example [3], if $G = SU_\mu(2)$ [8] and $\mu \in (-1,1) \setminus \{0\}$ then the space Γ_{inv} is infinite-dimensional, and can be naturally identified with the algebra of polynomial functions over the quantum 2-sphere S^2_μ [7].

(B) Classical principal bundles provide a natural mathematical framework for the study of gauge theories. It is interesting to see what will be the counterparts of these theories, in the context of quantum principal bundles [5] (M playing the role of space-time). Properties of such 'quantum gauge' theories essentially depend (besides on the 'symmetry group' G), on the following two prespecifications:

As first, it is necessary to fix a (bicovariant *-) calculus (Γ, d) over G. This determines kinematical degrees of freedom. Secondly, we have to choose a map $\delta\colon \Gamma_{inv} \to \Gamma_{inv}^{\otimes 2}$. This influences dynamical properties of the theory, because δ implicitely figures in the expression for the curvature.

Closely related with problematics of quantum gauge theories is the question of 'gauge transformations'. If M is a classical smooth manifold then the most direct way of defining gauge transformations as (vertical) automorphisms of the bundle P gives nothing new, because of the inherent geometrical inhomogeneity of the bundle P. More precisely, automorphism groups of P and its classical part P_{cl} are isomorphic. However, a proper quantum generalization of gauge transformations can be introduced via the concepts of quantum (infinitezimal) gauge bundles [4, 5]. These are bundles associated to P, relative to the adjoint actions of G on G and Γ_{inv} respectively.

(C) Interesting examples of quantum principal bundles can be obtained from quantum homogeneous spaces. A general construction is this. Let H be a compact matrix quantum group, represented by a *-Hopf algebra $\mathcal{B}$. Entities related to H will be endowed with a prime. Let us assume that G is a subgroup of H. At the formal level, this presumes a specification of a *-epimorphism $j:\mathcal{B} \rightarrow \mathcal{A}$ such that

$$(j \otimes j)\phi' = \phi j, \quad kj = jk'.$$

The *-homomorphism $F:\mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A}$ given by

$$F = (id \otimes j)\phi'$$

is interpretable as the right action of G on H. Let M be the corresponding 'orbit space'. This space is represented by the fixed point *-subalgebra $\mathcal{V}$. Let $i:\mathcal{V} \hookrightarrow \mathcal{B}$ be the inclusion map. The triplet $P = (\mathcal{B}, i, F)$ is a quantum principal G-bundle over M. Because of $\phi'(\mathcal{V}) \subseteq \mathcal{B} \otimes \mathcal{V}$ there exists a natural left action of H on M, represented by $\phi' i:\mathcal{V} \rightarrow \mathcal{B} \otimes \mathcal{V}$ (M is a quantum homogeneous H-space).

Let Ω be an arbitrary left-covariant graded-differential *-algebra over H, satisfying properties *diff1/2*. Let (Ψ, d) be the corresponding first-order calculus. We have $j(\mathcal{R}') \subseteq \mathcal{R}$ where $\mathcal{R}' \subseteq \ker(e')$ is the ideal corresponding to this calculus. Moreover, there exists the unique graded-differential *-homomorphism $j^\wedge:\Omega \rightarrow \Gamma^\wedge$ extending the map j. Explicitly,

$$j^\wedge = (j \otimes id)\pi_v$$

(and the identification $\Gamma^\wedge = \mathcal{A} \otimes \Gamma^\wedge_{inv}$ is assumed). We have $j^\wedge(\Psi_{inv}) = \Gamma_{inv}$.

Let us consider a splitting of the form

$$\Psi_{inv} = L \oplus \Gamma_{inv}$$

where the space Γ_{inv} is realized as a complement to $L = \ker(j^\wedge | \Psi_{inv})$, with the help of a hermitian right-G-covariant section $\epsilon:\Gamma_{inv} \rightarrow \Psi_{inv}$. Then the map $\omega:\Gamma_{inv} \rightarrow \Omega$ obtained by composing ϵ with the cannonical inclusion $\Psi_{inv} \hookrightarrow \Omega$ is a connection on P.

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ON BRAIDED TENSORCATEGORIES

THOMAS KERLER*
Department of Mathematics, Harvard University
Cambridge, MA, USA, kerler@math.harvard.edu

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Introduction

An important step in organizing selection rules and defining symmetry principles of Quantumtheories in algebraic terms has been the introduction of group theory into physics by Weyl, Wigner,Yang, Mills and others. Since the works of [4] and [2] it has become clear that the relevant data can be equivalently and more directly described by a *symmetric tensorcategory* (STC). Often in low dimensional physics the axiom that the commutativity constraint squares to one has to be relaxed so that we naturally obtain representation of the braid groups rather than the symmetric groups. The more general *braided tensorcategories* (BTC) are related to quasitriangular quasi-Hopfalgebras, but there is no one to one duality-correspondence as for STC's since BTC's are rarely Tannakian. Interestingly, they appear in many other areas of mathematical physics like the theory of subfactors of von Neumann algebras, two dimensional integrable lattice models, and low dimensional topology.

At generic points in the space of BTC's many uniqueness statements can be found by using deformation theory. They give some explanation about the relation of affine algebras and quantum groups at generic levels. For *rational* theories these methods break down. Nevertheless, one has identified equivalent rational BTC's coming from very different areas. An example of a family of related rational models includes $SU(2)$ and rank=2 WZW-models, the corresponding quantum groups at roots of unity, subfactors with Jones-index < 4, the Alexander or Jones polynomials, and the Q-state Pottsmodel. In order to explain these coincidences in terms of a classification we need to find reasonable constraints on the considered class of BTC's. Most conveniently they are imposed on the combinatorial part of the $\otimes$-category, i.e., the fusionrules.

In [10] (see also [6] for $k = 2$) it has been shown that if the entire fusionring of a BTC is equal to that of $\overline{Rep}(U_q(sl(k)))$ then the two categories

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themselves have to be isomorphic for suitable q. In this paper (which is in large parts a summary of results from [6]) we wish to impose a much weaker condition, namely that the category has a generating object X whose tensorsquare $X \otimes X$ is the sum of two simple objects. In this situation we face a much larger class of categories including those that are obtained as product-, orbit-, and subgrading-categories from the known ones. The mentioned constructions rely on the study of gradings and invertible objects of a BTC. Many of the resulting categories are inequivalent to any of the semisimplified representation categories of Hopfalgebras and those occurring in conformal field theory. We find a natural condition in terms of Hecke algebra representations for when this list of categories is complete. We prove it for the case where one of the summands of $X \otimes X$ is invertible, thereby yielding a complete classification.

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1. Braided Tensorcategories

In all our consideration we mean by a braided tensorcategory $\mathcal{C}$ an *abelian category* (see [12]) for which the morphism sets are finite dimensional vectorspaces over $\mathbf{C}$. In addition we have natural transformations $\epsilon \in Nat(\otimes, P\otimes)$ and $\alpha \in Nat(\otimes(id \times \otimes), \otimes(\otimes \times id))$. They yield the commutativity and associativity isomorphisms $\epsilon(X,Y) : X \otimes Y \to Y \otimes X$ and $\alpha(X,Y,Z) : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$ which have to obey the pentagonal and two hexagonal equations. For simplicity we shall omit α in the formulas although it can be a non trivial morphism. Also we shall only consider *rigid* categories. This means that to any object $X \in \mathcal{C}$ we find a conjugate object $X^\vee$ and morphisms $ev : X^\vee \otimes X \to 1$ and $coev : 1 \to X \otimes X^\vee$, with the usual pair of contraction identities. For details see, e.g., [13] for the symmetric and [11] for the braided case.

For any $\otimes$−category $\mathcal{C}$ we can define the *fusionring* $K_o^+(\mathcal{C})$, which is the ring over $\mathbf{Z}^+$ generated by the equivalence classes $[X]$ of objects subject to the relations $[X] = [Y] + [X/Y]$ whenever Y is included into X and $[X \otimes Y] = [X][Y]$. It is clear that with this definition every object can be written uniquely as the sum of the simple objects that appear in its composition series and the products of the simple objects determine all other products of the fusionring.

A notion that is very useful for our purposes is that of *grading*. For a BTC the set $\otimes\!-\!Nat(id_{\mathcal{C}})$ which consists of natural transformations $\xi(X) \in End(X)$ with $\xi(X \otimes Y) = \xi(X) \otimes \xi(Y)$ is an abelian group. This fact allows us to decompose every object uniquely into a direct sum $X = \bigoplus_{\nu \in Gr(\mathcal{C})} X_\nu$. Here X_ν is the maximal subobject such that the only eigenvalue of $\xi(X_\nu)$ is $\nu(\xi)$ for all ξ . $Gr(\mathcal{C})$ is the subgroup of all characters on $\otimes\!-\!Nat(id_\mathcal{C})$ of this form. This decomposition has the property that $(X \otimes Y)_\nu = \oplus_\eta X_{\nu\eta^{-1}} \otimes Y_\eta$ and that to any *simple* object X we can assign a unique $\nu \in Gr(\mathcal{C})$ with $X = X_\nu$. Thus $Gr(\mathcal{C})$ makes $K_o^+(\mathcal{C})$ into a graded algebra. We call $\mathcal{C}$ *locally rational* if every component $K_o^+(\mathcal{C})_\nu$ is finitely generated , i.e., if there are only finitely many inequivalent, simple objects of a given grading.

A special type of simple objects are the *invertible* ones, which satisfy $X \otimes X^\vee \cong 1$. They form an abelian group on $K_o^+(\mathcal{C})$ we shall call $Pic(\mathcal{C})$. Let us introduce two natural group homomorphisms:

$$\vartheta : Pic(\mathcal{C}) \longrightarrow Gr(\mathcal{C}) \tag{1.1}$$

$$\mu : Pic(\mathcal{C}) \longrightarrow \otimes\!-\!Nat(id_\mathcal{C}) \ , \tag{1.2}$$

where ϑ associates a grading to an irreducible element in $Pic(\mathcal{C})$ and μ is defined by $1_g \otimes \mu(g)(X) = \epsilon(X, g)\epsilon(g, X)$. A *balancing* of a tensorcategory is a natural transformation of $X \to X^{\vee\vee}$ to the identity functor. For a BTC a balancing is equivalently given by a transformation $\theta \in Nat(id_\mathcal{C})$ with

$$\epsilon(Y,X)\epsilon(X,Y) = \theta(X) \otimes \theta(Y)\theta(X \otimes Y)^{-1} \text{ and } \theta(X^\vee) = \theta(X)^t \ .$$

If such a balancing exists (there are plenty of examples where it does not) it is unique up to elements of order two in $\otimes\!-\!Nat(id_\mathcal{C})$. To a given balancing we can associate a family of traces $tr_X \in End(X)^*$ by

$$tr_X(f) : 1 \xrightarrow{coev} X \otimes X^\vee \xrightarrow{(f\theta(X))\otimes 1} X \otimes X^\vee \xrightarrow{\epsilon(X,X^\vee)} X^\vee \otimes X \xrightarrow{ev} 1 \ .$$

We call a *dimension* a function $d : K_o^+(\mathcal{C}) \to \mathbf{C}$ which respects sums and products and is invariant under conjugation. Since the trace is cyclic, also for pairs of morphisms between different objects, and factorizes w.r.t. tensorproducts, we can define a canonical dimension by $d_{tr}(X) = tr_X(1)$. Dimension functions can also be constructed in a different way by applying Perron-Frobenius theory to the fusion matrices of $K_o^+(\mathcal{C})$, representing the action of the ring on itself by multiplication.

THEOREM 1.1. *Assume that the fusionring $K_o^+(\mathcal{C})$ of a BTC $\mathcal{C}$ is locally rational, then*

1. *there is exactly one* <u>*positive*</u> *dimension $d_{PF} : K_o^+(\mathcal{C}) \to \mathbf{R}^+$,*
2. *$d_{PF} \geq 1$ and $d_{PF}(X) = 1$ if and only if $X \in Pic(\mathcal{C})$.*
3. *If $X = X_\eta$ then $\underline{X} : K_o^+(\mathcal{C})_\nu \to K_o^+(\mathcal{C})_{\nu\eta}$, defined by multiplication has norm $d_{PF}(X)$ independent of ν .*

In the last statement we assumed $K_o^+(\mathcal{C})$ to be equipped with the inner product for which the simple objects are an orthonormal basis.

In order to relate the positivity condition to properties of the categories themselves we introduce C^* structures which are known from applications in operator algebras and physics [4], but are also related to the *polarizations* in [13]. A **-structure* on a BTC is an antilinear, contravariant, coexact BTC-functor $* : \mathcal{C} \to \mathcal{C}$. For simplicity let us assume that $X^* \otimes Y^* \xrightarrow{\sim} (X \otimes Y)^*$ is the identity so that α and ϵ are unitary. We call the category of finite dimensional Hilbert spaces $\mathcal{H}$ and denote by Ω the class of all covariant, exact (not necessarily $\otimes$–) functors $\omega : \mathcal{C} \to \mathcal{H}$ which commute with $*$. We say that $\mathcal{C}$ is a *C^*-category* if for any morphism f there is some $\omega \in \Omega$ with $\omega(f) \neq 0$. In this case we can introduce a norm $\|f\| = sup_{\omega\in\Omega}\|\omega(f)\|$ which renders the category $\mathcal{C}$ semisimple and equips the algebras $End(X)$ with a C^*-structure in the usual sense. For C^*-categories we construct a balancing as follows. Define $\lambda_X \in End(X)$ by

$$X \xrightarrow{1\otimes ev^*} X \otimes X^\vee \otimes X \xrightarrow{\epsilon(X,X^\vee)\otimes 1} X^\vee \otimes X \otimes X \xrightarrow{ev\otimes 1} X\,.$$

From the positivity of $< f > = ev(1\otimes f)ev^* = tr_X(\lambda_X^* f)$ and the fact that $End(X)$ is a sum of type I factors with trace tr_X we infer that λ_X is central. Since tr_X is generally cyclic it follows that the unitary part $\theta_o(X) = U(\lambda_X)$ gives rise to a natural transformation.

THEOREM 1.2. *To any C^*-BTC $\mathcal{C}$ there exists precisely one balancing such that the associated traces tr_X are positive $\forall X \in ob(\mathcal{C})$. It is given by $\theta_o \in Nat(id_{\mathcal{C}})$.*

Clearly, for this choice, the dimension d_o associated to the balancing is positive. Thus by Theorem 1.1 we obtain for locally rational C^*-categories the remarkable identity

$$d_{PF} = d_o \tag{1.3}$$

where both quantities are defined in completely independent ways.

2. Hecke - and Temperley Lieb Type Categories

In many examples $K_o^+(\mathcal{C})$ is generated by a single object Π (e.g., a fundamental representation) meaning every object is the direct sum of subobjects of tensorpowers of $\Pi\oplus\Pi^\vee$. It is easy to see that in this situation $Gr(\mathcal{C}) \cong \mathbf{Z}/N$, generated by the character of Π, and the order $N \geq 1$ is the smallest number such that $Hom(\Pi^n, \Pi^{(n+N)}) \neq 0$ for some n.

In order to state a tractable classification problem we confine the class of BTC's further by restricting the dimension of $End(\Pi^{\otimes 2})$. The condition $End(\Pi^{\otimes 2}) = \mathbf{C}$ is by rigidity equivalent to $\Pi \in Pic(\mathcal{C})$ whereas $End(\Pi^{\otimes 2}) =$

$\mathbf{C}\oplus\mathbf{C}$ implies that $\Pi\otimes\Pi\cong A\oplus B$ for two inequivalent, simple objects A and B. The first is a special case of a θ-category which we classify in the next section. In the second case $\epsilon(\Pi,\Pi)$ has two eigenvalues γ_A and γ_B so that the rescaled natural representation of n-th braidgroup B_n on $\mathcal{E}_n = End(\Pi^{\otimes n})$, defined by $\rho(g_{i+1}) = -\gamma_A 1^{\otimes i}\otimes\epsilon(\Pi,\Pi)$ factors into a representation of the n-th Hecke algebra $\rho : H_n(q)\to\mathcal{E}_n$ with $q := -\gamma_B\gamma_A^{-1}$. (We choose conventions as in [14]). This sequence of morphisms is compatible with the inclusions $\mathcal{E}_n\hookrightarrow\mathcal{E}_{(n+1)} : f\mapsto f\otimes 1_\Pi$ and thus extends to $\rho : H_\infty\to\mathcal{E}_\infty$. If $\mathcal{C}$ is also a C^*-category we have $|q|=1$ and ρ is a *-representation on every $H_n(q)$. Henceforth we call BTC's with these properties *Hecke type categories.* For these $\tau|_{\mathcal{E}_n} = d(\Pi)^{-n}tr_{\Pi^n}$ defines a positive, normalized Markov trace on $\mathcal{E}_\infty$ with modulus $\eta = \tau(e_A) = d(A)d(\Pi)^{-2}$. Combining the above observations with results from [14] we find the following restrictions:

THEOREM 2.1. *For a Hecke type category with $\epsilon(\Pi,\Pi)^2$ nonscalar we have*

1. $q = -\gamma_B\gamma_A^{-1} = e^{\pm\frac{2\pi i}{l}}$ *for some* $l = 4,5,\ldots$
2. $\eta = \frac{d(A)}{d(\Pi)^2} = \frac{(1-q^{(-k+1)})}{(1+q)(1-q^{-k})}$ *for some* $k = 1,\ldots,l-1$.
3. *The morphism ρ factors through the semisimple quotient $H_n(q)\to H_n^{(k,l)}$ whose representations are labeled by (k,l)-diagrams.*

Since $H^{(k,l)}$ coincides with the GNS-quotient of the pullback $\rho^*\tau$, the factorized morphism $\bar\rho : H_\infty^{(k,l)}\to\mathcal{E}_\infty$ is an inclusion. It also yields a morphism of (non rigid) fusionrings $K_o(\rho)_n : K_o^+(H_n^{(k,l)})\to K_o^+(\mathcal{C})_n$ for positive gradings $n=0,1,\ldots$. Here $K_o^+(H_\infty^{(k,l)})$ has a unique, smallest extension into a rigid fusionring $F^{(k,l)}$ with $\mathbf{Z}$-grading which is shown in [9] to be isomorphic to the truncated subfusionring of $U_q(Gl(k))$ generated by the usual fundamental representation. If $\mathcal{C}$ is locally rational the norms of $\underline{[1]}|_{H_n^{(k,l)}}$ and and $\underline{\Pi}|_{K_o^+(\mathcal{C})_n}$ and hence of $\|K_o(\rho)_n\|$ are independent of n for large n. In this situation we find that $K_o(\rho)([1^k])$ has norm one, i.e., it is invertible, and thus can be used to extend $K_o(\rho)$ to a morphism of rigid fusionrings $\Psi : F^{(k,l)}\to K_o^+(\mathcal{C})$, defined also for negative gradings.

The embedding of the Hecke algebras gives us not only information on the fusionring but allows us to compute the balancing phases. In $H(q)$ the scalar α_λ by which the central braid group element $\Delta_N^2 = (g_1\ldots g_{(N-1)})^N$ acts in the irreducible representation associated to the diagram λ has been computed in [15] as a framing anomaly of link invariants. It is possible to factorize the product of ϵ's in $\mathcal{E}_N$ associated to Δ_N^2 into the expression $\theta(\Pi)^{\otimes N}\theta(\Pi^{\otimes N})^{-1}$. This observation enters the second part of the following theorem.

THEOREM 2.2. *If $\mathcal{C}$ is a locally rational Hecke type category, then*
1. there is a unique morphism of rigid fusionrings

$$\Psi \,:\, F^{(k,l)} \longrightarrow K_o^+(\mathcal{C}) \quad \text{with } \Psi([1]) = \Pi$$

2. if $X \in ob(\mathcal{C})$ is a subobject of $\Psi(\lambda)$ for some diagramm λ then

$$\theta(X) = \theta_\lambda 1_X \quad \text{where} \quad \theta_\lambda = \theta(\Pi)^{|\lambda|}(-\gamma_A)^{|\lambda|-|\lambda|^2}\alpha_\lambda^{-1}$$

By definition the image of Ψ generates additively $ob(\mathcal{C})$ so that every object is a sum of those considered in *b*). Hence the balancing of a Hecke type category is completely determined by $\theta(\Pi)$, γ_A and γ_B.

In order to explain the constraints on Ψ resulting from Theorem (2.2) we define the graph of a map of positive lattices $\Lambda : L_1 \to L_2$ as the bicolored graph whose vertices are the generators of L_1 and L_2 with respective coloration. The number of edges between them are given by the matrix elements of Λ. Denoting by Ψ_n, $\underline{[1]}_n$ and $\underline{\Pi}_n$ the respective restrictions to the n-th graded components the relation $\Psi_{(n+1)}\underline{[1]}_n = \underline{\Pi}_n\Psi_n$ means that pairs of neighboring simple objects in the graph of $\underline{[1]}_n$ are mapped by Ψ to sums of pairs of neighboring objects in the graph of $\underline{\Pi}_n$. By Theorem 1.1 Ψ is dimension preserving, i.e., $d_{PF}(\Psi(X)) = d_{PF}(X)$. From part *b*) of Theorem 2.2 we see that θ has to have the same value on every simple object of a connected component of the graph of Ψ_n. Knowing the specific values for one coloration namely the θ_λ on $F^{(k,l)}$ this imposes together with the neighborhood condition strong constraints on the structure of Ψ_n. In many cases the only remaining possibility is that the components of Ψ_n are pairs of different colorations so that Ψ_n is an isomorphism for every n. In this case we say that Ψ is a *local isomorphism.* A special subclass of such categories are *Temperley Lieb type categories* which are defined in the next theorem. Its proof is in part a direct consequence of Theorem 1.1 and identity (1.3).

THEOREM 2.3. *If $\mathcal{C}$ is a Hecke type category with $\epsilon(\Pi,\Pi)^2$ nonscalar, then the following four conditions are equivalent*

1.) $k = 2$	*2.) e_A and $1\otimes e_A$ generate $\mathcal{A}_\beta(3)$ with $\beta < 4$*
3.) $A \in Pic(\mathcal{C})$	*4.) $d(X) < 2$ and $d(A) \leq d(B)$.*

Here $\mathcal{A}_\beta(n)$ is the Temperley Lieb quotient of the Hecke algebra with modulus $\beta = q + q^{-1} + 2 = d(\Pi)^2$. The elements of $F^{(2,l)}$ are pairs $[\lambda_1, \lambda_2]$ with $\lambda_i \in \mathbf{Z}$ and $0 \leq \lambda_1 - \lambda_2 \leq l-2$. The graph associated to $[1]_n$ is A_{l-1}, where the gradation is $n = \lambda_1 + \lambda_2$ and two simple objects are adjacent if they coincide in one component. Specializing Theorem 2.2 to $k=2$ we can write the balancing as $\theta_\lambda = c_n t^{d^2}$ where $t^4 = q$ is the primitive l-th root of unity and $d = \lambda_1 - \lambda_2 + 1$.

The norm of $\underline{\Pi}_n$ has to be the same as the norm of $\underline{[1]}_n$ so that by an old result of Kronecker, see [8], the only possibilities for the graph of $\underline{\Pi}_n$ are A_{l-1}, $D_{l/2+1}$ or $E_{6,7,8}$ $(l = 12, 18, 30)$. One readily checks that the neighborhood and component condition discussed above exclude the D and E cases. In summary, we have the following result for $k = 2$:

THEOREM 2.4. *Suppose $\mathcal{C}$ is a Temperley Lieb type category with $\beta \neq 4$. Then there exists a local isomorphism of rigid, graded fusionrings $\Psi : F^{(2,l)} \to K_o^+(\mathcal{C})$ with $\Psi([1]) = \Pi$.*

3. Two Important Examples

A.) A class of braided tensorcategories that can be completely classified are semisimple BTC's for which all simple objects are invertible. We call them *θ-categories.* For a θ-category $\mathcal{C}$ we have in particular $K_o^+(\mathcal{C}) \cong \mathbf{Z}^+[Pic(\mathcal{C})]$ and the map ϑ from (1.1) yields an isomorphism $Pic(\mathcal{C}) \cong Gr(\mathcal{C})$. To a class of pairs (ϵ, α) of natural isomorphisms (considered as functions $\alpha \in C(Pic(\mathcal{C})^3)$ and $\epsilon \in C(Pic(\mathcal{C})^2)$ by specialization) that give rise to equivalent BTC's we can assign a unique class in $H^4(G, 2; \mathbf{C}^*)$, the cohomology group of the Eilenberg MacLane space $K_2(G)$. This correspondence results from the fact that the pentagonal and hexagonal equations translate to cocycle conditions and the transformations $g \otimes h \rightrightarrows g' \otimes h'$ of $\otimes$-isomorphisms give rise to coboundaries, see [6]. The function $\theta : Pic(\mathcal{C}) \to \mathbf{C}^*; \, g \mapsto \epsilon(g, g)$ is easily shown to be quadratic, only dependent on the cohomology class of ϵ and a possible balancing of $\mathcal{C}$. Combining these observations with results in [5] we find the following classification:

THEOREM 3.1. *To any quadratic fosm θ on a finitely generated abelian group G there exists one and up to isomorphism only one θ-category $\mathcal{P}(\theta, G)$ such that $Pic(\mathcal{P}(\theta, G)) \cong G$ and $\theta(g) = \epsilon(g, g)$.*

B.) It is well known that the category $Rep\Big(U_t(Sl(k))\Big)$ of quantum group representations, with $q = t^{-2k}$ a primitive l-th root of unity, is not semisimple. Nevertheless, it is possible to define a semisimple subquotient category. The morphisms are the quotients of $Hom_{\mathcal{C}}(X, Y)$ by the nullspaces $Hom(X, Y)^o$ of the trace pairing

$$Hom(Y, X) \otimes Hom(X, Y) \longrightarrow End(X) \xrightarrow{tr_X} \mathbf{C}^*$$

In this category we also discard objects with $End(X) = End(X)^o$ which for indecomposable X is equivalent to $d(X) = 0$.(For details of this construction see [11] and also [1] and [7]). The full subcategory generated by the image of $\Pi = [1]$ is a semisimple Hecke type category $R(t, k)$ without an apriori

*-structure. Let us call this an *indefinite Hecke type category.* As for $Sl(k)$ we label the simple objects by Young diagrams with the restrictions $0 \leq \lambda_1 - \lambda_k \leq l-k$ so that $A = [1,1]$, $B = [2]$, $\gamma_A = -t^{1+k}$ and $\gamma_B = t^{1-k}$. The group $Pic(R(t,k)) \cong \mathbf{Z}/k$ is generated by the $\alpha = [l-k]$. The grading group $Gr(R(t,k))$ is also cylic of order k and associates to a diagram λ the number of boxes $|\lambda| \, mod \, k$. Hence $\vartheta : \mathbf{Z}/k \to \mathbf{Z}/k$ from (1.1) is just multiplication with l. A possible balancing of $R(t,k)$ is given by $\theta_\lambda = t^{c(\lambda)}$ where

$$c(\lambda) = \sum_{i<j} (\lambda_i - \lambda_j)^2 + k(\lambda_i - \lambda_j).$$

The structure of the full subcategory over $Pic(\mathcal{C})$ is determined in the sense of sense of Theorem 3.1 by $\epsilon(\alpha,\alpha) = (-1)^{(l-k)} t^{(l-k)l}$. The map μ defined in (1.2) is given by $\mu(\alpha)([1]) = t^{2l}$. A deformation argument used in [6] (which should be extendable to general k) shows that the necessary constraint in Theorem 2.1 for the existence of *-structures is also sufficient:

THEOREM 3.2. *$R(t,2)$ is isomorphic to a C^*-category if and only if $t^4 = e^{\pm\frac{2\pi i}{l}}$.*

There is a remarkable uniqueness result on the categories with the same fusion ring as $R(t,k)$ due to [10] (for a proof for $k = 2$ using structure constants see [6]).

THEOREM 3.3. *Suppose for an indefinite Hecke type category $\mathcal{C}$ there is an isomorphism of fusionrings $\psi : K_o^+(\mathcal{C}) \xrightarrow{\sim} K_o^+(R(t,k))$ mapping generators to each other. If in addition the invariants γ_A and γ_B of $\mathcal{C}$ coincide with those of $R(t,k)$ then ψ extends to an isomorphism of categories $\mathcal{C} \cong R(t,k)$.*

4. Product and Orbit Categories

There are a number of natural operations between categories that allow us to produce new categories, e.g., from the examples in the previous section. A special class of $\otimes-$ subcategories of a given BTC $\mathcal{C}$ is obtained by picking a subgroup $H \subset Gr(\mathcal{C})$ and defining ${}_H\mathcal{C} \hookrightarrow \mathcal{C}$ to be the largest full subcategory for which all objects have grading in H. Of particular interest is the subcategory ${}_0\mathcal{C}$ which consists of objects with trivial grading. It is additively generated by the subobjects of all $j \otimes j^\vee$ with j simple. Also we denote by $\mathcal{C}_1 \cap \mathcal{C}_2$ the largest full subcategory which is contained in two full $\otimes-$subcategories $\mathcal{C}_i \hookrightarrow \mathcal{C}$.

Dual to the notion of direct products of Hopfalgebras we have the notion of a product of categories $\mathcal{C}_i$ which is a biexact functor $\odot : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_1 \odot \mathcal{C}_2$ onto the smallest additive completion of the ordinary product. The precise definition is given in [2]. Clearly, this functor induces an isomorphism $Gr(\mathcal{C}_1) \oplus Gr(\mathcal{C}_2) \cong Gr(\mathcal{C}_1 \odot \mathcal{C}_2)$.

The notion of quotients of BTC's related to branching of representations to sub-Hopfalgebras needs more explanation: To this end assume that P is a full $\otimes$−subcategory with a $\otimes$−fibre functor $\nu : P \to Vect(\mathbf{C})$ (or $\mathcal{H}$) of strict, symmetric categories. To any object $X \in ob(\mathcal{C})$ we have - up to isomorphism - a unique maximal subobject $X_P \hookrightarrow X$ with $X_P \in ob(P)$. We define a category $\mathcal{C}/P$ with $ob(\mathcal{C}/P) = ob(\mathcal{C})$ and morphisms $\widetilde{Hom}(X,Y) = \nu((Y \otimes X^\vee)_P)$. (see [2], [3] for Tannakian categories.) The canonical morphism in P, $(Z \otimes Y^\vee)_P \otimes (Y \otimes X^\vee)_P \to (Z \otimes X^\vee)$, obtained from ev, determines the composition of morphisms in $\mathcal{C}/P$. Using the natural braid isomorphisms we find two canonical isomorphisms in P

$$\oslash^{\pm} : (Y_1 \otimes X_1^\vee)_P \otimes (Y_2 \otimes X_2^\vee)_P \to ((Y_1 \otimes Y_2) \otimes (X_1 \otimes X_2)^\vee)_P \qquad (4.4)$$

both of which define tensorproducts of morphisms in $\mathcal{C}/P$.

Viewing the invariances as subobjects $Z_1 \hookrightarrow Z_P$ the map

$$Hom(X,Y) \to Hom(1,(Y \otimes X^\vee)_1) \xrightarrow{\nu} Hom_{\mathbf{C}}(1, \nu((Y \otimes X^\vee)_1))$$
$$\xrightarrow{\sim} \nu((Y \otimes X^\vee)_1) \hookrightarrow \nu((Y \otimes X^\vee)_P)$$

gives then rise to a $\otimes$−functor $p : \mathcal{C} \to \mathcal{C}/P$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{p} & \mathcal{C}/P \\ \uparrow & & \uparrow \otimes 1_{\mathcal{C}} \\ P & \xrightarrow{\nu} & Vect(\mathbf{C}) \end{array} \qquad (4.5)$$

Clearly, the images of the natural isomorphisms $\bar{\epsilon} = p(\epsilon)$ and $\bar{\alpha} = p(\alpha)$ satisfy the pentagonal and hexagonal equations and are natural with respect to morphisms in the image of p. But since the functor p is by definition not full for $P \neq Vect(\mathbf{C})$ there is a priori no reason for $\bar{\epsilon}$ and $\bar{\alpha}$ to be natural in $\mathcal{C}/P$. It turns out that naturality is equivalent to demanding that P *decouples*, i.e.,

$$\epsilon(Q,X)\epsilon(X,Q) = 1 \quad \text{for all} \quad Q \in ob(P), X \in ob(\mathcal{C}).$$

In this case the two morphisms $\oslash^{\pm}$ from (4.4) coincide. Suppose $j \in ob(\mathcal{C})$ is simple and $j \otimes j^\vee$ contains nontrivial subobjects from P. Then $\widetilde{End}(j) \neq \mathbf{C}$, and since kernels and cokernels have to stem from $\mathcal{C}$, $\mathcal{C}/P$ fails to be abelian. Also naïve abelian completions usually spoil naturality of $\bar{\epsilon}$. In order to avoid this situation we have to impose the condition $\mathcal{C}_o \cap P = Vect(\mathbf{C}) \otimes 1$. It is easily seen that the only subcategories with this property are θ-categories over subgroups $R \subset Pic(\mathcal{C})$ on which the grading $\vartheta|_R$ from (1.1) is injective. In this case the fusionring morphism associated to p is locally isomorphic and the inequivalent objects of $\mathcal{C}/P$ are identical with orbits of R. Hence we call $\mathcal{C}/R$ an *orbit category.* (In [6] the term *induced category* was used.)

Conversely, any local isomorphism ψ is of the form that it sends simple objects to their orbits under the action of $\psi^{-1}(1) \subset Pic(\mathcal{C})$. Moreover, we can pullback every category along such ψ by setting

$$Hom_{\mathcal{C}}(X,Y) := \bigoplus_{\nu \in Gr(\mathcal{C})} Hom_{\mathcal{C}/P}(\psi(X_\nu), \psi(Y_\nu)).$$

Note that the decoupling condition for R is that R lies in the kernel of the map μ from (1.2). We conclude with a survey of properties of orbit categories. For more details see [6].

THEOREM 4.1. *1. If $R \subset Pic(\mathcal{C})$ is a subgroup on which μ is trivial, ϑ is injective and the associsted θ-subcategory P is trivial then there exist a unique, abelian BTC $\mathcal{C}/P$, and functors ν and p such that (4.5) commutes.*

2. For any local isomorphism $\psi : F \to K_o^+(\bar{\mathcal{C}})$ of rigid fusionrings there is a unique BTC $\mathcal{C}$ with $K_o^+(\mathcal{C}) \cong F$, a functor $p : \mathcal{C} \to \bar{\mathcal{C}}$ and a fibre functor on the subcategory associated to $\psi^{-1}(1)$ extending ψ such that (4.5) commutes.

5. Hecke Categories and Temperley Lieb Categories

In this section we discuss a new family of Hecke categories and a classification of Temperley Lieb Categories. Combining the constructions and examples given in the previous sections we can define a class of indefinite Hecke type categories with fusionring $F^{(k,l)}$ by

$$D'(\theta,t,k) := {}_{\Delta}\Big(\mathcal{P}(\theta,\mathbf{Z}) \odot D(t,k)\Big)$$

where $\Delta \subset \mathbf{Z} \oplus \mathbf{Z}/k = Gr(\mathcal{P} \odot D)$ is the diagonal subgroup. The basic invariants with respect to the canonical generator $\Pi' = (1) \odot \Pi$ are $\gamma_A = -\theta(1)t^{1+k}$ and $\gamma_B = \theta(1)t^{1-k}$, where (1) is the generator of $\mathcal{P}$. In fact Theorem 3.3 and Theorem 4.1 show that D' is the only category with this fusionring and these invariants. We have an isomorphism

$$\varphi : \mathbf{Z} \oplus \mathbf{Z}/(k,l) \xrightarrow{\sim} Pic(D') \,;\, (i,j) \mapsto ((k,l)i) \odot \alpha^{jk'+il''} ,$$

where $k' = k/(k,l)$ and $ll'' = (k,l) \, mod \, k$. The grading ϑ is the projection onto the first factor $i(k,l)$. The θ-category P_{ij} associated to the infinite cyclic subgroup generated by an object $\varphi(i,j) = (n) \odot \alpha^m$ with $n \neq 0$ is trivial and decouples iff $\epsilon(\alpha,\alpha)^{m^2} = \theta(1)^{-n^2}$ and $t^{2lm} = \theta(1)^{2n}$. For these values we denote by

$$D''(\theta,t,k,i,j) := D'(\theta,t,k)/P_{ij} \tag{5.6}$$

the orbit category as defined in Theorem 4.1. In the list of the categories of the form (5.6) we recover the ones obtained from $\widehat{s\ell}(k)_{l-k}$ and $\widehat{s\ell}(l-k)_k$ and products of these with level one theories. Using that the group extension

$$0 \longrightarrow Pic({}_0\mathcal{C}) \longrightarrow Pic(\mathcal{C}) \longrightarrow Gr(\mathcal{C})$$

is an invariant of $\mathcal{C}$ we can easily check that the orbit construction yields categories inequivalent to any subcategories of the known representation categories of Hopfalgebras. The easiest such case is found for $l=6$, $k=2$, if we divide by the θ-subcategory generated by $\varphi(1,1) = (2)\odot[4]$. The set of simple objects $\{[\cdot],\ldots,[4]\}$ is the same as for the $U_t(sl_2)$ category, but we have modifies products $[1][1] = [3][3] = [2]+[4]$ and $[1][3] = [\cdot]+[2]$. In general the requirement *2.)* of local isomorphie from Theorem 4.1 is difficult to verify. However for $k=2$ we can use Theorem 2.4 and the uniqueness of the D'-categories to prove the following classification.

THEOREM 5.1. *Every Temperley Lieb type category with $\epsilon(\Pi,\Pi)^2$ non-scalar is of the form $D''(\theta,t,2,i,j)$ for $t^4 = e^{\pm\frac{2\pi i}{l}}$ and admissible θ, i and j.*

In the case where ϵ^2 is scalar we can consider products with suitable θ-categories and reduce the problem to the case where $\epsilon(\Pi,\Pi)^2 = 1$. Since Π is a generator this implies that the category is symmetric and we can apply the result of [4] to find a classification in terms of $U(2)$-subgroups.

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LIE ALGEBRAS AND BRAIDED GEOMETRY

SHAHN MAJID*
Department of Applied Mathematics & Theoretical Physics
University of Cambridge, Cambridge CB3 9EW, U.K.

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Abstract. We show that every Lie algebra or superLie algebra has a canonical braiding on it, and that in terms of this its enveloping algebra appears as a flat space with braided-commuting coordinate functions. This also gives a new point of view about q-Minkowski space which arises in a similar way as the enveloping algebra of the braided Lie algebra $gl_{2,q}$. Our point of view fixes the signature of the metric on q-Minkowski space and hence also of ordinary Minkowski space at $q = 1$. We also describe an abstract construction for left-invariant integration on any braided group.

Key words: Lie algebra – braided group – quantum group – q-Minkowski space – braided integration

1. Introduction

Braided geometry is a generalisation of ordinary geometry based on the idea of *braid statistics* between independent systems [1][2][3][4][5][6]. This includes as a special case the ideas of supergeometry but with the super-transposition $\Psi = \pm 1$ there replaced by a more general braiding where $\Psi^2 \neq \mathrm{id}$. Braided differentiation and integration on braided vector spaces, braided groups and braided Lie algebras are all known. Braided manifolds and braided Yang-Mills theory are in the pipeline. The main conclusion is that many constructions familiar in usual or supergeometry can be generalised to the braided case. Moreover, many constructions which are more commonly associated with quantum groups and the theory of q-deformations are more properly understood in these terms. There is a review article for physicists[7] as well as an introductory conference proceedings[8].

Here we would like to use some of this braided geometry to explore a basic conceptual problem that arises in quantum physics. The problem is that we think of a quantum algebra of observables on the one hand as a noncommutative version of the algebra of functions on phase space, or on the other hand as generated by the algebra of functions on configuration space and by the enveloping algebra $U(g)$ for g a generalised momentum symmetry. These points of view are contradictory unless it happens that we

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* Royal Society Research Fellow and Fellow of Pembroke College, Cambridge. This paper is in final form and no version of it will be submitted for publication elsewhere.

can view $U(g)$ as like the algebra of functions on some space, the momentum part of phase space.

We will see in Section 2 that for any Lie algebra g, one can indeed view $U(g)$ as the algebra of functions on a braided version of $\mathbb{R}^n$. So the non-commutativity of this algebra, which we normally associate with differential operators and quantisation, can be thought of equally well as statistical non-commutativity like that of Grassmann variables, albeit with a braiding Ψ rather than ± 1. We call this phenomenon in which a Lie algebra or enveloping algebra of operators is thought of instead as the coordinate functions of some space, a *quantum-geometry transformation.* The very simplest example is $U(\mathbb{R}^n) = \mathbb{C}[x_1, x_2, \cdots, x_n]$ where the enveloping algebra of an Abelian Lie algebra is thought of instead as polynomials in some bosonic position coordinates x_i. This is the idea behind Fourier transforms and our quantum-geometry transformation is a generalisation of this.

In fact, we have already explored this idea in the context of quantum groups in [9][10], where it is related to Hopf algebra duality. We proposed the ability to make this transformation, which reverses the role of quantum and gravitational physics, as a guiding principle for physics at the Planck scale. Now we want to touch upon these same ideas in the context of Lie algebras and their generalisations. In fact, the above remarks apply just as well to superLie algebras and the braided Lie algebras introduced in [11]. In each case the enveloping algebra can be viewed instead as a braided version of flat space. We develop this in Section 3. It provides a new way to think about the definition of Lie algebras and braided Lie algebras.

In Section 4 we focus on the example of the braided Lie algebra $gl_{2,q}$. Its enveloping algebra recovers a natural definition of q-Minkowski space. The quantum-geometry transformation takes the subalgebra $U_q(su_2)$ to the mass-shell in q-Minkowski space. The signature of the metric is also fixed as a deformation of the Lorentzian one in this approach. As far as I know, the Euclidean metric on $\mathbb{R}^4$ cannot be deformed in the same way. Thus the ability to q-deform spacetime provides in this way a kind of regularity principle that physics should not be too much an artifact of setting $q = 1$. This is in addition to the more usual motivation for q-deformation in terms of regularising infinities in physics[12] and quantum corrections to geometry.

It is hoped that this note will serve as an introduction for physicists to braided geometry and to some of its motivation. The Appendix demonstrates some of the mathematical techniques behind braided groups and braided geometry. We give a self-contained account of braided integration. This provides in principle the integration on many q-deformed spaces.

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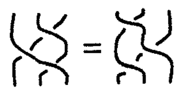
2. Canonical Braiding on any Lie Algebra

A braiding on a vector space V is, by definition, a map $\Psi : V \otimes V \to V \otimes V$ such that

$$\Psi_{23} \circ \Psi_{12} \circ \Psi_{23} = \Psi_{12} \circ \Psi_{23} \circ \Psi_{12}, \qquad \text{i.e.} \tag{1}$$



where the suffices refer to the copy of V in $V \otimes V \otimes V$. If one writes $\Psi = X$ then this equation expresses that the two sides are topologically the same braid as shown.

The simplest example is when V is $\mathbb{Z}_2$-graded and $\Psi(v \otimes w) = (-1)^{|v||w|} w \otimes v$ as in supersymmetry. Of course, in this example the exchange law is not truly braided since $\Psi^2 = \mathrm{id}$.

PROPOSITION 2.1. *Let $V = \mathbb{C} \oplus g$ and define the linear map*

$$\Psi(1 \otimes 1) = 1 \otimes 1, \quad \Psi(1 \otimes \xi) = \xi \otimes 1, \quad \Psi(\xi \otimes 1) = 1 \otimes \xi$$

$$\Psi(\xi \otimes \eta) = \eta \otimes \xi + [\xi, \eta] \otimes 1, \qquad \forall \xi, \eta \in g.$$

Then Ψ is a braiding iff $[\ ,\ ] : g \otimes g \to g$ *obeys the Jacobi identity. It has minimal polynomial*

$$(\Psi^2 - \mathrm{id})(\Psi + \mathrm{id}) = 0 \tag{2}$$

iff $[\ ,\ ]$ *is non-zero and antisymmetric.*

This is an elementary computation. It says that the definition of a Lie algebra is mathematically completely equivalent to looking for a braiding of a certain form. We will use this principle to give a new point of view on the definition of a braided Lie algebra in the next section.

Now in the theory of supergeometry, the simplest examples of superspaces are supercommutative superalgebras. Thus for $\mathbb{R}^{n|m}$ some of the variables (the bosonic ones) commute and some (the Grassmann ones) anticommute etc. So the algebra is not commutative in the ordinary sense, but it is commutative in the super sense

$$\cdot \circ \Psi = \cdot \tag{3}$$

where Ψ is included. Likewise, the universal enveloping algebra $U(g)$ for non-trivial Lie algebra g is of course not commutative.

PROPOSITION 2.2. *The braiding in Proposition 2.1 extends to a braiding* $\Psi : U(g)\otimes U(g) \to U(g)\otimes U(g)$ *and* $U(g)$ *is indeed braided commutative in the sense of (3).*

The proof of this is easy enough at degree 2 for there it says that $\cdot\circ\Psi(\xi\otimes\eta) = \eta\xi + [\xi,\eta]$ is to equal $\xi\eta$, which is the defining relation of the universal enveloping algebra. So imposing the relations of braided-commutativity at order two and for the above braiding is mathematically equivalent to the usual definition of the enveloping algebra. The easiest way to prove the result to all orders is to prove it in complete generality for any Hopf algebra, of which $U(g)$ is an example with coproduct $\Delta\xi = \xi\otimes 1 + 1\otimes\xi$. If H is a Hopf algebra then

$$\Psi(h\otimes g) = \sum \mathrm{Ad}_{h_{(1)}}(g)\otimes h_{(2)}, \qquad \mathrm{Ad}_h(g) = \sum h_{(1)} g S h_{(2)} \tag{4}$$

for all $h, g \in H$ is a braiding, and H is braided commutative with respect to it in the sense of (3). Here $\Delta h = \sum h_{(1)}\otimes h_{(2)}$ is the coproduct of the Hopf algebra and S is its antipode or 'inverse' operation.

We see that every enveloping algebra can be regarded as the algebra of functions on some braided space, and every quantum group too, with a suitable choice of braiding. This change in point of view in which an enveloping algebra gets regarded as a function algebra of some type is what we have called a quantum-geometry transformation in the introduction. Viewing a Lie algebra enveloping algebra in this way is significant for it means that the whole machinery of braided spaces and braided geometry[7], such as braided differential operators, etc can be applied. We will compute how one or two of these constructions look for our enveloping algebra.

In particular, given a braided algebra B one has the braided tensor product $B\underline{\otimes}B$ between two copies[2]. This is an algebra in which the two copies do not commute but rather enjoy braid statistics. The product rule is

$$(a\otimes b)(c\otimes d) = a\Psi(b\otimes c)d \tag{5}$$

where we braid b past c and then multiply up. This is like the supertensor product of superalgebras. Here is an example of what this is good for:

PROPOSITION 2.3. *Let* $B \doteq U(g)$ *be regarded as a braided space as above. There is an algebra homomorphism* $\underline{\Delta} : B \to B\underline{\otimes}B$ *given by* $\underline{\Delta}\xi = 1\otimes\xi - \xi\otimes 1$.

Just as the usual coproduct corresponds to addition (e.g. of angular momentum), so this map corresponds to subtraction. In a dynamical context the usual addition provides a realisation of the centre of mass system in the tensor product of two systems, whereas the above map is more like the realisation of the reduced mass system in the (braided) tensor product. It has properties that one would expect for subtraction in relation to the addition. It also generalises to any quantum group with $\Delta(h) = Sh_{(1)} \otimes h_{(2)}$.

Now we come to a matrix version of the above results, in which we shall do a few concrete calculations. If we choose a basis $V = \{x_\mu\}$ and write

$$\Psi(x_\mu \otimes x_\nu) = x_\beta \otimes x_\alpha \mathbf{R}^\alpha{}_\mu{}^\beta{}_\nu$$

then the requirement for Ψ to be a braiding is the celebrated Quantum Yang-Baxter Equation (QYBE) for $\mathbf{R}$.

Let $g = \{x_i\}$ for $i = 1, 2, \cdots, n-1$ and let $x_0 = 1$ so that $V = \mathbb{C} \oplus g$. We use greek indices when the whole range $0, \cdots, n-1$ is intended. Then the content of Proposition 2.1 is that

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & I & 0 & c \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \tag{6}$$

where I are identity matrices and $c^i{}_{jk}$ are the structure constants of g. The basis for $V \otimes V$ used here is $\{x_0 \otimes x_0, x_0 \otimes x_j, x_i \otimes x_0, x_i \otimes x_j\}$. Explicitly,

$$\mathbf{R}^0{}_i{}^k{}_j = c^k{}_{ij}, \ \mathbf{R}^i{}_j{}^k{}_l = \delta^i{}_j \delta^k{}_l, \ \mathbf{R}^0{}_0{}^i{}_j = \delta^i{}_j = \mathbf{R}^i{}_j{}^0{}_0, \ \mathbf{R}^0{}_0{}^0{}_0 = 1$$

and zero for the rest. This obeys the QYBE *iff* c obeys the Jacobi identity.

Next, given any R-matrix, the corresponding braided space $V\check{}(R)$ is the algebra with x_i and 1 as generators and relations

$$x_\mu x_\nu = x_\beta x_\alpha \mathbf{R}^\alpha{}_\mu{}^\beta{}_\nu.$$

This defines a braided version of $\mathbb{R}^n$. Such a structure arises in many areas in physics and is often called the Zamolodchikov or exchange algebra. Putting in the form of our R-matrix (6) we recover the commutation relations

$$[\lambda, x_i] = 0, \ [x_i, x_j] = \lambda x_k c^k{}_{ij}$$

so that the associated braided space is our enveloping algebra $U(g)$ in a homogenised form where we add the central element $\lambda = x_0$ on the right hand side. This is a concrete version of Proposition 2.2.

In the point of view of quantum or braided linear algebra[4], this is just one of many other constructions. If the $\{x_\mu\}$ are like a row vector, then another algebra $V(R)$ defined by generators 1 and $\{p^\mu\}$ and relations

$$\mathbf{R}^\mu{}_\alpha{}^\nu{}_\beta p^\beta p^\alpha = p^\mu p^\nu$$

is more like a column vector. For our R-matrix above, this comes out as

$$[p^\mu, p^\nu] = 0.$$

There is also a notion of braided-quantum mechanics generalising the one-dimensional case $px - qxp = \hbar$ to any R-matrix. It is generated by vector and covector algebras and cross relations

$$p^\mu x_\nu - x_\alpha \mathbf{R}^\alpha{}_\nu{}^\mu{}_\beta p^\beta = \hbar\delta^\mu{}_\nu$$

as studied by several authors[13][5]. See also the contribution of A. Kempf at this conference. For our R-matrix (6), this comes out as

$$[p^i, x_j] = \lambda c^i{}_{jk} p^k + \hbar\delta^i{}_j, \quad [p^i, \lambda] = 0, \quad [\pi, x_i] = 0, \ [\pi, \lambda] = \hbar$$

where $\pi = p^0$. Some natural xx and pp relations in this context are with a certain matrix R' rather than R, for in this case (or in the free case with no xx or pp relations) the general machinery in [5] says that one can represent p^μ by braided differentials $\frac{\partial}{\partial x_\mu}$ in analogy with usual quantum mechanics. One can likewise compute for our R-matrix (6) all the other R-matrix constructions for quantum groups and braided groups. On the quantum group side one has for example the usual quantum matrices $A(\mathbf{R})$. This comes out essentially as a matrix of n copies of the homogenised Lie algebra, one for each row, and with each copy transforming as an adjoint tensor operator with respect to the others.

Finally, we note that all the constructions above work equally well if we begin with a superLie algebra. Now the canonical braiding is

$$\Psi(\xi \otimes \eta) = (-1)^{|\xi||\eta|}\eta \otimes \xi + [\xi, \eta] \otimes 1$$

and obeys (1) *iff* [,] now obeys the superJacobi identity. It obeys (2) *iff* [,] is graded-antisymmetric. The superenveloping algebra is once again characterised by (3). More generally, if Ψ_0 is any other symmetric braiding in the sense that $\Psi_0^2 = \mathrm{id}$ then for

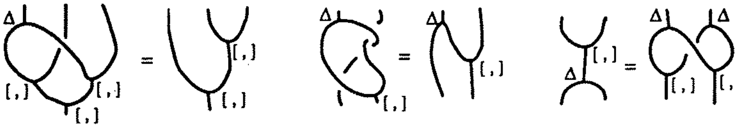
$$\Psi(\xi \otimes \eta) = \Psi_0(\xi \otimes \eta) + [\xi, \eta] \otimes 1$$

to obey (1) and (2) recovers the obvious axioms of a general Ψ_0-Lie algebra as in [14]. The corresponding matrix picture is

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & I & 0 & c \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & R_0 \end{pmatrix}$$

3. Braided-Lie Algebras

In this section we go beyond the super case and its obvious generalisations, to the case when our Lie algebra is of a type where the background Ψ_0 is itself truly braided. The axioms for such a braided Lie algebra have been introduced by the author in [11] and consist of a coalgebra $\mathcal{L}, \Delta, \epsilon$, a braiding $\Psi_0 = \times : \mathcal{L}\otimes\mathcal{L} \to \mathcal{L}\otimes\mathcal{L}$ and a map $[\ ,\ ] : \mathcal{L}\otimes\mathcal{L} \to \mathcal{L}$ such that



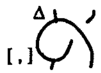
Here $\Delta : \mathcal{L} \to \mathcal{L}\otimes\mathcal{L}$ should be coassociative in an obvious sense and $\epsilon : \mathcal{L} \to \mathbb{C}$ should be a counit and obey $\epsilon \circ [\ ,\ ] = \epsilon\otimes\epsilon$. Note that an ordinary Lie algebra obeys these axioms if one puts $[1,\xi] = \xi$, $[\xi, 1] = 0$ and

$$\mathcal{L} = \mathbb{C}\oplus g, \quad \Delta 1 = 1\otimes 1,\ \epsilon 1 = 1, \quad \Delta\xi = \xi\otimes 1 + 1\otimes\xi,\ \epsilon\xi = 0.$$

So this structure Δ, ϵ is implicit for an ordinary Lie algebra but we never think about it because it has this standard form. The same is true for superLie algebras, etc. But for examples of the truly braided type we need to take a more general form.

THEOREM 3.1. *Let $\mathcal{L}, \Delta, \epsilon$ be a coalgebra and $\Psi_0 = \times$ a compatible braiding. Then $[\ ,\ ]$ defines a braided Lie algebra implies that*

$$\Psi =$$



is a braiding. The braided enveloping algebra $U(\mathcal{L})$ is generated by 1 *and* $\mathcal{L}$ *with the relations (3) of braided commutativity.*

The proof of this uses the same diagrammatic techniques as for braided groups[7]. We shall see some of these techniques in action in the Appendix. Here we content ourselves with the description of a general class of examples from [11]. They are of matrix type where

$$\mathcal{L} = \mathbb{C}^{n^2} = \{u^i{}_j\}, \quad \Delta u^i{}_j = u^i{}_k\otimes u^k{}_j, \quad \epsilon u^i{}_j = \delta^i{}_j.$$

The only data we need is a matrix solution $R \in M_n\otimes M_n$ of the QYBE which is bi-invertible. The 'second inverse' here is $\widetilde{R}$ and is characterised by

$$\widetilde{R}^i{}_a{}^b{}_l R^a{}_j{}^k{}_b = \delta^i{}_j\delta^k{}_l = R^i{}_a{}^b{}_l \widetilde{R}^a{}_j{}^k{}_b.$$

We write $I = (i_0, i_1)$ etc as multi-indices. Then[15][11]

$$\Psi_0(u_J\otimes u_L) = u_K\otimes u_I R_0{}^I{}_J{}^K{}_L, \quad [u_I, u_J] = u_K c^K{}_{IJ}$$

$$R_0{}^I{}_J{}^K{}_L = R^{j_0}{}_a{}^d{}_{k_0} R^{-1}{}^a{}_{i_0}{}^{k_1}{}_b R^{i_1}{}_c{}^b{}_{l_1} \widetilde{R}^c{}_{j_1}{}^{l_0}{}_d$$

$$c^K{}_{IJ} = \widetilde{R}^a{}_{i_1}{}^{j_0}{}_b R^{-1}{}^b{}_{k_0}{}^{i_0}{}_c R^{k_1}{}_e{}^c{}_d R^d{}_a{}^e{}_{j_1}$$

is a braided Lie algebra. We changed conventions here from [11] to lower indices for the $\{u_I\}$ in order to maintain compatibility with Section 2. The associated canonical braiding from Theorem 3.1 is

$$\Psi(u_J \otimes u_L) = u_K \otimes u_I \mathbf{R}^I{}_J{}^K{}_L$$

$$\mathbf{R}^I{}_J{}^K{}_L = R^{-1}{}^d{}_{k_0}{}^{j_0}{}_a R^{k_1}{}_b{}^a{}_{i_0} R^{i_1}{}_c{}^b{}_{l_1} \widetilde{R}^c{}_{j_1}{}^{l_0}{}_d.$$

The braided enveloping algebra $U(\mathcal{L})$ is given by taking $\mathbf{u} = \{u^i{}_j\}$ as generators and imposing $\cdot \circ \Psi = \cdot$. So this is the algebra

$$u_J u_L = u_K u_I \mathbf{R}^I{}_J{}^K{}_L, \quad \text{i.e.} \quad R_{21}\mathbf{u}_1 R_{12}\mathbf{u}_2 = \mathbf{u}_2 R_{21}\mathbf{u}_1 R_{12} \tag{7}$$

where the second puts two of the R's to the left and uses a popular notation.

Our construction of braided Lie algebras works over the whole moduli space of bi-invertible solutions R. Inside this moduli space is a subvariety of so-called triangular solutions where $R_{21}R = 1$. On this subvariety one has $\Psi_0^2 = \mathrm{id}$ and our braided Lie algebras are not truly braided. They reduce in this case to the more obvious notion of Ψ_0-Lie algebras as at the end of the last section after one takes a suitable scaling limit. To see this, we parametrise R in such a way that as a parameter $q \to 1$, we land on the triangular subvariety. We also change variables to $\chi_I = u_I - \delta_I$ where $\delta_I = \delta^{i_0}{}_{i_1}$. The braided enveloping algebra then looks like

$$\chi_J \chi_L - \chi_K \chi_I \mathbf{R}^I{}_J{}^K{}_L = \chi_K \left(\delta_I \mathbf{R}^I{}_J{}^K{}_L - \delta_J \delta^K{}_L\right) \tag{8}$$

and as $q \to 1$ the right hand side vanishes. But if we rescale χ to $\bar{\chi} = (q^2 - 1)^{-1}\chi$ say, then the effective structure constants for $\bar{\chi}$ can have a finite limit and indeed they become those of a usual, super, etc. Lie algebra depending on the point on the triangular subvariety that we are landing at. Meanwhile, the coproduct

$$\Delta\bar{\chi} = \bar{\chi} \otimes 1 + 1 \otimes \bar{\chi} + (q^2 - 1)\bar{\chi} \otimes \bar{\chi}, \quad \epsilon\bar{\chi} = 0$$

becomes our standard one. In this way, ordinary, super, etc. Lie algebras are the semiclassical limits of braided Lie algebras as we approach the triangular subvariety. They are therefore all unified and interpolated by our notion of braided Lie algebras. Incidentally, this shows why the classification of all solutions of the QYBE is such a hard problem: it includes the classification of all Lie algebras, superLie algebras and more generally, of braided-Lie algebras. Usual quantum enveloping algebras also fit into this picture[11].

So the braided enveloping algebra in the form (8) looks like an enveloping algebra but in the form (7) it looks like the coordinate functions on a braided commutative space. This is our quantum-geometry transformation again, in a braided form.

In fact, these quadratic algebras (7) and the matrices $R_0, \mathbf{R}$ were introduced by the author in [2] exactly as a braided analogue $B(R)$ of the algebra of functions on M_n. They are the *braided matrices* associated to R. We recall that the more well-known quantum matrices $A(R)$ have a matrix of non-commuting coordinate functions forming a bialgebra or quantum group[16]. Likewise, $B(R)$ is a braided-bialgebra or braided group. The difference is that the matrix coproduct above extends to an algebra homomorphism

$$\Delta : B(R) \to B(R)\underline{\otimes}B(R) \tag{9}$$

provided we take for $\underline{\otimes}$ the braided tensor product algebra (5). This is like the definition of a supermatrix, but with general braid statistics.

4. q-Minkowski Space

There are many approaches to what q-Minkowski space should be. Here we describe our own approach coming out of braided geometry[17]. Generally speaking, our approach to q-deforming physics is to introduce q as a parameter controlling braid statistics but with the geometry otherwise remaining commutative. Since usual Minkowski space can be thought of as 2×2 hermitian matrices, we naturally propose that q-Minkowski space should be the algebra of 2×2 braided hermitian matrices. This is broadly compatible with the pioneering approach of [18][19], who were motivated by the possibility of spinors when defining their q-Lorentz group. On the other hand, we understand directly the full structure of q-Minkowski space first and come to the q-Lorentz group etc. only later as a quantum group that acts covariantly on it.

We take the well-known R-matrix associated to the Jones knot polynomial and the quantum plane,

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{10}$$

and in this case we have the braided matrix algebra $BM_q(2)$ with generators and relations computed in [2] as $\mathbf{u} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$qd + q^{-1}a \text{ central}, \; ba = q^2 ab, \; ac = q^2 ca, \; bc = cb + (1 - q^{-2})a(d - a).$$

The braid statistics from Ψ_0 has $qd + q^{-1}a$ bosonic but the others mixing among themselves. The content of the braided matrix property (9) is that we can multiply two copies $\mathbf{u}, \mathbf{u}'$ as

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

provided we remember the corresponding braid statistics. We also showed in [2] that our algebra has a multiplicative braided determinant $\mathrm{BDET}(\mathbf{u}) = ad - q^2cb$. It is bosonic and central.

Next, we studied $*$-structures on braided matrices in [17]. For real q, we have

$$\begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

so that these matrices are naturally hermitian. One has also

$$\tau \circ (* \otimes *) \circ \Delta = \Delta \circ *$$

where τ denotes ordinary transposition. This is what one would expect since the coproduct corresponds to matrix multiplication and $(A \cdot B)^\dagger = B \cdot A$ for ordinary hermitian matrices A, B. We denote the braided matrix bialgebra $BM_q(2)$ with this $*$-structure by $BH_q(2)$, the algebra of *braided hermitian matrices.* Note that the situation here is in sharp contrast to the usual axioms of $*$-quantum groups, where hermitian quantum matrices cannot be formulated. BDET is self-adjoint.

All of this makes this particular algebra ideally suited to serve as q-Minkowski space. So we define q-Minkowski space as $BH_q(2)$. The generators

$$x_0 = qd + q^{-1}a, \quad x_1 = \frac{b+c}{2}, \quad x_2 = \frac{b-c}{2i}, \quad x_3 = d - a$$

are some natural self-adjoint spacetime coordinates while BDET becomes

$$\frac{q^2}{(q^2+1)^2}x_0^2 - q^2x_1^2 - q^2x_2^2 - \frac{(q^4+1)q^2}{2(q^2+1)^2}x_3^2 + \left(\frac{q^2-1}{q^2+1}\right)^2 \frac{q}{2} x_0 x_3$$

and provides a real q-deformed Lorentz metric.

This q-Minkowski space has plenty of geometry associated to it, some of which we describe now. It is evident from the description of braided matrices (7) that they can be viewed if we want as a 4-dimensional row vector algebra of the same general type as the $\{x_\mu\}$ in Section 2. They therefore transform as usual under the action of the corresponding quantum matrices $A(\mathbf{R})$. Thus,

$$u_J \to u_I \Lambda^I{}_J \tag{11}$$

is an algebra homomorphism (we have a right comodule algebra) under the 4×4 matrix quantum group

$$\mathbf{R}^I{}_A{}^K{}_B \Lambda^A{}_J \Lambda^B{}_L = \Lambda^K{}_B \Lambda^I{}_A \mathbf{R}^A{}_J{}^B{}_L, \quad \Delta\Lambda^I{}_J = \Lambda^I{}_A \otimes \Lambda^A{}_J$$

This quantum group provides the basis for a q-Lorentz group in our picture. It has a $*$-algebra structure

$$\Lambda^I{}_J{}^* = \Lambda^{(i_1, i_0)}{}_{(j_1, j_0)}$$

and the coaction and coproduct are $*$-algebra homomorphisms. We have taken the quantum group line here because it is more familiar. There is an equally good braided Lorentz group based on $B(\mathbf{R})$ acting in the same way as a braided comodule algebra.

Moreover, the quantum Lorentz group here maps into the dual of the Drinfeld quantum double[20] with the result that our approach is indeed compatible with other proposals based on spinors[18][21]. Thus, our $A(\mathbf{R})$ can be realised in the quantum group $A(R) \bowtie A(R)$ introduced in [22] and generated by two copies of the 2×2 quantum matrices. We take these in the form $\mathbf{t} \in A(R)$ and $\mathbf{t}^\dagger \in A(R_{21})$ say, with mutual relations and $*$-structure

$$t^i{}_a R^a{}_j{}^k{}_b t^{\dagger b}{}_l = t^{\dagger k}{}_b R^i{}_a{}^b{}_l t^a{}_j, \quad t^j{}_i{}^* = t^{\dagger i}{}_j, \qquad \text{i.e.,}\quad \mathbf{t}_1 R \mathbf{t}_2^\dagger = \mathbf{t}_2^\dagger R \mathbf{t}_1.$$

The abstract picture behind $A(R) \bowtie A(R)$ as a $*$-quantum group was found in [3] as well as its relation to the quantum double. One should use the inverse-transpose of the dual-quasitriangular structure found there in Proposition 12. The realisation and the resulting 2×2 matrix form of the Lorentz transformation (11) is

$$\Lambda^I{}_J = t^{\dagger j_0}{}_{i_0} t^{i_1}{}_{j_1}, \quad u^i{}_j \to u^a{}_b t^{\dagger i}{}_a t^b{}_j, \qquad \text{i.e.,}\quad \mathbf{u} \to \mathbf{t}^\dagger \mathbf{u}\mathbf{t}.$$

These constructions all work for any R-matrix of real type. For (10), one should think of our two copies of 2×2 quantum matrices as the analogue of the complexification $SL(2,\mathbb{C})$ of $SU(2)$. Then the diagonal action $\mathbf{u} \to \mathbf{t}^{-1}\mathbf{u}\mathbf{t}$ when $\mathbf{t}$ is unitary defines an action of the quantum group $SU_q(2)$. This in turn is the double-cover of rotations, which appears here as $SO_q(3) \subset SU_q(2)$, the subHopf algebra generated by expressions quadratic in the $\mathbf{t}$.

All the usual geometrical ideas likewise go though without difficulty. For example, the mass-shell or Lorentzian sphere in q-Minkowski space is defined by adding the relation

$$\mathrm{BDET}(\mathbf{u}) = 1 \tag{12}$$

and is preserved under the $SO_q(3)$ action as one would expect. There are also vector fields on q-Minkowski space for translation[11], and for Lorentz transformation from (11). The action of the rotational vectors generates the quantum group $U_q(su_2)$ as

$$X_+\triangleright\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} -q^{\frac{3}{2}}c & -q^{\frac{1}{2}}(d-a)\\ 0 & q^{-\frac{1}{2}}c\end{pmatrix} \to [\begin{pmatrix} a & b\\ c & d\end{pmatrix}, \begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}]$$

$$X_-\triangleright\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} q^{\frac{1}{2}}b & 0\\ q^{-\frac{1}{2}}(d-a) & -q^{-\frac{3}{2}}b\end{pmatrix} \to [\begin{pmatrix} a & b\\ c & d\end{pmatrix}, \begin{pmatrix} 0 & 0\\ 1 & 0\end{pmatrix}]$$

$$H\triangleright\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} 0 & -2b\\ 2c & 0\end{pmatrix} \to [\begin{pmatrix} a & b\\ c & d\end{pmatrix}, \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}]$$

where the limits are as $q \to 1$ and are as one would expect.

Another interesting feature is that this mass-shell or Lorentzian sphere forms a braided group. This parallels the way that the Euclidean sphere in the 2×2 quantum matrices $M_q(2)$ is the quantum group $SU_q(2)$. The big difference is the $*$-structure or signature. In fact, this is part of a general phenomenon. Just as most familiar groups have supergoup analogues, there is a general procedure in [1] called *transmutation* which turns a quantum group into a braided group in a systematic way. The formulae at the lowest level are

$$u^i{}_j = t^i{}_j,\ u^i{}_j u^k{}_l = t^a{}_b t^d{}_l R^i{}_a{}^c{}_d \widetilde{R}^b{}_j{}^k{}_c, \quad \text{i.e.,} \quad \mathbf{u} = \mathbf{t},\ \mathbf{u}_1 R \mathbf{u}_2 = R\mathbf{t}_1\mathbf{t}_2$$

etc. and come out of category theory. We also gave a direct quantum groups point of view to them in [15]. Finally we found in [17] that this transmutation from quantum geometry to braided geometry also has the side-effect in general of taking us from the unitary picture (our sphere in Euclidean space) to the hermitian picture (our Lorentzian sphere). This is the abstract reason why only braided matrices and not quantum matrices can serve in the q-deformed picture if we want the Lorentzian signature. One does not see this constraint at $q = 1$.

More recently, U. Meyer in [23] has found an addition law for q-Minkowski space by introducing a new braiding suitable for the coaddition $\Delta\mathbf{u} = \mathbf{u} \otimes 1 + 1 \otimes \mathbf{u}$. The R-matrix for this braiding is different from $\mathbf{R}$ above and provides for a better q-Lorentz group with the quantum double appearing as its double cover. The addition law also provides for braided differential calculus according to the framework of [5] and, in principle, a translation-invariant integration as we shall see in the Appendix below.

This completes our introduction to the braided geometry of q-Minkowski space. On the other hand, we have seen in the last section that these braided hermitian matrices are also the braided enveloping algebra of the braided Lie algebra associated to our R-matrix. In our case this is the 4-dimensional braided Lie algebra $gl_{2,q}$. It has basis h, x_+, x_-, γ with braided-Lie bracket

$$\begin{aligned}
&[h, x_+] = (q^{-2}+1)q^{-2}x_+ = -q^{-2}[x_+, h]\\
&[h, x_-] = -(q^{-2}+1)x_- = -q^2[x_-, h]\\
&[x_+, x_-] = q^{-2}h = -[x_-, x_+]\\
&[h, h] = (q^{-4}-1)h, \quad [\gamma, \begin{cases} h \\ x_+ \\ x_- \end{cases}] = (1-q^{-4}) \begin{cases} h \\ x_+ \\ x_- \end{cases}
\end{aligned}$$

with zero for the others. We see that as $q \to 1$ the γ mode decouples and we have the Lie algebra $su_2 \oplus u(1)$, but for $q \neq 1$ these are unified. There is also a braided Killing form[11] which is non-degenerate as long as $q \neq 1$. So $gl_{2,q}$ is an interesting braided-Lie algebra with potential applications in physics, such as in the unification of electroweak interactions in q-deformed Yang-Mills theory[24] with this as the gauge symmetry. Its su_2 part can also serve as differential operators of orbital angular momentum etc., along usual lines.

The quantum-geometry transformation thus connects these two conceptually quite distinct structures. Explicitly, it is

$$\begin{pmatrix} h \\ x_+ \\ x_- \\ \gamma \end{pmatrix} = (q^2-1)^{-1} \begin{pmatrix} a-d \\ c \\ b \\ q^{-2}a + d - (q^{-2}+1) \end{pmatrix}$$

and gives an isomorphism $U(gl_{2,q}) \cong BH_q(2)$. So, provided $q \neq 1$ there is only one braided group in the picture. From one point of view it is the algebra of functions on q-Minkowski space. From another point of view it is the enveloping algebra of a braided Lie algebra. But what we see at $q = 1$ is two structures, depending on how we take the limit. If we work with a, b, c, d then in the limit the algebra is the commutative algebra of functions on usual Minkowski space. If we work with h, x_+, x_-, γ then the limit is the highly non-commutative enveloping algebra $U(su_2 \oplus u(1))$.

The quantum-geometry transform here is valid for $q \neq 1$ and maps Lie algebras and their properties to geometry. For example, what from the geometrical point of view is the mass-shell constraint (12) in q-Minkowski space, comes out from the Lie algebra or differential operator point of view as the quantum enveloping algebra $U_q(su_2)$. Explicitly, the quantum-geometry transform at this level becomes

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} q^H & q^{-\frac{1}{2}}(q-q^{-1})q^{\frac{H}{2}}X_- \\ q^{-\frac{1}{2}}(q-q^{-1})X_+q^{\frac{H}{2}} & q^{-H} + q^{-1}(q-q^{-1})^2X_+X_- \end{pmatrix}.$$

This follows from some known results in the theory of quantum groups [16][25] by putting $\mathbf{u} = l^+Sl^-$. This connection with quantum groups is explained in full detail in [15], to which we refer the reader.

Likewise, what from the geometrical point of view is the time direction x_0 appears from the Lie algebra point of view as giving the $u(1)$ mode γ which could appear in a gauge theory or which, for example, acts via [,] on q-Minkowski space by scaling of the space coordinates $\{x_i\}$. On the mass-shell it appears as the quadratic Casimir. In summary, $U(gl_{2,q})$ is both a braided enveloping algebra, such as an internal symmetry or an algebra of differential operators acting on q-Minkowski space, and can be identified with q-Minkowski space itself. Only remnants of this unification are visible when $q = 1$. We have seen also that the ability to develop the q-deformed picture forces us from Euclidean space to Minkowski space.

We have not had room here to describe many other features of quantum and braided geometry. Notably, in [24] we introduced the theory of quantum group principal bundles and connections (gauge fields), including the example of a Dirac monopole on a q-sphere. Some of this machinery can be applied to q-Minkowski space. In short, a systematic q-deformed picture of the main ingredients of physics is emerging, as well as some unusual phenomena that are not very evident at the special point $q = 1$.

Appendix. Braided Integration

In this appendix we introduce the reader to some of the mathematical techniques of braided geometry by deriving here a formula for invariant integration. This is a problem that is of current interest and which was posed a couple of times at the conference. Since quantum planes, q-Minkowski space and many other q-deformed algebras are in fact braided groups, we can apply the general theory of braided groups. There are still some difficulties in interpreting and computing the formula for integration, which we offer as a challenge for the interested reader.

Our main goal is to demonstrate some diagrammatic techniques as used for the basic properties of braided groups in [7]. We refer there for full details of the methods and notation. As well as the result here, one can also prove Theorem 3.1 and the braided version of (4) using the same techniques.

Briefly, let us recall that a braided algebra B is an algebra with a braiding $\Psi = \times$ mapping $B \otimes B \to B \otimes B$. There should also be a unit element, which we view as a map $\eta : \mathbb{C} \to B$. The algebra, and indeed all our maps, should be compatible with the braiding in an obvious way. We view it as like functions on a braided space. A braided group is such a braided algebra equipped also with a coproduct $\Delta : B \to B \underline{\otimes} B$ and counit $\epsilon : B \to \mathbb{C}$. This is like the definition of a quantum group with the key difference that $B \underline{\otimes} B$ is defined with braid statistics as in (5). We saw some concrete examples in the form of the braided matrices in Sections 3 and 4. Likewise, some quantum planes are also braided groups with coaddition[3]. We are using the term 'braided group' quite loosely here. In general, there should also be an antipode $S : B \to B$ obeying axioms like the usual ones. One can also ask for some braided-commutativity as in [2] but we do not need this here.

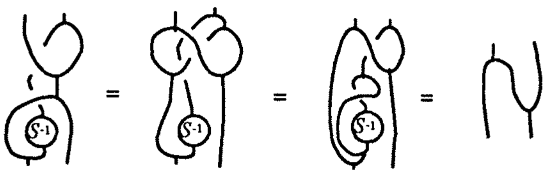
Crucial for us is the diagrammatic notation in which $\Delta = \curlywedge$ and $\cdot = \curlyvee$. We also suppose that our braided group has a dual B^* and denote the evaluation map ev : $B^* \otimes B \to \mathbb{C}$ and coevaluation map coev : $\mathbb{C} \to B \otimes B^*$ by ev $= \cup$ and coev $= \cap$. In concrete terms, ev is usual evaluation and $\mathrm{coev}(\lambda) = \lambda \sum e_a \otimes f^a$ for a basis $\{e_a\}$ and dual basis $\{f^a\}$.

Our goal is to find a map $\int : B \to \mathbb{C}$ which assigns to a 'function' in B a number, and which is translation invariant under the group law. Classically this means $\int b(h(\ )) = \int b$ for all h in our group. We find correspondingly

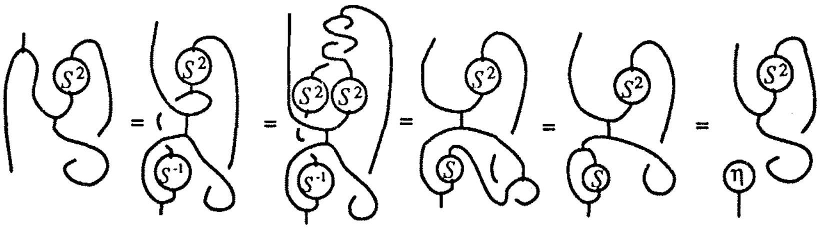
$$\int = \underline{\mathrm{Tr}}\, L \circ S^2 =$$  $$(\mathrm{id} \otimes \int)\Delta = 1 \int$$

where the first is our definition of $\int$ and the second is its translation-invariance property. Here $\underline{\mathrm{Tr}}$ is the braided trace as in [11] and L is left multiplication, which gives the diagrammatic form shown.

A similar formula applies for ordinary quantum groups, and we will use a similar strategy of proof. We note that braided integrals have also been studied in [26] but our proof will be different. Our first step in the proof is a lemma. We assume that S is invertible, then



where the first equality is the property that Δ is an algebra homomorphism to the braided tensor product algebra $B\underline{\otimes}B$. The second equality uses associativity and coassociativity of the product and coproduct. The last equality then cancels the inverse-antipode as explained in [7]. Then



where the first equality is our lemma and the second uses that S is a braided antialgebra homomorphism. Now pick up the coproduct at the top of the third expression and push it down and to the left (not changing the topology), giving the fourth expression. Now we use coassociativity and cancel the antipode loop. We obtain the desired left-invariance of the integral.

Thus we have a nice formula for the invariant integral on a braided group. The braided trace plays the role of 'averaging'. The formula should, however, be viewed with care because it could easily happen that it gives identically zero or infinity and may well require a renormalisation to get a finite answer. To see the nature of this problem, let G be an ordinary finite group and take a basis of delta-functions $\{\delta_g\}$. The dual basis is the the set of group elements themselves. Then the formula gives

$$\int b = \sum_g \langle g, b\delta_g \rangle = \sum_g b(g)\delta_g(g).$$

In the continuous case this gives $\delta(0)$ times the usual integral. One can evaluate the trace in any convenient basis. It would be interesting to find a suitable basis in the case of the quantum plane or q-Minkowski space and likewise evaluate this integral. This is a direction for further work.

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3D CHERN-SIMONS THEORIES AND THEIR RELATION TO QUANTUM GROUPS

CUPATITZIO RAMÍREZ
Fac. de Ciencias Físico Matemáticas
Universidad Autónoma de Puebla
Ap. Postal 1364, Puebla, Pue. México

and

LUIS URRUTIA
Departamento de Física
U.A.M.-I, México, D.F. México

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Abstract. The relation of 3D Chern-Simons theories to quantum groups is studied, it turns out that besides the already known quantum group realization for the quantized theory, a similar realization exists for the classical theory. The classical limit of the theory is considered in detail.

1. Introduction

In the last years 3D Chern-Simons theories have been studied due to their multiple applications [1, 2, 3, 4].

As topological field theories, Chern-Simons theorie do not depend on the metric of space-time manifold M.

If A is an algebra valued connexion of the group G on the manifold M, then the Chern-Simons action is given by:

$$I = \frac{k}{4\pi} \int_M d^3x \epsilon^{ijk} Tr \left(A_i \partial_j A_k + A_i A_j A_k \right) \tag{1}$$

where k is the coupling constant and Tr is the bilineal form of the algebra of the group G.

The action (1) is invariant under spacetime reparametrizations and under gauge transformations it is invariant up to an additive constant given by the winding number of the transforming group element.

The equations of motion following from (1) are gauge covariant:

$$F_{ij} = \partial_i A_j - \partial_j A_i - [A_i, A_j] = 0 \tag{2}$$

Thus, C-S theories will describe only "trivial" motions given by flat connexions.

If the group G is $ISO(2,1)$, then it was shown in [1] that the resulting theory is equivalent to 3D Einstein gravity.

For quantized theories the expectation value of Wilson lines along knotted closed curves will give the corresponding Jones polynomial. Moreover the quantum Hilbert space of a 2D spacelike section punctured by the intersection of the contained Wilson loops, describes the space of conformal blocks of 2D conformal field theories.

An interesting issue is the one of computing the commutator algebra of Wilson loops along spacelike curves [5]. As usual a foliation of $M = R\times\Sigma$ has been chosen. Thus, it is enough to study Wilson loops on Σ. In [5], $ISO(2,1)$ has been considerated in some detail; this study has been further pursued in [6] for the spinorial representations of $SO(3,1)$ and $SO(2,2)$ where after quantization the resulting algebra has been identified with $SL(2)_q$. Further, the Wilson loop algebra of Poincaré and conformal groups [7], and for de Sitter supergravity [8] have been calculated with similar results. Generalizations for $g > 1$ have been pursued in [9].

In this contribution $SU(2)$ C-S theory is considerated. In Sec. 2 it is shown that although the Poisson bracket algebra of integrated connexions is of braid type, the Jacobi identities are trivially satisfied. In Sec. 3 it is shown that the Poisson bracket algebra of traces, i.e. Wilson loops, has the structure of $SL(2)_q$. In Sec. 4 different quantization schemes are discussed. Conclusions are drawn in Sec. 5.

2. Quantum Symmetry of Classical Chern-Simons Theory

If $\gamma : R \to \Sigma$ is a noncontractible closed curve on Σ, then an integrated connexion

$$\Psi(\gamma) = Pe^{\int_\gamma A dx} \tag{3}$$

is a solution of the differential equation [5]:

$$\frac{d\Psi}{dt} = A_s\Psi \tag{4}$$

where A_s is the connexion tangent to γ at s. From the action (1) the canonical Poisson bracket relations can be derived:

$$\left[A_{a\alpha}(t,\mathbf{x}), A^b_\beta(t,\mathbf{x}')\right] = \frac{2\pi}{k}\epsilon_{\alpha\beta}\delta^a_b\delta^2(\mathbf{x}-\mathbf{x}') \tag{5}$$

where $\alpha, \beta = 1,2$ and a corresponds to the adjoint representation of G.

In order to compute the Poisson brackets of integrated connexions let us consider two crossing closed curves γ and σ [5, 6]. We take them as independent nontrivial homotopy classes, e.g., the cycles of a thorus. Both curves are decomposed into three pieces, the central one being in the neighborhood of the crossing point. (fig. 1).

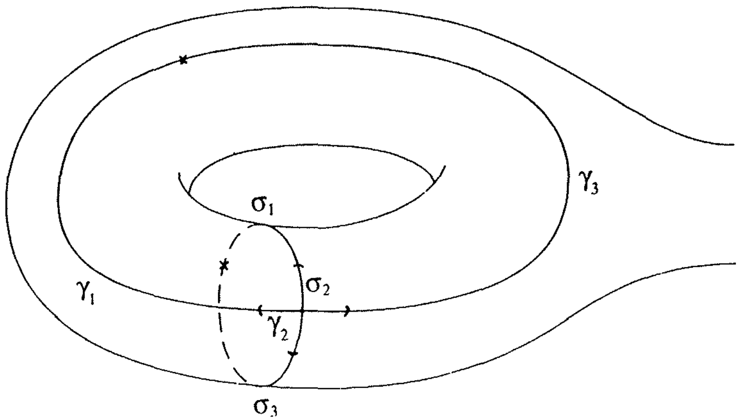

Figure 1

Thus $\Psi(\gamma) = \Psi(\gamma_3)\Psi(\gamma_2)\Psi(\gamma_1)$ and $\Psi(\sigma) = \Psi(\sigma_3)\Psi(\sigma_2)\Psi(\sigma_1)$. Taking (5) into account we obtain:

$$\begin{aligned}&\{\Psi_1(\gamma), \Psi_2(\sigma)\}_{PB} = \\ &\Psi_1(\gamma_3)\Psi_2(\sigma_3)\,\{\Psi_1(\gamma_2), \Psi_2(\sigma_2)\}_{PB}\,\Psi_1(\gamma_1)\Psi_2(\sigma_1)\end{aligned} \tag{6}$$

where, as usual, the notations are:

$$\Psi_1 = \Psi \otimes 1 \;\; \textit{and} \;\; \Psi_2 = 1 \otimes \Psi \tag{7}$$

Further, we have:

$$\begin{aligned}\Psi(\gamma_2) &= 1 + \int_{s_0-\epsilon}^{s_0+\epsilon} ds\, A_\alpha\,[\mathbf{x}(s)]\, x'^\alpha(s) + \mathcal{O}(\epsilon^2) \\ \Psi(\sigma_2) &= 1 + \int_{u_0-\epsilon}^{u_0+\epsilon} du\, A_\alpha\,[\mathbf{x}(u)]\, x'^\alpha(u) + \mathcal{O}(\epsilon^2)\end{aligned} \tag{8}$$

thus

$$\{\Psi_1(\gamma_2), \Psi_2(\sigma_2)\}_{PB} = -\frac{2\pi}{k} s(\gamma,\sigma)\,(T^a \otimes T_a) \tag{9}$$

where $s(\gamma,\sigma) = \pm 1$ is the signature of the relative orientation of γ and σ.

Therefore, in the limit $\epsilon \to 0$:

$$\begin{aligned}&\{\Psi_1(\gamma), \Psi_2(\sigma)\}_{PB} = \\ &-\tfrac{2\pi}{k} s(\gamma,\sigma)\Psi_1(\gamma_3)\Psi_2(\sigma_3)\,(T^a \otimes T_a)\,\Psi_1(\gamma_1)\Psi_2(\sigma_1)\end{aligned} \tag{10}$$

If we restrict ourselves to curves γ and σ with a common base point, the algebra (2.8) can be put in a closed form, so that for example in the limit $\epsilon \to 0$ we have $\Psi(\gamma_3) = \Psi(\sigma_3)$, then we can fix the gauge in such a way that we obtain the braid-like algebra:

$$\{\Psi_1(\gamma), \Psi_2(\sigma)\}_{PB} = r_{12}(\gamma,\sigma)\Psi_1(\gamma)\Psi_2(\sigma) \tag{11}$$

where

$$r_{12}(\gamma,\sigma) = -\frac{2\pi}{k}s(\gamma,\sigma)(T^a \otimes T_a) \tag{12}$$

which obviously does not satisfy the classical Yang-Baxter equation. In fact, in order to satisfy the Jacobi identities of (11), we need three different, but equally based elements, say $\Psi(\gamma)$, $\Psi(\sigma)$ and $\Psi(\sigma')$ as in fig.1.2 (the fact that γ, σ and σ' are equally based is not explicitly shown). The point is that the gauge (11) cannot be implemented simultaneously for all possible brackets, for each of these brackets we must do separatedly a partition of the curves. fig. 1.1.

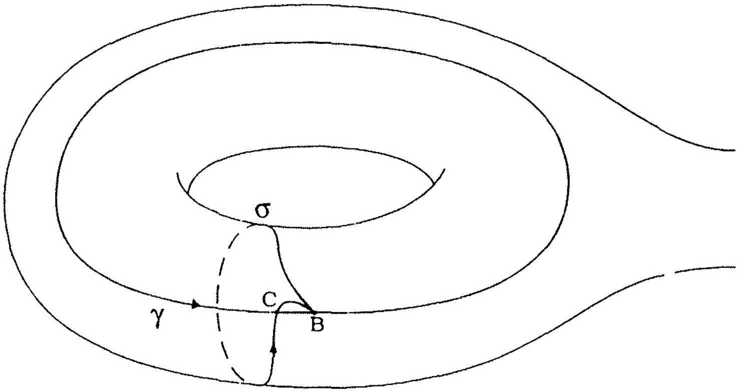

Figure 2

Taking that into account, it is easy to show that:

$$\begin{aligned}&\{\{\Psi_1(\gamma),\Psi_2(\sigma)\}_{PB},\Psi_3(\sigma')\}_{PB} +\\ &\{\{\Psi_1(\sigma),\Psi_2(\sigma')\}_{PB},\Psi_3(\gamma)\}_{PB} +\\ &\{\{\Psi_3(\sigma'),\Psi_1(\gamma)\}_{PB},\Psi_2(\sigma)\}_{PB} \equiv 0\end{aligned} \tag{13}$$

where the second term vanishes identically due to the fact that

$$\{\Psi_1(\sigma),\Psi_2(\sigma')\}_{PB} = 0 \tag{14}$$

Now we consider the Poisson bracket algebra of traces of integrated connexions (Wilson loops) for $SU(2)$:

$$C(\gamma) = Tr(\gamma) \tag{15}$$

$$\{C(\gamma),C(\sigma)\}_{PB} = -\frac{2\pi}{k}s(\gamma,\sigma)Tr\left[T^a\Psi(\gamma)\right]Tr\left[T_a\Psi(\sigma)\right] \tag{16}$$

where the Casimir element is given by:

$$T^a{}_{m'_1}{}^{m_1}T_{am'_2}{}^{m_2} = -\frac{1}{4}\delta_{m'_1}{}^{m_1}\delta_{m'_2}{}^{m_2} + \frac{1}{2}\delta_{m'_1}{}^{m_2}\delta_{m'_2}{}^{m_1} \tag{17}$$

resulting the algebra [5, 6]:

$$\{C(\gamma), C(\sigma)\}_{PB} = \frac{\pi}{k} s(\gamma,\sigma)\left[-\frac{1}{2}C(\gamma)C(\sigma) + C(\gamma\sigma)\right] \tag{18}$$

which closes due to the trace identities for 2x2 matrices [5, 6]:

$$Tr(AB) = Tr(A)Tr(B) - \det A\, Tr(A^{-1}B) \tag{19}$$

In our case the determinant is one and we have:

$$\begin{aligned} C(\gamma^2\sigma) &= -C(\sigma) + C(\gamma)C(\gamma\sigma) \\ C(\sigma^2\gamma) &= -C(\gamma) + C(\sigma)C(\gamma\sigma) \end{aligned} \tag{20}$$

and so on. Thus, the only independent generators are: $X_1 = C(\gamma)$, $X_2 = C(\sigma)$ and $X_3 = C(\gamma\sigma)$ with the resulting algebra [6]:

$$\{X_i, X_j\}_{PB} = \frac{\pi}{k}\left(\epsilon_{ij}X_iX_j + \epsilon_{ijk}X_k\right) \tag{21}$$

where $\epsilon_{ij} = -\epsilon_{ji}$, $\epsilon_{12} = \epsilon_{23} = \epsilon_{31} = 1$ and ϵ_{ijk} is the 3D Levi-Civita symbol. Relations similar to (21) arise for the monodromies of groups elements of $SU(2)$ WZW model in [10] where the resulting algebra has been interpreted as the semiclassical version of $SL(2)_q$. In the following we will show that in fact they constitute an exact representation of $SL(2)_q$.

Indeed, if we do the nonlinear reparametrization similar to the one used for the quantized theory in [6], see also e.g. [11]:

$$K^{\pm} = X_1 \pm iX_2 e^{\mp\frac{\pi}{k}H} \tag{22}$$

$$X_3 = \frac{i}{2}\left(e^{-\frac{\pi}{k}H} - e^{\frac{\pi}{k}H}\right) \tag{23}$$

We obtain:

$$\begin{aligned} \{K^+, K^-\}_{PB} &= \frac{\pi}{k}\left(e^{-\frac{\pi}{k}H} - e^{\frac{\pi}{k}H}\right) \\ \{H, K^{\pm}\}_{PB} &= \pm\frac{\pi}{k}K^{\pm} \end{aligned} \tag{24}$$

where the deformation parameter is given by:

$$q = e^{-\frac{\pi}{k}} \tag{25}$$

3. Quantization

In this section, we wish to obtain the algebra corresponding to (2.22) in the quantized theory. Due to the lack of regularization criteria like operator ordering, quantization of Chern-Simons theories imply a certain degree of arbitrariness.

In our case the indicated thing to do is canonical quantization. However, it would imply complicated operator manipulations to achieve (21). Instead of it we choose a way similar to [6]. We start from the following naive ansatz:

$$[X_i, X_j] = i\hbar\frac{\pi}{k}\left[\epsilon_{ij}\mathcal{O}(X_iX_j) + \epsilon_{ijk}X_k\right] \tag{26}$$

where

$$\mathcal{O}(X_iX_j) = \begin{cases} X_iX_j, & \epsilon_{ij} = 1 \\ X_jX_i, & \epsilon_{ij} = -1 \end{cases} \tag{27}$$

takes into account the nonconmutativity of X_i and X_j. In this case the nonlinear reparametrization is given by:

$$\begin{aligned} K^{\pm} &= \sqrt{1 - i\frac{\pi\hbar}{k}X_1 \pm iX_2e^{\pm\mu}} \\ X_3 &= i\frac{\sqrt{1-i\frac{\pi\hbar}{k}}}{2-i\frac{\pi\hbar}{k}}\left(e^{\mu} - e^{-\mu}\right) \end{aligned} \tag{28}$$

with the resulting algebra:

$$[K^+, K^-] = \frac{i\pi\hbar}{2k - i\pi\hbar}\left(e^{\mu} - e^{-\mu}\right) \tag{29}$$

$$[\mu, K^{\pm}] = \pm\ln\left(1 - i\frac{\pi\hbar}{k}\right)K^{\pm} \tag{30}$$

so that after some rescalings the canonical form turns out to be:

$$[K^+, K^-] = \frac{\left(q^{2H} - q^{-2H}\right)}{q^2 - q^{-2}} \tag{31}$$

$$[H, K^{\pm}] = \pm i\hbar K^{\pm} \tag{32}$$

where the quantized deformation parameter is given by:

$$q = \left(1 - i\frac{\pi\hbar}{k}\right)^{\frac{1}{i\hbar}} \tag{33}$$

such that the limit $\hbar \to 0$

$$\lim_{\hbar \to 0} q = \lim_{\hbar \to 0} \left[\left(1 - i\frac{\pi\hbar}{k} \right)^{-\frac{k}{i\pi\hbar}} \right]^{-\frac{\pi}{k}} = e^{-\frac{\pi}{k}} \tag{34}$$

gives the deformation parameter of the classical theory.

Now, we wish to make an ansatz of regularization for the operator product on the r.h.s. of (26) as follows:

$$X_i X_j \to \frac{1}{1+a} \left(X_i X_j + a X_j X_i \right) \tag{35}$$

Therefore

$$[X_1, X_2] = i\hbar u \left[\frac{1}{1+a} \left(X_1 X_2 + a X_2 X_1 \right) + X_3 \right] \tag{36}$$

hence

$$[X_1, X_2] = \frac{i\hbar u}{1 + ia\frac{\hbar u}{1+a}} \left(X_1 X_2 + X_3 \right) = i\hbar\tilde{u} \left(X_1 X_2 + X_3 \right) \tag{37}$$

which has the same form as (26).

Therefore the deformation parameter will be:

$$q = (1 - i\hbar\tilde{u})^{\frac{1}{i\hbar}} = \left(\frac{1 - i\frac{\pi\hbar/k}{1+a}}{1 + ia\frac{\pi\hbar/k}{1+a}} \right)^{\frac{1}{i\hbar}} \tag{38}$$

where we substituted $u \to \pi/k$.

It is interesting to expand (38) in power series on $\hbar$.

We obtain:

$$q = e^{-\frac{\pi}{k}} e^{-\sum_{n=2}^{\infty} \frac{1}{n} (i\hbar)^{n-1} \left(\frac{\pi/k}{1+a} \right)^n [1-(-a)^n]} \tag{39}$$

For example, if we take a symmetric ordering, i.e. $a = 1$, only even powers of $\hbar$ will survive and the deformation parameter will be real:

$$q_s = e^{-\frac{\pi}{k}} e^{-2\sum_{m=1}^{\infty} \frac{(-1)^m}{2m+1} \hbar^{2m} \left(\frac{\pi/k}{2a} \right)^{2m+1}} = e^{-\frac{\pi}{k}} \left[1 + \mathcal{O}\left(\hbar^2 \right) \right] \tag{40}$$

Our results are based on a heuristic quantization of the trace algebra (11). Nevertheless, the resulting deformation parameter is consistent as far as the classical limit ($\hbar \to 0$) concerns.

It would be interesting to quantize (11) instead of (21). However in this case the noncommutativity of the operators leads to considerably complications, for example the trace identity (19) is not fulfiled anymore. Work is in progress in this direction.

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ON POSITION AND MOMENTUM OPERATORS IN QUANTUM MECHANICS WITH QUANTUM GROUP SYMMETRY

ACHIM KEMPF*
Department of Applied Mathematics & Theoretical Physics
University of Cambridge, Cambridge CB3 9EW, U.K.

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Abstract. Within the framework of quantum group symmetric Heisenberg algebras and their (Bargmann-) Fock representations, we study the position and momentum operators: Their commutation relations, uncertainty relations and spectra. As an effect of the underlying noncommutative geometry, a scale appears, leading to the existence of minimal uncertainties in the positions and momenta. The usual quantum mechanical behaviour is recovered as a limiting case for not too small and not too large distances and momenta.

1. Introduction

Quantum groups, i.e. dual quasitriangular Hopf algebras, considered as deformations of function algebras on group manifolds, are examples of noncommutative geometry [1, 2, 3, 4]. It is interesting to examine, whether the introduction of noncommutative geometry into quantum theory can regularise its short distance behaviour or even lead to a link to gravity. Here we study the effects of noncommutative geometry in quantum mechanics.

The commutation relations of the following generalised bosonic Heisenberg algebra are conserved under the action of the quantum group $SU_q(n)$ [5, 6]: (See also [7] and also compare with [8, 9, 10, 11])

$$\begin{aligned} a_i a_j - q a_j a_i &= 0 \quad \text{for} \quad i<j \\ a_i^\dagger a_j^\dagger - q a_j^\dagger a_i^\dagger &= 0 \quad \text{for} \quad i>j \\ a_i a_j^\dagger - q a_j^\dagger a_i &= 0 \quad \text{for} \quad i \neq j \\ a_i a_i^\dagger - q^2 a_i^\dagger a_i &= 1 + (q^2-1)\sum_{j<i} a_j^\dagger a_j \end{aligned} \tag{1}$$

Here i runs from 1 to n and q is real. One obtains for the scalar product:

$$\langle 0|(a_n)^{r_n}\cdot\ldots\cdot(a_1)^{r_1}(a_1^\dagger)^{r_1}\cdot\ldots\cdot(a_n^\dagger)^{r_n}|0\rangle \quad = \prod_{i=1}^{n}[r_i]_q! \tag{2}$$

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with

$$[r]_q! := [1]_q \cdot [2]_q \cdot [3]_q \cdot \ldots \cdot [r]_q \qquad \text{and} \qquad [p]_q := \frac{q^{2p}-1}{q^2-1} \tag{3}$$

Although quantum groups do in general have more than one free parameter, no further parameters enter in these commutation relations [12, 13]. The Hilbert space $\mathcal{H}$, completed using the induced norm, is as usual isomorphic to l^2. The Poincaré series of the a's and the $a^\dagger$'s remain unchanged. The quantum group $SU_q(n)$ is a symmetry of the Heisenberg algebra. Nevertheless arbitrary Hamiltonians can be studied which not neccesarily have this symmetry. The usual quantum mechanical programme, representation of the Heisenberg algebra on a positive definite (Bargmann Fock-) Hilbert space of wave functions and the definition of integral kernels like Green functions etc., could be performed and some dynamical systems, using the undeformed Schrödinger equation (and leading to unitary time evolution) were worked out [14].

Since the above given commutation relations are respected by formal hermitean conjugation, the natural candidates for position and momentum operators $x \propto a + a^\dagger$ and $p \propto i(a^\dagger - a)$ are representable as symmetric operators on a suitable domain. Let us now try to reveal some features of the underlying noncommutative geometry by studying these observables in more detail.

2. Commutation relations of positions and momenta

We start with the following ansatz for the position and momentum operators: $(r = 1, \ldots, n)$

$$x_r := L_r(a_r^\dagger + a_r) \qquad , \qquad p_r := iK_r(a_r^\dagger - a_r) \tag{4}$$

Defining their domain D to be

$$D := \{v \in \mathcal{H} | v = \text{Polynomial}(a_1^\dagger, \ldots, a_n^\dagger)|0\rangle\} \tag{5}$$

which is dense in $\mathcal{H}$, we insure that all x_r and p_r are represented as symmetric operators with images that lie in their domain. Since the a's and $a^\dagger$'s do not carry units, the newly introduced constants L and K do.

It is reasonable to require the existence of a physical region in which the usual quantum mechanics is recovered as a limiting case[1], even if $q \neq 1$. This would be achieved if the commutation relations come out in the form $[x,p] = i\hbar + f(q,x,p)$ with uncertainty relation $\Delta x \Delta p \geq \frac{\hbar}{2} + \frac{1}{2}\langle f(q,x,p)\rangle$ which then reduce to the usual relations where $\langle f \rangle$ is negligible.

[1] Weakening this restriction, the ansatz Eqs.4 is generalisable.

Explicitly the commutation relations Eqs.1 read in terms of the x's and p's:

$$[x_r, p_r] = i\frac{4K_rL_r}{q^2+1} \tag{6}$$
$$+i\frac{4K_rL_r(q^2-1)}{q^2+1}\left[\sum_{s\leq r}\left(\frac{x_s^2}{4L_s^2}+\frac{p_s^2}{4K_s^2}\right)-\sum_{t<r}\frac{1}{4K_tL_t}[x_t,p_t]\right]$$

Working out the induction and setting

$$K_rL_r := \frac{\hbar}{2}\left(\frac{q^2+1}{2}\right)^r \tag{7}$$

the commutation relations do indeed take the desired form:

$$[x_r, p_r] = i\hbar + i\hbar(q^2-1)\sum_{s\leq r}\left(\frac{q^2+1}{2}\right)^{s-1}\left(\frac{x_s^2}{4L_s^2}+\frac{p_s^2}{4K_s^2}\right) \tag{8}$$

The mixed commutation relations read for $r < s$:

$$[x_s, p_r] = i\frac{L_r}{K_r}\frac{q-1}{q+1}\{x_s, x_r\} \tag{9}$$

$$[x_s, x_r] = i\frac{K_r}{L_r}\frac{q-1}{q+1}\{x_s, p_r\} \tag{10}$$

For $r > s$ one gets:

$$[x_s, p_r] = i\frac{K_s}{L_s}\frac{q-1}{q+1}\{p_s, p_r\} \tag{11}$$

$$[p_s, p_r] = -i\frac{L_s}{K_s}\frac{q-1}{q+1}\{x_s, p_r\} \tag{12}$$

If $q = 1$ the constants K and L drop out of the commutation relations, reflecting that in ordinary quantum mechanics a length or a momentum scale can only be set by the Hamiltonian i.e. by chosing a particular system. Here, for $q \neq 1$ the K and L appear in the commutation relations, thus these scales become a property of the quantum mechanical formalism itself.

3. Uncertainty relation

Let us consider for simplicity the 1 dimensional case where Eq.8 reads:

$$[x, p] = i\hbar + i\hbar(q^2-1)\left(\frac{x^2}{4L^2}+\frac{p^2}{4K^2}\right) \tag{13}$$

with

$$K = \frac{\hbar}{4L}(q^2+1) \tag{14}$$

We will now study the situation for $q^2 > 1$. The case $q^2 < 1$ is quite different and will be discussed elsewhere. The following (standard) derivation of the uncertainty relation holds on every domain D' of x and p, on which both operators are symmetric and have their images in the domain. The above given domain D is an example.

We start with the trivial statement that the following norm is positive:

$$|((x - \langle v, x.v\rangle) + i\alpha(p - \langle v, p.v\rangle))\, v| \geq 0 \qquad \forall v \in D' \quad \forall \alpha$$

Using Eq.13, that x and p are symmetric on D' and choosing α such as to get the most restrictive inequality this yields for all v in D' the uncertainty relation

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left(1 + (q^2-1)\left(\frac{(\Delta x)^2 + \langle x\rangle^2}{4L^2} + \frac{(\Delta p)^2 + \langle p\rangle^2}{4K^2}\right)\right) \tag{15}$$

with the notation:

$$(\Delta x)^2 := \langle v|(x - \langle v, x.v\rangle)^2|v\rangle \quad \text{and} \quad \langle x\rangle := \langle v, x.v\rangle\ .$$

In 'polar coordinates'

$$\Delta x := 2Lr\cos\alpha \quad \text{and} \quad \Delta p := 2Kr\sin\alpha \tag{16}$$

the uncertainty relation reads:

$$\sin 2\alpha > \frac{q^2-1}{q^2+1} \tag{17}$$

and

$$r^2 \geq \frac{1 + (q^2-1)\left(\frac{\langle x\rangle^2}{4L^2} + \frac{\langle p\rangle^2}{4K^2}\right)}{(q^2+1)\sin 2\alpha - (q^2-1)} \tag{18}$$

From Eq.17 follows that the minimal α is larger than 0 and the maximal α is smaller than $\pi/2$. Thus the hyperbola of the ordinary uncertainty relation, having the Δx and the Δp axes as asymptotes has turned into a graph with asymptotes that are no longer parallel to the axes. From Eq.18 follows that r is always larger than 0, thus there are minimal uncertainties in the positions and the momenta[2]. They are calculated to be:

$$\Delta x_0 = L\sqrt{\frac{q^2-1}{q^2}} \quad \text{and} \quad \Delta p_0 = K\sqrt{\frac{q^2-1}{q^2}} \tag{19}$$

[2] They depend on $\langle x\rangle$ and $\langle p\rangle$, the absolutely smallest values are given.

Due to Eq.14 there were two free parameters: the length L and q. Instead we can now use Δx_0 and Δp_0 as the free parameters and express L,K and q in terms of these:

$$L = \Delta x_0 \sqrt{\frac{2\Delta x_0 \Delta p_0 + \hbar + \sqrt{4(\Delta x_0)^2(\Delta p_0)^2 + (\hbar)^2}}{4\Delta x_0 \Delta p_0}} \tag{20}$$

$$K = \Delta p_0 \sqrt{\frac{2\Delta x_0 \Delta p_0 + \hbar + \sqrt{4(\Delta x_0)^2(\Delta p_0)^2 + (\hbar)^2}}{4\Delta x_0 \Delta p_0}} \tag{21}$$

$$q = \sqrt{\left(2\Delta x_0 \Delta p_0 + \sqrt{4(\Delta x_0)^2(\Delta p_0)^2 + \hbar^2}\right) / \hbar} \tag{22}$$

The commutation relation Eq.13 then takes the form:

$$[x,p] = i\hbar + ig(\Delta x_0, \Delta p_0)\left(\frac{x^2}{(\Delta x_0)^2} + \frac{p^2}{(\Delta p_0)^2}\right) \tag{23}$$

where

$$g(\Delta x_0, \Delta p_0) := 4\frac{\Delta x_0 \Delta p_0}{\hbar} \frac{2\Delta x_0 \Delta p_0 + \sqrt{4(\Delta x_0 \Delta p_0)^2 + \hbar^2} - \hbar}{2\Delta x_0 \Delta p_0 + \sqrt{4(\Delta x_0 \Delta p_0)^2 + \hbar^2} + \hbar} \tag{24}$$

Let us now identify the physical region where the ordinary quantum mechanical behaviour is recovered:

Since physically we know that Δx_0 and Δp_0 can only be very small, say $\Delta x_0 \Delta p_0 \ll \hbar/2$, we expand g to the first nonzero order and arrive at the simplified commutation relation:

$$[x,p] = i\hbar + \frac{4i}{\hbar}\left(x^2(\Delta p_0)^2 + p^2(\Delta x_0)^2\right) \tag{25}$$

Now it becomes clear that in our formalism not only the behaviour for small distances and momenta is altered: Also for expectation values of x^2 or p^2 large enough to make the second term on the rhs of the order $\hbar$ or larger, the behaviour will be significantly changed. The region of approximately ordinary quantum mechanical behaviour is thus specified through:

$$(\Delta x_0)^2 \ll x^2 \ll \frac{\hbar^2}{4(\Delta p_0)^2} \tag{26}$$

$$(\Delta p_0)^2 \ll p^2 \ll \frac{\hbar^2}{4(\Delta x_0)^2} \tag{27}$$

From the point of view of wave-particle dualism, meaning high momenta are needed to measure small distances etc. this is of course a reasonable result.

4. Functional analysis of x and p

The above derived uncertainty relation holds on every domain D' on which both x and p are symmetric and have their images in D'. It implied minimal uncertainties in the positions and momenta. Now if there was a $v_\lambda \in D'$ that is eigenvector e.g. of x: $x.v_\lambda = \lambda v_\lambda$, one would then of course have $(\Delta x)^2 = \langle v_\lambda|(x - \langle v_\lambda, x v_\lambda\rangle)^2|v_\lambda\rangle = 0$, which would be a contradiction. We thus conclude that there is no domain on which x and p are symmetric *and* have eigenvectors. Let us now study the functional analysis of x in more detail, the analysis for p is completely analogous.

We start be choosing for x the domain $D_x := D$ (the finite linear combinations of the vectors $(a^\dagger)^r|0\rangle$ with $r = 0, 1, 2, ...$), on which x and p are obviously symmetric and have their image in D_x. We can thus already conclude from above that x has no eigenvectors in D_x. Indeed, the eigenvalue problem

$$x.v_\lambda = \lambda v_\lambda \qquad \text{with} \qquad v_\lambda = \sum_{r=0}^{\infty} f_r(\lambda) \frac{(a^\dagger)^r}{\sqrt{[r]_q!}}|0\rangle \tag{28}$$

can be solved for all complex λ, but from the recursion formula that we obtain for the coefficients $f_r(\lambda)$ of v_λ it is clear that infinitely many of them are nonzero, thus $v_\lambda \notin D_x$.

Let us now consider the adjoint x^* of x, which has the domain:

$$D_{x^*} = \{v \in \mathcal{H} | \quad \exists w \in \mathcal{H} \quad \forall a \in D_x : \quad \langle v, x.a\rangle = \langle w, a\rangle\} \tag{29}$$

Of course $D_x \subset D_{x^*}$ and, using the above mentioned recursion formula one proves that actually all v_λ are normalisable and are contained in the domain D_{x^*}, i.e. they are eigenvectors of x^*. Since there are nonreal eigenvalues we conclude that x^* is not symmetric. An analytic expression for the interesting scalar product of two normalised eigenvectors $\langle \hat{v}_\lambda, \hat{v}_{\lambda'}\rangle$ has not yet been worked out (the numerical approximation converges as quickly as a geometrical series).

x^{**} is a much better behaved operator since it is closed and symmetric. Its domain

$$D_{x^{**}} = \{v \in \mathcal{H} | \quad \exists w \in \mathcal{H} \quad \forall a \in D_{x^*} : \quad \langle v, x^*.a\rangle = \langle w, a\rangle\} \tag{30}$$

is in between those of x and x^*: $D_x \subset D_{x^{**}} \subset D_{x^*}$ and it can easily be checked that it does not contain any eigenvectors v_λ.

We now apply the standard procedure, see e.g. [15, 16][3], for checking for self-adjoint extensions of closed symmetric operators:

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[3] Note that [15] defines 'hermitean' as synonymous to self-adjoint while [16] uses it as synonymous to symmetric.

The dimensions of the spaces (we use $x^{***} = x^*$)

$$L^{\perp}_{\pm i,x^{**}} := ker(x^* \mp i).D_{x^*} \tag{31}$$

i.e. the deficiency indices, are both equal to 1 (there is only one v_i and one v_{-i}). We can thus define the following one-parameter family of self adjoint extensions:

$$x_{sa}(\phi).a := i(b + U.b) \quad \text{for all} \quad a = b - U.b \tag{32}$$

with the isometric operator U defined on $(x^{**} + i).D_{x^{**}} \oplus \mathbf{C}v_i$ as

$$U.v := (x^{**} - i)(x^{**} + i)^{-1}.v \qquad \forall v \in (x^{**} + i).D_{x^{**}} = L_{+i,x^{**}} \tag{33}$$

and

$$U.\alpha v_i := \alpha e^{i\phi} v_{-i} \tag{34}$$

Here ϕ is a free real parameter, labeling the self-adjoint extensions. For the eigenvalues one can stay with the 'Cayley transform' U, calculate its eigenvalues, and an inverse Möbius transform then maps them onto the eigenvalues of $x_{sa}(\phi)$.

The analysis for p analogously leads to a one-parameter family of self-adjoint extensions $p_{sa}(\psi)$. One may now be tempted to try to fix the choice of the extension parameters ϕ and ψ by requiring that $x_{sa}(\phi)$ and $p_{sa}(\psi)$ be defined on the same domain. One would then like to diagonalise $x_{sa}(\phi)$ to obtain a coordinate space representation or to diagonalise $p_{sa}(\psi)$ to obtain a momentum space representation.

However, we know from section 3 that x and p cannot be extended to a common domain on which they are both diagonalisable.

We thus arrive at the following picture:

While in classical mechanics the states can have exact positions and momenta, in quantum mechanics there is the well known uncertainty principle, not allowing x and p to have common eigenvectors. Nevertheless x and p seperately do have eigenvectors, though nonnormalisable ones.

From the above discussion we conclude that the 'noncommutative geometry' or quantum group generalisation of the Heisenberg algebra has further consequences for x and p: It is not only that they have no common eigenvectors, they even do not have a common domain on which they are symmetric and have eigenvectors. Nevertheless x and p seperately do have self-adjoint extensions, and can even have normalisable eigenvectors. It remains to determine the maximal common domain on which they are symmetric.

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ON $SO_q(3,1)$ q-REGULARIZATION

SUEMI RODRÍGUEZ-ROMO
Centro de Investigaciones Teoricas
Facultad de Estudios Superiores-Cuautitlán, UNAM,
Apdo. 142 Cuautitlan Izcalli, Estado de México,
54700 México
suemi@fis.cinvestav.mx & suemi@unamvm1.dgsca.unam..mx

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Abstract. We replace invariant integration over momentum space by invariant integration over the vector representation of $SO_q(3,1)$. Our invariant measure is reduced to an $SU_q(2)$ invariant one by imposing the q-time zero projection. Finally, we show the null directions in the $SO_q(3,1)$ Hopf algebra that lead to a quantum Galilei group.

Key words: $SO_q(3,1)$ - q-regularization - q-Minkowski space - q-Galilei group.

1. Introduction

By introducing non-commutative algebraic geometry, we can regulate relevant quantities in field theory before renormalizing [Rodríguez-Romo, to appear]. In this context, we propose a scheme called q-regularization, where the deformation is parametrized by $q \in R$ (being R the reals), and $q^2 \neq -1$, in which relevant quantities in quantum field theories are finite for $q \neq 1$, and reduce to the unregulated, divergent, physically meaningful quantity as $q \to 1$. Namely, as well as in dimensional regularization we interpolate consistently to dimension $4-\epsilon$ where the relevant quantities are finite (these would be infinite at dimension four); in q-regularization we extend a quantum field theory to the non-commutative framework (by introducing the parameter q) where the relevant quantities are finite (these would be infinite at $q=1$).

We want to preserve the desired Lorentz invariance, so we present a q-regularization invariant under the q-Lorentz group. The q-Lorentz group that we use as symmetry has been constructed from the tensor product representation of $SL_q(2,C)$ [Carow-Watamura, Schlieker, Scholl and Watamura 1990]. Since $SL_q(2,C)$ is view as the general (coordinate) transformation of the q-spinor (two dimensional object with the generators of $A_q^{2/0}$, Manin's quantum plane [Manin 1988], as entries) and the q-deformed Minkowski space is obtained as a tensor product representation of pairs of two independent copies of q-spinors, the quantum group of transformation matrices acting on the quantum Minkowski space is identified as the quantum Lorentz group $SO_q(3,1)$. In other words, a four dimensional real representation of $SL_q(2,,C)$ generates the quantum Minkowski space with $SO_q(3,1)$ as its symmetry. We work out our approach in the light cone q–deformed coordinates of this quantum Minkowski space-time.

In order to learn more about the symmetries of our $SO_q(3,1)$ invariant measure and link them with physically meaningful problems, we study the zero q-time projection in q-deformed Minkowski space-time and its relation to the Woronowicz's $SU_q(2)$ measure [Woronowicz 1988]. Moreover, we show how, some null directions of the quantum group $SO_q(3,1)$, lead us to a q-deformed Galilei group.

This paper is organized as follows; in Section 2 we introduce the q-spinors from the non-commutative Heisenberg algebra and present the $SO_q(3,1)$ Hopf algebra we use. In Section 3, we obtain the $SO_q(3,1))$ invariant q-regularisation in terms of light-cone coordinates (out from q-deformed Minkowski space) and q-spinors. In Section 4, we study the zero time projection of the $SO_q(3,1)$ invariant measure in terms of the $SU_q(2)$ measure given by Woronowicz (1988) and the null directions of $SO_q(3,1)$ that lead to a q-deformed Galilei group. The quantum Galilei group has been found as symmetry in condensed matter [Bonechi, Celeghini, Giachetti, Sorace et al. 1992].

2. From Heisenberg algebra to q-spinors and $SO_q(3,1)$.

To start with, consider the fundamental Heisenberg commutator algebra on phase space $(\mathbf{r},\mathbf{p})$;

$$\begin{aligned}\left[r^i,p^j\right] &= i\hbar\delta^{ij} \\ \left[r^i,r^j\right] &= \left[p^i,p^j\right] = 0.\end{aligned} \tag{1}$$

and the translator operator on phase space;

$$U(\mathbf{a},\mathbf{b}) = e^{i(\mathbf{a}\cdot\mathbf{p}-\mathbf{b}\cdot\mathbf{r})/\hbar} \text{ where } \mathbf{a} \text{ and } \mathbf{b} \in R^n . \tag{2}$$

In a ray or projective representation eq.(2) obeys the following composition law;

$$U(\mathbf{a}_2,\mathbf{b}_2)\cdot U(\mathbf{a}_1,\mathbf{b}_1) = e^{[2\pi i\alpha_2(\mathbf{r};(\mathbf{a}_1,\mathbf{b}_1),(\mathbf{a}_2,\mathbf{b}_2))]}\cdot U(\mathbf{a}_1+\mathbf{a}_2,\mathbf{b}_1+\mathbf{b}_2), \tag{3}$$

where $\mathbf{a}_1,\mathbf{b}_1,\mathbf{a}_2,\mathbf{b}_2 \in R^n$ and, for a free particle in quantum mechanics, the two-co-cycle α_2 for translations in the phase space is given by

$$2\pi\alpha_2(\mathbf{r};(\mathbf{a}_1,\mathbf{b}_1),(\mathbf{a}_2,\mathbf{b}_2)) = \frac{1}{2\hbar}(\mathbf{a}_1\cdot\mathbf{b}_2 - \mathbf{a}_2\cdot\mathbf{b}_1). \tag{4}$$

Let us consider the following infinitesimal Galilei transformation

$$\begin{aligned}\mathbf{r}' &= \mathbf{r}+\mathbf{a}_1 = \mathbf{r}+\hbar\mathbf{u}, &\qquad \mathbf{r}'' &= \mathbf{r}+\mathbf{a}_2 = \mathbf{r}, \\ \mathbf{p}' &= \mathbf{p}+\mathbf{b}_1 = \mathbf{p}, & \mathbf{p}'' &= \mathbf{p}+\mathbf{b}_2 = \mathbf{p}+\hbar\mathbf{u},\end{aligned} \tag{5}$$

where $\mathbf{u}$ is a unit vector in R^n. Define $q = e^{-i\hbar}$, impose eq.(5) as symmetry in eq.(4) and substitute the result in eq.(3), thus

$$U(\hbar\mathbf{u},0)U(0,\hbar\mathbf{u}) = qU(0,\hbar\mathbf{u})U(\hbar\mathbf{u},0) \tag{6}$$

is a realization of $A_q^{(2/0)}$.

Following Carow-Watamura et al. (1990 and 1991), let us define

$$Z^\rho = \begin{bmatrix} Z^1 \\ Z^2 \end{bmatrix} = \begin{bmatrix} U(\hbar\mathbf{u},0) \\ U(0,\hbar\mathbf{u}) \end{bmatrix}, \text{ i.e. } \rho = 1,2 \,. \tag{7}$$

as a q-spinor, introduce the tensor product representation of two q-spinor spaces, called $(Z^i, \tilde{Z}^i)$, with a pair of q-spinors $(i = 1,2)$ in each space. Hereafter greek indices are for spinor suffix and roman ones for different spinors. Besides, it is required that

$$Z^i \tilde{Z}^j = \hat{R}^{ij}_{j'i'} \tilde{Z}^{j'} Z^{i'} \tag{8}$$

where $\hat{R}^{ij}_{j'i'}$ is the Yang-Baxter matrix for $SL_q(2,C)$.

After projecting the real part of q, as above given, the identification $\tilde{Z}^\sigma = \epsilon^{\rho\sigma}\bar{Z}_\rho$ is made. Here $\epsilon^{\rho\sigma} = \begin{pmatrix} 0 & q^{1/2} \\ -q^{-1/2} & 0 \end{pmatrix}$, $\bar{Z}_\rho \in A^{2/0}_{1/q^*}$ is the hermitian conjugate of the q-spinor Z^ρ.

It is straightforward to see that

$$X^{ij} = \tilde{Z}^i Z^j \in A_q^{2/0} \otimes A_q^{2/0} \tag{9}$$

yields to the generators of a four-dimensional real representation corresponding to the q-deformed Minkowski space and its quantum group of transformation matrices is $SO_q(3,1)$ [Carow-Watamura et al. 1990 and 1991].

3. On $SO_q(3,1)$ invariant q-regularisation.

From the X^{ij} algebra take the q-light cone coordinates $(A,\bar{A},B,\bar{B})$, where

$$A = X + Y, \quad \bar{A} = X - Y, \quad B = Z + T, \quad \bar{B} = Z - T \tag{10}$$

and (X,Y,Z,T) is the q-Lorentz vector [Carow-Watamura et al. 1990]. Rewrite the generators $(A,\bar{A},B,\bar{B})$ as follows

$$A = a, \quad \bar{A} = \bar{a}, \quad B = q^{\frac{b}{Q'}}, \quad \bar{B} = q^{\frac{\bar{b}}{Q'}} \quad \text{where } Q' = \sqrt{1-q}. \tag{11}$$

Let us define the Haar measure $\int\int$ as a map $A_q^{2/0} \otimes A_q^{2/0} \to$ C, being C the complex, such that

$$\int\int f = f_{(1)} \int\int f_{(2)} \quad \forall f \in A_q^{2/0} \otimes A_q^{2/0} \tag{12}$$

where Δ is a linear map called coproduct

$$\Delta : (A_q^{2/0} \otimes A_q^{2/0}) \to (A_q^{2/0} \otimes A_q^{2/0}) \otimes (A_q^{2/0} \otimes A_q^{2/0}). \tag{13}$$

We have expressed the action of Δ on f as

$$\Delta f = f_{(1)} \otimes f_{(2)}. \tag{14}$$

By analogy with the case of finite dimensional Hopf algebras [Larson and Radford 1988], we use the following formal expression for eq.(14)

$$\int\int f = Tr_{A_q^{2/0} \otimes A_q^{2/0}} L_f S^2 \tag{15}$$

where L_f stands for f acting by left multiplication on $A_q^{2/0} \otimes A_q^{2/0}$ and S is the antipode map. The deformed additive structure of the algebra generated by the unit and $(a, \bar{a}, b, \bar{b})$ is such that [Rodríguez-Romo, to appear, Taft 1971]

$$S^2(a - \bar{a}) = w^{-1}(a - \bar{a}); \quad S^2((a + \bar{a}) = w^{-1}(a + \bar{a});$$

$$S^2(b) = b; \qquad S^2(\bar{b}) = \bar{b} \tag{16}$$

where $w = f(q)$ and $lim_{q \to 1} w^{-1} = 1$. Exact expressions for Δ, ϵ and S as well as the $*$-algebra are given in [Rodríguez-Romo, to appear].

To compute $\int\int$ we propose the following basis in $A_q^{2/0} \otimes A_q^{2/0}$,

$$F^{\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_5\lambda_6} = \left(e^{i\lambda_1\bar{b}} e^{i\lambda_2\frac{(a-\bar{a})}{2}}, e^{i\lambda_3\bar{b}} e^{i\lambda_4\frac{(a+\bar{a})}{2}}, e^{i\lambda_5\bar{b}} e^{i\lambda_6 b} \right), \tag{17}$$

where

$$F^{\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_5\lambda_6} \in A_q^{2/0} \otimes A_q^{2/0} \text{ and } (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \in R.$$

We associate to $F^{\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_5\lambda_6}$ a dual basis $F_{\lambda_1'\lambda_2',\lambda_3'\lambda_4',\lambda_5'\lambda_6'} \in (A_q^{2/0} \otimes A_q^{2/0})^!$ where $(\tilde{A}_{q*}^{2/0} \otimes A_{q*}^{2/0})^!$ is the dual Hopf algebra of $\tilde{A}_q^{2/0} \otimes A_q^{2/0}$, such that

$$F^{\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_5\lambda_6} F_{\lambda_1'\lambda_2',\lambda_3'\lambda_4',\lambda_5'\lambda_6'} = (\delta(\lambda_1' - \lambda_1)\delta(\lambda_2' - \lambda_2), \tag{18}$$

$$\delta(\lambda_3' - \lambda_3)\delta(\lambda_4' - \lambda_4), \delta(\lambda_5' - \lambda_5)\delta(\lambda_6' - \lambda_6))$$

where, as usual, the Dirac delta functions δ are defined with respect to the Lebesgue integration on R. The basis $F^{\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_5\lambda_6}$, admits a q-spinor representation, see [Rodríguez-Romo, to appear]. Furthermore, the Haar measure $\int\int$ defined on $A_q^{2/0} \otimes A_q^{2/0}$ can be written in terms of ordinary integration [Rodríguez-Romo, to appear].

Theorem 1 [Rodríguez-Romo, to appear] For a suitable $f \in A_q^{2/0} \otimes A_q^{2/0}$ that can be expressed on the $F^{\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_5\lambda_6}$ basis, $\int\int f$ contains a component that can be q-regularized, i.e. it is finite provided $q \neq 1$, but infinite in the limit $q = 1$.

Proof. To start with, let

$$f =: f' := \int_{-\infty}^{\infty} d\lambda_1 d\lambda_2 \tilde{f}(\lambda_1, \lambda_2) F^{\lambda_1\lambda_2} + \tag{19}$$

$$\int_{-\infty}^{\infty} d\lambda_3 d\lambda_4 \tilde{f}(\lambda_3, \lambda_4) F^{\lambda_3\lambda_4} + \int_{-\infty}^{\infty} d\lambda_5 d\lambda_6 \tilde{f}(\lambda_5, \lambda_6) F^{\lambda_5\lambda_6}$$

where $F^{\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_5\lambda_6} = (F^{\lambda_1\lambda_2}, F^{\lambda_3\lambda_4}, F^{\lambda_5\lambda_6})$ and we express f as a normal ordered form of f', in terms of the generators. Namely, putting $\bar{b}$ to the left of a, $\bar{a}$ and b. Additionally, $\tilde{f}$ is the Fourier transform of f', i.e.

$$\tilde{f}(\lambda_i, \lambda_j) = (2\pi)^{-2} \int_{-\infty}^{\infty} d\mu_i d\mu_j f'(\mu_i \mu_j) e^{-i\mu_i\lambda_i} e^{-i\mu_j\lambda_j} \tag{20}$$

$$i, j = (1,2), (3,4), (5,6).$$

Then we obtain

$$\int\int f = \int_{-\infty}^{\infty} d\lambda_1' d\lambda_2' d\lambda_2 \tilde{f}(0, \lambda_2) \delta(\lambda_2'(1 - w^{-1}) - \lambda_2 e^{-2i\lambda_1' Q'}) + \tag{21}$$

$$\int_{-\infty}^{\infty} d\lambda_3' d\lambda_4' d\lambda_4 \tilde{f}(0, \lambda_4) \delta(\lambda_4'(1 - w^{-1}) - \lambda_4 e^{-2i\lambda_3' Q'}) +$$

$$\int_{-\infty}^{\infty} d\lambda_5' d\lambda_6' \tilde{f}(0,0)$$

or

$$\int\int f = \int_{-\infty}^{\infty} d\lambda_1' d\lambda_2' e^{\lambda_1' Q'} \tilde{f}(0, \lambda_2'(1 - w^{-1}) e^{2i\lambda_1' Q'}) + \tag{22}$$

$$\int_{-\infty}^{\infty} d\lambda_3' d\lambda_4' e^{-\lambda_3' Q'} \tilde{f}(0, \lambda_4'(1 - w^{-1}) e^{-2i\lambda_3' Q'}) +$$

$$\int_{-\infty}^{\infty} d\lambda_5' d\lambda_6' \tilde{f}(0,0)$$

The last term in eq.(22) corresponds to the ordinary divergent term that appears in the commutative algebraic formulation of quantum field theory; there is no way we can recover a finite term out of this in the limit $q \to 1$. Checking the non-commutative algebra of $SO_q(3,1)$ we find the reason why this happens to be so; the light cone q-coordinates $b, \tilde{b}$ commute; i.e. T is central with respect to (X, Y, Z), so this part of the Haar measure is not really defined on a non-commutative algebraic variety. Therefore we can extract out of $\int\int f$ the following q-regularizable component

$$\int\int f - \int_{-\infty}^{\infty} d\lambda_5' d\lambda_6' \tilde{f}(0,0) = \tag{23}$$

$$(2\pi\delta(0))\left\{\int_{-\infty}^{\infty} d\lambda_2' \tilde{f}(0,\lambda_2'(1-w^{-1})) + \int_{-\infty}^{\infty} d\lambda_4' \tilde{f}(0,\lambda_4'(1-w^{-1})\right\}.$$

But $w^{-1} = f(q)$ is such that $lim_{q\to 1} w^{-1} = 1$ [Rodríguez-Romo, to appear]. Thus, as $q \to 1$, $\int\int f - \int_{-\infty}^{\infty} d\lambda_5' d\lambda_6' \tilde{f}(0,0)$ diverges, by contrast at $q \neq 1$ and assuming suitable analicity and decay of $\tilde{f}$ to allow contour integration, eq.(23) can be made finite for suitable f; moreover, this is proportional to $(1-w^{-1})^{-1}$. Q.E.D.

Summarizing, in this section we have extracted out of a suitable $f \in A_q^{2/0} \otimes A_q^{2/0}$, written on the basis $F^{\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_5\lambda_6}$, a component that is made finite as $q \neq 1$ but diverge as $q \to 1$. This can be written in terms of light cone q-Minkowski coordinates and q-spinors (Weyl or Majorana type) [Rodríguez-Romo, to appear]. An example in two dimensional $\lambda\phi^4$ theory can be seen in [Rodríguez-Romo, to appear].

4. q-Time zero-projection and q-Galilei group.

Theorem 2. The q-Time zero-projection in q-Minkowski space-time, reduces $\int\int f - \int_{-\infty}^{\infty} d\lambda_5' d\lambda_6' \tilde{f}(0,0)$ to the Haar weight on the vector representation of $SO_{q^2}(3)$ written in terms of $SU_q(2)$ [Rodríguez-Romo, to appear].

Proof

a) The identification $\tilde{M}^\dagger = M^{-1}$, that leads Λ^μ_ν to be written in terms of $SU_q(2)$, corresponds to $T = 0$ in q-Minkowski space-time.

b) The algebra generated by $(A, \bar{A}, B)$ is isomorphic to the q-space relations of the 3-dimensional vector representation for $SO_{q^2}(3)$, given by Faddeev et al. (1987), (i.e. q from Fadeev corresponds to q^2 here). Let us call $X^1 = \frac{1}{2}(A+\bar{A})$, $X^2 = \frac{1}{2}(A-\bar{A})$ and $X^3 = B$.

c) From the $F^{\lambda_1\lambda_2,\lambda_3\lambda_4,\lambda_5\lambda_6}$ basis, we project;

$$F^{\lambda_1\lambda_2,\lambda_3\lambda_4} = \left(e^{i\lambda_1 x^3} e^{i\lambda_2 x^1}, e^{i\lambda_3 x^3} e^{i\lambda_4 x^2}\right) \tag{24}$$

where $X^1 = x^1$, $X^2 = x^2$, and $X^3 = q^{\frac{x^3}{Q'}}$.

From this and from the work done on the category of representations of a Hopf algebra follow the proof.

Finally, let us now show how null directions in the $SO_q(3,1)$ Hopf algebra can lead us to obtain a quantum mechanical Galilei group.

Theorem 3 [Rodríguez-Romo, to appear] By imposing the following null bi-ideals

$$u_2^1 = 0, \quad u_1^2 = 0, \quad u_1^1 u_2^2 = u_2^2 u_1^1 = 1 \tag{25}$$

on $M \in SL_q(2,C)$ and $\tilde{M} \in \tilde{S}L_q(2,C)$ (factors in $SO_q(3,1)$) we obtain from $\lambda^{(ij)}_{(i'j')} = \tilde{M}^i_{i'} M^j_{j'} \in SO_q(3,1)$ a direct product representation of the quantum Galilei group.

Proof. By requiring the quantum matrix $M = \begin{pmatrix} u_1^1 & u_2^1 \\ u_1^2 & u_2^2 \end{pmatrix} \in SL_q(2)$ to belong to the quantum mechanical Galilei group; i.e. M must fulfill eq.(5) then the null bi-ideals in eq.(25) have to be imposed on M, so to end up with a group that has only one generator, as should be.

If we impose the null-directions given by eq.(25) in $\Lambda \in SO_q(3,1)$ we obtain the representation of the quantum Galilei group as symmetry on a 4-dimensional real representation of $SL_q(2,C)$. Namely

$$\Lambda = \begin{pmatrix} (\bar{u}_1^1)^{-1} u_1^1 & 0 & 0 & 0 \\ 0 & \frac{\bar{u}_1^1 u_1^1 + q^2 (\bar{u}_1^1 u_1^1)^{-1}}{1+q^2} & 0 & \frac{q^2(\bar{u}_1^1 u_1^1 - (\bar{u}_1^1 u_1^1)^{-1})}{1+q^2} \\ 0 & 0 & \bar{u}_1^1 (u_1^1)^{-1} & 0 \\ 0 & \frac{\bar{u}_1^1 u_1^1 - (\bar{u}_1^1 u_1^1)^{-1}}{1+q^2} & 0 & \frac{q^2(\bar{u}_1^1 u_1^1 + (\bar{u}_1^1 u_1^1)^{-1})}{1+q^2} \end{pmatrix} \tag{26}$$

where $\bar{u}_1^1$ is the generator for the Galilei group that comes from $\tilde{M} = M^{-1}$, the transformation matrix for $\tilde{Z}^i$. Q.E.D.

5. Summary and Conclusions

In this paper we have used the projective representation of the non commutative Heisenberg algebra to construct the Manin quantum plane, thereby defining q-spinors. Using this as a building block we present a $SO_q(3,1)$ invariant q-regularisation in terms of q-deformed light-cone coordinate , we show how to extract, from relevant quantities, finite components (provided $q \neq 1$) that can become infinite at $q = 1$. To compute the Haar weight, we propose a basis projected from the q-deformed Minkowski space-time in light cone coordinates, so the functions to be q-regularized are to be considered on this frame of reference. Additional work must be done to generalize our scheme to arbitrary functions on the full q-Minkowski space-time basis˙

Finally, in order to learn about this programme and its symmetries, we study the T=0 (in q-Minkowski space-time) projection of our $SO_q(3,1)$ invariant Haar measure in terms of th $SU_q(2)$ measure and the null directions in the $SO_q(3,1)$ Hopf algebra that lead to a quantum mechanical Galilei group.

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CHAPTER II

NONCOMMUTATIVE DIFFERENTIAL GEOMETRY

Q-DEFORMED QUANTIZATION

JULIUS WESS
Sektion Physik, LMU München, Theresienstr. 37,
80333 München, Germany

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A concept of quantized spaces has emerged from the study of quantum groups. Inhomogeneous quantum groups as the q-Poincaré group or the q-Euclidean group lead to a q-deformation of the Heisenberg algebra. A one-dimensional version of it is:

$$px - q\,xp = -i. \tag{1}$$

$q \neq 0$ is supposed to be a real number, characterising the deformation. It is the simplest example of this type and in this lecture I want to show some of the characteristic features of the above-mentioned quantized spaces by way of this example.

In a quantum mechanical model based on the algebra (1), x and p have to be represented by linear operators in a Hilbert space. For real q, x and p cannot be both hermitean. Denoting by x^* and p^* the conjugate of x and p respectively we find:

$$p^*x^* - \frac{1}{q}x^*p^* = -\frac{i}{q}. \tag{2}$$

We shall assume p to be hermitean ($p^* = p$) and we will introduce x^* as a new variable.

To complete the algebra, the x,x^* relations have to be specified as well. A consistent relation is:

$$xx^* = q\,x^*x. \tag{3}$$

The algebra defined that way has a Casimir operator (central element):

$$C = (q-1)xpx^* + i(x - x^*). \tag{4}$$

It can be used to eliminate x^* in an irreducible representation with C having a unique eigenvalue:

$$x^* = r^{-1}(iC + x), \quad r = 1 + i(q-1)xp = i[p,x]. \tag{5}$$

It is convenient to rewrite the algebra in terms of hermitean variables. We introduce the hermitean "space"-variable

$$\xi = \frac{q^{-1} - q}{q^{1/2} + q^{-1/2}} \left\{ (rr^*)^{-1/2} x + x^* (rr^*)^{-1/2} \right\}, \tag{6}$$

and a unitary operator u:

$$u = (rr^*)^{-1/2} r, \qquad u^* u = u u^* = 1. \tag{7}$$

The algebra is:

$$p\xi - q\,\xi p = i(q^{3/2} - q^{-1/2})u \tag{8}$$

$$up = q\,pu, \qquad u\xi = q^{-1}\xi u,$$

with the conjugation properties:

$$p^* = p, \qquad \xi^* = \xi, \qquad u^* = u^{-1}. \tag{9}$$

A Hilbert space representation can be constructed. We choose p to be diagonal and find:

$$\begin{aligned} p \mid n >^{\pi_0} &= \pi_0\, q^n \mid n >^{\pi_0}, \\ \xi \mid n >^{\pi_0} &= -\frac{i}{\pi_0 q^n} \left\{ q^{1/2} \mid n-1 >^{\pi_0} - q^{-1/2} \mid n+1 >^{\pi_0} \right\}, \\ u \mid n >^{\pi_0} &= \mid n-1 >^{\pi_0} . \end{aligned} \tag{10}$$

The real number $\pi_0 \neq 0$ characterizes the representation. The Hilbert space H_{π_0} is defined as follows:

$$^{\pi_0} < n \mid m >^{\pi_0} = \delta_{nm}, \tag{11}$$

$$\sum_n c_n \mid n >^{\pi_0} \in H_{\pi_0} \iff \sum_n \mid c_n \mid^2 < \infty.$$

The operators p and ξ in the Hilbert space representation (10) are hermitean. The operator p is diagonal and self-adjoint. The operator ξ is not essentially self-adjoint. This can be seen in that ξ does not possess a complete orthogonal set of eigenvectors with real eigenvalues. Assume ξ has an eigenstate $\mid x_0 >$ with real eigenvalue x_0. Applying u and u^{-1} to such a state gives a state with eigenvalue qx_0 and $q^{-1}x_0$ respectively. Assuming that these states with different eigenvalues are orthogonal leads to a contradiction. From (8) follows:

$$p\xi = -i \left\{ q^{-1/2} u - q^{1/2} u^{-1} \right\}. \tag{12}$$

The orthogonality assumption leads to the equation $< x_0 \mid p\xi \mid x_0 >= 0$. If $\mid x_0 >$ is in H_{π_0} and has expansion coefficients c_n we conclude:

$$\pi_0 \sum_n q^n x_0 \mid c_n \mid^2 = 0. \tag{13}$$

A clear contradiction. This short argument shows that for a representation of p and ξ in terms of essentially self-adjoint and therefore diagonalizable operators, the operator p has to admit eigenvalues of both signs. The mathematical requirement of essentially self-adjoint operators leads to the physically very reasonable consequence that there should be left and right movers in the model. In the direct sum of Hilbert spaces $H_{\pi_0} \oplus H_{-\pi_0}$, essentially self-adjoint operators p and ξ exist.

$$\begin{aligned} p \mid n >^{\pm\pi_0} &= \pm\pi_0 q^n \mid n >^{\pm\pi_0}, \\ \xi \mid n >^{\pm\pi_0} &= \mp\frac{i}{\pi_0 q^n}\left\{q^{1/2} \mid n-1 >^{\pm\pi_0} - q^{-1/2} \mid n+1 >^{\pm\pi_0}\right\}, \\ u \mid n >^{\pm\pi_0} &= \mid n-1 >^{\pm\pi_0}. \end{aligned} \tag{14}$$

A q-deformed Fourier transformation transforms the momentum basis into a coordinate basis and vice versa. The q-deformed cos and sin functions have been defined by Koornwinder and Swarttouw [5]. With a slight change in notation, they are:

$$\begin{aligned} \cos[n] &= \cos(q^{2n}; q^{-4}), \qquad \sin[n] = \sin(q^{2n}; q^{-4}), \\ \cos(z; q^{-4}) &= \sum_{k=0}^{\infty}(-1)^k \frac{q^{-2k(k+1)}}{(q^{-2}; q^{-2})_{2k}} z^{2k}, \\ \sin(z; q^{-4}) &= \sum_{k=0}^{\infty}(-1)^k \frac{q^{-2k(k+1)}}{(q^{-2}; q^{-2})_{2k+1}} z^{2k+1}, \\ (a; q)_k &= \prod_{m=0}^{k-1}(1 - aq^m). \qquad N = \frac{(q; q^{-4})_\infty}{(q^{-4}; q^{-4})_\infty}. \end{aligned} \tag{15}$$

These functions satisfy the orthogonality and completeness relations:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} q^{2n} \cos[k+n]\cos[l+n] &= N^{-2} q^{-2l} \delta_{kl} \\ \sum_{n=-\infty}^{\infty} q^{2n} \sin[k+n]\sin[l+n] &= N^{-2} q^{-2l} \delta_{kl}. \end{aligned} \tag{16}$$

The momentum basis of (14) is transformed into the coordinate basis with these functions:

$$\begin{aligned}
\mid 2k >^{\pm} &= \frac{N}{2}\sum_{n=-\infty}^{\infty} q^{k+n}\left\{\cos[k+n]\left(\mid 2n >^{\pi_0} + \mid 2n >^{-\pi_0}\right)\right. \\
&\qquad \left.\pm i\sin[k+n]\left(\mid 2n+1 >^{\pi_0} - \mid 2n+1 >^{-\pi_0}\right)\right\} \\
\mid 2k+1 >^{\pm} &= \frac{N}{2}\sum_{n=-\infty}^{\infty} q^{k+n}\left\{\sin[k+n]\left(\mid 2n >^{\pi_0} + \mid 2n >^{-\pi_0}\right)\right. \\
&\qquad \left.\mp iq\cos[k+n+1]\left(\mid 2n+1 >^{\pi_0} + \mid 2n >^{-\pi_0}\right)\right\}.
\end{aligned} \tag{17}$$

These states form a complete and orthogonal set of eigenvectors of the operator ξ:

$$\xi \mid k >^{\pm} = \mp \frac{1}{\pi_0 q^{1/2}} q^k \mid k >^{\pm} . \tag{18}$$

The action of p on the eigenstates of ξ can be calculated:

$$p \mid k >^{\pm} = \pi_0(-q)^{-k}\left\{\mid k+1 >^{\pm} - q \mid k-1 >^{\pm}\right\}. \tag{19}$$

Let us introduce the wave function $\psi(k)$ for an arbitrary state:

$$\mid \psi > = \sum_{k=-\infty}^{\infty}\left\{\psi_+(k) \mid k >^+ + \psi_-(k) \mid k >^-\right\}. \tag{20}$$

The probability of finding the "particle" at the point $\pm\frac{1}{\pi_0 q^{1/2}} q^k$ is given by $\mid \psi_\pm(k) \mid^2$. The normalization condition is:

$$\sum_{k=-\infty}^{\infty}\left(\mid \psi_+(k) \mid^2 + \mid \psi_-(k) \mid^2\right) = 1. \tag{21}$$

The momentum acts on the wave function:

$$p\psi_\pm(k) = (-q)^{-k}\pi_0\left\{\psi_\pm(k+1) - q\psi_\pm(k-1)\right\}. \tag{22}$$

A dynamic is defined by a Hamiltonian and the corresponding Schroedinger equation. It is natural to study the Hamiltonian of a "free particle":

$$H = \frac{1}{2}p^2, \tag{23}$$

with the Schroedinger equation:

$$\begin{aligned}
i\frac{\partial}{\partial t}\psi_\pm(k,t) &= H\psi_\pm(k,t) \\
&= -\tfrac{1}{2}\pi_0^2 q^{-2k+1}\left\{q^{-2}\psi_\pm(k+2,t) - (q+\tfrac{1}{2})\psi_\pm(k,t) + q^2\psi_\pm(k-2,t)\right\}.
\end{aligned} \tag{24}$$

This Schroedinger equation finds its solution in terms of the momentum eigenfunctions with eigenvalues of the Hamiltonian:

$$E_n = \frac{1}{2}\pi_0^2 q^{2n}. \tag{25}$$

Let us calculate how the probability of finding the "particle" at the space point $\pm\frac{1}{\pi_0 q^{1/2}} q^k$ changes in time due to the Schroedinger equation (24). We find

$$\frac{\partial}{\partial t}\left\{\psi_\pm^\dagger(k,t)\psi_\pm(k,t)\right\} = j_\pm(k+1,t) - j_\pm(k-1,t), \tag{26}$$

with the "current density":

$$j_\pm(k-1,t)$$
$$= \frac{i}{2}\pi_0^2 q^{-2k-1}\left\{\psi_\pm^\dagger(k,t)\psi_\pm(k+2,t) - \psi_\pm^\dagger(k+2,t)\psi_\pm(k,t)\right\} \tag{27}$$

The probability is conserved, equ. (26) is the "continuity" equation.

After having studied the one-particle states, let us try to generalize the model to a many-particle system by imposing a second quantization. The one-particle states are created by creation operators from a vacuum state:

$$| k >^\pm = c_{k,\pm}^\dagger \mid 0 >, \qquad | n >^\pm = a_{n,\pm}^\dagger \mid 0 > . \tag{28}$$

The operators $c^\dagger$, $a^\dagger$ create the particles in the coordinate or momentum representation, respectively. From (17) follows how these operators are connected:

$$\begin{aligned} c_{2k,\pm}^\dagger &= \frac{1}{2}N\sum_n q^{(k+n)}\left\{\cos[k+n](a_{2n,+}^\dagger + a_{2n,-}^\dagger)\right. \\ &\qquad \left.\pm\, i\sin[k+n](a_{2n+1,+}^\dagger - a_{2n+1,-}^\dagger)\right\} \\ c_{2k+1,\pm}^\dagger &= \frac{1}{2}N\sum_n q^{(k+n)}\left\{\sin[k+n](a_{2n,+}^\dagger - a_{2n,-}^\dagger)\right. \\ &\qquad \left.\mp i\cos[k+n](a_{2n+1,+}^\dagger + a_{2n+1,-}^\dagger)\right\}. \end{aligned} \tag{29}$$

The operators c, and therefore the operators a as well, are assumed to annihilate the vaccuum:

$$c_{k,r} \mid 0 > = a_{n,r} \mid 0 > = 0, \tag{30}$$

r takes the values $+, -$. In accordance with a locality requirement we assume:

$$c_{k,r}c_{k',r'}^\dagger \mid 0 > = \delta_{k,k'}\delta_{r,r'} \mid 0 > . \tag{31}$$

From (29) follows the same relation for the a operators:

$$a_{n,r}a^{\dagger}_{n',r'} \mid 0 >= \delta_{n,n'}\delta_{r,r'} \mid 0 > . \tag{32}$$

We define a momentum operator and a Hamilton operator, both bilinear in the respective creation and annihilation operators in such a way that they have the right one-particle spectrum:

$$P = \pi_0 \sum_{n,r} q^n r a^{\dagger}_{n,r} a_{n,r}, \qquad H = \frac{\pi_0^2}{2} \sum_{n,r} q^{2n} a^{\dagger}_{n,r} a_{n,r}. \tag{33}$$

These operators can be expressed in terms of the operators c, c^* as well:

$$\begin{aligned} P &= \sum_{k,r} r q^{-2k} \left\{ c^{\dagger}_{2k,r}(c_{2k+1,r} - q c_{2k-1,r}) + (c^{\dagger}_{2k+1,r} - q c^{\dagger}_{2k-1,r}) c_{2k,r} \right\} \\ H &= \frac{\pi_0^2}{2} \sum_{k,r} q^{-2k} c^{\dagger}_{k,r}(c_{k,r} - q^3 c_{k-1,r} - q^{-1} c_{k+1,r}). \end{aligned} \tag{34}$$

The simplest way to define second quantization is by imposing the "canonical" quantization condition:

$$[c_{k,r}, c^{\dagger}_{k',r'}] = \delta_{k,k'}\delta_{r,r'}, \tag{35}$$

which implies:

$$[a_{n,r}, a^{\dagger}_{n',r'}] = \delta_{n,n'}\delta_{r,r'}. \tag{36}$$

This renders the momentum and energy density of (34) as "quasi" local objects:

$$\begin{aligned} \mathcal{P} &= \sum_{r} r q^{-2k} \left\{ c^{\dagger}_{2k,r}(c_{2k+1,r} - q c_{2k-1,r}) + (c^{\dagger}_{2k+1,r} - q c^{\dagger}_{2k-1,r}) c_{2k,r} \right\} \\ \mathcal{H} &= \frac{\pi_0^2}{2} \sum_{r} q^{-2k} c^{\dagger}_{k,r}(c_{k,r} - q^3 c_{k-1,r} - q^{-1} c_{k+1,r}). \end{aligned} \tag{37}$$

By "quasi local" we mean that they commute when they have no lattice points in common. If the system has an additional quantum symmetry, commutation relations like (35) or (36) will not be covariant.

The commutation relations will have to be deformed as well. Again, the simplest example of this kind is:

$$\begin{aligned} &c_k c_l = c_l c_k, \quad c_k c^{l\dagger} = \hat{R}^{ls}_{kt} c^{t\dagger} c_s \\ &\hat{R}^{ls}_{kt} = \delta^l_t \delta^s_k (1 + (\tilde{q} - 1)\delta^l_k). \end{aligned} \tag{38}$$

The matrix $\hat{R}$ is a solution of the Yang-Baxter equation. The index of the creation operator has been raised to make the formal structure of equ. (38)

agree with the q-deformed differential calculus, as it was introduced in ref.[1]. All the consistency conditions of this calculus are satisfied with the $\hat{R}$ of (38). We introduced the $\tilde{q}$ parameter as there is no connection with the q-parameter from before in this model. The relations (35) are obtained for $\tilde{q} = 1$. With this $\tilde{q}$-deformation of the canonized commutation relations, momentum and energy density remain quasi local.

The a,$a^\dagger$ relations are not local in momentum space any more. An explicit calculation, using (17), gives:

$$
\begin{aligned}
&\mathrm{a}_{2\mathrm{m},\mathrm{r}}\mathrm{a}^{\dagger 2\mathrm{n},\mathrm{s}} = \delta^{\mathrm{n}}_{\mathrm{m}}\delta^{\mathrm{s}}_{\mathrm{r}} + \mathrm{a}^{\dagger 2\mathrm{n},\mathrm{s}}\mathrm{a}_{2\mathrm{n},\mathrm{r}} + \tfrac{1}{8}\mathrm{N}(\tilde{\mathrm{q}}-1)\textstyle\sum_{\mathrm{k},\mu,\nu}\mathrm{q}^{4\mathrm{k}+\mathrm{m}+\mathrm{n}+\mu+\nu} \qquad (39)\\
&\Big\{ [a^{\dagger 2\nu,+}a_{2\mu,+} + a^{\dagger 2\nu,-}a_{2\mu,-}][\cos[k+m]\cos[k+n]\cos[k+\mu]\cos[k+\nu]\\
&\qquad + rs[\sin[k+m]\sin[k+n]\sin[k+\mu]\sin[k+\nu]]\\
&\quad + [a^{\dagger 2\nu,+}a_{2\mu,-} + a^{\dagger 2\nu,-}a_{2\mu,+}][\cos[k+m]\cos[k+n]\cos[k+\mu]\cos[k+\nu]\\
&\qquad - rs[\sin[k+m]\sin[k+n]\sin[k+\mu]\sin[k+\nu]]\\
&\quad + [a^{\dagger 2\nu+1,+}a_{2\mu+1,+} + a^{\dagger 2\nu+1,-}a_{2\mu+1,-}]\\
&[\cos[k+m]\cos[k+n]\sin[k+\mu]\sin[k+\nu]\\
&\qquad + q^2 rs[\sin[k+m]\sin[k+n]\cos[k+\mu]\cos[k+\nu]]\\
&\quad + [a^{\dagger 2\nu+1,+}a_{2\mu+1,-} + a^{\dagger 2\nu+1,-}a_{2\mu+1,+}]\\
&[-\cos[k+m]\cos[k+n]\sin[k+\mu]\sin[k+\nu]\\
&\qquad + q^2 rs[\sin[k+m]\sin[k+n]\cos[k+\mu]\cos[k+\nu]]\Big\}\,.
\end{aligned}
$$

and similar relations for a, $a^\dagger$ with odd indices. These relations can again be written in $\hat{R}$-matrix notation. It yields again a solution of the Yang-Baxter equation and meets all the consistency conditions of ref.[1].

If we had started from "locally deformed" relations in momentum space, we would have obtained non-local relations in coordinate space - there would have been no local expression for the momentum and energy density. This example clearly shows how difficult it will be to handle the locality requirement in q-deformed theories - a problem far from being solved.

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NONCOMMUTATIVE DIFFERENTIAL CALCULUS: QUANTUM GROUPS, STOCHASTIC PROCESSES AND ANTIBRACKET

ARISTOPHANES DIMAKIS
Department of Mathematics, University of Crete
GR-71409 Iraklion, Greece

and

FOLKERT MÜLLER-HOISSEN
Institut für Theoretische Physik, Universität Göttingen
Bunsenstrasse 9, D-37073 Göttingen, Germany

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Abstract. We explore a differential calculus on the algebra of C^∞-functions on a manifold. The former is 'noncommutative' in the sense that functions and differentials do not commute, in general. Relations with bicovariant differential calculus on certain quantum groups and stochastic calculus are discussed. A similar differential calculus on a superspace is shown to be related to the Batalin-Vilkovisky antifield formalism.

Key words: Noncommutative geometry, quantum groups, stochastic differential equations, antifield formalism

1. Introduction

Since Connes' work on noncommutative geometry, the notion of differential calculus on algebras has entered the realm of physics through numerous publications. As the commutative algebra of ($\mathbb{C}$-valued) functions on a topological space carries all the information about the space in its algebraic structure, certain noncommutative algebras may be regarded as a generalization of the notion of a 'space'. If the algebra $\mathcal{A}$ is associative, one can enlarge it to a differential algebra, a kind of analogue of the algebra of differential forms on a differentiable manifold.

More precisely, this is a $\mathbb{Z}$-graded associative algebra $\bigwedge(\mathcal{A}) = \bigoplus_{r\geq 0} \bigwedge^r(\mathcal{A})$ where $\bigwedge^0 = \mathcal{A}$. The spaces $\bigwedge^r(\mathcal{A})$ of *r-forms* are generated as $\mathcal{A}$-bimodules via the action of an *exterior derivative* $d : \bigwedge^r(\mathcal{A}) \to \bigwedge^{r+1}(\mathcal{A})$ which is a linear operator acting in such a way that $d^2 = 0$ and $d(\omega\omega') = (d\omega)\,\omega' + (-1)^r \omega\, d\omega'$ (where ω and ω' are r- and r'-forms, respectively). Without further restrictions, $\bigwedge(\mathcal{A})$ is the so-called *universal differential envelope* of $\mathcal{A}$. It associates, for example, independent differentials with $f \in \mathcal{A}$ and f^2.

What we would rather like to have is a closer analogue of the algebra of differential forms on a manifold. In particular, if $\mathcal{A}$ is generated by a set of n elements (e.g., coordinate functions x^i on a manifold), we might want the space of 1-forms to be generated as a left- (or right-) $\mathcal{A}$-module by the differentials dx^i. In order to achieve this, one has to add commutation rules for functions and differentials to the differential algebra structure defined above. In case of the commutative algebra of C^∞-functions on a manifold, the ordinary calculus of differential forms simply assumes that 1-forms and functions commute. If, however, $\mathcal{A}$ is the algebra of functions on a discrete set, this assumption cannot be kept. The algebra of functions on a two-point set, for example, is generated by a function y such that $y^2 = 1$. Acting with d on this relation yields $y\, dy = -dy\, y$ and thus *anti*-commutativity. In this example the commutation relation is *not* an additional assumption, but follows from the general rules of differential calculus. This is a special feature of the two-point space. This example plays a crucial role in models of elementary particle physics [1]. Here we just take it to illustrate what we mean by 'noncommutative differential calculus', namely noncommutativity between functions and differentials.

Let $\mathcal{A}$ be the set of functions on $\mathbb{R}$ generated by a coordinate function x (and a unit element which we identify with $1 \in \mathbb{C}$). The simplest consistent deformation of the ordinary differential calculus is then determined by $[x, dx] = a\, dx$ where a is a positive real constant. If we define partial derivatives by $df = \overleftarrow{\partial} f\, dx = dx\, \overrightarrow{\partial} f$, they turn out to be (left- and right-) *discrete* derivatives. An integral is naturally associated with d and (for the higher-dimensional generalization of the calculus) it turns out that the deformation from $a = 0$ to $a > 0$ transforms continuum theories (like a gauge theory) to the corresponding lattice theory (where a plays the role of the lattice spacing) [2]. A simple coordinate transformation brings the above commutation relation into the form $y\, dy = q\, dy\, y$ with $q \in \mathbb{C}$, the differential calculus underlying q-calculus [3]. This noncommutative differential calculus is the best understood and most complete example so far. We can also introduce it on the space of functions on a lattice with spacings a instead of $\mathcal{A}$. More generally, differential calculus on discrete sets is supposed to be of relevance for approaches towards discrete field theory and geometry (see [4] and the references given there).

Another interesting example of a noncommutative differential calculus on a commutative algebra is the following [5, 6]. Let $\mathcal{A}$ be the algebra of C^∞-functions on a manifold $\mathcal{M}$ and let us assume the following commutation relations expressed in terms of local coordinates x^i:

$$[x^i, dx^j] = \gamma\, g^{ij}\, dt \tag{1.1}$$

where γ is a constant, g a real symmetric tensor (e.g., a metric) on $\mathcal{M}$, and t an 'external' (time) parameter. The above commutation relation is actually coordinate independent. The differential calculus based on it is related to quantum mechanics [5] and stochastics [6] (depending on whether γ is imaginary or real), and to 'proper time' (quantum) theories [5]. A generalization of (1.1) is obtained by replacing $\gamma\, dt$ by a 1-form τ, i.e.

$$[x^i, dx^j] = \tau\, g^{ij} \tag{1.2}$$

where τ should have the following properties,

$$[x^i, \tau] = 0 \quad , \quad \tau\,\tau = 0 \quad , \quad d\tau = 0\,. \tag{1.3}$$

This structure in fact shows up in the classical limit ($q \to 1$) of (bicovariant [7]) differential calculus on certain quantum groups [8]. For functions $f, h \in \mathcal{A}$, we have

$$[f, dh] = \tau\,(f,h)_g \quad , \quad (f,h)_g := g^{ij}\, \partial_i f\, \partial_j h \tag{1.4}$$

where $\partial_i := \partial/\partial x^i$. In sections 2-5, a brief introduction to various aspects of this differential calculus is given. Some of the results, in particular in sections 3 and 5, have not been published before.

Sections 6 and 7 present basically new results. We introduce a differential calculus on a superspace and show that the antibracket and the Δ-operator of the Batalin-Vilkovisky formalism [9] (developed for quantization of gauge theories) appear naturally in this framework. A corresponding generalization of gauge theory is also formulated. The differential calculus is a kind of superspace counterpart of the abovementioned differential calculus on manifolds.

Our work establishes relations between noncommutative differential calculus and various mathematical structures which play a role in physics. The latter are thus put into a new perspective which will hopefully contribute to an improved understanding and handling of these structures.

2. The classical limit of bicovariant differential calculi on the quantum groups $GL_q(2)$ and $SL_q(2)$

Let us denote the entries of a $GL(2)$-matrix as follows,

$$M = \begin{pmatrix} x^1 & x^2 \\ x^3 & x^4 \end{pmatrix} . \tag{2.1}$$

Let $\mathcal{A}$ be the algebra of polynomials in x^i. The quantum group $GL_q(2)$ is a noncommutative deformation of $\mathcal{A}$ as a Hopf algebra. The structure of a quantum group allows to narrow down the many possible differential calculi on it. This results in the notion of *bicovariant* differential calculus [7]. For $GL_q(2)$ there is a 1-parameter set of bicovariant differential calculi. In the classical limit $q \to 1$ they lead [8, 6] to the commutation relations (1.2) with

$$g^{ij} = (\det M)^{-1} x^i x^j + 4 (\delta_2^{(i} \delta_3^{j)} - \delta_1^{(i} \delta_4^{j)}) \tag{2.2}$$
$$\tau = s (dx^1 x^4 - dx^2 x^3 - dx^3 x^2 + dx^4 x^1) \tag{2.3}$$

where s is a free parameter. The ordinary differential calculus on $GL(2)$ is only obtained when $s = 0$.

The condition for the matrix M to be in $SL(2)$ is the quadratic equation

$$\det M = x^1 x^4 - x^2 x^3 = 1 . \tag{2.4}$$

Compatibility of the analogous condition for the quantum group $SL_q(2)$ with bicovariant differential calculus restricts the parameter s to only two values (both different from zero) [8]. There are thus only two bicovariant differential calculi on $SL_q(2)$ and for both the classical limit is not the ordinary differential calculus. We will only consider one of them here. In a cordinate patch where $x^1 \neq 0$ we can use x^a, $a = 1,2,3$, as coordinates. The differential calculus is then determined by (1.2) with

$$g^{ab} = x^a x^b + 4 \delta_2^{(a} \delta_3^{b)} \tag{2.5}$$
$$\tau = \frac{1}{3} (dx^1 x^4 - dx^2 x^3 - dx^3 x^2 + dx^4 x^1) \tag{2.6}$$

where $x^4 = (1 + x^2 x^3)/x^1$. Although we only have *three* independent coordinates in this case, the space of 1-forms (as a left or right $\mathcal{A}$-module) is *four*-dimensional since τ cannot be expressed as $\tau = \sum_{a=1}^3 dx^a f_a$ with $f_a \in \mathcal{A}$. What's going on here is explained in more detail in the following section, using a simple example.

3. Differential calculi on quadratic varieties

Let x^i, $i = 1, \ldots, n$, be real variables, α_{ij} a nondegenerate symmetric constant form with inverse α^{ij}. We want to construct a noncommutative differential calculus with (1.2) and (1.3), compatible with the quadratic relation

$$\alpha_{ij} x^i x^j = 1 . \tag{3.1}$$

The $SL(2)$-condition (2.4) provides us with a particular example. Acting with d on (3.1) and using (1.2), we obtain

$$\tau = dx^i (-2 \alpha_{ij} x^j / a) =: dx^i \tau_i \tag{3.2}$$

where we have assumed that $a := \alpha_{ij} g^{ij} \neq 0$. The condition $[x^i, \tau] = 0$ implies

$$g^{ij} \tau_j = 0 . \tag{3.3}$$

It is natural to look for an expression for g^{ij} in terms of α^{ij} and the coordinates x^i. We are then led to the following solution of the last equation:

$$g^{ij} = x^i x^j - \alpha^{ij} . \tag{3.4}$$

From this we find $a = 1 - n$. In the $SL(2)$ case, we recover (2.2) and (2.3) with the correct restriction on the parameter, i.e. $s = 1/3$.

Example: Consider two variables x, y subject to the quadratic relation

$$x\, y = 1 . \tag{3.5}$$

We thus have $n = 2$, $\alpha_{ij} = (1/2)(\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1})$ and

$$(g^{ij}) = \begin{pmatrix} x^2 & -1 \\ -1 & y^2 \end{pmatrix} . \tag{3.6}$$

Furthermore, $\tau = dx\, y + dy\, x$. In the case under consideration, (1.2) is a system of four equations. Three of them are redundant, however, since they are consequences of

$$[x, dx] = \tau\, x^2 . \tag{3.7}$$

Although we have only one free coordinate (x), the 1-forms dx and τ are independent in the sense that $\tau = dx\,(1/x) - (1/x)\,dx$ cannot be expressed as $f(x)\,dx$ or $dx\, f(x)$. The space of 1-forms is therefore two-dimensional (as a left or right $\mathcal{A}$-module, where $\mathcal{A}$ is now the algebra of functions of x). We can use the expression for τ to eliminate τ from (3.7). This results in the equation $x\, dx - 2\, dx\, x + (1/x)\, dx\, x^2 = 0$ which is insufficient to transform the $\mathcal{A}$-bimodule of 1-forms into a left (or right) $\mathcal{A}$-module.

4. A generalized gauge theory and 'second order differential geometry'

It is rather straightforward to formulate a generalization of gauge theory and differential geometry using the 'deformed' differential calculus on $\mathcal{A} = C^\infty(\mathcal{M})$ with (1.2) and (1.3) (see also [5]). It should be noticed, however, that – as a consequence of the deformation – the differential of a function f is now given by

$$df = \tau\, \frac{1}{2}\, g^{ij}\, \partial_i \partial_j f + dx^i\, \partial_i f \tag{4.1}$$

and involves a *second* order differential operator. If a (space-time) metric is given, it is natural to identify it with g^{ij}.

Let ψ be an element of $\mathcal{A}^n$ which transforms as $\psi \mapsto \psi' = U\,\psi$ under a representation of a Lie group G. For local transformations we can construct a covariant derivative in the usual way,

$$D\psi = d\psi + A\,\psi . \tag{4.2}$$

This is indeed covariant if the 1-form A transforms according to the familiar rule

$$A' = U\,A\,U^{-1} - dU\,U^{-1}\,. \tag{4.3}$$

In the following we will only consider the case where the coordinate differentials dx^i and the 1-form τ are linearly independent and form a basis of the space of 1-forms (as a left or right $\mathcal{A}$-module). A can then be written in a unique way as

$$A = \tau\,\frac{1}{2}\,A_\tau + dx^i\,A_i\,. \tag{4.4}$$

Inserting this expression in (4.3), we find that A_i behaves as an ordinary gauge potential and

$$A_\tau = g^{ij}\,(\partial_i A_j - A_i A_j) + M \tag{4.5}$$

where M is an arbitrary tensorial part ($M' = UMU^{-1}$). Since U depends on x^i, in general, it does not commute with dx^j. It is convenient to introduce the gauge-covariant differential $Dx^i := dx^i - \tau\,A^i$. The covariant derivative of ψ can now be written as

$$D\psi = \tau\,\frac{1}{2}\,(g^{ij}\,D_i D_j + M)\,\psi + Dx^i\,D_i\psi \tag{4.6}$$

where D_i denotes the ordinary covariant derivative (with A_i). The field strength of A is

$$F = dA + A^2 = \tau\,\frac{1}{2}\,(D^*F - DM) + \frac{1}{2}\,Dx^i\,Dx^j\,F_{ij} \tag{4.7}$$

where $D^*F = dx^i\,D^jF_{ji}$ involves the Yang-Mills operator (when g^{ij} is identified with the space-time metric). F_{ij} is the (ordinary) field strength of A_i.

If τ behaves as a scalar and g^{ij} as a contravariant tensor under coordinate transformations, the defining relations of our differential calculus – and in particular (1.2) – are coordinate independent [5, 6]. The coordinate differentials dx^i do not transform covariantly, however, since

$$dx'^k = \tau\,\frac{1}{2}\,g^{ij}\partial_i\partial_j x'^k + dx^\ell\,\partial_\ell x'^k \tag{4.8}$$

as a consequence of (4.1). For a vector field Y^i we introduce a (right-) covariant derivative

$$DY^i := dY^i + Y^j\,{}_j\Gamma^i\,. \tag{4.9}$$

This is indeed *right*-covariant iff the generalized connection ${}_j\Gamma^i$ is given by

$${}_j\Gamma^i = \tau\,\frac{1}{2}\,\left[g^{k\ell}(\partial_k\Gamma^i{}_{j\ell} + \Gamma^i{}_{mk}\Gamma^m{}_{j\ell}) + M^i{}_j\right] + dx^k\,\Gamma^i{}_{jk} \tag{4.10}$$

where ${\Gamma^i}_{jk}$ are the components of an ordinary linear connection on $\mathcal{M}$ and ${M^i}_j$ is a tensor. Let us introduce the *right*-covariant 1-forms

$$Dx^k := dx^k + \tau \, \frac{1}{2} \, {\Gamma^k}_{ij} g^{ij} \, . \tag{4.11}$$

(4.1) can now be rewritten as

$$df = \tau \, \frac{1}{2} \, g^{ij} \nabla_i \nabla_j f + Dx^i \, \partial_i f \tag{4.12}$$

where ∇_i denotes the ordinary covariant derivative. Also the covariant exterior derivative of Y^i can now be written in an explicitly right-covariant form,

$$DY^i = \tau \, \frac{1}{2} \, (g^{k\ell} \nabla_k \nabla_\ell Y^i + {M^i}_j Y^j) + Dx^j \, \nabla_j Y^i \, . \tag{4.13}$$

It is interesting that the (covariant) exterior derivative of a field contains in its τ-part the corresponding part of the field equation to which it is usually subjected in physical models. We refer to [5] for further results.

5. Stochastic differential calculus

When $\tau = \gamma \, dt$ as in (1.1), we may consider (smooth) functions $f(x^i, t)$ depending also on the parameter t. (4.1) then has to be replaced by

$$df = dt \, (\partial_t + \frac{\gamma}{2} \, g^{ij} \, \partial_i \partial_j) \, f + dx^i \, \partial_i f \, . \tag{5.1}$$

Such a formula is wellknown in the theory of stochastic processes (Itô calculus) [10] and suggests that our noncommutative differential calculus provides us with a convenient framework to deal with stochastic processes on manifolds. There is indeed a kind of translation [6] to the (Itô) calculus of stochastic differentials. This can be used to carry the *expectation* map from the latter over to our calculus. In this section, we introduce an expectation $\mathbf{E}$ on the (first order) differential calculus in a more formal way. It is then shown for a specific example, that our rules reproduce familiar results.

Let us consider the equation (1.1) in one dimension (for simplicity). We write it in the form

$$[X_t, dX_t] = dt \tag{5.2}$$

viewing X_t as a process on $\mathbb{R}$, a map $\mathbb{R} \times [0, \infty) \to \mathbb{R}$. $\mathcal{A}$ denotes the algebra of smooth functions of X_t and t, and $\mathcal{F}$ the subalgebra of functions of t only. Let $\mathbf{E}$ be an $\mathcal{F}$-linear map $\mathcal{A} \to \mathcal{F}$ which is the identity on $\mathcal{F}$. We extend it to 1-forms as an $\mathcal{F}$-linear map via

$$\mathbf{E}\, df_t = d(\mathbf{E} f_t) \quad , \quad \mathbf{E}(dX_t\, f_t) = 0 \qquad (\forall f_t \in \mathcal{A}) \,. \tag{5.3}$$

On the rhs of the first equation in (5.3), d is the ordinary exterior derivative. The second equation can be interpreted by saying that, given f_t, a further increment dX_t is statistically independent (i.e., f_t is 'nonanticipating'). Then, as a consequence of (5.2), $\mathbf{E}(f_t\, dX_t)$ does *not* vanish, in general. Here we should view f_t as evaluated *after* a time step dt with increment dX_t in X_t.

Example: (Ornstein-Uhlenbeck process)
Let us consider the differential equation

$$dY_t = -k\, dt\, Y_t + \sigma\, dX_t \tag{5.4}$$

with constants k, σ. For $\mathbf{E}Y_t$ we obtain from (5.4) the ordinary differential equation

$$d\mathbf{E}Y_t = -k\, \mathbf{E}Y_t\, dt \tag{5.5}$$

with the solution $\mathbf{E}Y_t = \mathbf{E}Y_0\, e^{-kt}$. Let us now show how to calculate higher moments. With

$$[Y_t, dY_t] = \sigma\, [Y_t, dX_t] = \sigma\, [X_t, dY_t] = \sigma^2\, dt \,. \tag{5.6}$$

we find

$$\begin{aligned} d(Y_t^2) &= dY_t\, Y_t + Y_t\, dY_t = 2\, dY_t\, Y_t + \sigma^2\, dt \\ &= 2\,\sigma\, dX_t\, Y_t + dt\,(\sigma^2 - 2k\, Y_t^2) \end{aligned} \tag{5.7}$$

and, using $\mathbf{E}(dX_t Y_t) = 0$, the ordinary differential equation

$$d(\mathbf{E}Y_t^2) = dt\,(\sigma^2 - 2\,k\, \mathbf{E}Y_t^2) \tag{5.8}$$

for the second moment. The solution is

$$\mathbf{E}Y_t^2 = e^{-2kt}\, \mathbf{E}Y_0^2 + \frac{\sigma^2}{2k}\,(1 - e^{-2kt}) \,. \tag{5.9}$$

If the moments $\mathbf{E}Y_0^n$ are given, we obtain in this way the moments $\mathbf{E}Y_t^n$, $t > 0$. The results are the same as if we treat (5.4) as an (Itô) stochastic differential equation, which is the Ornstein-Uhlenbeck equation (see [10], for example). We have used rather unusual techniques, however, namely a noncommutative differential calculus.

6. A differential calculus on superspace

So far we dealt with a commutative algebra generated by coordinate functions x^i, $i = 1, \dots, n$. In this section we enlarge it to an algebra $\mathcal{A}$ of functions on a superspace by adding odd variables ξ_i and η. Again, we associate with $\mathcal{A}$ a differential algebra $\bigwedge(\mathcal{A})$ via the action of an exterior derivative d. In the case of superalgebras a different version of the Leibniz rule is usually adopted [11],

$$d(\omega\,\omega') = d\omega\,\omega' + \hat{\omega}\,d\omega' \tag{6.1}$$

where the hat denotes the grading involution. This is defined on $\bigwedge(\mathcal{A})$ by $\hat{x}^i = x^i$, $\hat{\xi}_i = -\xi_i$, $\hat{\eta} = -\eta$, $\widehat{d\omega} = -d\hat{\omega}$, $\widehat{\omega\omega'} = \hat{\omega}\hat{\omega}'$ and linearity. In particular, the dx^i are odd and $d\eta$, $d\xi_i$ are even. In the even sector of $\mathcal{A}$, (6.1) coincides with our previous rule, however. We write $[\,,\,]$ for the graded commutator (i.e., $[\omega, \omega'] = \omega\omega' - \omega'\omega$ for ω even and $[\omega, \omega'] = \omega\omega' - \hat{\omega}'\omega$ for ω odd). The universal differential calculus is now restricted by the following relations,

$$[x^i, d\xi_j] = -[\xi_j, dx^i] = d\eta\,\delta^i_j \,. \tag{6.2}$$

The remaining graded commutators between superspace coordinates and their differentials are taken to be zero (so that we have the standard rules in the pure even and odd sectors). This defines a consistent differential calculus where the space of 1-forms is generated as a right (or left) $\mathcal{A}$-module by $dx^i, d\xi_j, d\eta$. The differential of a function f on the superspace can then be expressed as

$$df = d\eta\,\tilde{\partial}_\eta f + dx^i\,\tilde{\partial}_i f + d\xi_i\,\tilde{\zeta}^i f \tag{6.3}$$

where $\tilde{\partial}_\eta$, $\tilde{\partial}_i$, $\tilde{\zeta}^i$ are operators on $\mathcal{A}$. Using (6.1) and the basic commutation relations, we find

$$[dx^i, f] = -d\eta\,\tilde{\zeta}^i f \quad , \quad [d\xi_i, f] = -d\eta\,\tilde{\partial}_i f \,. \tag{6.4}$$

With the help of these relations, the Leibniz rule (6.1) for d now implies

$$\tilde{\partial}_i(fh) = (\tilde{\partial}_i f)\,h + f\,(\tilde{\partial}_i h) \quad , \quad \tilde{\zeta}^i(fh) = (\tilde{\zeta}^i f)\,h + \hat{f}(\tilde{\zeta}^i h) \tag{6.5}$$

$$\tilde{\partial}_\eta(fh) = (\tilde{\partial}_\eta f)\,h + \hat{f}\,(\tilde{\partial}_\eta h) + (\tilde{\zeta}^i f)\,\tilde{\partial}_i h + (\tilde{\partial}^i \hat{f})\,\tilde{\zeta}_i h \,. \tag{6.6}$$

Together with $\tilde{\partial}_i x^j = \delta^j_i = \tilde{\zeta}^j \xi_i$, $\tilde{\partial}_\eta \eta = 1$ (a consequence of (6.3)), this leads to

$$\tilde{\partial}_i = \partial_i := \frac{\partial}{\partial x^i} \,, \; \tilde{\zeta}^i = \zeta^i := \frac{\partial_{(\ell)}}{\partial \xi_i} \,, \; \tilde{\partial}_\eta = \partial_\eta + \Delta := \frac{\partial_{(\ell)}}{\partial \eta} + \zeta^i \partial_i \tag{6.7}$$

(where a subscript (ℓ) indicates that the derivative is taken from the left). Hence

$$df = d\eta\,(\partial_\eta f + \Delta f) + dx^i\,\partial_i f + d\xi_i\,\zeta^i f \,. \tag{6.8}$$

Using (6.4), we obtain

$$[f, dh] = d\eta \, (f, h) \tag{6.9}$$

where on the rhs appears the *antibracket* [9]

$$(f,h) := (\partial_i f)\, \zeta^i h + (\zeta^i \hat{f})\, \partial_i h = \Delta(\hat{f} h) - (\Delta \hat{f})\, h - f \, \Delta h \, . \tag{6.10}$$

The operator Δ satisfies $\Delta^2 = 0$.

The relation (6.9) is very much analogous with the relation (1.4). Of course, we may consider both deformations of the ordinary differential calculus on the superspace simultaneously. In a sense, η is the odd counterpart of t in (1.1).

7. Generalized gauge theory on superspace

We consider again the superspace differential calculus introduced in the preceeding section. Let ψ transform under the action of a (super) group G according to $\psi \mapsto \psi' = U\psi$. With respect to local transformations on the superspace, an exterior covariant derivative can be defined in the usual way as

$$D\psi := d\psi + A\,\psi \tag{7.1}$$

with a connection 1-form A. It is indeed covariant, i.e. $D'\psi' = \hat{U}\, D\psi$, if

$$A' = \hat{U}\, A\, U^{-1} - dU\, U^{-1} \, . \tag{7.2}$$

Inserting the decomposition

$$A = d\eta\, \alpha + dx^i\, A_i + d\xi_i\, \Lambda^i \tag{7.3}$$

we find

$$A'_i = U\, A_i\, U^{-1} - (\partial_i U)\, U^{-1} \quad , \quad \Lambda'^i = \hat{U}\, \Lambda^i\, U^{-1} - (\zeta^i U)\, U^{-1} \tag{7.4}$$

and

$$\mu' = \hat{U}\, \mu\, U^{-1} - (\partial_\eta U)\, U^{-1} \quad , \quad \mu := \alpha + \hat{A}_i\, \Lambda^i - \zeta^i A_i \, . \tag{7.5}$$

In order to read off gauge covariant components from covariant (generalized) differential forms, we need the following covariantized differentials (cf also section 4),

$$Dx^i := dx^i - d\eta\, \Lambda^i \quad , \quad D\xi_i := d\xi_i - d\eta\, \hat{A}_i \, . \tag{7.6}$$

Their transformation rule is

$$D'x^i = \hat{U}\, Dx^i\, U^{-1} \quad , \quad D'\xi_i = \hat{U}\, D\xi_i\, \hat{U}^{-1} \, . \tag{7.7}$$

Now we find

$$D\psi = d\eta\,(D_\eta\psi + \Gamma^i D_i\psi) + Dx^i\, D_i\psi + D\xi_i\, \Gamma^i\psi \tag{7.8}$$

where

$$D_\eta := \partial_\eta + \mu \quad , \quad D_i := \partial_i + A_i \quad , \quad \Gamma^i := \zeta^i + \Lambda^i\,. \tag{7.9}$$

The operator $\Gamma^i D_i$ (the covariantized Δ) which appears in (7.8) is a ***generalization of the Dirac operator***. If a metric tensor g^{ij} is given and $\zeta^i U = 0$, we can choose $\Lambda^i = g^{ij}\xi_j = \xi^i$ so that $\Gamma^i = \zeta^i + \xi^i$ and

$$\Gamma^i\Gamma^j + \Gamma^j\Gamma^i = 2\,g^{ij} \tag{7.10}$$

which is the Clifford algebra relation. In this case, $\Gamma^i D_i$ is indeed the Dirac operator.

More generally, we have the following relations between transformation properties and exterior covariant derivatives,

$$\begin{array}{lllll}
\psi \mapsto U\psi & \Rightarrow & D\psi = d\psi + A\,\psi & \mapsto & \hat{U}\,D\psi \\
\psi \mapsto \hat{U}\psi & \Rightarrow & D\psi = d\psi - \hat{A}\,\psi & \mapsto & U\,D\psi \\
\psi \mapsto \psi U^{-1} & \Rightarrow & D\psi = d\psi - \hat{\psi}\,A & \mapsto & D\psi\,U^{-1} \\
\psi \mapsto \psi\hat{U}^{-1} & \Rightarrow & D\psi = d\psi + \hat{\psi}\,\hat{A} & \mapsto & D\psi\,\hat{U}^{-1}\,.
\end{array} \tag{7.11}$$

The curvature 2-form of the connection A is given by

$$F := dA - \hat{A}\,A\,. \tag{7.12}$$

We will leave the further investigation of this calculus to a separate work.

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CARTAN CALCULUS ON QUANTUM LIE ALGEBRAS

PETER SCHUPP, PAUL WATTS and BRUNO ZUMINO
Department of Physics
and
Theoretical Physics Group
Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720

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Abstract. A generalization of the differential geometry of forms and vector fields to the case of quantum Lie algebras is given. In an abstract formulation that incorporates many existing examples of differential geometry on quantum spaces we combine an exterior derivative, inner derivations, Lie derivatives, forms and functions all into one big algebra, the "Cartan Calculus".

Key words: Quantum Groups — Differential Geometry — Lie Algebras

1. Introduction

The central idea behind Connes' Universal Calculus (Connes, 1985) in the context of non-commutative geometry was to retain the classical differential geometric properties of $\mathbf{d}$, *i.e.* nilpotency and the undeformed Leibniz rule: $\mathbf{d}\alpha = \mathbf{d}(\alpha) + (-1)^p \alpha \mathbf{d}$ for any p-form α.

We use parentheses to delimit operations like $\mathbf{d}$, $\boldsymbol{i}_x$ and $\mathcal{L}_x$, *e.g.* $\mathbf{d}a = \mathbf{d}(a) + a\mathbf{d}$. However, if the limit of the operation is clear from the context, we will suppress the parentheses, *e.g.* $\mathbf{d}(\boldsymbol{i}_x \mathbf{d}a) \equiv \mathbf{d}(\boldsymbol{i}_x(\mathbf{d}(a)))$.

Here we want to base the construction of a differential calculus on quantum groups on two additional classical formulas: to extend the definition of a Lie derivative from functions and vector fields to forms we postulate

$$\mathcal{L} \circ \mathbf{d} = \mathbf{d} \circ \mathcal{L}; \tag{1}$$

this is essential for a geometrical interpretation of vector fields. The second formula that we can — somewhat surprisingly — keep undeformed in the quantum case is

$$\mathcal{L}_{\chi_i} = \boldsymbol{i}_{\chi_i}\mathbf{d} + \mathbf{d}\boldsymbol{i}_{\chi_i}, \qquad \textit{(Cartan Identity)} \tag{2}$$

where $\{\chi_i\}$ are the generators of some quantum Lie algebra.

2. Quantum Lie Algebras

A quantum Lie algebra is a Hopf algebra $\mathcal{U}$ with a finite-dimensional biinvariant subvector space $\mathcal{T}_q$ spanned by generators $\{\chi_i\}$ with coproduct

$$\Delta\chi_i = \chi_i \otimes 1 + O_i{}^j \otimes \chi_j. \tag{3}$$

More precisely we will call this a quantum Lie algebra of **type II.** Let $\{\omega^j \in \mathcal{T}_q{}^*\}$ be a dual basis of 1-forms corresponding to a set of functions $b^j \in \mathcal{A}$ via $\omega^j \equiv Sb^j_{(1)}\mathrm{d}b^j_{(2)}$; *i.e.*

$$\begin{aligned}
{}_{\mathcal{A}}\Delta(\chi_i) &= 1 \otimes \chi_i, &(4)\\
\Delta_{\mathcal{A}}(\chi_i) &= \chi_j \otimes T^j{}_i, \quad T^j{}_i \in \mathrm{Fun}(G_q), &(5)\\
i_{\chi_i}(\omega^j) &= -< \chi_i, Sb^j >= \delta^j_i, &(6)\\
{}_{\mathcal{A}}\Delta(\omega^i) &= 1 \otimes \omega^i, &(7)\\
\Delta_{\mathcal{A}}(\omega^i) &= \omega^j \otimes S^{-1}T^i{}_j. &(8)
\end{aligned}$$

If the functions b^i also close under adjoint coaction $\Delta^{Ad}(b^i) = b^j \otimes S^{-1}T^i{}_j$, we will call the corresponding quantum Lie algebra one of **type I.**

We can derive two alternate expressions for the exterior derivative of a function from the Cartan identity (2) in terms of these bases

$$\mathbf{d}(f) = \omega^j \pounds_{\chi_j}(f) = -\pounds_{S\chi_j}(f)\omega^j. \tag{9}$$

Combining the two expressions for $\mathbf{d}$ one easily derives the well-known $f-\omega$ commutation relations

$$f\omega^i = \omega^j \pounds_{O_j{}^i}(f). \tag{10}$$

The classical limit is given by $O_j{}^i \rightarrow 1\delta^i_j$, so that forms commute with functions.

3. Generators, Metrics and the Pure Braid Group

How does one go about finding the basis of generators $\{\chi_i\}$ and the set of functions $\{b^i\}$ that define the basis of 1-forms $\{\omega^i\}$? Here we would like to present a method that utilizes pure braid group elements as introduced in (Schupp *et al.*, 1992).

Let us recall that a pure braid element Υ is an element of $\mathcal{U}\hat{\otimes}\mathcal{U}$ that commutes with all coproducts of elements of $\mathcal{U}$, *i.e.*

$$\Upsilon\Delta(y) = \Delta(y)\Upsilon, \qquad \forall y \in \mathcal{U}. \tag{11}$$

Υ maps elements of $\mathcal{A}$ to elements of $\mathcal{U}$ with special transformation properties under the right coaction:

$$\begin{aligned}
&\Upsilon : \mathcal{A} \rightarrow \mathcal{U} : \quad b \mapsto \Upsilon_b \equiv < \Upsilon, b \otimes \mathrm{id} >;\\
&\Delta_{\mathcal{A}}(\Upsilon_b) = \Upsilon_{b_{(2)}} \otimes S(b_{(1)})b_{(3)} = < \Upsilon \otimes \mathrm{id}, \tau^{23}(\Delta^{Ad}(b) \otimes \mathrm{id}) > .
\end{aligned} \tag{12}$$

An element Υ of the pure braid group defines furthermore a bilinear quadratic form on $\mathcal{A}$

$$(\ ,\ ) : \mathcal{A} \otimes \mathcal{A} \rightarrow k : \quad a \otimes b \mapsto (a,b) = - < \Upsilon, a \otimes S(b) > \in k, \tag{13}$$

with respect to which we can construct orthonormal $(b_i, b^j) = \delta_i^j$ bases $\{b_i\}$ and $\{b^j\}$ of functions that in turn will give generators $\chi_i := \Upsilon_{b_i}$ and 1-forms $\omega^j := S(b^j_{(1)})\mathrm{d}b^j_{(2)}$. Typically, one can choose $\mathrm{span}\{b_i\} = \mathrm{span}\{b^j\}$; then one starts by constructing one set, say $\{b_i\}$, of functions that close under adjoint coaction

$$\Delta^{Ad} b_i = b_j \otimes T^j{}_i. \tag{14}$$

If the numerical matrix

$$\eta_{ij} := - < \Upsilon, b_i \otimes S b_j > \qquad \textit{(metric)} \tag{15}$$

is invertible, *i.e.* $\det(\eta) \neq 0$, then we can use its inverse $\eta^{ij} := (\eta^{-1})_{ij}$ to raise indices

$$b^i = b_j \eta^{ji}. \tag{16}$$

This metric is invariant — or T is orthogonal — in the sense

$$\eta_{ji} = \eta_{kl} T^k{}_j T^l{}_i. \tag{17}$$

Once we have obtained a metric η, we can truncate the pure braid element Υ and work instead with:

$$\Upsilon \rightarrow \Upsilon_{trunc} = -S(\chi_i) \otimes \chi^i = -S(\chi_i) \otimes \chi_j \eta^{ji}, \tag{18}$$

which also commutes with all coproducts. Casimir operators can also be constructed from elements of the pure braid group. The truncated pure braid element gives for instance the quadratic casimir:

$$[\cdot \circ \tau \circ (S^{-1} \otimes \mathrm{id})](\Upsilon_{trunc}) = \eta^{ji} \chi_j \chi_i. \qquad \textit{(casimir)} \tag{19}$$

Now we would like to show that we have actually obtained a quantum Lie algebra of type I:

$$- < \chi_i, S b^j > = - < \Upsilon, b_i \otimes S b_k > \eta^{kj} = \eta_{ik} \eta^{kj} = \delta_i^j, \tag{20}$$

$$\Delta_{\mathcal{A}}(\chi_i) = \Upsilon_{b_{i(2)}} \otimes S(b_{i(1)}) b_{i(3)} = \Upsilon_{b_j} \otimes T^j{}_i = \chi_j \otimes T^j{}_i \tag{21}$$

and

$$\Delta^{Ad}(b^i) = b_k \otimes T^k{}_j \eta^{ji} = b_k \otimes \eta^{kl} \eta_{ln} T^n{}_j \eta^{ji} = b^k \otimes S^{-1} T^i{}_k. \tag{22}$$

Note, that Υ has to be carefully chosen to insure the correct number of generators. Furthermore, we still have to check the coproduct of the generators. If they are not of the form $\Delta \chi_i = \chi_i \otimes 1 + O_i{}^j \otimes \chi_j$ then we might still consider a calculus with deformed Leibniz rule.

3.1. Examples

3.1.1. The R-matrix approach

Often one can take $b_i \in \text{span}\{t^n{}_m\}$, where $t^n{}_m$ is a quantum matrix in the defining representation of the quantum group under consideration. If we are dealing with a quasitriangular Hopf algebra with universal $\mathcal{R} \equiv \alpha_i \otimes \beta^i$, a natural choice for the pure braid element is

$$\Upsilon_R = \frac{1}{\lambda}\left(1 \otimes 1 - \mathcal{R}^{21}\mathcal{R}^{12}\right), \tag{23}$$

where the term $\mathcal{R}^{21}\mathcal{R}^{12}$ has been introduced and extensively studied by Reshetikhin & Semenov-Tian-Shansky (1990) and later by Jurčo (1991), Majid (1993) and Schupp, Watts & Zumino (1992). These choices of b_i's and Υ lead to the R-matrix approach to differential geometry on quantum groups. The metric is

$$\eta = - < X_1, St_2 > = \frac{1}{\lambda}\left(1 - \left[\left(R_{21}{}^{-1}\right)^{t_2}\left(R_{12}{}^{t_2}\right)^{-1}\right]^{t_2}\right), \tag{24}$$

where $X_1 =< \Upsilon_R, t_1 \otimes \text{id} >$ and $R_{12} =< \mathcal{R}, t_1 \otimes t_2 >$. In the case of $\text{GL}_q(2)$ we find.

$$\eta_{\text{GL}_q(2)} = \begin{pmatrix} q^{-3} & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & q^{-3} & 0 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix} \tag{25}$$

In its reduced form, this matrix agrees with the metric obtained from quantum traces (see next section) except in the casimir sector $X^1{}_1 + q^{-2}X^2{}_2$. The formulation in terms of the pure braid element has the advantage that it does not require the existence of an element like u that implements the square of the antipode.

Using this metric we recover — as expected — the well-known (Zumino, 1992 and Schupp *et al.*(2), 1992) expression of the exterior derivative $\mathbf{d}$ on functions in terms of the quantum trace over X and the Cartan-Maurer form $\Omega = t^{-1}\mathrm{d}t$:

$$\mathbf{d} = \omega^i \chi_i = \text{tr}_q(\Omega \cdot X) \qquad \textit{(on functions)}. \tag{26}$$

(This follows essentially from $D_2^{-1}\eta_{12} = P_{12}$, where $D =< u, t >$ with $u = S(\beta^i)\alpha_i$ and P is the permutation matrix.)

3.1.2. Trace formula for the metric

Again, in the case where $\mathcal{U}$ is a quasitriangular Hopf algebra, there exists an alternate way of defining a Killing form; let $\rho : \mathcal{U} \to M_n(k)$ be an $n \times n$

matrix representation of $\mathcal{U}$ with entries in k. Define the map $\eta^{(\rho)} : \mathcal{U}\otimes\mathcal{U} \to k$ as

$$\eta^{(\rho)}(x \otimes y) := \mathrm{tr}_\rho(uxy), \tag{27}$$

where $x, y \in \mathcal{U}$, tr_ρ is the trace over the given representation, and u (see above) implements the square of the antipode. The map $\eta^{(\rho)}$ has the following properties:

$$\eta^{(\rho)}(y \otimes x) = \eta^{(\rho)}(x \otimes S^2(y)), \tag{28}$$

$$\eta^{(\rho)}((z_{(1)} \overset{\mathrm{ad}}{\triangleright} x) \otimes (z_{(2)} \overset{\mathrm{ad}}{\triangleright} y)) = \eta^{(\rho)}(x \otimes y)\epsilon(z), \tag{29}$$

for all $x, y, z \in \mathcal{U}$. Respectively, these express the symmetry of $\eta^{(\rho)}$ and its invariance under the adjoint action. In the case when $\mathcal{U}$ is a quantum Lie algebra with generators $\{\chi_i\}$, we can define the Killing metric for the representation ρ as

$$\eta_{ij}^{(\rho)} := \eta^{(\rho)}(\chi_i \otimes \chi_j). \tag{30}$$

3.1.3. The 2-dim quantum euclidean group

This is an example of a quantum Lie algebra that seems to have no universal $\mathcal{R}$ and where the set of functions $\{b_i\}$ does not arise from the matrix elements of some quantum matrix. In (Schupp *et al.*, 1992) we constructed such a set of functions b_0, b_+, b_-, b_1 and a pure braid element $\Upsilon_e = \frac{1}{\lambda}(\Delta c - c \otimes 1)$ from the casimir $c := P_+P_-$ of $\mathrm{e}_q(2)$. Now we can put the new machinery to work and calculate the (invertible) metric

$$\eta_{\mathrm{E}_q(2)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -q^{-2} & 0 \end{pmatrix}, \tag{31}$$

which immediately gives an expression for $\mathbf{d}$ on functions:

$$\mathbf{d} = \omega_0\chi_1 + \omega_1\chi_0 - q^2\omega_+\chi_- - \omega_-\chi_+. \tag{32}$$

4. Calculus of Functions, Vector Fields and Forms

Here we will generalize the Cartan calculus of ordinary *commutative* differential geometry to the case of quantum Lie algebras.

As in the classical case, the Lie derivative of a function is given by the action of the corresponding vector field, *i.e.*

$$\begin{aligned} \pounds_x(a) &= x \triangleright a = a_{(1)} < x, a_{(2)} >, \\ \pounds_x a &= a_{(1)} < x_{(1)}, a_{(2)} > \pounds_{x_{(2)}}. \end{aligned} \tag{33}$$

The action on products is given through the coproduct of x:

$$x \triangleright ab = (x_{(1)} \triangleright a)(x_{(2)} \triangleright b). \tag{34}$$

The Lie derivative along x of an element $y \in \mathcal{U}$ is given by the adjoint action in $\mathcal{U}$:

$$\pounds_x(y) = x \overset{\text{ad}}{\triangleright} y = x_{(1)} y S(x_{(2)}). \tag{35}$$

To find the action of i_{χ_i} we can now attempt to use the Cartan identity (2):

$$\chi_i \triangleright a = \pounds_{\chi_i}(a) = i_{\chi_i}(\mathrm{d}a) + \mathrm{d}(i_{\chi_i} a). \tag{36}$$

The idea is to use this identity as long as it is consistent and modify it if needed.

As the inner derivation i_{χ_i} contracts 1-forms and is zero on 0-forms like a, we find

$$i_{\chi_i}(\mathrm{d}a) = \chi_i \triangleright a = a_{(1)} < \chi_i, a_{(2)} > . \tag{37}$$

Next consider that for any form α,

$$\pounds_{\chi_i}(\mathrm{d}\alpha) = \mathrm{d}(i_{\chi_i}\mathrm{d}\alpha) + i_{\chi_i}(\mathrm{d}\mathrm{d}\alpha) = \mathrm{d}(\pounds_{\chi_i}\alpha) + 0, \tag{38}$$

which shows that Lie derivatives commute with the exterior derivative; $\pounds_{\chi_i}\mathrm{d} = \mathrm{d}\pounds_{\chi_i}$. We will later need to extend this equation to all elements of $\mathcal{U}$: $\pounds_x \mathrm{d} = \mathrm{d}\pounds_x$. From this and (33) we find

$$\pounds_x \mathrm{d}(a) = \mathrm{d}(a_{(1)}) < x_{(1)}, a_{(2)} > \pounds_{x_{(2)}}. \tag{39}$$

To find the complete commutation relations of i_{χ_i} with functions and forms rather than just its action on them, we next compute the action of $\pounds_{\chi_i}$ on a product of functions a, $b \in \mathcal{A}$, *i.e.*

$$\pounds_{\chi_i}(ab) = i_{\chi_i}\mathrm{d}(ab) = i_{\chi_i}(\mathrm{d}(a)b + a\mathrm{d}(b)), \tag{40}$$

and compare with equation (34). Recalling that the χ_i have coproducts of the form $\Delta\chi_i = \chi_i \otimes 1 + O_i{}^j \otimes \chi_j$, $O_i{}^j \in \mathcal{U}$, we obtain

$$i_{\chi_i} a = (O_i{}^j \triangleright a)\, i_{\chi_j} = \pounds_{O_i{}^j}(a)\, i_{\chi_j} \tag{41}$$

if we assume that the commutation relation of i_{χ_i} with $\mathrm{d}(a)$ is of the general form

$$i_{\chi_i}\mathrm{d}(a) = \underbrace{i_{\chi_i}(\mathrm{d}a)}_{\in\mathcal{A}} + \text{“braiding term”} \cdot i_{\chi_i} . \tag{42}$$

A calculation of $\pounds_{\chi_i}(\mathrm{d}(a)\mathrm{d}(b))$ along similar lines gives in fact

$$\begin{aligned} i_{\chi_i}\mathrm{d}(a) &= (\chi_i \triangleright a) - \mathrm{d}(O_i{}^j \triangleright a)\, i_{\chi_j} \\ &= i_{\chi_i}(\mathrm{d}a) - \pounds_{O_i{}^j}(\mathrm{d}a)\, i_{\chi_j}, \end{aligned} \tag{43}$$

and we propose for any p-form α:

$$i_{\chi_i}\alpha = i_{\chi_i}(\alpha) + (-1)^p \pounds_{O_i{}^j}(\alpha)\, i_{\chi_j}. \tag{44}$$

Using the Cartan identity we can derive commutation relations for (Lie) derivatives and functions from equation (41), which can be written in Hopf algebra language as

$$\chi a = a_{(1)} < \chi_{(1)}, a_{(2)} > \chi_{(2)}. \tag{45}$$

This actually defines the product in the cross-product algebra $\mathcal{A}\rtimes\mathcal{U}$ of general vector fields that one obtains by combining the Hopf algebras $\mathcal{A}$ and $\mathcal{U}$; see *e.g.* (Schupp *et al.*, 1992).

4.1. Maurer-Cartan Forms

The most general left-invariant 1-form can be written (Woronowicz 1989)

$$\omega_b := S(b_{(1)})\mathbf{d}(b_{(2)}) = -\mathbf{d}(Sb_{(1)})b_{(2)}, \tag{46}$$

$$\textit{left-invariance:}\quad {}_{\mathcal{A}}\Delta(\omega_b) = S(b_{(2)})b_{(3)} \otimes S(b_{(1)})\mathbf{d}(b_{(4)}) = 1 \otimes \omega_b, \tag{47}$$

corresponding to a function $b \in \mathcal{A}$. If this function happens to be $t^i{}_k$, where $t \in M_m(\mathcal{A})$ is an $m \times m$ matrix representation of $\mathcal{U}$ with $\Delta(t^i{}_k) = t^i{}_j \otimes t^j{}_k$ and $S(t) = t^{-1}$, we obtain the well-known Cartan-Maurer form $\omega_t = t^{-1}\mathbf{d}(t)$. Here is a nice formula for the exterior derivative of ω_b:

$$\mathbf{d}(\omega_b) = -\omega_{b_{(1)}}\omega_{b_{(2)}}. \tag{48}$$

The Lie derivative is

$$\pounds_\chi(\omega_b) = \omega_{b_{(2)}} < \chi, S(b_{(1)})b_{(3)} > . \tag{49}$$

The contraction of left-invariant forms with i_χ is

$$i_\chi(\omega_b) = - < \chi, S(b) > \in k. \tag{50}$$

4.2. Tensor Product Realization of the Wedge

From (49) and (50) we find commutation relations for i_{χ_i} with ω^j,

$$\begin{aligned} i_{\chi_i}\omega^j &= \delta^j_i - \pounds_{O_i{}^k}(\omega^j) i_{\chi_k} \\ &= \delta^j_i - \omega^m < O_i{}^k, S^{-1}(T^j{}_m) > i_{\chi_k}, \end{aligned} \tag{51}$$

which can be used to define the wedge product $\wedge$ of forms as some kind of antisymmetrized tensor product. So far we have suppressed the $\wedge$-symbol; to avoid confusion we will reinsert it in this paragraph. As in the classical case we make an ansatz for the product of two forms in terms of tensor products

$$\omega^i \wedge \omega^j = \omega^i \otimes \omega^j - \hat{\sigma}^{ij}{}_{mn}\omega^m \otimes \omega^n, \tag{52}$$

with as yet unknown numerical constants $\hat{\sigma}^{ij}{}_{mn} \in k$, and define $\boldsymbol{i}_{\chi_i}$ to act on this product by contracting in the first tensor product space, *i.e.*

$$\boldsymbol{i}_{\chi_i}(\omega^j \wedge \omega^k) = \delta^j_i \omega^k - \hat{\sigma}^{jk}{}_{mn}\delta^m_i \omega^n. \tag{53}$$

But from (51) we already know how to compute this, and we find

$$\hat{\sigma}^{ij}{}_{mn} = < O_m{}^j, S^{-1}(T^i{}_n) >, \tag{54}$$

or

$$\begin{aligned} \omega^i \wedge \omega^j &= (I - \hat{\sigma})^{ij}{}_{mn}\omega^m \otimes \omega^n \\ &= \omega^i \otimes \omega^j - \omega^k \otimes \pounds_{O_k{}^j}(\omega^i). \end{aligned} \tag{55}$$

These equations give implicit (anti)commutation relations between the ω^is. Note that $(1 - \hat{\sigma})$ has a sensible classical limit — it becomes $(1 - P)$ where P is the permutation matrix. Using the same method as for ω we can also obtain a tensor product decomposition of products of inner derivations.

Example: Maurer-Cartan Equation

$$\begin{aligned} \mathbf{d}\omega^j &= \mathbf{d}\omega_{b^j} = -\omega_{b^j_{(1)}} \wedge \omega_{b^j_{(2)}} \\ &= -\omega_{S^{-1}(Sb^j_{(1)}b^j_{(3)})} \otimes \omega_{b^j_{(2)}} \\ &= -\omega^k \otimes \omega^l < -S\chi_k, S^{-1}(Sb^j_{(1)}b^j_{(3)}) >< -S\chi_l, b^j_{(2)} > \\ &= -\omega^k \otimes \omega^l < \underbrace{(S^{-1}\chi_k)_{(1)}\chi_l S(S^{-1}\chi_k)_{(2)}}_{S^{-1}\chi_k \overset{\text{ad}}{\triangleright} \chi_l}, Sb^j > \\ &= -f'^{j}_{k}{}_l \omega^k \otimes \omega^l. \end{aligned} \tag{56}$$

In the previous equation we have introduced the adjoint action of a left-invariant vector field on another vector field. A short calculation gives

$$S^{-1}\chi_k \overset{\text{ad}}{\triangleright} \chi_l = \chi_b\chi_c(\delta^c_k\delta^b_l - \hat{\sigma}^{cb}{}_{kl}) = \chi_a < S^{-1}\chi_k, T^a{}_l >= \chi_a f'_k{}^a{}_l \tag{57}$$

as compared to

$$\chi_k \overset{\text{ad}}{\triangleright} \chi_l \stackrel{\iota}{\equiv} \pounds_{\chi_k}(\chi_l) = \chi_b\chi_c(\delta^c_k\delta^b_l - \hat{R}^{cb}{}_{kl}) = \chi_a f_k{}^a{}_l, \tag{58}$$

with $\hat{R}^{cb}{}_{kl} =< O_k{}^b, T^c{}_l >$. The two sets of structure constants are related by $< \chi_k, T^a{}_l >= f_k{}^a{}_l = -f'^{a}_{i}{}_l R^{ij}{}_{kl}$. See (Castellani *et al.* 1993) for a detailed discussion of such structure constants.

4.2.1. The "Anti-Wedge" Operator.

There is actually an operator W that recursively translates wedge products into the tensor product representation:

$$\begin{aligned} &W : \Lambda_q^p \rightarrow \mathcal{T}_q^* \otimes \Lambda_q^{p-1}, \quad p \geq 1, \\ &W(\alpha) = \omega^n \otimes i_{\chi_n}(\alpha), \end{aligned} \tag{59}$$

for any p-form α. For example,

$$\begin{aligned} \omega^j \wedge \omega^k &= \omega^n \otimes i_{\chi_n}(\omega^j \wedge \omega^k) \\ &= \omega^n \otimes (\delta_n^j \omega^k - \mathcal{L}_{O_n{}^m}(\omega^j)\delta_m^k). \end{aligned} \tag{60}$$

4.3. Summary of Relations in the Cartan Calculus

Commutation Relations

For any p-form α:

$$\mathbf{d}\alpha = \mathbf{d}(\alpha) + (-1)^p \alpha \mathbf{d} \tag{61}$$
$$i_{\chi_i}\alpha = i_{\chi_i}(\alpha) + (-1)^p \mathcal{L}_{O_i{}^j}(\alpha) i_{\chi_j} \tag{62}$$
$$\mathcal{L}_{\chi_i}\alpha = \mathcal{L}_{\chi_i}(\alpha) + \mathcal{L}_{O_i{}^j}(\alpha)\mathcal{L}_{\chi_j} \tag{63}$$

Actions

For any function $f \in \mathcal{A}$, 1-form $\omega_f \equiv Sf_{(1)}\mathbf{d}f_{(2)}$ and vector field $\phi \in \mathcal{A} \rtimes \mathcal{U}$:

$$i_{\chi_i}(f) = 0 \tag{64}$$
$$i_{\chi_i}(\mathbf{d}f) = \mathbf{d}f_{(1)} < \chi_i, f_{(2)} > \tag{65}$$
$$i_{\chi_i}(\omega_f) = - < \chi_i, Sf > \tag{66}$$
$$\mathcal{L}_{\chi}(f) = \chi(f) = f_{(1)} < \chi, f_{(2)} > \tag{67}$$
$$\mathcal{L}_{\chi}(\omega_f) = \omega_{f_{(2)}} < \chi, S(f_{(1)})f_{(3)} > \tag{68}$$
$$\mathcal{L}_{\chi}(\phi) = \chi_{(1)}\phi S(\chi_{(2)}) \tag{69}$$

Graded Quantum Lie Algebra of the Cartan Generators

$$\mathbf{dd} = 0 \tag{70}$$
$$\mathbf{d}\mathcal{L}_{\chi} = \mathcal{L}_{\chi}\mathbf{d} \tag{71}$$
$$\mathcal{L}_{\chi_i} = \mathbf{d}i_{\chi_i} + i_{\chi_i}\mathbf{d} \tag{72}$$
$$[\mathcal{L}_{\chi_i}, \mathcal{L}_{\chi_k}]_q = \mathcal{L}_{\chi_l} f_i{}^l{}_k \tag{73}$$
$$[\mathcal{L}_{\chi_i}, i_{\chi_k}]_q = i_{\chi_l} f_i{}^l{}_k \tag{74}$$

The quantum commutator $[\,,\,]_q$ is here defined as follows:

$$[\mathcal{L}_{\chi_i}, \Box]_q := \mathcal{L}_{\chi_i}\Box - \mathcal{L}_{O_i{}^j(\Box)}\mathcal{L}_{\chi_j}. \tag{75}$$

This quantum Lie algebra becomes infinite-dimensional as soon as we introduce derivatives along general vector fields.

4.4. Lie Derivatives Along General Vector Fields

So far we have focused on Lie derivatives and inner derivations along *left-invariant* vector fields, *i.e.* along elements of $\mathcal{T}_q$. The classical theory allows functional coefficients, *i.e.* the vector fields need not be left-invariant. Here we may introduce derivatives along elements in the $\mathcal{A}\rtimes\mathcal{T}_q$ plane by the following set of equations valid on forms: (note: $\epsilon(\chi) = 0$ for $\chi \in \mathcal{T}_q$)

$$\begin{aligned} i_{f\chi} &= f i_\chi, & (76)\\ \mathcal{L}_{f\chi} &= \mathrm{d} i_{f\chi} + i_{f\chi}\mathrm{d}, & (77)\\ \mathcal{L}_{f\chi} &= f\mathcal{L}_\chi + \mathrm{d}(f) i_\chi, & (78)\\ \mathcal{L}_{f\chi}\mathrm{d} &= \mathrm{d}\mathcal{L}_{f\chi}. & (79)\end{aligned}$$

Equation (78) can be used to define Lie derivatives recursively on any form.

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INTEGRALS AND FOURIER TRANSFORMS IN QUANTUM PLANE

CHRYSSOMALIS CHRYSSOMALAKOS and BRUNO ZUMINO
Department of Physics and Theoretical Physics Group Lawrence Berkeley Laboratory University of California, Berkeley, California 94720

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Abstract. We present a formulation of covariant translations in the quantum plane. We are led to an extension of the algebra of the coordinate functions and their dual derivatives by the quantum analogue of their eigenvalues. Jackson exponentials emerge as the corresponding eigenfunctions. An integral invariant under quantum translations is introduced and is used to define quantum Fourier transforms.

Key words: Quantum Plane – Integrals – Quantum Fourier Transform

1. Introduction

Since its inception, the quantum plane has been envisioned by many as a paradigm for the general program of q-deformed physics. Such an endeavour presupposes the availability of adequate mathematical tools, integration being one of the most indispensable among them. The aim of this paper is to address aspects of the problem of integration in the quantum plane in a manner that will keep the results accessible to physicists.

The paper is structured as follows: section 2 introduces translations for the coordinates and the derivatives which are shown to be implemented by a translation operator in the form of a Jackson exponential. Section 3 discusses integration, invariant under the above translations, while in section 4 we define the quantum Fourier transform. We close, in section 5, with the introduction of vacuum projectors which allow us to recover the integration prescription introduced earlier in a constructive way.

2. Translations in the q-plane

We recall now the construction of the quantum plane (Wess *et al.* 1990). We deal in the following with the (non-commutative) algebra of functions on the quantum plane enlarged so as to also include derivatives that operate on these functions. We choose as generators of $\mathcal{A}$ the coordinate functions x^i, $i = 1, \ldots, n$ (together with the unit function 1_x) and the derivatives dual to them, ∂_j, $j = 1, \ldots, n$ (together with the unit 1_∂). A set of consistent commutation relations among the above generators is known:

$$\begin{aligned} x^i x^j &= q^{-1} \hat{R}^{ij}_{kl} x^k x^l \\ \partial_l \partial_k &= q^{-1} \hat{R}^{ij}_{kl} \partial_j \partial_i \\ \partial_k x^i &= \delta^i_k + q \hat{R}^{ij}_{kl} x^l \partial_j . \end{aligned} \tag{1}$$

Here, $\hat{R}$ is an invertible solution of the quantum Yang-Baxter equation:

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}$$

and satisfies the characteristic equation:

$$\hat{R}^2 - \lambda \hat{R} - 1 = 0, \qquad \lambda \equiv q - q^{-1} \tag{2}$$

(this is the $GL_q(n)$ $\hat{R}$-matrix of (Reshetikhin *et al.* 1990). The above commutation relations permit unambiguous ordering of an arbitrary monomial in the x's and ∂'s into any desired order. One can now write down, if one wishes, differential equations for functions of the x's and study for example quantum mechanical systems by solving Schroedinger's equation in deformed space. In doing so, as well as in many other applications, one is sooner or later bound to be confronted with the problem of (spatially) translating functions of the x's. One place, in particular, where this question would certainly manifest itself, would be in the statement of finite translation invariance of any sort of integral one adopts for the quantum plane. One has then to first make more precise the notion of translation - it is natural, for example, to require a certain covariance. In its simplest form, that would be the requirement that the translated coordinates obey the same algebra as the original ones (we would also need, of course, reduction to the correct classical limit $x^i \mapsto x^i + a^i$ as the deformation parameter approaches its classical value). We introduce then a set of "displacements" a^i, $i = 1, \ldots, n$ and require that the $x + a$'s obey the same commutation relations as the

x's. We would also like the displacements to be of "coordinate nature" *i.e.* we postulate $a - a$ commutation relations identical to those of the x's. We are then forced to introduce non-trivial $a - x$ commutation relations. The resulting algebra is:

$$\begin{aligned} a^i a^j &= q^{-1} \hat{R}^{ij}_{kl} a^k a^l \\ x^i a^j &= q \hat{R}^{ij}_{kl} a^k x^l. \end{aligned} \tag{3}$$

One easily checks, with the help of (2), that

$$(x_1 + a_1)(x_2 + a_2) = q^{-1} \hat{R}_{12}(x_1 + a_1)(x_2 + a_2)$$

i.e. translation of the coordinates preserves their algebra (given by (1)). It is interesting to compare (3) with the commutation relations between coordinates and differentials introduced in (Wess *et al.* 1990):

$$x_1 \xi_2 = q \hat{R}_{12} \xi_1 x_2 \,;$$

the displacements a^i are the bosonic analogue of the ξ's (the algebra (3) has been introduced by Majid, in the context of braided Hopf algebras, in (Majid, 1992). We can also give consistent $\partial - a$ commutation relations:

$$\partial_k a^i = q^{-1} (\hat{R}^{-1})^{ij}_{kl} a^l \partial_j \tag{4}$$

(again, similar to the $\partial - \xi$ ones). Consider now the translation generator T defined by:

$$T \equiv a^i \partial_i \equiv a \cdot \partial.$$

Using (3), (4), we easily find:

$$[T, x^i] = a^i, \quad T\partial_i = q^2 \partial_i T, \quad T a^i = q^{-2} a^i T. \tag{5}$$

These allow us to build a finite translation operator by "q-exponentiation". We have:

$$T^n x^i = x^i T^n + [n]_q T^{n-1} a^i \tag{6}$$

where $[n]_q \equiv (1 - q^{2n})/(1 - q^2)$ and therefore:

$$x^i e_q(T) = e_q(T)(x^i - a^i) \tag{7}$$

where:

$$e_q(T) \equiv \sum_{n=0}^{\infty} \frac{1}{[n]_q!} T^n, \quad [n]_q! \equiv [1]_q [2]_q \ldots [n]_q$$

is the Jackson exponential (see for example (Exton, 1983) and references therein). Alternatively, we can write (6) in the form:

$$T^n x^i = x^i T^n + [n]_{q^{-1}} a^i T^{n-1}$$

which gives:

$$e_{q^{-1}}(T) x^i = (x^i + a^i) e_{q^{-1}}(T) \tag{8}$$

or, more generally:

$$e_{q^{-1}}(T) f(x) = f(x + a) e_{q^{-1}}(T). \tag{9}$$

One can regard (7) (or (8)) as an eigenvalue equation for the operator x^i. To make this more precise, we introduce coordinate and derivative vacua, denoted by $|\Omega_x\rangle$ and $|\Omega_\partial\rangle$ respectively, which satisfy:

$$x^i|\Omega_x\rangle = 0, \quad 1_x|\Omega_x\rangle = |\Omega_x\rangle, \quad \partial_i|\Omega_\partial\rangle = 0, \quad 1_\partial|\Omega_\partial\rangle = |\Omega_\partial\rangle$$

with similar relations for x, ∂ acting from the right:

$$\langle\Omega_x|x^i = 0, \quad \langle\Omega_x|1_x = \langle\Omega_x|, \quad \langle\Omega_\partial|\partial_i = 0, \quad \langle\Omega_\partial|1_\partial = \langle\Omega_\partial|.$$

The action of x^i on a function $f(\partial, a)$, denoted by $x^i(f(\partial, a))$, is expressed in terms of the coordinate vacuum as:

$$x^i(f(\partial, a))|\Omega_x\rangle = x^i f(\partial, a)|\Omega_x\rangle. \tag{10}$$

In words, to compute the left hand side of (10), we order it with all the x's on the right, where they anihilate the vacuum, and what remains is termed "the action of x^i on $f(\partial, a)$". We can define the (more familiar) action of the derivatives on functions of x, a in a similar manner. Actions from the right are also obviously defined via "left vacua" $\langle\Omega_x|$, $\langle\Omega_\partial|$. With these (standard) definitions, (7) gives:

$$(e_q(T)) x^i = e_q(T) a^i \tag{11}$$

which suggests the interpretation of a^i as the eigenvalue of x^i (x^i acts here from the right). In the classical limit $q \to 1$, the a's commute with everything. Notice that $e_q(T)$ is a common eigenfunction for all x^i, the noncommutativity of the latter being reflected in the non-trivial $a - a$ commutation relations. It is interesting to note that one can interpret the derivatives ∂_i

the way one does in classical analysis: $\partial_i(f(x))$ is the coefficient of a^i in the expansion of $f(x+a)$ around x (Majid 1993). Indeed, from (9) we have:

$$e_{q^{-1}}(T)(f(x)) = f(x+a) \tag{12}$$

which, by expanding the Jackson exponential, gives:

$$f(x+a) = f(x) + a^i\partial_i(f(x)) + \mathcal{O}(a^2), \tag{13}$$

the only difference in the quantum case being that one has to specify an ordering before identifying the derivative (above, we took the a's to stand to the left of the x's).

The interpretation given to the a^i above naturally leads to the question whether a similar construction is possible for the derivatives. To this end, we introduce the momentum-space analogue of the a's, which we call p_i, $i = 1,\ldots,n$ and find that the following commutation relations are consistent with the rest of the algebra:

$$\begin{aligned} p_l p_k &= q^{-1}\hat{R}^{ij}_{kl}p_j p_i \\ p_l\partial_k &= q\hat{R}^{ij}_{kl}\partial_j p_i \\ p_k x^i &= q^{-1}(\hat{R}^{-1})^{ij}_{kl}x^l p_j \\ p_k a^i &= q^{-1}(\hat{R}^{-1})^{ij}_{kl}a^l p_j. \end{aligned} \tag{14}$$

Several useful identities can now be computed. We give a list involving those that we will need later ($\alpha\cdot\beta \equiv \alpha^i\beta_i$):

$$\begin{aligned} Tp_i &= p_i T \\ x\cdot\partial\, x^i &= x^i + q^2 x^i\, x\cdot\partial \\ x\cdot\partial\, a^i &= a^i\, x\cdot\partial \\ \partial_i\, x\cdot\partial &= \partial_i + q^2 x\cdot\partial\,\partial_i \\ x\cdot\partial\, p_i &= p_i\, x\cdot\partial \\ (x\cdot p)(a\cdot p) &= q^2(a\cdot p)(x\cdot p). \end{aligned} \tag{15}$$

We easily find now how ∂_i commutes with $e_q(x\cdot p)$:

$$\partial_i e_q(x\cdot p) = e_q(x\cdot p)\partial_i + p_i e_q(x\cdot p). \tag{16}$$

Also:

$$\partial_i e_{q^{-1}}(x\cdot p) = e_{q^{-1}}(x\cdot p)\partial_i + e_{q^{-1}}(x\cdot p)p_i. \tag{17}$$

We can therefore interpret the p's as (non-commuting) eigenvalues of the derivatives:

$$\partial_i(e_q(x \cdot p)) = p_i e_q(x \cdot p), \tag{18}$$

Notice that being eigenvalues of derivatives, rather than momenta, the p's become real commuting quantities in the classical limit.

A couple of remarks are in order here. The first regards the covariance of the scheme described above under the coaction of $GL_q(n)$. The commutation relations given in (1), (3), (4) and (14) go into themselves when x, ∂, a and p transform according to:

$$\begin{aligned} x^i \mapsto (x')^i &= T^i_j x^j \\ a^i \mapsto (a')^i &= T^i_j a^j \\ \partial_i \mapsto (\partial')_i &= \partial_j M^j_i \\ p_i \mapsto (p')_i &= \partial_j M^j_i \end{aligned}$$

where T^i_j is a $GL_q(n)$ matrix, $M^t = (T^t)^{-1}$ (M^t denotes the transpose of M) and we take, as in (Wess *et al.* 1990), the elements of T to commute with all the variables and derivatives above. A second point that deserves attention is the fact that derivations do not commute with translations. In general:

$$\partial_i(f(x+a)) \neq \partial_i(f(x))|_{x \mapsto x+a}. \tag{19}$$

This can be traced to the fact that ∂_i does not commute with $a \cdot \partial$. Indeed, in order for (19) to be an equality, we would need (using (13)):

$$\begin{aligned} \partial_i(f(x) + a \cdot \partial(f(x))) &= \partial_i(f(x)) + a \cdot \partial(\partial_i(f(x))) \Rightarrow \\ \Rightarrow \partial_i(a \cdot \partial(f(x))) &= a \cdot \partial(\partial_i(f(x))) \Rightarrow \\ \Rightarrow \partial_i\, a \cdot \partial &= a \cdot \partial\, \partial_i \end{aligned}$$

while, in our case, (5) holds: $a \cdot \partial\, \partial_i = q^2 \partial_i\, a \cdot \partial$. One can make a different choice of commutation relations which will make (19) into an equality but then (13) is not satisfied - we will not explore this further here.

3. Invariant Integration

We wish now to turn our attention to the problem of integration. The integral we are looking for is a linear map from functions on the quantum plane to complex numbers. Keeping in mind the classical limit, we expect it to be defined only for a class $\mathcal{A}_x^I$ of elements of $\mathcal{A}_x$ - we will use the notation $\langle f \rangle$ for the average, or integral, of f in that class. Such a map acquires interest when endowed with specific covariance properies. In our case, it is natural to require invariance under translations. This can be expressed in infinitesimal form as the requirement that the integral of a derivative vanish:

$$\langle\, \partial_i(f(x))\,\rangle = 0, \quad f \in \mathcal{A}_x^I. \tag{20}$$

A prescription for computing such an integral is known (Wess *et al.* 1990). One first expands $f(x)$ in a sum of monomials in the x's and uses the commutation relations to bring each such monomial into some standard ordering (the same for all monomials). Then one performs the classical integral (from minus infinity to plus infinity) of the ordered function - the result is the quantum integral $\langle f \rangle$. Different standard orderings of the x's change only the overall normalization and the result satisfies (20)(notice that ∂_i is the *quantum* derivative). We would like though to be able to talk about finite translation invariance, *i.e.* we would like our integral to satisfy an equation like

$$\langle f(x+a) \rangle = \langle f(x) \rangle. \tag{21}$$

To make this precise, we ought to generalize the prescription for integration given above to the case of a function of x *and* a (since x, a do not commute, such a generalization is not trivial). Nevertheless, the natural approach works: to compute $\langle f(x+a)\rangle$, expand in monomials of x, a, use the commutation relations to move all the a's, say, to the left of each monomial and out of the integral, and then compute the quantum integral of the x's as before (notice that the a's need not be brought into any standard order). That such an integral satisfies (21) can easily be seen as follows. From (12) we have:

$$\langle f(x+a) \rangle \;=\; \langle e_{q^{-1}}(a\cdot\partial)(f(x))\rangle \;=\; \sum_{n=0}^{\infty} \frac{1}{[n]_{q^{-1}}!}\langle (a\cdot\partial)^n (f(x))\rangle\,.$$

We may now use the commutation relations given in (4) to move all the a's in $(a\cdot\partial)^n$ to the left (and then out of the integral). The form of (4) ensures that each term left in the integrand, except for $n=0$, will be the derivative of some function of the x's and the integral of these vanishes by (20); (21) then follows. We should emphasize here that the integral of $f(x,a)$ is not, in general, translationally invariant (*i.e.* while $\langle f(x+a)\rangle = \langle f(x)\rangle$ holds, $\langle f(x+a,a)\rangle \neq \langle f(x,a)\rangle$ in general). In the same spirit, we define the integral $\langle f(x,a,p)\rangle$: we move a and p to the left and then perform quantum integration on the x's - we'll need this in defining the Fourier transform in the next section.

4. Fourier Transform

Armed with the tools developed in the previous section, we are now (almost) ready to discuss Fourier transforms in the quantum plane. The only ingredient missing is the observation that

$$\langle f(x+a,p)\rangle = \langle f(x,p)\rangle, \tag{22}$$

which one can show by noticing that p commutes with T and that the p, x commutation relations are identical to those between p and $x+a$.

We define now the Fourier transform $\tilde{f}(p)$ of a function $f(x)$ by:

$$\tilde{f}(p) \equiv \langle e_q(-ix\cdot p) f(x)\rangle\ . \tag{23}$$

We will need the properties: $e_q(\alpha+\beta) = e_q(\beta)e_q(\alpha)$ for $\alpha\beta = q^2\beta\alpha$ and $e_q(\alpha)e_{q^{-1}}(-\alpha) = 1$ of the Jackson exponential to derive the analogue of a property of Fourier transforms, familiar in the classical case. Setting $f_a(x) \equiv f(x+a)$ we have:

$$\begin{aligned}
\tilde{f}(p) &= \langle e_q(-ix\cdot p)f(x)\rangle \\
&= \langle e_q(-i(x+a)\cdot p)f_a(x)\rangle \\
&= \langle e_q(-ia\cdot p)e_q(-ix\cdot p)f_a(x)\rangle \\
&= e_q(-ia\cdot p)\tilde{f}_a(p). \\
\Rightarrow \tilde{f}_a(p) &= e_{q^{-1}}(ia\cdot p)\tilde{f}(p)
\end{aligned} \tag{24}$$

In the third line we used the last of (15). Notice that, as in the classical case, the factor in front of $\tilde{f}(p)$ is actually a one dimensional representation of translations. Under $x \mapsto x+a$:

$$e_{q^{-1}}(ix\cdot p) \ \mapsto\ e_{q^{-1}}(ix\cdot p + ia\cdot p) \ =\ e_{q^{-1}}(ix\cdot p)e_{q^{-1}}(ia\cdot p).$$

5. Vacuum Projectors

In this section we introduce a "vacuum projector" E which realizes the operator $|\Omega_\partial\rangle\langle\Omega_x|$ (up to a possible normalization factor) in terms of coordinates and derivatives (a similar object, in a Hopf algebra context, has been introduced in (Chryssomalakos *et al.* 1992). It is given by the formal expansion:

$$E = \sum_{k=0}^{\infty} \frac{(-1)^k}{[k]_{q^{-1}}!} E_k. \tag{25}$$

where $E_k \equiv x^{i_1} \dots x^{i_k} \partial_{i_k} \dots \partial_{i_1}$ (x^i is the i-th coordinate). Indeed, one can show, using the easily verifiable commutation relation:

$$E_k x^i = [k]_q x^i E_{k-1} + q^{2k} x^i E_k,$$

that:

$$E x^i = 0, \qquad \partial_i E = 0.$$

As a result, $E^2 = E$. We can now easily realize the projector $|\Omega_x\rangle\langle\Omega_\partial|$ as well. We know from (Wess *et al.* 1990) that $\mathcal{A}$ admits the $*$-involution (which we denote by a bar):

$$\bar{x^i} = x^i, \qquad \bar{\partial}_i = -q^{2(n+1-i)}\partial_i, \qquad \bar{q} = q^{-1}$$

(corresponding to a real quantum plane). It then follows immediately that $\bar{E}$, given explicitly by:

$$\bar{E} = \sum_{k=0}^{\infty} \frac{1}{[k]_{q^{-1}}!} q^{2k(n+1)} q^{-2(i_1+i_2+\dots+i_k)} \bar{E}_k \tag{26}$$

where $\bar{E}_k \equiv \partial_{i_1} \dots \partial_{i_k} x^{i_k} \dots x^{i_1}$, realizes the operator $|\Omega_x\rangle\langle\Omega_\partial|$. An alternative form for E_k, as a function of $x \cdot \partial$, is:

$$E_k = q^{-k(k-1)}(x \cdot \partial)(x \cdot \partial - [1]_q)(x \cdot \partial - [2]_q) \dots (x \cdot \partial - [n-1]_q).$$

One can easily show that $\bar{E}_k$ can also be expressed in terms of $x \cdot \partial$.

The above objects allow us to approach the problem of integration from an alternative point of view. We can use the vacua introduced earlier to define the integral of a function $f(x)$ via:

$$\langle f(x) \rangle = \langle \Omega_\partial | f(x) | \Omega_\partial \rangle. \tag{27}$$

This definition automatically satisfies $\langle \partial_i(f(x)) \rangle = 0$ and therefore it also satisfies $\langle f(x+a) \rangle = \langle f(x) \rangle$. Notice however that in deriving this last invariance property we do not need any ad-hoc rules about how to commute a's (or, for that matter, p's) through the "integral sign". Indeed, choosing the normalization $E = |\Omega_\partial\rangle\langle\Omega_x|$, $\bar{E} = |\Omega_x\rangle\langle\Omega_\partial|$ (which, in turn, implies $\langle\Omega_x|\Omega_\partial\rangle = 1$) (27) gives:

$$\begin{aligned} \bar{E} f(x) E &= |\Omega_x\rangle\langle\Omega_\partial|f(x)|\Omega_\partial\rangle\langle\Omega_x| \\ &= \langle f(x)\rangle|\Omega_x\rangle\langle\Omega_x| \\ &\equiv \langle f(x)\rangle\delta(x). \end{aligned}$$

However, as we have seen, E_k and $\bar{E}_k$ can be expressed as functions of $x \cdot \partial$ only. Refering back to the list given in (15), we see that a^i, p^j commute with $x \cdot \partial$ and this justifies postulating the integration procedure described in section 3.

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$GL_q(N)$-COVARIANT NONCOMMUTATIVE GEOMETRY

ALEXEI P. ISAEV*
Bogolubov Theoretical Laboratory, JINR, Dubna,
SU-101 000 Moscow, Russia

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Abstract. The algebraic formulation of the quantum group noncommutative geometry in the framework of the R-matrix approach to the theory of quantum groups is given. We consider structure groups taking values in quantum groups and introduce the notion of noncommutative connections and curvatures transformed as comodules under the coaction of the structure quantum group $GL_q(N)$. These noncommutative connections and curvatures generate $GL_q(N)$-covariant quantum algebras. For the special case of these algebras we find $GL_q(N)$-invariant composite elements that can be interpreted as noncommutative analogs of the Chern characters. We also present an explicit realization of such covariant algebras considering the coset construction $GL_q(N+1)/GL_q(N)$. In this report, we generalize some results presented in [1].

Noncommutative geometry [2] has started to play a significant role in mathematical physics for the last few years. One of the nontrivial examples of the noncommutative geometry is given by quantum groups [3,4]. The differential geometric aspects of the theory of quantum groups have been intensively investigated recently in the papers [5,6,7]. Using these investigations many approaches to formulate quantum group gauge theories have been developed [1,8,9,10]. In this report we continue the investigations presented in the letter [1] and describe how it is possible to generalize usual commutative geometry and to introduce noncommutative $GL_q(N)$-covariant derivative, $GL_q(N)$-connections (or $GL_q(N)$-gauge fields) and curvature 2-forms. We use the notation of the paper [3] in which the R-matrix formulation of quantum groups has been elaborated. We note also that according to the results of the paper [10] one can reformulate our algebraic constructions for the case of the unitary groups $U_q(N)$.

Let us consider a Z_2-graded finite dimensional Zamolodchikov algebra (denoted by Ω_Z) generated by the operators $\{e^i,\ (de)^j\}$, $(i,j=1,2,\dots,N)$

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with the following commutation relations:

$$\mathbf{R}ee' = cee' \ , \ \ (\pm)c\mathbf{R}(de)e' = e(de)' \ , \ \ \mathbf{R}(de)(de)' = -\frac{1}{c}(de)(de)', \quad (1)$$

where $e = e_1$ is a q-vector in the first space, $e' = e_2$ is a q-vector in the second space, $\mathbf{R} = P_{12}R_{12}$ is a matrix which acts in the first and second spaces simultaneously, $P_{12} = \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}$ is the permutation matrix and R_{12} is the $GL_q(N)$ R–matrix satisfying the Hecke relation: $\mathbf{R}^2 = \lambda\mathbf{R}+1$ ($\lambda = q-q^{-1}$). We imply the wedge product in the multiplication of the differential forms in formulas (1) (we also omit $\wedge$ in all formulas below). One can recognize in the relations (1) (for $(\pm) = +1$) the Wess-Zumino formulas of the covariant differential calculus on the bosonic ($c = q$) and fermionic ($c = -1/q$) quantum hyperplanes [11] where e^i are the coordinates of the quantum hyperplane while $(de)^i$ are the associated differentials. The choice $(\pm) = -1$ corresponds to the case when e^i are bosonic ($c = -1/q$) and fermionic ($c = q$) veilbein 1-forms. Note, that there is a second version of the algebra (1) that can be obtained by the replacement $\mathbf{R} \to \mathbf{R}^{-1}$, $c \to c^{-1}$. Below, we concentrate only on the consideration of the algebra (1) (an other type can be treated analogously).

It has been suggested in [12,1] that the algebra Ω_Z (1) should be considered as a comodule with respect to the coaction of the Z_2-graded quantum group $\Omega_{GL_q(N)}$ with the $GL_q(N)$-generators $\{T^i_j\}$ and additional generators $\{(dT)^k_l\}$ $(i,j,k,l = 1,2,...,N)$ which are the basis of the differential 1-forms on the quantum group $GL_q(N)$. This coaction $\Omega_z \xrightarrow{g_l} \Omega_{GL_q(N)} \otimes \Omega_Z$ conserves the grading and can be written down as a homomorphism:

$$e^i \xrightarrow{g_l} \tilde{e}^i = T^i_j \otimes e^j, \quad (2)$$

$$(de)^i \xrightarrow{g_l} (\widetilde{de})^i = (dT)^i_j \otimes e^j + T^i_j \otimes (de)^j. \quad (3)$$

Here $\otimes$ denotes the graded tensor product: $a\otimes b = (-1)^{\hat{a}\hat{b}}(1\otimes b)(a\otimes 1)$, where $\hat{f} = deg(f)$ and $a \in \Omega^{(\hat{a})}_{GL_q(N)}$, $b \in \Omega^{(\hat{b})}_Z$. We recall that the algebra Ω_Z with the generators (1) has the following expansion $\Omega_Z = \bigoplus_{n=0} \Omega^{(n)}_Z$, where $\Omega^{(n)}_Z$ denotes the subspace of the differential n-forms and there exists a similar expansion for the Z_2-graded quantum group $\Omega_{GL_q(N)} = \bigoplus_{n=0} \Omega^{(n)}_{GL_q(N)}$. Substituting the transformed algebra $\{\tilde{e}^i, \ (\widetilde{de})^i\}$ into the commutation relations (1) we obtain the following equations for the generators $\{T^i_j, \ (dT)^i_j\}$

$$(\mathbf{R} - c)TT'(\mathbf{R} + \frac{1}{c}) = 0 \ , \ \ (\mathbf{R}(dT)T' - T(dT)'\mathbf{R}^{-1})(\mathbf{R} + \frac{1}{c}) = 0 \ , \quad (4)$$

$$(\mathbf{R} + \frac{1}{c})(dT)(dT)'(\mathbf{R} + \frac{1}{c}) = 0 \ , \ \ (\mathbf{R} + \frac{1}{c})((dT)T'\mathbf{R} - \mathbf{R}^{-1}T(dT)') = 0 \ , (5)$$

where $T = T_1 = T \otimes I$ while $T' = T_2 = I \otimes T$ and I is a $(N \times N)$ unit matrix. The relations (4), (5) have to be fulfilled both for $c = q$ and $c = -q^{-1}$; therefore, we deduce from them the following q-commutation relations for the bicovariant differential complex on $GL_q(N)$ (see [12,6,7]):

$$\mathbf{R}TT' = TT'\mathbf{R}\,, \tag{6}$$

$$\mathbf{R}(dT)T' = T(dT)'\mathbf{R}^{-1}\,, \tag{7}$$

$$\mathbf{R}(dT)(dT)' = -(dT)(dT)'\mathbf{R}^{-1}\,. \tag{8}$$

We stress that (8) follows from (7) if the differential d is nilpotent $d^2 = 0$ and obeys the graded Leibnitz rule $d(fg) = d(f)g + (-1)^{\hat{f}} f d(g)$. It is interesting to note (see [1]) that the algebra $\Omega_{GL_q(N)}$ (6)-(8) is the Hopf algebra. The comultiplication Δ, the counit ϵ and the antipode S are defined by

$$\begin{array}{c}\Delta(T) = T \otimes T\ ,\ \ \epsilon(T) = 1\ ,\ \ \mathcal{S}(T) = T^{-1}\ , \\ \Delta(dT) = dT \otimes T + T \otimes dT\ ,\ \ \epsilon(dT) = 0\ ,\ \ \mathcal{S}(dT) = -T^{-1}dTT^{-1},\end{array} \tag{9}$$

and satisfy all the axioms of the Hopf algebra. One can show that it is possible to extend the action of the differential d over the tensoring and apply d to the algebra $\Omega_{GL_q(N)} \otimes \Omega_Z$ in such a way that: $d(g \otimes \Omega_Z) = d(g) \otimes \Omega_Z + (-1)^k g \otimes d(\Omega_Z)$, where $g \in \Omega^{(k)}_{GL_q(N)}$ and $d^2 = 0$.

Now we would like to interpret formulas (2) and (3) as a structure (gauge) quantum group transformation of the comodule e^i. Here, the matrix T^i_j is interpreted as a noncommutative analog of a structure (gauge) group element. In view of this, it is natural to consider the appearing of the additional term $(dT)^i_j \otimes e^j$ in (3) as a noncovariance of the comodule $(de)^i$ under the gauge rotation (2) (or as an indication that the differentials $(de)^i$ describe "nonparallel transporting" of the vector e^i). To restore the covariance, let us define a covariant differential ∇ so that the transformations (2), (3) are rewritten in the form

$$e^i \xrightarrow{g_l} \widetilde{e}^i = T^i_j \otimes e^j, \tag{10}$$

$$(\nabla e)^i \xrightarrow{g_l} \left(\widetilde{\nabla e}\right)^i = T^i_j \otimes (\nabla e)^j\ . \tag{11}$$

In general $(\nabla e)^i \notin \Omega_Z$ and, hence, the action of the operator ∇ enlarges the algebra Ω_Z up to some new algebra $\Omega_{\bar{Z}}$. Then, we assume that the operator d can be induced (as a differential) onto the whole algebra $\Omega_{\bar{Z}}$ and this algebra is naturally decomposed as $\Omega_{\bar{Z}} = \bigoplus_{n=0} \Omega^{(n)}_{\bar{Z}}$, where $\Omega^{(n)}_{\bar{Z}}$ is the subspace of n-forms. We postulate that the elements $(\nabla e)^i$ belong to the space $\Omega^{(1)}_{\bar{Z}}$ and there is the following expansion of $(\nabla e)^i$ over the generators $\{e^i,\ (de)^j\}$

$$(\nabla e)^i = (de)^i - A^i_j e^j, \tag{12}$$

It is clear that the coefficients $A^i_j \in \Omega^{(1)}_{\bar{Z}}$ and it is natural to consider them as noncommutative analogs of the connection 1-forms. Under the transformations (10) and (11) 1-forms A^i_j are transformed as

$$A^i_k \xrightarrow{g_l} \widetilde{A^i_k} = T^i_j(T^{-1})^l_k \otimes A^j_l + dT^i_j T^{-1}{}^j_k \otimes I \equiv (TAT^{-1})^i_k + (dTT^{-1})^i_k . \tag{13}$$

Here $\widetilde{A}^i_j \in \Omega_{GL_q(N)} \otimes \bar{Z}$. In the last part of (13), a short notation is introduced to be used below. The second action of the covariant derivative ∇ to the expression (12) leads to the definition of the curvature 2-forms $F^i_j \in \Omega^{(2)}_{\bar{Z}}$:

$$\nabla(\nabla e) = -\left(d(A) - A^2\right) e = -Fe. \tag{14}$$

The quantum group gauge transformation (13) for the curvature 2-forms F^i_j is represented as the adjoint coaction

$$F^i_j \xrightarrow{g_{ad}} \widetilde{F}^i_j = \left(T^i_k T^{-1}{}^l_j\right) \otimes F^k_l \equiv T^i_k F^k_l T^{-1}{}^l_j \, . \tag{15}$$

We note that the tensor F^i_j is a reducible adjoint representation of $GL_q(N)$ and it is possible to decompose it into the scalar $F^0 = Tr_q(F)$ and the q-traceless tensor: $\tilde{F}^i_j = F^i_j - \delta^i_j Tr_q(F)/Tr_q(1)$. Here, we have introduced the q-deformed trace [3,6,1,13]

$$Tr_q(F) \equiv Tr(DF) \equiv \sum_{i=0}^{N} q^{-N-1+2i} F^i_i. \tag{16}$$

The next action of the covariant derivative on the formula (14) yields the Bianchi identities that are represented in the classical form: $d(F) = [A, \, F]$.

To complete the definition of the algebra $\Omega_{\bar{Z}}$, we have to deduce the commutation relations of the new generators $\{A^i_j, \, F^i_j, \ldots\}$ and the old ones $\{e^i, \, (de)^j\}$. First of all, let us note that the choice of the connection A^i_j in the pure gauge form (see (13))

$$A^i_j = dT^i_k(T^{-1})^k_j \otimes I \, , \tag{17}$$

leads to the conclusion that the generators A^i_j could satisfy the following q-deformed anticommutation relations:

$$\mathbf{RARA} + \mathbf{ARAR}^{-1} = 0 \, , \tag{18}$$

where $\mathbf{A} = A_1 = A \otimes I$. These relations for the noncommutative gauge fields have been postulated in the context of the quantum group gauge theories in [1,9]. Note, however, that in the right-hand side of Eq.(18) one may add an arbitrary linear combination of the curvature 2-form $F = dA - A^2$ which vanishes on the solution (17). Thus, the general covariant commutation relations for A^i_j are

$$\mathbf{RARA} + \mathbf{ARAR}^{-1} = a(\mathbf{R})(\mathbf{FR} + \mathbf{R}^{-1}\mathbf{F}) + \alpha(\mathbf{R})F^0 , \tag{19}$$

where $\mathbf{F} = F_1 = F \otimes I$ and $a(\mathbf{R}) = a_1 + a_2\mathbf{R}$, $\alpha(\mathbf{R}) = \alpha_1 + \alpha_2\mathbf{R}$. Further, for simplicity, we will consider the case when $\alpha(\mathbf{R}) = 0$. We stress that the anticommutation relations (19) are covariant under the transformations (13) and (15). The special form of the right-hand side of Eq.(19) is dictated by the symmetry properties of the q-anticommutator appearing in the left-hand side of this equation ($c = \pm q^{\pm 1}$): $(\mathbf{R}-c)(\mathbf{RARA}+\mathbf{ARAR}^{-1})(\mathbf{R}+c^{-1}) = 0$. Arbitrary parameters a_i, α_i introduced in Eq.(19) depend on the choice of the noncommutative geometry and have to be fixed partially by the consistency conditions (with respect to the two ways of reordering of any cubic monomial) for the algebra $\Omega_{\tilde{Z}}$. It is amusing to note that the additional nonzero term included into the right hand side of (19) looks similar to the quantum anomaly terms arising in the (anti)commutators of fields (or currents) in certain conventional quantum field theories.

In order to find commutation relations A^i_j with the generators $\{e^i, (de)^j\}$, we postulate that the coordinates of the comodule (12) commute in the same way as the components of 1-forms $(de)^i$ (see (1))

$$\mathbf{R}(\nabla e)(\nabla e)' = -\frac{1}{c}(\nabla e)(\nabla e)' \tag{20}$$

$$(\pm)(c - b)\mathbf{R}(\nabla e)e' = e(\nabla e)' . \tag{21}$$

where b is a constant to be fixed below. From (1) and (21) we deduce covariant commutation relations for A and e:

$$(\pm)e\mathbf{A}' = \mathbf{RAR}e + b\mathbf{R}(\nabla e) \tag{22}$$

and the consistency condition for reordering (in two different ways) the monomials $ee'\mathbf{A}'' \equiv e_1e_2A_3$ leads to the only two solutions for the parameter b: $A.)b = 0$, and $B.)b = \lambda$. Thus, we have two variants for Eq.(22)

$$A.)\ (\pm)e\mathbf{A}' = \mathbf{RAR}e\ ,\quad B.)\ (\pm)e\mathbf{A}' = \mathbf{RAR}^{-1}e + \lambda\mathbf{R}(de)\ . \tag{23}$$

Recall that in the paper [1] we have considered only the first case A.): $b = 0$. Taking into account (20) one can obtain the corresponding commutation relations for (de) and A

$$(\pm)(de)\mathbf{A}' = -\mathbf{R}^{-1}\mathbf{AR}(de) + (b-\lambda)\mathbf{AR}(\nabla e) + \tilde{a}(\mathbf{R})(\mathbf{RF}+\mathbf{FR}^{-1})e\ , \tag{24}$$

where

$$\tilde{a}(\mathbf{R}) = \frac{1 + \gamma(\mathbf{R} - c)}{1 + c^2}a(\mathbf{R}) \tag{25}$$

and γ is a new arbitrary parameter. Type A.) and type B.) commutation relations (22), (24) are covariant under the gauge coactions (2), (3) and (13) and both cases lead to the same covariant commutation relation for (∇e) and A:

$$(\pm)(\nabla e)\mathbf{A}' = -\mathbf{RAR}(\nabla e) + (\tilde{a}(\mathbf{R}) - a(\mathbf{R}))(\mathbf{RF} + \mathbf{FR}^{-1})e \ , \tag{26}$$

Differentiating (22) and then using (24), one can derive

$$\begin{aligned} e\mathbf{F}' &= \mathbf{RF}(\mathbf{R} - b)e + \tilde{\tilde{a}}(\mathbf{R})(\mathbf{RF} + \mathbf{FR}^{-1})e = \\ &= (\mathbf{R} + \tilde{\tilde{a}}(\mathbf{R}))\mathbf{FR}e + (\tilde{\tilde{a}}(\mathbf{R})\mathbf{R}^{-1} - b\mathbf{R})\mathbf{F}e \end{aligned} \tag{27}$$

where we define

$$\tilde{\tilde{a}}(\mathbf{R}) = -(1 + b\mathbf{R})\tilde{a}(\mathbf{R}) + (b - \lambda)\mathbf{R}a(\mathbf{R}). \tag{28}$$

Considering the reordering of the monomials $ee'\mathbf{F}''$ in two possible ways and comparing the results we obtain for both types A.) and B.) ($b = 0, \ \lambda$) the restriction

$$1.) \ \tilde{\tilde{a}}(\mathbf{R}) = 0, \tag{29}$$

which leads to the commutation relation:

$$e\mathbf{F}' = \mathbf{RF}(\mathbf{R} - b)e. \tag{30}$$

Note that for the type A.) ($b = 0$) we have an additional solution 2.) $\tilde{\tilde{a}}(\mathbf{R}) = -\lambda$ equivalent to the relation: $e\mathbf{F}' = \mathbf{R}^{-1}\mathbf{FR}^{-1}e$. This relation, however, contradicts the algebra (19), (22) and (26) when we consider the consistency of the reordering of the cubic monomial $e\mathbf{R}'\mathbf{A}'\mathbf{R}'\mathbf{A}'$, where $\mathbf{R}' = P_{23}R_{23}$.

Substituting the definitions (28) and (25) into the condition (29), we obtain the following solutions for the parameters $a(\mathbf{R})$ and γ:

$$\begin{aligned} &1.) \ a(\mathbf{R}) = 0 \Rightarrow \tilde{a}(\mathbf{R}) = 0 \\ &2.) \ a(\mathbf{R}) = a_0(\mathbf{R} - c), \ \gamma = \tfrac{1}{c + c^{-1}} + (b - \lambda) \Rightarrow \\ &\Rightarrow \tilde{a}(\mathbf{R}) = \tfrac{a_0(\lambda - b)}{c}(\mathbf{R} - c), \end{aligned} \tag{31}$$

where $a_0 \neq 0$ is a constant.

Now, postulating the natural quantum hyperplane condition:

$$(\mathbf{R} - c)(\mathbf{F}e)(\mathbf{F}'e') = 0$$

and using Eq.(30) we find the following closed relations for the generators F^i_j

$$(\mathbf{R} - c)\mathbf{FRF}(\mathbf{R} + c^{-1}) = 0 \ . \tag{32}$$

We assume that the commutation relations for the curvature 2-form F^i_j have to be independent of the choice of the parameter $c = \pm q^{\pm 1}$. So, we deduce from Eq.(32) the commutation relations **RFRF** = **FRFR**. These relations are known, first, as reflection equations [14], second, as the commutation relations for invariant vector fields on $GL_q(N)$ [6,7] and, third, as the defining relations for the braided algebras [15].

To complete the definition of the algebra $\Omega_{\bar{Z}}$ we postulate the following commutation relation for F and A: **FRAR** = **RARF**. This is the simplest relation that covariant under the coactions (13), (15) and allowing one to push the operators F through the operators A.

Thus, leaving aside the commutation relations with the generators $\{e, de\}$, we obtain the following algebra with the generators A (1-form connection) and $F = dA - A^2$ (2-form curvature):

$$\begin{gathered} \mathbf{FRAR} = \mathbf{RARF}\ , \quad \mathbf{RFRF} = \mathbf{FRFR}\ , \\ \mathbf{RARA} + \mathbf{ARAR}^{-1} = a(\mathbf{R})(\mathbf{FR} + \mathbf{R}^{-1}\mathbf{F}) + \alpha(\mathbf{R})F^0\ , \end{gathered} \tag{33}$$

where $a(\mathbf{R}) = a_0(\mathbf{R} - c)$ or $a(\mathbf{R}) = 0$ (see Eqs.(31)) and $\alpha(\mathbf{R}) = 0$. Note that for the case $a_0 \neq 0$, the associativity conditions for the whole covariant algebra $\Omega_{\bar{Z}}$ give some additional constraints on the generators of this algebra. In particular, one can deduce

$$(\mathbf{R} - c)\mathbf{FR}e = 0. \tag{34}$$

Now, we present an explicit realization of such a covariant algebra $\Omega_{\bar{Z}}$ where the parameter a_0 and additional relations on the generators will be fixed. We consider the differential geometry of the group $GL_q(N+1)$ and interpret it as a noncommutative geometry on the total space of the principal fibre bundle with the base space $GL_q(N+1)/GL_q(N)$ and the structure group being $GL_q(N)$.

Let us introduce Z_2-graded extension of the $GL_q(N+1)$ quantum group generated by elements $\{T^I_J,\ dT^I_J\}$ $(I, J = 0, 1, \ldots N)$ satisfying the commutation relations (6)-(8) with the $GL_q(N+1)$ R-matrix acting in the space $Mat(N+1) \times Mat(N+1)$. Then, we consider the following left coaction of the group $GL_q(N)$ on the group $GL_q(N+1)$:

$$T^I_J \to \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & T^i_k \end{array}\right) \otimes \left(\begin{array}{c|c} T^0_0 & T^0_j \\ \hline T^k_0 & T^k_j \end{array}\right) \tag{35}$$

where as usual $i, j, k = 1, 2, \ldots N$. It is evident (from the commutation relations for the $GL_q(N+1)$-generators) that the elements T^i_j generate the quantum group $GL_q(N)$. For the Cartan 1-forms on the $GL_q(N+1)$-group

$$\Omega^I_J = dT^I_K (T^{-1})^K_J = \left(\begin{array}{c|c} \omega & \Omega^0_j = <\bar{e}|_j \\ \hline \Omega^i_0 = |e>^i & A^i_j \end{array}\right) \tag{36}$$

the coaction (35) is rewritten in the form:

$$\left(\begin{array}{c|c} \omega & <\bar{e}| \\ \hline |e> & A \end{array}\right) \rightarrow \left(\begin{array}{c|c} \omega & <\bar{e}|T^{-1} \\ \hline T|e> & TAT^{-1} + dTT^{-1} \end{array}\right) \qquad (37)$$

where the short notation has been used (see e.g. (13)). Comparing these transformations with the transformations (10) and (13) it becomes clear that the Cartan 1-forms $|e>$ and A can be interpreted as veilbein 1-forms and connection 1-forms respectively. Then, the generators $<\bar{e}|$ are nothing but contragradient veilbein 1-forms. The Maurer-Cartan equation $d\Omega^I_J = \Omega^I_K \Omega^K_J$ leads to the following constraints on the noncommutative differential 1-forms Ω^I_J:

$$\left(\begin{array}{c|c} d\omega - \omega^2 - <\bar{e}|e> & d<\bar{e}| - <\bar{e}|A - \omega<\bar{e}| \\ \hline d|e> - A|e> - |e>\omega & dA - A^2 - |e><\bar{e}| \end{array}\right) = 0 \, . \qquad (38)$$

Now, we deduce the commutation relations for the noncommutative Cartan 1-forms (36) using the $N+1$-dimensional analog of the commutation relations presented in (18). Taking into account the Maurer-Cartan equations (38) one can rewrite these relations in terms of the notation (36) in the form:

$$\mathbf{RARA} + \mathbf{ARAR}^{-1} = -\lambda(\mathbf{RF} + \mathbf{FR}^{-1}) \qquad (39)$$

$$-e\mathbf{A}' = \mathbf{RAR}e + \lambda\mathbf{R}(de - \mathbf{A}e)\,, \quad -\mathbf{A}'\bar{e} = \bar{e}\mathbf{RAR} + \lambda(d\bar{e} - \bar{e}\mathbf{A})\mathbf{R} \qquad (40)$$

$$\bar{e}\mathbf{R}e = -qe'\bar{e}', \quad \mathbf{R}ee' = -q^{-1}ee', \quad \bar{e}'\bar{e}\mathbf{R} = -q^{-1}\bar{e}'\bar{e}, \qquad (41)$$

$$\omega^2 = 0, \quad \{\omega, e\} = \{\omega, \bar{e}\} = 0, \quad \{A, \omega\} = q\lambda|e><\bar{e}| = q\lambda F. \qquad (42)$$

Here, we have also introduced the notation for the curvature 2-form:

$$F = dA - A^2 = |e><\bar{e}| \, . \qquad (43)$$

The last equality follows from Eqs.(38). Note, that for the curvature (43) one can prove the identity (34) using the relations (41). Then, we obtain, from the commutation relations (39)-(41) and Eq.(43), that the following commutation relations for F and A hold:

$$\mathbf{RFRF} = \mathbf{FRFR}\,, \quad \mathbf{RARF} = \mathbf{FRAR} + \lambda(\mathbf{RF}\omega - \mathbf{F}\omega\mathbf{R}) \qquad (44)$$

To exclude from these relations the noncommutative scalar generator ω we introduce a new connection 1-form: $A_t = A - \omega \cdot I$, where the corresponding curvature 2-form is

$$F_t = q^2 F - <\bar{e}|e> I = q^2 F + q^{1-N} F^0 \cdot I. \qquad (45)$$

The scalar $F^0 = Tr_q(F)$ is defined in (16) and invariant under the adjoint coaction (15). Finally, we obtain from Eqs.(40)-(41) and (44) that the elements $\{e,\ A_t,\ F\}$ generate the following closed algebra:

$$\begin{gathered}\mathbf{RFRF} = \mathbf{FRFR},\ \ \mathbf{RA}_t\mathbf{RF} = \mathbf{FRA}_t\mathbf{R}, \\ \mathbf{RA}_t\mathbf{RA}_t + \mathbf{A}_t\mathbf{RA}_t\mathbf{R}^{-1} = a_0(\mathbf{FR}^{-1} + \mathbf{RF})(\mathbf{R} - c), \\ -e\mathbf{A}'_t = \mathbf{RA}_t\mathbf{R}e,\ \ e\mathbf{F}' = \mathbf{RFR}e\end{gathered} \tag{46}$$

where $a_0 = 1 - q^2$ and $c = -q^{-1}$.

Comparing the commutation relations (41) and (46) with the relations (1), (22) and (33) one can infer that we have explicitly realized the covariant quantum algebra $\Omega_{\bar{Z}}$ of the type A.) ($b = 0$) in terms of the algebraic objects related to the $GL_q(N+1)/GL_q(N)$-geometry.

At the end of this report, we present the noncommutative analogs of the Chern characters. For this, let us consider the special case of the closed algebra (33) with the generators A and F where the parameters $a(\mathbf{R}) = 0$. Here, as we have explained above, A^i_j are noncommutative analogs of connection 1-forms, while F^i_j are interpreted as curvature 2-forms. In analogy with the classical case (see e.g. [16]), we consider as invariant characters the following expressions:

$$C_k = Tr_q(F^k) = D^i_j F^j_{j_1} \cdots F^{j_{k-1}}_i, \tag{47}$$

where we have used the q-deformed trace introduced in (16). By definition, the q-trace possesses the invariant property

$$Tr_q(TET^{-1}) = Tr_q(E) \tag{48}$$

for $T^i_j \in GL_q(N)$ and arbitrary quantum matrix E^i_j satisfying $[T,\ E] = 0$. In particular, we have

$$Tr_{q2}(\mathbf{R}E\mathbf{R}^{-1}) = Tr_{q2}(\mathbf{R}^{-1}E\mathbf{R}) = Tr_q(E) \tag{49}$$

Here $Tr_{q2}(.)$ denotes quantum trace over the second space. One can obtain also the following identities

$$Tr_{q2}(\mathbf{R}^{\pm 1}) = q^{\pm N} I_1\ ,\ \ Tr_q(I) = \frac{q^N - q^{-N}}{q - q^{-1}} = [N]_q. \tag{50}$$

Using (48) we immedeatly obtain that $2k$-forms C_k (47) are invariant under the adjoint coaction (15). Moreover, C_k are the closed $2k$-forms. Indeed, from the Bianchi identities $dF = [A, F]$ we deduce

$$dC_k = Tr_q(AF^k - F^k A) = 0, \tag{51}$$

where we have taken into account (see Eqs.(33), (49) and (50))

$$\begin{gathered}Tr_q(AF^k) = q^{-N} Tr_{q1}(Tr_{q2}(\mathbf{R}^{-1}\mathbf{RARF}^k)) = \\ q^{-N} Tr_{q1}(Tr_{q2}(\mathbf{F}^k\mathbf{RA})) = Tr_q(F^k A).\end{gathered}$$

We believe that C_k have to be presented as the exact form $C_k = dL^{(k)}_{CS}$, where the Chern-Simons $(2k-1)$-forms $L^{(k)}_{CS}$ are represented as

$$L^{(k)}_{CS} = Tr_q\{A(dA)^{k-1} + \frac{1}{h^{(k)}_1} A^3 (dA)^{k-2} + \ldots + \frac{1}{h^{(k)}_{\ldots}} A^{2k-1}\} \qquad (52)$$

and the constants $h^{(k)}_i$ depend on the deformation parameter q. We do not have explicit formulas for all parameters $h^{(k)}_{\ldots}$ (in the classical case $q = 1$ these formulas are known [17]), but for the case $k = 2$ one can obtain a noncommutative analog of the three-dimensional Chern-Simons term in the form:

$$L^{(2)}_{CS} = Tr_q\{AdA - \frac{1}{h^{(2)}_1} A^3\}, \quad h^{(2)}_1 = 1 + \frac{1}{q^2 + q^{-2}} . \qquad (53)$$

To conclude this report, we would like to note that it is extremely interesting to write the Chern characters for the general case of the algebra (33) when the parameters $a(\mathbf{R}) \neq 0$ and $\alpha(\mathbf{R}) \neq 0$.

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ON EVEN COLLECTIVE EFFECTS IN PURELY ODD SUPERSPACES

VLADIMIR G. PESTOV
Department of Mathematics, Victoria University of Wellington, P.O. Box 600, Wellington, New Zealand

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Abstract. We discuss a mathematical approach to collective even effects in infinite-dimensional odd geometry, which is based on nonstandard hulls, or ultraproducts. Our construction leads to previously unknown examples of geometries (graded Lie-Cartan pairs) with substantial even sector. All results are illustrated with a particular example of a purely odd l_∞-supermanifold.

Key words: Infinite-dimensional supergeometry, purely odd superspaces, ultraproducts, nonstandard hulls, Lie-Cartan pairs.

1. Introduction

Infinite-dimensional supergeometry still remains a land of mystery: mathematical theory of it is but rudimentary, which fact makes it difficult to approach a few appealing problems in the area. We address one of such problems suggested by Manin [13]: to represent even geometry as a collective effect in infinite-dimensional odd geometry.

One can handle infinite-dimensional objects of supergeometry by extrapolating from finite dimensions the functor of points [12][3]. A far-reaching theory of Banach supermanifolds has been constructed along those lines [14]. However, for many needs of supergeometry – the above Manin's problem notwithstanding – the functor of points approach is insufficient and what one actually needs, is a genuine geometric object representing such a functor [1]. One solution was proposed by Schmidt [23]; his holomorphic supermanifolds modelled on graded locally convex spaces are geometric, or locally ringed, superspaces [13] with a certain additional structure. Khrennikov [11] attempted to extend to infinite dimensions the "naïve" view of supermanifolds [5][22]; it appears, however, that Khrennikov's theory was mathematically shaky at some points [15][21].

In this paper we consider one particular example of a $(0,\infty)$-dimensional superspace which may be justly termed a supermanifold modelled on the purely odd l_∞. Namely, we produce a representing object for the functor of points determined by the purely odd Banach space $l_\infty^{odd} = (0) \oplus l_\infty$. To do that, we enlarge the category of finite-dimensional Grassmann algebras (the usual realm of functor of points) to a wider category of locally convex graded-commutative algebras. The opposite category can be thought of as a category of purely odd superspaces.

Our approach to Manin's problem is based on the existence of preferred topology on "function algebras" of the above superspaces. Our construction can be presented in either of two different languages: that of ultraproducts, or that of nonstandard analysis. The latter one provides a nice conceptual understanding of what is happening. Suppose we allow for infinitely large and infinitely small quantities to dwell in topological vector spaces. Imagine a finite observer viewing a superspace, $\operatorname{Spec}\Lambda$, of dimension $(0,\infty)$. Being finite, he or she can only see finite elements of the function algebra Λ – infinitely large elements are non-observable. At the same time, our observer cannot distinguish between elements which are infinitely close to each other. Therefore, the function algebra on the geometric superspace appears to him or her as the quotient of a subalgebra of all finite elements of Λ modulo the ideal of infinitesimals. This quotient is a reputed object of nonstandard analysis, termed the *nonstandard hull* of Λ (alias the *ultrapower* of it). A remarkable fact is that the nonstandard hull of a nilpotent algebra can possess a highly nontrivial semisimple quotient, which means that the observed geometric superspace has a nontrivial spatial sector.

After being rewritten in the language of ultraproducts, the above construction becomes functorial.

In this paper we show that in the case of purely odd l_∞-supermanifold, the derivations of the nonstandard hull algebra are abundant. This is important for existence of a differential geometry substantial in its even sector, in the spirit of Lie-Cartan pairs approach.

Throughout the paper, the basic field is $\mathbf{C}$. The symbol $\bigwedge(q)$ denotes the exterior algebra on $\mathbf{C}^q$.

2. A purely odd l_∞-supermanifold

The functor of points, consciously transplanted from algebraic geometry to supergeometry and advertised since then by Leites [12], has been re-invented in different guises [5][24].

If M is a geometric superspace and q a natural number, then a q-point of M is any geometric superspace morphism $\operatorname{Spec}\bigwedge(q) \to M$. Denote by $\mathcal{P}_M(q)$ the set of all q-points of M; the correspondence $\bigwedge(q) \mapsto \mathcal{P}_M(q)$ is a contravariant functor from the category of finite-dimensional Grassmann algebras and even algebra homomorphisms, which we denote by $\mathcal{G}$, to $\mathcal{S}ets$.

Let $E = E^0 \oplus E^1$ be a graded locally convex space. One normally associates to E the functor of points of the form $\bigwedge(q) \mapsto (\bigwedge(q) \otimes E)^0$ [14]. This functor of points is assumed to represent (in fact, serve as a surrogate of) a supermanifold naturally associated to E.

We shall construct a representing object for the functor of points associated to the graded Banach space $l_\infty^{odd} = (0) \oplus l_\infty$.

Denote by $\bigwedge l_1$ the exterior algebra on l_1, endowed with the strongest locally convex topology inducing the given topology on l_1, and completed thereafter. The algebra $\bigwedge l_1$ can be otherwise described as the graded-commutative complete locally convex algebra generated by a bounded countable subset of the odd part in a universal way. Namely, if one denotes by $\Xi = \{\xi_1, \dots, \xi_n, \dots\}$ the set of canonical coordinate vectors for l_1, then every map f from Ξ to the odd part of a complete locally convex graded-commutative algebra Λ, such that $f(\Xi)$ is a bounded subset in the LCS Λ^1, extends in a unique fashion to a continuous even algebra homomorphism $\hat{f}: \bigwedge l_1 \to \Lambda$ [20].

Let $\mathcal{GO}$ stand for the category of all complete locally convex graded-commutative algebras topologically generated by the odd part (the *algebras of Grassmann origin*, or *GO*-algebras [2]) and continuous even algebra homomorphisms. This category plainly includes the category $\mathcal{G}$ as a full subcategory in a canonical way. The opposite category $\mathcal{GO}^{op}$ will be interpreted as a category of purely odd superspaces. For an object $\Lambda \in Ob\,\mathcal{GO}$ we denote by $\operatorname{Spec}\Lambda$ the corresponding object in $Ob\,\mathcal{GO}^{op}$, realized as a geometric superspace over a one-pointed underlying topological space, $\{*\}$, with Λ as the algebra of sections of a constant structure sheaf.

Theorem 1. *The object* $\operatorname{Spec}\bigwedge l_1$ *of the category* $\mathcal{GO}^{op}$ *represents the functor of points determined by* l_∞^{odd} *on the full subcategory* $\mathcal{G}^{op}$ *of* $\mathcal{GO}^{op}$.

Proof. Let $q \in \mathbf{N}$. The set of all geometric superspace morphisms $\mathbf{C}^{0,q} \to \operatorname{Spec}\bigwedge l_1$, that is, even continuous homomorphisms $\bigwedge l_1 \to \bigwedge(q)$, can be identified, by virtue of the universality property of $\bigwedge l_1$, with the collection of all bounded sequences of elements of $\bigwedge(q)^1$, that is, the l_∞ type sum of countably many copies of the finite-dimensional vector space $\bigwedge(q)^1$. But it is nothing other than $\bigwedge(q)^1 \otimes l_\infty$ (under a canonical identification), or, just the same, $[\bigwedge(q) \otimes l_\infty^{odd}]^0$.

Remark that the functor of points determined by l_∞^{odd} does not admit a canonical extension to the whole of category $\mathcal{GO}^{op}$. For a discussion, see [16][19].

Even as we do not name here what other fragments of a general construction we are aware of, the above example suggests that the presence of a distinguished topology on the algebra of functions can be of significance.

3. General construction

Let I be an infinite index set and $\mathcal{U}$ be a free non-δ-complete ultrafilter on I. For a locally convex space E, form the *ultrapower* of E. This is the Hausdorff quotient of a locally convex space of all bounded families $x = (x_i)_{i\in I}$ of elements of E, where the topology is determined by seminorms ${}^*p(x) = \lim_{\mathcal{U}} p(x_i)$. This object is denoted either by $E^I_{\mathcal{U}}$, or, in the context of nonstandard analysis, by $\hat{E}$. In the latter case it is called the ***nonstandard hull*** of E and interpreted as the quotient of a subspace, $\operatorname{fin} E$, of all finite elements of the nonstandard enlargement ${}^*E$ of E by the ideal, $\mu_E(0)$, of all infinitesimals. Here $\operatorname{fin} E$ is the union of all sets of the form ${}^*B$, where $B \subset E$ is bounded, and $\mu_E(0)$ is the intersection of all sets of the form ${}^*U$, where $U \subset E$ is a neighbourhood of zero. (See [6] for theory of ultraproducts of LCS, [25] for theory of nonstandard hulls, and [17][18][20] for more on the present construction.)

The correspondence $E \mapsto E^I_{\mathcal{U}}$ is functorial in E. If A is a locally convex topological algebra, then so is $A^I_{\mathcal{U}}$.

The core of the suggested approach is an idea to view the nonstandard hull, $\hat{\Lambda}$, of the algebra of functions on a purely odd superspace, $\operatorname{Spec}\Lambda$, as the algebra of functions on some new geometric superspace, which is an "observable form" of $\operatorname{Spec}\Lambda$, or the shadow $\operatorname{Spec}\Lambda$ throws into the finite world.

The whole construction can be given a form of a covariant functor from $\mathcal{GO}^{op}$ to a suitable category of geometric superspaces. However, we will discuss only some aspects of this functor now.

Recall that the Gelfand space, $\Sigma(A)$, is defined in case where A is a non-Banach LC algebra as the set of all *continuous* characters on A endowed with the weak* topology. The underlying topological space of the "observable part" of $\operatorname{Spec}\Lambda$ is the Gelfand space of the nonstandard hull $\hat{\Lambda}$, and this way a covariant functor emerges, $\mathcal{GO}^{op} \to \mathcal{T}ych$.

The geometry of the nostandard hull algebra, $\hat{\Lambda}$, turns out to be in some cases nontrivial in the even sector, unlike that of Λ. The following results show that for $\Lambda = \bigwedge l_1$ the underlying topological space of the "shadow" geometric superspace is rich, and a nontrivial analytic structure dwells in it.

Theorem 2. ([20]; cf. [18]) *The Gelfand space $\Sigma(\widehat{\bigwedge l_1})$ is an inseparable Tychonoff topological space. This space contains a topological copy of the cube I^n for each natural number n, therefore the topological dimension of $\Sigma(\widehat{\bigwedge l_1})$ is infinite in any sense.*

Theorem 3. [20] *There exists a homeomorphism Φ from the unit disc $D \subset \mathbf{C}$ into $\Sigma(\widehat{\bigwedge l_1})$ such that for every element $a \in \widehat{\bigwedge l_1}$ the composition $\hat{a} \circ \Phi$ is a holomorphic function from D to $\mathbf{C}$.*

4. Derivations and Lie-Cartan pairs

The aim of this Section is to show that the algebra of the form $\hat{\Lambda}$, where $\Lambda \in Ob\,\mathcal{GO}$, in some cases possesses an adundance of (continuous) derivations. Specifically, the following is true.

Theorem 4. *Let $x \in \widehat{\bigwedge l_1}$. If for all continuous graded derivations, d, of $\widehat{\bigwedge l_1}$ one has $dx = 0$, then $x \in \mathbf{C}$.*

Or, in an equivalent form,

Theorem 5. *Let $x \in \widehat{\bigwedge l_1} \setminus \{0\}$. There exists a continuous differential operator $\mathcal{P}$ of order ≥ 0 on $\widehat{\bigwedge l_1}$ such that $\mathcal{P}x = 1$.*

This result gives a hope that differential geometry of the nonstandard hull $\widehat{\bigwedge l_1}$ (as well as the corresponding geometric superspace, which we do not discuss here) is substantial in even sector.

We start with a general construction, hopefully of independent interest for noncommutative differential geometry.

A *graded Lie-Cartan pair* [10][7][8] (L, A) consists of an associative unital graded algebra A, a Lie superalgebra L which is a left unital A-module, and a Lie homomorphism $L \to \mathcal{D}er\, A$, satisfying the following axioms

(i) $a(\xi b) = (a\xi)b$ for all $a, b \in A$ and $\xi \in L$;
(ii) $[\xi, a\eta] = (-1)^{\tilde{\xi}\tilde{a}} a[\xi, \eta] + (\xi a)\eta$ for all $\xi, \eta \in L$ and $a \in A$.

(Here $[,]$ stands for the supercommutator in L, and the usual parity conventions are assumed.)

The two fundamental examples are 1) $A = C^\infty(X)$ is the algebra of smooth functions on a finite-dimensional manifold X, and $L = \mathrm{vect}(X)$ is the Lie algebra of smooth vector fields on X (*classical differential calculus* [7]), and 2) $A = \bigwedge(q)$, and $L = \mathcal{D}er \bigwedge(q)$ (*fermionic differential calculus* [8]).

Consider a Lie-Cartan pair (L, A) and assume that A carries a locally convex topology such that every derivation $d \in L$, $d\colon A \to A$ is continuous. (As is invariably the case in all "classical" examples.) We shall also assume, without much loss in generality, that the fixed homomorphism $L \to \mathcal{D}er\, A$ is a monomorphism.

Denote by $\mathrm{fin}\, L$ the set of all $\xi \in {}^*L$ such that $\xi(\mathrm{fin}\, A) \subseteq \mathrm{fin}\, A$, and by $\mu_L(0)$ the set of all $\xi \in {}^*L$ with $\xi x \in \mu_A(0)$ for all $x \in \mathrm{fin}\, A$. One can check that $\mathrm{fin}\, L$ is a Lie subalgebra of L, and $\mu_L(0)$ is a Lie ideal in $\mathrm{fin}\, L$. Denote by $\hat{L}$ the quotient Lie algebra $\mathrm{fin}\, L/\mu_L(0)$. The monomorphism $\mathrm{fin}\, L \to \mathcal{D}er\, \mathrm{fin}\, A$ factors through $\mu_L(0)$, giving rise to a monomorphism $\hat{L} \to \mathcal{D}er\, \hat{A}$.

Theorem 6. *The pair $(\hat{L}, \hat{A})$ is a Lie-Cartan pair.*

We call $(\hat{L}, \hat{A})$ the *nonstandard hull* of the pair (L, A).

The Lie superalgebra of graded derivations of $\bigwedge(q)$ is well-known [9][4]: it is generated, as a free graded $\bigwedge(q)$-module, by $\partial/\partial\xi_1, \partial/\partial\xi_2, \ldots, \partial/\partial\xi_q$, where $\xi_1, \xi_2, \ldots, \xi_q$ is any system of free odd generators for $\bigwedge(q)$ and $\partial/\partial\xi_i$ stands for the formal odd derivation by ξ_i.

One can describe in a similar way the algebra $\mathcal{D}er \bigwedge l_1$ of all ***continuous*** derivations of $\bigwedge l_1$. Remark that for every odd generator ξ_i of $\bigwedge l_1$, the odd derivation of the latter algebra $\partial/\partial\xi_i$ is well-defined and continuous.

Theorem 7. *The algebra $\mathcal{D}er \bigwedge l_1$ is isomorphic, as a $\bigwedge l_1$-module, to the l_∞-type sum of countably many copies of $\bigwedge l_1$ under the correspondence $e_i \mapsto \partial/\partial\xi_i$.*

Proof. The l_∞-type sum of countably many copies of $\bigwedge l_1$ is formed by all bounded sequences $y = (y_i)$ of elements of $\bigwedge l_1$, and e_i is the i-th standard basic vector $(0, 0, \ldots, 0, 1_i, 0, \ldots)$. For any such y and any element $x \in \bigwedge l_1$, the rule $yx = \sum_i y_i \partial x/\partial\xi_i$ correctly defines an element of $\bigwedge l_1$; using structural results on $\bigwedge l_1$ from [20], one can show that the emerging derivation, y, is continuous; therefore, an even homomorphism from the l_∞-type sum to $\mathcal{D}er \bigwedge l_1$ mentioned in Theorem 7 is well-defined. This homomorphism is onto, because any derivation $d \in \mathcal{D}er \bigwedge l_1$ is the image of a sequence $y = (d\xi_i)$.

Theorem 8. *The nonstandard hull of the Lie-Cartan pair $(\mathcal{D}er \bigwedge l_1, \bigwedge l_1)$ is canonically isomorphic to the pair $(\widehat{\mathcal{D}er \bigwedge} l_1, \widehat{\bigwedge l_1})$, where the nonstandard hull of $\mathcal{D}er \bigwedge l_1$ is formed as of a LCS under the identification of Theorem 7.*

Proof. A direct application of relevant definitions.

Proof of Theorem 4. We adopt the notation of [18]. Let $x = \sum_\mu x_\mu \xi^\mu$ be an arbitrary element of fin $\bigwedge l_1 \setminus \mu_{\bigwedge l_1}(0)$. One can assume that the set of indices with non-vanishing coefficients is $*$-finite, and that all multi-indices μ are of the same (standard) finite length $n \in \mathbf{N}$. If there exists an i with $\partial x/\partial\xi_i$ non-infinitesimal, then set $d = \partial/\partial\xi_i$. Otherwise, one can assume (by proceeding to a sub-sum, if necessary) that the distinct multi-indices in the representation of x are disjoint. Select a $*$-finite subset $A \subseteq {}^*\mathbf{N}$ which intersects every such multi-index exactly once, and set $y = (y_i)$, where $y_i = \epsilon_i \chi_A(i)$ and $\epsilon_i = \pm 1$. According to Theorem 8, the element y is in fin $\mathcal{D}er \bigwedge l_1$. Since x is finite, one can choose ϵ_i so as to make the element yx finite. If one denotes by $\hat{y}$ the image of y in $\widehat{\mathcal{D}er \bigwedge} l_1$, and by $\hat{x}$ the image of x in $\widehat{\bigwedge l_1}$, then $\hat{y}\hat{x} \neq 0$.

5. Conclusion

Here are some problems not entirely devoid of interest.

1. The comultiplication and antipode can be extended by continuity over completions of infinite-dimensional Grassmann algebras at least for some locally convex topologies on them. (Such an extension is possible, e.g., for $\bigwedge l_1$, and impossible for the Banach-Grassmann algebra B_∞ studied in [22].) Does this structure – at least, in some cases – give rise to a Hopf algebra structure on the nonstandard hull, and thereby, an abelian topological group structure on the Gelfand space?

2. Can one construct a genuine measure and integral on the Gelfand space of the nonstandard hull of a locally convex Grassmann algebra, starting from the formal Berezin integral in the latter algebra?

3. Suggested after my talk by Achim Kempf. Is it possible that for some $\Lambda \in Ob\,\mathcal{GO}$ the Gelfand space of the nonstandard hull $\widehat{\Lambda}$ is both nondiscrete and topologically finite-dimensional?

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CHAPTER III

QUANTUM FIELD THEORY

TOPOLOGICAL FIELD THEORIES, STRING BACKGROUNDS AND HOMOTOPY ALGEBRAS

A. A. VORONOV*
Department of Mathematics
University of Pennsylvania
Philadelphia, PA 19104-6395
*USA***

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Abstract. String backgrounds are described as purely geometric objects related to moduli spaces of Riemann surfaces, in the spirit of Segal's definition of a conformal field theory. Relations with conformal field theory, topological field theory and topological gravity are studied. For each field theory, an algebraic counterpart, the (homotopy) algebra satisfied by the tree level correlators, is constructed

Key words: Frobenius algebra – Homotopy Lie algebra – Homotopy commutative algebra – Gravity algebra – Moduli space – Topological field theory – Conformal field theory – String theory – String background – Topological gravity

1. Introduction

The usual way of describing a string background as some construction on top of a conformal field theory involving the Virasoro operators, the antighost fields and the BRST operator appears too eclectic to be seriously accepted by the general mathematical public. Here we make an attempt to include string theory in the framework of geometric/topological field theories such as conformal field theory and topological field theory. Basically, we describe all two-dimensional field theories as variations on the theme of Segal's conformal field theories [10]. Our definition is in some sense dual to Segal's definition of a string background, also known as a topological conformal field theory, via differential forms and operator formalism, see Segal [11] and Getzler [1].

In this paper, each geometric field theory is followed by a leitmotif, the structure of an algebra built on the state space of the theory. Whereas it is commonly known that two-dimensional quantum field theories comprise very interesting geometrical structures, related algebraic structures have emerged

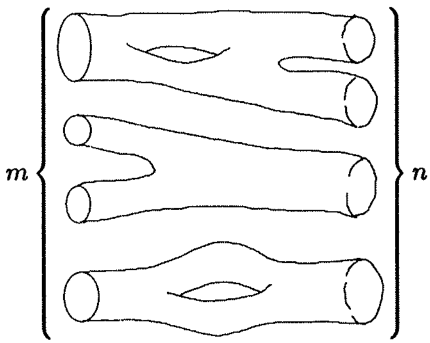
* Research supported in part by NSF grant DMS-9108269.A03
** Affiliated to Department of Mathematics, Princeton University, Princeton, NJ 08544-1000, USA

very recently and are still experiencing a very active period of growth. A list of references, perhaps, already outdated, can be found in the recent paper [5].

2. Topological Field Theory and Frobenius Algebras

Note. The field theories we are going to consider here will all have the total central charge zero. The general case can be done by involving the determinant line bundles over the moduli spaces.

A *topological field theory* (*TFT*) is a complex vector space V, called the *state space*, together with a correspondence



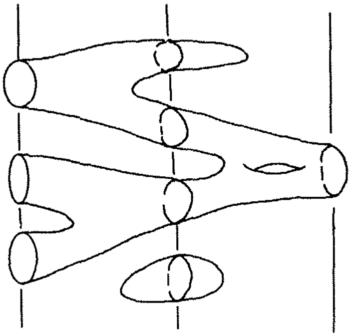
$$\longmapsto \quad |\Sigma\rangle : V^{\otimes m} \to V^{\otimes n} \tag{1}$$

An orientable surface Σ bounding $m+n$ circles — A linear operator $|\Sigma\rangle$

Here a surface is not necessarily connected. Its boundary circles are enumerated and parameterized. The first $m \geq 0$ circles are called *inputs* and the remaining $n \geq 0$ circles are called *outputs*. The linear operator $|\Sigma\rangle$ is called the *state* corresponding to the surface Σ.

This correspondence should satisfy the following axioms.

1. **Topological invariance**: The linear mapping $|\Sigma\rangle$ is invariant under diffeomorphisms of the surface Σ.
2. **Permutation equivariance**: The correspondence $\Sigma \mapsto |\Sigma\rangle$ commutes with the action of the symmetric groups S_m and S_n on surfaces and linear mappings by permutations of inputs and outputs.
3. **Factorization property**: Sewing along the parameterizations of the boundary corresponds to composing:



$\longmapsto \qquad V^{\otimes m} \to V^{\otimes n} \to V^{\otimes k}$

The sewing of the outputs of a surface with inputs of another surface

The composition of the corresponding linear operators

4. **Normalization:**

$\longmapsto \qquad \mathrm{id} : V \to V$

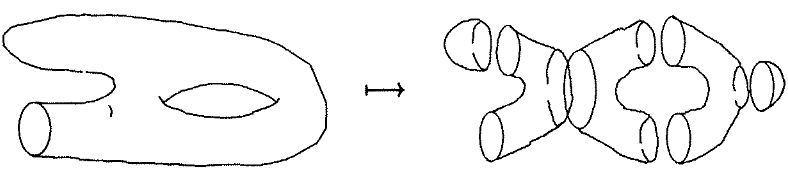
A cylinder

The identity operator

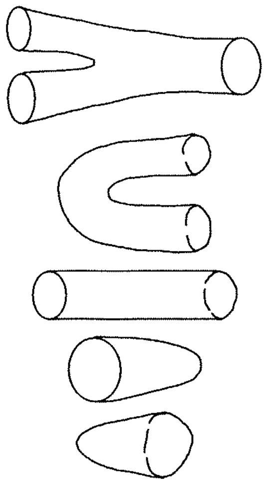
These data and axioms can be formulated equivalently using functors. Within this approach, a TFT is a multiplicative functor from a "topological" tensor category *Segal* to a "linear" tensor category *Hilbert*. An object of the category *Segal* is a diffeomorphism class of parameterized one-dimensional compact manifolds, i.e., disjoint unions of circles. A morphism between two collections of circles is a diffeomorphism class of orientable surfaces bounding the circles. The identity morphism of an object is the cylinder over it. The operation of disjoint union of collections of circles introduces the structure of a tensor category on *Segal*.

The other category *Hilbert* is the category of complex vector spaces (Hilbert in real examples), not necessarily finite dimensional, with the usual tensor product. Then the space V is the vector space corresponding to the single circle and it is easily checked that the functoriality plays the role of the factorization property and that the two definitions are equivalent.

Any orientable surface can be cut into pants and caps:



In fact, observe that orientable surfaces have the following generators with respect the sewing operation. And respectively, the space V is provided with an algebraic structure generated by the operations below with respect to composition of linear mappings.



$\mapsto \quad V \otimes V \to V$

$\mapsto \quad \mathbb{C} \to V \otimes V$

$\mapsto \quad V \to V$

$\mapsto \quad V \to \mathbb{C}$

$\mapsto \quad \mathbb{C} \to V$

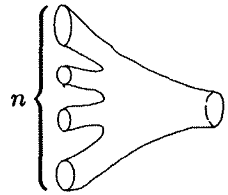
THEOREM 1 (Folklore). *A TFT is equivalent to a Frobenius algebra, i.e., a commutative algebra V with a unity and a nondegenerate symmetric bilinear form $\langle , \rangle : V \otimes V \to \mathbb{C}$ which is invariant with respect to the multiplication:*

$$\langle ab, c \rangle = \langle a, bc \rangle$$

and has an "adjoint" $\mathbb{C} \to V \otimes V$.

An "adjoint" to a mapping $\phi : V \otimes V \to \mathbb{C}$ is a mapping $\psi : \mathbb{C} \to V \otimes V$, such that the compositions $V \xrightarrow{\mathrm{id}\otimes\psi} V \otimes V \otimes V \xrightarrow{\phi_{12}\otimes\mathrm{id}} V$ and $V \xrightarrow{\psi\otimes\mathrm{id}} V \otimes V \otimes V \xrightarrow{\mathrm{id}\otimes\phi_{23}} V$ are identities. When the space V is finite dimensional, an inner product establishes an isomorphism $V \to V^*$, and an adjoint mapping gives a mapping $V^* \to V$, which is nothing but its inverse. Thus, in the finite dimensional case, a Frobenius algebra is just an algebra with an invariant nondegenerate inner product. The theorem follows from the remark above about decomposing a surface into pants, caps and cylinders and the obvious fact that the symmetric form $\langle , \rangle$ in a Frobenius algebra V can be obtained from a linear functional $f : V \to \mathbb{C}$ as $\langle a, b \rangle = f(ab)$.

An important substructure is observed for a TFT at the *tree level*, when we restrict our attention to surfaces of genus zero and with exactly one output:



$\longmapsto \quad V^{\otimes n} \to V$

$\longmapsto \quad \mathbb{C} \to V$

Topological invariance, permutation equivariance, the factorization and the normalization axioms make sense for such surfaces and are assumed.

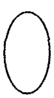
The following fact is worth mentioning, because we are aiming to study similar algebraic structures occurring in string theory at the tree level.

COROLLARY 2. *A TFT at the tree level is equivalent to a commutative algebra V with a unity.*

3. Conformal Field Theory

A *conformal field theory* (*CFT*) is a device very similar to a TFT, except that

1. the correspondence (1) is defined on Riemann surfaces bounding holomorphic disks and the state $|\Sigma\rangle$ depends smoothly on the Riemann surface Σ,
2. topological invariance is replaced by conformal invariance,
3. when two Riemann surfaces are sewn, the result is provided with a unique complex structure, and
4. normalization is slightly different:

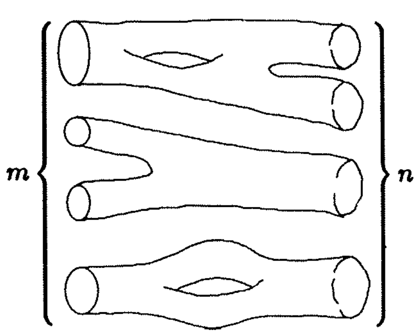


$\longmapsto \quad \mathrm{id} : H \to H$

A cylinder of zero length — The identity operator

In other words, a CFT is a smooth mapping

$$\mathcal{P}_{m+n} \quad \to \quad \mathrm{Hom}(H^{\otimes m}, H^{\otimes n}), \qquad (2)$$



$$\longmapsto \quad |\Sigma\rangle : H^{\otimes m} \to H^{\otimes n},$$

A point Σ in the moduli space $\mathcal{P}_{m+n}$

A linear operator $|\Sigma\rangle$

where $\mathcal{P}_{m+n}$ is the moduli space of Riemann surfaces (one-dimensional complex compact manifolds) bounding $m+n$ holomorphic disks. The surfaces can have arbitrary genera, the disks are holomorphic mappings from the unit disk to a closed Riemann surface and they are enumerated. The mapping (2) must be equivariant with respect to permutations, transform sewing of Riemann surfaces into composition of the corresponding linear operators and must be normalized as above.

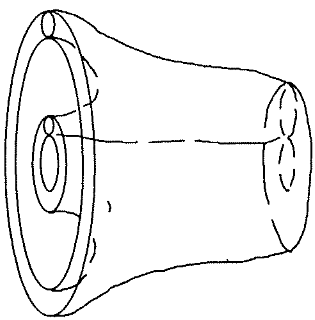
There is an evident reformulation of the CFT data as a functor from a suitable category *Segal* to the category *Hilbert* analogous to the one for TFT's.

4. String Theory and Homotopy Lie Algebras

4.1. String Backgrounds

Let H be a graded vector space with a differential Q, $Q^2 = 0$, i.e., H be a complex. A *string background* is a correspondence

$$C_\bullet \mathcal{P}_{m+n} \quad \to \quad \mathrm{Hom}(H^{\otimes m}, H^{\otimes n}), \tag{3}$$



$$\longmapsto \quad |C\rangle : H^{\otimes m} \to H^{\otimes n},$$

Chains C in $\mathcal{P}_{m+n}$

Linear operators $|C\rangle$

which satisfies the axioms below. [On the figure, the surface is nothing but a pair of pants (so $m = 2$, $n = 1$) and the chain is just a circle. The pants moving along the circle in the moduli space sweep out a "surface of revolution", which I attempted to sketch above.] By chains here we mean the (complex) vector space generated by (oriented) singular chains.

1. **Smoothness**: The mapping (3) is smooth.
2. **Equivariance**: The mapping (3) is equivariant with respect to permutations of inputs and outputs.
3. **Factorization**: The sewing of outputs of a chain with inputs of another chain (namely, outputs of each Riemann surface in the first chain are sewn with inputs of each Riemann surface in the second chain, each time producing a new Riemann surface) transforms under (3) into the composition of the corresponding linear operators.
4. **Homogeneity and Q-∂-Invariance**: The mapping (3) is a morphism of complexes. That means that it maps a chain of dimension k to a linear mapping of degree $-k$ (with respect to the natural grading on the Hom) and that the boundary of a chain in $\mathcal{P}_{m+n}$ transforms into the differential of the corresponding mapping,
   $$|\partial C\rangle = Q|C\rangle,$$
   where Q acts on each of the $m+n$ components H of Hom, as usual.
5. **Normalization**: The point {Riemann sphere with two unit disks around 0 and ∞ cut out} $\in \mathcal{P}_{m+n}$ maps to the identity operator id : $H \to H$.

This correspondence can also be axiomatized as a functor, like in the cases of TFT and CFT. The corresponding category *Segal* will still have disjoint unions of circles as objects, but its morphisms will be chains in the moduli spaces. In the category *Hilbert*, one has to consider graded spaces with differentials (i.e., complexes), but still all linear mappings as morphisms.

The Virasoro semigroup of cylinders (including the group of diffeomorphisms of the circle, represented by cylinders of zero width) acts on H via the degree 0 states $\exp tT(v) = |\exp tv\rangle$ corresponding to cylinders, regarded as points in $\mathcal{P}_{1+1}$:

$\exp tv$

$$\longmapsto \quad \exp tT(v) : H \to H, \tag{4}$$

where v is the generating complex vector field on the circle. The so-called *antighost* operators $b(v)$ on H can also be easily identified in our picture. They are the derivatives

$$b(v) = \frac{d}{dt} B(tv)|_{t=0} \tag{5}$$

of the operators $B(tv)$ of degree -1 obtained when the same cylinder corresponding to v is regarded as a one-chain in $\mathcal{P}_{1+1}$. At time t, the cylinder

$\exp(tv)$ is a point in $\mathcal{P}_{1+1}$. When t changes, these points sweep out a path in $\mathcal{P}_{1+1}$. Note that

$$[T(v_1), T(v_2)] = T([v_1, v_2]),$$

because the operators $\exp tT(v)$ define a representation of the Virasoro semigroup, and

$$\{b(v_1), b(v_2)\} = 0,$$

because the two-chains $\exp(sv_1) \times \exp(tv_2)$ and $\exp(tv_2) \times \exp(sv_1)$ differ only by orientation. In particular, $b^2(v) = 0$. Moreover,

$$\{Q, b(v)\} = T(v),$$

because the boundary of the cylinder $\exp(tv)$ viewed as a one-chain is equal to the same cylinder viewed as a point minus the trivial zero-width cylinder.

String theories are also referred to as topological, because of the following fact.

THEOREM 3. *The cohomology of the state space H of a string background with respect to the differential Q forms a TFT. Thus, the cohomology of H has a natural structure of a Frobenius algebra.*

Proof. Two Riemann surfaces Σ_1 and Σ_2 which are diffeomorphic can be connected by a smooth path C in the moduli space. Hence, for the corresponding states we have

$$|\Sigma_2\rangle - |\Sigma_1\rangle = |\partial C\rangle = Q|C\rangle,$$

which means that their Q-cohomology classes are equal. □

4.2. Higher Brackets

The state $|C\rangle$ is an operator from H^m to H^n, which for $n = 1$ may be thought of as an m-ary operation on the space H. By the factorization axiom, the operation of sewing of chains C in the moduli spaces corresponds to compositions of the corresponding operations on the space H. Respectively, any relation (involving compositions and boundaries) between chains in the moduli spaces produces an identity (involving compositions and the differential Q) for the corresponding operations on H. At the tree level, when we consider Riemann surfaces of genus 0 only, this algebraic structure on H is rather tamable. This is because the topology of the finite dimensional moduli spaces $\mathcal{M}_{0,m+1}$ of isomorphism classes of $m+1$ punctured Riemann spheres takes over the situation.

Consider the following brackets:

$$[x_1, \dots, x_m] = \text{“}|\mathcal{M}_{0,m+1}\rangle\text{”}(x_1, \dots x_m), \qquad m \geq 2, \tag{6}$$

where $x_1, \dots, x_m \in H$ are substituted on the right-hand side as arguments of the $\mathrm{Hom}(H^m, H)$, where the state $|\mathcal{M}_{0,m+1}\rangle$ lives. The quotes are due to the fact that the space $\mathcal{M}_{0,m+1}$ is not really a chain in $\mathcal{P}_{m+1}$: there is no natural mappings from $\mathcal{M}_{0,m+1}$ to $\mathcal{P}_{m+1}$. A standard escape is to impose these mappings as extra part of data. To preserve nice properties, this is achieved in the following two steps.

Step 1. Push the correspondence $C_\bullet \mathcal{P}_{m+1} \to \mathrm{Hom}(H^m, H)$ down to a mapping $C_\bullet \mathcal{P}'_{m+1} \to \mathrm{Hom}((H^{\mathrm{rel}})^m, H^{\mathrm{rel}})$ from chains on the quotient space $\mathcal{P}'_{m+1}$ of $\mathcal{P}_{m+1}$ by rigid rotations of the holomorphic disks to the space of multilinear operators on H^{rel}. The latter is the subspace H^{rel} of vectors in H which are *rotation-invariant*, i.e., stable under the operators $\exp(tT(v))$ of (4) and annihilated by the operators $B(tv)$ of (5) corresponding to rigid rotations $v \in S^1$. The pushdown is performed by pulling a chain C in $\mathcal{P}'_{m+1}$ back to a chain $\widetilde{C}$ of the same dimension in $\mathcal{P}_{m+1}$, restricting the operator $|\widetilde{C}\rangle$ to $(H^{\mathrm{rel}})^m$ and projecting the value of the operator $|\widetilde{C}\rangle$ onto H^{rel} via the mapping $h \mapsto b(\partial/\partial\theta)h_0$, where θ is the phase parameter on the circle S^1 and h_0 is the rotation-invariant part of h (which exists provided the action of S^1 on H is diagonalizable).

Step 2. Map the finite dimensional moduli spaces $\mathcal{M}_{0,m+1}$ to the infinite dimensional quotient spaces $\mathcal{P}'_{m+1}$, so that gluing Riemann spheres in $\mathcal{M}_{0,m+1}$'s at punctures corresponds to sewing of Riemann spheres in $\mathcal{P}'_{m+1}$'s. Sewing in $\mathcal{P}'_{m+1}$'s can only be performed provided at least relative phases at sewn disks are given. The corresponding gluing operation should also be of this kind. Thus, the gluing operation takes us actually out of the spaces $\mathcal{M}_{0,m+1}$ to certain real compactifications of them, [5]. Such mappings $\mathcal{M}_{0,m+1} \to \mathcal{P}'_{m+1}$ exist. Zwiebach's string vertices [12] make up an example of those. Here we thereby allow certain freedom of their choice.

These additional data have been called a *closed string-field theory* in [5] after Zwiebach gave this title to the choice of his string vertices. After these modifications, we obtain brackets $[x_1, \dots, x_m]$ defined on H^{rel}.

THEOREM 4. *These brackets define the structure of a homotopy Lie algebra (see next section) on the space* H^{rel}.

This result was obtained by Zwiebach in [12]. A mathematically rigorous proof of this theorem with the use of operads was given in [5]. This algebraic structure generalizes the trivial one of Corollary 2 to the case of string theory.

4.3. Homotopy Lie Algebras

A *homotopy Lie algebra* is a graded vector space H, together with a differential Q, $Q^2 = 0$, of degree 1 and multilinear graded commutative brackets $[x_1, \dots, x_m]$ of degree $3 - 2m$ for $m \geq 2$ and $x_1, \dots, x_m \in H$, satisfying the identities

$$Q[x_1, \ldots, x_m] + \sum_{i=1}^{m} \epsilon(i)[x_1, \ldots, Qx_i, \ldots, x_m]$$
$$= \sum_{\substack{k+l=m+1 \\ k,l \geq 2}} \sum_{\substack{\text{unshuffles } \sigma: \\ \{1,2,\ldots,m\} = I_1 \cup I_2, \\ I_1 = \{i_1, \ldots, i_k\}, \\ I_2 = \{j_1, \ldots, j_{l-1}\}}} \epsilon(\sigma)[[x_{i_1}, \ldots, x_{i_k}], x_{j_1}, \ldots, x_{j_{l-1}}],$$

where $\epsilon(i) = (-1)^{|x_1|+\ldots+|x_{i-1}|}$ is the sign picked up by taking Q through x_1, $\ldots, x_{i-1}$, $|x|$ denoting the degree of $x \in H$, $\epsilon(\sigma)$ is the sign picked up by the elements x_i passing through the x_j's during the unshuffle of $x_1, \ldots, x_m$, as usual in graded algebra.

Note that for m=2, we have

$$\begin{aligned} Q[x_1, x_2, x_3] &+ (\pm[Qx_1, x_2, x_3] \pm [x_1, Qx_2, x_3] \pm [x_1, x_2, Qx_3]) \\ &= [[x_1, x_2], x_3] \pm [[x_1, x_3], x_2] \pm [[x_2, x_3], x_1], \end{aligned}$$

which means that the graded Jacobi identity is satisfied up to a null-homotopy, the Q-exact term on the left-hand side.

5. Topological Gravity

A *topological gravity* is the same as a string background, except that it is based on a graded vector space V which is not required to have a differential Q and that the correspondence $C_\bullet \mathcal{P}_{m+n} \to \mathrm{Hom}(H^{\otimes m}, H^{\otimes n})$ is replaced with

$$H_\bullet \overline{\mathcal{M}}_{m+n} \to \mathrm{Hom}(V^{\otimes m}, V^{\otimes n}),$$

where $\overline{\mathcal{M}}_{m+n}$ is the Deligne-Knudsen-Mumford compactification of the moduli space and $H_\bullet$ stands for homology. Sewing in the factorization property should be replaced with gluing at punctures to form double points similar to Step 2 in Section 4.2, but with no relative phases. This notion of a topological gravity is essentially the same as the notion of a homotopical field theory of Morava [9].

Another notion closely related to topological gravity is in certain sense a *dual topological gravity*, where the real compactification of [5] replaces the Deligne-Knudsen-Mumford one. At the tree level, when the Riemann surfaces have genus 0 and $n = 1$, this theory is dual to the one above in the sense that the underlying operads are Koszul dual, see Getzler and Jones [3].

THEOREM 5. 1. *A string background based on a state space H yields the structure of a dual topological gravity on the space V of Q-cohomology of H^{rel}.*

2. *A dual topological gravity implies a TFT on its state space V.*

Proof. 1 becomes evident if we observe that the quotient of $\mathcal{P}_{m+n}$ modulo rigid rotations at each puncture is homotopically equivalent, respecting sewing, to (the real compactification of) $\mathcal{M}_{m+n}$.
2. A TFT is obtained by restricting a dual toplogical gravity to a correspondence $H_0(\mathcal{M}_{m+n}) \to \mathrm{Hom}(V^m, V^n)$. □

5.1. Gravity Algebras

At the tree level, a dual topological gravity also gives rise to a remarkable algebraic structure on V. This structure is called a *gravity algebra* and was introduced by Getzler [2]. It consists of an infinite number of multilinear brackets, staisfying quadratic equations. It would be interesting to describe the algebraic structure corresponding to a topological gravity, i.e., the structure of an algebra over the operad $H_\bullet \overline{\mathcal{M}}_{m+1}$, in similar terms.

5.2. Homotopy Commutative Algebras

According to general ideology, cf. Kontsevich [7] and Ginzburg-Kapranov [4], there are three principal types of homotopy algebras: homótopy Lie, homotopy commutative and homotopy associative. The first two types are dual in certain sense, the third one is self-dual. It is remarkable that this duality is implemented in algebraic geometry by passing from the usual Deligne-Knudsen-Mumford compactification of the moduli space to the real version of it.

More precisely, suppose we are given a complex V of vector spaces and a correspondence

$$\begin{aligned} C_\bullet \overline{\mathcal{M}}_{m+n} &\to \mathrm{Hom}(V^{\otimes m}, V^{\otimes n}), \\ C &\mapsto |C\rangle, \end{aligned}$$

which is a multiplicative functor between the corresponding tensor categories, i.e., compatible with gluing at punctures, permutations, differentials, etc. Such a theory may be regarded as topological gravity lifted to the chain level from homology. Suppose it satisfies the additional condition $|C\rangle = 0$ whenever $\dim C > (1/2) \dim \overline{\mathcal{M}}_{m+n}$. This condition is a kind of chirality, not literally, though: no holomorphicity is assumed.

If we consider $m-2$-cycles (more exactly, half-dimensional cycles relative to the boundary) in $\mathcal{M}_{0,m+1}$ instead of the fundamental cycle to define m-ary products as in (6), the operad approach of [5] will lead to the structure of a homotopy commutative algebra [3, 4]. This is the matter of the forthcoming paper [6].

Acknowledgements

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Note. Since this paper was written up, our question (see Section 5.1) of describing the algebraic structure of topological gravity has been answered by Kontsevich-Manin [8] and Dijkgraaf-Getzler. This structure may be described as a family of graded commutative associative multiplications on a vector space V parameterized by the very space V. Following Getzler, it is reasonable to call it a *WDVV-algebra*, after Witten-Dijkgraaf-Verlinde-Verlinde, who observed it in quantum field theory.

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CONFORMAL FIELD THEORY, DILOGARITHMS, AND THREE DIMENSIONAL MANIFOLDS

WERNER NAHM
Physikalisches Institut der Universität Bonn
Nussalle 12 D-5300 Bonn 1
nahm@pib1.physik.uni-bonn.de

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The dilogarithm $Li_2(z)$ is defined by

$$\frac{d}{dz}Li_2(z) = -\frac{\log(1-z)}{z}, \quad \text{for non-real } z,$$

$$Li_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \quad \text{for} \quad |z| < 1 .$$

In particular $Li_2(1) = \pi^2/6$. Its analytic continuation is multivalued and will be considered below.

A q-deformed version of the dilogarithm is given by

$$-\sum_{n=1}^{\infty} \log(1-uq^n) , \quad |q| < 1.$$

When the absolute value of q is close to 1, one can approximate the sum by an integral, which yields $Li_2(u)/\log(q)$.

Dilogarithms have a long history both in physics and in mathematics, which would take to long to describe. In physics, they appear in the evaluation of Feynman graphs, which at present has no relations to the new applications considered below. The latter seem to have originated first in the investigations of Faddeev's Leningrad/ St.Petersburg group. On one hand, q-deformed dilogarithms describe the S-matrix of the sine-Gordon model [FK 1978, eq. 5.3], on the other hand the magnetization of the XXY model was linked to the central charge of the corresponding conformal theory and later calculated in dilogarithmic form, see e.g. [KR 1987]. Parts of the group dispersed, but the investigations were taken up elsewhere. In particular, much evidence was produced which links dilogarithms to conformal dimensions, see [FNO 1992]. We shall see that the corresponding dilogarithm identities correspond to elements of finite order of the so called Bloch group. The fact that the conformal dimensions are rational is closely linked to the finite order property.

On the other hand, infinite order elements of the Bloch group seem to be essential for the classification of three dimensional manifolds, one of the most interesting problems of present day mathematics. Some relation to the classification of two dimensional conformal field theories can be expected, since the latter yield topological field theories in three dimensions and those in turn yield invariants of manifolds of three (real) dimensions. These invariants are of the type of the well known Chern-Simons and η-invariants. Again, they are rational and related to the real part of dilogarithms. At least as important, however, is the volume invariant of Thurston's classification program for three-manifolds [Thurston 77,82], [DS 1982], [NZ 85]. This invariant yields imaginary values of the dilogarithm function, for Bloch group elements of infinite order.

For an elementary introduction to Thurston's program see [Meyerhofer 1992]. For convenience of the reader, I repeat some of the essential points. Three dimensional manifolds have canonical decompositions with respect to cuts along spheres and tori. For manifolds which are indecomposable with respects to such cuts, Thurston argued that they can be given a geometric structure. In other words, they can be written as the quotient of a homogeneous space with respect to the action of a discrete group. Thurston proved his conjecture under various conditions, but the general problem is still open.

In two dimensions it is easy to write any manifold in such a way, namely just as the quotient of its covering space by its fundamental group. In fact, the covering space of any compact two dimensional Riemannian manifold is either the sphere, the plane or the hyperbolic space, which all can be given a homogeneous metric. Apart from the torus, the curvature can be normalized to ± 1, which gives the manifolds a canonical volume. Due to the Gauss-Bonnet theorem

$$\int R dV = 4\pi(1-g) \; ,$$

the classification by the genus g and the one by the normalized volume are equivalent.

For almost all values of the genus, indeed for $g \neq 0, 1$, the geometric structure is hyperbolic. In three dimensions, there is a sense in which the generic manifolds have a hyperbolic structure, too. Note first that every three dimensional manifolds can be constructed by Dehn twists around some link in the sphere S^3. To perform such twists, one cuts out a small tubular neighborhood around each knot component of the link, transforms the surface of each resulting solid torus by a diffeomorphism of the mapping class group $SL(2,Z)$, and glues it back in. More precisely, consider a solid torus $D \times S^1$, where D is the unit disk with polar coordinates r, ϕ. On S^1 we have an angle coordinate θ. The mapping class group of the torus surface is given by the transformations

$$\begin{pmatrix} \phi \\ \theta \end{pmatrix} \to \begin{pmatrix} p & r \\ q & s \end{pmatrix} \begin{pmatrix} \phi \\ \theta \end{pmatrix}, \qquad \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in SL(2,Z) .$$

Thus it can be identified with the modular group $SL(2,Z)/Z_2$. The transformations generated by $\binom{11}{01}$ can be continued to the whole solid torus, such that the corresponding Dehn twists do not change the topology of the manifold. Thus the resulting manifolds are given by the cosets of $\binom{pr}{qs}$ modulo this subgroup, in other words by the first matrix column (p,q). Except for special links or special small values of the (p,q), the manifold constructed by Dehn twists will be hyperbolic. In this sense, generic manifolds are hyperbolic.

The Dehn twist construction is very far from a classification, since it is highly non-unique and since there is no effective classification of knots. Nevertheless, the construction easily generates many hyperbolic manifolds of small volume. In particular, take the Dehn twists around the figure-of-eight knot. The volume is an increasing function of the partially ordered labels (p,q). Its minimum at (1,5) is conjectured to be the smallest value in the set $\mathcal{V}$ of all possible volumes of hyperbolic three-manifolds of curvature -1.

To calculate such a volume, one cuts the manifold into tetrahedral pieces. The volumes of hyperbolic tetrahedra first were calculated by Lobatchevsky. His formula is a bit complicated, but it simplifies a lot for ideal tetrahedra, for which the vertices lie at infinity. Disregarding lower dimensional submanifolds, any hyperbolic manifold can be cut up into such ideal tetrahedra. For compact manifolds, one just has to cut out a circular geodesic, which comes to lie at the infinite points of the hyperbolic space. The sides of the tetrahedra all wind around this geodesic and converge towards it.

In terms of the quaternions $1, i, j, k$, the points of hyperbolic space can be written in the form $X = x + iy + jz$, $z > 0$. Consider the matrices $\binom{ab}{cd} \in SL(2,C)$, where the complex numbers C are given by the linear combinations of the quaternions $1, i$. Their group acts on hyperbolic space by the transformations

$$X' = (aX + b)(cX + d)^{-1} .$$

To see this, note that

$$X' = (aX + b)(X^+ c^+ + d^+)|cX + d|^{-2} .$$

The j, k part of X' is proportional to

$$ajd^+ - bjc^+ = (ad - bc)j = j \ .$$

In particular,

$$z' = z|cX + d|^{-2} > 0 \ .$$

Since

$$dX' = -c^{-1}(cX + d)^{-1} c\, dX\, (cX + d)^{-1} \ ,$$

the metric

$$(dx^2 + dy^2 + dz^2)/z^2 = dX dX^+/z^2$$

is conserved under $SL(2, C)$. The boundary at infinity of hyperbolic space is given by the plane $z = 0$ plus a point at infinity, thus it is isomorphic to the Riemann sphere. Its transformations under $SL(2, C)$ are the usual rational linear ones.

The volume of an ideal tetrahedron depends on its four vertices $z_i \in C$, $i = 1, 2, 3, 4$. As we have seen it is invariant under rational linear transformations. Invariant functions of the z_i only depend on the double ratio

$$z = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} \ .$$

Let the volume function be denoted by $V(z_1, z_2, z_3, z_4) = D(z)$. For real z, all vertices lie on one line, such that the volume vanishes. It is convenient to incorporate the tetrahedron orientation by the sign of the volume, such that $D(\bar{z}) = -D(z)$. Permuting the vertices yields the symmetry properties $D(z) = -D(1 - z) = -D(1/z)$. As long as no vertices coincide, D is a real analytic function of its argument.

The union of two tetrahedra joined along a face can be cut into three tetrahedra by using the body diagonal as a new edge. Similarly, each tetrahedron can be cut up into four tetrahedra by choosing another vertex in the interiour. This yields

$$\sum_{i=0}^{5} (-)^i V(\hat{z}_i) = 0 \ ,$$

where $\hat{z}_i$ denotes z_0, z_1, z_2, z_3, z_4 with z_i omitted. Equivalently one has

$$D(x) + D(y) + D(1 - xy) + D\left(\frac{1 - x}{1 - xy}\right) + D\left(\frac{1 - y}{1 - xy}\right) = 0 \ .$$

This is the five term identity, which has been discovered and rediscovered by several famous mathematicians. With the convention $D(\infty) = 0$, it implies the symmetry properties of D.

Under very weak regularity conditions, which are obviously true for the tetrahedron volumes, the five term identity and the relatity properties yield

$$D(z) = \Im(Li_2(z)) + arg(1-z)\log|z| \ .$$

The dilogarithm Li_2 has a multivalued analytic continuation, but $D(z)$ becomes a continuous function on the whole plane. Away from the singularities at $z = 0$ and $z = 1$, it is real analytic.

Splitting the five term identity for $D(z)$ into its holomorphic and antiholomorphic parts, one obtains a five term identity for the Rogers dilogarithm

$$L(x) = Li_2(x) + \frac{1}{2}\log(x)\log(1-x) \ ,$$

namely

$$L(x) + L(y) + L(1-xy) + L\left(\frac{1-x}{1-xy}\right) + L\left(\frac{1-y}{1-xy}\right) = \pi^2/2 \ .$$

The constant $3L(1) = \pi^2/2$ on the right hand side is determined by putting $x = y = 0$. The identity is valid for $x, y \in (0,1)$.

Thus volumes of hyperbolic three-manifolds have the form

$$V = \sum_k D(z_k) \ .$$

The fact that the tetrahedra fit together to form a manifold without boundary yields the closure condition

$$\sum_k [z_k] \wedge [1-z_k] = 0 \ ,$$

where the symbol $[z]$ fulfils the single relation $[xy] = [x] + [y]$ and the wedge product is defined by bilinearity and antisymmetry. Replacing adjoining tetrahedra as in the five term relation conserves this condition, since

$$[x] \wedge [1-x] + [y] \wedge [1-y] =$$
$$[xy] \wedge [1-xy] + \left[\tfrac{x(1-y)}{1-xy}\right] \wedge \left[\tfrac{1-x}{1-xy}\right] + \left[\tfrac{y(1-x)}{1-xy}\right] \wedge \left[\tfrac{1-y}{1-xy}\right] \ ,$$

as can be checked easily.

The Bloch group [Bloch 78] is defined by the formal sums $\sum_k n_k(z_k)$, $z_k \in C$, $n_k \in Z$, which satisfy the closure condition

$$\sum_k n_k [z_k] \wedge [1-z_k] = 0 \ ,$$

modulo the formal sums coming from the five term identity, for which this condition always is satisfied. Moreover, one uses the convention $(\infty) = 0$, which implies $(0) = (1) = 0$. This is no significant restriction, since $15(\infty)$

vanishes by the five term identity. The map $D : \sum_k n_k(z_k) \to \sum_k n_k D(z_k)$ is a well defined map from the Bloch group elements to the real numbers. The volume set $\mathcal{V}$ belongs to the image of this map.

Since the closure equation is essentially algebraic, this implies that $\mathcal{V}$ is a countable set. Admitting disjoint unions and manifolds with boundaries, this set becomes closed and additive. Moreover, one can show that it is well ordered. In other words, for every volume there is a unique next larger volume. Accumulation points only arise by convergence towards upper limits, not towards lower ones. Let $\mathcal{V}'$ be the subset of accumulation points of $\mathcal{V}$ and iterate this procedure to obtain the sets $\mathcal{V}^{(n)}$ of n-fold accumulation points. One finds that all of these sets are non-empty, though they have empty intersection. In the language of ordinal numbers, this is expressed by the fact that the ordinal number of $\mathcal{V}$ is ω^ω.

Many elements of the Bloch group can be produced by the equations

$$\log(1 - z_i) = \sum_j B_{ij} \log(z_i) \ ,$$

$i = 1, \ldots, r$. If one makes Dehn twist around the figure-eight knot one finds $r = 2$ and

$$B = -\frac{1}{p+q} \begin{pmatrix} p & q \\ q & p \end{pmatrix} \ .$$

So far, we have considered some standard manifold mathematics. Now let us cosider the partition functions of some conformal field theories, which yield new, but apparently related features. On the Hilbert space of such a theory one has the action of left and right Virasoro algebras with generators L_n, L'_n and central extension c. The Hamiltonian is given by $H = L_0 + L'_0$ is the momentum by $P = L_0 - L'_0$. For our purposes it is convenient to shift these generators, such that $\tilde{L}_0 = L_0 - c/24$ and analogously for L'_0. Consider the partition function

$$Z(\tau, \bar{\tau}) = tr(\exp(2\pi i(\tilde{L}_0 \tau - \tilde{L}'_0 \bar{\tau})))$$

of such a theory, such that the imaginary part of τ can be identified with the inverse temperature.

The partition functions of conformal theories are invariant under modular transformations

$$\tau \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \phi \\ \theta \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z) \ .$$

Using $\tau \to -1/\tau$, one reads off the high temperature behaviour

$$Z \sim \exp\left(\frac{\pi i}{12} c_{\text{eff}} (1/\tau - 1/\bar{\tau})\right) \ .$$

The quantity c_{eff} is called left and right effective central charge of the theory. Since the partition function diverges at large temperature, the effective central charges have to be non-negative. For a unitary theory, they coincide with the central charge c of the Virasoro algebra. More generally, c_{eff} is given by the maximum of $c - 24h$, where h runs over eigenvalues of L_0.

The set $\mathcal{C}$ of values of c_{eff} for all possible conformal theories is conjectured to share many properties of $\mathcal{V}$. It is additive, since the tensor product of theories yields the sum of the effective central charges. At least for rational theories, the effective central charges are rational, and this may be generally true. In fact, $\mathcal{C}$ is conjectured to be well ordered, with the same ordinal number ω^ω as $\mathcal{V}$.

For rational theories one has

$$Z(\tau,\bar{\tau}) = \sum_i Z_i(\tau) Z_i'(\bar{\tau}) ,$$

where the sum is finite and runs over the superselection sectors of the theory. Asymptotically, all the Z_i are proportional. The proportionality constants are called conformal dimensions. With $q = \exp(2\pi i\tau)$ the asymptotic behaviour is obtained for $|q|$ close to 1. One finds

$$Z_i(\tau) \sim \exp(-\frac{\pi^2}{6} c_{\text{eff}} / \log(q)).$$

The normalization is chosen such that $\delta_0 = 1$ for the vacuum character Z_0.

For $c_{\text{eff}} < 1$, the rational conformal theories are classified by pairs (p, q) of natural numbers > 1 without common prime divisors. One has $c_{\text{eff}} = 1 - 6/pq$, such that the possible values in $\mathcal{C}$ are given by $c_{\text{eff}} = 1 - 6/n$, where n runs over those natural numbers which are not prime powers. For $c_{\text{eff}} > 1$ not much is known.

Let us calculate partition functions and effective central charges for a few simple theories. The partition function of a free boson is proportional to

$$P(\tau) = \prod_k (1 - q^k)^{-1} ,$$

One has

$$P(\tau) = \sum_k p(k) q^k ,$$

where $p(k)$ counts the additive partitions of k into natural numbers. If we denote the number of partitions into exactly n natural numbers by $p_n(k)$, we find

$$Z(\tau) = \sum_n \sum_k p_n(k) q^k .$$

Any partition can be written in the form $k = m_1 + m_2 + \ldots + m_n$, where $m_i \geq m_{i+1}$. Writing $l_i = m_i - m_{i+1}$, the l_i are independent variables. Since $k = \sum_i il_i$, one finds

$$\sum_k p_n(k)q^k = (q)_n^{-1} ,$$

where

$$(q)_n = (1-q)(1-q^2)\ldots(1-q^n) .$$

For a free fermion, we have a very similar partition function, except that the m_i have to be different due to the Pauli principle. To get independent variables, we have to write $l_i = m_i - m_{i+1} - 1$. Moreover, we can use integral or half-integral k. In the latter case we obtain

$$Z_0(\tau) = q^{\tilde{h}_0} \sum_n q^{n^2/2}/(q)_n ,$$

in the former

$$Z_1(\tau) = q^{\tilde{h}_1} \sum_n q^{n(n+1)/2}/(q)_n .$$

The values of the $\tilde{h}_i$ are given by the lowest eigenvalues of $\tilde{L}_0$ in the coresponding superselection sector. Their minimal value is $-c_{\text{eff}}/24$.

Finally let us consider the conformal theory given by the Lee-Yang edge singularity. The theory is minimal, such that all holomorphic fields are generated by the energy momentum density $T(z)$. The normal ordered product $:TT:$ is proportional to the second derivative of T and does not yield an independent state. Taking Fourier coefficients, one sees that products of the form L_nL_n and L_nL_{n+1} can be reduced to simpler ones, which looks like an extended Pauli principle. For the partition function this means $m_i \geq m_{i+1} + 2$. With the independent variables $l_i = m_i - m_{i+1} - 2$ one finds

$$Z_0(\tau) = q^{\tilde{h}_0} \sum_n q^{n(n+1)}/(q)_n$$

and

$$Z_1(\tau) = q^{\tilde{h}_1} \sum_n q^{n^2}/(q)_n .$$

in the two superselection sectors.

When we tensorize r theories of this kind, the characters are multiplied. To write the product characters in analogous form, we consider n as a vector with r components and define $(q)_n = (q)_{n_1} \ldots (q)_{n_r}$. The exponent in the numerator becomes a quadratic form

$$Q(n) = \frac{1}{2} nBn + bn + \tilde{h} ,$$

with a diagonal $r \times r$ matrix B and a vector b which depends on the representation.

Other conformal theories have characters of analogous form, but with more general symmetric matrices B. Thus we consider characters of the form

$$\sum_{n_1,\ldots,n_r} q^{Q(n)}/(q)_n \ .$$

To calculate c_{eff}, one has to evaluate all these sums for q close to 1. This can be done by a saddle point calculation. First one interpolates the summands by a continuous function of n, using

$$(q)_m^{-1} = (q)_\infty^{-1} \prod_k^\infty (1 - q^m q^k) \ .$$

The logarithm of the product is essentially the q-deformed dilogarithm of q^m. In leading order it can be replaced by $-Li_2(q^m)/\log(q)$. Now $(q)_\infty$ is essentially the Dedekind η-function, whose modular behaviour is well known. In particular, we have

$$\log(q)_\infty \sim Li_2(1)/\log(q)$$

in leading order.

Varying the n_i yields a stationary point for $x_i = q^{n_i}$ with

$$\log(1 - x_i) = \sum_j B_{ij} \log(x_i) \ .$$

For the effective central charge one obtains

$$c_{\text{eff}} = \sum_i L(1 - x_i)/L(1) \ ,$$

where we used the Rogers dilogarithm and its properties $L(1) = \pi^2/6$, $L(1)-L(x) = L(1-x)$.

For the free boson, $B = 0$ and $c_{\text{eff}} = 1$. For the free fermion $B = 1/2$, which yields $c_{\text{eff}} = 1/2$. For the Lee-Yang edge singularity $B = 2$, such that $1-x = x^2$, which yields the golden ratio. The five term identity immediately yields $5L(x) = 3L(1)$ and $c_{\text{eff}} = 2/5$.

The characters of conformal field theories are modular, which essentially means that the saddle point approximation gets no perturbative correction, in the sense of a power series expansion in τ. To find these corrections, one uses the expansion

$$\begin{aligned}\textstyle\sum_{k=1}^\infty \log(1 - uq^k) &= Li_2(u)/2\pi i\tau - \tfrac{1}{2}\log(1-u) \\ &+ \textstyle\sum_m \sum_k \frac{u^m}{m}(2\pi i m\tau)^{2k+1}\frac{B_{2k+2}}{(k+2)!}\end{aligned}$$

with $u = q^n$. Here the B_k are the Bernoulli numbers, which apart from B_1 vanish for odd k. Finally one uses the fact that $q^{1/24}(q)_\infty$ is a modular form and has no perturbative corrections [NRT 93].

An alternative expression for these corrections can be obtained by the following method. First one writes the character in the form

$$\oint \sum_n u^{-n} q^{Q(n)} \sum_m u^m/(q)_m du/2\pi i u\ .$$

The first sum can be transformed by Poisson summation. For the second sum one uses

$$\sum_m u^m/(q)_m = \prod_{k=0}^{\infty}(1-uq^k)^{-1}$$

and its expansion given in the previous paragraph.

The perturbation expansion has exactly the same form as before, except for the substitution of τ by $-\tau$ and of $Q(n) = \frac{1}{2}nBn + bn + \tilde{h}$, by

$$Q'(n) = \frac{1}{2}nB^{-1}n + nB^{-1}b + \tilde{h}'\ ,$$

where

$$\tilde{h}' = \tilde{h} - \frac{r}{24} + \frac{1}{2}bB^{-1}b\ .$$

In other words, when a pair B, b yields partition functions without power law correction, then $B^{-1}, B^{-1}b$ does the same. The duality between B and B^{-1} generalizes the level rank duality known from the characters of Kac-Moody algebras, as we shall see.

First, however, let us study non-perturbative corrections. Using the Jacobi triple product identity

$$(1-u)\prod_n(1-uq^n)(1-q^n)(1-u^{-1}q^n) = \sum_n(-)^n u^n q^{n(n-1)/2}$$

and Poisson summation of the right hand side plus reverse application of the Jacobi triple product identity one finds

$$q^{\frac{1}{12}}\prod_n(1-uq^n)(1-u^{-1}q^n) = \frac{iu^{1/2}}{u-1}\tilde{u}^{-1/2}\tilde{q}^{\frac{1}{12}}e^{-\pi i z^2/\tau}F(\tilde{u},\tilde{q})\ ,$$

where $u = exp(2\pi i z)$, $\tilde{u} = exp(-2\pi i z/\tau)$, $q = exp(2\pi i\tau)$, $\tilde{q} = exp(-2\pi i/\tau)$,

$$F(\tilde{u},\tilde{q}) = (1-\tilde{u})\prod_n(1-\tilde{u}\tilde{q}^n)(1-\tilde{u}^{-1}\tilde{q}^n)\ .$$

We split the right hand side into a factor regular at zero and one regular at ∞. In particular, the dilogarithm symmetry yields

$$\frac{iu^{1/2}}{u-1}\tilde{u}^{-1/2}\tilde{q}^{\frac{1}{12}}e^{-\pi i z^2/\tau}$$
$$= \exp((Li_2(u) + Li_2(u^{-1}))/2\pi i\tau - \tfrac{1}{2}\log(1-u) - \tfrac{1}{2}\log(1-u^{-1}) .$$

By the residue theorem we have

$$\log F = -\frac{1}{2}\int_{-i\infty}^{i\infty}\log(1-e^{-2\pi i s/\tau})\left(\frac{e^{2\pi i s}u+1}{e^{2\pi i s}u-1}+\frac{e^{2\pi i s}+u}{e^{2\pi i s}-u}\right)ds .$$

This yields the unique splitting

$$\textstyle\sum_{k=1}^{\infty}\log(1-uq^k) = Li_2(u)/2\pi i\tau - \frac{1}{2}\log(1-u) -$$
$$\textstyle -\frac{1}{2}\int_0^{\infty}\log(1-e^{-2\pi i s/\tau})\left(\frac{e^{2\pi i s}u+1}{e^{2\pi i s}u-1}+\frac{e^{2\pi i s}+u}{e^{2\pi i s}-u}\right)ds ,$$

which is exact for $|u| < 1$ and takes care of non-perturbative terms.

Contour integration yield the contributions of characters with non-minimal $\tilde{h}$. One must be careful with its analytic continuation, since the dilogarithm has cuts at arguments $0, 1$. To handle these difficulties, it is convenient to define the Rogers dilogarithm $L(x)$ only modulo $4\pi^2$ and as a function of U, V, such that

$$e^U = 1 - e^V = x.$$

Starting from $0 < x < 1$ and real U, V, one obtains by analytic continuation a function $L(U, V)$ which is one valued modulo $4\pi^2$. The five term identity now takes the form

$$\sum_{i=1}^{5} L(U_i, V_i) = \pi^2/2 \qquad \text{mod } 4\pi^2 ,$$

if $U_{i-1} + U_{i+1} = V_i$ and $U_{i+5} = U_i$, $V_{i+5} = V_i$. Note that U, V are well defined on the Riemann surface of the dilogarithm.They are found by deforming the integration contour into the Riemann surface of the dilogarithm and using the variables U, V instead of x. One finds stationary points for

$$V_i = \sum_j B_{ij} U_i .$$

For the $\tilde{h}$ we obtain

$$-\tilde{h} = \sum_i L(U_i, V_i)/(4\pi^2) \qquad \text{mod } 1 .$$

For any given stationary point we can find others by adding $2\pi i n_i$ to V_i and $2\pi i m_i$ to U_i, as long as the m_i, n_i are integral and $n = Bm$. This changes the value of $\tilde{h}$ by $nm/2$. For bosonic theories, the eigenvalues of L_0 are integrally space. Thus B must have a form such that nm only takes even values. Such matrices may be called even. As long as the matrix elements are integral, this terminology coincides with the usual condition that even matrices have even diagonal entries. For fermionic theories, half-integral spacing is allowed in the Neveu-Schwarz sector.

The sum $\sum_i [x_i]$ belongs to the Bloch group, since one checks easily $\sum_i [x_i] \wedge [1-x_i] = 0$. Moreover, $\sum_i D(x_i) = 0$, since the eigenvalues $\tilde{h}$ of L_0 are real. It has been proven that this property implies that $\sum_i [x_i]$ is a torsion element of the Bloch group. In other words, for some N, $\sum_i N[x_i]$ vanishes by the five term relation.

Already at the present stage, we obtain a simple interpretation of level rank dualities. In the simplest case, this duality relates the $SU(N)$ Kac-Moody algebra at level M and the $SU(M)$ Kac-Moody algebra at level N. In particular, the central charges of the two theories add up to the integer $NM-1$. In our formalism, every torsion element $\sum_{i=1}^{r}[x_i]$ of the Bloch group has a dual element $\sum_{i=1}^{r}[1-x_i]$. The corresponding matrix B' is just the inverse of B, as expected. The effective central charges add up to the matrix rank r.

In terms of $L(U,V)$, the Bloch group is given by equivalence classes of sums $\sum_i n_i(U_i, V_i)$,

$$\exp(U_i) + \exp(V_i) = 1 \ ,$$

with the closure condition $\sum_i U_i \wedge V_i = 0$, using the ordinary wedge product over the vector space of complex numbers. If $N \sum_i U_i \wedge V_i$ is a sum of five term relations in U, V, the five term identity for $L(U,V)$ implies that the denominators of $\tilde{h}$ have to divide $8N$. This is obviously true for the examples considered above. Since the free fermion yields $N = 2$ and in particular $h = 1/16$, the result is the best possible of this form.

There is some hope to classify the matrices B which yield torsion elements. For $r = 1$, the only relevant elements are $[1/2]$, $[1-\tau]$ and the golden ratio defined by $\tau + \tau^2 = 1$. The corresponding matrices for the first two cases are $B = 1$ and $B = 2$. The theories are the ones described above and the effective central charges are $1/2$, $2/5$. The first case is self-dual, the second one has a dual theory with $B = 1/2$ and effective central charge $3/5$. The free fermion is given by the (3,4) minimal model, the two others by the (2,5) and (3,5) models.

More generally, the vanishing of perturbative corrections to the saddle point approximation yields strong and calculable constraints on $B, b, \tilde{h}$.

For $r = 2$, M. Terhoeven made a classification under the plausible assumption that every allowed matrix B admits $b = 0$. In this case, one always finds $\tilde{h} = -c_{\text{eff}}/24$, such that the level rank duality between the conformal dimensions just yields $c_{\text{eff}} + c'_{eff} = r$.

For $r = 2$, this assumption yields three exceptional cases and one series. The exceptional cases are given by the (2,7), (3,7) and (3,8) minimal models and correspond to $B = 2\binom{21}{11}$, $B = \frac{1}{2}\binom{32}{24}$ and $B = \binom{21}{11}$. The series is of the form

$$B = \frac{1}{p+q}\begin{pmatrix} p & q \\ q & p \end{pmatrix}.$$

The corresponding characters are theta functions and the central extension is 1.

A relation of this series to the Dehn twists of the figure-eight knot seems evident, but so far the close formal analogies cannot yet be explained. In particular, it would be very interesting to relate the renormalization flow of the conformal models to the continuous interpolation between Dehn twist is considered in [NZ 85] and in [Yoshida 85].

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2-D GRAVITIES AS GAUGE THEORIES WITH EXTENDED GROUPS

DANIEL CANGEMI*
Department of Physics, University of California at Los Angeles
405 Hilgard Avenue, Los Angeles, CA 90024, U.S.A.

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Abstract. The interaction of matter with gravity in two dimensional spacetimes can be supplemented with a geometrical force analogous to a Lorentz force produced on a surface by a constant perpendicular magnetic field. In the special case of constant curvature, the relevant symmetry does not lead to the de Sitter or the Poincaré algebra but to an extension of them by a central element. This richer structure suggests to construct a gauge theory of 2-D gravity that reproduces the Jackiw-Teitelboim model and the string inspired model. Moreover matter can be coupled in a gauge invariant fashion. Classical and quantized results are discussed.

Introduction

The beautiful success of General Relativity and the key role played by gauge theories in the description of fundamental interactions are two main reasons leading physicists to be interested in differential geometry. On the one hand, particles follow geodesics of spacetime, on the other hand, gauge potentials are identified with connections on some principal bundle. Moreover, it is tempting to exploit the local symmetries of General Relativity to write it as a gauge theory. Attempts in this direction turn out to be rather successful in lower dimensional gravities. In 2+1 dimensions, it is recognized (Achùcarro and Townsend 1986, Witten 1988/89) that planar gravity is described by a Chern-Simons model. In this note, I will consider the even simpler case of 1+1 dimensions, where a gauge theoretical formulation of lineal gravity has a natural setting using an extended (Cangemi and Jackiw 1992) Poincaré (Verlinde, eds. 1992, Grignani and Nardelli 1993) group or, more generally, an extended (Kim, Soh and Yee 1993, Cangemi and Dunne 1993) de Sitter (Fukiyama and Kamimura 1985, Jackiw 1992) group; the extension is related to a geometrical force (Cangemi and Jackiw 1993) which exists only in that particular dimension.

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Gravity in 1+1 dimensions

The reduction of General Relativity to 1+1 dimensions is not straightforward because of the vanishing of the Einstein tensor. There are two main proposals for lineal gravities.

One is obtained with a dimensional reduction of the Einstein-Hilbert action in 2+1 dimensions (Teitelboim 1983-1985),

$$I_{\mathrm{JT}} = \frac{1}{2\pi k} \int d^2x \sqrt{-g}\,\eta(R - \Lambda). \tag{1}$$

The Lagrange multiplier η enforces constant curvature, $R = \Lambda$.

The other proposal (Callan, Giddings, Harvey and Strominger 1992) is inspired by string theory on a two dimensional target space (it can alternatively be viewed as an s-wave approximation of 3+1 gravity (Harvey and Strominger 1992).

$$\bar{I}_{\mathrm{SI}} = \frac{1}{2\pi k} \int d^2x \sqrt{-\bar{g}}\, e^{-2\phi}(\bar{R} + 4\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \lambda) \tag{2}$$

Its classical solutions are $\bar{g}_{\mu\nu} = h_{\mu\nu}/(M - \lambda(x-\bar{x})^2)$, where $h_{\mu\nu} = \mathrm{diag}(1,-1)$ is the flat spacetime metric. The value $M = 0$ corresponds to a flat metric (vacuum solution), whereas the cases $M \neq 0$ have the characteristics of a black hole. The action (2) takes a simpler form with a change of variables (Verlinde 1992, Grignani and Nardelli 1993), $g_{\mu\nu} = \exp(-2\phi)\bar{g}_{\mu\nu}$, $\eta = \exp(-2\phi)$.

$$I_{\mathrm{SI}} = \frac{1}{2\pi k} \int d^2x \sqrt{-g}(\eta R - \lambda) \tag{3}$$

The Lagrange multiplier, η, now enforces zero curvature, $R = 0$. Proposals (1) and (2) suggest the more general action (Kim, Soh and Yee 1993, Cangemi and Dunne 1993)

$$I_{\mathrm{g}} = \frac{1}{2\pi k} \int d^2x \sqrt{-g}\Big(\eta(R - \Lambda) - \lambda\Big) \tag{4}$$

In view of the string inspired model (2), the "stringy" metric $\bar{g}_{\mu\nu}$ is conformally related to $g_{\mu\nu}$, $\bar{g}_{\mu\nu} = g_{\mu\nu}/\eta$. However, there is no definite reason to prefer one or the other as the physical metric (Fujiwara, Igarashi and Kubo 1993).

Let us end this section by recalling an equivalent formulation of geometry where $(g_{\mu\nu}, R)$ is substituted with (e^a_μ, ω_μ). The *Zweibein*, e^a_μ, is related to the metric, $g_{\mu\nu} = e^a_\mu h_{ab} e^b_\nu$, and the spin-connection, ω_μ, to the curvature, $d\omega = R\,\mathrm{vol}/2$ (vol is the volume two-form). Moreover, a space without torsion implies a relation between the *Zweibein* and the spin-connection, $de^a + \epsilon^a{}_b \omega e^b = 0$ (ϵ_{ab} is the antisymmetric two-tensor with value $\epsilon^{01} = 1$).

Point particle motion on the line

The gauge symmetry hidden in the action (4) becomes obvious if one studies the motion of a particle on the line. The interaction of a point particle in a background geometry is usually described by the geodesic equation. However, in two dimensions (and only in this dimension), the right side of that equation may be supplemented by a force term of a geometrical nature (Cangemi and Jackiw 1992),

$$\frac{d}{d\tau}\frac{m\,\dot{x}^\mu}{\sqrt{\dot{x}^\alpha g_{\alpha\beta}\dot{x}^\beta}} + \frac{1}{\sqrt{\dot{x}^\alpha g_{\alpha\beta}\dot{x}^\beta}}\Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = \mathcal{F}(R)g^{\mu\nu}\sqrt{-g}\epsilon_{\nu\rho}\dot{x}^\rho. \tag{5}$$

This equation is still general covariant and invariant under reparametrization provided $\mathcal{F}(R)$ is a scalar function. We will restrict ourself to linear examples, $\mathcal{F}(R) = -\mathcal{B} - \mathcal{A}R/2$. Due to its similarity with electromagnetism (which is *not* included here), the generalized geodesic equation (5) is obtained from the variation of the action,

$$I_m = -\int d\tau \Big[m\sqrt{\dot{x}^\mu(\tau)g_{\mu\nu}(x(\tau))\dot{x}^\nu(\tau)} + \dot{x}^\mu(\tau)\Big(\mathcal{A}\omega_\mu(x(\tau)) + \mathcal{B}a_\mu(x(\tau))\Big)\Big] \tag{6}$$

where ω is the spin-connection and a a one-form satisfying the exactness condition $da = \mathrm{vol}$.

It is easy to check that for constant curvature this action is invariant under a change of coordinates defined by a Killing vector field. Constant curvature spacetimes (with trivial topology, which we assume here) are maximally symmetric and thus possess three independent Killing vectors fields, $\xi^\mu_{(J)}, \xi^\mu_{(0)}, \xi^\mu_{(1)}$. By Noether's theorem, they generate three conserved currents.

$$\begin{aligned} \xi^\mu_{(J)} &= \epsilon^\mu{}_\nu x^\nu \longrightarrow J \\ \xi^\mu_{(a)} &= \delta^\mu_a(1 - \tfrac{\Lambda}{8}x^2) + \tfrac{\Lambda}{4}h_{a\nu}x^\nu x^\mu \longrightarrow P_a \qquad (a = 0,1) \end{aligned} \tag{7}$$

With the canonical symplectic structure $\left[\frac{\delta L}{\delta \dot{x}^\mu}, x^\nu\right] = \delta^\nu_\mu$, these currents fulfill the algebra,

$$\begin{aligned} [P_a, J] &= \epsilon_a{}^b P_b, \\ [P_a, P_b] &= \epsilon_{ab}(\tfrac{\Lambda}{2}J + \mathcal{B}_\Lambda I), \qquad \mathcal{B}_\Lambda \equiv \mathcal{B} + \tfrac{1}{2}\mathcal{A}\Lambda, \end{aligned} \tag{8}$$

where I is a central element acting by 1 in the representation (7).

Due to the presence of a geometrical force, we do not get the de Sitter algebra in its expected form; more specifically, in the flat case, $\Lambda = 0$, we do not recover the Poincaré algebra but a central extension of it. For $\mathcal{B} \neq 0$, this algebra possesses a non-degenerate, invariant inner product,

$$h_{AB} = \langle Q_A, Q_B \rangle = \begin{bmatrix} h_{ab} & 0 & 0 \\ 0 & \frac{(m/\mathcal{B}_\Lambda)^2}{1-\frac{\Lambda}{2}(m/\mathcal{B}_\Lambda)^2} & -\frac{1/\mathcal{B}_\Lambda}{1-\frac{\Lambda}{2}(m/\mathcal{B}_\Lambda)^2} \\ 0 & -\frac{1/\mathcal{B}_\Lambda}{1-\frac{\Lambda}{2}(m/\mathcal{B}_\Lambda)^2} & \frac{\Lambda/2\mathcal{B}_\Lambda^2}{1-\frac{\Lambda}{2}(m/\mathcal{B}_\Lambda)^2} \end{bmatrix} \tag{9}$$

($A, B = 0,1,2,3$; $Q_a = P_a$; $Q_2 = J$; $Q_3 = I$), which depends on a real parameter m. The Casimir $Q_A h^{AB} Q_B$ in the representation (7) coincides with the Hamiltonian for a particle of mass m. It can be shown that the freedom in the parameter m corresponds in the case $\Lambda = 0$ to a global symmetry (Jackiw 1992) also found in the dilaton model (Russo, Susskind and Thorlacius 1992) where its anomaly plays a crucial role in the existence of Hawking radiation (Fujiwara, Igarashi and Kubo 1993).

Gauge formulation of the gravity sector

We suggest to use this enhanced group structure for a gauge description of gravity. A connection will be thus a one-form of the type

$$A = e^a P_a + \omega J + \mathcal{B}_\Lambda a I \tag{10}$$

with curvature two-form

$$\begin{aligned} F &= dA + A^2 \\ &= (de^a + \epsilon^a{}_b \omega e^b) P_a + (d\omega + \tfrac{\Lambda}{4} e^a \epsilon_{ab} e^b) J + \mathcal{B}_\Lambda (da + \tfrac{1}{2} e^a \epsilon_{ab} e^b) I \end{aligned} \tag{11}$$

The components of F reproduce geometrical quantities if we interpret e^a as a *Zweibein* and ω as a spin-connection: The two first components are the torsion relating the *Zweibein* to the spin-connection, the third one equals $(R - \Lambda)\text{vol}/2$ and the last one $(da - \text{vol})$. Using a scalar function with value in the adjoint representation of the gauge group, $\eta = \eta^a P_a + \eta^2 J + \eta^3 I$, and the non-degenerate inner product (9), we build a gauge invariant action,

$$\begin{aligned} I'_g &= \frac{1}{2\pi k} \int \langle \eta, F \rangle \\ &= \frac{1}{2\pi k} \int \Big[\eta_a (de^a + \epsilon^a{}_b \omega e^b) \\ &\qquad - \tfrac{1}{1-\frac{\Lambda}{2}(m/\mathcal{B}_\Lambda)^2} \Big((m/\mathcal{B}_\Lambda)^2 \eta^2 - \tfrac{1}{\mathcal{B}_\Lambda} \eta^3 \Big) (d\omega + \tfrac{\Lambda}{4} e^a \epsilon_{ab} e^b) \\ &\qquad + \tfrac{1}{1-\frac{\Lambda}{2}(m/\mathcal{B}_\Lambda)^2} \Big(-(\eta^2 + \tfrac{\Lambda}{2\mathcal{B}_\Lambda} \eta^3 \Big) (da + \tfrac{1}{2} e^a \epsilon_{ab} e^b) \Big] \end{aligned} \tag{12}$$

which not only reproduces the action (4) with

$$\begin{aligned} \eta &= \tfrac{1}{1-\frac{\Lambda}{2}(m/\mathcal{B}_\Lambda)^2} \Big(\tfrac{1}{2} (m/\mathcal{B}_\Lambda)^2 \eta^2 - \tfrac{1}{2\mathcal{B}_\Lambda} \eta^3 \Big), \\ \lambda &= \tfrac{1}{1-\frac{\Lambda}{2}(m/\mathcal{B}_\Lambda)^2} \Big(-\eta^2 + \tfrac{\Lambda}{2\mathcal{B}_\Lambda} \eta^3 \Big), \end{aligned} \tag{13}$$

but also provides a one-form, whose classical value, da = vol, is the one needed to construct the matter action (6).

Besides the zero curvature condition, $F = 0$, we also get an equation for the scalar function, $D_\mu \eta = 0$. This set of equations is easily solved by the general solution

$$A = U^{-1}dU, \quad \eta = U^{-1}\eta_{(0)}U \tag{14}$$

for any group element U and constant gauge algebra element $\eta_{(0)}$. Of course, U has to be chosen carefully in order to reproduce a geometric solution associated to a non-degenerate metric $g_{\mu\nu}$ (Cangemi and Dunne 1993, Jackiw 1992). The "stringy" metric $\bar{g}_{\mu\nu} = g_{\mu\nu}/\eta$ then takes the form of a static black hole, for $\Lambda = 0$, $\bar{g}_{\mu\nu} = h_{\mu\nu}/(M - \lambda(x-\bar{x})^2)$. Nevertheless the physical content of the model will not depend on this choice and $U = \mathbb{1}$ *i.e.*, $e^a = \omega = a = 0$, is perfectly admissible. This is sometimes referred as the unbroken phase. The physics should be contained in the gauge invariant part of η,

$$\begin{aligned} \langle \eta, \eta \rangle &= \langle \eta_{(0)}, \eta_{(0)} \rangle = M, \\ \langle \eta, I \rangle &= \langle \eta_{(0)}, I \rangle = \lambda/\mathcal{B}_\Lambda. \end{aligned} \tag{15}$$

The gauge theoretical approach relates the number of free parameters in the classical solutions $(M, \lambda, \bar{x}^0, \bar{x}^1)$ to the dimension (four) of the gauge group. It introduces also the cosmological constant λ as a dynamical variable. The parameters M and λ are gauge invariant quantities and describe the physical content of the theory, as we will see in the next section.

Quantization of the gravity sector

A gauge theoretical setting allows a more tractable way to deal with quantization. We present here the canonical quantum structure of gravity without matter; it is simple and interesting, even if, in the absence of matter, there are no propagating degrees of freedom. We write the action (4) in its Hamiltonian form.

$$\begin{aligned} I'_g &= \frac{1}{2\pi k}\int d^2x \epsilon^{\mu\nu}\langle \eta, F_{\mu\nu}\rangle \\ &= \frac{1}{2\pi k}\int dtdx\Big(\langle \eta, \partial_0 A_1\rangle + \langle A_0, D_1\eta\rangle\Big) - \frac{1}{2\pi k}\int dtdx\partial_1\langle \eta, A_0\rangle \end{aligned} \tag{16}$$

The Hamiltonian is a sum of constraints

$$G^A = -(\partial_1 \eta^A + f_{BC}{}^A A_1^B \eta^C) \tag{17}$$

($A, B, C = 0, 1, 2, 3$ are the gauge group indices, which are raised and lowered with the inner product h_{AB}, and $f_{BC}{}^A$ are the structure constants of the gauge group). The spatial component of the gauge connection is canonically conjugate to η and we postulate the usual commutation relations

$$[\eta_A(x), A_1^B(y)] = i\,2\pi k\,\delta_A^B \delta(x-y). \tag{18}$$

With these commutation relations, the algebra of constraints coincides, as usual in gauge theories, with the original gauge algebra.

$$[G_A(x), G_B(y)] = i f_{AB}{}^C G_C(x)\delta(x-y). \tag{19}$$

In a Schroedinger picture, we consider states as functionals of $\eta_A(x)$, $\Psi[\eta_A]$, on which $A_1^A(x)$ acts by functional derivation, $(2\pi k/i)(\delta/\delta\eta_A(x))$. Physical states are those annihilated by the constraints G_A and they satisfy the differential equations

$$\begin{aligned} \left(\partial_1\eta_a - i\,2\pi k\,\epsilon_a{}^b\eta_b\frac{\delta}{\delta\eta^2} + i\,2\pi k\,\eta^2\epsilon_{ab}\frac{\delta}{\delta\eta_b}\right)\Psi &= 0, \\ \left(\partial_1\eta^2 + i\,2\pi k\,\frac{\Lambda}{4}\epsilon^a{}_b\eta_a\frac{\delta}{\delta\eta_b}\right)\Psi &= 0, \\ \left(\partial_1\eta^3 + i\,2\pi k\,\mathcal{B}_\Lambda\epsilon^a{}_b\eta_a\frac{\delta}{\delta\eta_b}\right)\Psi &= 0. \end{aligned} \tag{20}$$

These equations are solved by the functionals

$$\Psi[\eta_A] = \exp\left(\frac{i}{2\pi k}\int dx\,\eta^2\frac{\epsilon^{ab}\partial_1\eta_a\eta_b}{\eta_c\eta^c}\right)\psi(M,\lambda)\Bigg|_{\substack{\langle\eta,\eta\rangle=M \\ \langle\eta,I\rangle=\lambda/\mathcal{B}_\Lambda}}, \tag{21}$$

with support on the constant gauge invariant combinations $\langle\eta,\eta\rangle = M$ and $\langle I,\eta\rangle = \lambda/\mathcal{B}_\Lambda$; ψ is a function of the variables M and λ. The physical states depend on the two values M and λ, which coincide for classical solutions with the two parameters of the black hole configuration. Let us now couple matter to this gravity.

Coupling to matter

The coupling to matter follows the one discussed before, see Eq. (6). It is possible to find a gauge invariant formulation of it either for point particle or for fields, *cf.* Ref. (Cangemi and Jackiw 1993). The gauge invariant actions are of the form

$$I'_m[A_\mu, p(\tau), \xi^a], \quad I'_m[A_\mu, \bar\psi, \psi, \xi^a], \quad \ldots \tag{22}$$

where the additional field ξ^a acts like a Higgs field that insures the gauge invariance of the action. The essential feature of this coupling is that it does not involve η. In this gauge formulation, the matter is coupled to the metric $g_{\mu\nu}$, whereas in the geometrical point of view people use mainly a coupling to $\bar{g}_{\mu\nu}$. But, since their coupling is conformal, it is not really different at the classical level. Nevertheless, this difference could have its importance once we proceed to the quantization (Fujiwara, Igarashi and Kubo 1993). Notice that our coupling breaks conformal invariance at the classical level even in the massless case. Namely, the trace of the energy-momentum is proportional to the additional force strength, $\mathcal{B}$ and at the quantum level its vacuum expectation picks up an additional term, $R/24\pi$ (Cangemi and Jackiw 1993).

The equations of motion are modified in the following way

$$F = 0, \quad D_\mu \eta = 2\pi k \, J^5_\mu, \tag{23}$$

where $(J^5_\mu)_A = -\epsilon_{\mu\nu}(\delta I'_m/\delta A^A_\nu)$ is the axial current. Let us consider the point particle. Outside the particle trajectory, J^5_μ is zero and the equations are those of pure gravity. We have two sets of four constant parameters on each side of the trajectory, whose differences are fixed by the particle characteristics. The shift in M and λ implies a transition from a pure gravity state to another when crossing the particle line; this is usually interpreted (Callan, Giddings, Harvey and Strominger 1992) as a black hole created by an in-falling particle. The shift in $\bar{x}$ is a basic ingredient in deriving a Hawking radiation (Callan, Giddings, Harvey and Strominger 1992) for the "stringy" metric, $\bar{g}_{\mu\nu}$.

Our formulation reproduces interesting features of lineal gravity. But being a gauge theory, we are able to discuss in a straightforward manner issues concerning gauge charges or quantization.

A gauge definition of mass

The definition of mass and angular-momentum is an ill-defined concept in General Relativity. Different methods lead to different results (Bak, Cangemi and Jackiw 1993). However, when one has a gauge invariance, Noether's procedure uniquely define conserved currents and charges. In our model, $I'_g + I'_m$, an infinitesimal gauge transformation θ generates an explicit conserved current

$$j^\mu_\theta = \frac{1}{\pi k}\epsilon^{\mu\nu}\partial_\nu\langle\eta, \theta\rangle \tag{24}$$

and a conserved charge.

$$Q_\theta = \int dx^1 j^0_\theta = \frac{1}{\pi k}\langle\eta, \theta\rangle\Big|^{x^1=+\infty}_{x^1=-\infty}. \tag{25}$$

The question is which θ define energy. Obviously, energy should be related with infinitesimal diffeomorphisms in a time-like Killing direction.

But, in topological field theory ($F = 0$), infinitesimal diffeomorphisms are equivalent to infinitesimal gauge transformations (Jackiw 1978).

$$L_f A_\mu = f^\alpha \partial_\alpha A_\mu + \partial_\mu f^\alpha A_\alpha = D_\mu(f^\alpha A_\alpha) + f^\alpha F_{\alpha\mu}. \tag{26}$$

An infinitesimal diffeomorphism, f^α, is identified with an infinitesimal gauge transformation, $f^\alpha A_\alpha$. It is thus associated to the conserved charge

$$Q_f = \frac{1}{\pi k}\langle \eta, f^\alpha A_\alpha \rangle |_{x^1=-\infty}^{x^1=+\infty} \tag{27}$$

and energy E is defined for a time-like Killing vector f^α.

In the absence of matter, the contributions at $x^1 = +\infty$ and $x^1 = -\infty$ are identical, which implies $E = 0$. When matter is included, due to the jump of the value of η across the particle trajectory, the contributions are different and gives a non zero energy, $E = \langle \eta, \eta \rangle = M$, in full agreement with the ADM definition.

Conclusions

In this brief note, I have shown how General Relativity and gauge theory can be combined in 1+1 dimensional spacetime. Once the gauge group is recognized, we are able to produce a gauge theory, which encompasses the Jackiw-Teitelboim and the string inspired models. The inclusion of matter in a gauge invariant way is possible and provides a model, which not only reproduces previous results but also provides a natural way to define gauge invariant and conserved quantities, as energy, and to deal with quantization. Another interesting feature of the model is the introduction of the cosmological constant as a dynamical variable (Izawa 1993). Supersymmetric extensions have been studied in relation to a positive energy theorem (Park and Strominger 1993) and for a topological description of supergravity (Cangemi and Leblanc 1993). The quantization of pure gravity has shown how the physical states depend on gauge invariants. The quantization of the full model deserves further study. It would also be interesting to consider topological effects occuring in the definition of the one-form a and in the resolution of $F = 0$ (e.g. Hwang, Kim, Soh and Yee 1993).

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CRITICAL O(N) - VECTOR NONLINEAR SIGMA - MODELS: RÉSUMÉ OF THEIR FIELD STRUCTURE

KLAUS LANG AND WERNER RÜHL
Fachbereich Physik Universität Kaiserslautern
67653 Kaiserslautern, Germany, lang@gypsy.physik.uni-kl.de

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Abstract. The classification of quasi - primary fields is outlined. It is proved that the only conserved quasi - primary currents are the energy - momentum tensor and the O(N)-Noether currents. Derivation of all quasi - primary fields and the resolution of degeneracy is sketched. Finally the limits $d = 2$ and $d = 4$ of the space dimension are discussed. Whereas the latter is trivial the former is only almost so.

1. Some general remarks

We have studied only a very special example of a critical field theory at dimensions $2 < d < 4$. Nevertheless we believe that the results are relevant for many critical field theories, in particular sigma models in a neighbourhood of a free theory. Our neighbourhood is defined by a $\frac{1}{N}$ expansion.

In this résumé we extract results from a series of papers (Lang and Rühl 1991-1993) and from earlier literature on conformal field theory in general (Dobrev, Mack, Petkova, Petrova and Todorov 1977; Ferrara, Gatto and Grillo 1973) or conformal sigma models in particular (Vasil'ev, Pismak and Khonkonen 1981-1982). These results may have different status but we condense them equally into "theorems" which should not be considered as mathematical theorems but as tested conjectures. General statements of quantum field theory and group theory are thus mixed up with conclusions from low order perturbative expansions. Let us start with such a theorem which certainly disappoints many of the readers:

Theorem 0: Almost none of the structures of conformal field theory at $d = 2$ can be rediscovered at $2 < d < 4$.

2. Definition of the model

We start with the partition function

$$\mathcal{Z} = \int D[\mathbf{S}]D[\alpha]\exp\left\{-\int dx\left[\frac{1}{2}(\partial_\mu \mathbf{S})^2(x) + z^{\frac{1}{2}}\alpha(x)(\mathbf{S}^2(x)-1)\right]\right\} \quad (1)$$

where

$$\begin{aligned} \mathbf{S} &: \mathrm{O}(N)\text{- vector, } \mathrm{O}(d) \text{ - scalar;} \\ \alpha &: \mathrm{O}(N)\text{- } \mathrm{O}(d) \text{ - scalar;} \\ d = 2\mu &: \text{space - time dimension} \end{aligned}$$

If $\mathbf{S}$ and α are normalized in a standard fashion

$$\left\langle S_a(x)S_b(0)\right\rangle = \delta_{ab}\left(x^2\right)^{-\alpha} \quad (2)$$

$$\left\langle \alpha(x)\alpha(0)\right\rangle = \left(x^2\right)^{-\beta} \quad (3)$$

the critical coupling constant z becomes a computable function of N, and

$$N \to \infty : \qquad z = \mathcal{O}(\tfrac{1}{N}) \quad (4)$$

The limit $N \to \infty$ is a free field limit

$$\lim_{N\to\infty} \mathbf{S}(x) = \mathbf{s}(x) \quad (5)$$

but $\mathbf{s}(x)$ possesses infinitely many components which leads to problems sometimes. A saddle point expansion of (1) gives the $\frac{1}{N}$ - expansion.

A critical theory such as this is conformally covariant. Operator product expansions (OPE) generate a field algebra $\mathcal{A}(\mathbf{S},\alpha)$ of the two fundamental fields $\mathbf{S}$ and α which is associative and possesses a commutation property connected with the crossing behaviour of n - point functions. The building blocks of $\mathcal{A}(\mathbf{S},\alpha)$ are the conformal or quasiprimary fields (qp - fields).

Theorem 1: All qp - fields belong to representations of the conformal group characterized by two quantum numbers only: δ, the scaling dimension under dilatations and l, the tensor rank under space - time rotations.

These are the elementary representations. In addition the qp - fields transform irreducibly under $\mathrm{O}(N)$. We ascribe to them a Young frame Y.

Consider the dimension δ_ϕ of the qp - field ϕ

$$\delta_\phi = [\delta_\phi] + \eta(\phi) \quad (6)$$

$$[\delta_\phi] : \text{ the normal dimension}$$

$$\eta(\phi) = \mathcal{O}(\tfrac{1}{N}) : \text{ the anomalous dimension} \quad (7)$$

By definition

$$[\delta_\phi] = p(\mu - 1) + q, \; p, q \in I\!N_0 \tag{8}$$
$$[\delta_S] = \mu - 1 \tag{9}$$

So we expect that in the limit $N \to \infty$ ϕ tends to a normal product of p fields $\mathbf{S}$ with not more than q derivatives (see below).

Each elementary representation $[\delta, l]$ of a qp - field possesses a dual representation $[\delta', l']$ ('shadow representation')

$$\delta' = d - \delta + 2\,l\mathrm{a} \tag{10}$$
$$l' = l\mathrm{b} \tag{11}$$

The two - point functions

$$\left\langle \phi_{[\delta,l]}(x)\phi_{[\delta,l]}(0) \right\rangle \quad \text{and} \quad \left\langle \phi_{[\delta',l']}(x)\phi_{[\delta',l']}(0) \right\rangle$$

are as kernels and up to a normalization inverse to each other. An n - point function of $\phi_{[\delta,l]}$ is transformed into an n - point function of $\phi_{[\delta',l']}$ by amputation. Therefore we have

Theorem 2: The fields $\phi_{[\delta,l]}$ and $\phi_{[\delta',l']}$ are dynamically equivalent.

So from each pair $\phi_{[\delta,l]}, \phi_{[\delta',l']}$ we would like to choose only one representative as basis element of $\mathcal{A}(\mathbf{S}, \alpha)$. We will in fact be able to do that but in an unexpected fashion.

From

$$\begin{aligned} [\delta'] &= d - \Big(p(\mu - 1) + q\Big) + 2l \\ &= (2 - p)(\mu - 1) + 2 - q + 2l \end{aligned} \tag{12}$$

we see that the α - field can be considered as the shadow field of

$$\Big(\mathbf{S}^2(x)\Big)_{\text{ren.}} \tag{13}$$

since

$$p = 2, \; q = 0, \; l = 0 \text{ implies } [\delta'] = 2 \tag{14}$$

Inspection of the action in (1) also suggests this interpretation of α.

Next we decompose q in (8) as

$$q = l + t = l + 2r \qquad (t\text{: twist}) \tag{15}$$

where r is the number of α fields bound into ϕ at $N \to \infty$ and l is the number of derivatives. p, l and r (or t) serve as quantum numbers in a neighbourhood of $N \to \infty$.

3. Classes of qp - fields

Construction of the qp - fields goes by OPE and harmonic analysis. This automatically orders the qp - fields according to increasing dimensions δ. From the interpretation of the quantum numbers p, l, t in (8), (15) we can naturally expect these numbers to be bounded by

$$p \geq 0,\ l \geq 0,\ t \geq 0 \tag{16}$$

In fact, this is fulfilled by our construction. Most of the shadow fields are forbidden by (16) but a few of them are still permitted.
We put all qp - fields with the same Y and p into a class (Y, p). A generic class looks graphically as Fig.1.

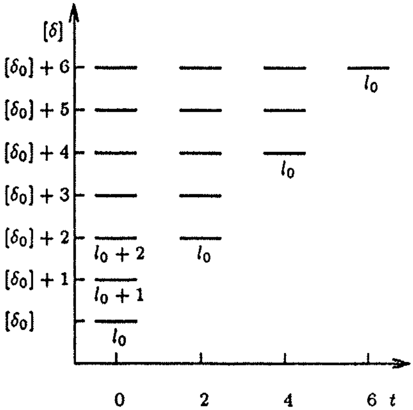


Figure 1: A generic class (Y, p)

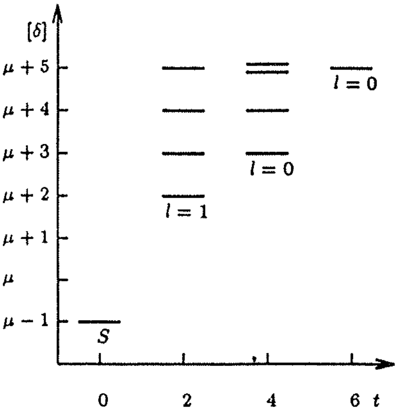


Figure 2: The class $(\square, 1)$

Labels may be multiply occupied by qp -fields, which are distinguished by their anomalous dimensions ("degeneracy"). Some of the simplest classes look different indeed.

(A) The class $(\square, 1)$ containing the fundamental field **S**. At $t = 0$ there is only the scalar field **S**. At the level $t = 2$, $l = 0$ we would expect the shadow field S' of **S**. But it is not found, this level is empty. The level $t = 4$, $l = 2$ is twofold degenerate.

(B) The class $(\emptyset, 0)$ containing the fundamental field α. At $t = 2$ we have only the α field (we start counting from $t = 2$ in this case). At $t = 4$ we have only even l and at $t \geq 6$, $l = 1$ is empty.
Indeed, fusion of two qp - fields into a third one by OPE

$$A(x)B(0) = \left(x^2\right)^{\frac{1}{2}(\delta_C - \delta_A - \delta_B)} C(0) + \ldots \tag{17}$$

abbreviated as

$$A \otimes B \rightarrow C \tag{18}$$

is analogous with the formation of bound states. Two bosonic α's cannot be bound together to a state with odd l and for more than three α's $l = 1$ is also excluded by bose symmetrization.

(C) The class $(\emptyset, 2)$ containing the energy - momentum tensor $T_{\mu\nu}$.
The level $t = 0$, $l = 0$ has been found unoccupied. The shadow field of α should appear on this level, or, according to our remark above, the field $\left(\mathrm{S}^2(x)\right)_{\text{ren.}}$. Thus the sigma - model constraint works and this field has been eliminated. The energy - momentum tensor field lies at

$$t = 0, \quad l = 2, \quad \delta = [\delta] = 2\mu = d \tag{19}$$

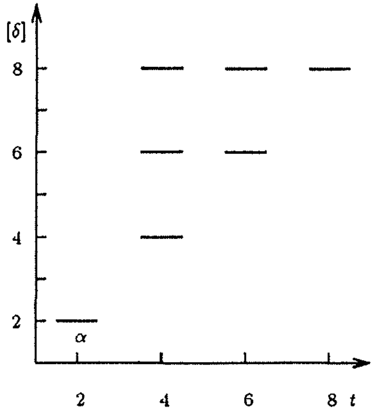


Figure 3: The class $(\emptyset, 0)$

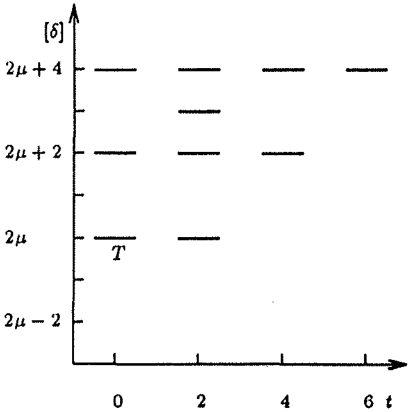
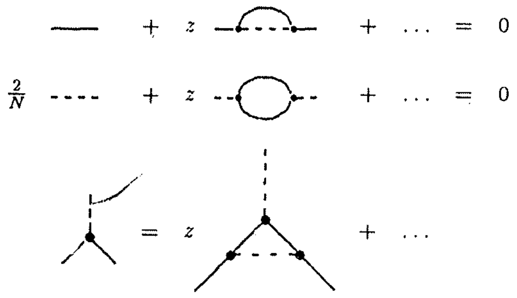


Figure 4: The class $(\emptyset, 2)$

Looking through the classes more carefully, we recognize that the elimination of shadow fields has been completed.

In (Lang and Rühl 1992b) we showed that elimination of the shadow field of α was directly related with a renormalization condition. Using dressed propagators and vertices (represented as Polyakov triangles $\bullet$) we have three such conditions



(S)

(α)

(Γ)

These three conditions suffice to determine $\eta(S)$, $\eta(\alpha)$ and z. A generalization of the argument in (Lang and Rühl 1992b) shows validity of

Theorem 3: The requirement that one (two) shadow field(s) of the fundamental fields do(es) not show up replaces one (two) renormalization condition(s).

The status of the proof is still not satisfactory: $\mathcal{O}(\frac{1}{N^2})$ calculations at best. The theorem ('equivalence theorem') is very powerful in practice.

The α - field produces a field algebra $\mathcal{A}(\alpha)$ which is a subalgebra of $\mathcal{A}(\mathbf{S},\alpha)$. It contains only O($N$)- scalars, among them the energy - momentum tensor $T_{\mu\nu}$

$$T_{\mu\nu} \in (\emptyset, 2) \tag{20}$$

Indeed

$$\alpha \otimes \alpha \rightarrow T \tag{21}$$

at $\mathcal{O}(\frac{1}{N})$, so p is not conserved at this order. Moreover

$$T \otimes T \rightarrow \alpha \tag{22}$$

so all $\mathcal{A}(\alpha)$ can be generated from T (at $d = 2$ T generates not only $Vir \times \overline{Vir}$ but W algebras as well!)..

Theorem 4: The only conserved qp - currents in $\mathcal{A}(\mathbf{S},\alpha)$ are $T_{\mu\nu}$ and $J_{\mu,ab}$, the Noether currents of O(N)- symmetry from the class $(\begin{smallmatrix}\square\\\square\end{smallmatrix}, 2)$.

Sketch the proof. Denote by $\#Y$ the number of blocks in the Young frame Y. Then

$$p - \#Y = 2n, \quad n \in \mathbb{N}_0 \tag{23}$$

This is obvious at $N = \infty$ since n is the number of contractions applied to the normal product of p vector fields $\mathbf{s}$. But in a neighbourhood of $N = \infty$ it remains valid due to standard arguments of harmonic analysis.

Next we use a classical lemma of conformal field theory (Lang and Rühl 1993a, Appendix A) for qp - fields which are symmetric tensors in spacetime. In fact for $2 < d < 4$ we have the situation of $d = 3$: symmetric tensors are sufficient. The lemma says that a qp - current is conserved if and only if

$$l \geq 1,\ \delta = [\delta] = 2\mu - 2 + l \tag{24}$$

i.e.

$$p = 2,\ l \geq 1,\ t = 0 \tag{25}$$

and

$$\eta(\phi) = 0 \tag{26}$$

This leaves as candidates the classes

$$(\Box\Box, 2), \quad \left(\begin{smallmatrix}\Box\\\Box\end{smallmatrix}, 2\right), \quad (\emptyset, 2) \tag{27}$$

In each case the $t = 0$ towers are nondegenerate with the following anomalous dimensions at leading order

$$\left.\begin{array}{ll} (\Box\Box, 2) : & \dfrac{\eta(M_l)}{\eta(S)} \\ \left(\begin{smallmatrix}\Box\\\Box\end{smallmatrix}, 2\right) : & \dfrac{\eta(J_l)}{\eta(S)} \end{array}\right\} = 2\frac{(l-1)(2\mu - 2 + l)}{(\mu - 1 + l)(\mu - 2 + l)} \qquad \begin{array}{ll} l & \text{even} \\ l & \text{odd} \end{array} \tag{29}$$

$$(\emptyset, 2) : \frac{\eta(T_l)}{\eta(S)} = \begin{cases} 0, & (l = 2) \\ 2(l-1) + \displaystyle\sum_{p=1}^{\frac{1}{2}l-2} \left((p+1)!\right)^2 \frac{(2\mu + 1 + p)_{l-4-2p}}{(2\mu + 1)_{l-4}}, & (l \geq 4, \text{even}) \end{cases} \tag{30}$$

The curves for the expression (29) are presented in (Lang and Rühl 1992c, Fig. 6). None of these functions changes sign. They vanish identically for J_1 and T_2 and are otherwise different from zero for all $2 < d < 4$. It is also important to guarantee that no empty levels are filled up at higher orders of $\frac{1}{N}$ or that degeneracy appears this way. The first is made sure by crossing symmetry, the second possibility can at present not be excluded.

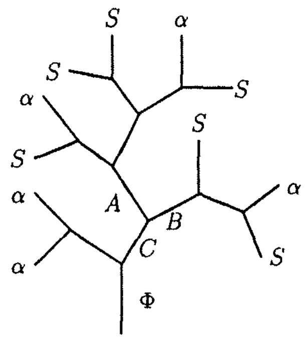


Figure 5 A pedigree of fusion

4. Fusion

Each qp - field has a pedigree of fusion. Fig. 5.

The internal lines are arbitrary qp - fields which can be produced from the parents. This means that the fusion coefficients effective at a vertex must be nonzero:

$$f^{C}_{AB} \neq 0 \tag{30}$$

Theorem 5: Fusion coefficients vanish only if the corresponding Littlewood - Richardson coefficients of $O(N)$ are zero or if this follows from a crossing symmmetry selection rule.

As an example let $A = B$ scalar and the Littlewood - Richardson coefficient be symmetric (antisymmetric) under exchange of A and B. Then odd (even) l are forbidden for C. Another example is the fusion

$$\alpha \otimes \mathbf{S} \tag{31}$$

which leads to any level of the class $(\square, 1)$ at $\mathcal{O}(\frac{1}{N})$ already, but in the class of degenerate levels only to one linear combination of qp - fields . So to resolve degeneracies we have to consider different pedigrees with the same final level.

Theorem 6: The qp - fields with $l = 0$ are never degenerate.

This corresponds to the uniqueness of a ground state in QM.

We introduce the concept of 'dominant channel fusion' (DCF). This kind of fusion acts already at $\mathcal{O}(1)$ and produces scalar qp - fields of the type

$$(Y, p; [\delta], l) = (\square\!\square\!\square\!\square_{\,p}\ , p; p(\mu - 1) + 2r, 0) \tag{32}$$

from qp - fields of the same type. Let two such qp - fields with labels $\{p_1, r_1\}$, $\{p_2, r_2\}$ be given. The resulting field has labels $\{P, R\}$ with

$$P = p_1 + p_2 \tag{33}$$
$$R = r_1 + r_2 \tag{34}$$

For DCF normal dimensions are additive and degeneracy does not occur. Only symmetric O(N) tensors are produced by definition. Pedigrees with DCF at each vertex produce a qp - field of type (32) which depends only on the numbers p of S fields and r of α fields entering and not on the form of the pedigree. In other words: DCF is abelian.

We denote the qp - fields (32) by $M_0^{\{p,r\}}$. Any qp - field on the level

$$(\square\!\square\!\!\rule{2em}{0.4pt}_{\,p}\ , p; p(\mu - 1) + 2r + l, l) \tag{35}$$

is denoted $M_{l,k}^{\{p,r\}}$ where k is introduced to take account of the degeneracy. We are interested in the fusion process

$$M_0^{\{p_1,r_1\}} \otimes M_0^{\{p_2,r_2\}} \;\rightarrow\; M_{l,k}^{\{p_1+p_2,r_1+r_2\}} \tag{36}$$

If we keep

$$P = p_1 + p_2, \qquad R = r_1 + r_2 \tag{37}$$

fixed but let p_1, r_1 run, we obtain different combinations of $M_{l,k}^{\{P,R\}}$ which can be resolved.

Technically one considers the four - point functions

$$\left\langle M_0^{\{p_1,r_1\}}(y_1)\, M_0^{\{p_2,r_2\}}(y_2)\, M_0^{\{p_1',r_1'\}}(y_3)\, M_0^{\{p_2',r_2'\}}(y_4)\right\rangle \tag{38}$$

with fixed

$$\begin{aligned} P &= \quad p_1 + p_2 \quad = p_1' + p_2' \\ R &= \quad r_1 + r_2 \quad = r_1' + r_2' \end{aligned} \tag{39}$$

On the one hand these four - point functions (38) are calculated from a $2(P + R)$ - point function involving $2P$ S fields and $2R$ α fields by OPE reduction via DCF. This is mainly a combinatorical task bringing in the "replica parameters" $p_1, r_1, p_2, r_2, p_1', r_1', p_2', r_2'$ and, at $\mathcal{O}(\frac{1}{N})$, the connected four - point functions

$$\left\langle SSSS\right\rangle_{\text{conn}}, \qquad \left\langle \alpha\alpha\alpha\alpha\right\rangle_{\text{conn}}, \qquad \left\langle \alpha S\alpha S\right\rangle_{\text{conn}} \tag{40}$$

which are explicitly known (Lang and Rühl 1992a, 1992b, 1993a). Crossing between the unprimed factors exchanges

$$p_1 \leftrightarrow p_2, \qquad r_1 \leftrightarrow r_2 \tag{41}$$

so that we can use the crossing symmetric combinations

$$t_1 = r_1 r_2, \quad t_2 = p_1 p_2, \quad t_3 = p_1 r_2 + p_2 r_1 \tag{42}$$

On the other hand we compare the four - point function (38) with conformal exchange amplitudes (this is an element of harmonic analysis).

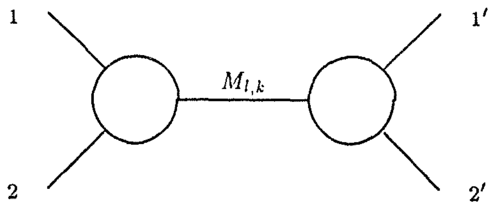


Figure 6 Conformal exchange amplitude

This allows us to extract expressions for

$$\sum_k f_{12}^{M_{l,k}} f_{1'2'}^{M_{l,k}} \tag{43}$$

and

$$\sum_k f_{12}^{M_{l,k}} f_{1'2'}^{M_{l,k}} \eta(M_{l,k}) \tag{44}$$

The fusion constants $f_{12}^{M_{l,k}}$ are functions of the replica parameters

$$f_{12}^{M_{l,k}} = F_{l,k}(p_1, r_1; p_2, r_2) \tag{45}$$

By a simultanous diagonalization procedure for the two expressions obtained for (43), (44) we can extract the fusion coefficients and the anomalous dimensions . The fusion coefficients are obtained in the form

$f_{12}^{M_{l,k}}$ = polynomial in the replica parameters giving $(-1)^l$ under crossing times an algebraic function depending homogenously on t_1, t_2, t_3 (46)

We have in fact solved the following cases (Lang and Rühl 1993b)

$P = 0$, R arbitary > 0: levels $0 \leq l \leq 6$ and $t = 2R$ in the class $(\emptyset, 0)$. Degeneracy sets in at $R \geq 4$ and $l \geq 4$,

$R = 0$, P arbitary > 0: levels $0 \leq l \leq 6$ and $t = 0$ in the classes $(\square\!\square\!\square_P\,, P)$. Degeneracy sets in at $P \geq 4$ and $l \geq 4$.

In both cases the anomalous dimensions are

$$\frac{\eta(M_{l,k})}{\eta(S)} = \text{rational functions of } \mu \text{ at leading order.} \tag{47}$$

and the algebraic function in (46) reduces to a (nonhomogeneous) polynomial of either t_1 or t_2.

If $RP \neq 0$, degeneracy starts already at $R + P \geq 3$, $l \geq 2$. We resolved only the cases $0 \leq l \leq 3$. Moreover we find

$$\frac{\eta(M_{l,k})}{\eta(S)} = \text{algebraic (irrational) function of } \mu \text{ at leading order.} \tag{48}$$

Many infinite sequences of anomalous dimensions are known now and in these sequences we can study limits. Consider a tower of nondegenerate qp - fields $M_l^{\{P,R\}}$, P, R fixed, l running. Then in the DCF process (36) the pair of qp - fields on the left hand side is uniquely determined. At leading order in $\frac{1}{N}$ we find

$$\lim_{l\to\infty} \eta(M_l^{\{P,R\}}) = \eta(M_0^{\{p_1,r_1\}}) + \eta(M_0^{\{p_2,r_2\}}) \tag{49}$$

Instead in the case of degeneracy

$$\eta(M_l^{\{P,R\}}) = \mathcal{O}\Big(\frac{l^2}{N}\Big) \tag{50}$$

which makes the $\frac{1}{N}$ expansion asymptotic only if $N \gg l^2$. We could also think of keeping l fixed and letting P, R run. Then

$$\eta(M_l^{\{P,R\}}) = \mathcal{O}\Big(\frac{1}{N} \times \text{second order polynomial in } P \text{ and } R\Big) \tag{51}$$

imposing a similiar restriction on N.

We emphasize that our method of constructing the states $M_l^{\{P,R\}}$ by forcing all internal qp - fields of the pedigree to have tensor rank zero may be too restrictive for large l. In a forthcoming article we will study an alternative algorithm which remains correct at large l as well.

5. The limits $d \searrow 2$ and $d \nearrow 4$

For any $2 < d < 4$ the limit $N \to \infty$ leads to a free field theory. In this limit each qp - field $\phi \in \mathcal{A}(\mathbf{S}, \alpha)$ possesses a corresponding qp - field φ in the free field algebra $\mathcal{A}_0(\mathbf{s})$. In Green functions involving α fields we may first amputate them and perform the limit afterwards. At the boundaries $d = 2$, $d = 4$ the behaviour of coupling constant and critical indices

$$\eta(\phi) = \sum_{k=1}^{\infty} \frac{\eta_k(\phi)}{N^k}, \tag{52}$$

$$\eta_1(S) = 2\frac{\sin\pi\mu}{\pi}\frac{\Gamma(2\mu-2)}{\Gamma(\mu+1)\Gamma(\mu-2)}, \tag{53}$$

$$z = \sum_{k=1}^{\infty}\frac{z_k}{N^k}, \tag{54}$$

concerning their zero orders in d is listed in the following table

	$d=2$	$d=4$
z_1	0	2
z_2	0	2
$\eta_1(S)$	1	2
$\eta_2(S)$	1	2
$\eta_1(\phi)$, $\phi \neq S$	1	1

All critical exponents vanish at both limits. These limits are therefore connected with free field theory.

At $d = 4$ we obtain a free field theory in the trivial sense that

$$\lim_{d\nearrow 4}\mathbf{S}(x) = \mathbf{s}(x), \quad \Delta\mathbf{s}(x) \doteq 0$$

$$\mathbf{s}(x) \ : \ N\text{ - component } O(N) \text{ - vector field} \tag{55}$$

As a test we can calculate the limit of $\left\langle\alpha\mathbf{S}\alpha\mathbf{S}\right\rangle$ after amputation. This limit $d = 4$ is assumed fieldwise and is an isomorphism of field algebras in the straightforward sense. Let $A, B, C \in \mathcal{A}(\mathbf{S},\alpha)$

$$A(x)\,B(0) = \left(x^2\right)^{\frac{1}{2}(\delta_C-\delta_A-\delta_B)(\mu)} f^C_{AB}(\mu)\,C(0) + \ldots \tag{56}$$

Then if a, b, c are the corresponding free fields

$$a(x)\,b(0) = \left(x^2\right)^{\frac{1}{2}(\delta_C-\delta_A-\delta_B)(2)} f^C_{AB}(2)\,c(0) + \ldots \tag{57}$$

The limit $\mu \to 2$ is performed termwise.

This is not true at the other limit $d = 2$. First we consider the two conserved qp - currents

$$\phi \ : \ T_{\mu\nu} \quad \text{or} \quad J_{\mu,ab}$$

which have well defined local field limits

$$\varphi \ : \ t_{\mu\nu} \quad \text{or} \quad j_{\mu,ab}$$

Both T and J can be constructed from fusion of $\mathbf{S}\otimes\mathbf{S}$. We introduce the Ward identities in any ad hoc normalization and normalize the fields $\phi \ni$

$\{T, J\}$, $\varphi \ni \{t, j\}$ relative to the same Ward identities. Conformal invariance implies the same scaling dimension and tensor structure for ϕ and φ so that

$$\left\langle \phi(x)\,\phi(0)\right\rangle = C_\phi(\mu)\left\langle \varphi(x)\,\varphi(0)\right\rangle \tag{58}$$

By explicit calculation we find

$$\lim_{\mu \searrow 1} C_\phi(\mu) = 1 - \frac{l+1}{N} + \mathcal{O}\Big(\frac{1}{N^2}\Big) \tag{59}$$
$$l = \text{tensor degree of } \phi \text{ (1 or 2)}$$

Ward identities can be derived from the two - point functions. Instead of normalizing fields by three - point functions and comparing the two - point functions we can introduce a standard normalization of two - point functions

$$\left\langle \phi(x)\,\phi(0)\right\rangle = \left(x^2\right)^{-\delta_\phi} \cdot \text{tensor}(x) \tag{60}$$

with the tensor factors connected to Gegenbauer polynomials which can be submitted to an ad hoc normalization, say $C_l^{\mu-1}(1) = 1$, too. Doing that, the factors $C_\phi(\mu)$ appear in the three - point functions as fusion coefficients. It becomes clear that the appearance of such factors is quite general. Consider the fusion of n fields S by DCF into the field $M_0^{\{n,0\}}$. In the free field limit this corresponds to taking the Wick normal product

$$:\mathbf{s}_\otimes^n: \tag{61}$$

Two such fields multiply as

$$:\mathbf{s}_\otimes^{p_1}:(x)\ :\mathbf{s}_\otimes^{p_2}:(0) = \ :\mathbf{s}_\otimes^{p_1+p_2}:(0) + \dots \tag{62}$$

whereas DCF yields

$$M_0^{\{p_1,0\}}(x)\,M_0^{\{p_2,0\}}(0) = f\left(x^2\right)^{\mathcal{O}(\mu-1)} M_0^{\{p_1+p_2,0\}}(0) + \dots \tag{63}$$

The exponent of x^2 contains only anomalous dimensions and tends to zero at $\mu = 1$. Computation of f gives

$$f(\mu) = 1 + \frac{\mu}{(\mu-1)(\mu-2)}\eta(S)\,p_1 p_2 + \mathcal{O}\Big(\frac{1}{N^2}\Big) \tag{64}$$

so that, with (53)

$$\lim_{\mu \searrow 1} f(\mu) = 1 - \frac{p_1 p_2}{N} + \mathcal{O}\Big(\frac{1}{N^2}\Big) \tag{65}$$

Then we end up with a final

Theorem 7: The $d = 2$ limit is into the universality class of the polynomial algebra of free fields. Fusion coefficients are $\mathcal{O}(\frac{1}{N})$ deformed with respect to free field theory.

In particular this implies that exponential expressions of free fields ("vertex operators") cannot arise. Moreover the $\epsilon = d - 2$ expansions (which are in the literature since about 1976) are correct only if applied to critical indices and not to amplitudes. To our knowledge this restriction has never been clearly expressed before.

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THE GEOMETRY OF QUANTUM CORRECTION FOR TOPOLOGICAL SIGMA MODELS: A SIMPLE EXAMPLE.

DAVIDE FRANCO and CESARE REINA *
SISSA - Via Beirut 4 - Trieste, Italy

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1. Introduction

There are well known examples of classical field theories for which the minima (instantons) of the euclidean action come into families (moduli). The basic examples are the euclidean versions of Yang Mills theory, gravity with cosmological constant and some classes of σ-models. In the semiclassical approximation to such theories, there are observables which correspond to cohomology classes on the moduli spaces of instantons and whose expectation values coincide with the associated intersection numbers. This is the "topological sector" of the theory. As the intersection numbers do not depend on the covariance, the expectation values of the topological observables is not perturbative and to compute them one can safely set the covariance to zero (i.e. pick up a purely topological action) and work with the intersection theory of moduli spaces (see e.g. [3] for more details).

For non linear σ-models with values in compact Kahler manifold, however, the folklore is that what matters is the cohomology ring of the target space modulo the "quantum correction". This is somewhat surprising because:

1) what really matters is the intersection ring of the instanton moduli spaces
2) the quantum correction is generically inomogeneous (e.g. $\omega^2 = 1$ with $\deg\omega = 1$ for $\mathbf{P}^1$ models) and breaks the grading which is typical of intersection rings.

This inconsistency is actually only apparent, as everything can be explained and understood geometrically as we did in [2]. There we first considered the case of σ-models $\mathbf{P}^1 \to Gr(s;n)$ from the Riemann sphere $\mathbf{P}^1$ to

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a Grassmannian and checked that the geometrical construction yielded the right answer alredy proposed by Gepner and Intriligator on purely algebraic grounds. We then studied the generalization to flag-manifold valued models. The case of models with values on toric manifolds is presentely under study.

The general picture we get is as follows:

i) the moduli spaces M_d of instantons of (multi)-degree d have projective compactifications $\overline{M}_d = M_d \cup \Delta$,

ii) whenever one has to compute intersections of the form

$$(\beta_1 ... \beta_n . \alpha; \overline{M}_d),$$

one can find classes $\beta'_1 ... \beta'_n$in the Chow ring of $\overline{M}_{d-1}$ such that

$$(\beta_1 ... \beta_n . \alpha; \overline{M}_d) = (\beta'_1 ... \beta'_n; \overline{M}_{d-1})$$

which formally amounts to putting $\alpha = 1$, i.e. to applying the quantum correction.

Understsanding these results in general is a nice exercise of algebraic geometry, which requires some technical tools which may be not widely known among physicists. We feel better to concentrate hereinafter on the simplest example where all the details can be explicitely worked out. We hope that this may help a wider understanding of the basic geometrical ideas underlying "quantum corrections". For the approach to more general examples we refer to [2].

The topological P^1-model

As it is well known, the minima (instantons) of the Dirichlet action for maps $f : \mathbf{P}^1 \to \mathbf{P}^1$, $\mathbf{P}^1$ being the Riemann sphere, are simply holomorphic maps. The parameter space of such instantons is then the disjoint union $M = \coprod_{d \geq 0} M_d$ of the spaces M_d of holomorphic maps of degree d. Every $f \in M_d$ can be explicitely given as

$$x_i = P_i(z_0, z_1), \ (i = 0, 1)$$

where $(z_0, z_1), (x_0, x_1)$ are homogeneous coordinates of the source and the target and $P_i(z_0, z_1)$ are homogeneous polynomials of degree d. Notice that this gives a well defined map of degree d if and anly if the P_i's have no no common zeroes. As homogeneous coordinates are defined up to scalar multiplication, the parameter space of such maps is the space of coefficients occurring in the polynomial P_i modulo scalar multiplication. We get then that $M_d \simeq \mathbf{P}^{2d+1} \setminus \Delta$ is an open subvariety of $\mathbf{P}^{2d+1}$, the degeneracy locus Δ being the set of parameter where P_0 and P_1 have common zeroes.

There are in principle several ways of compactifying M_d into a projective variety $\overline{M}_d$. In the present case, it is natural to set $\overline{M}_d \simeq \mathbf{P}^{2d+1} \simeq M_d \cup \Delta$.

Notice that Δ has (complex) codimension 1 in $\overline{M}_d$; indeed, it is a divisor given by the vanishing of the resultant

$$R = \begin{vmatrix} a_0 & a_1 & . & . & a_d & 0 & . & 0 \\ 0 & a_0 & a_1 & . & a_{d-1} & a_d & . & 0 \\ . & . & . & . & . & . & . & . \\ 0 & . & 0 & a_0 & a_1 & . & a_{d-1} & a_d \\ b_0 & b_1 & . & . & b_d & 0 & . & 0 \\ 0 & b_0 & b_1 & . & b_{d-1} & b_d & . & 0 \\ . & . & . & . & . & . & . & . \\ 0 & . & 0 & b_0 & b_1 & . & b_{d-1} & b_d \end{vmatrix}.$$

of the two polynomials $P_0(z_0, z_1) = a_0 z_0^d + ... + a_{d+1} z_1^d$ and $P_1(z_0, z_1) = b_0 z_0^d + ... + b_{d+1} z_1^d$. For instance, when d=1 $d = 1$, the two polynomials $P_0 = az_0 + bz_1$, $P_1 = cz_0 + dz_1$ are parametrized by $m := (a : b : c : d) \in \mathbf{P}^3$, $(a : ... : d)$ denoting the line spanned by $(a, ..., d)$, and have a common zero whenever m belongs to the quadric $Q \subset \mathbf{P}^3$given by the equation $ad - bc = 0$. So $\Delta \simeq Q$ and $M_1 \simeq \mathbf{P}^3 \setminus Q$.

The "universal" instanton of degree d

$$f : \mathbf{P}^1 \times \overline{M}_d \longrightarrow \mathbf{P}^1$$

$$(z_0 : z_1, m) \longrightarrow f_m(z_0 : z_1),$$

where f_m denotes the map parametrized by $m \in \overline{M}_d$, is holomorphic on $\mathbf{P}^1 \times M_d \longrightarrow \mathbf{P}^1$. For $m \in \Delta$ instead, there is a finite number of points of $\mathbf{P}^1$ where f_m degenerates and hence the universal instanton is not defined on a locus of codimension two in $\mathbf{P}^1 \times \overline{M}_d$. It follows that f is actually a rational map from $\mathbf{P}^1 \times \overline{M}_d$ to $\mathbf{P}^1$. For every point $p \in \mathbf{P}^1$, the map

$$f(p) : p \times \overline{M}_d \longrightarrow \mathbf{P}^1$$

$$(p, m) \longrightarrow f_m(p)$$

is again a rational map, with degeneracy locus

$$\Delta_p = \{m \in \overline{M}_d \mid P_0(p, m) = P_1(p, m) = 0\}.$$

Let us next work out in full details the simplest case of $d = 1$. The universal instanton reads

$$f : \mathbf{P}^1 \times \overline{M}_1 \longrightarrow \mathbf{P}^1$$

$$(z_0 : z_1, a : b : c : d) \longrightarrow (az_0 + bz_1 : cz_0 + dz_1)$$

and and we know that it is not defined where $ad - bc = 0$ and $az_0 + bz_1 = 0$.

We want to see in this simple example how the homology (actually the Chow) ring of the target space is related to intersection theory on the $\overline{M}_1$. Recall that the intersection ring of $\mathbf{P}^1$ is generated by the class $[p]$ of a point $p \in \mathbf{P}^1$ (i.e. the Poincaré dual of the Kahler form ω), with the relation $[p].[p] = 0$. The idea is to study the "preimage" under f of p in the target $\mathbf{P}^1$ The map f, where defined, is completely determined by its graph $\Gamma \subset \mathbf{P}^1 \times \overline{M}_1 \times \mathbf{P}^1$. To define a suitable preimage of a point p belonging to the target $\mathbf{P}^1$ we have to compactify Γ into $\overline{\Gamma} \subset \mathbf{P}^1 \times \overline{M}_1 \times \mathbf{P}^1$ so that we can define $f^{-1}(p) =: \overline{\Gamma} \cap (\mathbf{P}^1 \times \overline{M}_1 \times \{p\})$. Clearly

$$\overline{\Gamma} = V(x_0(cz_0 + d_1) - x_1(az_0 + bz_1)) \subset \mathbf{P}^1 \times \overline{M}_1 \times \mathbf{P}^1,$$

where $V(...)$ denotes the zero locus of ..., and

$$f^{-1}(p) = V(\overline{x}_0(cz_0 + dz_1) - \overline{x}_1(az_0 + bz_1)) \subset \mathbf{P}^1 \times \overline{M}_1,$$

where $(\overline{x}_0 : \overline{x}_1)$ are the homogeneous coordinate of $p \in \mathbf{P}^1$. To find classes in $\overline{M}_1$ corresponding to local observables, we have to fix also a point (0, say) of the source $\mathbf{P}^1$ and consider the map

$$f^0 : \overline{M}_1 \to \mathbf{P}^1$$

given by $f^0(a : b : c : d) := f(0 : 1; a : b : c : d)$. Similarly to what we have done before, we can look at the primage in $\overline{M}_1$ under f^0 of the fundamental class of $p \in \mathbf{P}^1$, getting

$$(f^0)^{-1}(p) =: (\{0\} \times \overline{M}_1 \times \{p\}) \cap \overline{\Gamma} \subset \overline{M}_1,$$

which is the zero locus of

$$\overline{x}_0 d = \overline{x}_1 b,$$

i.e. a hyperplane $H \subset \mathbf{P}^3 = \overline{M}_1$. Let us calculate the intersection $(f^0)^{-1}(p) \cap (f^0)^{-1}(p')$, setting for simplicity $p = (0 : 1), p' = (1 : 0)$. We have $(f^0)^{-1}(p) = V(b)$, $(f^0)^{-1}(p') = V(d)$ and their intersection is the line

$$L =: (f^0)^{-1}(p) \cap (f^0)^{-1}(p') = \{(a : 0 : c : 0) \in \overline{M}_1\}.$$

Clearly L is contained into Δ. To give a closer look to L we need a more concrete understanding of what kind of graphs are represented by the points of Δ. If $m = (a : b : c : d) \in \overline{M}_1 \setminus \Delta$ then f_m has as graph $\Gamma_m = V(x_0(cz_0 + dz_1) - x_1(az_0 + bz_1)) \subset \mathbf{P}^1 \times \mathbf{P}^1$. Now, Γ_m is a graph of an actual map if and only if the first projection $\Gamma_m \to \mathbf{P}^1$ an isomorphism. Then, if $m \in \Delta$, Γ_m does not represent any function, because the point at which both $az_0 + bz_1$ and $cz_0 + dz_1$ vanish does not belong to the graph. Nevertheless if we define $\overline{\Gamma}_m$ as before, we have a subset of $\mathbf{P}^1 \times \mathbf{P}^1$ of the same degree of a real graph (in the present case, this is a consequence of the fact that $\overline{\Gamma}$ is defined by a unique equation, for more general targets this depends on the fact that M_d is equipped with a "flat family" of graphs). Whenever $m \in L$ whe have that $\overline{\Gamma}_m = V(cx_0z_0 - ax_1z_0) = V(z_0(cx_0 - ax_1))$. It is clear that $\overline{\Gamma}_m$ is the union of two lines: the first, which is given by the equation $z_0 = 0$, does not represent any function; the second, which is given by $cx_0 - ax_1 = 0$ is the graph of a constant function.

We have then an isomorphism

$$f^{0-1}(p) \cap f^{0-1}(p') = L = M_0,$$

given by $L \ni (a:0:c:0) \mapsto (a:c) \in M_0 = \overline{M}_0 \simeq \mathbf{P}^1$. which, as explained in the introduction, gives the quantum correction $f^{0-1}(p) \cap f^{0-1}(p') = 1$, at least at the level of degree-one instantons. Let see more closely how this works. Since $\overline{M}_1 \simeq \mathbf{P}^3$, the unique non vanishing expectation value of local observables can be computed as an intersection of the form

$$[i] = [(f^0)^{-1}(p_1)].[(f^0)^{-1}(p_2)].[(f^0)^{-1}(p_3)]$$

of three cycles classes [...] in the intersection ring of $\mathbf{P}^3$.

Unfortunately, the representatives of the cycles classes explicitely occourring above do note intersect transversally, but we can argue as follows. The action of $Sl(2, \mathbf{C})$ on the source $\mathbf{P}^1$ sends cycles into equivalent cycles, and therefore we can compute $[i]$ by computing

$$i = (f^{q_1})^{-1}(p_1) \cap (f^{q_2})^{-1}(p_2) \cap (f^{q_3})^{-1}(p_3)$$

i.e.

$$i = \cap_k V_k, \quad V_k =: V(-\bar{x}_1^{(k)}\bar{z}_0^{(k)}a - \bar{x}_1^{(k)}\bar{z}_1^{(k)}b + \bar{x}_0^{(k)}\bar{z}_0^{(k)}c + \bar{x}_0^{(k)}\bar{z}_1^{(k)}d)$$

where $p_k = (\bar{x}_0^{(k)} : \bar{x}_1^{(k)})$ and $q_k = (\bar{z}_0^{(k)} : \bar{z}_1^{(k)})$. One easily checks that the sistem of 3 linear equations in (a,b,c,d) equivalent to the intersection above has maximal rnk for a generic choice of the p_k, q_k and therefore there is a unique line $(a:b:c:d)$ in the intersection.

It is also obvious that $V_2 \cap V_3$ is isomorphic to a copy of $\overline{M}_0 = \mathbf{P}^1$ and therefore $i \subset \overline{M}_0$. Accordingly, for the intersection numbers, we have

$$< [V_1].[V_2].[V_3]; \bar{M}_1 >=< [V_1].[M_0]; M_0 >,$$

and this is exactly what is called the quantum correction in the physical literature.

Returning to the general situation there is a map

$$f : \mathbf{P}^1 \times (\overline{M}_d \setminus \Delta) \to \mathbf{P}^1$$

$$(z_0 : z_1 : a_0 : ... : a_d : b_0 : ... : b_d) \to (a_0 z_0^d + ... + a_d z_1^d : b_0 z_0^d + ... b_d z_1^d)$$

(Δ being the zero locus of the resultant) from which we get a divisor

$$\overline{\Gamma} = V(x_0(b_0 z_0^d + ... + b_d z_1^d) - x_1(a_0 z_0 + ... + a_d z_1^d)) \subset \mathbf{P}^1 \times \overline{M}_1 \times \mathbf{P}^1.$$

Fixing a point 0 of the source and two points $p = (0 : 1), p' = (1 : 0)$ of the target as we made in the example, we can compute the intersection $L = (f^0)^{-1}(p) \cap (f^0)^{-1}(p') \subset \overline{M}_d$ where again $f^0 : M_d \to \mathbf{P}^1$ is given by $f^0(m) := f(0; m)$. We find

$$L = V(a_d) \cap V(b_d) = \{(a_0 : ... : a_{d-1} : 0 : b_0 : ... : b_{d-1} : 0) \in \overline{M}_d\} \subset \Delta.$$

To every $m \in L$ we can associate a subset $\Gamma_m \subset \mathbf{P}^1 \times \mathbf{P}^1$, which cannot represent any function, by setting

$$\Gamma_m = V(z_0[x_0(b_0 z_0^{d-1} + ... + b_{d-1} z_1^{d-1}) + x_1(a_0 z_0^{d-1} + ... + a_{d-1} z_1^{d-1})]).$$

Again Γ_m is the union of two components:the first, which is given by the equation $z_0 = 0$, does not correspond to any function; the second, which is given by $x_0(b_0 z_0^{d-1} + ... + b_{d-1} z_1^{d-1}) + x_1(a_0 z_0^{d-1} + ... + a_{d-1} z_1^{d-1}) = 0$, represents an instanton of degree $d - 1$.

Finally, similarly to what happens for degree-one maps, we have an isomorphism

$$(f^0)^{-1}(p) \cap (f^0)^{-1}(p') = L \simeq M_{d-1}.$$

which gives the quantum correction

$$(f^0)^{-1}(p) \cap (f^0)^{-1}(p') = 1.$$

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SOLITONS IN TOPOLOGICAL FIELD THEORIES

MARK TEMPLE-RASTON*
Department of Physics, Bishop's University,
Lennoxville, Quebec Canada

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Abstract. We present a topological Lagrangian field theory that is geometrically similar to the Yang-Mills(-Higgs) Lagrangian, and study the Bogomol'nyi solitons contained within this theory. The topological field theory may provide an example of a dual field theory to Yang-Mills(-Higgs). The existence of a dual field theory to Yang-Mills(-Higgs) theory was conjectured by Montonen and Olive.

1. Introduction

Recently a class of Lagrangian topological field theories possessing a 'minimizing' Bogomol'nyi structure has been introduced on oriented, compact, connected four-manifolds [8]. The associated Bogomol'nyi equations are reminiscent of the self-duality equations in Yang-Mills theory, and the solutions to the topological Bogomol'nyi equations share much in common with solutions to the self-duality equations (instantons). Like the Yang-Mills instanton, for example, solutions to the topological Bogomol'nyi equations can be translated into geometrical structure on an appropriate holomorphic vector bundle, and, the moduli space of solutions forms a Hausdorff differentiable manifold. We shall call solutions to the topological Bogomol'nyi equations 'topological instantons'. The topological field theories studied in [8] achieve these results with relatively little hard analysis and algebraic geometry when compared with the Yang-Mills instanton theory [2]. The reason for this is that topological instantons are essentially equivalent to the differential geometric formulation of 'stable vector bundles' due to Kobayashi [6]. The differences between Yang-Mills instantons and topological instantons are also significant. We mention three differences. First, non-trivial topological instantons can exist on pseudo-Riemannian space-times, while Yang-Mills instantons are trivial on space-times. Second, topological instantons have a larger gauge group, $U(n)$. Third, topological instantons by virtue of their non-triviality on space-times have a space-of-motions equivalent to the moduli space; Yang-Mills instantons are pseudo-particles and do not possess a

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space-of-motions. It is well-known that self-dual instantons in Yang-Mills theory and BPS magnetic monopoles in Yang-Mills-Higgs theory are closely related [1]. BPS magnetic monopoles are non-singular, finite-energy solutions to the self-duality equations reduced to three spatial dimensions with a gauge symmetry in the (imaginary) time direction. A similar process can be applied to the topological instanton, leading to the theory of topological monopoles. The topological instanton and the topological monopole obtained by dimensional reduction are the subject of this paper.

In the next section we discuss the differential geometry of the class of topological field theories on four-manifolds introduced in [8], and expose the Bogomol'nyi structure. Solutions to the Bogomol'nyi equations (topological instantons) are shown to be projectively flat. The physical stability of the topological instanton field configuration is argued from the topology of the underlying four-manifold. In section three, we dimensionally reduce the four-dimensional topological field theory to three spatial dimensions. The Bogomol'nyi structure survives the dimensional reduction. Topological monopoles are the solutions to the Bogomol'nyi equations in three dimensions. Although the theory of topological monopoles is very similar to the theory of BPS magnetic monopoles, there is an interesting difference between the Bogomol'nyi structures of the two theories. In the theory of BPS magnetic monopoles the Bogomol'nyi equations appear as a completed square in the Lagrangian, while in the theory of topological monopoles they do not. The Bogomol'nyi equations in our class of TFTs consist of two equations, either of which will saturate the Bogomol'nyi energy. This added flexibility in saturating the Bogomol'nyi energy allows greater freedom in constructing solitonic particles with either an electric or magnetic charge.

2. Instantons in topological field theories

The Lagrangian theories in [8] are defined by the Lagrangian Action functional:

$$\mathcal{L}(A,B) = \int_M < (H^A \otimes I_E) \wedge (I_E \otimes K^B) > -\frac{1}{2} < (I_E \otimes K^B)^2 > \quad (1)$$

defined on the product space $\mathcal{A}(P) \times \mathcal{A}(P)$. Interpreting H^A and K^B as curvatures in the Lagrangian Action requires that the real dimension of M be four. I_E is the identity transformation on the adjoint bundle, E. The brackets $< >$ remind us that a choice of adjoint-invariant, real-valued inner product on the adjoint bundle is needed. The Action functional introduces an artificial asymmetry in H^A and K^B which is not supported by a physical argument; we will return to this later. The variational field equations for (1) are

$$D^A K^B = 0, \qquad D^B H^A = 0, \quad (2)$$

where we have made use of the Bianchi identity $D^B K^B = 0$. The set of solutions is clearly neither empty nor entirely trivial. The physical stability of a class of nontrivial, nonsingular, finite-Action solutions to the variational equations (3) can be demonstrated by a topological argument. The Lagrangian (1) can be rewritten as

$$2\mathcal{L}(A,B) = -\int_M < (H^A \otimes I_E - I_E \otimes K^B)^2 > + \int_M < (H^A \otimes I_E)^2 > \quad (3)$$

The inner product structure defines a Weyl polynomial of degree two. Let E_A and E_B be the vector bundle E equipped with either the connection A or B, respectively. The first term in the Lagrangian $\mathcal{L}$ in equation (3) is a topological invariant for the tensor product bundle $E_A \otimes E_B^*$. Recall that the curvature of $E_A \otimes E_B^*$ is given by $\Omega_{E_A \otimes E_B^*} = H^A \otimes I_E - I_E \otimes K^B$. The Bogomol'nyi equations,

$$H^A \otimes I_E = I_E \otimes K^B, \quad (4)$$

are therefore a vanishing curvature condition on the tensor product bundle $E_A \otimes E_B^*$. Solutions to (4) automatically satisfy the variational field equations (2). An indice computation for (4), $H^A_{ab}\ \delta_{cd} = \delta_{ab}\ K^B_{cd}$, shows that the curvature forms H^A and K^B are projectively flat. That is,

$$H^A = K^B = iFI_r, \quad (5)$$

where F is a real-valued two form on M, and I_r is the identity endomorphism for the vector bundle, E, of rank r. The Bianchi identity imposes a simple condition on F, that $dF = 0$, so that $F \in H^2(M, \mathbf{R})$. Since M is compact, $H^2(M, \mathbf{R})$ is of finite dimension. If F is a curvature on M, then the second term in (3) is a topological invariant of the underlying four-manifold, M. Topologically non-trivial solutions to the Bogomol'nyi equations will be said to be 'physically stable' if F is a curvature of M and if the solutions have a fixed non-zero Action given by

$$2\mathcal{L} = -\int_M F \wedge F = -24\pi\ \mathrm{sgn}(M) \neq 0,$$

where the topological signature of the manifold, M, is denoted by $\mathrm{sgn}(M)$. Physically stable, non-trivial solutions to the Bogomol'nyi equations (5) on the vector bundle $(E, < >)$ are called topological instantons [8].

3. Monopoles in topological field theories

We now examine static, non-singular solutions to the Bogomol'nyi equations (5). By assuming a gauge symmetry in the direction of time, X_t, we can dimensionally reduce the four-dimensional theory on $\mathbf{R}^4$ defined by (1), to a theory on $\mathbf{R}^3$. The reductions are performed using the gauge symmetry equations,

$$\begin{aligned} H^A(X_t,\cdot) &= -D^A\Phi_A, \\ K^B(X_t,\cdot) &= -D^B\Phi_B. \end{aligned} \tag{6}$$

Dimensional reduction introduces the equivariant Lie algebra valued fields, $\Phi_A, \Phi_B \in \Lambda^0(R^3, \mathrm{End}(E))$, defined by $\Phi_A = A(X_t)$ and $\Phi_B = B(X_t)$ [4]. We can either reduce the Bogomol'nyi field equations (5) directly, or, reduce the full variational field equations. In the first case, the Bogmol'nyi equations locally reduce to

$$\begin{aligned} D^A\Phi_A = D^B\Phi_B &= EI_E, \\ H^A|_{\mathbf{R}^3} = K^B|_{\mathbf{R}^3} &= FI_E. \end{aligned} \tag{7}$$

E is the one-form obtained by contracting F in (5) on the infinitesimal time displacement. In the second equation in (7) F denotes the restriction of F in four-dimensions restricted to the leaves in the foliation defined by X_t. Alternatively, the Lagrangian Action (1) after reduction becomes

$$\begin{aligned} \mathcal{E}(A,B) = \textstyle\int_{M_3} &< (I_E \otimes K^B) \wedge (I_E \otimes D^B\Phi_B) > \\ &- < (I_E \otimes K^B) \wedge (D^A\Phi_A \otimes I_E) > \\ &- < (H^A \otimes I_E) \wedge (I_E \otimes D^B\Phi_B) >, \end{aligned} \tag{8}$$

and the dimensionally reduced field equations become

$$\begin{aligned} D^BH^A &= 0, \qquad & D^BD^A\Phi_A &= [H^A, \Phi_B], \\ D^AK^B &= 0, \qquad & D^AD^B\Phi_B &= [K^B, \Phi_A]. \end{aligned} \tag{9}$$

The energy functional (8) can be rewritten as

$$\begin{aligned} &\mathcal{E}(A,B) = \\ &\textstyle\int_{M_3} < (H^A \otimes I_E - I_E \otimes K^B) \wedge (D^A\Phi_A \otimes I_E - I_E \otimes D^B\Phi_B) > \\ &- \textstyle\int_{M_3} < (D^A\Phi_A \otimes I_E) \wedge (H^A \otimes I_E) > \end{aligned} \tag{10}$$

It is clear from (10) that the reduced topological instanton equations (7) continue to saturate the Bogomol'nyi bound, given by the second integral in (10). In Yang-Mills theory, solutions to the time-reduced instanton equations are called BPS magnetic monopoles. We call solutions to the time-reduced topological instanton equations: topological monopoles. Unlike Yang-Mills theory, however, the energy functional (10) is saturated at the Bogomol'nyi bound with either equation in (7). We need not insist that both equations in (7) be satisfied in order to saturate the bound, although of course the field configurations must still satisfy the second-order variational field equations.

To be observable to conventional detectors, $U(n)$ field configurations must be broken. The symmetry breaking mechanism for BPS magnetic monopoles is very attractive [3], so we will use it here. Imagine a solitonic core region at the origin. Let G and H be compact and connected gauge groups, where the group H is assumed to be embedded in G. The gauge group of the core region G is spontaneously broken to H outside of the core region when the Higgs field is covariantly constant, $D\Phi = 0$. In regions far from the core $(r \to \infty)$ where we assume that $D^A\Phi_A = 0$, it can be shown that

$$H^A = \Phi_A F_A, \tag{11}$$

where $F_A \in \Lambda^2(M_3, E_H)$, is any closed two-form on M_3 taking values in the H-Lie algebra bundle, denoted by E_H here. A similar expression to (11), $K^B = \Phi_B F_B$, can be written when $D^B\Phi_B = 0$. We assume that $<\Phi_A\Phi_A>= 1$ when $r >> 1$ and where spontaneous symmetry breaking has occurred. When $G = U(n)$ and $H = U(1)$, F_A becomes a pure imaginary two-form on M_3. Consider the Bogomol'nyi solitons defined by (7). Solutions to (7) have an energy topologically fixed by

$$\begin{aligned} \mathcal{E} &= -\int_{M_3} < (D^A\Phi_A \otimes I_E) \wedge (H^A \otimes I_E) > \\ &= -\int_{M_3} d < \Phi_A H^A >= -\int_{S^2} < \Phi_A H^A >, \end{aligned} \tag{12}$$

where S^2 is a large sphere surrounding the monopole core and lying completely in a region where $D^A\Phi_A = 0$. Substituting (11) into (12) and using the normalisation condition $<\Phi_A\Phi_A>= 1$, the energy is fixed by $\int F_A$. As in the case of the BPS magnetic monopole, $\int F_A$ would be interpreted as the magnetic charge.

4. Conclusion

In this short contribution we have introduced a class of topological field theories in three and four dimensions, exposed their Bogomol'nyi structures, and argued the physical stability of solutions. But we believe that the theories presented here are incomplete because there is a physical asymmetry in the gauge fields present in (1). Symmetry in the Action is easily regained, however, by exchanging H^A and K^B, and adding it to the Lagrangian (1). The variational field equations and the Bogomol'nyi equations are unchanged by the symmetrization. In four dimensions, the stability of the topological instanton is only slightly different—the symmetrized Action is twice that of the asymmetric Action. In three dimensions, the saturated energy functional becomes

$$\begin{aligned} \mathcal{E} = &-\int_{M_3} < (D^A\Phi_A \otimes I_E) \wedge (H^A \otimes I_E) > \\ &-\int_{M_3} < (D^B\Phi_B \otimes I_E) \wedge (K^B \otimes I_E) > . \end{aligned} \tag{13}$$

We argued stability from the topological interpretation that can be given to (13). The symmetrization of the topological field theory implies that the solitonic particle is topologically stable if either of the integrals in (13) is non-vanishing. The integrals should correspond to the magnetic and electric charge of the soliton given by $\int_{S^2} F_A$ and $\int_{S^2} F_B$, respectively.

A particularly glaring omission in the solitonic particle spectrum in YMH theory is the electric monopole. The Montonen-Olive conjecture addresses this by proposing with some compelling evidence that there exists a dual field theory to YMH theory which would replace the BPS magnetic monopole with solitonic intermediate vector bosons: $W^{\pm}$, Z^0 [5]. Although there is still much study needed, we believe that theory of topological monopoles may be an example of a dual field theory [9]. If so, then in order to ensure the stability of the Z_0 particle, the Z_0 must be a magnetic monopole.

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CLASSIFICATION OF NON-ABELIAN CHERN-SIMONS VORTICES

GERALD V. DUNNE*
Department of Physics, University of Connecticut
2152 Hillside Road Storrs, CT 06269 USA, dunne@hep.phys.uconn.edu

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Abstract. The two-dimensional self-dual Chern-Simons equations are equivalent to the conditions for static, zero-energy vortex-like solutions of the (2+1) dimensional gauged nonlinear Schrödinger equation with Chern-Simons matter-gauge coupling. The finite charge vacuum states in the Chern-Simons theory are shown to be gauge equivalent to the finite action solutions to the two-dimensional chiral model (or harmonic map) equations. The Uhlenbeck-Wood classification of such harmonic maps into the unitary groups thereby leads to a complete classification of the vacuum states of the Chern-Simons model. This construction also leads to an interesting new relationship between $SU(N)$ Toda theories and the $SU(N)$ chiral model.

The study of the nonlinear Schrödinger equation in $2+1$-dimensional space-time is partly motivated by the well-known success of the $1+1$-dimensional nonlinear Schrödinger equation. Here we consider a *gauged* nonlinear Schrödinger equation in which we have not only the nonlinear potential term for the matter fields, but also we have a coupling of the matter fields to the gauge fields. Furthermore, this matter-gauge dynamics is chosen to be of the Chern-Simons form rather than the conventional Yang-Mills form. With this choice, the nonlinear term in the Schrödinger equation may also be viewed as a Pauli interaction, due to the Chern-Simons relation between the magnetic field and the charge density.

The theory with an Abelian gauge field was analyzed by Jackiw and Pi [7] who found static, zero energy solutions which arise from a two-dimensional notion of self-duality. The static, self-dual matter density satisfies the Liouville equation, which is known to be integrable [10]. The gauged nonlinear Schrödinger equation with *non-Abelian* Chern-Simons matter-gauge dynamics has also been considered [5, 3, 4], and once again static, zero energy solutions (referred to as "*self-dual Chern-Simons vortices*") have been found to arise from an analogous, but much richer, two-dimensional self-duality condition. These two-dimensional self-duality equations are formally integrable

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and in special cases they reduce to the classical and affine Toda equations, both known integrable systems of nonlinear partial differential equations [8, 9].

Here, I classify all finite charge solutions to the self-dual Chern-Simons equations by first showing that the self-duality equations are equivalent to the classical equations of motion of the Euclidean two-dimensional chiral model (also known as the harmonic map equations), and then using a deep classification theorem due to Uhlenbeck [11] which classifies all $U(N)$ and $SU(N)$ chiral model solutions with finite chiral model action. The chiral model action is in fact proportional to the net gauge invariant *charge* Q in the matter-Chern-Simons system, and so the classification of all finite charge solutions is complete. I also present the explicit "*uniton*" decomposition of a special class of solutions to the $SU(N)$ chiral model equations which have the remarkable property that when the matter density for these solutions is diagonalized, it satisfies the classical $SU(N)$ Toda equations. Such a direct correspondence between the Toda equations and the chiral model equations is surprising.

The $2+1$-dimensional nonlinear Schrödinger equation reads [1]

$$iD_0\Psi = -\frac{1}{2}\mathbf{D}^2\Psi + \frac{1}{\kappa}[\,[\Psi,\Psi^\dagger],\Psi] \tag{1}$$

where the covariant derivative is $D_\mu \equiv \partial_\mu + [A_\mu,\ \ ]$, and both the gauge potential A_μ and the matter field Ψ are Lie algebra valued: $A_\mu = A_\mu^a T^a$, $\Psi = \Psi^a T^a$. The main results of this paper are for the Lie algebra of $SU(N)$, but the formulation generalizes straightforwardly to any simple Lie algebra (the noncompact case has been studied in [1]). The matter and gauge fields are coupled dynamically by the Chern-Simons equation

$$F_{\mu\nu} = \frac{i}{\kappa}\epsilon_{\mu\nu\rho}J^\rho \tag{2}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ is the gauge curvature, κ is a coupling constant and J^ρ is the covariantly conserved ($D_\mu J^\mu = 0$) nonrelativistic matter current

$$\begin{aligned} J^0 &= [\Psi^\dagger, \Psi] \\ J^i &= -\frac{i}{2}\left([\Psi^\dagger, D_i\Psi] - [(D_i\Psi)^\dagger, \Psi]\right) \end{aligned} \tag{3}$$

The Schrödinger equation (1) and the Chern-Simons equation (2) are invariant under the gauge transformation

$$\begin{aligned} \Psi &\to g^{-1}\Psi g \\ A_\mu &\to g^{-1}A_\mu g + g^{-1}\partial_\mu g \end{aligned} \tag{4}$$

[1] Note that there is a typographical error in this equation in [4].

where $g \in SU(N)$.

In [3, 4] it has been shown that the minimum (in fact *zero*) energy solutions to (1,2) are given by the *self-dual Ansatz*

$$D_-\Psi = 0 \tag{5}$$

combined with the remaining Chern-Simons equation

$$\partial_- A_+ - \partial_+ A_- + [A_-, A_+] = \frac{2}{\kappa}\,[\Psi^\dagger, \Psi]. \tag{6}$$

Here $A_\pm = A_1 \pm iA_2$, $D_\pm = D_1 \pm iD_2$ and with antihermitean Lie algebra generators we have $A_\pm = -(A_\mp)^\dagger$. Equations (5,6) are collectively referred to as the *self-dual Chern-Simons equations.* The self-dual solutions provide *static* solutions to the gauged nonlinear Schrödinger equation, as can be seen from a Hamiltonian formulation [3]. Alternatively, this follows directly from the equations of motion (1,2). To see this, note that if $D_-\Psi = 0$, then the currents take the simple form

$$J^+ \;\equiv\; J^1 + iJ^2 = -\frac{i}{2}[\Psi^\dagger, D_+\Psi]. \tag{7}$$

It then follows from the Chern-Simons equation (2) that $A_0 = \frac{i}{2\kappa}[\Psi^\dagger, \Psi]$. The identity

$$\mathbf{D}^2\Psi \;\equiv\; D_+D_-\Psi + i[F_{12}, \Psi] = D_+D_-\Psi - \frac{1}{\kappa}[[\Psi^\dagger, \Psi], \Psi] \tag{8}$$

then implies that the Schrödinger equation reduces to

$$i\partial_0\Psi = -\frac{1}{2}D_+D_-\Psi = 0. \tag{9}$$

In fact, owing to a remarkable dynamical $SO(2,1)$ symmetry of the gauged nonlinear Chern-Simons-Schrödinger equations (1,2), it is possible to show that the implication holds in the reverse direction: *all* static solutions are self-dual [3].

Before classifying the general solutions to the self-dual Chern-Simons equations, it is instructive to consider certain special cases in which simplifying algebraic *Ansätze* for the fields reduce (5,6) to familiar integrable nonlinear equations. First, choose the fields to have the following Lie algebra decomposition

$$A_i = \sum_\alpha A_i^\alpha H_\alpha, \qquad\qquad \Psi = \sum_\alpha \psi^\alpha E_\alpha, \tag{10}$$

where the sum is over all positive, simple roots α of the Lie algebra, and H_α and E_α are the Cartan subalgebra and step operator generators (respectively) in the Chevalley basis [6]. Then the self-dual Chern-Simons equations (5,6) combine to yield the classical Toda equations

$$\nabla^2 \log\rho_\alpha = -\frac{2}{\kappa}K_{\alpha\beta}\rho_\beta \tag{11}$$

where $\rho_\alpha \equiv |\psi^\alpha|^2$, and $K_{\alpha\beta}$ is the classical Cartan matrix for the Lie algebra. For $SU(2)$, (11) becomes the Liouville equation $\nabla^2 \log\rho = -\frac{4}{\kappa}\rho$, which Liouville showed to be integrable and indeed "solvable" [10] - in the sense that the general real solution can be expressed in terms of a single holomorphic function $f = f(x^-)$:

$$\rho = \frac{\kappa}{2}\nabla^2 \log\left(1 + f(x^-)\bar{f}(x^+)\right) \tag{12}$$

Kostant [8], and Leznov and Saveliev [9] have shown that the classical Toda equations (11) are similarly integrable, with the general real solutions for ρ_α being expressible in terms of r arbitrary holomorphic functions, where r is the rank of the algebra. For $SU(N)$ it is possible to adapt the Kostant-Leznov-Saveliev solutions to a simpler form more reminiscent of the Liouville solution (12):

$$\rho_\alpha = \frac{\kappa}{2}\nabla^2 \log\det\left(M_\alpha^\dagger(x^+)M_\alpha(x^-)\right) \tag{13}$$

where M_α is the $N\times\alpha$ *rectangular* matrix $M_\alpha = (u\ \partial_- u\ \partial_-^2 u\ \ \ldots \partial_-^{\alpha-1} u)$, with u being an N-component column vector

$$u = \begin{pmatrix} 1 \\ f_1(x^-) \\ f_2(x^-) \\ \vdots \\ f_{N-1}(x^-) \end{pmatrix} \tag{14}$$

For a radially symmetric $SU(3)$ example see Figure 1. An alternative, extended, *Ansätz* involves the matter field choice

$$\Psi = \sum_\alpha \psi^\alpha E_\alpha + \psi^M E_{-M} \tag{15}$$

where E_{-M} is the step operator corresponding to minus the maximal root. With the gauge field still as in (10), the self-dual Chern-Simons equations then combine to give the *affine* Toda equations

$$\nabla^2 \log\rho_a = -\frac{2}{\kappa}\tilde{K}_{ab}\rho_b, \quad \tilde{K} \text{ is the affine Cartan matrix.} \tag{16}$$

The affine Toda equations (16) are also known to be integrable.

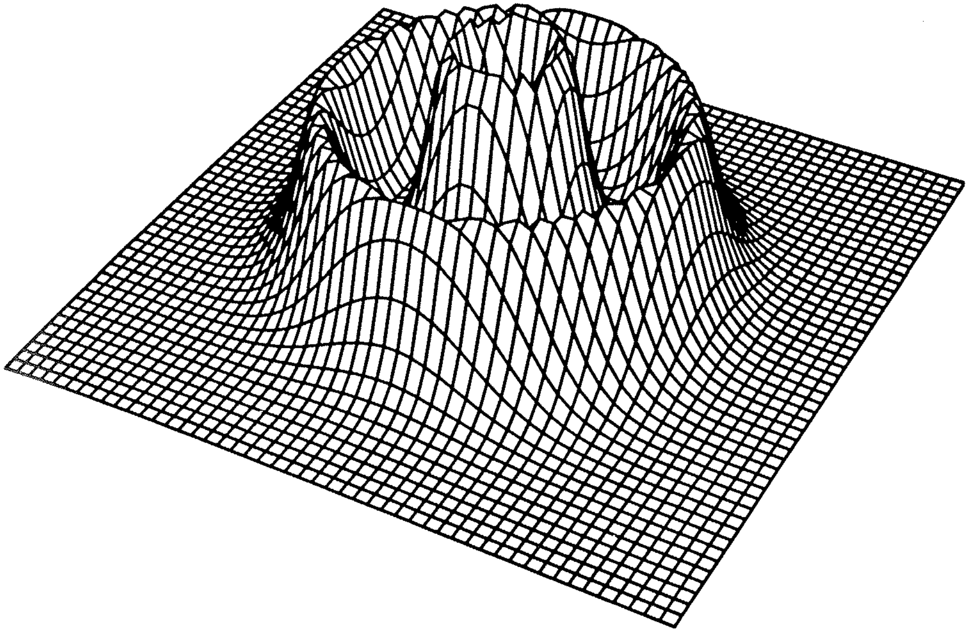

Fig. 1. A plot of the nonAbelian charge density ρ_1 for a radially symmetric SU(3) Toda-type vortex solution (13) to the self-dual Chern-Simons equations (5,6). For a radially symmetric solution, the functions $f_\alpha(x^-)$ appearing in (14) are chosen to be powers of x^-.

Having considered some special cases of solutions to the self-dual Chern-Simons equations, we now consider the general solutions by first making a gauge transformation to convert the equations (5,6) into the *single* equation

$$\partial_- \chi = [\chi^\dagger, \chi] \tag{17}$$

where χ is the gauge transformed matter field $\chi = \sqrt{\frac{2}{\kappa}} g \Psi g^{-1}$. The existence of such a gauge transformation g^{-1} follows from the following zero-curvature formulation of the self-dual Chern-Simons equations [3, 4]. Define

$$\mathcal{A}_+ \equiv A_+ - \sqrt{\frac{2}{\kappa}}\Psi, \qquad \mathcal{A}_- \equiv A_- + \sqrt{\frac{2}{\kappa}}\Psi^\dagger. \tag{18}$$

Then the self-dual Chern-Simons equations imply that the gauge curvature associated with $\mathcal{A}_\pm$ vanishes: $\partial_- \mathcal{A}_+ - \partial_+ \mathcal{A}_- + [\mathcal{A}_-, \mathcal{A}_+] = 0$. Therefore, locally, one can write $\mathcal{A}_\pm$ as pure gauge

$$\mathcal{A}_\pm = g^{-1}\partial_\pm g \quad \text{for some}\, g \in SU(N). \tag{19}$$

Gauge transforming the self-dual Chern-Simons equations (5,6) with this group element g^{-1} leads to the single equation (17).

Equation (17) can be converted into the chiral model equation by defining $\chi = \frac{1}{2}h^{-1}\partial_+ h$ for some $h \in SU(N)$ (the fact that it is possible to write χ in this manner is a consequence of (17)). The chiral model equation [15] reads:

$$\partial_+(h^{-1}\partial_- h) + \partial_-(h^{-1}\partial_+ h) = 0. \tag{20}$$

Given any solution h of the chiral model equations, or alternatively any solution χ of (17), we automatically obtain a solution of the original self-dual Chern-Simons equations:

$$\Psi^{(0)} = \sqrt{\frac{\kappa}{2}}\chi, \qquad A_+^{(0)} = \chi, \qquad A_-^{(0)} = -\chi^\dagger. \tag{21}$$

The *global* condition which permits the classification of solutions to the chiral model equation (20) is that the chiral model "action functional" (also referred to in the literature as the "energy functional")

$$\mathcal{E}[h] = -\frac{1}{2}\int d^2x \ \mathrm{tr}(h^{-1}\partial_- h h^{-1}\partial_+ h) \tag{22}$$

be *finite*. This finiteness condition is directly relevant in the related matter-Chern-Simons system because $\mathcal{E}[h] = 2\int d^2x \ \mathrm{tr}(\chi\chi^\dagger) = \frac{4}{\kappa}\int d^2x \ \mathrm{tr}(\Psi\Psi^\dagger) = \frac{4}{\kappa}Q$ where Q is the net gauge invariant matter charge. As well as being physically significant, this finiteness condition is mathematically crucial because it permits the chiral model solutions on $\mathbb{R}^2$ to be classified by conformal compactification to the sphere S^2 [11, 13].

Theorem (Uhlenbeck [11]; see also Wood [14]): *Every finite action solution h of the $SU(N)$ chiral model equation (20) may be uniquely factorized as a product of "uniton" factors*

$$h = \pm h_0 \prod_{i=1}^{m}(2p_i - 1) \tag{23}$$

where:

a) $h_0 \in SU(N)$ is constant;

b) each p_i is a Hermitean projector ($p_i^\dagger = p_i$ and $p_i^2 = p_i$);

c) defining $h_j = h_0\prod_{i=1}^{j}(2p_i - 1)$, the following linear relations must hold:

$$\begin{aligned} (1-p_i)\left(\partial_+ + \frac{1}{2}h_{i-1}^{-1}\partial_+ h_{i-1}\right) p_i &= 0 \\ (1-p_i)\, h_{i-1}^{-1}\partial_- h_{i-1}\, p_i &= 0 \end{aligned}$$

d) $m \leq N-1$.

The $\pm$ sign in (23) has been inserted to allow for the fact that Uhlenbeck and Wood actually considered $U(N)$ rather than $SU(N)$.

An important implication of this theorem is that for $SU(2)$ *all* finite action solutions of the chiral model have the "single uniton" form

$$h = -h_0(2p-1) \tag{24}$$

where p is a holomorphic projector satisfying

$$(1-p)\,\partial_+ p = 0 \tag{25}$$

These solutions are essentially the $\mathbb{C}P^1$ model solutions of Din and Zakrzewski [2, 15].

At this point, it is not at all obvious how these types of solutions to the chiral model equations (and therefore by (21) of the self-dual Chern-Simons equations) are related to the special Toda-type solutions discussed previously. The key observation is that while the algebraic *Ansätze* (10,15) each lead to a non-Abelian charge density $\rho = [\Psi^\dagger, \Psi]$ which is *diagonal*, the chiral model solutions (21) have charge density $\rho^{(0)} = \frac{\kappa}{2}[\chi^\dagger, \chi]$ which need not be diagonal. However, ρ is always hermitean, and so it can be diagonalized by a gauge transformation. It is still an algebraically nontrivial task to implement this diagonalization, but this is achieved below for the solutions of $SU(N)$ Toda type.

It is instructive to illustrate this procedure with the $SU(2)$ case first. Since $p^2 = p$, the holomorphic projector condition (25) is equivalent to the condition $\partial_+ p\, p = 0$. All such projectors may be written as

$$p = M(M^\dagger M)^{-1}M^\dagger \tag{26}$$

where $M(x^-)$ is any rectangular matrix depending only on the x^- variable. For $SU(2)$ we can only project onto a *line* in $\mathbb{C}^2$, so we take

$$M = \begin{pmatrix} 1 \\ f(x^-) \end{pmatrix}. \tag{27}$$

This then leads to

$$p = \frac{1}{1+f\bar{f}}\begin{pmatrix} 1 & \bar{f} \\ f & f\bar{f} \end{pmatrix}, \qquad \chi = \partial_+ p = \frac{f\partial_+\bar{f}}{(1+f\bar{f})^2}\begin{pmatrix} -1 & 1/f \\ -f & 1 \end{pmatrix}. \tag{28}$$

The corresponding matter density is

$$[\chi^\dagger, \chi] = -\frac{\partial_+\bar{f}\partial_- f}{(1+f\bar{f})^3}\begin{pmatrix} 1-f\bar{f} & 2\bar{f} \\ 2f & -1+f\bar{f} \end{pmatrix}, \tag{29}$$

which may be diagonalized by the unitary matrix

$$g = \frac{1}{\sqrt{1+f\bar{f}}}\begin{pmatrix} -\bar{f} & 1 \\ 1 & f \end{pmatrix},$$

$$g^{-1}[\chi^\dagger, \chi]g = \partial_+\partial_- \log\det(M^\dagger M)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{30}$$

This is precisely Liouville's solution (12) to the classical $SU(2)$ Toda equation.

For the $SU(N)$ chiral model with $N \geq 3$ it becomes increasingly difficult to describe systematically all possible uniton factorizations consistent with the linear relations listed in Uhlenbeck's theorem, but Wood [14] has given an explicit construction and parametrization of all $SU(N)$ solutions in terms of sequences of Grassmannian factors.

Another useful result from the chiral model literature is due to Valli:

Theorem (Valli [12]): *Let h be a solution of the chiral model equation (20). Then the action $\mathcal{E}$ defined in (22) is quantized in integral multiples of 8π.*

As a consequence, the gauge invariant Chern-Simons charge $Q \equiv \int \mathrm{tr}(\Psi^\dagger\Psi)$ is quantized in integral multiples of $2\pi\kappa$. A related quantization condition has been noted in [3], where the *non-Abelian* charges $Q_\alpha \equiv \int \rho_\alpha$ are quantized in integral multiples of $2\pi\kappa$ for the $SU(N)$ Toda-type solutions (13). In this case, $Q = \sum_\alpha Q_\alpha$.

The relationship between the $SU(2)$ uniton solutions and the $SU(2)$ Toda solutions illustrated above (26-30) can be generalized to $SU(N)$ as follows:

Theorem [4]: *The following matrix*

$$h = (-1)^{\frac{1}{2}N(N+1)} \prod_{\alpha=1}^{N-1} (2p_\alpha - 1) \tag{31}$$

where p_α is the hermitean holomorphic projector $p_\alpha = M_\alpha(M_\alpha^\dagger M_\alpha)^{-1}M_\alpha^\dagger$ for the matrix M_α in (13,14), is a solution of the SU(N) chiral model equation (20). Furthermore, defining $\chi = \frac{1}{2}h^{-1}\partial_+ h$, there exists a unitary transformation g which diagonalizes the charge density matrix $[\chi^\dagger, \chi]$ so that

$$g^{-1}[\chi^\dagger, \chi]g = \sum_{\alpha=1}^{N-1} \{\partial_+\partial_- \log\, \det(M_\alpha^\dagger M_\alpha)\} H_\alpha \tag{32}$$

where H_α are the Cartan subalgebra generators of $SU(N)$ in the Chevalley basis. This diagonal form is precisely the $SU(N)$ Toda solution (13).

This theorem is proved [4] by expressing the projectors p_α in terms of an orthonormal basis for the space spanned by the columns of M_N. The diagonalizing matrix g is also constructed from this orthonormal basis.

In conclusion, I mention some open problems suggested by these results.

1. The most important physical problem is now to make use of this complete description of the vacuum of these Chern-Simons-matter theories in order to develop a second quantized theory.
2. The fact that this quantization is possible for the $1 + 1$-dimensional nonlinear Schrödinger equation (NLSE) is intimately related to the integrability of the $1 + 1$-dimensional NLSE. Here, in $2 + 1$-dimensional, the situation is less clear. Is the $2 + 1$-dimensional gauged nonlinear Schrödinger equation (1) with Chern-Simons coupling (2) integrable?
3. Can one find time-dependent (i.e. positive energy) solutions other than those obtained via the action of the dynamical $SO(2,1)$ symmetry acting on the static solutions?
4. The work of Uhlenbeck, Wood and Ward gives a beautiful geometrical picture of the chiral model solutions for the unitary group. What is the *geometrical* interpretation of self-dual Chern-Simons solutions for other Lie groups? Some solutions, in the Toda form, are known, but the geometrical understanding of the corresponding chiral model solutions is not clear. This should be particularly interesting for the self-dual Chern-Simons solutions of the *affine* Toda form.

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FREE FERMION CONSTRUCTIONS OF SUPER VIRASORO AND SUPER KAC-MOODY ALGEBRAS

LOUISE DOLAN
Department of Physics, University of North Carolina
Chapel Hill, North Carolina 27599-3255, USA

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Abstract. Free fermion constructions of the superconformal and Kac-Moody algebras are discussed. Coset representations provide examples for the $N = 1$, $c < \frac{3}{2}$ discrete series. They generalize the Kac-Todorov construction of the supercurrent which was valid for $N = 1$, $c \geq \frac{3}{2}$, and differ by the terms mixing the SuperVirasoro and Kac-Moody algebras. They thus provide a guide for searching for new forms for the lower and upper components of the superfields in one-to-one correspondence with the untwisted states in a twisted superconformal field theory, and may be useful in discussing the low energy phenomenology of superstring theory.

1. Introduction

Conformal, superconformal and extended superconformal algebras play a role in string theory. In this paper we investigate constructions of the supercurrent generator $F(z)$ of the two $N = 1$ world sheet supersymmetric extensions of the Virasoro algebra, i.e., the Ramond and Neveu-Schwarz sectors. The critical dimension of this system (determined by the absence of negative norm ghost states) is $D = 10$. Unitary representations of the $N = 1$ superconformal algebra with critical central charge $c = 15$ are constructed from the matter superfields. The superconformal BRST ghost system provides a non-unitary representation of the $N = 1$ superVirasoro algebra (SVA) with $c = -15$. Although the representation is non-unitary, the SVA generators still satisfy the hermiticity conditions $L_n^{\text{ghost}\dagger} = L_{-n}^{\text{ghost}}$, $F_n^{\text{ghost}\dagger} = F_{-n}^{\text{ghost}}$. In addition, the superconformal ghost system also carries a representation of the extended $N = 2$ world sheet algebra.[1]

The matter superconformal fields of conformal weight one-half close to form a super Kac-Moody algebra (SKMA). The mixing between the SVA and the SKMA differs depending on the particular construction of the supercurrent. The $N = 1$ algebra is given by operator products where the right hand side holds for $|z| > |\zeta|$ up to terms regular as $z \to \zeta$.

$$L(z)L(\zeta) = \frac{\frac{c}{2}}{(z-\zeta)^4} + \frac{2L(\zeta)}{(z-\zeta)^2} + \frac{\frac{dL(\zeta)}{d\zeta}}{(z-\zeta)}$$

$$L(z)F(\zeta) = \frac{\frac{3}{2}F(\zeta)}{(z-\zeta)^2} + \frac{\frac{dF(\zeta)}{d\zeta}}{(z-\zeta)}$$

$$F(z)F(\zeta) = \frac{\frac{2c}{3}}{(z-\zeta)^3} + \frac{2L(\zeta)}{(z-\zeta)} \tag{1}$$

In component form we have

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m}$$

$$[L_n, F_m] = (\frac{n}{2} - m)F_{n+m}$$

$$[F_n, F_m] = 2L_{n+m} + \frac{c}{3}(n^2 - \frac{1}{4})\delta_{n,-m} \tag{2}$$

The super Kac-Moody algebra is

$$T^a(z)T^b(z) = \frac{k\delta_{ab}}{(z-\zeta)^2} + \frac{if_{abc}T^c(\zeta)}{(z-\zeta)}$$

$$T^a(z)d^b(z) = \frac{if_{abc}T^c(\zeta)}{(z-\zeta)}$$

$$d^a(z)d^b(z) = \frac{\delta_{ab}}{(z-\zeta)}. \tag{3}$$

Here $f_{abc}f_{abe} = c_\psi \delta_{ce}$; the level of the KMA is $x = \frac{2k}{\psi^2} = \frac{2k}{c_\psi}\tilde{h}$, where $\tilde{h}$ is the dual Coxeter number of the compact Lie algebra with structure constants f_{abc}.

Constructions of the matter supercurrent are given by the following.

1)The Kac-Todorov construction extends to a super Kac-Moody algebra and has a mixing between the SVA and SKMA which reflects the fact that the SKMA generators are conformal weight one-half superfields. The Virasoro generators form a Sugawara construction and $\frac{3}{2} \leq c < \frac{3dimg}{2}$.

2)The coset constructions have SVA generators which are seen to be a modification of the Kac-Todorov construction. This constructions also extends to a SKMA, but now the mixing between the SVA and the SKMA is different from the Kac-Todorov case. The central charge satisfies $0 \leq c < \frac{dimg}{2}$.

3)Complex free fermions provide a construction similar to Kac-Todorov, but now the supercurrent can carry automorphisms of groups other than $SU(2)^6$ in the presence of massless fermions. Here $c = \frac{dimg}{2}$.

2. The Kac-Todorov construction

This construction provides the general free real fermion representations of the internal space SVA and SKMA algebras with the following mixing characteristic of a weight one-half superfield.

$$
\begin{aligned}
L(z)T^a(\zeta) &= \frac{T^a(\zeta)}{(z-\zeta)^2} + \frac{\frac{dT^a(\zeta)}{d\zeta}}{(z-\zeta)} \\
L(z)d^a(\zeta) &= \frac{\frac{1}{2}d^a(\zeta)}{(z-\zeta)^2} + \frac{\frac{dd^a(\zeta)}{d\zeta}}{(z-\zeta)} \\
F(z)T^a(\zeta) &= \sqrt{k}[\frac{d^a(\zeta)}{(z-\zeta)^2} + \frac{\frac{dd^a(\zeta)}{d\zeta}}{(z-\zeta)}] \\
F(z)d^a(\zeta) &= \frac{1}{\sqrt{k}}\frac{T^a(\zeta)}{(z-\zeta)}.
\end{aligned} \tag{4}
$$

A realization is given by

$$
\begin{aligned}
\check{L}(z) &= \frac{1}{c_\psi}({}^x_x\check{T}^a(z)\check{T}^a(z){}^x_x) = \frac{1}{2} : \frac{dd^a(z)}{dz}d^a(z) : + \frac{\epsilon}{16z^2} \\
\check{T}^a(z) &= \frac{-i}{2}f_{abc}d^b(z)d^c(z) \\
\check{F}(z) &= \frac{1}{3\sqrt{\frac{c_\psi}{2}}}d^a(z)\check{T}^a(z) = \frac{-i}{6\sqrt{\frac{c_\psi}{2}}}f_{abc}d^a(z)d^b(z)d^c(z).
\end{aligned} \tag{5}
$$

Here $1 \leq a \leq \epsilon$. This representation has level $\check{x} = \frac{2k}{c_\psi}\tilde{h} = \tilde{h}$ and $\check{c} = \frac{dim g}{2}$, *i.e.* $\frac{3}{2} \leq \check{c}$. The most general realization is given by

$$
\begin{aligned}
L(z) &= \check{L}(z) + \frac{1}{2k^q + c_\psi}({}^x_x q^a(z)q^a(z){}^x_x) = \check{L}(z) + L^q_{Sug}(z) \\
T^a(z) &= \check{T}^a(z) + q^a(z) \\
F(z) &= \sqrt{\frac{\frac{c_\psi}{2}}{k^q + \frac{c_\psi}{2}}}\check{F}(z) + \frac{1}{\sqrt{k^q + \frac{c_\psi}{2}}}d^a(z)q^a(z)
\end{aligned} \tag{6}
$$

with level $x = \tilde{h} + x^q$ and $c = \frac{dim g}{2} + \frac{x^q dim g}{x^q + \tilde{h}} = \frac{3\dim g}{2} - \frac{\tilde{h} dim g}{x}$, *i.e.* $\frac{3}{2} \leq \check{c} \leq c < \frac{3 dim g}{2}$. The abelian SKMA is

$$
L(z) = \frac{1}{2} : \frac{dd^a(z)}{dz}d^a(z) : + \frac{\epsilon}{16z^2} + \frac{1}{2k^q}({}^x_x q^a(z)q^a(z){}^x_x)
$$

$$T^a(z) = q^a(z) = \frac{-i}{2} f_{aIJ} b^I(z) b^J(z)$$

$$F(z) = \frac{1}{\sqrt{k^q}} d^a(z) q^a(z), \tag{7}$$

Here $f_{aIJ} f_{bIJ} = 2k^q \delta_{ab}$ and $c = \frac{1}{2}(\mathrm{dim} d^a + \mathrm{dim} b^I) \geq \frac{3}{2}$.

3. Coset construction

We now modify the Kac-Todorov construction of the supercurrent to be of the general form

$$F(z) = A\check{F}(z) + B d^a(z) q^a(z). \tag{8}$$

It follows that

$$L(z) = C\check{L}(z) + D L^q_{Sug}(z) + E\check{T}^a(z) q^a(z)$$

$$T^a(z) = \check{T}^a(z) + q^a(z). \tag{9}$$

Case 1: for $E = 0$, we regain the Kac-Todorov forms: a) minimal $B = D = 0, A = C = 1$ and b) maximal $B = \frac{1}{\sqrt{\frac{c_\psi}{2}}} A = \frac{1}{\sqrt{k^q + \frac{c_\psi}{2}}}$.

Case 2: for $E = \frac{-2\tilde{h}}{c_\psi(x^q + 2\tilde{h})}$; $C = \frac{x^q}{x^q + 2\tilde{h}}$; $D = \frac{\tilde{h}}{x^q + 2\tilde{h}}$; where $A = \frac{-x^q}{\sqrt{(x^q + \tilde{h})(x^q + 2\tilde{h})}}$; $B = \frac{\tilde{h}\sqrt{\frac{2}{c_\psi}}}{\sqrt{(x^q + \tilde{h})(x^q + 2\tilde{h})}}$, the mixing between the SVA and SKMA is given by

$$L(z) T^a(\zeta) = 0$$

$$L(z) d^a(\zeta) =$$

$$= \left(\frac{x^q}{x^q + 2\tilde{h}}\right)\left(\frac{d^a(\zeta)}{2(z-\zeta)^2} + \frac{\frac{dd^a(\zeta)}{d\zeta}}{(z-\zeta)}\right) + \left(\frac{-2\tilde{h}}{(x^q + 2\tilde{h}) c_\psi}\right) \frac{i f_{abc} q^b(z) d^c(z)}{(z-\zeta)}$$

$$F(z) T^a(\zeta) = 0$$

$$F(z) d^a(\zeta) = \frac{1}{(z-\zeta)}\left[\frac{\sqrt{\frac{2}{c_\psi}}}{\sqrt{(x^q + \tilde{h})(x^q + 2\tilde{h})}}\right]\left[-x^q T^a(\zeta) + \tilde{h} q^a(\zeta)\right]. \tag{10}$$

This representation has level $x = \tilde{h} + x^q$ and $c = \frac{dimg}{2}\left(1 - \frac{2\tilde{h}^2}{(x^q + \tilde{h})(x^q + 2\tilde{h})}\right)$, *i.e.* $0 \leq c < \frac{dimg}{2}$. For $g = SU(2)$ (so $\tilde{h} = 2$), *we see this is just the discrete series for unitary representations of the* $N = 1$ SVA (let $x^q \equiv m$):

$$c = \frac{3}{2}\left(1 - \frac{8}{(m+2)(m+4)}\right) = 0, \frac{7}{10}, 1, \ldots, \frac{3}{2}. \tag{11}$$

Case 2 is seen to be equivalent to the coset construction[3],

$$L(z) = L^G(z) - L^H(z) = \check{L}(z) + L^q_{Sug}(z) -$$

$$-\frac{1}{2k^q + 2c_\psi}[{}^x_x(\check{T}^a(z) + q^a(z))(\check{T}^a(z) + q^a(z)){}^x_x]T^a(z) =$$

$$= \check{T}^a(z) + q^a(z) \tag{12}$$

$$F(z) = \frac{-x^q}{\sqrt{(x^q + \tilde{h})(x^q + 2\tilde{h})}}\check{F}(z) + \frac{\tilde{h}\sqrt{\frac{2}{c_\psi}}}{\sqrt{(x^q + \tilde{h})(x^q + 2\tilde{h})}}d^a(z)q^a(z)$$

The coset here corresponds to $G = SU(2) \otimes SU(2)$ and $H = SU(2)$.

4. Complex fermions

For complex fermions satisfying twisted boundary conditions $f^a(e^{2\pi i}z) = e^{2\pi i\nu}f^a(z)$, $\tilde{f}^a(e^{2\pi i}z) = e^{-2\pi i\nu}\tilde{f}^a(z)$, the supercurrent construction generalizes the Kac-Todorov expression to be given by

$$F(z) = \frac{-i}{6\sqrt{\frac{c_\psi}{2}}}f_{abc}h^a(z)h^b(z)h^c(z). \tag{13}$$

The space-time fermi fields satisfy the periodicity condition $h^\mu(e^{2\pi i}z) = \delta_\alpha h^\mu(z)$ where $\delta_\alpha = \mp 1$ for R an NS fields respectively, so $F(e^{2\pi i}z) = \delta_\alpha F(z)$. In a given sector, all the fermionic boundary conditions can be specified by a matrix ω^a_b: so $h^a(e^{2\pi i}z) = \omega^a_b h^b(z)$ and

$$f_{def}\omega^d_a\omega^e_b\omega^f_c = \delta_\alpha f_{abc}, \tag{14}$$

i.e. $\mp\omega^a_b$ is an automorphism [4] of the Lie algebra g with structure constants f_{abc} used to define the supercurrent in a Ramond (Neveu-Schwarz) sector. The Virasoro generator is then given by

$$L(z) = \frac{1}{2} : \frac{d\tilde{f}(z)}{dz}f(z) : + \frac{1}{2} : \frac{df(z)}{dz}\tilde{f}(z) : + \frac{1}{16z^2}\mathrm{tr}([\frac{1}{i\pi}\log(-\omega)]^2) \tag{15}$$

where we are ultimately interested in the automorphisms ω of g for which the coboundary term $\frac{1}{16}\mathrm{tr}([\frac{1}{i\pi}\log(-\omega)]^2) = \frac{dim g}{16\cdot 3}$, *i.e.* the automorphisms for which the coboundary term takes its minimum value on the Ramond sector. In D space-time dimensions, the mass operator is

$$m^2 = -\frac{1}{2} + \frac{D-2}{16} + \frac{1}{16}\mathrm{tr}([\frac{1}{i\pi}\log(-\omega)]^2) \tag{16}$$

and the structure constants require $dimg = 3(10 - D)$. So for massless space-time fermions, $m^2 = 0$, the coboundary term realizes its minimum value, and in $D = 4$, *the dimension of g is 18, where g is the algebra of the structure constants occuring in the supercurrent (13).* This mass formula is to be contrasted with that of the Kac-Todorov construction where massless fermions require

$$m^2 = -\frac{1}{2} + \frac{D-2}{16} + \dim d^a = 0, \tag{17}$$

so for $D = 4$, *g is $U(1)^6$*, which is the gauge group of the states in string models using the Kac-Todorov supercurrent construction necessary for sectors with massless Ramond states.

In order to examine which gauge groups occur as the relevant internal gauge symmetry of the spectrum of states for the complex fermion form of the supercurrent given in (13), we investigate the SKMA (which mixes as a weight one-half superfield) with this representation of the internal superVirasoro algebra, $c = 9$. The modified Cartan Weyl basis for the SKMA diagonalizes inner automorphisms. The fermions in this basis are $h^i(z), h^\alpha(z)$, for $1 \leq i \leq \text{rank(g)}$ and $\alpha \in \text{roots(g)}$, thus h^i are real (R, NS) and $h^{\alpha *} = h^{-\alpha}$ are complex. The SKMA generators now form a twisted SKMA where $H^i(e^{2\pi i}z) = H^i(z)$ and the step operators $E^\alpha(z) = e^{2\pi i\alpha\cdot\lambda}E^\alpha(e^{-2\pi i}z)$. So for *inner* automorphisms, the zero mode subalgebra which is the gauge symmetry of the spectrum is $U(1)^{rank(g)}$. For outer automorphisms, one can check for the relevant groups $SU(3)$, $SU(4)$ and $SO(5)$ that the zero mode subalgebra is again a product of $U(1)$ factors.

5. Non-free fermion representations of the supercurrent

Not all known representations of the superconformal algebra can be expressed as free fermion constructions. In particular, the Waterson boson[5,6] provides a representation for $N = 2$, $c = 1$. The $N = 1$ subalgebra is generated by

$$L(z) = \frac{1}{2} : a(z) \cdot a(z) :$$

$$F(z) = \frac{1}{\sqrt{2}}(: e^{i\sqrt{3}X(z)} : + : e^{-i\sqrt{3}X(z)} :). \tag{18}$$

6. BRST superconformal ghost system

Non-unitary representations of the superVirasoro algebra are provided by the BRST superconformal ghost system[1,7]. The ghost superfields are $B(z) = \beta(z) + \theta b(z)$ and $C(z) = c(z) + \theta\gamma(z)$ with conformal spin $h_\beta = \frac{3}{2}$, $h_c = -1$, etc. The commutation relations on the Ramond sector are $\{b_n, c_m\} = \delta_{n,-m}$,

$[\beta_n, \gamma_m] = -\delta_{n,-m}$. The superVirasoro representation has $c = -15$, but the generators still satisfy the hermitian property $F_n^\dagger = F_{-n}$ and $L_n^\dagger = L_{-n}$:

$$L(z) = -2 : b(z)\frac{dc(z)}{dz} : - : \frac{db(z)}{dz}c(z) : -\frac{3}{2} : \beta(z)\frac{d\gamma(z)}{dz} : -$$

$$-\frac{1}{2} : \frac{d\beta(z)}{dz}\gamma(z) : -\frac{1}{2z^2}$$

$$F(z) =: b(z)\gamma(z) : -3 : \beta(z)\frac{dc(z)}{dz} : -2 : \frac{d\beta(z)}{dz}c(z) : \tag{19}$$

An alternative form for the supercurrent is given by Schwarz[8]:

$$F(z) = -2 : b(z)\gamma(z) : +\frac{3}{2} : \beta(z)\frac{dc(z)}{dz} : + : \frac{d\beta(z)}{dz}c(z) : \tag{20}$$

The ghost number current forms an abelian SKMA:

$$T(z) = - : b(z)c(z) : - : \beta(z)\gamma(z) : +\frac{1}{2z} \tag{21}$$

The supercurrent in (19) can be identified as $F^+ + F^-$ and can be used to construct a second $h = \frac{3}{2}$ supercurrent $-F^+ + F^-$ as the upper component of the superfield whose lower component is

$$H(z) = 2 : b(z)c(z) : +3 : \beta(z)\gamma(z) : \tag{22}$$

We find

$$-F^+(z) + F^-(z) =: b(z)\gamma(z) : +3 : \beta(z)\frac{dc(z)}{dz} : +2 : \frac{d\beta(z)}{dz}c(z) : \tag{23}$$

The set L, G^+, G^-, H form an $N = 2$ superconformal algebra with $c = -15$.

We include here for completeness, the unitarity restrictions on the central for the $N = 0, 1, 2$ superVirasoro algebras[9,10]. For $N = 0$, unitary representations occur for all values of $c \geq 1, h \geq 0$ and for discrete values below 1 given by $c = 1 - \frac{6}{(m+2)(m+3)} = 0, \frac{1}{2}, \frac{7}{10}, \frac{4}{5}, \ldots, 1$. The critical value of the central charge is $c = 26$. The $N = 1$ system provides representations of two supersymmetric extensions of the Virasoro algebra, i.e. the Ramond and the Neveu-Schwarz. The critical dimension is $D = 10$. The only possible unitary highest weight representations, i.e. representations generated from a state $|h\rangle$, satisfy $L_n|h\rangle = 0, n \geq 0$; $L_0|h\rangle = h|h\rangle; F_n|h\rangle = 0, n \geq 0$; are characterized by (c, h) where either $c \geq \frac{3}{2}$, $h \geq 0$; or for the discrete values $0 \leq c < \frac{3}{2}$ given by $c = \frac{3}{2}[1 - \frac{8}{(m+2)(m+4)}] = 0, \frac{7}{10}, 1, \ldots, \frac{3}{2}$. The critical value of the central charge is $c = 15$. For $N = 2$, the critical dimension is $D = 2$ complex or $D = 4$ real. Unitary representations occur for all values of $c \geq 3$ and for discrete values below 3 given by $c = \frac{3m}{m+2} = 0, 1, \ldots, 3$. The critical central charge is $c = 6$.

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QUANTIZED DE SITTER GAUGE THEORY WITH CLASSICAL METRIC AND AXIAL TORSION

WOLFGANG DRECHSLER
Max-Planck-Institut für Physik, Werner-Heisenberg-Institut
P.O.Box 40 12 12, 80712 Munich, Germany

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Abstract. Geometro-stochastically quantized fields are introduced as sections on a first quantized Hilbert bundle, $\mathcal{H}$, over Riemann-Cartan space-time with axial vector torsion representing quantized elementary matter in a gauge theory based on the (4, 1)-de Sitter group. $\mathcal{H}$ is a soldered bundle with built-in fundamental length parameter R typical for hadron physics carrying a spin zero phase space representation of $G = SO(4,1)$ belonging to the principal series of unitary irreducible representations. In a nonlinear realization of G the Lorentz subgroup may be related to a gauge formulation of gravitation. Bilinear currents are introduced through G-invariant integration over the local fibers in $\mathcal{H}$, and covariant field equations are set up for the quantum fiber dynamics (QFD) describing the coupling of quantized material sources to the underlying bundle geometry in the presence of gravitation.

1. Introduction

It is well known that Einstein's metric theory of gravitation may be formulated as a Lorentz gauge theory by reducing the original linear frame bundle $P'(B, G' = GL(4,R))$ over the space-time base B, in the presence of a pseudo-Riemannian metric $g_{\mu\nu}(x)$ with Lorentz signature, to the Lorentz frame bundle $P_L(B, H = SO(3,1))$ over $B = V_4$. Also the connection on P' reduces to a connection on P_L provided $g_{\mu\nu}(x)$ is covariant constant, i.e. satisfies $\bar{\nabla}_\rho g_{\mu\nu}(x) = 0$, which just defines the Levi-Civita connection denoted by $\bar{\Gamma}^\rho_{\mu\nu} = \left\{ {\rho \atop \mu\nu} \right\}$. [Purely metric quantities will be denoted by a bar in the following]. The metric $g_{\mu\nu}(x)$ may thus be regarded as a parallel section on a bundle over space-time with ten-dimensional homogeneous fiber $G'/H = GL(4,R)/SO(3,1)$.

The pull back of a connection on P_L with respect to a local section defining a gauge will be called $\bar{\omega}(x)$ which is a Lorentz Lie algebra-valued matrix of one-forms, $\bar{\omega}_{ij}(x) = -\bar{\omega}_{ji}(x)$; $i,j = 0,1,2,3$, $x \in V_4$, with $\bar{\omega}_{ij}(x) = \theta^k \bar{\Gamma}_{kij}(x)$, where $\theta^k = \lambda^k_\mu(x) dx^\mu$ are the fundamental one-forms on the base V_4 of P_L [providing an orthonormal basis for the dual tangent space $T^*_x(V_4)$ at x], and $\bar{\Gamma}_{kij}(x)$ are the Ricci rotation coefficients. A local orthonormal basis of the tangent space $T_x(V_4)$ at x, which is provided by a local section on P_L, will be denoted by e_i; $i = 0,1,2,3$; with $e_i = \lambda_i^\mu(x)\partial_\mu$, where $\lambda_i^\mu(x)$ are the vierbein fields and $\lambda_\mu^k(x)$ their inverse, obeying

$$\lambda_\mu^k(x)\lambda_j^\mu(x) = \delta_j^k \ ; \qquad g_{\mu\nu}(x) = \lambda_\mu^i(x)\lambda_\nu^k(x)\eta_{ik} \ . \tag{1.1}$$

$\eta_{ik} = diag(1,-1,-1,-1)$ is the Minkowski metric, and local Lorentzian indices will be written with Latin letters $(i,k,j\ldots)$, while Greek indices $(\mu,\nu,\rho\ldots)$ refer to a natural basis ∂_μ in $T_x(V_4)$ and dx^μ in $T_x^*(V_4)$, respectively. [For repeated covariant and contravariant Greek and Latin indices the summation convention applies. Greek indices are lowered with $g_{\mu\nu}(x)$ and raised with its inverse $g^{\mu\nu}(x)$; Latin indices are raised and lowered with η^{ik} and η_{ik}, respectively.]

The problem of the theoretical description of atomic, nuclear or subnuclear particles in the presence of gravitational fields raises the question of how to extend classical general relativity – relating the geometry of space-time to the distribution of energy and momentum of *classical macroscopic matter* – to the domain of quantum physics obeying the laws of quantum mechanics for the description of matter at the atomic and "elementary" particle level and requiring a treatment in terms of wave functions and field operators. We are aiming here at a unified geometric formulation of gravitational *and* subnuclear hadronic forces in the presence of classical *as well as* quantized material sources. [The electroweak interaction will, for simplicity, be disregarded in the following discussion. It may be included by enlarging the principal bundles introduced below by an additional $U(1)\otimes SU(2)$ fiber.]

Although gravitational effects are negligibly small in particle physics the structure of Einstein's metric theory of gravitation is so unique in its dualism between the metric of the ambient space and the distribution of matter therein that it may legitimately be asked whether a similar dualism may also be invoked for the theoretical description of interections in the subnuclear world, i.e. being relevant for hadrons at distances of the order of a Fermi or below ($\sim 10^{-13} - 10^{-15}$ cm). With this aim in mind we shall investigate here a model based on a higher dimensional bundle raised over space-time characterized by a structural group G which is bigger than the Lorentz group but, in fact, contains the Lorentz group $H = SO(3,1)$ as a closed subgroup with $SO(3,1) \equiv O(3,1)^{++}$ (proper isochronous Lorentz group) being related to a gauge formulation of gravitation [1]. However, there will appear further contributions at the level of the connection $\tilde{\omega}(x)$ on P_L introduced above when one considers the induced Lorentz gauge degrees of freedom appearing in this enlarged bundle formalism; i.e. there may appear torsion or Weyl degrees of freedom related to the additional motions present or possible in the internal spaces [the local fibers] on which the group G acts and which may come into play at small (subnuclear) distances in the space-time base.

We shall base the following discussion on the (4,1)-de Sitter group, $G = SO(4,1)$, as the bigger gauge or structural group containing the Lorentz group as a gauged subgroup, and introduce the de Sitter frame bundle, $P(B = U_4, G = SO(4,1))$, over a Riemann-Cartan space-time U_4 as a geometric arena for the unification of general relativity describing classical (macrophysical) gravitation, and strong subnuclear interactions modifying Einstein's theory at small distances due to the presence of quantized elementary (microphysical) sources. We shall disregard Weyl degrees of freedom in the following and shall specialize later to axial vector torsion (i.e. to a completely antisymmetric torsion tensor). Compare Ref. [2] for a Weyl rescaling of the metric in the fiber in the de Sitter gauge theory. It will be seen in this context that Einstein's metric of general relativity *remains* a classical field describing macroscopic gravitation despite the presence of quantized elementary sources in the geometry. Gravitation need thus not be quantized in this unified theory.

Nonlinear field equations for the additional nonmetric geometric fields are set up establishing a further feed back mechanism between matter (i.e. elementary hadronic matter described in a quantum mechanical manner) and the underlying bundle geometry raised over space-time. These additional source equations have the consequence that despite the presence of quantized matter – represented in the form of generalized wave functions (sections on a Hilbert bundle $\mathcal{H}$) transforming under an irreducible phase space representation of $SO(4,1)$ – which induce in the bundle geometry the additional geometric fields through certain bilinear currents, the metric continues to play a classical rôle as the potential for a classical part of the connection on $P(U_4, SO(4,1))$ [more exactly, its Lorentz part in the so-called nonlinear gauge (see below)]. For details see Ref. [3]. After introducing in the next section the Hilbert bundle $\mathcal{H}$ over space-time and discussing the generalized wave functions representing quantized spinless matter in the theory, we investigate various generalized gauge currents as source currents for the geometry and discuss, finally, two sets of covariant nonlinear field equations (current-curvature and Einstein-type equations) for a gauge dynamics on $\mathcal{H}$ which we call quantum fiber dynamics (QFD).

2. Representation of Quantized Matter

In order to describe quantized elementary matter in the presence of gravitational fields generated by distant macroscopic classical masses one introduces a Hilbert bundle $\mathcal{H}$ over a Riemann-Cartan space-time U_4 carrying a system of covariance of the (4,1)-de Sitter group [4]. $\mathcal{H}$ is a " first quantized" bundle (in the terminology used in [4]), which is associated to $P(U_4, G = SO(4,1))$, possessing a standard fiber, $\mathcal{H}^{(\rho)}_{\tilde{\eta}}$, being a resolution kernel Hilbert space with resolution generator $\tilde{\eta}$ and generalized coherent state basis. The bundle $\mathcal{H}$ carries a spin zero *phase space representation* of $SO(4,1)$ belonging to the principal series of UIR's (unitary irreducible representation) determined by the parameter ρ. The quantum bundle $\mathcal{H}$ with local fiber $\mathcal{H}_{\tilde{\eta}}^{(\rho)}(x)$ provides a geometric arena for the propagation of the de Sitter quantum fields $\Psi_x^{(\rho)}(\xi,\zeta)$, as described below. Moreover, $\mathcal{H}$ possesses a built-in fundamental length parameter R of geometric origin chosen, as mentioned, to be of the order of 10^{-13} cm typical for hadron physics [4,1]. We call $\Psi_x^{(\rho)}(\xi,\zeta)$ a *generalized quantum mechanical wave function* of de Sitter type which is square-integrable, for any $x \in U_4$, with respect to a G-invariant measure $d\tilde{\Sigma}(\xi,\zeta)$ [see below] in the local de Sitter phase space variables $(\xi,\zeta) \in \tilde{\Sigma}_x^{\pm} \subset \mathcal{N}_x^{\pm}$, where

$$\mathcal{N}^{\pm} = V_4' \times \mathcal{C}^{\pm} \tag{2.1}$$

with $[\eta_{ab} = diag(1,-1,-1,-1,-1)]$:

$$V_4' : \; [\xi,\xi] = \xi^a \xi^b \eta_{ab} = -R^2 \; ,$$

$$\mathcal{C}^{\pm} : \; [\zeta,\zeta] = \zeta^a \zeta^b \eta_{ab} = 0; \;\; \zeta^5 = \frac{1}{R} \; . \tag{2.2}$$

The summations in (2.2) run over $a,b = 0,1,2,3,5$. Here $\mathcal{N}^{\pm}$ denotes the de Sitter phase space: $V_4' \simeq G/H = SO(4,1)/SO(3,1)$ is (4,1)-de Sitter space [a single-shell hyperboloid of radius R in a Lorentzian embedding space $R_{4,1}$], and $\mathcal{C}^{\pm}$ is the intersection of the light cone in $R_{4,1}$ with the surface $\zeta^5 = \frac{1}{R}$. The superscript of $\mathcal{C}^{\pm}$ stands for $sign\zeta^0 = \pm$ with the vector $\zeta^a = (\zeta^i, \zeta^5 = \frac{1}{R}); i = 0,1,2,3$, characterizing a so-called horosphere or horocycle [5] through the origin $\xi^o = (0,0,0,0,-R)$ of V_4'. $\zeta \in \mathcal{C}^{\pm}$ plays the rôle of the wave vector or momentum variable for a wave phenomenon in de Sitter space (a space of constant curvature with curvature radius R). $\tilde{\Sigma}^{\pm} = H \times \mathcal{C}^{\pm}$ denotes a six-dimensional horospherical submanifold of $\mathcal{N}^{\pm}$ composed of a horosphere H (a space-like hypersurface) in V_4' and the cone $\mathcal{C}^{\pm}$.

For later use we, furthermore, introduce the de Sitter phase space bundle over space-time U_4,

$$\tilde{E} = \tilde{E}(U_4, \tilde{F} = V_4' \times \mathcal{C}^{\pm}, G = SO(4,1)) \tag{2.3}$$

which is a soldered bundle associated to P. [The soldering is performed here through the local subspace $V_4'(x)$ of $\mathcal{N}_x^{\pm}$ being tangent to the space-time base U_4 for each x [6,7].] Moreover, we introduce the de Sitter bundle over U_4,

$$E = E(U_4, F = V_4' \simeq G/H, G = SO(4,1)) \tag{2.4}$$

which is a soldered bundle associated to P with "curled up" four-dimensional fiber of definite (fixed) radius R which is isomorphic to the noncompact coset space $G/H = SO(4,1)/SO(3,1)$ [7].

We now construct a *phase space representation* of the de Sitter group for spinless particles, denoted by $\tilde{U}(A_g) = \tilde{U}^{(\rho)}(A_g); A_g \in SO(4,1)$, which is related to the spin zero UIR of $SO(4,1)$ of the principal series characterized by the parameter ρ, $0 \leq \rho < \infty$. The value of ρ determines the mass of the particle in question. [The eigenvalue of the Laplace-Beltrami operator, $\Box_\xi$, on V_4' has eigenvalues $\kappa(\kappa+3)/R^2$ with $\kappa = -\frac{3}{2} + i\rho$; $0 \leq \rho < \infty$, leading to the following relation between ρ, the radius R of de Sitter space and the mass m of the particle: $[\frac{mc}{\hbar}]^2 R^2 = \rho^2 + \frac{1}{4}$ (compare Ref. [8]).]

$\mathcal{H}_{\tilde{\eta}}^{(\rho)}$ is the Hilbert space $L^2(\tilde{\Sigma}^\pm)$ of square-integrable functions in the variables $(\xi,\zeta) \in \tilde{\Sigma}^\pm \subset \mathcal{N}^\pm$ with respect to the G-invariant measure [4]

$$d\tilde{\Sigma}(\xi,\zeta) = \frac{1}{R^2}\frac{1}{[\xi,\zeta]^2}\delta(|[\xi,\zeta]| - c)d\mu(\xi)\delta([\zeta,\zeta])d^4\zeta \ , \tag{2.5}$$

where $d\mu(\xi) = \frac{R}{|\xi^5|}d\xi^0 d\xi^1 d\xi^2 d\xi^3$ is the invariant measure on V_4'. $d\tilde{\Sigma}(A_g\xi, A_g\zeta) = d\tilde{\Sigma}(\xi,\zeta)$. In (2.5) c is a positive constant determining a particular horosphere H_ζ^c in V_4' characterized by ζ being parallel to a horosphere H_ζ^1 through the origin ξ^o characterized by the same vector ζ. One can construct a coherent state basis of $\mathcal{H}_{\tilde{\eta}}^{(\rho)}$ in terms of horospherical waves [8,4] (which are analogous to plane waves in flat space) from $SO(3)$-invariant resolution generators $\tilde{\eta}(\zeta')$ yielding a parametrization of the basis of $\mathcal{H}_{\tilde{\eta}}^{(\rho)}$ in terms of the coset space $SO(4,1)/SO(3)$. $\mathcal{H}_{\tilde{\eta}}^{(\rho)}$ is a single-particle resolution kernel Hilbert space with decomposition $\mathcal{H}_{\tilde{\eta}}^{(\rho)} = \mathcal{H}^+ \bigoplus \mathcal{H}^-$, where the superscripts $+$ and $-$ stand for the sign of ζ^0 with $\mathcal{H}^+$ and $\mathcal{H}^-$ denoting the one-particle and one-antiparticle Hilbert spaces, respectively. For the discussion of second quantized Hilbert bundles with Fock space fibers constructed in terms of tensor products of the spaces $\mathcal{H}^+$ and $\mathcal{H}^-$ compare [4] and also the recent book by Prugovečki [9] Here we shall confine the discussion to the spaces $\mathcal{H}^\pm$ and the resulting first quantized Hilbert bundle.

Having introduced the Hilbert space $\mathcal{H}_{\tilde{\eta}}^{(\rho)}$ we now consider as the geometric arena for the description of spinless quantized matter the following soldered (first quantized) Hilbert bundle over Riemann-Cartan space-time with standard fiber $\mathcal{H}_{\tilde{\eta}}^{(\rho)}$ and structural group provided by the unitary irreducible phase space representation $\tilde{U}(A_g)$:

$$\mathcal{H} = \mathcal{H}(U_4, \mathcal{F} = \mathcal{H}_{\tilde{\eta}}^{(\rho)}, \tilde{U}(A_g)). \tag{2.6}$$

$\mathcal{H}$ is associated to $P(U_4, SO(4,1))$ and carries a system of covariance of the $(4,1)$-de Sitter group. The variables (ξ,ζ) in each local fiber $\mathcal{N}_x^\pm$ of $\tilde{E}$

play the rôle of local geometro-stochastic variables on $\mathcal{H}$ determining the quantum-kinematical localization properties of spinless quantum particles (possessing internal de Sitter gauge degrees of freedom) on curved space-time [4]. Denoting now the generalized coherent state basis in the local fibers $\mathcal{H}_{\tilde{\eta}}^{(\rho)}(x)$ of $\mathcal{H}$ which is adapted to a particular choice of gauge $\sigma = \mathbf{u}(x)$ on P by $\Phi_{\xi,\zeta}^{\mathbf{u}(x)}$, where $x \in U_4$ and $(\xi,\zeta) \in \tilde{\Sigma}_x^{\pm}$, one obtains the following resolution of unity at $x \in U_4$:

$$\int_{\tilde{\Sigma}_x^{\pm}} |\Phi_{\xi,\zeta}^{\mathbf{u}(x)}\rangle d\tilde{\Sigma}(\xi,\zeta)\langle\Phi_{\xi,\zeta}^{\mathbf{u}(x)}| = \mathbf{1}_x^{\pm} \ . \tag{2.7}$$

Here $d\tilde{\Sigma}(\xi,\zeta)$ is the measure (2.5) on the local (horospherical) hypersurface $\tilde{\Sigma}_x^{\pm}$ in $\mathcal{N}_x^{\pm}$. Any state vector $\Psi_x^{(\rho)\pm}$ belonging to the principal series of UIR's of $SO(4,1)$ with zero spin may be expanded with respect to the local quantum frame basis, $\Phi_{\xi,\zeta}^{\mathbf{u}(x)}$ according to:

$$\Psi_x^{(\rho)\pm} = \int_{\tilde{\Sigma}_x^{\pm}} d\tilde{\Sigma}(\xi,\zeta)\Psi_x^{(\rho)}(\xi,\zeta)\Phi_{\xi,\zeta}^{\mathbf{u}(x)} \ . \tag{2.8}$$

The coefficient $\Psi_x^{(\rho)}(\xi,\zeta)$ in the expansion (2.8) is the *scalar de Sitter coordinate wave function*, called for short the *generalized wave function*, which may be regarded as a section on the first quantized bundle $\mathcal{H}$ and represents first quantized matter in the theory.

One can adopt a convenient bracket notation for the G-invariant integration with measure (2.5) over the local hypersurface $\tilde{\Sigma}_x^{\pm}$ and solve (2.8) for $\Psi_x^{(\rho)}(\xi,\zeta)$ yielding

$$\Psi_x^{(\rho)}(\xi,\zeta) = \langle\Phi_{\xi,\zeta}^{\mathbf{u}(x)} \mid \Psi_x^{(\rho)\pm}\rangle_{\tilde{\Sigma}_x^{\pm}}. \tag{2.9}$$

$\Psi_x^{(\rho)}(\xi,\zeta)$ has the following transformation property under gauge transformations (i.e. changes of section on $\mathcal{H}$)[4]:

$$\left(\tilde{U}(A_g)\Psi_x^{(\rho)}\right)(\xi,\zeta) = \Psi_x^{(\rho)}\left(A_{g(x)}^{-1}\xi, A_{g(x)}^{-1}\zeta\right) \ . \tag{2.10}$$

A G-invariant scalar product of two sections $\Psi_{1,x}^{(\rho)}(\xi,\zeta)$ and $\Psi_{2,x}^{(\rho)}(\xi,\zeta)$ is defined by

$$\langle\Psi_{1,x}^{(\rho)} \mid \Psi_{2,x}^{(\rho)}\rangle_{\tilde{\Sigma}_x^{\pm}} = \int_{\tilde{\Sigma}_x^{\pm}} \Psi_{1,x}^{(\rho)*}(\xi,\zeta)\Psi_{2,x}^{(\rho)}(\xi,\zeta)d\tilde{\Sigma}(\xi,\zeta). \tag{2.11}$$

The covariant derivative of a section $\Psi_x^{(\rho)}(\xi,\zeta)$ on $\mathcal{H}$ is given by

$$D^R\Psi_x^{(\rho)}(\xi,\zeta) = [d + i\Gamma^R]\Psi_x^{(\rho)}(\xi,\zeta)$$

$$= [d + \frac{i}{2}[\omega^R(x)]_{ab}\tilde{M}^{ab}]\Psi_x^{(\rho)}(\xi,\zeta) \tag{2.12}$$

where $[\omega^R(x)]_{ab} = -[\omega^R(x)]_{ba}$ is the pull back of a connection on P, and $\tilde{M}_{ab}$ denote the generators of the spin zero phase space representation $\tilde{U}(A_g)$ of $SO(4,1)$ given by

$$\tilde{M}_{ab} = -\tilde{M}_{ba} = L_{ab}(\xi) + L_{ab}(\zeta) \tag{2.13}$$

with

$$L_{ab}(\xi) = i\Big(\xi_a\frac{\partial}{\partial\xi^b} - \xi_b\frac{\partial}{\partial\xi^a}\Big), \qquad L_{ab}(\zeta) = i\Big(\zeta_a\frac{\partial}{\partial\zeta^b} - \zeta_b\frac{\partial}{\partial\zeta^a}\Big)\,. \tag{2.14}$$

Local de Sitter indices $a, b, c, \ldots$ running over $0,1,2,3,5$ are raised and lowered with the de Sitter metric η^{ab} and η_{ab}, respectively.

We, finally, introduce the kernel for the propagation from (ξ,ζ) to (ξ',ζ') in the local fiber over $x \in U_4$ in $\mathcal{H}$ which is determined by the following overlap of the coherent state basis $\Phi^{\mathbf{u}(x)}_{\xi,\zeta}$ at x:

$$\tilde{K}^{(\rho)}_{\tilde{\eta},x}(\xi',\zeta';\xi,\zeta) = \langle\Phi^{\mathbf{u}(x)}_{\xi',\zeta'} \mid \Phi^{\mathbf{u}(x)}_{\xi,\zeta}\rangle_{\tilde{\Sigma}^\pm_x}\,. \tag{2.15}$$

Eq. (2.15) defines a reproducing kernel in $\mathcal{H}^{(\rho)}_{\tilde{\eta}}(x)$ with the reproducing property following from (2.7), i.e.

$$\tilde{K}^{(\rho)}_{\tilde{\eta},x}(\xi',\zeta';\xi,\zeta) = \int_{\tilde{\Sigma}^\pm_x} \tilde{K}^{(\rho)}_{\tilde{\eta},x}(\xi',\zeta';\xi'',\zeta'')\tilde{K}^{(\rho)}_{\tilde{\eta},x}(\xi'',\zeta'',\xi,\zeta)d\tilde{\Sigma}(\xi'',\zeta'')\,. \tag{2.16}$$

The kernel $\tilde{K}^{(\rho)}_{\tilde{\eta},x}(\xi',\zeta';\xi,\zeta)$ determines the propagation of the generalized wave functions $\Psi_x^{(\rho)}(\xi,\zeta)$ in the local fiber variables. For the discussion of the (strongly and weakly) causal geometro-stochastic propagation on the bundle $\mathcal{H}$ we refer to Refs. [4] and [9].

We, moreover, require that the generalized wave function $\Psi_x^{(\rho)}(\xi,\zeta)$ satisfies a de Sitter gauge covariant and U_4-covariant second order wave equation on $\mathcal{H}$ with real eigenvalue α. Specializing to axial vector torsion in the U_4 base this equation may be written, with $D^R_\rho = \partial_\rho + i\Gamma^R_\rho(x)$, as [1]

$$(\Box_{\mathcal{H}} + \alpha)\Psi_x^{(\rho)}(\xi,\zeta) = \Big(\frac{1}{\sqrt{-g}}D^R_\rho\sqrt{-g}g^{\rho\sigma}D^R_\sigma + \alpha\Big)\Psi_x^{(\rho)}(\xi,\zeta) = 0 \tag{2.17}$$

where $g = det\ g_{\mu\nu}(x)$, and α is a constant of dimension L^{-2} (L=length) characterizing the wave motion on $\mathcal{H}$.

Using the operators $D^R_k = \lambda^\mu_k(x)D^R_\mu$ with $D^R = \theta^k D^R_k$ as defined in (2.12) and the generators $\tilde{M}_{ab}$ of the phase space representation $\tilde{U}(A_g)$ of $SO(4,1)$

one can construct, by G-invariant integration over the local fiber variables, the following set of hermitean gauge covariant currents, antisymmetric in a,b, and bilinear in the matter fields $\Psi_x^{(\rho)}(\xi,\zeta)$ and their adjoints for a fixed value of ρ:

$$J_{kab}^{(\rho)}(x) = \frac{i}{2}\int_{\tilde{\Sigma}_x^{\pm}} \Psi_x^{(\rho)}(\xi,\zeta)^* [\overrightarrow{\tilde{M}}_{ab}\overrightarrow{D}_k^R - \overleftarrow{D}_k^R \overleftarrow{\tilde{M}}_{ab}]\Psi_x^{(\rho)}(\xi,\zeta) d\tilde{\Sigma}(\xi,\zeta) , \quad (2.18)$$

with

$$\overrightarrow{D}_k^R = \overrightarrow{\partial}_k + i \overrightarrow{\Gamma}_k^R (x); \qquad \overleftarrow{D}_k^R = \overleftarrow{\partial}_k - i \overleftarrow{\Gamma}_k^R (x) , \quad (2.19)$$

and analogously for $\overrightarrow{\tilde{M}}_{ab}$ and $\overleftarrow{\tilde{M}}_{ab} = \overrightarrow{\tilde{M}}_{ab}^{\dagger}$. As a result of (2.17) the currents (2.18) are covariantly conserved. The equations (2.18), (2.12) and (2.13), (2.14) show that the currents $J_{kab}^{(\rho)}(x)$ result from an averaged internal motion taking place in the local fibers on $\mathcal{H}$. For $a,b = i,j$ it is an internal rotational motion (Lorentz rotation); for $a,b = i,5$ it is a generalized translation (de Sitter boost) in the fiber. It is thus apparent that our formalism describes quantized material objects possessing internal gauge degrees of freedom and extension. We shall use the currents (2.18) as source currents for the bundle geometry tying thereby the quantized motion in the fiber to the geometry of the entire space.

3. Nonlinear gauge and field equations

In order to recover gravitation in a G-invariant manner as a gauge theory of the Lorentz subgroup $H = SO(3,1)$ of $G = SO(4,1)$ we introduce a new Higgs-type field in the formalism given as a section, $\xi(x)$, of the soldered bundle E defined in (2.4) obeying $\xi^a(x)\xi^b(x)\eta_{ab} = -R^2$ [compare (2.2)]. Global sections on E always exist. The "zero section", $\xi(x) = \xi^o$, may be identified with the space-time base of E. The field $\xi(x)$ acts as a symmetry reducing field in the bundle framework: If $\xi(x)$ is *parallel* with respect to $\omega^R(x)$, i.e. satisfies

$$D^R\xi^a(x) = d\xi^a(x) + [\omega^R(x)]_b{}^a\xi^b(x) = 0, \quad (3.1)$$

the $SO(4,1)$ gauge symmetry reduces to the $SO(3,1)$ gauge symmetry describing pure (metric) gravitation. We assume that the full de Sitter gauge symmetry does not reduce everywhere to the Lorentz subsymmetry, but that this reduction of symmetry indeed occurs far outside the quantized material sources present in the geometry. There are, however, regions in space-time, denoted by $D_{(i)}; i = 1,\ldots N$, where the G-symmetry does *not* reduce, i.e. where $D^R\xi^a(x) \neq 0$. In these regions ω^R (or W^R, see (3.2) below) takes values in the Lie algebra $\mathbf{g}$ of $SO(4,1)$ while for regions where (3.1) is true

ω^R (or W^R) reduces to a set of one-forms with values in the Lie algebra $\mathbf{g}'$ of the subgroup $SO(3,1)$. In general $\mathbf{g}$ may be decomposed as $\mathbf{g}=\mathbf{g}'\oplus\mathbf{t}$, where $\mathbf{t}$ is a vector space generating the homogeneous space G/H isomorphic to V_4'.

One may now consider de Sitter boost transformations, $A(\xi(x)) \in SO(4,1)$, transforming the origin ξ^o in V_4' into $\xi(x)$ at $x \in U_4$ and use these transformations to go over to a nonlinear realization of the de Sitter transformations in terms of Lorentz transformations in ξ^o yielding for ω^R the following form:

$$[\omega^R(x)]^a{}_b \overset{A^{-1}(\xi(x))}{\longrightarrow} [W^R(x,\xi(x))]^a{}_b = \begin{pmatrix} [W^R(x,\xi(x))]^i{}_j & [\theta^R(x,\xi(x))]^i \\ [\theta^R(x,\xi(x))]_j & 0 \end{pmatrix}, \tag{3.2}$$

$$\text{with} \qquad [W^R(x,\xi(x))]^i{}_j = [\bar{\omega}(x)]^i{}_j + [\tau^R(x,\xi(x))]^i{}_j, \tag{3.3}$$

$$\text{and} \qquad [W^R(x,\xi(x))]^i{}_5 = \frac{1}{R}[A^{-1}(\xi(x))]^i{}_a D^R\xi^a(x). \tag{3.4}$$

We call the form $W^R(x,\xi(x))$ in (3.2) the nonlinearly transforming form of the connection on P (the nonlinear gauge denoted by N.L.). It transforms under gauge transformations, $\xi'(x) = A_{g(x)}\xi(x)$, with a matrix $A(\Lambda(\xi'(x),\xi(x))) \in H$ leaving the form of the r.-h. side of (3.2) unchanged (for details see [1] and [3]). The first term on the r.-h. side of (3.3) is the metric part defining a connection on P_L. However, the Lorentz part (3.3) of (3.2) has a torsion addition denoted by $\tau^R(x,\xi(x))$. (3.4) defining the soldering forms $[\theta^R(x,\xi(x))]^i$ of the de Sitter connection shows explicitly that (3.2) is Lorentz valued for $D^R\xi(x) = 0$, i.e. outside the domains $D_{(i)}$ where the G-gauge symmetry reduces to the H-gauge symmetry. A form analogous to (3.2) is obtained for the curvature two-forms $[\Omega^R(x,\xi(x))]^a{}_b$ in the N.L. gauge.

As field equations for the bundle geometry we now introduce, besides (2.17) for $\Psi_x^{(\rho)}(\xi,\zeta)$, the following two sets of de Sitter gauge covariant and U_4-covariant nonlinear source equations

$$R^R_{ik}(x,\xi(x)) = \frac{1}{2}\eta_{ik}R^R(x,\xi(x)) = \kappa \overset{N.L.}{T_{ik}} (x,\xi(x)) \, , \tag{3.5}$$

$$\overset{N.L.}{D^i} \; R^R_{ijab}(x,\xi(x)) = \bar{\kappa} \overset{N.L.}{J_{jab}} (x,\xi(x)) \, . \tag{3.6}$$

A further equation for the reducing section $\xi(x)$ is introduced and discussed in [1]. Here κ and $\bar{\kappa}$ are two independent coupling constants; κ is Einstein's gravitational constant, and $\bar{\kappa}$ is a new coupling constant characterizing the quantum fiber dynamics (QFD), i.e. the dynamical relation between quantized matter described on $\mathcal{H}$ and the full uncontracted bundle curvature tensor. The operator $\overset{N.L.}{D^i}$ in (3.6) denotes the full covariant derivative of the Lorentz *and* de Sitter indices [the latter taken with respect to the N.L. form, $W^R(x,\xi(x))$, of the connection]. The r.-h. side of (3.6) is the current (2.18) tranformed to N.L. form with the help of $A^{-1}(\xi(x))$. $R^R_{ij\,ab}(x,\xi(x))$ is the full curvature tensor with the Lorentz part (for $a,b = k,l$ composed of metric, torsion, and quadratic de Sitter boost contributions (see [1]), and with the de Sitter boost part (for $a,b = i,5$). Eqs. (3.5) are of Einstein type involving the contracted Lorentz curvature tensor, $R^R_{ik}(x,\xi(x)) = \eta^{jl} R^R_{ijkl}(x,\xi(x))$, and the corresponding curvature scalar $R^R(x,\xi(x))$ again composed of three parts (metric, torsion and boost). On the r.-h. side of (3.5) appears the total energy-momentum tensor decomposing into the classical symmetric part, $\bar{T}_{ik}(x)$, of general relativity representing classical matter, and a quantum part induced by $\Psi_x^{(\rho)}(\xi,\zeta)$ possessing no symmetry in the indices, i,k :

$$\overset{N.L.}{T_{ik}}(x,\xi(x)) = \bar{T}_{ik}(x) + \overset{N.L.}{T_{ik}}(\Psi) \; . \tag{3.7}$$

A detailed investigation of (3.5) and (3.6) is presented in [1] and [3], in the latter reference with particular emphasis of the rôle played by the metric of Einstein's theory in this context. It is shown there that the $g_{\mu\nu}$-field of classical general relativity survives unchanged in this theory in the presence of quantized matter which, on the other hand, determines the additional fields characterizing the bundle geometry: axial vector torsion in the base and de Sitter boost contributions related to the soldering forms of the $SO(4,1)$ connection in P.

The rôle of axial torsion outside the region $D_{(i)}$ and outside the sources may be studied solving the vacuum torsion equation contained in (3.6). It reads when one neglects classical gravitational forces, i.e. for a metrically flat space-time base,

$$\Box^* K^R_s - \frac{1}{12}\partial^*_s P^R - \varepsilon_s{}^{ijl*} K^R_i \partial_j{}^* K^R_l = 0 \tag{3.8}$$

where ${}^*P^R = -6\partial^{i*} K^R_i$, and the axial vector torsion field, ${}^*K^R_s$, is given by [compare (3.3)]

$${}^*K^R_s = -\frac{1}{6}\varepsilon_s{}^{kij} K^R_{kij}(x,\xi(x)) \; ; \qquad \tau^R_{ij}(x,\xi(x)) = \theta^k K^R_{kij}(x,\xi(x)) \; . \tag{3.9}$$

Despite serious efforts no truely nonlinear solution of the equations (3.8) has yet been found except the trivial solution ${}^*K^R_s = k_s exp(\pm ik\cdot x)$ with $k_s k^s = k\cdot k = 0$, for which each term on the l.-h. side of (3.8) is separately zero.

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BRST QUANTIZATION OF PREGEOMETRY AND TOPOLOGICAL PREGEOMETRY

KEIICHI AKAMA
Department of Physics, Saitama Medical College
Kawakado, Moroyama, Saitama, 350-04, Japan

and

ICHIRO ODA
Department of Physics, Ochanomizu University
2-1-1, Otsuka, Bunkyo-ku, Tokyo, 112, Japan

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Abstract. We investigate the Dirac bracket algebra of the scalar pregeometry including topological pregeometry with BRST formalism, as the first step to its quantization. We derive the precise expression for induced gravity, and discuss how the gravity is induced from the topological theory.

1. Introduction

The quantum theory and the relativity are two of the greatest successes of physics in this century. The problem is, however, that we have no realistic quantum theory of general relativity. Tremendous efforts have been devoted to this subjects, canonical quantization approaches, superstring theories, searches for fundamental clues in lower dimensional systems, etc. Faced with a so difficult problem, we may come to the question "Is the general relativity really fundamental?" "Should it really be quantized as the fundamental object?" In fact, there exist theoretical schemes, called "pregeometry" by Wheeler, where Einstein's general relativity is not fundamental but is induced from more fundamental ingredients [1]. In Sakharov's idea, the Einstein gravity is induced through quantum fluctuations of the matters. The metric is a composite of the fundamental matter fields, and the quantum properties of gravitation are secondary effects due to those of the fundamental matters, just like the quantum properties of the hadrons stem from those of the quarks. Then we have first to establish the quantized theory of pregeometric matters rather than the gravity itself. In this talk, we would like to make a first step towards the quantization by applying the BRST formalism to the scalar pregeometry [2].

2. Scalar pregeometry

The fundamental action for pregeometry should be invariant under diffeomorphisms, and be written without metric, but with the matter fields only. If the matters are the scalar fields, the fundamental action is given by the Nambu-Goto type one [3],

$$\mathcal{L}_\phi = \sqrt{-\det_{\mu\nu} \partial_\mu \phi^I \partial_\nu \phi^J \eta_{IJ}} F(\phi). \tag{1}$$

where ϕ^I is the fundamental scalar field, $F(\phi)$ is a function of ϕ^I, and η_{IJ}=diag(1,-1,-1,$\cdots$,-1). In (1), $\mu,\nu=0,1,\cdots,D-1,D$, and $I,J=0,1,\cdots N-1$, where D is the number of the spacetime dimensions, and N is the number of the field ϕ. Now we briefly review how the Einstein gravity is induced in this system. The Lagrangian $\mathcal{L}_\phi$ is equivalent to the following Lagrangian with the auxiliary field $g_{\mu\nu}$:

$$\mathcal{L}_{\phi g} = \sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi^I\partial_\nu\phi^J\eta_{IJ} - G(\phi)\right), \tag{2}$$

where $g_{\mu\nu}$ plays the role of the metric, $g = \det g_{\mu\nu}$, and $G(\phi) = (D/2 - 1)(F(\phi))^{-2/(D-2)}$. They are equivalent because their equations of motion as well as their commutator algebras of the fields coincide with those of each other. The quantum effects of this $\mathcal{L}_{\phi g}$ give rise to this effective Lagrangian of Einstein gravity,

$$\mathcal{L}_R = \sqrt{-g}\left(\lambda + \frac{1}{16\pi G_{\rm N}}R\right), \tag{3}$$

where R is the scalar curvature, and λ and $G_{\rm N}^{-1}$ are divergent coefficients, which plays the roles of the cosmological and the Newtonian constants, respectively. We introduce a momentum cutoff which we take as realistic one connected with the fundamental scale. Thus the Einstein gravity is induced with a composite metric.

3. BRST Formalism

We should quantize the pregeometric matter Lagrangian $\mathcal{L}_\phi$ or $\mathcal{L}_{\phi g}$ rather than the Lagrangian $\mathcal{L}_R$ for the Einstein gravity.

$$\mathcal{L}_{\phi g} = \sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\cdot\partial_\nu\phi - G(\phi)\right). \tag{4}$$

The action $S_{\phi g} = \int \mathcal{L}_{\phi g} {\rm d}^D x$ is invariant under diffeomorphisms

$$\delta\phi^I = \varepsilon^\lambda\partial_\lambda\phi^I, \quad \delta g_{\mu\nu} = g_{\mu\lambda}\partial_\nu\varepsilon^\lambda + g_{\nu\lambda}\partial_\mu\varepsilon^\lambda + \varepsilon^\lambda\partial_\lambda g_{\mu\nu}. \tag{5}$$

where ε^λ is an arbitrary infinitesimal function of x^μ.

Our strategy of quantization is as follows. 1) we fix the gauge by adding the gauge fixing term, 2) add the Faddeev-Popov ghost term to make the action invariant under BRST transformations, 3) exhaust the constraints of the system, 4) work out their Poisson-bracket algebra, 5) define the Dirac bracket, and assign the commutators.

We fix the gauge by the de Donder condition:

$$\mathcal{L}_{\mathrm{GF}} = b_\mu \partial_\nu(\sqrt{-g}g^{\mu\nu}). \tag{6}$$

where b_ν is an auxiliary field. The BRST transformations of ϕ^I and $g_{\mu\nu}$ are given by replacing ε^λ in (5) by the Faddeev-Popov ghost c^λ:

$$\delta_{\mathrm{B}}\phi^I = c^\lambda \partial_\lambda \phi^I, \quad \delta_{\mathrm{B}} g_{\mu\nu} = g_{\mu\lambda}\partial_\nu c^\lambda + g_{\nu\lambda}\partial_\mu c^\lambda + c^\lambda \partial_\lambda g_{\mu\nu}. \tag{7}$$

To make the total action invariant under the BRST transformation, we add to the Lagrangian the Faddeev-Popov term

$$\mathcal{L}_{\mathrm{FP}} = i\sqrt{-g}g^{\mu\nu}\partial_\mu \bar{c}_\lambda \partial_\nu c^\lambda, \tag{8}$$

and define the BRST transformations for c^μ, $\bar{c}_\mu$ and b_μ by

$$\delta_{\mathrm{B}} c^\mu = c^\lambda \partial_\lambda c^\mu, \quad \delta_{\mathrm{B}} \bar{c}_\mu = ib_\mu + c^\lambda \partial_\lambda \bar{c}_\mu, \quad \delta_{\mathrm{B}} b_\mu = c^\lambda \partial_\lambda b_\mu. \tag{9}$$

The BRST transformations δ_{B} are nilpotent.

4. Constraints

Now the system is described by the Lagrangian (with $\tilde{g}^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu}$)

$$\begin{aligned}\mathcal{L} &\equiv \mathcal{L}_{\phi g} + \mathcal{L}_{\mathrm{GF}} + \mathcal{L}_{\mathrm{FP}} \\ &= -\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi \cdot \partial_\nu\phi - \sqrt{-g}G + \partial_\mu \tilde{g}^{\mu\nu} b_\nu + i\tilde{g}^{\mu\nu}\partial_\mu \bar{c}_\lambda \partial_\nu c^\lambda. \end{aligned} \tag{10}$$

The canonical conjugate variables of ϕ_J, c^ν, $\bar{c}_\nu$, and $\tilde{g}^{\mu\nu}$, are, respectively,

$$\begin{aligned}&\pi_J^\phi = -\tilde{g}^{0\mu}\partial_\mu\phi_J, \quad \pi_\nu^c = i\tilde{g}^{0\mu}\partial_\mu \bar{c}_\nu, \quad \pi_{\bar{c}}^\nu = -i\tilde{g}^{0\mu}\partial_\mu c^\nu, \\ &\pi_b^\mu = 0, \quad \tilde{p}_{mn} = 0, \quad \tilde{p}_{00} = b_0, \quad \tilde{p}_{0m} = b_m/2. \end{aligned} \tag{11}$$

where $\mu, \nu \cdots = 0, 1, \cdots, D-1$, and $m, n, \cdots = 1, \cdots, D-1$. Among them, the first three are solved for the time derivatives, $\partial_0\phi^I$, $\partial_0\bar{c}_\mu$, and $\partial_0 c^\mu$, while the last four give the constraints

$$\pi_b^\mu \approx 0, \quad \tilde{p}_{mn} \approx 0, \quad \chi_0 \equiv \tilde{p}_{00} - b_0 \approx 0, \quad \chi_m \equiv \tilde{p}_{0m} - b_m/2 \approx 0. \tag{12}$$

The only non-vanishing Poisson bracket among the constraints is

$$[\chi_\mu(x), \pi_b^\nu(y)]_{\mathrm{P}} = \delta_\mu^\nu \delta(\mathbf{x} - \mathbf{y}). \tag{13}$$

At this stage, π_b^μ, and χ_μ belong to the second class, while $\widetilde{p}_{mn}$ belongs to the first class.

The Hamiltonian density reads

$$\begin{aligned}\mathcal{H} &= -\frac{1}{2\widetilde{g}^{00}}(\pi^\phi + \widetilde{g}^{0m}\partial_m\phi)\cdot(\pi^\phi + \widetilde{g}^{0n}\partial_n\phi) \\ &+\frac{1}{2}\widetilde{g}^{mn}\partial_m\phi\cdot\partial_n\phi + (-\widetilde{g})^{\frac{1}{D-2}}G - \partial_m\widetilde{g}^{m\nu}b_\nu \\ &+i\frac{1}{\widetilde{g}^{00}}(\pi_\lambda^{\bar{c}} - i\widetilde{g}^{0m}\partial_m\bar{c}_\lambda)(\pi_c^\lambda + i\widetilde{g}^{0n}\partial_n c^\lambda) - i\widetilde{g}^{mn}\partial_m\bar{c}_\lambda\partial_n c^\lambda, \end{aligned} \tag{14}$$

which governs the time evolution of the physical quantities. For consistency, the constraints should remain vanishing during the time evolution. As for the second class constraints, we can make them so by adding appropriate constraints to the Hamiltonian. On the other hand, the condition that the first class constraint $\widetilde{p}_{mn}$ remain vanishing implies the secondary constraint

$$\Phi_{mn} \equiv \frac{D-2}{G}\left(-\frac{1}{2}\partial_m\phi\cdot\partial_n\phi - \partial_{(m}b_{n)} + i\partial_{(m}\bar{c}_\lambda\partial_{n)}c^\lambda\right) - g_{mn} \approx 0. \tag{15}$$

At this stage, the constraints are χ_μ, π_b^μ, Φ_{mn}, and $\widetilde{p}_{mn}$. Though the Poisson bracket algebra of them are complicated, it is diagonalized by introducing $\widehat{\chi}_\mu$ and $\widehat{\pi}_b^\mu$ defined by

$$\begin{aligned}&\widehat{\chi}_0 = \chi_0 + \widetilde{p}_{mn}\widetilde{g}^{mn}g_{00}, \qquad \widehat{\chi}_k = \chi_k + 2\widetilde{p}_{mn}\widetilde{g}^{mn}g_{0k}, \\ &\widehat{\pi}_b^0 = \pi_b^0, \qquad \widehat{\pi}_b^k = \pi_b^k - (D-2)\partial_l(\widetilde{p}_{mn}\overline{C}^{mnkl}/G),\end{aligned} \tag{16}$$

where

$$\overline{C}^{mnkl} = \sqrt{-g}\left(\overline{g}^{m(k}\overline{g}^{l)n} - \overline{g}^{mn}\overline{g}^{kl}\right), \quad \overline{g}^{mn} = g^{mn} - g^{m0}g^{n0}/g^{00}. \tag{17}$$

The only non-vanishing Poisson brackets among them are

$$[\widehat{\chi}_\mu(x), \widehat{\pi}_b^\nu(y)]_{\rm P} \approx \delta_\mu^\nu\delta(\mathbf{x}-\mathbf{y}), \tag{18}$$

$$[\Phi_{kl}(x), \widetilde{p}_{mn}(y)]_{\rm P} = C_{klmn}\delta(\mathbf{x}-\mathbf{y}), \tag{19}$$

where

$$C_{klmn} = \frac{1}{\sqrt{-g}}\left(g_{k(m}g_{n)l} - \frac{1}{D-2}g_{kl}g_{mn}\right). \tag{20}$$

Now all the constraints belong to the second class, and can be kept vanishing during the time evolution. Then, it is convenient to define the Dirac bracket for arbitrary fields $A(x)$ and $B(x)$ by

$$\begin{aligned}[A(x), B(y)]_{\rm D} &= [A(x), B(y)]_{\rm P} \\ &- \int\Big([A(x), \widehat{\pi}_b^\mu(z)]_{\rm P}[\widehat{\chi}_\mu(z), B(y)]_{\rm P}\end{aligned}$$

$$
\begin{aligned}
&-[A(x), \widehat{\chi}_\mu(z)]_{\mathrm{P}}[\widehat{\pi}_b^\mu(z), B(y)]_{\mathrm{P}} \\
&+[A(x), \widetilde{p}_{kl}(z)]_{\mathrm{P}}\overline{C}^{klmn}(z)[\Phi_{mn}(z), B(y)]_{\mathrm{P}} \\
&-[A(x), \Phi_{kl}(z)]_{\mathrm{P}}\overline{C}^{klmn}(z)[\widetilde{p}_{mn}(z), B(y)]_{\mathrm{P}} \Big) d\mathbf{z}
\end{aligned} \tag{21}
$$

which vanishes if $A(x)$ or $B(x)$ is a constraint.

5. Quantization

We assign the equal-time (anti-)commutators in terms of the Dirac bracket as

$$
[A(x), B(y)] \text{ or } \{A(x), B(y)\} = i[A(x), B(y)]_{\mathrm{D}}. \tag{22}
$$

Then we obtain for ϕ^I, c^μ, $\overline{c}_\mu$, $\widetilde{g}^{0\mu}$, and the conjugates π_I^ϕ, π_μ^c, $\pi_{\overline{c}}^\mu$, $b_\mu(=\widetilde{p}_{00}$ or $=2\widetilde{p}_{0m})$

$$
\begin{aligned}
&[\phi^I(x), \pi_J^\phi(y)] = i\delta_I^J\delta(\mathbf{x}-\mathbf{y}), \quad [\widetilde{g}^{0\mu}(x), b_\nu(y)] = i\delta_\nu^\mu\delta(\mathbf{x}-\mathbf{y}), \\
&\{c^\lambda(x), \pi_\kappa^c(y)\} = i\delta_\kappa^\lambda\delta(\mathbf{x}-\mathbf{y}), \quad \{\overline{c}_\lambda(x), \pi_{\overline{c}}^\kappa(y)\} = i\delta_\lambda^\kappa\delta(\mathbf{x}-\mathbf{y}),
\end{aligned} \tag{23}
$$

and $\widetilde{g}^{mn}$ depend on ϕ^I, c^μ, $\overline{c}_\mu$, b_μ through the constraint $\Phi_{mn} \approx 0$, and obey

$$
\begin{aligned}
&[\widetilde{g}^{kl}(x), \pi_I^\phi(y)] = i\widetilde{C}^{klmn}(x) \\
&\quad \times \left(\partial_m\phi_I(x)\partial_n\delta(\mathbf{x}-\mathbf{y}) + \frac{1}{D-2}g_{mn}\frac{\partial G}{\partial\phi^I}\delta(\mathbf{x}-\mathbf{y})\right), \\
&[\widetilde{g}^{kl}(x), \pi_\mu^c(y)] = \widetilde{C}^{klmn}(x)\partial_m\overline{c}_\mu(x)\partial_n\delta(\mathbf{x}-\mathbf{y}), \\
&[\widetilde{g}^{kl}(x), \pi_{\overline{c}}^\mu(y)] = -\widetilde{C}^{klmn}(x)\partial_m c^\mu(x)\partial_n\delta(\mathbf{x}-\mathbf{y}), \\
&[\widetilde{g}^{kl}(x), \widetilde{p}_{0\mu}(y)] = -\widetilde{g}^{kl}g_{0\mu}\delta(\mathbf{x}-\mathbf{y}), \\
&[\widetilde{g}^{kl}(x), \widetilde{g}^{0m}(y)] = -\widetilde{C}^{klmn}(x)\partial_n\delta(\mathbf{x}-\mathbf{y}),
\end{aligned} \tag{24}
$$

with $\widetilde{C}^{klmn} = (D-2)\overline{C}^{klmn}/G$. The BRST charge is given by $Q_{\mathrm{B}} = \int J_{\mathrm{B}}^0 d\mathbf{x}$ with

$$
\begin{aligned}
J_{\mathrm{B}}^0 = c^0\Bigg(&-\frac{1}{2\widetilde{g}^{00}}(\pi^\phi + \widetilde{g}^{0m}\partial_m\phi)\cdot(\pi^\phi + \widetilde{g}^{0n}\partial_n\phi) \\
&+\frac{i}{\widetilde{g}^{00}}(\pi_\lambda^c - i\widetilde{g}^{0m}\partial_m\overline{c}_\lambda)(\pi_{\overline{c}}^\lambda + i\widetilde{g}^{0n}\partial_n c^\lambda) \\
&+\frac{1}{2}\widetilde{g}^{mn}\partial_m\phi\cdot\partial_n\phi - \sqrt{-g}G - i\widetilde{g}^{mn}\partial_m\overline{c}_\rho\partial_n c^\rho\Bigg) \\
+c^n &\left(\pi_\phi\cdot\partial_n\phi + \pi_\mu^c\partial_n c^\mu + \pi_{\overline{c}}^\mu\partial_n\overline{c}_\mu\right) \\
+b_\mu &\left(-i\pi_{\overline{c}}^\mu + \partial_n(c^n\widetilde{g}^{0\mu} - c^0\widetilde{g}^{n\mu})\right).
\end{aligned} \tag{25}
$$

6. Quantum transition amplitude

The transition amplitude from the state Ψ_i to Ψ_f is given by

$$\begin{aligned} T_{fi} = & \int \mathcal{D}\phi^I \mathcal{D}\pi_I^\phi \mathcal{D}\tilde{g}^{\mu\nu} \mathcal{D}\tilde{p}_{\mu\nu} \mathcal{D}b_\mu \mathcal{D}\pi_b^\mu \mathcal{D}c^\mu \mathcal{D}\pi_\mu^c \mathcal{D}\bar{c}_\mu \mathcal{D}\pi_{\bar{c}}^\mu \\ & \Psi_f^* \Psi_i \prod_x [\delta(\hat{\pi}_b^\mu)\delta(\hat{\chi}_\mu)\delta(\tilde{p}_{mn})\delta(\Phi_{mn}) \det C_{ijkl}] \\ & \exp i \int \mathrm{d}^D x \left(\pi^\phi \cdot \dot{\phi} + \tilde{p}_{\mu\nu}\dot{\tilde{g}}^{\mu\nu} + \pi_b^\mu \dot{b}_\mu + \pi_\mu^c \dot{c}^\mu + \pi_{\bar{c}}^\mu \dot{\bar{c}}_\mu - \mathcal{H}\right), \end{aligned} \tag{26}$$

Using the explicit expression of $\det C_{ijkl}$, we get

$$\det C_{ijkl} = (\det_{mn} g_{mn})^D (\sqrt{-g})^{-D(D-1)/2} \times \text{constant}. \tag{27}$$

The integrations by π_b^μ, $\tilde{p}_{0\mu}$, and $\tilde{p}_{mn}$ are trivial. We perform the integrations by π_I^ϕ, π_μ^c, and $\pi_{\bar{c}}^\mu$

$$\begin{aligned} & \int \mathcal{D}\pi_I^\phi \mathcal{D}\pi_\mu^c \mathcal{D}\pi_{\bar{c}}^\mu \exp i \int \mathrm{d}^D x \left(\pi^\phi \cdot \dot{\phi} + \pi_\mu^c \dot{c}^\mu + \pi_{\bar{c}}^\mu \dot{\bar{c}}_\mu + b_\mu \dot{\tilde{g}}^{0\mu} - \mathcal{H}\right) \\ & \qquad = (\tilde{g}^{00})^{-D+N/2} \exp i \int \mathrm{d}^D x \mathcal{L}, \end{aligned} \tag{28}$$

where $\mathcal{L} = \mathcal{L}_{\phi g} + \mathcal{L}_{\mathrm{GF}} + \mathcal{L}_{\mathrm{FP}}$. We rewrite the $\delta(\Phi_{mn})$ as

$$\prod_x \delta(\Phi_{mn}) = \int \mathcal{D}(\sqrt{-g}u^{mn}) \exp i \int \mathrm{d}^D x \sqrt{-g} u^{mn} \Phi_{mn}, \tag{29}$$

where u^{mn} is the Lagrange multiplier. Then we obtain

$$\begin{aligned} T_{fi} = & \int \mathcal{D}\phi^I \mathcal{D}\tilde{g}^{\mu\nu} \mathcal{D}b_\mu \mathcal{D}c^\mu \mathcal{D}\bar{c}_\mu \mathcal{D}u^{mn} \Psi_f^* \Psi_i \\ & \prod_x \left[(\tilde{g}^{00})^{N/2} (\sqrt{-g})^D\right] \exp i \int \mathrm{d}^D x \left(\mathcal{L} + \sqrt{-g} u^{mn} \Phi_{mn}\right). \end{aligned} \tag{30}$$

7. Induction of gravity

To get the effective Lagrangian $\mathcal{L}_{\mathrm{eff}}$ for induced gravity we perform the integration by ϕ in T_{fi}

$$\int \mathcal{D}\phi^I \exp i \int \mathrm{d}^D x \left(\mathcal{L} + \sqrt{-g} u^{mn} \Phi_{mn}\right) = \exp i \int \mathrm{d}^D x \mathcal{L}_{eff}. \tag{31}$$

We use the stationary phase approximation, i. e. in the integration by ϕ, we neglect $O(\phi^3)$ terms in $\mathcal{L} + \sqrt{-g} u^{mn} \Phi_{mn}$. After a lengthy calculation, we finally obtain the effective Lagrangian

$$\mathcal{L}_{\mathrm{eff}} = \sqrt{-g}\Bigg[\frac{1}{16\pi G_{\mathrm{N}}} R + \frac{1}{\sqrt{-g}} b_\mu \partial_\nu (\sqrt{-g} g^{\mu\nu}) + i g^{\mu\nu} \partial_\mu \bar{c}_\lambda \partial_\nu c^\lambda$$

$$+\frac{1}{2}\Big\{\frac{1}{2}(\mathcal{D}_\lambda U^{mn})^2 - (\mathcal{D}_m U^{mn})^2 + \mathcal{D}_m U^{mn}\mathcal{D}_n U + \frac{1}{4}(\mathcal{D}_m U)^2\Big\}$$
$$+\frac{1}{2}M^2\Big\{(U^{mn})^2 + \frac{1}{2}U^2\Big\} + k_1 U + k_2\left(U^{mn}R_{mn} - \frac{1}{2}UR\right)$$
$$+k_3 U^{mn}\left(-\partial_m b_n + i\partial_m \bar{c}_\lambda \partial_n c^\lambda\right)\Bigg] \tag{32}$$

with the divergent coefficients

$$\frac{1}{16\pi G_{\mathrm{N}}} = \rho^2 \Lambda^{D-2}, \quad M = \sqrt{\frac{3(D-2)}{D}}\Lambda,$$
$$k_1 = 3\rho\Lambda^{D/2+1} \quad k_2 = \rho\Lambda^{D/2-1}, \quad k_3 = \frac{1}{\rho}\left(1+\frac{D}{2}\right)\Lambda^{-D/2+1},$$
$$\left(\rho = \sqrt{N/6(D-2)(D/2)!(4\pi)^{D/2}}\right), \tag{33}$$

where Λ is the ultraviolet cut off (Pauli-Villars mass), and the cosmological constant λ is fine-tuned to be 0. Since $G_{\mathrm{N}} > 0$, the induced gravity is attractive. Roles of the field U is not yet fully investigated.

8. Conclusions and discussions

If the number N of the fundamental scalar fields coincides with the number of dimensions D, the scalar pregeometry becomes topological [4]. In this case, we can show that no local physical mode exists. Only the topological invariant quantities are observable. It is interesting to see that the gravity is still induced. This is because the fundamental scale breaks the topological symmetry.

In summary, if the Einstein gravity is an induced effect (pregeometry), we have first to quantize the pregeometric matters rather than the gravity itself. To make a first step towards the quantization we applied the BRST formalism to the scalar pregeometry. We derived the Dirac bracket algebra, but the problem of the operator ordering is not solved. We derived the precise expression for induced gravity. It depends on the ultraviolet cutoff Λ, which we take as the physical fundamental scale.

Finally, we would like to emphasize that, the quantum gravity has another possibility that the gravity is not fundamental, and we should first quantize the pregeometric matter.

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EXCHANGE RELATIONS AND CORRELATION FUNCTIONS FOR AN $SO(4)$ INVARIANT QUANTUM MECHANICAL MODEL

PAOLO FURLAN
Dipartimento di Fisica Teorica dell' Universitá di Trieste and I.N.F.N. Sezione di Trieste, Italy

and

YASSEN S. STANEV and IVAN T. TODOROV*
The Erwin Schrödinger International Institute for Mathematical Physics (ESI), Vienna, Austria

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Abstract. We consider the SO_4 invariant quantum dynamics of a point particle moving on the 3-sphere. Quantum exchange relations for different times are derived with an "R matrix" depending on the time difference and on the conserved angular momentum. Their implications for correlation functions are worked out.

1. Introduction

In studying the G current algebra models (G standing in general for a simple, compact Lie group) special attention is devoted to the analysis of the so-called zero modes, which can be described in terms of a point particle moving on the group manifold G itself (for a sample of references on the Hamiltonian approach to such models – see [1,2,3]). In this context Alekseev and Faddeev [4] presented an R-matrix treatment of the phase space $\Gamma = T^*SU_2$ with an emphasis on its splitting into chiral parts which admit a natural quantum group deformation. Here we present a manifestly SO_4 invariant solution of the corresponding quantum mechanical model.

Our main result is the derivation of generalized "exchange relations"

$$g(t_2) \otimes g(t_1) = R_{12}\, g(t_1) \otimes g(t_2) = g(t_1) \otimes g(t_2)\, \tilde{R}_{12}$$

where the "R-matrix" (or rather "$6-j$ symbol") depends on the time difference $t_{12} = t_1 - t_2$ (playing the role of a spectral parameter) and on the conserved right (or left) invariant angular momentum . The representatives of the resulting R-matrices with operator valued entries on a set of

* On leave from the Institute for Nuclear Research, Bulgarian Academy of Sciences, Tsarigradsko Chaussee 72, BG-1784 Sofia, Bulgaria

(permuted) n-point correlation functions satisfy a generalized Yang-Baxter equation . $\widetilde{R}_{12}$ appears to provide an example for an R-matrix depending on a spectral parameter in the framework of quasi-coassociative bialgebras (cf. [5]).

A more detailed version of this work appears elsewhere [6].

2. Classical approach

We write the SU_2 group element $g = (g_\alpha^\beta)$ as a pair of conjugate 2-spinors

$$g = \begin{pmatrix} w^1 & w^2 \\ -w_2^* & w_1^* \end{pmatrix}. \tag{1.1}$$

Since gg^* is a multiple of the unit matrix

$$gg^* = \det(g)\cdot 1, \quad \det g = w^* w \ \ (= w_\alpha^* w^\alpha), \tag{1.2}$$

the configuration space $SU_2 \approx S^3$ appears as a real hypersurface in $\mathbb{C}^2$ given by the (primary) constraint equation

$$S^3 : ww^* - 1 = 0 \quad (w \in \mathbb{C}^2). \tag{1.3}$$

We shall derive the PB structure on Γ from the canonical PB on $T^*\mathbb{C}^2$. Let

$$p = \begin{pmatrix} \pi^{*1} & \pi^{*2} \\ -\pi_2 & \pi_1 \end{pmatrix} \tag{1.4}$$

be the canonical momentum matrix. The non-zero PB on $T^*\mathbb{C}^2$ are

$$\{w^\alpha, \pi_\beta\} = \delta_\beta^\alpha = \{w_\beta^*, \pi^{*\alpha}\}. \tag{1.5}$$

Primary constraints generate gauge transformations. Since π, π^* are the only gauge-dependent quantities, we impose the gauge condition (secondary constraint)

$$2\mu \equiv \mathrm{tr}(gp^*) = w\pi + w^*\pi^* = 0, \quad \{w^*w - 1, \mu\} = 1. \tag{1.6}$$

At this point we have a pair of second class constraints. Rather than computing the Dirac brackets for $\pi^{(*)}$ and $w^{(*)}$ we shall single out a subalgebra $\mathcal{A}(\Gamma)$ of the algebra of functions on $T^*\mathbb{C}^2$ whose Dirac brackets coincide with the original PB. To this end we introduce the right invariant angular momentum

$$\ell = \begin{pmatrix} \ell_3 & \ell_- \\ \ell_+ & -\ell_3 \end{pmatrix} = i\,pg^* \tag{1.7}$$

and its left invariant counterpart

$$\tilde{\ell} = -ig^*p \ \ (= ip^*g) = -g^*\ell g \ \ (gg^* = 1); \tag{1.8}$$

ℓ_a and $\tilde{\ell}_a$ generate left and right infinitesimal SU_2 shifts:

$$i\{\ell_a, g\} = -\frac{1}{2}\sigma_a g, \quad i\left\{\tilde{\ell}_a, g\right\} = \frac{1}{2} g \sigma_a. \tag{1.9}$$

They imply that the angular momenta have vanishing PB with the constraints,

$$\{\ell_a, w^* w\} = 0 = \{\ell_a, \mu\}, \tag{1.10}$$

(and similar relations for $\tilde{\ell}$), thus appearing as gauge invariant observables corresponding to vector fields tangent to Γ. They span among themselves the $SU_2 \times SU_2$ PB Lie algebra:

$$\{\ell_a, \ell_b\} = \epsilon_{abc}\, \ell_c, \quad \left\{\tilde{\ell}_a, \tilde{\ell}_b\right\} = \epsilon_{abc}\, \tilde{\ell}_c, \tag{1.11}$$

$$\left\{\ell_a, \tilde{\ell}_b\right\} = 0. \tag{1.12}$$

The similarity relation (1.8) between $-\ell$ and $\tilde{\ell}$ implies that left and right angular momentum squares coincide:

$$\frac{1}{2}\operatorname{tr} \ell^2 = L^2 \, (= \ell_a \ell^a) = \frac{1}{2}\operatorname{tr} \tilde{\ell}^2. \tag{1.13}$$

The subalgebra $\mathcal{A}(\Gamma)$ is generated by ℓ, $\tilde{\ell}$ and g subject to the relations (1.9) (1.11-13) and

$$\{g \overset{\otimes}{,} g\} = 0. \tag{1.14}$$

In order to get the time evolution of our mechanical model we have to introduce a Hamiltonian which has to be $SU_2 \times SU_2$ invariant and depending on the constraint (1.3). The simplest choice is

$$H = L^2 + \lambda(w^* w - 1). \tag{1.15}$$

Angular momenta ℓ,$\tilde{\ell}$ are conserved and the Lagrangean

$$\mathcal{L} = i \operatorname{tr}(\ell \dot{g} g^*) - H = \frac{1}{2}\operatorname{tr}(\dot{g}\dot{g}^* - \lambda(g g^* - 1)) \tag{1.16}$$

allows to identify the linear momentum (1.4) with $\dot{g}$ and the angular momenta ℓ and $\tilde{\ell}$ with (the traceless parts of) $i\dot{g}g^*$ and $-ig^*\dot{g}$.

We see that the position observables g (or $w^{(*)}$) and the gauge-invariant angular momenta ℓ and $\tilde{\ell}$ have linear PB. We would like instead to have quadratic expressions in the r.h.s. of the PB, since such a kind of quadratic relations are a point of departure for the R-matrix approach to the study of completely integrable systems. We shall see that the different-time PB for $g(t)$ appears as a quadratic expression in $g(t_1) \otimes g(t_2)$ with coefficients depending on the constants of motion ℓ (or $\tilde{\ell}$).

Instead of going on with the classical formulation we shall pass now to the quantum one, coming back again to the classical case at the end.

3. Quantum approach

We define the quantum algebra $\mathcal{A}_\hbar = \mathcal{A}_\hbar(\Gamma)$ as the algebra generated by $\ell, \tilde{\ell}$ and g with the PB (1.9) (1.11) (1.12) and (1.14) represented by commutators according to the standard rule

$$i\hbar\{\ ,\ \} \to [\ ,\ ].$$

Since

$$\ell^2 = (\ell_1^2 + \ell_2^2 + \ell_3^2)1 - \hbar\ell, \tag{2.1}$$

we define the invariant (quantum) Hamiltonian by

$$H = \ell^2 + \hbar\ell = (\ell_1^2 + \ell_2^2 + \ell_3^2)1 = L(L+\hbar)1 = \tilde{\ell}^2 + \hbar\tilde{\ell}. \tag{2.2}$$

The equations of motion

$$i\dot{g}(t) = \hbar^{-1}[g(t), H] = -\frac{1}{2}\left[g(t), \tilde{\ell}_a\right]_+ \sigma_a = -g(t)\left(\tilde{\ell} + \frac{3}{4}\hbar\right) = \left(\ell + \frac{3}{4}\hbar\right) g(t) \tag{2.3}$$

have the solutions ($e(x) \equiv e^{ix}$)

$$g(t) = g\, e\left(\left(\tilde{\ell} + \frac{3}{4}\hbar\right)t\right) \ \left(= e\left(-\left(\ell + \frac{3}{4}\hbar\right)t\right)g\right), \quad g \equiv g(0) \tag{2.4a}$$

or

$$w(t) = w\, e\left(\left(\tilde{\ell} + \frac{3}{4}\hbar\right)t\right), \quad w^*(t) = e\left(-\left(\tilde{\ell} + \frac{3}{4}\hbar\right)t\right)w^*. \tag{2.4b}$$

$\mathcal{A}_\hbar(\Gamma)$ admits an antilinear involution ϑ, "the TCP symmetry", such that

$$\vartheta(w(t)) = w^*(-t), \quad \vartheta(w^*(t)) = w(-t), \tag{2.5}$$

$$\vartheta(\ell_a) = -\ell_a, \quad \vartheta(\tilde{\ell}_a) = -\tilde{\ell}_a. \tag{2.6}$$

We shall view the elements of $\mathcal{A}_\hbar(\Gamma)$ as operators in the vacuum Hilbert space $\mathcal{H}$ with a unique $SU_2 \times SU_2$ invariant state $\langle 0 \mid \ \mid 0 \rangle$ such that $\mid 0 \rangle$ is a cyclic vector with respect to $\mathcal{A}_\hbar(\Gamma)$.The involution $\vartheta(A)$ is then implemented by an antiunitary operator Θ such that

$$\Theta \mid 0 \rangle = \mid 0 \rangle, \quad \Theta A \Theta^{-1} = \vartheta(A), \quad \Theta^2 = 1. \tag{2.7}$$

Charge conservation implies that only even point correlation functions with an equal number of w and w^* can be nonvanishing. Θ invariance and antiunitarity allow to relate correlation functions with opposite order of factors, e.g.

$$\begin{aligned} &\langle 0 \mid \overset{1}{w}(t_1)(\overset{2}{w})^*(t_2)\cdots \overset{2n-1}{w}(t_{2n-1})(\overset{2n}{w})^*(t_{2n}) \mid 0 \rangle = \\ &= \langle 0 \mid (\overset{2n}{w})^*(-t_{2n})\ \overset{2n-1}{w}(-t_{2n-1})\cdots(\overset{2}{w})^*(-t_2)\ \overset{1}{w}(-t_1) \mid 0 \rangle. \end{aligned} \tag{2.8}$$

Using the solution of the equations of motion (2.4) together with Θ and time translation invariance, we get

$$\langle 0 | g^{\beta_1}_{\alpha_1}(t_1)\, g^{\beta_2}_{\alpha_2}(t_2) | 0 \rangle = \frac{1}{2}\, \epsilon_{\alpha_1\alpha_2}\, \epsilon^{\beta_1\beta_2}\, e\left(-\frac{3}{4}\hbar t_{12}\right), \tag{2.9}$$

where ϵ is the unit antisymmetric tensor. The 2n-point function is uniquely determined from the initial (equal time) condition

$$\langle 0 | g^{\beta_1}_{\alpha_1} \ldots g^{\beta_{2n}}_{\alpha_{2n}} | 0 \rangle = \frac{1}{(n+1)!} \sum \prod_{i<j} \epsilon_{\alpha_i\alpha_j}\, \epsilon^{\beta_i\beta_j}. \tag{2.10}$$

As an explicit example we give the 4-point function

$$\langle 0 | \overset{1}{g}(t_1)\, \overset{2}{g}(t_2)\, \overset{3}{g}(t_3)\, \overset{4}{g}(t_4) | 0 \rangle = \langle 0 | \overset{1}{g}\overset{2}{g}\overset{3}{g}\overset{4}{g} | 0 \rangle\, e\left(-\frac{\hbar}{4}(3t_{14} + (1 + 4P_{34})t_{23})\right) \tag{2.11}$$

where P_{34} is the operator permuting the indices $\beta_3\beta_4$ and the equal-time 4-point function is given by

$$\langle 0 | \overset{1}{g}\overset{2}{g}\overset{3}{g}\overset{4}{g} | 0 \rangle =$$

$$= \frac{1}{6}\left(\epsilon_{\alpha_1\alpha_2}\epsilon_{\alpha_3\alpha_4}\epsilon^{\beta_1\beta_2}\epsilon^{\beta_3\beta_4} + \epsilon_{\alpha_1\alpha_3}\epsilon_{\alpha_2\alpha_4}\epsilon^{\beta_1\beta_3}\epsilon^{\beta_2\beta_4} + \epsilon_{\alpha_1\alpha_4}\epsilon_{\alpha_2\alpha_3}\epsilon^{\beta_1\beta_4}\epsilon^{\beta_2\beta_3}\right). \tag{2.12}$$

4. A quantum R-matrix with operator valued entries

The operators $g(t)$ satisfy for different times a generalized exchange relation of the type

$$\overset{2}{g}(t_2)\, \overset{1}{g}(t_1) = R_{12}\, \overset{1}{g}(t_1)\, \overset{2}{g}(t_2) = \overset{1}{g}(t_1)\, \overset{2}{g}(t_2)\, \widetilde{R}_{12}, \tag{3.1}$$

where R_{12} and $\widetilde{R}_{12}$ depend on the time difference t_{12} and on the conserved right and left invariant angular momenta ℓ_a and $\tilde{\ell}_a$, respectively. To construct $\widetilde{R}_{12}$ and thus derive (3.1) we express $\overset{1}{g}(t_1)$ in terms of $\overset{1}{g}(t_2)$ and use the commutatativity of g at equal times:

$$\overset{2}{g}(t_2)\, \overset{1}{g}(t_1) = \overset{1}{g}(t_2)\, \overset{2}{g}(t_2)\, e\left(\left(\overset{1}{\tilde{\ell}} + \frac{3}{4}\hbar\right) t_{12}\right) =$$

$$= \overset{1}{g}(t_1)\, e\left(-\overset{1}{\tilde{\ell}}\, t_{12}\right) \overset{2}{g}(t_2)\, e\left(\overset{1}{\tilde{\ell}}\, t_{12}\right) = \overset{1}{g}(t_1)\, \overset{2}{g}(t_2)\, U_{12}\, e\left(\overset{1}{\tilde{\ell}}\, t_{12}\right)$$

where U_{12} is a unitary operator defined by

$$e\left(-\overset{1}{\tilde{\ell}}\,t_{12}\right)\overset{2}{g}(t_2)=\overset{2}{g}(t_2)\,U_{12}.$$

Analysing the explicit form of U_{12}, we can see that the structure of

$$\mathcal{F}_{12}\equiv e\left(-\frac{\hbar}{2}t_{12}\right)\widetilde{R}_{12} \tag{3.2}$$

is given by

$$\mathcal{F}_{12} = F_1+F_2P+F_3\left(\overset{1}{\tilde{\ell}}+\overset{2}{\tilde{\ell}}\right)+F_4\,\overset{2}{\tilde{\ell}}\,P+F_5\,\overset{2}{\tilde{\ell}}\overset{1}{\tilde{\ell}} \tag{3.3}$$

where

$$F_i=F_i(t_{12},N^2),\qquad N^2\hbar^2\equiv(2L+\hbar)^2. \tag{3.4}$$

To compute F_i we differentiate (3.1) with respect to t_1, finding

$$\overset{1}{g}(t_1)\,\overset{2}{g}(t_2)\left\{\dot{\mathcal{F}}_{12}+i\left[\overset{1}{\tilde{\ell}},\mathcal{F}_{12}\right]+i\hbar P\mathcal{F}_{12}\right\}=0, \tag{3.5}$$

and we add the initial condition

$$\widetilde{R}_{12}(0)=1\Leftrightarrow F_i(0,N^2)=\delta_{i1}. \tag{3.6}$$

The unique solution is given by

$$\hbar^2F_5 = \frac{2}{N^2-1}\left(e^{-i\hbar t}-\frac{1}{N^2}\cos N\hbar t+\frac{i}{N}\sin N\hbar t\right)-\frac{2}{N^2}, \tag{3.7a}$$

$$\hbar F_4 = i\hbar\dot{F}_5=\frac{2}{N^2-1}\left(-\cos N\hbar t+\frac{i}{N}\sin N\hbar t+e^{-i\hbar t}\right), \tag{3.7b}$$

$$\hbar F_3=\frac{\cos N\hbar t-1}{N^2}, \tag{3.7c}$$

$$F_2=-i\,\frac{\sin N\hbar t}{N}, \tag{3.7d}$$

$$F_1 = 1+\frac{\cos N\hbar t-1}{N^2}+\hbar^2F_5\frac{N^2-1}{4}. \tag{3.7e}$$

One can derive a similar relation for $R_{12}(t_{12};\ell)=\widetilde{R}_{12}(t_{12};-\tilde{\ell})$.

We can compute the action of $R_{i\,i+1}$ or $\mathcal{F}_{i\,i+1}$ on correlation functions. The explicit expressions for the 4-point functions are given by

$$\langle 0 \,|\, \mathcal{F}_{12}(t_{12})\ \overset{1}{g}(t_1)\ \overset{2}{g}(t_2)\ \overset{3}{g}(t_3)\ \overset{4}{g}(t_4) \,|\, 0 \rangle =$$

$$= e(-\hbar P_{12} t_{12}) \langle 0 \,|\, \overset{1}{g}(t_1)\ \overset{2}{g}(t_2)\ \overset{3}{g}(t_3)\ \overset{4}{g}(t_4) \,|\, 0 \rangle, \tag{3.8a}$$

$$\langle 0 \,|\ \overset{1}{g}(t_1)\ \overset{2}{g}(t_2)\ \overset{3}{g}(t_3)\ \overset{4}{g}(t_4)\, \mathcal{F}_{34}(t_{34}) \,|\, 0 \rangle =$$

$$e(-\hbar P_{34} t_{34}) \langle 0 \,|\ \overset{1}{g}(t_1)\ \overset{2}{g}(t_2)\ \overset{3}{g}(t_3)\ \overset{4}{g}(t_4) \,|\, 0 \rangle, \tag{3.8b}$$

while (with shorthand notation)

$$\langle \mathcal{F}_{23} \rangle_4 = e(\hbar\, t_{23}\, P_{12})\, e(\hbar\, t_{23}\, P_{13}) \tag{3.8c}$$

$$\langle \mathcal{F}_{13} \rangle_4 = e(\frac{\hbar}{2}\, t_{12}\, P_{23})\, e(\frac{\hbar}{2}\, t_{23}\, P_{12}). \tag{3.8d}$$

The R_{ij} (and $\mathcal{F}_{ij}$) so defined are verified to satisfy the relations

$$R_{i\,i+1}(t)\, R_{i\,i+1}(-t) = 1, \tag{3.9}$$

$$\begin{aligned} R_{12}^{(123)}(t_{12})\, R_{13}^{(213)}(t_{13})\, R_{23}^{(231)}(t_{23}) &= R_{13}^{(123)} \\ = R_{23}^{(123)}(t_{23})\, R_{13}^{(132)}(t_{13})\, R_{12}^{(312)}(t_{12}), & \end{aligned} \tag{3.10}$$

where the upper indices (ijk) stand for the order of $\overset{i}{g}(t_i)$ to which R is applied. We note that such generalized Yang-Baxter equations that reflect the operator dependence of R are reminiscent to the relations found by Mack and Schomerus in their study of quasi co-associative quantum symmetries [5].

In the limit $\hbar \to 0$ we obtain the classical counterpart of the "quantum R matrix".Setting

$$\left\{\overset{1}{g}(t_1),\, \overset{2}{g}(t_2)\right\} = \lim_{\hbar\to 0} \frac{i}{\hbar} \left[\overset{2}{g}(t_2),\, \overset{1}{g}(t_1)\right] \tag{3.11}$$

we find

$$\left\{\overset{1}{g}(t_1),\, \overset{2}{g}(t_2)\right\} = \overset{1}{g}(t_1)\ \overset{2}{g}(t_2)\, \tilde{r}(t_{12}, \tilde{\ell}), \tag{3.12}$$

where

$$\tilde{r}(t, \tilde{\ell}) = tP - \frac{i}{2} t^2 \left(\overset{1}{\tilde{\ell}} + \overset{2}{\tilde{\ell}}\right) + i\, t^2\, \overset{2}{\tilde{\ell}}\, P + \frac{1}{3} t^3\, \overset{2}{\tilde{\ell}}\, \overset{1}{\tilde{\ell}}, \tag{3.13}$$

or a similar expression with $r(t, \ell) = \tilde{r}(t, -\ell)$ acting on the left. One can also write a linear in t_{12} expression for the PB with operators acting on both sides of $\overset{1}{g}\overset{2}{g}$ (see [6]).

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CURRENT ALGEBRA AND RENORMALIZATION

JOUKO MICKELSSON
Theoretical Physics, Royal Institute of Technology,
Stockholm S-100 44, Sweden
jouko theophys.kth.se

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Abstract. In this talk I want to explain the operator substractions needed to renormalize gauge currents in a second quantized theory. The case of space-time dimensions $3+1$ is considered in detail. In presence of chiral fermions the renormalization effects a modification of the local commutation relations of the currents by local Schwinger terms. In $1+1$ dimensions one gets the usual central extension (Schwinger term does not depend on background gauge field) whereas in $3+1$ dimensions one gets an anomaly linear in the background potential.

We extend our method to the spatial components of currents. Since the bose-fermi interaction hamiltonian is of the form $j^k A_k$ (in the temporal gauge) we get a new renormalization scheme for the interaction. The idea is to define a field dependent conjugation for the fermi hamiltonian in the one-particle space such that after the conjugation the hamiltonian can be quantized just by normal ordering prescription. We also discuss the regularization of vector fields in Fock space.

1. Introduction

Algebraic techniques have been proven to be very powerful when solving many quantum field theory models in $1+1$ space-time dimensions. The best understood and applied algebras are the affine Kac-Moody algebras, which are related to central extensions of loop groups, and the Virasoro algebra which is a central extension of the Lie algebra of vector fields on the circle. The latter algebra emerge for example in the construction of the energy momentum tensor in conformal field theory in two dimensions. On the other hand, the energy momentum tensor can be constructed by the Sugawara method as a quadratic expression in the components of the affine Lie algebra. In gauge theories the affine algebra is the Lie algebra of the group of gauge transformations, broken by the chiral anomaly. The Virasoro algebra is also important in understanding certain solvable statistical mechanics models in two dimensions. For a review on these topics see [3].

A natural question is how far we can go with some suitable generalizations of the above algebraic structures in dimensions higher than $1+1$. It is clear that for a succesful theory we need at least two things to begin with. First, what are the algebras. Second, what are the representations of the algebras. In this talk I will mainly address the questions related to generalized affine (chirally broken gauge symmetry) algebras. In the last section 5 I will briefly address the problems arising from quantization of vector fields in dimensions higher than 1.

The appropriate group (and Lie algebra) extensions replacing the affine algebra in higher dimensions has been found in the study of anomalous gauge theories, [7]. The difficulty is that no unitary faithful representations are known (and they probably do not exist, [12]). But there is a way around this problem. A representation is replaced by a *operator valued cocycle* in a Hilbert space (which in this talk is the Fock space of chiral fermions). Alternatively, one can view our construction (in 3 physical space dimensions) as a representation, not of the original Mickelsson-Faddeev-Shatasvili extension in [7], but as a representation of the gauge algebra extended by certain pseudo-differential operators of degree -2. The degree -2 comes from the fact that these operators are Hilbert-Schmidt and can be canonically quantized in the Fock space.

Chiral fermions in a nonabelian external gauge field are quantized as follows. Let G be a compact gauge group, $\mathbf{g}$ its Lie algebra, M the physical space, and $\mathcal{A}$ the space of smooth $\mathbf{g}$ valued vector potentials in M. For each $A \in \mathcal{A}$ one constructs a fermionic Fock space $\mathcal{F}_A$ containing a Dirac vacuum ψ_A. The Hilbert space $\mathcal{F}_A$ carries an irreducible representation of the canonical anticommutation relations (CAR)

$$a^*(u)a(v) + a(v)a^*(u) = (u,v) \quad \text{all other anticommutators} = 0.$$

The representation is characterized by the property

$$a^*(u)\psi_A = 0 = a(v)\psi_A \quad \text{for}\, u \in H_-(A)\, \text{and}\, v \in H_+(A) \tag{1.1}$$

where $H_+(A)$ is the subspace of the one-particle fermionic Hilbert space H spanned by the eigenvectors of the Dirac-Weyl Hamiltonian

$$D_A = i\gamma_k(\nabla_k + A_k) \tag{1.2}$$

belonging to nonnegative eigenvalues and $H_-(A)$ is the orthogonal complement of $H_+(A)$. Here ∇_k's are covariant derivatives in directions given by a (local) orthonormal basis, with respect to a fixed Riemannian metric on M. In the following we shall concentrate to the physically most interesting case $\dim M = 3$ and the γ-matrices can be chosen as the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ with $\sigma_1\sigma_2 = i\sigma_3$ (and similarly for cyclic permutations of the indices) and $\sigma_k^2 = 1$.

The group $\mathcal{G} = Map(M, G)$ of smooth gauge transformations acts on $\mathcal{A}$ as $g \cdot A = gAg^{-1} + dgg^{-1}$. The Fock spaces $\mathcal{F}_A$ form a vector bundle over $\mathcal{A}$.

A natural question is then: How does $\mathcal{G}$ act in the total space $\mathcal{F}$ of the vector bundle? Since the base base $\mathcal{A}$ is flat there obviously is a lift of the action on the base to the total space. However, we have the additional physical requirement that

$$\hat{g}\hat{D}_A\hat{g}^{-1} = \hat{D}_{g\cdot A} \tag{1.3}$$

where $\hat{D}_A$ is the second quantized Hamiltonian and $\hat{g}$ is the lift of g to $\mathcal{F}$. This condition has as a consequence that $\hat{g}\psi_A$ should be equal, up to a phase, to the vacuum $\psi_{g\cdot A}$.

A complication in all space-time dimensions higher than $1+1$ is that the representations of CAR in the different fibers of $\mathcal{F}$ are inequivalent, [1]. The effect of this is that a proper mathematical definition of the infinitesimal generators of $\mathcal{G}$ (current algebra) involves further renormalizations in addition to the normal ordering prescription. In one space dimensions the situation is simple. The current algebra is contained in a Lie algebra $\mathbf{gl}_1$ which by definition consists of all bounded operators X in H satisfying $[\epsilon, X] \in L_2$, where ϵ is the sign operator $\frac{D_0}{|D_0|}$ associated to the free Dirac operator and L_2 is the space of Hilbert-Schmidt operators. In general, we denote by L_p the Schatten ideal of operators T with $|T|^p$ a trace-class operator. Let $a_n^* = a^*(u_n)$, where $D_0 u_n = \lambda_n u_n$ and the eigenvales are indexed such that $\lambda_n \geq 0$ for $n \geq 0$ and $\lambda_n < 0$ for $n < 0$. Denoting the matrix elements of a one-particle operator X by (X_{nm}), the second quantized operator $\hat{X}$ is

$$\hat{X} = \sum X_{nm} : a_n^* a_m : \tag{1.4}$$

where the normal ordering is defined by

$$: a_n^* a_m : = \begin{cases} -a_m a_n^* & \text{if } n = m < 0 \\ a_n^* a_m & \text{otherwise.} \end{cases}$$

The commutation relations are

$$[\hat{X}, \hat{Y}] = \widehat{[X, Y]} + c(X, Y) \tag{1.5}$$

where c is the Lundberg's cocycle, [5],

$$c(X, Y) = \frac{1}{4}\mathrm{tr}\epsilon[\epsilon, X][\epsilon, Y]. \tag{1.6}$$

When X, Y are infinitesimal gauge transformations on a circle the right-hand-side is equal to the central term of an affine Kac-Moody algebra, [13],

$$c(X, Y) = \frac{i}{2\pi}\int_{S^1} \mathrm{tr} X'Y. \tag{1.7}$$

In this talk I want to explain the regularizations needed in $3+1$ space-time dimensions and the generalization of (1.4) through (1.7). In section 4 we shall use the same regularization to define a finite bose-fermi interaction hamiltonian for QCD. (We shall not attack problems associated to vector boson self-interactions.)

2. Action of the group of gauge transformations in the Fock bundle

Let $\epsilon(A) = \frac{D_A}{|D_A|}$; if D_A has zero modes define $\epsilon(A)$ to be +1 in the zero mode subspace. For $A \in \mathcal{A}$ denote by P_A the set of unitary operators $h : H \to H$ such that

$$[\epsilon, h^{-1}\epsilon(A)h] \in L_2. \tag{2.1}$$

If $h \in P_A$ then also $hs \in P_A$ for any $s \in U_1$, where U_1 is the group of unitary operators s with the property $[\epsilon, s] \in L_2$. The spaces P_A form a principal bundle over $\mathcal{A}$ with the structure group U_1.

Since $\mathcal{A}$ is flat the bundle P is trivial and we may choose a section $A \mapsto h_A \in P_A$. Define

$$\omega(g;A) = h_{g\cdot A}^{-1} T(g) h_A. \tag{2.2}$$

where $T(g)$ is the one-particle representation of $g \in \mathcal{G}$. By construction, ω satisfies the 1-cocycle condition

$$\omega(gg';A) = \omega(g;g'\cdot A)\omega(g';A). \tag{2.3}$$

Using $T(g)D_A T(g)^{-1} = D_{g\cdot A}$ we get $T(g)\epsilon(A)T(g)^{-1} = \epsilon(g\cdot A)$ which implies

$$\begin{aligned} h_{g\cdot A}[\epsilon, \omega(g;A)]h_A^{-1} &= (h_{g\cdot A}\epsilon h_{g\cdot A}^{-1})T(g) - T(g)(h_A \epsilon h_A^{-1}) \\ &\equiv \epsilon(g\cdot A)T(g) - T(g)\epsilon(A) \quad mod\, L_2 \\ &= 0. \end{aligned}$$

Since L_2 is an operator ideal this equation implies

$$[\epsilon, \omega(g;A)] \in L_2. \tag{2.4}$$

Thus the 1-cocycle ω takes values in the group U_1.

Remark. In one space dimensions we can set $h_A \equiv 1$ since $[\epsilon, T(g)]$ is already Hilbert-Schmidt. In d space dimensions the off-diagonal blocks of $T(g)$ are only in the Schatten ideal L_p, $p > d$, [11].

The group valued cocycle ω gives rise to a Lie algebra cocycle θ by

$$theta(X;A) = \frac{d}{dt}\omega(e^{tX};A)\mid_{t=0}$$

$$= h_A^{-1} dT(X) h_A + h_A^{-1}\mathcal{L}_X h_A. \tag{2.5}$$

It satisfies the Lie algebra cocycle condition

$$\theta([X,Y];A) - [\theta(X;A), \theta(Y;A)] - \mathcal{L}_X\theta(Y;A) + \mathcal{L}_Y\theta(X;A) = 0, \tag{2.6}$$

where $\mathcal{L}_X$ is the Lie derivative in the direction of the infinitesimal gauge transformation X, $\mathcal{L}_X f(A) = \frac{d}{dt} f(e^{-tX}\cdot A)\mid_{t=0}$. We denote by dT the Lie algebra representation in H corresponding to the representation T of finite gauge transformations. For each $A \in \mathcal{A}$ and $X \in \mathrm{Map}(M, \mathrm{g})$ the operator $\theta(X; A) \in \mathbf{gl}_1$.

The section h_A of P can be used to trivialize the bundle of Fock spaces over $\mathcal{A}$. Each fiber $\mathcal{F}_A$ is identified as the free Fock space $\mathcal{F}_0$. The Hamiltonian D_A is quantized as

$$\hat{D}_A = q(h_A^{-1} D_A h_A), \tag{2.7}$$

that is, we first conjugate the one-particle operator D_A by h_A and then canonically quantize $h_A^{-1} D_A h_A$. The conjugated operator has a Dirac vacuum ψ_A contained in F_0 (but differing from the free vacuum ψ_0). The CAR algebra in the background A is represented in $\mathcal{F}_0$ through the automorphism $a^*(u) \mapsto a_A^*(u) = a^*(h_A^{-1}u)$, $a(u) \mapsto a_A(u) = a(h_A^{-1}u)$ and using the free CAR representation for the operators on the right. The Hamiltonian $\hat{D}_A$ is then

$$\hat{D}_A = \sum \lambda_n(A) : a_A^*(u_n) a_A(u_n) : \tag{2.8}$$

where the u_n's for nonnegative (negative) indices are the eigenvectors of D_A belonging to nonnegative (negative) eigenvalues. The normal ordering is defined with respect to the free vacuum.

Sections of the Fock bundle are now ordinary $\mathcal{F}_0$ valued functions. The effect of an infinitesimal gauge transformation consists of two parts: The Lie derivative $\mathcal{L}_X$ acting on the argument A of the function and an operator acting in $\mathcal{F}_0$,

$$\hat{X} = \mathcal{L}_X + \sum \theta(X; A)_{nm} : a_n^* a_m :, \tag{2.9}$$

where the $\theta(X; A)_{nm}$'s are matrix elements of $\theta(X; A)$ in the eigenvector basis (v_n) of D_0. The commutation relations of the second quantized operators are modified by the Lundberg's cocycle, [9],

$$[\hat{X}, \hat{Y}] = \widehat{[X, Y]} + c(\theta(X; A), \theta(Y; A)). \tag{2.10}$$

In the next section we want to compute the right-hand side of (2.10) more explicitly. We shall denote by $c_n(X, Y; A)$ (n=dim$M$) the second term on the right. It is a Lie algebra 2-cocycle in the following sense:

$$c_n([X, Y], Z; A) + \mathcal{L}_X c_n(Y, Z; A) + \text{ cyclic perm.} = 0.$$

Remark. In the case of massive Dirac fermions the cocycle vanishes in cohomology. Namely, there is a mass gap $[-m, m]$ in the spectrum of the Hamiltonian D_A. For this reason the spaces $H_+(A)$ form a smooth vector bundle over $\mathcal{A}$. Since $\mathcal{A}$ is flat this bundle can be trivialized. It means that one can define a continuous family of operators h_A such that $\epsilon = h_A^{-1} \epsilon(A) h_A$. With this choice it is easy to see that actually $[\epsilon, \omega(g; A)] = 0$ and therefore the cocycle c_n is identically zero. This does not work for massless chiral fermions because there is no mass gap and in fact there is a spectral flow across any point in the spectrum, that is, one can always choose a continuous path in the space $\mathcal{A}$ such that along the path the eigenvalues of D_A crosses any given point in the spectrum.

3. A computation of the cocycle

Let us recall first some basic facts about pseudodifferential operators (PSDO's). A (classical) PSDO P is represented through its *symbol*. The symbol is a smooth function in the cotangent space T^*M which has an *asymptotic expansion* of the form

$$p(x,\xi) = p_k(x,\xi) + p_{k-1}(x,\xi) + p_{k-2}(x,\xi) + \dots \tag{3.1}$$

where n is an integer and the p_j's are functions which are smooth outside of the zero section in T^*M and are homogeneous of degree j in the momentum variables $\xi = (\xi_1, \dots, \xi_n)$,

$$p_j(x,t\xi) = t^j p_j(x,\xi) \text{for } t > 0. \tag{3.2}$$

The degree k of the *principal symbol* p_k is the degree of the PSDO P. We shall consider PSDO's acting on vector valued functions. In that case the symbols are $N \times N$ matrix valued functions. For simplicity we shall consider only the case when the cotangent bundle is trivial; in general, one has to cover T^*M with coordinate charts and the symbol is given by a collection of local symbols in the coordinate charts, with appropriate rules for a change of coordinates in the overlap sets; see [6], III.3 for details.

A PSDO P is a partial differential operator if the symbol p is a polynomial in the coordinates ξ_j. In that case the operator P is simply obtained from p by replacing the coordinates ξ_j by the partial derivatives $-i\partial_j^x$ and inserting the derivatives to the right-hand-side of the coefficient x-space functions.

A PSDO P is defined by its asymptotic expansion up to an *infinitely smoothing* operator. An infinitely smoothing PSDO is an operator with a symbol approaching zero faster than any power $\frac{1}{|\xi|^k}$ as $|\xi| \to \infty$. In particular, an infinitely smoothing operator is trace class. A PSDO on a compact manifold of dimension n is trace class if and only if its degree $k \leq -n-1$. The product of a pair P, Q of PSDO's is represented by the symbol

$$(p * q)(\xi, x) = \sum_m \frac{(-i)^{|m|}}{m!} \partial_\xi^m p \partial_x^m q \tag{3.3}$$

where the sum is over multi-indices $m = (m_1, \dots, m_n) \in \mathbf{N}^n$, $|m| = m_1 + \dots + m_n$, $m! = m_1! \dots m_n!$ and $\partial_x^m = (\frac{\partial}{\partial x_1})^{m_1} \dots (\frac{\partial}{\partial x_n})^{m_n}$. In particular, the principal symbol of the product is just the (matrix) product of the principal symbols of the factors.

In the euclidean case $M = \mathbf{R}^n$ a PSDO P with symbol p acts on sections ψ of a trivial $\mathbf{C}^N$ bundle over M in the following way:

$$(P\psi)(x) = \int p(x,\xi)\hat{\psi}(\xi)e^{ix\cdot\xi}d^n\xi \tag{3.4}$$

where $\hat{\psi}$ is the Fourier transform of ψ,

$$\hat{\psi}(\xi) = \frac{1}{(2\pi)^n}\int \psi(x)e^{-ix\cdot\xi}d^nx. \tag{3.5}$$

The adjoint of P (in the Hilbert space of square-integrable sections, the measure defined by a Riemannian metric on M) is in general a complicated expression in terms of the symbol p. We shall give the formula only in the euclidean case:

$$P^* \sim p^* + \Omega p^* + \frac{1}{2!}\Omega^2 p^* + \dots \tag{3.6}$$

where

$$\Omega = -i\sum_j \partial_j^x \partial_j^\xi$$

and p^* is the matrix adjoint of the matrix valued symbol p.

We shall construct the section h_A explicitly as a function of the vector potential when $\dim M = 3$. We shall define h_A through its symbol, as a pseudodifferential operator in the spin bundle over M. I claim that an operator with the following asymptotic expansion satisfies the requirement (2.1):

$$h_A = 1 - \frac{i}{4}\frac{[\xi, A]}{|\xi|^2} + \text{ terms of lower order in } |\xi|. \tag{3.7}$$

In order to make the discussion as simple as possible we assume that M is the one-point compactification of $\mathbf{R}^3$ and we use standard coordinates in $\mathbf{R}^3$. We also use the notation $A = \sum A_k\sigma_k$.

An example of an unitary operator with the asymptotic expansion (3.1) is the operator

$$h_A = exp\left(\frac{i}{4}(D_0^2 + l)^{-1/2}[D_0, A](D_0^2 + l)^{-1/2}\right) \tag{3.8}$$

where we have added a small positive constant λ to the denominator in order to cancel the infrared singularity at $\xi = 0$; this has an effect in the asymptotic expansion only on terms of order -2 and lower in the momentum ξ. It is clear that the lower order terms do not have any effect on the condition (2.1) since any operator of order ≤ -2 is automatically Hilbert-Schmidt when the dimension of M is 3. Thus we have

$$\theta(X; A) = h_A^{-1}dT(X)h_A + h_A^{-1}\mathcal{L}_X h_A = X + \frac{i}{4}\frac{[\xi, dX]}{|\xi|^2} + O(-2) \tag{3.9}$$

where $O(-p)$ denotes terms of order $\leq -p$. The symbol of the PSDO ϵ is $\frac{\xi}{|\xi|}$ and it is a simple computation to check that indeed $[\epsilon, \theta(X;A)] \in L_2$ using the product rule of symbols.

The term of order -2 in θ is important in computing the actual value of the Schwinger term. It is equal to

$$\theta_{-2} = -\frac{1}{4}\frac{[\sigma_k, A]}{|\xi|^2}\partial_k X + \frac{1}{2}\frac{[\xi, A]}{|\xi|^4}\xi_k\partial_k X$$
$$+\frac{1}{16}\frac{[\xi, A]}{|\xi|^4}[\xi, dX]. \tag{3.10}$$

Note that all terms are linear in the vector potential A. The computation of $c_3(X, Y; A) = c(\theta(X;A), \theta(Y;A))$ is greatly simplified when we keep in mind that it is only the cohomology class of the cocycle c_3 we are interested in. Another simplification is the following: Formally,

$$\frac{1}{4}\mathrm{tr}\epsilon[\epsilon, P][\epsilon, Q] = -\frac{1}{2}\mathrm{tr}[\epsilon, P]Q \tag{3.11}$$

when P, Q are in $\mathbf{gl}_1$. However, the operator on the right is not quite trace-class; only its diagonal blocks are trace-class. For this reason the trace is only conditionally convergent. It is convergent when evaluated with respect to a basis compatible with the polarization $H = H_+ \oplus H_-$, for example, one can choose a basis of eigenvectors of D_0. The trace of an operator P with symbol $p(\xi, x)$ on a n-dimensional manifold is

$$\mathrm{tr}P = (\frac{1}{2\pi})^n \int_{\xi,x} \mathrm{tr}\, p(\xi, x) d^n\xi d^n x \tag{3.12}$$

As an exercise, let us compute (3.11) when $M = S^1$ and P, Q are multiplication operators (infinitesimal gauge transformations). In that case the symbols are just smooth functions of the coordinate x on the circle. Now $\epsilon = \frac{\xi}{|\xi|}$ is a step function on the real line, its derivative is twice the Dirac delta function located at $\xi = 0$. It follows that the symbol of the commutator $\frac{1}{2}[\epsilon, P]$ is

$$(-i)\delta_\xi p'(x) + \frac{(-i)^2}{2!}\delta'_\xi p''(x) + \dots$$

Applying the formula (3.12) to (3.11) we get

$$\frac{1}{4}\mathrm{tr}\,\epsilon[\epsilon, P][\epsilon, Q] = \frac{i}{2\pi}\int_{S^1} \mathrm{tr}\, p'(x) q(x) dx,$$

where the trace under the integral sign is an ordinary matrix trace. If one feels uneasy with singular symbols, one can approximate ϵ by a differentiable function $\frac{\xi}{|\xi|+\lambda}$ and at the very end let $\lambda \mapsto 0$.

In the 3-dimensional case we have to insert $P = \theta(X;A), Q = \theta(Y;A)$ in (3.11). Using the asymptotic expansions for P and Q, $p = \sum p_{-k}(\xi, x)$ one has

$$c_3(X,Y;A) = \sum_k \mathrm{tr}\left(\frac{\xi}{|\xi|} * p * q - p * \frac{\xi}{|\xi|} * q\right)_k \tag{3.13}$$

In fact, one needs to take into account only finite number of terms. The sum of terms with $k \leq -4$ is a coboundary of the 1-cochain

$$\sum_{k \geq 4} \mathrm{tr}\,(\epsilon * \theta(X;A))_{-k} \tag{3.14}$$

Thus we may restrict the sum in (3.13) to indices $k > -4$, so we have only a finite number of terms to check. To take care of the infrared singularity in the integration in (3.12) we replace all denominators $|\xi|^{-k}$ by $(|\xi| + \lambda)^{-k}$. One can then check by a direct computation that, modulo coboundaries, the result of the computation in (3.13) does not depend on the value of λ (i.e., one may take the limit $\lambda \mapsto 0$ in cohomology). The final result is in accordance with the cohomological [7], [10], [14] and perturbative arguments, [4],

$$c_3(X,Y;A) = \frac{1}{24\pi^2}\int_M \mathrm{tr} A[dX, dY]. \tag{3.15}$$

What we have constructed here is an action of a Lie algebra $\hat{\mathbf{g}}$, which is an extension of Map$(M, \mathbf{g})$ by the abelian Lie algebra of complex valued functions on $\mathcal{A}$, in the space of smooth fucntions $\mathcal{A} \to \mathcal{F}$. The operators acting in this space are the generators $\mathcal{L}_X + \theta(X;A)$. We have not really constructed a unitary representation of $\hat{\mathbf{g}}$ because we do not have a quasi-invariant measure in the space $\mathcal{A}$ of smooth vector potentials. However, our construction can be viewed as a true old fashioned representation of a different extension $\tilde{\mathbf{g}}$ of Map$(M, \mathbf{g})$.

Consider the Lie algebra of all pseudodifferential operators of the form

$$\theta = X + \frac{i}{4}\frac{[\xi, dX]}{|\xi|^2} + \zeta, \tag{3.16}$$

where X is an infinitesimal gauge transformation (multiplication operator by $X \in$ Map$(M, \mathbf{g})$) and ζ is an arbitrary PSDO of degree -2 acting in H. As we saw above, the commutator $[\epsilon, \theta]_*$ is Hilbert-Schmidt and so θ is canonically quantizable. It is easely seen that the commutator of two operators of the form (3.16) is again of the same type and therefore they indeed close a Lie algebra. This Lie algebra $\mathbf{g}'$ is an extension of Map$(M, \mathbf{g})$ by the Lie algebra $\mathcal{P}_{-2}$ of PSDO's of degree -2 acting in H,

$$0 \longrightarrow \mathcal{P}_{-2} \longrightarrow \mathbf{g}' \longrightarrow \mathrm{Map}(M, \mathbf{g}) \longrightarrow 0,$$

defined by the natural inclusion $\mathcal{P}_{-2} \to \mathbf{g}'$ and by the projection $\theta \mapsto X$.

In the second quantization the Lie algebra $\mathbf{g}'$ is centrally extended to a Lie algebra $\tilde{\mathbf{g}}$ because of the Lundberg's cocycle. Thus the algebra of second quantized operators θ can be vieved as a Hilbert space representation of $\tilde{\mathbf{g}}$. It would be an interesting task to pursue in greater generality the representation theory of $\tilde{\mathbf{g}}$.

4. The interaction hamiltonian

Up to this we have discussed the regularization of the time component j_0 (= charge density) of the nonabelian gauge current. However, in renormalized perturbation theory one needs also the space components

$$j_k^a(x) =: \overline{\psi}(x)\gamma_k T^a \psi(x) : \tag{4.1}$$

where the T^a's are generators of $\mathbf{g}$. This is because the interaction Hamiltonian contains the term

$$H_I = \int A_k^a(x) j_k^a(x) d^3x. \tag{4.2}$$

Actually, in the abelian case the hamiltonian is the free quadratic Dirac & Maxwell hamiltonian + the interaction H_I. Thus in the abelian case it is sufficient to renormalize H_I such that it becomes a well-defined operator in the Fock space of fermions and photons.

In this section I shall explain only the renormalizations needed to make H_I well-defined in the background quantization.

The aim is achieved through a sharpening of the regularization used for the time component. We want to define an operator valued function h_A such that

$$h_A^{-1} D_A h_A = D_0 + W_A \tag{4.3}$$

where W_A is a PSDO of degree 0 with the additional property that

$$[\epsilon, W_A] \in L_2. \tag{4.4}$$

The condition (4.4) guarantees that the matrix elements

$$< \phi | W_{A^1} \dots W_{A^n} | 0 > \tag{4.5}$$

are finite, when ϕ is a state in the fermionic Fock space containing a finite number of particles. Here $A^1 \dots A^n$ are any given values for the external gauge field (smooth and with appropriate vanishing conditions at spatial infinity when the physical space is noncompact). But the finiteness of the matrix elements (4.5) is precisely what is needed in the perturbation expansion, based on the Dyson expansion of the time evolution operator; see any standard quantum field theory text book, e.g. [2].

The choice of h_A in the previous section is not quite sharp enough to achieve (4.4). A correct modified expression is the following:

$$h_A = 1 - \frac{i}{4|\xi|^2}[\xi, A] - \frac{1}{32|\xi|^4}[\xi, A]^2 - \frac{1}{8}\left[\frac{\sigma_k}{|\xi|^2} - 2\frac{\xi\xi_k}{|\xi|^4}, \partial_k A\right]$$

$$-\frac{1}{8|\xi|^4}[\xi, A](A\cdot\xi) - \frac{1}{8|\xi|^4}(A\cdot\xi)[\xi, A] + O(-3) \qquad (4.6)$$

After a tedious computation we obtain

$$W_A = h_A^*(\xi + iA)h_A - \xi = \frac{i\xi}{|\xi|^2}A\cdot\xi$$

$$-\frac{1}{8}\left[\xi, [\frac{\sigma_k}{|\xi|^2} - 2\frac{\xi\xi_k}{|\xi|^4}, \partial_k A]\right] - \frac{\sigma_k}{4|\xi|^2}[\xi, \partial_k A]$$

$$+\frac{i}{2|\xi|^2}\epsilon_{ljk}\xi_j[A_l, A_k] - \frac{\xi}{|\xi|^2}A_m A_m + \frac{\xi}{|\xi|^4}(A\cdot\xi)^2 + O(-2). \qquad (4.7)$$

It is then a simple computation to show that $[\epsilon, W_A]$ is of degree -2. There is no magic in the derivation of the formula (4.6) for h_A. It is a simple recursive procedure. Writing

$$h_A = 1 + h_{-1} + h_{-2} + \ldots$$

in the asymptotic expansion, one gets

$$h_A^*(\xi + \alpha_0 + \alpha_{-1} + \ldots)h_A = \xi + \alpha_0' + \alpha_{-1}' + \ldots,$$

where

$$\alpha_0' = \alpha_0 + [\xi, h_{-1}]$$

$$\alpha_{-1}' = \alpha_{-1} + [\alpha_0, h_{-1}] + \xi h_{-2} + (h_A^*)_{-2}\xi - i\sigma_k\partial_k h_{-1}. \qquad (4.8)$$

The condition (4.4) is equivalent to the pair of equations

$$[\epsilon, \alpha_0'] = 0 \quad and\, [\epsilon, \alpha_{-1}'] - i(\partial_k^{(\xi)}\epsilon)(\partial_k^{(x)}\alpha_0') = 0$$

which together with (4.8) gives a set of linear equations for h_{-1} and h_{-2}. One can then determine the lower order terms h_k, $k < -2$, from the unitarity condition for h. This is again a set of recursive linear relations obtained from the formula (3.6) for the adjoint of a PSDO.

5. Vector fields and second quantization

One parameter groups of diffeomorphisms on a manifold M are generated by smooth vector fields. For this reason, although the exponential mapping is not quite onto, the algebra of vector fields (with respect to the Lie bracket) can be considered as the Lie algebra of the diffeomorphism group $Diff(M)$. Instead of considering the action of the finite group transformations in the Fock space formalism we shall restrict here to the action of (regularized) vector fields.

It is easy to see that the commutator $[\epsilon, f]$ is not even compact when $f = f_k \partial_k$ is a vector field on a manifold with $\dim M > 1$. It is only in dimension 1 that the commutator is actually Hilbert-Schmidt. We shall show that for any f there is a PSDO $F = F(f)$ with the following properties:
(1) $F(f) - f$ is of order zero
(2) $[\epsilon, F(f)]$ is Hilbert-Schmidt
(3) for any vector fields f, g also $[F(f), F(g)]$ has the properties (1), (2).
The Lie algebra of operators in the one-particle space generated by the $F(f)$'s is called the algebra of regularized vector fields, to be denoted by $D_{reg}(M)$.

We shall define $F(f)$ through its expansion $F(f) = f + \theta_0 + \theta_{-1} + \ldots$ where θ_n is homogeneous and of degree n. Note first that the first order term in the commutator $F(f)*F(g) - F(g)*F(f)$ is just the ordinary commutator $[f, g]$ of vector fields, so the algebra $D_{reg}(M)$ is an extension of $D(M)$ by the algebra $\mathbf{gl}_{res}$ of zeroth order operators ω such that $[\epsilon, \omega]$ is Hilbert-Schmidt.

In order to find θ_0 and θ_{-1} (for the regularization, we are not interested in the lower order terms because they are Hilbert-Schmidt) we first compute $[\epsilon, F(f)]_0$,

$$(\epsilon * F(f) - F(f) * \epsilon)_0 = -i \frac{\partial \epsilon}{\partial \xi_k} \frac{\partial f_j}{\partial x_k} \xi_j + [\epsilon, \theta_0] \equiv \eta_0 + [\epsilon, \theta_0].$$

where the last commutator on the right is just the matrix commutator involving the Pauli matrices.

Since ϵ is a unit spinor, its derivative in any direction in momentum space is orthogonal with respect to $\xi = \xi_k \sigma_k$. Consequently the matrix η_0 can be written as a commutator in spin space, $\eta_0 = -[\epsilon, \theta_0]$ for some symbol θ_0 of order zero. Actually, by the algebra of Pauli matrices, we get a solution

$$\theta_0 = -\frac{1}{4}[\epsilon, \eta_0].$$

Having made this choice, we compute the next term in the starcommutator,

$$(\epsilon * F(f) - F(f) * \epsilon)_{-1} = [\epsilon, \theta_{-1}] - i \frac{\partial \epsilon}{\partial \xi_k} \frac{\partial \theta_0}{\partial x_k}$$

$$- \sum_{k \neq j} \frac{\partial^2 \epsilon}{\partial \xi_j \partial \xi_k} \frac{\partial^2 f_m}{\partial x_k \partial x_j} \xi_m - \frac{1}{2} \sum_k \frac{\partial^2 \epsilon}{\partial \xi_k^2} \frac{\partial^2 f_m}{\partial x_k^2} \xi_m$$
$$\equiv [\epsilon, \theta_{-1}] + \eta_{-1}.$$

One can check by a completely straight-forward algebra that η_{-1} has vanishing trace and also $\mathrm{tr}\epsilon\eta_{-1} = 0$. From this follows that η_{-1} can be written as a commutator of some θ_{-1} with ϵ. We can choose $\theta_{-1} = -\frac{1}{4}[\epsilon, \eta_{-1}]$.

With these choices $[\epsilon, F(f)]_*$ is of order -2 and thus Hilbert-Schmidt. In an analogous manner as in section 3, in the case of infinitesimal gauge transformations, we can view $D_{reg}(M)$ as an extension of the Lie algebra $D(M)$ of smooth vector fields by the nonabelian Lie algebra $\mathbf{gl}_{res}$

$$0 \longrightarrow \mathbf{gl}_{res} \longrightarrow D_{reg}(M) \longrightarrow D(M) \longrightarrow 0$$

with the obvious maps. This Lie algebra is then modified in the second quantization by the Lundberg's cocycle.

We have all the time assumed that M is compact in order to avoid discussing infrared divergencies. In the case of a noncompact physical space (like $\mathbb{R}^3$) one has to choose suitable boundary conditions for the Dirac and gauge fields and vector fields in order to preserve the Hilbert-Schmidt property of the commutator with the sign operator ϵ.

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CHAPTER IV

CLASSICAL FIELD THEORY AND SPACE-TIME

MATTER AND GEOMETRY IN A UNIFIED THEORY

LEOPOLD HALPERN
Florida State University, Department of Physics, Tallahassee, Florida 32306

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The conditions which are imposed by mathematical axioms can in general only within limits be fulfilled by physical objects. The integers which occur in arithmetics may still rather well be in harmony with atomistic physics. Points, lines, planes, etc., defined by the continuum in geometry obey however definite relations which can at best in crude approximation be identified with measurable physical systems. This is apparent from the one to one mapping of sets of the continuum on subsets. One can expect from the foregoing that any good and therefore clear physical theory involving a continuum will lead eventually to extreme results where physics can no longer do justice to the axioms so that no reasonable person can believe in the absurdity of its predictions. Riemann had already recognized the problem of the continuum in the complementarity of geometry and physics for the description of nature. He devoted a section of his habilitation work to a discrete description. One can not expect that in the sophisticated spacetime continuum of the general theory of relativity the consequences of Riemann's critique of such sharply contoured geometric constructions as points, lines – and even the light cone, will not come to the light when describing extreme physical situations. This is already known in microscopic physics where the uncertainty relation rules out the identification of points with physical objects. The persistence of the curse of the thirteenth fairy – (with which Schrödinger poetically compares the continuum because it proceeded the birth of our science) – results strangely enough from macroscopic physics. The Einstein-Hilbert equations of general relativity predict inevitably the gravitational collapse of a sufficiently large cloud of dust to a point, irrespective of the nature of the short range interaction between the dust particles. The point of view that this extreme result is a manifestation of the predicted absurdity and has not the character of a physical law, is not shared today by many physicists. Einstein himself and also Schrödinger did however not advocate the last mentioned trend. This is witnessed by the article [1] which introduces a modified interpretation of the field equations to abandon the domain beyond the horizon. One sees in the apparent inevitability of gravitational collapse rather one of the greatest revolutions in our physical world picture. The argument for this collapse is based on the fact that the curvature near the formation of a horizon can remain small. The principle of equivalence seems then to rule out any reason why physics should be different in this domain than in others so that a given solution of the Einstein-Hilbert field equations does apply everywhere.

The author's counter-argument is based on macroscopic quantum effects induced by the curvature. The earliest discovered of these effects is Schrödinger's alarming phenomenon of elementary particle pairs created in the time dependent metric of an expanding universe [2]. Associated with it are the contributions of virtual elementary particle pair effects, which became known as the gravitational analogues of the Uehling term of the Lamb shift [3] and of the Casimir effect [4]. There exist other more complicated contributions of quantized fields in classical gravitational fields, even in low order approximations. The terms due to virtual particle contributions are in general divergent and non-renormalizable. Every quantum field contributes to an additional source term of the gravitational field equations. The gravitational field itself has also to be considered – but we can hardly do more than speculate about the microscopic manifestations of the gravitational field. Solutions of the classical Einstein-Hilbert equations can not account for the appearance of such source terms.

We summarize the conclusions we draw from the foregoing considerations:

1. The gravitational collapse of dust to a geometrical point predicted by classical general relativity is tentatively considered as an absurdity of the kind discussed.
2. Our knowledge about quantum effects in classical gravitational fields excludes a *rigorous* macroscopic description of extreme situations in terms of the Einstein-Hilbert equations alone. These cannot produce the Schrödinger phenomenon and thus also not its virtual manifestations which ought to be considered *before* conclusions about horizon formation are drawn.
3. We lack empirical knowledge about the structure of the gravitational interaction in microscopic regions and lack adequate knowledge of particle – and field theory to even estimate the magnitude of the mentioned macroscopic quantum effects in classical gravitational fields.
4. We have no direct observational criterium to distinguish a highly collapsed system from a true black hole with a horizon.
5. To avoid the fallacies cited under point 3 we search for modified macroscopic equations which are hoped to include the macroscopic quantum effects in average and tend to eliminate the absurdity. These equations must be of higher order than the second and must give a good approximation to general relativity in less extreme situations. The coupling of matter with gravitation should contain nonminimal terms to produce

the Schrödinger phenomenon.
What about the principle of equivalence, Einstein's ingenious bridge between physics and geometry? Einstein and Rosen [1] conclude with its help that something different than the vacuum must be found (latest) at the horizon – if the rest of physics is to remain valid. They postulate there a source in accordance with Einsteins geometrization program. On the other hand the ultrarelativists – those who for one or the other reason follow an orthodox course without much consideration about results of quantum and particle physics – they conclude from the principle of equivalence that space-time has to be extended unaltered beyond the horizon until it ends in a singularity. Their use of the prescribed mathematics is certainly correct – but they risk to fall just because of this onto the mentioned absurdity. The Schrödinger phenomenon demands modifications of the classical equations already before and outside the horizon.

Equations with an admixture of fourth order terms, derivable from a Lagrangian of the form:

$$\mathcal{L} = \sqrt{g}\,(R + aR^2 + bR_{hijk}\,R^{hijk}) \tag{1}$$

a,b constants of dimension (length) have early been considered [2]. Their vacuum solutions include all of those of general relativity. Other physically significant vacuum solutions are not known. The presence of matter requires here solutions different from general relativity but none are known either. Other lower order effects of quantum field theory are even more difficult to incorporate into classical equations.

The search for modified equations need not to be restricted to the perturbation formalism of quantum field theory. The approach from a gauge principle and in particular from Kałuża-Klein models appears promising. The latter achieve a quasi-unification of general relativity in interaction with a gauge field of vanishing rest mass in a (somewhat mutilated) metrical space of $4+n$ dimensions. The Schrödinger phenomenon, of particular interest for a massless gauge field, does not appear in the classical theory. A nonmininal interaction is required to obtain it classically. There are too many possibilities to arrive at such equations. We shall follow one way led by an early attempt of the author to describe the inner quantum number of spin by a higher dimensional Kałuża-Klein generalization. The gauge group is in the simplest case that of the tetrade rotations. The theory has the unique feature of convertability of the inner quantum number (spin) into a dynamical variable (angular momentum) [5,6].

The theory is formulated on the ten-dimensional manifold of the anti-De Sitter group $G = SO(3,2)$. The subgroup $H = SO(3,1)$ is the gauge group. The principal fibre bundle $P(G,\ H,\ G/H,\ \pi)$ has the anti-De Sitter universe with the topology of G/H as base manifold and the natural projection π : $G \to G/H$. The Cartan-Killing metric γ of every semi-simple Lie group G_r,

$$\gamma_{uv} = \mathrm{Tr}\{(\mathrm{Ad}A_u) \circ (\mathrm{Ad}A_v)\}, \tag{2}$$

with A_u, A_v left invariant vectors of G_r, fulfills Einstein equations,

$$R_{uv} - \tfrac{1}{2}\gamma_{uv}\left(R - \tfrac{r-2}{4}\right) = 0. \tag{3}$$

A metric $g = \pi' \gamma$ is then defined on the base. It is in this case the anti-De Sitter metric which fulfills:

$$B_{ik} - \tfrac{1}{2}g_{ik}(B+1) = 0 \tag{4}$$

with B_{ik} the Ricci tensor of the base space. The left invariant vectors A_R ($R = 1 \ldots 10$) are Killing vectors of γ. We shall label henceforth indices pertaining to the base space by letters $A \ldots L$ running from $1 \ldots 4$ and those pertaining to the fibre by letters $M \ldots Q$ running from $5 \ldots 10$. General indices $R \ldots Z$ run from $1 \ldots 10$. This rule will be applied without further warning also to the Einstein summation convention.

We consider more general metrics γ which are solutions of the Einstein equations (3) and keep the six Killing's vectors with unaltered commutation relations on each fibre,

$$[A_P, A_Q] = c^M_{PQ}\, A_M. \tag{5}$$

The structure of the principal fibre bundle P and of the subgroup H on the fibres thus still exists. In the space perpendicular to the A_M there exist four orthonormal vector fields A_E with the unaltered commutation relations of the group G:

$$[A_E, A_M] = c^H_{EM}\, A_H \tag{6}$$

only the commutation relations:

$$[A_E, A_F] = C^R_{EF}(x)\, A_R \tag{7}$$

are modified to base point dependent general structure constants.

The metric γ defines a connection on P with horizontal vectors A_E perpendicular to the fibre. The generalized structure constants C^M_{EH}, C^J_{EH} determine respectively curvature and torsion two forms over the base. The topology of the base remains that of the anti- De Sitter universe, but the metric $g = \pi'\gamma$ is now generalized.

The construction constitutes a generalized classical Kałuża-Klein theory with a gauge field F^M which is determined by the C^M_{EH}. The geometry on the base is non-Riemannian. The torsion two-form is in general not vanishing. The gauge group H is a pseudo orthogonal subgroup of $GL(4, \mathbb{R})$ which allows the decomposition of the connection into a Riemannian part and contortion,

$$\Gamma^H{}_{EF} = \left\{ \begin{matrix} H \\ EF \end{matrix} \right\} - K^H_{EF}, \tag{8}$$

$$K^H_{EF} = \frac{1}{2}\left(T^H_{EF} + T_{FE}{}^H + T_{EF}{}^H\right) \tag{9}$$

with T the torsion tensor.

The components of the curvature tensor F of the two form F^M can likewise be decomposed:

$$F_{AEIJ} = B_{AEIJ} + Q_{AEIJ}, \tag{10}$$

$$Q_{AEIJ} = K_{AEI;J} - K_{AEJ;I} + K_{AHI}\, K^H_{EJ} - K_{AHJ}\, K^H_{EI} \tag{11}$$

with the Riemann tensor B and contortion K. The semicolon denoting the Riemannian covariant derivative.

Such a decomposition cannot be achieved with the full $GL(4,\mathbb{R})$ as gauge group. The assumptions about Riemannian curvature found in the literature [7] in connection with this gauge group can thus in general not be right. See ref. 6.

The purely vertical component of the ten-dimensional equation (3) is eliminated with Lagrange multipliers to restrict only to such solutions for which the natural metric on the fibres is preserved and the Planck length (in units with $\hbar = c = 1$ the square root of the gravitational constant G) is introduced on the base manifold as physical unit of length instead of that of the radius of the universe. The theory cannot yield a relationship between these two lengths without altering the topology of the manifolds.

The mixed horizontal-vertical components of equation (3) are

$$F^A{}_{HI}{}^J{}_{;J} = 0 \tag{12}$$

this becomes if torsion vanishes

$$B^A{}_{HI}{}^J{}_{;J} = 0 \tag{13}$$

and due to the Bianchi identities:

$$B^A{}_{H;I} - B^A{}_{I;H} = 0 \tag{14}$$

related by Yang to a gauge theory of $GL(4,\mathbb{R})$ [7]. The absence of torsion which can in this case not be separated, is not accounted for in Yang's paper and equation (14) alone also admit unphysical solutions. Yet the term (12) is the Riemannian analog of Maxwell's equations. It is supplemented in equation (11) by a source formed out of torsion and by the purely horizontal components of equation (3),

$$B^I_E - \tfrac{3}{2}GF_{AHDE}\, F^{AHDI} - \tfrac{1}{2}\delta^I_E \left(B - \tfrac{3}{4}GF_{AHDJ}\, F^{AHDJ} + 1\right) = 0 \quad (15)$$

We are inclined to relate the torsion term of equation (12) to a nonminimal interaction of torsion with elementary particle spin. Equation (13) admits all vacuum solutions of general relativity. Equation (14) consists of the Einstein's term with cosmological member and the energy-momentum tensor of the Yang-Mills field, which can be decomposed again into metric curvature and torsion; it is of vanishing trace. Vanishing torsion leaves this term bilinear in the metric curvature – apparently an additional vacuum energy of virtual matter fields which remains small with the curvature. The real field part is bilinear in Q and the term linear in B and Q constitutes the nonminimal interaction which can give rise to particle creation by gravitation, the Schrödinger phenomenon, of which even the virtual part appears. Einstein's request for the geometric expression of the matter tensor is fulfilled – yet as its vanishing trace shows, the model describes only very special matter. The spherically symmetric vacuum solution of general realtivity satisfies also equations (12,13,14) but other solutions of Einstein's theory in general do not, due to the nonlinear term.

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GENERAL PROPERTIES OF CLASSICAL W ALGEBRAS

F. DELDUC*, L. FRAPPAT**, E. RAGOUCY*** and PAUL SORBA‡

Laboratoire de Physique Théorique ENSLAPP
URA 14-36 du CNRS, associée à l'E.N.S. de Lyon
et au L.A.P.P. (IN2P3-CNRS) d'Annecy-le-Vieux

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Abstract. In the first part we review the construction and classification of classical W (super)algebras symmetries of Toda theories. The second part deals with recently obtained properties. We show that chains of W algebras can be obtained by imposing constraints on some W generators. We call secondary reduction such a gauge procedure on W algebras. Then we emphasize the role of the Kac-Moody part, when it exists, in a W (super) algebra. Factorizing out this spin 1 subalgebra gives rise to a new W structure which we interpret either as a rational finitely generated W algebra, or as a polynomial non linear W_∞ realization.

1. Introduction

W algebras constitute today a rather broad subject: on the one hand they play a role in different parts of 2 dimensional Conformal Field Theories (CFT), on the other hand much has still to be done for a complete knowledge of these algebras and their algebraic properties. First it was thought that they can be used to facilitate the analysis of rational CFT (i.e. theories in which the main parameters, namely central charge c and conformal dimensions h_i are all rational numbers): this extra symmetry, bigger than the conformal one, could help to characterize degeneracies, and to classify in a simpler way the physical states. After that it was realized that they show up in several places. We currently talk nowadays about W gravity. W algebras appear in the quantum Hall effect, black holes models, in lattice models of statistical mechanics at criticality, and in Toda models (Leznov and Saveliev 1989) as symmmetry algebras (Feher, O'Raifeartaigh, Ruelle, Tsutsui and Wipf 1992).

After some definitions (Section 2), we will concentrate on classical W algebras and superalgebras which are finitely generated -we generically denote

* Groupe de Lyon, ENS Lyon, 46 Allée d'Italie, F-69364 Lyon Cedex 07, France.
** Groupe d'Annecy, LAPP, BP 110, F - 74941 Annecy-le-Vieux Cedex, France.
*** Groupe d'Annecy.
‡ Groupe de Lyon and groupe d'Annecy.

them W_n-. Two remarkable facts can then be mentioned (Section 3):

-i) The constants of motion of a Toda theory form a W_n algebra, and such a Toda theory can be seen as a gauged WZW model, on which constraints have been imposed (Feher, O'Raifeartaigh, Ruelle, Tsutsui and Wipf 1992).

-ii) As a consequence, one can explicitly construct such W_n algebras, and give a group theoretical classification of them (Frappat, Ragoucy and Sorba CMP 1993).

Two comments:

- this classification is based on the $Sl(2)$ embeddings in a simple Lie (super)algebra $\mathcal{G}$ and on the $OSp(1|2)$ embeddings in a simple superalgebra $\mathcal{SG}$. We will try to insist on the property of $Sl(2)$ to be intimately linked to a W_n algebra from its definition: this is important for our construction, but also allows to think that the classification of W_n algebras symmetries of Toda models hereafter given is "not far" from exhausting the set of W_n algebras.

- there are two main types of W_n algebras: those that we will call the Abelian ones because they are related to Abelian Toda models: for example, if the underlying group of the Toda model is $Sl(n)$, one gets the algebra generated by $W_2, W_3, ...W_n$.

There is a second type of W_n algebra, less well-known: they are associated to non Abelian Toda models (Leznov and Saveliev 1989), and we call them non Abelian W_n algebras, and we will come back to this class of algebras.

The above classification can be simplified using two interesting features, directly suggested by properties of simple Lie algebras and superalgebras, namely:

- deduction of W_n algebras related to non simply laced algebras B_n, C_n... from W_n algebras related to A_n series by "foldings" (Frappat, Ragoucy and Sorba NP 1993) analogous to the folding technics which produce B_n, C_n... algebras from A_n ones (Section 4).

- existence of chains of W_n algebras mimicking chains of embeddings of subalgebras in a simple Lie Algebra (Delduc, Frappat, Ragoucy and Sorba 1994). Imposing constraints, when possible, on a the W algebra itself, one can reduce W into another algebra W: we will call this technics a secondary reduction (Section 5).

Finally coming back to the non Abelian W_n algebras, one can remark that most of them contain a Kac Moody part. Such a Kac Moody subalgebra should play a particular role. In particular, we will see that factorizing out this "spin one" part in the W_n algebra gives rise to an algebra which can be seen either as an W_∞ algebra, that is an infinitely generated W algebra, or as a finitely generated W algebra but of a new type; we will call it "rational" W_n algebras (Delduc, Frappat, Ragoucy, Sorba and Toppan 1993). This problem as well as its supersymmetric generalisation is the subject of Section 6. which ends up by a comparative study of the factorizations of spin 1/2 fermions and spin 1 bosons in a W algebra.

We have chosen to illustrate each property which is introduced on an example instead of presenting general proofs. We hope that this approach will make the reading as easy for the non experts as for those familiar with W algebras, these last ones being invited to directly go to the three last sections.

2. Definitions

We know from $d = 2$ CFT that the stress energy tensor has a short-distance O.P.E. of the form, with z, w complex variables:

$$T(z).T(w) = \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)^2} + \frac{c/2}{(z-w)^4} + \dots \tag{2.1}$$

Expressing $T(z)$ into Laurent modes

$$T(z) = \sum_{m \in Z} z^{-m-2} L_m, \qquad L_m = \oint \frac{dz}{2i\pi z}\, z^{m+2} T(z) \tag{2.2}$$

the integral being understood around the origin clockwise, we have the C.R. of the Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}. \tag{2.3}$$

Note that $\{L_{+1}, L_{-1}, L_0\}$ generate an $Sl(2,R)$ algebra, while c is the central charge.

In a CFT, primary fields are those which transform as tensors of weight $(h, \bar{h})$ under conformal transformations:

$$z \to w(z), \quad \bar{z} \to \bar{w}(\bar{z})$$

$$\phi'_{h,\bar{h}}(z,\bar{z}) = \phi_{h,\bar{h}}\left(w(z), \bar{w}(\bar{z})\right) \left(\frac{dw}{dz}\right)^h \left(\frac{d\bar{w}}{d\bar{z}}\right)^{\bar{h}}. \tag{2.4}$$

$T(z)$ being the generator of local scale transformations, one gets the O.P.E., after restricting to the z-part:

$$T(z).\phi_h(w) = \frac{h\phi_h(w)}{(z-w)^2} + \frac{\partial\phi_h(w)}{(z-w)} + \dots \tag{2.5}$$

h is called the conformal spin of the primary field $\phi_h(z)$. One can deduce from eq. (2.5) the CR

$$[L_m, \phi_h(z)] = (m+1)hz^m\phi_h(z) + z^{m+1}\partial\phi_h(z). \tag{2.6}$$

Now let us add to the Virasoro algebra some primary fields. With some precautions, we can obtain a W algebra.

As an example, let us consider the $N = 1$ superconformal algebra: it is made from the (conformal spin 2) stress energy tensor $T(z)$ and a conformal spin 3/2 fermionic field $G(z)$. Developing $T(z)$ and $G(z)$ in Laurent modes:

$$G(z) = \sum z^{-3/2-r}\, G_r \tag{2.7}$$

with $r \in \mathbf{Z}$ or $r \in \mathbf{Z} + \frac{1}{2}$ following we are in the Ramond or Neveu-Schwarz sector, we get the (anti) C.R.:

$$\begin{aligned} [L_m, G_r] &= (\frac{1}{2}m - r)G_{m+r} \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{c}{3}(r^2 - 1/4)\delta_{r+s,0} \end{aligned} \tag{2.8}$$

We have a W (super)algebra. It is specially simple since it closes linearly on the generators L_m and G_r. Let us add two remarks which will be relevant for the future.

First $\left\{L_{+1}, L_{-1}, L_0, G_{+1/2}, G_{-1/2}\right\}$ generate the $OSp(1|2)$ superalgebra, that is the "supersymmetric" $Sl(2)$ extension. In the following $OSp(1|2)$ will play for W_n superalgebras the role of $Sl(2,R)$ for W_n algebras.

Secondly $\{G_{\pm 1/2}\}$ constitutes a spin 1/2 representation of the algebra $\{L_{\pm 1}, L_0\}$. More generally (Bowcock and Watts 1992) if $W_h(z)$ is a h primary field under $T(z)$ the modes W_n with $-h+1 \le n \le h-1$ will form a spin $(h-1)$ representation of $\{L_{\pm 1}, L_0\}$.

The above definitions and properties stand for the above OPE to be radially ordered. We will relax this last feature in the following and restrict ourselves to the classical case.

Then a classical finitely generated W_n algebra will be defined as a Lie algebra with a Poisson bracket $\{,\}_{P.B.}$, and a set of generators involving a stress-energy tensor T as well as a finite number of primary fields $W_{h_i}(i = 1, \ldots n-1)$ under T satisfying:

$$\begin{aligned} \{T(z), T(w)\}_{P.B.} &= -2T(w)\delta'(z-w) + \partial T(w)\delta(z-w) \\ &\quad + \frac{c}{2}\delta'''(z-w) \end{aligned} \tag{2.9}$$

$$\{T(z), W_{h_i}(w)\}_{P.B.} = -h_i W_{h_i}(w)\delta'(z-w) + \partial W_{h_i}(w)\delta(z-w) \tag{2.10}$$

$$\{W_{h_i}(z), W_{h_j}(w)\} = \sum_\alpha P_{i,j;\alpha}(w)\delta^{(\alpha)}(z-w) \tag{2.11}$$

where $P_{i,j;\alpha}(w)$ are polynomials in the primary fields W_{h_i}, T and their derivatives.

Let us remark that the property of a primary field W_h of conformal spin h to be connected to the representation D_{h-1} of the $Sl(2,R)$ algebra $\{L_\pm, L_0\}$ limitates through the tensorial product $D_{h_i-1} \times D_{h_j-1}$ the allowed conformal spin of the $P_{i,j;\alpha}$ polynomials.

3. From a WZW model to a Toda theory

3.1. The method

It has been elegantly shown that, starting from a WZW model, the action of which is $S(g)$ and the fields $g(x)$ belong to the group G, and imposing some of the components of the conserved currents to be constant or zero leads to a Toda model (Feher, O'Raifeartaigh, Ruelle, Tsutsui and Wipf 1992).

Let us denote $S_{WZW}(g)$ the action of the WZW model based on a real connected Lie group G, and $g \in G$. Then from the Kac-Moody invariance $G_1 \times G_2$ with $G_1 \cong G_2 \cong G$ of the model

$$g(x) \to g_1(x^-)g(x)g_2(x^+) \tag{3.1}$$

with $x = (x^+, x^-)$ denoting the two-dimensional variable, we get the currents:

$$J_+ = g^{-1}\partial_+ g \quad \text{and} \quad J_- = \partial_- g g^{-1} \tag{3.2}$$

which, due to the equations of motion, are conserved:

$$\partial_\pm J_\mp = 0 \tag{3.3}$$

In order to perform the gauge theory approach which will be relevant, we need G to be non compact: let us consider as an example the $Sl(n, R)$ group. We decompose its Lie algebra $\mathcal{G}$ as follows:

$$\mathcal{G} = \mathcal{G}_- \oplus \mathcal{H} \oplus \mathcal{G}_+ \tag{3.4}$$

where $\mathcal{G}_+(\mathcal{G}_-)$ is the subalgebra of positive (negative) root generators and $\mathcal{H}$ the Cartan part, i.e.:

$$\begin{pmatrix} * & & & \\ & \ddots & \mathcal{G}_+ & \\ & \mathcal{G}_- & \ddots & \\ & & & * \end{pmatrix} \tag{3.5}$$

Note that the generators $E_{\alpha_i}(i = 1...n-1)$ associated to the (positive) simple roots are in the positions $E_{12}, E_{23}, ...E_{n-1,n}$ in the above matrix, while $E_{-\alpha_i}$ occupy the position $E_{21}, ..., E_{n,n-1}$ (E_{ij} being the $n \times n$ matrix with 1 in position (i,j) only).

The basic idea is to impose constraints on some components of these $J_\pm$ currents. Let us impose the restriction of J_- to its $\mathcal{G}_-$ components to be:

$$J_-|_{\mathcal{G}_-} = M_- = \sum_{i=1}^{n-1} \mu_i E_{-\alpha_i}, \qquad J_+|_{\mathcal{G}_+} = \sum_{i=1}^{n} \nu_i E_{\alpha_i} \tag{3.6}$$

with μ_i and ν_i real positive constants.

Such constraints can be obtained as a part of the equations of motion of a new model resulting from a Lagrange multiplier treatment on the WZW action. More precisely, it is a gauge theoretical approach involving as gauge group the (non compact) part G_+ in G_1 and G_- in G_2, associated to the Lie $\mathcal{G}$ subalgebra $\mathcal{G}_+$ and $\mathcal{G}_-$ respectively with elements $g_+(x) \in G_+$ and $g_-(x) \in G_-$ which will lead to the Euler equations (3.3) and (3.6). The use of the local Gauss decomposition

$$g = g_+ \cdot h \cdot g_- \text{ with } h(x) = \exp \sum_{i=1}^{r} \phi_i(x) H_i \tag{3.7}$$

provides in the Euler equations the differential equations of the Toda theory based on the group G, the ϕ_i's being the corresponding fields.

$$\partial_+ \partial_- \phi_i = \mu_i \nu_i \ \exp \sum_j K_{ij} \phi_j \tag{3.8}$$

where K_{ij} is the Cartan matrix associated to the Lie algebra $\mathcal{G}$ of G.

Two remarks can be made at this point.

i) The above G Toda theory involves r = rank $\mathcal{G}$ fields in one-to-one correspondence with the Cartan part $\mathcal{H}$ of $\mathcal{G}$, and it is usually called the "Abelian" Toda theory on $\mathcal{G}$.

ii) The above construction actually involves the principal $Sl(2)$ subalgebra of $\mathcal{G}$ with generators:

$$H = \sum_{i,j=1}^{r} K^{ij} H_j \qquad E_- = \sum_{i=1}^{r} E_{-\alpha_i} \qquad E_+ = \sum_{i,j=1}^{r} K^{ij} E_{\alpha_i} \tag{3.9}$$

(note that a rescaling in Eq.(3.6) allows to take all the $\mu_i = 1$; K^{ij} is the inverse Cartan matrix).

Moreover the currents J_- (resp. J_+) are not invariant under the gauge transformations generated by the constraints (3.6). Focussing on J_-, these transformations read:

$$\begin{aligned} J_-(x_-) \to \quad & J_-^g(x_-) \\ & = g_+(x_-) J_-(x_-) g_+(x_-)^{-1} + \partial_- g_+(x_-) \cdot g_+(x_-)^{-1} \end{aligned} \tag{3.10}$$

where $g_+(x_-) \in G_+$. This will allow to bring the currents to the gauge-fixed form:

$$J^g = M_- + \sum_{j \geq 0} W_{j+1}(J) M_j \tag{3.11}$$

where the W_{j+1} are polynomials in the currents J_- and their derivatives $\partial_-^n J_-$. In the so-called "Drinfeld-Sokolov highest weight gauge" each generator M_j is the highest weight in the $Sl(2)_{ppal}$ representation $\mathcal{G}_j$ space obtained by reducing with respect to $Sl(2)_{ppal}$ the Lie algebra $\mathcal{G}$: considered as a vector space, $\mathcal{G}$ writes

$$\mathcal{G} = \oplus_{j=1}^{k} D_j \tag{3.12}$$

with D_j of dimension $(2j+1)$. The Poisson brakets among the W_j's can be obtained from the Poisson-Lie algebra satisfied by the current components:

$$\{J_{-}^{a}(x_-), J_{-}^{b}(x'_-)\}_{PB} = i f_c^{ab} J_{-}^{c}(x'_-)\delta(x_- - x'_-) + k\delta^{ab}\delta'(x_- - x'_-) \tag{3.13}$$

where f_c^{ab} are the structure constants for a given basis of $\mathcal{G}$.

Then each W_{j+1} is associated to a D_j and its conformal spin is $(j+1)$ with respect to the stress energy tensor itself relative to the D_1 representation spanned by the generators of $Sl(2)_{ppal}$:

$$T = T_0 + trH.\partial J \tag{3.14}$$

with

$$T_0 = \frac{1}{2k} tr(J.J). \tag{3.15}$$

Note also that each W_{j+1} can always be seen as a primary field with respect to T, after adjunction of an extra term in the $J's$ and derivatives.

Before going to examples, let us remark that, in this approach, a classical W-algebra is a subalgebra of the enveloping algebra of (3.13), itself symmetry of a WZW model: the constraints reduce the symmetry in such a way that only some polynomials in the J^a's and their derivatives generate the residual symmetry.

3.2. Examples

Let us take for $\mathcal{G}$ the $Sl(3)$ algebra. The Abelian Toda theory is obtained by imposing on the J currents the constraints:

$$J_- = \begin{bmatrix} \varphi_1 & \varphi_3 & \varphi_4 \\ 1 & \varphi_2 & \varphi_5 \\ 0 & 1 & -\varphi_1 - \varphi_2 \end{bmatrix} \quad \begin{matrix} \text{leading by the} \\ \text{gauge action of} \\ g_+(x_-) \in G_+ \text{ to} \end{matrix} \quad J_-^g = \begin{bmatrix} 0 & T & W_3 \\ 1 & 0 & T \\ 0 & 1 & 0 \end{bmatrix} \tag{3.16}$$

Involving $Sl(2)_{ppal}$ generated by:

$$E_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad E_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{3.17}$$

$\mathcal{G}$ decomposes under the (adjoint) action of $Sl(2)_{ppal}$ as:

$$\mathcal{G}/_{Sl(2)} = D_1 \oplus D_2 \tag{3.18}$$

to which are associated resp. with the spin 2 and 3 quantities T and W_3 generating the well known Zamolodchikov (Zamolodchikov 1985) $\{T, W_3\}$ algebra.

But still with $Sl(3)$ there exists another kind of constraints which allows for a similar treatment of the WZW model. It reads

$$J_- = \begin{bmatrix} \varphi_1 & \varphi_3 & \varphi_4 \\ 1 & \varphi_2 & \varphi_5 \\ 0 & \varphi_6 & -\varphi_1 - \varphi_2 \end{bmatrix} \tag{3.19}$$

Now the $Sl(2)$ subalgebra which is involved is the following:

$$E_{-\alpha_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad E_{+\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.20}$$

with respect to this $Sl(2)$, $\mathcal{G}$ decomposes as:

$$\mathcal{G} = D_1 \oplus D_{1/2} \oplus D_{1/2} \oplus D_0 \tag{3.21}$$

and the gauge invariant matrix current takes the form:

$$J_-^g = \begin{bmatrix} W_1 & W_2 & W_{3/2}^+ \\ 1 & W_1 & 0 \\ 0 & W_{3/2}^- & -2W_1 \end{bmatrix} \tag{3.22}$$

The algebra $\{W_2, W_{3/2}^+, W_{3/2}^-, W_1\}$ is usually called the classical Bershadsky algebra (Bershadski 1991). It is the symmetry algebra of the "non Abelian" Toda model constructed from the $Sl(2)$ algebra defined in (3.20).

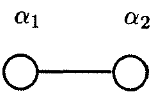
There are only two different $Sl(2)$ subalgebras in $Sl(3)$; therefore we have exhausted the different Toda models and the associated W-algebras relative to $Sl(3)$. More generally, starting from a simple algebra $\mathcal{G}$, each admissible choice of J components which can be set to constant (i.e. first class constraints in Dirac terminology) will correspond to an $Sl(2)$ in $\mathcal{G}$ and vice-versa. Then to determine all the different W-algebras symmetries of Toda theories associated to $\mathcal{G}$, one has first to consider all the different $Sl(2)$ in $\mathcal{G}$. (This mathematical problem has been solved by Dynkin). In each case, the decomposition of $\mathcal{G}$ with respect to $Sl(2)$ representations will provide the conformal spin of the associated W algebra (Frappat, Ragoucy and Sorba CMP 1993).

Supersymmetric Toda theories can also be considered. A supersymmetric treatment of the WZW models, based on simple superalgebras $S\mathcal{G}$ has to be done, constraints being written in a superspace formulation (Delduc, Ragoucy and Sorba 1992). Then $Sl(2)$ is replaced by its supersymmetric extension $OSp(1|2)$. The classification of $OSp(1|2)$ subsuperalgebras in simple superalgebras followed by the reduction for each $\mathcal{SG}$ of its adjoint representation with respect to each $OSp(1|2)$ subpart provide the conformal superspin content of the W superalgebras symmetries of Super Toda theories (Frappat, Ragoucy and Sorba CMP 1993).

From such a classification, general properties of the W (super)algebras, allowing a simplified and synthetic overview, can be deduced: this will be the object of the two next sections.

4. Folding the W (super)algebras

Using the properties of a non simply laced simple algebra to appear as a subalgebra of $Sl(n)$ after a suitable identification of $Sl(n)$ simple roots, one can obtain W algebras related to B-C-D series from W algebras related to unitary ones (Frappat, Ragoucy and Sorba NP 1993). Let us give an example, based again on the $Sl(3)$ group. Its Dynkin diagram (DD) is :



(4.1)

α_1 and α_2 representing the simple roots, to which are associated the generators E_{α_1} and E_{α_2}. It is known that the transformation τ such that: $\tau(\alpha_i) = \alpha_j \quad i \neq j = 1,2$ which is a symmetry of DD can be lifted up to an (outer) automorphism on the Lie algebra of $Sl(3)$ by defining:

$$\hat{\tau}\,(E_{\pm\alpha_i}) = E_{\pm\tau(\alpha_i)} \quad i = 1,2 \tag{4.2}$$

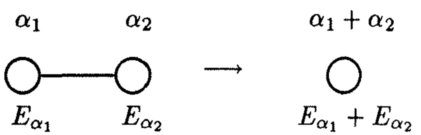
with

$$\hat{\tau}[E_{\alpha_i}, E_{-\alpha_i}] = \tau(\alpha_i)H \tag{4.3}$$

The $Sl(3)$ subalgebra $\mathcal{G}$ invariant under $\hat{\tau}$ is then generated from:

$$E_{\pm\alpha_1} + E_{\pm\alpha_2} \tag{4.4}$$

That is, by "folding" the root α_1 onto α_2, $Sl(3)$ reduces to the Lie algebra $\mathcal{G}^F$ of the (non compact) 3 dimensional orthogonal group:



(4.5)

On the 3×3 matrix representation, where E_{α_1} is identified with E_{12} and E_{α_2} with E_{23}, it will result that from the $\mathcal{G}$ matrices $M = m^{ij}E_{ij}$, m^{ij} being real numbers satisfying the traceless condition $\sum_{i=1}^{3} m^{ii} = 0$, one obtains a representation of $\mathcal{G}^F$ by imposing the conditions:

$$m^{ij} = (-1)^{i+j+1} m^{4-j,4-i} \tag{4.6}$$

Identifying in the Abelian Toda theory on $Sl(3)$ the J^a current components as in (4.6), it is not a surprise to get, by Hamiltonian reduction:

$$J^g_{Sl(3)} = \begin{pmatrix} 0 & T & W_3 \\ 1 & 0 & T \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow J^g_{SO(3)} = \begin{pmatrix} 0 & T' & 0 \\ 1 & 0 & T' \\ 0 & 1 & 0 \end{pmatrix} \tag{4.7}$$

as can be expected in a rank 1 algebra.

Of course, this simple example can be generalized, the foldings of $A_{2n-1} = Sl(2n)$ and $A_{2n} = Sl(2n+1)$ providing the symplectic $C_n = Sp(2n)$ and $B_n = SO(2n+1)$ algebras respectively. If one notes that $SO(2n)$ can be obtained from $SO(2n+1)$ by a regular embedding, one realizes that the W algebras associated to the A_n series can be "folded" into the W algebras relative to the other infinite series (note also that for the exceptional cases, the G_2 ones can be deduced from $D_4 \equiv SO(8)$ and F_4 W-algebras from the E_6 ones). The same procedure can be applied to superalgebras (see (Frappat, Ragoucy and Sorba NP 1993)).

An useful consequence of this technics is to get identities between structure constants of W-algebras relative to different simple algebras: denoting by C_{ij}^k the general structure constant of the "fusion rule":

$$[W_i] \cdot [W_j] = \delta_{ij} \frac{c}{2} [I] + C_{ij}^k(\mathcal{G})[W_k] \tag{4.8}$$

We have as examples, in the Abelian case:

$$C_{ij}^k(D_n) = C_{ij}^k(A_{2n}) \qquad i,j,k \neq n \tag{4.9}$$

$$C_{ij}^k(C_n) = C_{ij}^k(A_{2n-1}) \quad C_{ij}^k(B_n) = C_{ij}^k(A_{2n}), \tag{4.10}$$

such relations being sometimes precious, due to the difficulty to obtain explicit commutation relations.

5. Secondary reductions

Let us consider again $\mathcal{G} = SL(3)$ and the two W-algebras which can be constructed, via Toda theories, from such an underlying simple algebra; they are the Zamolodchikov algebra $\{T, W_3\}$ and the Bershadsky algebra generated by $\{W_2, W_{3/2}^+, W_{3/2}^-, W_1\}$. The corresponding J^g matrices read (see Eq. (3.16) and (3.22)):

$$J^g_{\text{Abel}} = \begin{pmatrix} 0 & T & W_3 \\ 1 & 0 & T \\ 0 & 1 & 0 \end{pmatrix} \qquad J^g_{\text{Non Abel}} = \begin{pmatrix} W_1 & W_2 & W_{3/2}^+ \\ 1 & W_1 & 0 \\ 0 & W_{3/2}^- & -2W_1 \end{pmatrix} \tag{5.1}$$

One remarks that the constraints imposed in the Non Abelian case

$$\{trJ_- \cdot E_{-\alpha_1} = 1 \ ; \ trJ_- \cdot E_{-(\alpha_1+\alpha_2)} = 0\} \tag{5.2}$$

form a subset of the constraints corresponding to the Abelian case:

$$\left\{trJ_- \cdot E_{-\alpha_1} = trJ_- \cdot E_{-\alpha_2} = 1 \ ; \ trJ_- \cdot E_{-(\alpha_1+\alpha_2)} = 0\right\} \tag{5.3}$$

It is time to give explicitly the P.B. of the Classical Bershadsky algebra: let us, for convenience, make a little change in the notations and denote W_1 by J and $W_2 + \frac{1}{3c} J \cdot J$ by T.

$$\begin{aligned}
\{J(z), J(w)\} &= -\frac{3}{2} c\delta'(z-w) \\
\{J(z), W^{\pm}_{3/2}(w) &= \pm\frac{3}{2} W^{\pm}_{3/2}\delta(z-w) \\
\{T(z), W^{\pm}_{3/2}(w)\} &= -\frac{3}{2} W^{\pm}_{3/2}(w)\delta'(z-w) + \partial W_{\pm}(w)\delta(z-w) \\
\{T(z), J(w)\} &= -J(w)\delta'(z-w) + \partial J(w)\delta(z-w) \\
\{T(z), T(w)\} &= -2T(w)\delta'(z-w) + \partial T(w)\delta(z-w) \\
&\quad +\frac{c}{2}\delta'''(z-w) \\
\{W^{+}_{3/2}(z), W^{-}_{3/2}(w)\} &= 2J(w)\delta'(z-w) - c\delta''(z-w) + \\
&\quad +(T - \frac{4}{3c}J^2 - \partial J)(w)\delta(z-w) \\
\{W^{\pm}_{3/2}(z), W^{\pm}_{3/2}(w)\} &= 0
\end{aligned} \tag{5.4}$$

The last relation, which expresses the nilpotency of $W^{-}_{3/2}$ (and $W^{+}_{3/2}$), allows to consider the constraint

$$W^{-}_{3/2} = 1 \tag{5.5}$$

as a gauge constraint (first class constraint).

With the help of $J(z)$, it is possible to redefine the energy momentum tensor T in such a way that the constraint becomes conformally invariant, that is, shifting T into

$$\hat{T} = T - \partial J_+ \tag{5.6}$$

$W^{-}_{3/2}$ behaves as a spin 0 field:

$$\begin{aligned}
\{\hat{T}(z), W^{-}_{3/2}(w)\} &= \partial W^{-}_{3/2}(w)\delta(z-w) \\
&\simeq 0 \qquad \text{using Eq.(5.5)} \quad .
\end{aligned} \tag{5.7}$$

Then one can look at the reduced W algebra obtained by constructing the polynomials invariant under the gauge transformations associated to $W^-_{3/2}$. Therefore, let us consider the finite gauge transformations on the currents:

$$\begin{aligned} X(w) \rightarrow & \hat{X}(w) \\ = & X(w) + \int dz\ \alpha(z)\{W^-_{3/2}(z), X(w)\} \\ & + \frac{1}{2!}\int dz\ dz'\alpha(z)\alpha(z')\left\{W^-_{3/2}(z), \{W^-_{3/2}(z'), X(w)\}\right\} \\ & + \ldots \end{aligned} \tag{5.8}$$

where $X = J, T, W^+_{3/2}$, the constraint (5.5) being used on the r.h.s. of the P.B., following Dirac prescriptions on constraints ("weak equations"). Then the J current transforms as:

$$\hat{J}(w) = J(w) + \int dz\ \alpha(z)\{W^-_{3/2}(z), W_1(w)\} + 0 \tag{5.9}$$

since

$$\{W^-_{3/2}(z), J(w)\} \simeq \left(\frac{1}{2}\delta(z-w)\right) \tag{5.10}$$

that is:

$$\hat{J}(w) = J(w) + \frac{3}{2}\alpha(w) \tag{5.11}$$

Then, it is clear that a global gauge fixing is given by

$$\hat{J}(w) = 0 \tag{5.12}$$

that is, by taking:

$$\alpha = -\frac{2}{3}J \tag{5.13}$$

It follows for T:

$$\begin{aligned} T(w) \rightarrow \hat{T}(w) &= T(w) - \frac{3}{2}\int dz \cdot \alpha(z) \cdot \delta'(z-w) + 0 \\ &= T(w) + \frac{3}{2}\partial\alpha \\ &= T - \partial J \end{aligned} \tag{5.14}$$

as expected from Eq.(5.6) !

In the same way:

$$W^+_{3/2} \rightarrow \hat{W}_3 = W^+_{3/2} + \frac{2}{3}J \cdot T + \frac{2}{3}J \cdot \partial J - \frac{8}{27c}J^3 - \frac{2c}{3}\partial^2 J \tag{5.15}$$

the notation $\hat{W}_3$ being justified by the property of $\hat{W}_3$ to behave as a spin 3 field under $\hat{T}$.

At this point, it is not a surprise to realize that the $\hat{T}$ and $\hat{W}_3$ quantities generate a (algebra isomorphic to) Zamolodchikov algebra.

The above illustrated method with W algebras based on $\mathcal{G} = SL(3)$ can be applied to any simple algebra $\mathcal{G}$ up to some obvious technical difficulties. Starting from the weakest constraints and adding new ones on a W algebra relative to some Lie algebra $\mathcal{G}$, one can then obtain chains of W algebras, the "smallest" one being relative to the Abelian Toda case (highest number of constraints). As could be expected by Lie algebra experts, there also exist cases with $\mathcal{G}$ non simply laced, i.e. B_n or C_n, for which such a secondary reduction towards the Abelian case cannot be obtained. Finally, in the same way one gets Toda equations by gauging WZW models, a gauging of the Toda action in which a (Non Abelian) W algebra stands as the current algebra of the theory could be performed, leading to a new (more constrained) Toda action. Such an approach for a generalized gauge Toda field theory, as well as a more complete discussion on secondary reductions will soon be available (Delduc, Frappat, Ragoucy and Sorba 1994).

6. Rational W algebras

6.1. Commutant of the spin 1 part

Now let us turn our attention to the particular role of the spin one part, when it is present, in a W algebra. One can easily check, by dimensional arguments, that these fields generate a Kac-Moody algebra W_1. Moreover the set of W generators decomposes into irreducible representations under the adjoint action of this Kac Moody algebra. Let us study what happens when factorizing out the spin one part in a W algebra, that is by computing the commutant in W of the W_1 Kac-Moody subalgebra (Delduc, Frappat, Ragoucy, Sorba and Toppan 1993).

Most of W algebras associated to Non Abelian Toda theories contain spin-one fields. Let us perform our calculations on the Bershadsky algebra already considered in the previous sections (see in particular Eq. (5.4)).

First, by the following shift on T,

$$\bar{T} = T - \frac{1}{3c}J^2 \tag{6.1}$$

one gets the P.B.:

$$\begin{aligned} \{\bar{T}(z), J(w)\} &= 0 \\ \{\bar{T}(z), W_\pm(w)\} &= -\frac{3}{2}W_\pm(w)\delta'(z-w) + (\mathcal{D}W_\pm)(w)\delta(z-w) \\ \{W_+(z), W_-(w)\} &= (\bar{T} - c\mathcal{D}^2)(w)\delta(z-w) \end{aligned} \tag{6.2}$$

while $\bar{T}$ satisfies the usual Virasoro P.B.:

$$\{\bar{T}(z), \bar{T}(w)\} = -2\bar{T}(w)\, \delta'(z-w) + \partial\bar{T}(w)\delta(z-w) + \frac{c}{2}\delta'''(z-w) \quad (6.3)$$

In the above equations, one has used the covariant derivative $\mathcal{D}$ such that

$$\mathcal{D}W_\pm = (\partial \mp \frac{1}{c}J)W_\pm \quad (6.4)$$

while the D^2 showing up in the r.h.s. of $\{W_+, W_-\}$ is relative to w. The appearance of a covariant derivative may open new perspectives in the field of integrable models. It is here particularly convenient in order to construct the commutant of J. Indeed the set of fields commuting with J is generated by the stress energy tensor $\bar{T}$ and the bilinear products:

$$W^{(p,q)} = (\mathcal{D}^p W_+)(\mathcal{D}^q W_-) \quad (6.5)$$

with p, q non negative integers.

Actually, the fields $W^{(p,q)}$ and $\bar{T}$ are the building blocks from which one can construct an infinite tower of primary fields of spin 3,4,...

$$\begin{aligned} W_3 &= W_+W_- \\ W_4 &= W_+\mathcal{D}W_- - W_-\mathcal{D}W_+ \\ &\vdots \\ W_{3+n} &= W_+\mathcal{D}^nW_- - (\mathcal{D}^nW_+)W_- + \ldots \qquad \text{for} \qquad n > 2 \end{aligned} \quad (6.6)$$

these fields being created by the P.B. of fields of lower conformal spin, for ex.:

$$\{W_3(z), W_3(w)\} = 2W_4(w)\delta'(z-w) - \partial W_4(w)\delta(z-w) \quad (6.7)$$

and so on.

At this point, one may say that by looking at the commutant of the spin one generator J in the Bershadsky W algebra, one has obtained a polynomial non linear <u>W_∞ realization</u>.

But the primary fields W_{3+n} with $n \geq 2$ are not independent, and can be expressed as <u>rational</u> -and not polynomials- functions of T, W_3, W_4: for example W_5 can be written in terms of W_3 and W_4 as follows:

$$W_5 = \frac{1}{4W_3}\left[7\left(W_4^2 - (\partial W_3)^2\right) + 6W_3(\partial^2 W_3) + \bar{T}W_3)\right] \quad (6.8)$$

Therefore, the commutant of J exhibits a new structure with respect to the standard W algebras, which can be seen either as a rational finitely generated W algebra or as a polynomial non linear W_∞ realization.

The above example is the simplest one exhibiting such a structure. Of course a general approach with a non Abelian W_1 part can be performed (see (Delduc, Frappat, Ragoucy, Sorba and Toppan 1993)).

6.2. Supersymmetric extension

The supersymmetric extension of this problem can be considered in an analogous way. Again, let us illustrate the method on an example, the $N = 3$ superconformal algebra $\mathcal{SC}(N = 3)$ generated by a spin 2 generator $T(z)$, 3 spin $\frac{3}{2}$ components $G^a_{3/2}$ $(a = 1,2,3)$, 3 spin 1 elements $J^a(z)$, constituting an $Sl(2)$ Kac-Moody algebra and a spin $\frac{1}{2}$ fermion $\psi(z)$. The C.R. in the classical case can be deduced from the formulas (15) of (Goddard and Schwimmer 1988), in which we identify the O.P.E. with the P.B. and the singular terms $\frac{1}{(z-w)^k}$ with $(-1)^{k-1}\frac{1}{(k-1)!}\delta^{(k-1)}(z-w)$. After defining:

$$\begin{array}{ll} G^{\pm}(z) = \frac{1}{\sqrt{2}}(G^1 \pm iG^2)(z) \text{ and } & J^{\pm}(z) = \frac{1}{\sqrt{2}}(J^1 \pm iJ^2)(z) \\ G^0(z) = G^3(z) & J^0(z) = J^3(z) \end{array} \tag{6.9}$$

we will adopt the superfield formalism (Delduc, Ragoucy and Sorba 1992) and define:

$$\begin{array}{lll} \mathcal{T}(z) = \frac{1}{2}\, G^0(z) + \theta T(z) & \text{of superspin} & \frac{3}{2} \\ \mathcal{J}^{\pm}(z) = \pm J^{\pm}(z) + \theta G^{\pm}(z) & \text{of superspin} & 1 \\ \Phi(z) = \psi(z) + \theta J^0(z) & \text{of superspin} & \frac{1}{2} \end{array} \tag{6.10}$$

using the supervariable notations:

$$Z = (z,\theta), \quad W = (w,\eta) \text{ and } Z - W = z - w - \theta\eta \tag{6.11}$$

then the P.B. can be "compactly" written as (keeping in mind from above that: $\frac{\theta-\eta}{Z-W} = (\theta-\eta)\delta(Z-W) \doteq \delta(Z-W)$ and so on for their derivatives, and the O.P.E. being in place of the P.B.):

$$\begin{aligned} \mathcal{T}(Z)\cdot\Theta_s(W) = \quad & s\frac{\theta-\eta}{(Z-W)^2}\Theta_s(W) + \frac{1}{2}\,\frac{D\theta_s(W)}{Z-W} \\ & + \frac{\theta-\eta}{Z-W}\,\partial\Theta_s(W) + \dots \end{aligned} \tag{6.12}$$

if $\Theta_s(W)$ denotes the superspin $\mathcal{J}^{\pm}(W)$ or $\Phi(W)$ of superspin $s = 1$ or $\frac{1}{2}$, and as usual: $D = \partial_\eta + \eta\partial_w$

$$\begin{aligned} \mathcal{T}(Z)\mathcal{T}(W) &= \frac{3}{2}\,\frac{\theta-\eta}{(Z-W)^2}\mathcal{T}(W) + \frac{1}{2}\,\frac{D\mathcal{T}(W)}{Z-W} + \frac{\theta-\eta}{Z-W}\partial\mathcal{T}(W) \\ &\quad + \frac{c/6}{(Z-W)} + \dots \\ \Phi(Z)\mathcal{J}^{\pm}(W) &= \pm\frac{\theta-\eta}{Z-W}\mathcal{J}^{\pm}(W) + \dots \\ \Phi(Z)\Phi(W) &= \frac{c/3}{Z-W} + \dots \end{aligned}$$

$$\begin{aligned}\mathcal{J}^+(Z)\mathcal{J}^-(W) &= -\frac{\theta-\eta}{(Z-W)^2}\Phi(W) - \frac{1}{Z-W}D\Phi(W) - \frac{\theta-\eta}{Z-W}\partial\Phi \\ &\quad -2\frac{\theta-\eta}{Z-W}\mathcal{T}(W) - \frac{c/3}{(Z-W)^2} + \dots \end{aligned} \tag{6.13}$$

We wish to factorize out the superspin $\frac{1}{2}$ superfield $\Phi(Z)$. As in the nonsupersymmetric case, we can operate a shift on $\mathcal{T}(Z)$

$$\mathcal{T}_0(Z) = \mathcal{T}(Z) - \frac{3}{2c}\Phi(Z)D\Phi(Z) \tag{6.14}$$

such that:

$$\mathcal{T}_0(Z)\cdot\Phi(W) = 0 \tag{6.15}$$

We can expect the covariant derivative of Eq.(6.4) to become:

$$\mathcal{D} = D - \frac{3q}{c}\Phi \tag{6.16}$$

if q is the super $U(1)$ charge carried by the primary superfield, i.e.:

$$\mathcal{D}\mathcal{J}^\pm = (D \mp \frac{3}{c}\Phi)\mathcal{J}^\pm \tag{6.17}$$

Now the spin 2 superfield $W_2(Z) = \mathcal{J}^+(Z)\cdot\mathcal{J}^-(e)$ is a primary superfield under $\mathcal{T}_0(Z)$ in the commutant of $\Phi(Z)$. The properties above obtained with W algebras generalize here with W superalgebras. Computing for example the P.B. of W_2 with itself one gets:

$$\begin{aligned}W_2(Z)W_2(W) &= -\frac{c}{3}\left(\frac{2W_2(W)}{(Z-W)^2} + \frac{\partial W_2(W)}{Z-W} + \frac{\theta-\eta}{(Z-W)^2}DW_2(W)\right. \\ &\quad \left.+\frac{3}{5}\,\frac{\theta-\eta}{Z-W}D\partial W_2(W)\right) - \frac{36}{5}\,\frac{\theta-\eta}{Z-W}(\mathcal{T}_0\cdot W_2)(W) \\ &\quad +\frac{c}{3}\,\frac{\theta-\eta}{Z-W}W_{7/2}(W) + \dots \end{aligned} \tag{6.18}$$

where $W_{7/2}(W)$ is the (new!) 7/2 superspin primary superfield defined as:

$$W_{7/2} = \mathcal{J}^+\mathcal{D}^3\mathcal{J}^- + \mathcal{J}^-\mathcal{D}^3\mathcal{J}^+ - \frac{3}{5}D\partial W_2 - \frac{48}{5c}\,\mathcal{T}_0\cdot W_2 \tag{6.19}$$

6.3. Spin 1/2 versus spin 1 fields

The superalgebra $\mathcal{SC}(N=3)$ was the first example considered by the authors of (Goddard and Schwimmer 1988) to illustrate their result about the factorization of the spin 1/2 part in a superconformal field theory, more precisely that a meromorphic field theory can be decomposed into the tensor product of a spin 1/2 part and a conformal field theory without spin 1/2 field. We would like to stress that this property can easily be proved, at least at the classical level, by the use of finite gauge transformations already introduced in the previous section (see Eq.(5.8). Indeed, leaving to the reader the general proof (which will also be found in (Delduc, Frappat, Ragoucy and Sorba 1994)) let us stay with the $\mathcal{SC}(N=3)$ algebra and perform on its generators $X(w)$ the transformation:

$$X(w) \to \hat{X}(w) = X(w) + \int dz\ \alpha(z)\psi(z).X(w) + 0 \tag{6.20}$$

where $\psi(z)$ is the fermion field (we do not use any more the superfield formalism, since we wish to only factorize the $\psi(z)$ fermion and not the superspin 1/2 field).

Owing to the OPE relation:

$$\psi(z) \cdot \psi(w) = \frac{c/3}{z-w} \tag{6.21}$$

one directly gets, imposing the "gauge fixing":

$$\alpha(w) = -\psi(w) \tag{6.22}$$

the transformed fields ($a = 1,2,3$)

$$\hat{\psi} = 0 \ \ ; \ \ \hat{T} = T - \frac{1}{2}\psi\partial\psi \ \ ; \ \ \hat{G}^a = G^a - T^a\psi \quad \hat{J}^a = J^a. \tag{6.23}$$

In accordance with the results of (Goddard and Schwimmer 1988), the O.P.E. among the transformed fields are identical, except for the central charge to the ones relative to the non transformed fields, and as expected such that:

$$\hat{T} \cdot \psi = \hat{G}^a \cdot \psi = \hat{J}^a \cdot \psi = 0 \tag{6.24}$$

Note that this gauge transformation can also be done with spin 1/2 bosons, and leads to the same conclusion (Delduc, Frappat, Ragoucy and Sorba 1994). It has also be shown that the action of such a super-Toda model can be rewritten as the sum of two terms, one relative to the spin 1/2 part and the other to the factorized W part (Ragoucy 1993).

It is natural to wonder what happens if, instead of performing a gauge transformation associated with a 1/2 fermion, one involves a spin 1 field. Let us take once more as an example the Bershadsky algebra (see Eq.(5.4)): its (simple) Kac Moody generator $J(z)$ satisfies:

$$J(z) \cdot J(w) = \frac{3/2c}{(z-w)^2} \tag{6.25}$$

In order to obtain $\hat{J} = 0$ in the transformation:

$$J(w) \to \hat{J}(w) = J(w) + \int dz\ \alpha(z)J(z) \cdot J(w) + \ldots \tag{6.26}$$

We would have to impose α such that

$$\partial\alpha(w) = J(w) \tag{6.27}$$

The pathology created by this relation appears in different places. In particular, one would get:

$$\alpha(z) \cdot W^{\pm}_{3/2}(w) = \pm\frac{3}{2}W^{\pm}_{3/2}(w)\ln(z-w) \tag{6.28}$$

and some trouble to compute, from:

$$\hat{W}^{\pm}_{3/2}(w) = e^{\pm 3/2\alpha(w)}W^{\pm}_{3/2}(w) \tag{6.29}$$

the quantity:

$$\hat{W}^{+}_{3/2}(z) \cdot \hat{W}^{-}_{3/2}(w) \tag{6.30}$$

Thus, gauge transformations relative to spin 1/2 fields allow to recover the result of Ref (Goddard and Schwimmer 1988), namely the property that spin 1/2 fermions can be eliminated in a super W algebra, but such a technics does not appear suitable for the factorization of spin 1 fields, as could be expected from the results presented in the first part of this section.

Note that the above discussion has to be compared with the factorization at quantum level, of spin 1/2 and 1 fields considered in (Deckmyn and Thielesmans 1993): the projection used there appears as a quantum version of our gauge transformation.

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SKYRMION-SKYRMION SCATTERING

THOMAS GISIGER and MANU B. PARANJAPE*
Laboratoire de physique nucléaire,
Université de Montréal, Montréal, Quebec Canada

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Abstract. We study the scattering of Skyrmions at low energy and large separation using the method proposed by Manton of truncation to a finite number of degrees of freedom. We calculate the induced metric on the manifold of the union of gradient flow curves, which for large separation, to first non-trivial order is parametrised by the variables of the product ansatz.

1. Introduction

Scattering of solitons in a non-integrable, non-linear classical or quantum field theory remains an intractable and difficult problem, however, it concerns one of the most interesting aspects of the nature of the corresponding physics. Numerical methods have given reasonable ideas on how the scattering proceeds but they are still unsatisfactory for uncovering the detailed dynamics governing the scattering.

A method has been proposed by Manton (1988) for truncating the degrees of freedom from the original infinite number to a relevant finite number of variables. The idea first considers the case of theories of the Bogomolnyi type, those theories which admit *static* soliton solutions, usually in the topological two soliton sector, which asymptotically describe two single solitons at arbitrary positions and relative orientations. The configuration at small separation contains, in general, strong deformations of the individual solitons and in fact they lose their identity. However the set of configurations have the same energy since they correspond to the continuous variation of a finite number of parameters, the moduli. Otherwise they could not be stationary points of the potential. In general, for solitons corresponding to a topological quantum number, the moduli space corresponds to the sub-manifold of minimum energy configurations within the given topological sector. Manton suggests that the low energy scattering of solitons, with initial configuration on this sub-manifold corresponding to asymptotic, single solitons, with arbitrarily small initial velocity tangent to the sub-manifold, will self-consistently be constrained to remain on the sub-manifold. Since the potential energy is a constant on the sub-manifold the resulting dynamics

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reduces to geodesic motion on the sub-manifold in the induced metric on the sub-manifold from the kinetic term. It is a difficult task to prove such a truncation of degrees of freedom in a mathematically rigorous fashion, however, it does seem intuitively correct. The non-linearity of the theory implies the coupling of the degrees of freedom corresponding to the sub-manifold with all other excitations through the potential. We are assuming that these are negligible. Manton and Gibbons (1986) applied this program with remarkable success to the case of magnetic monopoles in the BPS limit and it has also been applied to vortex scattering in a similar limit (Samols 1992).

The generalization to the more common situation where the set of static solutions correspond to a finite set of critical points proceeds as follows. The critical points are typically a minimum energy configuration which is essentially a bound state of two solitons, an asymptotic critical point which corresponds to two infinitely separated solitons and possibly a number of unstable non-minimal critical points of varying energies of the same order. These critical points are degenerate with a finite number of degrees of freedom. They are connected by special paths, the paths of steepest descent or equivalently the gradient flow curves. In this case Manton proposes that the dynamics will be constrained to lie on the sub-manifold comprising of the union of all these curves. This again is intuitively reasonable. If we think of the space of all configurations as a large bag, the bottom surface of the bag will correspond to this sub-manifold, and a slow moving marble rolling on the bottom will tend to stay there.

The Skyrme model falls into the second case. We identify the corresponding sub-manifold for well-separated Skyrmions and we calculate the induced metric to lowest non-trivial inverse order in the separation from the kinetic term. This is the first step towards calculating the scattering of Skyrmions in this formalism.

2. The Skyrme model

The Skyrme model is described by the langrangian,

$$\mathcal{L} = \frac{f_\pi^2}{4} tr(U^\dagger \partial_\mu U U^\dagger \partial^\mu U) + \frac{1}{32e^2} tr([U^\dagger \partial_\mu U, U^\dagger \partial_\nu U]^2) \quad (1)$$

where $U(x)$ is a unitary matrix valued field. We take

$$U(x) \in SU(2). \quad (2)$$

The Skyrme langrangian corresponds to first terms of a systematic expansion in derivatives of the effective langrangian describing low energy interaction of pions. It is derivable from QCD hence f_π and g are in principle calculable from QCD. What is even more surprising is that it includes the baryons as well which arise as topological solitonic solutions of the equations of motion.

The original proposal of this by Skyrme (1961) was put on solid footing by Witten (1983).

The topological solitons, called Skyrmions, correspond to non-trivial mappings of $\mathbb{R}^3$ plus the point at infinity into $SU(2)$:

$$U(x) : \mathbb{R}^3 + \infty \to SU(2) = S^3. \tag{3}$$

But

$$\mathbb{R}^3 + \infty = S^3 \tag{4}$$

thus the homotopy classes of mappings

$$U(x) : S^3 \to S^3 \tag{5}$$

which define

$$\Pi_3(S^3) = \mathbb{Z} \tag{6}$$

characterize the space of configurations.

The topological charge of each sector is given by

$$N = \frac{1}{24\pi^2} \int d^3\mathbf{x}\, \epsilon^{ijk} tr(U^\dagger \partial_i U U^\dagger \partial_j U U^\dagger \partial_k U) \tag{7}$$

which is identified with the baryon number. Thus for the scattering of two Skyrmions, we are looking at the sector of baryon number equal to 2. In this sector the minimum energy configuration should correspond to the bound state of two Skyrmions, which must represent the deuteron. The asymptotic critical point corresponds to two infinitely separated Skyrmions. There exist, known, non-minimal critical points, corresponding to a spherically symmetric configuration, the di-baryon solution (Kutschera and Pethick 1985). The energy of this configuration is about three times the energy of a single Skyrmion. There are also, possibly, other non-minimal critical points with energy less than two infinitely separated Skyrmions (Isler, LeTourneux and Paranjape 1991). The scattering of two Skyrmions will take place on the union of the paths of steepest descent which connect the various critical points.

3. Skyrmion-Skyrmion scattering

We consider the scattering only for large separation. In this way we do not have to know the structure of this manifold in the complicated region where the two Skyrmions interact strongly and consequently are much deformed. In the region of large separation the product ansatz corresponds to

$$\begin{aligned} U(\mathbf{x}) &= U_1(\mathbf{x} - \mathbf{R}_1)U_2(\mathbf{x} - \mathbf{R}_2) \\ &= A_1^\dagger U(\mathbf{x} - \mathbf{R}_1)A_1 A_2^\dagger U(\mathbf{x} - \mathbf{R}_2)A_2 \end{aligned} \tag{8}$$

where $U(\mathbf{x}-\mathbf{R}_1)$ and $U(\mathbf{x}-\mathbf{R}_2)$ correspond to the field of a single Skyrmion solution centered at R_1 and R_2 respectively. The full Skyrme model dynamics implies a deformation of each Skyrmion. This deformation, from a numerical studies, is found to be unimportant already at a separation of 1.5 fermi (Walhout and Wambach 1991). We will neglect this deformation.

It remains to calculate the metric on the sub-manifold parametrized by the product ansatz. We find the interesting result that the metric behaves like $1/d$ where d is the separation (Schroers 1993). We find the kinetic energy:

$$\begin{aligned} T = & - 2M + \tfrac{1}{2}M\dot{\mathbf{R}}_1^{\,2} + \tfrac{1}{2}M\dot{\mathbf{R}}_2^{\,2} \\ & - \Lambda\, tr(A_1^\dagger \dot{A}_1 A_1^\dagger \dot{A}_1) - \Lambda\, tr(A_2^\dagger \dot{A}_2 A_2^\dagger \dot{A}_2) + T_{int} \end{aligned} \tag{9}$$

where

$$M = 4\pi\int_0^\infty r^2 dr\left\{\tfrac{1}{8}f_\pi^2\left[\left(\frac{\partial f}{\partial r}\right)^2 + 2\,\frac{\sin^2 f}{r^2}\right] + \frac{1}{2e^2}\frac{\sin^2 f}{r^2}\left[\frac{\sin^2 f}{r^2} + 2\left(\frac{\partial f}{\partial r}\right)^2\right]\right\}, \tag{10}$$

$$\Lambda = (ef_\pi)^3\int r^2 dr \sin^2 f\left[1 + \frac{4}{(ef_\pi)^2}\left(f'^2 + \frac{\sin^2 f}{r^2}\right)\right]$$

and finally the interesting term

$$T_{int} = 2f_\pi^2\kappa^2\, F_{ia}^1\, F_{jb}^2\, \frac{4\pi}{d}(\delta^{ij} - \hat{d}^i\hat{d}^j)\, D_{ab}(A_1^\dagger A_2) \tag{11}$$

where

$$F_{ia}^1 = -\beta_i^1\dot{\beta}_a^1 + \dot{\beta}_i^1\beta_a^1 - \epsilon_{iab}\,(\dot{\beta}_b^1\alpha^1 - \beta_b^1\dot{\alpha}^1) \tag{12}$$

$$A_1 = \alpha^1 + i\,\boldsymbol{\beta}^1\cdot\boldsymbol{\tau} \tag{13}$$

$$(\alpha^1)^2 + |\boldsymbol{\beta}^1|^2 = 1 \tag{14}$$

(correspondingly for A_2), and κ is determined by

$$f(r) \sim \frac{\kappa}{r^2} \quad \text{and} \quad \mathbf{d} = \mathbf{R}_1 - \mathbf{R}_2, \quad d = |\mathbf{d}|.$$

The metric can be be obtained from this expression by choosing local coordinates on the product ansatz manifold $(\mathbf{R}_1, \mathbf{R}_2, \boldsymbol{\beta}^1, \boldsymbol{\beta}^2)$ and extracting the quadratic form relating their time derivatives.

The potential (Isler, LeTouurneux and Paranjape 1991) between two Skyrmions can be calculated to give

$$V = 4\pi f_\pi^2\kappa^2\frac{(1-\cos\theta)(3\,(\hat{n}\cdot\hat{d})^2 - 1)}{d^3}$$

where θ, $\hat{n}$ pick out the element of $SU(2)$ given by $A_1 A_2^\dagger$.

The potential is of higher order than the metric, hence the dominant contribution to the scattering at large separation comes only from the metric. Thus to leading order we may even neglect the potential, and then the problem reduces to calculating the geodesics on the product ansatz manifold. We are presently working this out.

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NEW RESULTS ON THE CANONICAL STRUCTURE OF CLASSICAL NON-LINEAR SIGMA MODELS

MICHAEL FORGER*
Departamento de Matemática Aplicada,
Instituto de Matemática e Estatística,
Universidade de São Paulo,
Cx. Postal 20570, BR-01452-990 São Paulo, SP / Brazil
forger@ime.usp.br

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The material presented in this talk is based on recent work of the author (done in collaboration with colleagues from the University of Freiburg and from the University of São Paulo) which has produced new insight into the algebraic structure of classical non-linear sigma models as (infinite-dimensional) Hamiltonian systems [1-5]. After some introductory remarks intended to place this line of research into its appropriate context, the two main results obtained so far were discussed. The first, valid for general sigma models (defined on arbitrary Riemannian manifolds), is the explicit calculation of their extended current algebra, i.e., the algebra generated, under Poisson brackets, by the components of the Noether current referring to a given internal symmetry and by the components of the energy-momentum tensor; this calculation can be carried out in closed form by introducing a single new composite scalar field. The second, valid for integrable sigma models (defined on Riemannian symmetric spaces), is the identification of a new algebra which, in this class of models, should be regarded as the substitute for the classical Yang-Baxter algebra. To put this result into its proper perspective, a brief summary of basic definitions from the theory of two-dimensional integrable field theories was given, including that of ultralocal vs. non-ultralocal models.
A full presentation will be published in the journal "Resenhas", edited by the Instituto de Matemática e Estatística da Universidade de São Paulo.

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POINCARÉ-CARTAN SUBBUNDLES

MAGDALENA GUSIEW-CZUDŻAK*
University of Wrocław, Institute of Theoretical Physics
Plac Maxa Borna 9, 50204 Wrocław, Poland, magc@proton.ift.uni.wroc.pl

and

ZBIGNIEW OZIEWICZ**
Centro de Investigaciones Teoricas
Facultad de Estudios Superiores Cuatitlán
Universidad Nacional Autonoma de México
Apartado Postal # 25, Cuautitlán Izcalli,
54700 Estado de México, oziewicz@redvax1.dgsca.unam.mx

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Abstract. A multisymplectic framework for phenomenology like Newtonian mechanics or the Maxwell electrodynamics is said to be a phenomenological field theory. We consider a vector bundle of exterior forms, in mechanics of dimension $1+4n$, in electromagnetism of dimension $4+20n$. A phenomenological differential pseudoform Ω determines four Poincaré-Cartan subbundles $\{\mathcal{P}\}$ on which Ω is presymplectic, $d\Omega|\mathcal{P}=0$. This leads to twelwe Legendre's transforms among these subbundles, of which two transforms are well known.

CONTENTS

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** On leave of absence from University of Wrocław, Poland.

Foreword

Newtonian mechanics and Maxwell's electrodynamics are phenomenologies related to a diverse phenomena and presented as disconnected in a separate courses and textbooks. The one aim of this paper is to exibits a formal analogies among them in a framework of multisymplectic geometry.

We consider a vector bundle E of exterior forms over oriented manifold, $E \xrightarrow{\pi} \{M, vol\}$. In mechanics $\dim M = 1$ and $\dim E = \dim M + 4n$, in electromagnetism $\dim M = 4$ and $\dim E = \dim M + 20n$. A phenomenological (not unique) differential pseudoform Ω on E (vol-dependent) determines four Poincaré-Cartan subbundles $\{\mathcal{P}\}$: hamiltonian, lagrangian and two new not-named subbundles on which Ω is presymplectic, $\mathcal{P}^* d\Omega = 0$. This leads to twelve Legendre's transforms among these subbundles, of which two are well known.

A field equations of considered bivertical theory for $\dim M = 1$ reduce to the Newton equations and for $\dim M = 4$ to the Maxwell equations. This unification allows to see analogies. In particular, *force field* $\leftrightarrow$ *current*, the London equation in electromagnetism is an analogy of the harmonic oscilator force in the Newton dynamics, one can pose the Kepler problem in the Maxwell electrodynamics by formal analogy to the Kepler problem in mechanics, etc.

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History. Multisymplectic geometry in a classical field theory was initiated by Dedecker in 1953 and was developed in Warsaw by Tulczyjew around 1968, and by Kijowski (1973), Gawędzki (1972), Szczyrba and Kondracki (1979). In Chechia by Krupka since 1975. See Kijowski and Tulczyjew (1979).

Notations.
$\Lambda \equiv \Lambda_E \equiv \oplus\Lambda^k$ is de Rham complex of differential forms on a manifold E, $\mathcal{F} \equiv \Lambda^0$. A cocycles are denoted by $Z \equiv \{\alpha \in \Lambda,\ d\alpha = 0\}$.
$W \equiv W_E \equiv \operatorname{der}\mathcal{F}$ is the Lie $\mathcal{F}$-module of (**one**)-vector fields on E, such that $W^\wedge \equiv \oplus W^{\wedge k}$ is the Grassmann $\mathcal{F}$-algebra and graded Lie $\mathbb{R}$-algebra of multivector fields; $W^{\wedge 0} \equiv \mathcal{F}$.
$|\alpha| \equiv \operatorname{grade}\alpha \in \mathbb{N}$ and ψ denote an automorphism of Grassmann algebras, $\psi\alpha \equiv (-1)^{|\alpha|}\alpha$.
An inner product is denoted by $i \in \operatorname{alg}(W^\wedge, \operatorname{End}\Lambda)$.

I. Axiomatique classique

Vertical distribution and filtration of forms

Let E be fibered over oriented manifold $E \xrightarrow{\pi} \{M, \mathrm{vol}\}$. Then $\theta \equiv \pi^*\mathrm{vol} \in Z$ is a decomposable cocycle on E.

Let Ver be the associative distribution of θ, which is said to be a *vertical* distribution,

$$\mathrm{Ver} \equiv \mathrm{Ver}\,\theta \equiv \{X \in W;\ i_X\theta = 0\} \subset W.$$

There is one to one correspondence between a set of (vertical) distributions and a set of one-dimensional modules of decomposable forms. We will identify

$$\{E, \theta\} \equiv \{E, \mathrm{Ver}\}.$$

DEFINITION 1. *Let* Λ *be* $\mathcal{F}$*-algebra of differential forms on* E. *A* $\mathcal{F}$*-submodule*

$$\Lambda_{(k)} \equiv \{\alpha \in \Lambda,\ i_Z\alpha = 0 \quad \forall\, Z \in \mathrm{Ver}^{\wedge(k+1)}\},$$

is said to be a submodule of k-vertical *forms. The factor module is denoted by* $\Lambda_{[k]} \equiv \Lambda_{(k)}/\Lambda_{(k-1)}$ *and if* $\alpha \in \Lambda$ *then* $\alpha/(k) \in \Lambda/\Lambda_{(k)}$.

COROLLARY 2. $\Lambda_{(k)} \wedge \Lambda_{(l)} \subset \Lambda_{(k+l)}$, $\Lambda_{[k]} \wedge \Lambda_{[l]} \subset \Lambda_{[k+l]}$ *and we have a* filtration *of forms*

$$\Lambda_{(0)} \subset \ldots \subset \Lambda_{(k)} \subset \Lambda_{(k+1)} \subset \ldots \subset \Lambda.$$

COROLLARY 3. $\{\Lambda_{(k)}^{|\theta|+j}\}/(k-1) \neq 0$ *iff* $j \le k \le |\theta| + j$. *In particular the following implication holds* $|\Omega| = 1 + |\theta| \implies \Omega \cap \Lambda_{(0)} = 0$.

Let $\mathcal{V}er$ be a $\mathcal{F}$-submodule of a differential one-forms anihilating Ver

$$\mathcal{V}er \equiv \{\alpha \in \Lambda^1;\ \alpha(\mathrm{Ver}) = 0\}.$$

DEFINITION 4. *A differential form* α *on* $\{E, \mathrm{Ver}\}$ *is said to be* vertical *if* $\alpha \in \mathcal{V}er^{\wedge}$; $\mathcal{V}er^{\wedge 0} \equiv \mathcal{F}_E$.

A form θ is decomposable iff $\dim E = |\theta| + \dim \mathrm{Ver}\,\theta$.

LEMMA 5. *A form* α *is* 0*-vertical iff* α *is a vertical,* $\Lambda_{(0)} = \mathcal{V}er^{\wedge}$,

$$i_{\mathrm{Ver}}\alpha = 0 \iff \alpha \in \mathcal{V}er^{\wedge}.$$

LEMMA 6. *The following are equivalent*

$$\left\{ \begin{array}{l} \theta \in \mathcal{V}er^{\wedge} \\ |\theta| = \dim \mathcal{V}er \end{array} \right\} \iff \{\theta \text{ is decomposable}\}$$

Classical field theory

DEFINITION 7. *Let E be a vector bundle over oriented manifold $E \xrightarrow{\pi} \{M, vol\}$.*

(i) A classical field theory is a triple $\{E, \theta \equiv \pi^* vol, \Omega\}$, where Ω is a differential vol-dependent pseudoform on E such that $|\Omega| = 1 + \dim M$.

(ii) A subbundle ϕ of E is said to be a solution of $\{E, \theta, \Omega\}$ if for every vector field Z on E,

$$\phi^*\theta \neq 0 \quad \textit{and} \quad \phi^* i_Z \Omega = 0. \tag{1}$$

(iii) A field theory $\{E, \theta, \Omega\}$ is said to be regular *if every integrable distribution* Hor *tangent to solutions of equation (1) is complementary to* Ver*,*

$$\text{Hor} \cap \text{Ver} = 0 \quad \textit{and} \quad W = \text{Hor} \cup \text{Ver}.$$

Comment. The field theory is regular if every solution ϕ of (1) is transversal to Ver and $\dim \phi = |\theta|$.

A pseudoform Ω determine a $\mathcal{F}$-linear map $\Omega : W^{\wedge|\theta|} \longrightarrow \Lambda^1$. A pseudoform Ω can be viewed as a retrangular matrix $\binom{\dim E}{|\theta|} \times (\dim E)$.

PROPOSITION 8. *Let* $\ker \Omega \subset W^{\wedge|\theta|}$. *Then*

(i) $\dim \ker \Omega = 1 \quad \Longrightarrow \quad \{E, \text{Ver}, \Omega\}$ *is regular.*

(ii) Let $\{E, \text{Ver}, \Omega\}$ be regular, $\text{codim}\text{Ver} \equiv |\theta| = 1$ *and let Ω be a cocycle (so Ω is symplectic). Then* $\dim \ker \Omega = 1$, *($\Longrightarrow$* $\dim E =$*odd).*

Comment. If $|\theta| = 1$, then a cocycle Ω is regular iff $\dim \ker \Omega = 1$. The $|\theta| = 1$ refers to mechanics and the property to be regular is said to be the *classical determinism.* A symplectic mechanics, $d\Omega = 0$ with $\dim \ker \Omega = 1$, on jet manifolds of arbitrary order is presented in Thesis by Olga Krupkova (1992).

Example. Regular field theory $\{E, \theta, \Omega\}$ need not imply that $\dim \ker \Omega = 1$. Let $\dim E = 1 + 4n$ with a chart $\{t, q^A, v^A, p_A, f_A\}$ and $\theta \equiv dt$. Let $\Omega \equiv (dp_A - f_A dt) \wedge (dq^A - v^A dt)$, then $d\Omega \neq 0$, $\dim \ker \Omega = 1 + 2n$ and this mechanics $\{E, \theta, \Omega\}$ is *regular.*

Proof of Proposition 8. An integrable distribution $\text{Hor} \subset W$ tangent to solutions of field equations (1) needs to satisfy two conditions

– $\theta(\text{Hor}^{\wedge|\theta|}) \neq 0 \quad (\Longrightarrow \quad \dim \text{Hor} \geq |\theta| + \dim(\text{Hor} \cap \text{Ver})),$

– $\text{Hor}^{\wedge|\theta|} \subset \ker \Omega \quad \left(\Longrightarrow \quad \dim(\text{Hor}^{\wedge\theta|}) = \binom{\dim \text{Hor}}{|\theta|} \leq \dim \ker \Omega\right).$

The last condition imply

$$\dim\ker\Omega = 1 \quad\Longrightarrow\quad \dim\mathrm{Hor} \leq |\theta|.$$

It follows that $\mathrm{Hor}\cap\mathrm{Ver}=0$ and $\dim\mathrm{Hor} = |\theta| \equiv \mathrm{codim}\,\mathrm{Ver}$, which complete the proof of (i).

Let Ω be a cocycle. Then the associated distribution $\ker\Omega \subset W$ is integrable. If $|\theta| = 1$, then $\mathrm{Hor}=\ker\Omega$ is integrable and the regularity of Ω imply that $\dim\ker\Omega = 1$. □

COROLLARY 9 (Gawędzki 1972). *Let $\{E,\mathrm{Ver},\Omega\}$ be regular field theory. Then it is sufficient to consider the field equations (1) for a vertical vector fields only,*

$$\phi^* i_{\mathrm{Ver}}\Omega = 0 \quad\Longrightarrow\quad \phi^* i_W\Omega = 0.$$

Proof. Let a distribution Hor be as in the proof of Proposition 8. We must show an implication $\Omega(\mathrm{Ver}\wedge\mathrm{Hor}^{\wedge|\theta|}) = 0 \implies \Omega(W\wedge\mathrm{Hor}^{\wedge|\theta|}) = 0$. This is the case if $W = \mathrm{Hor}\cup\mathrm{Ver}$ and $\mathrm{Hor}^{\wedge(1+|\theta|)} = 0$. □

Subbundles

DEFINITION 10. *Let $\{E,\theta,\Omega\}$ be a classical field theory as in definition 7(i).*

(i) A subbundle $\Psi \hookrightarrow E$ is said to be a pre-symplectic *for $\{E,\theta,\Omega\}$ if Ψ is a* maximal *subbundle anihilating $d\Omega$,*

$$\Psi^*\theta \neq 0 \quad \textit{and} \quad \Psi^* d\Omega = 0.$$

A presymplectic bundle Ψ is said to be symplectic *if a field theory $\{\Psi,\Psi^*\theta,\Psi^*\Omega\}$ is regular. A regular cocycle $\Psi^*\Omega$ is said to be a* symplectic *form on $\{\Psi,\Psi^*\theta\}$.*

(ii) A subbundle $\mathcal{P} \hookrightarrow \Psi \hookrightarrow E$ is said to be the Poincaré-Cartan subbundle *(exact presymplectic) if $\mathcal{P}$ is a* maximal *subbundle on which Ω is exact,*

$$\mathcal{P}^*\theta \neq 0 \quad \textit{and} \quad \mathcal{P}^*\Omega = d\alpha.$$

A presymplectic potential α is said to be the Poincaré-Cartan form. If a field theory $\{\mathcal{P},\mathcal{P}^*\theta,d\alpha\}$ is regular then α is said to be a regular *Poincaré-Cartan form.*

(iii) A subbundle $\mathcal{L} \hookrightarrow \Psi \hookrightarrow E$ is said to be a lagrangian *for $\{E,\theta,\Omega\}$ if $\mathcal{L}$ is a* maximal *subbundle anihilating Ω,*

$$\mathcal{L}^*\theta \neq 0 \quad \textit{and} \quad \mathcal{L}^*\Omega = 0.$$

(iv) *A subbundle $\mathcal{J} \hookrightarrow \mathcal{P} \hookrightarrow \Psi \hookrightarrow E$ is said to be the* Hamilton-Jacobi *bundle if $\mathcal{J}$ is a* maximal *subbundle on which a Poincaré-Cartan form is exact,*

$$\mathcal{J}^*\theta \neq 0 \quad and \quad \mathcal{J}^*\alpha = dS.$$

A potential S is said to be the Hamilton-Jacobi form, $|S| = |\theta| - 1$. The equations in (iii-iv) are said to be the Hamilton-Jacobi equations.

A Poincaré-Cartan form strictly speaking is *vol*-dependent and therefore is a pseudoform.

On presymplectic subbundle $\Psi^*\Omega$ is a cocycle, the action integral is well defined (see e.g. Oziewicz 1992) and a field equation of definition 7 (ii) is the Euler-Lagrange equation.

A (pre)symplectic Ψ and Poincaré-Cartan $\mathcal{P}$ subbundles are known as a phenomenological material relations, $p = mv$, Kepler problem $f_A = -q^{-3}q_A$, $D = \varepsilon_0 E$, $B = \mu_0 H$, London equation $J_\mu = A_\mu$, etc.

Jacobi in 1838 proved that in mechanics the lagrangian and the Hamilton-Jacobi subbundles, $\mathcal{L}$ and $\mathcal{J}$, are families of solutions, $\phi \hookrightarrow \mathcal{L}$ and $\phi \hookrightarrow (\mathcal{P} \circ \mathcal{L}) \hookrightarrow \mathcal{J}$. A coordinate-free proof is in (Oziewicz and Gruhn 1983). An extension of the Jacobi theorem beyond mechanics is not known.

The following table gives dimensions of subbundles for a *phenomenological field theory*, formula (10) below, and follows from the considerations in the part II, see definition (14) and formulas (15-17).

	E	$\hookleftarrow \Psi \hookleftarrow \mathcal{P} \hookleftarrow$	$\mathcal{J} \hookleftarrow$	ϕ
	$\downarrow$dim	$\downarrow$dim	$\downarrow$dim	dim$\downarrow$
mechanics	1+4n	1+2n	1+n	1
strings	2+6n	2+3n	2+n	2
electrostatics	3+8n	3+4n	3+n	3
magnetostatics	3+12n	3+6n	3+3n	3
Klein-Gordon's n-fields	4+10n	4+5n	4+n	4
electromagnetism	4+20n	4+10n	4+4n	4
	God's choice	material relations	"quantum space"	time space space-time

Hamilton-Lagrange field theory

DEFINITION 11. *A field theory* $\{E,\theta,\Omega\}$ *is said to be* k-vertical *if* $0 \neq d\Omega \in \Lambda_{(k)}$ *and* $d\Omega \notin \Lambda_{(k-1)}$ *or if* $d\Omega = 0$, $\Omega \in \Lambda_{(k)}$ *and* $\Omega \notin \Lambda_{(k-1)}$.

Comment. For a cocycle $\Omega \in Z$ the definition 11 was introduced by Kondracki (1978). The notion of the k-vertical field theory is essential for the theory of the Poincaré-Cartan forms if $\Omega \notin Z$ and for the Hamilton-Jacobi theory if $\Omega \in Z$.

Because a distribution Ver is integrable threfore

$$\left\{ \begin{array}{l} d\Omega \in \Lambda_{(k)} \\ d\Omega \notin \Lambda_{(k-1)} \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{l} \Omega \in \Lambda_{(k-1)} \oplus Z \\ \Omega \notin \Lambda_{(k-2)} \oplus Z \end{array} \right\}.$$

A form $d\Omega \neq 0$ is k-vertical iff Ω can be decomposed (not uniquely) as the sum of $(k-1)$-vertical form and a cocycle. Two fibrations of de Rham complex Λ are involved in this decomposition: first over a factor $\mathcal{F}$-module $\Lambda/\Lambda_{(k-1)}$, second over a factor $\mathbb{R}$-space Λ/Z. A form $d\Omega$ is k-vertical if exist splittings

$$\begin{array}{ccc} \Lambda/Z & \xrightarrow{\mu} & \Lambda \\ & & \uparrow \nu \\ & & \Lambda/\Lambda_{(k-1)} \end{array}$$

such that

$$\mu(\Omega/Z) \in \Lambda_{(k-1)}, \qquad \nu(\Omega/(k-1)) \in Z$$

$$\text{and} \qquad \Omega = \mu(\Omega/Z) + \nu(\Omega/(k-1)). \tag{2}$$

Above splittings are not unique, they are determined up to the $(k-1)$-vertical cocycles

$$\Lambda_{(k-1)} \cap Z \overset{loc}{\simeq} d\Lambda_{(k-2)}.$$

Let a field theory $\{E,\theta,\Omega\}$ be k-vertical. Then a splitting μ determine a splitting ν and vice versa. Locally

$$\nu(\Omega/(k-1)) \overset{loc}{\simeq} d\omega,$$

$$\text{and} \quad \Omega = f + d\omega, \text{ where } f \equiv \Omega - d\omega \in \Lambda_{(k-1)}. \tag{3}$$

In (3) a differential form ω is determined modulo $(k-2)$-vertical forms. On Poincaré-Cartan subbundle $\mathcal{P} \hookrightarrow E$, Ω and f are exact,

$$dF \equiv \mathcal{P}^* f, \qquad d\alpha_F \equiv \mathcal{P}^*\Omega, \tag{4}$$

$$\text{and} \qquad \alpha_F = F + \mathcal{P}^*\omega \mod Z_{\mathcal{P}}.$$

In the last section we shall show that exists a correlation between decompositions (2-3) and Poincaré-Cartan subbundles.

Depending on choice of f in (3), a potential F could coincide (up to sign) with a hamiltonian H or with a lagrangian L (see the next sections), however a freedom in the decomposition (2-3) allows to see more possibilities.

If f is $(k-1)$-vertical on $\{E, \text{Ver}\}$, then F in (4) is $(k-2)$-vertical on $\{\mathcal{P}, \mathcal{P}^*\theta\}$,

$$(\Lambda^{|\theta|}_{(k-2)} \oplus Z_{\mathcal{P}}) \ni F \quad \longmapsto \quad dF \in \Lambda^{|\theta|+1}_{(k-1)}. \tag{5}$$

The Poincaré-Cartan equation $dF = \mathcal{P}^* f$ (4), allows to express $\mathcal{P}^*\omega$ in terms of partial derivatives of F wrt a basis of a $\mathcal{F}_{\mathcal{P}}$-module $\Lambda^{|\theta|+1}_{(k-1)} \subset \Lambda_{\mathcal{P}}$. Therefore a differential form F determine a Poincaré-Cartan form,

$$\Lambda^{|\theta|}_{(k-2)}/Z_{\mathcal{P}} \ni F \quad \longmapsto \quad \alpha_F \in \Lambda^{|\theta|}_{\mathcal{P}} \mod Z_{\mathcal{P}}.$$

This motivate the definition

DEFINITION 12. *Let a distribution* Ver *be integrable and* $\Omega \notin Z$.

(i) Let $2 \leq k \leq |d\Omega|$ *and let a field theory* $\{E, \text{Ver}, \Omega\}$ *be* k *vertical,* $d\Omega \in \Lambda_{(k)}$ *and* $d\Omega \notin \Omega_{(k-1)}$. *Then a* $\mathcal{F}_{\mathcal{P}}$*-module* $\Lambda^{|\theta|}_{(k-2)}$ *is said to be a module* generating *Poincaré-Cartan forms.*

(ii) A field theory $\{E, \theta, \Omega\}$ *is said to be a* Hamilton-Lagrange *field theory, abbreviated by* HL, *if the* $\mathcal{F}_{\mathcal{P}}$*-dimension of a generating module of Poincaré-Cartan forms is 1.*

For HL field theory a Poincaré-Cartan form is determined by *one* (pseudo)-scalar function, (lagrangian, hamiltonian, ...).

LEMMA 13. *A field theory* $\{E, \text{Ver}, \Omega \notin Z\}$ *is* HL *iff* $d\Omega$ *is bi-vertical,* $d\Omega \in \Lambda_{(2)}$.

Proof.

$$\dim\{\Lambda^{|\theta|}_{(k-2)}\} = \sum_{i=0}^{min(k-2,|\theta|)} \binom{\dim \text{Ver}}{i}.$$

Therefore

$$\dim\{\Lambda^{|\theta|}_{(k-2)}\} = 1 \quad \text{iff} \quad \{E, \text{Ver}, \Omega \notin Z\} \text{ is bi-vertical.}$$

Comment. In HL field theory a (local) hamiltonian and lagrangian F (4) are *vertical* differential forms on $\mathcal{P}$. For not HL theories analogous hamiltonian or lagrangian forms are no more vertical and therefore can not be expressed by means of one pseudoscalar function. Analogous considerations are valid for the Hamilton-Jacobi theory if $\Omega \in Z$.

Partial derivatives of vertical forms. Note that

$$\dim\{\Lambda_{(1)}^{|\theta|+1}\} = \dim \text{Ver}.$$

We will suppose that a modul $\Lambda_{(1)}^{|\theta|+1}$, on E as well as on subbundle $\mathcal{P}$, is generated by differentials of homogeneous vertical forms (in general of different degrees),

$$\Lambda_{(1)}^{|\theta|+1} \equiv \text{gen}\{dw^A,\ w^A \in \Lambda_{(0)}\}.$$

This means that $\forall\, \alpha \in \Lambda_{(1)}^{|\theta|+1}$ has unique decomposition

$$\alpha = dw^A \wedge \alpha_A, \qquad w^A, \alpha_A \in \Lambda_{(0)}.$$

In particular a generating set $\{dw^A\}$ determines partial derivatives of highest degree vertical forms

$$\Lambda_{(0)}^{|\theta|} \ni F \longmapsto dF = dw^A \wedge \frac{\partial F}{\partial w^A} \in \Lambda_{(1)}^{|\theta|+1}.$$

Example

$$dL \equiv dq^A \wedge \frac{\partial L}{\partial q^A} + dv^A \wedge \frac{\partial L}{\partial v^A}.$$

II. Phenomenology

Phenomenological field theory

Let E be a vector bundle over oriented manifold $E \xrightarrow{\pi} \{M, vol\}$. Let $\{q^A, v^A;\ A \in I \subset I\!N\}$ be a collection of vertical differential forms on a bundle E and let $\{p_A, f_A;\ A \in I \subset I\!N\}$ be a collection of *vol*-dependent vertical differential pseudoforms on E. Newton's and Maxwell's phenomenological equations as well as of electrostatics and of magnetostatics have the following form

$$\begin{aligned} \vartheta^A &\equiv dq^A - v^A, & \phi^*\vartheta^A &= 0,\\ \omega_A &\equiv dp_A - f_A, & \phi^*\omega_A &= 0. \end{aligned} \tag{6}$$

A differential forms $\{q^A, v^A, p_A, f_A\}$ in (6) are *independent*, as they are determined by independent experiments. A phenomenological material relations among these fields are consequence of further independent measurements. This was stressed by Newton (1686) and Maxwell. After Lagrange it became customary to present the Newton equations (as well as of electrodynamics

$\Box A = j$) as a second order from the begining, contrary to original Newton's presentation. That the equations (6) should not presuppose a material relations was stressed by Piron (e.g. in Piron's *Lectures on electrodynamics*, 1989).

A phenomenological material relations are equations for a (pre)symplectic subbundle of a field theory $\{E, \theta, \Omega\}$ (definition 10 (i)), and are given in the last section.

A strategy is to determine a most general *regular* field theory $\{E, \theta, \Omega\}$ which field equations (1) coincide with the experimental one (6). A different field theories with the same set of a *first* order equations (6) will lead to a different (pre)symplectic subbundles and therefore to a different *second* order equations. A solutions ϕ of equations (6) anihilate an ideal generated by $\{\vartheta^A, \omega_A\}$ therefore for a regular field theory we need an equality of ideals,

$$\text{gen}\{i_{\text{Ver}}\Omega\} = \text{gen}\{\vartheta^A, \omega_A\}. \tag{7}$$

Because a distribution Ver is integrable then $\{\vartheta^A, \omega_A\}$ are 1-vertical. A most general pseudoform Ω compatible with (7) needs to be 2-vertical of the form

$$\Omega \equiv \sum_{A,B} (K^A_B \wedge \omega_A \wedge \vartheta^B + \Gamma_{AB} \wedge \vartheta^A \wedge \vartheta^B + \chi^{AB} \wedge \omega_A \wedge \omega_B), \tag{8}$$

where $\{K^A_B, \Gamma_{AB}, \chi^{AB}\}$ is a collection of vertical (pseudo)forms such that

$$\Gamma_{AB} = (-1)^{|\vartheta^A||\vartheta^B|}\Gamma_{BA}, \qquad \chi^{AB} = (-1)^{|\omega_A||\omega_B|}\chi^{BA}.$$

DEFINITION 14. *A field theory* $\{E, \text{Ver}, \Omega\}$ *with* Ω *of the form (8) is said to be a* phenomenological field theory.

Because Ω is $\mathbb{N}$-homogeneous then

$$\begin{aligned} |v^A| &= 1 + |q^A|, \\ |f_A| &= 1 + |p_A|, \\ |q^B| + |f_A| + |K^A_B| &= |\theta|, \\ |v^B| + |p_A| + |K^A_B| &= |\theta|. \end{aligned} \tag{9}$$

In mechanics $|K| = |\Gamma| = |\chi| = 0$. A conditions

$$\forall\, A \quad |\omega_A| \geq |\vartheta^A| \quad \text{and} \quad |K| = |\chi| = 0,$$

determine unique grades for mechanics and string theory and n possibilities for $|\theta| = 2n - 1$ and $2n$. In this case χ can contribute in mechanics and magnetostatics only.

$\lvert\theta\rvert$	$\lvert q\rvert$	$\lvert v\rvert$	$\lvert p\rvert$	$\lvert f\rvert$	$\lvert\Gamma\rvert$	
1	0	1	0	1	0	mechanics
2	0	1	1	2	1	strings
3	0	1	2	3	2	electrostatics
	1	2	1	2	0	magnetostatics
4	0	1	3	4	3	Klein-Gordon scalar fields
	1	2	2	3	1	electromagnetic field

If $\{K, \Gamma, \chi\}$ are vertical cocycles then field theory (8) is HL, $d\Omega$ is bivertical, and

$$\Omega = \{d(K_A^B \wedge p_B + \psi\Gamma_{AB} \wedge q^B) - (K_A^B \wedge f_B + \Gamma_{BA} \wedge v^B)\} \wedge \vartheta^A + \omega^A \wedge \omega_A.$$

Effectively "momenta-induction" and "force-current" are rotated and translated by "connection Γ", a natural description of velocity-dependent forces,

$$\begin{aligned} p_A &\longmapsto K_A^B \wedge p_B + \psi\Gamma_{AB} \wedge q^B, \\ f_A &\longmapsto K_A^B \wedge f_B + \ \Gamma_{BA} \wedge v^B. \end{aligned}$$

A phenomenological *symplectic mechanics* (8) without χ-terms has been considered by Jadczyk and Modugno (1992).

Consider HL field theory

$$\Omega \equiv \sum \omega_A \wedge \vartheta^A \quad \in \Lambda_{(1)} \oplus Z \quad \subset \Lambda_{(2)}. \tag{10}$$

The following decompositions, like (2-3), define 1-vertical differential forms $\{h, l, s, t\}$,

$$\begin{aligned} \Omega &= -h + d(p_A \wedge dq^A) \\ &= +l + d\{p_A \wedge (dq^A - v^A)\} \\ &= +s + d\{q^A \wedge \psi^{|\theta|}(dp_A - f_A)\} \\ &= +t + d\{q^A \wedge \psi^{|\theta|}(dp_A - f_A) - p_A \wedge v^A\}. \end{aligned} \tag{11}$$

Therefore

$$\begin{aligned} h &\equiv dp_A \wedge v^A \qquad + dq^A \wedge \psi^{|\theta|} f_A, \\ l &\equiv dv^A \wedge \psi^{1+|\theta|} p_A - dq^A \wedge \psi^{|\theta|} f_A, \\ s &\equiv dp_A \wedge v^A \qquad - df_A \wedge (-)^{|\theta|}\psi q^A, \\ t &\equiv dv^A \wedge \psi^{1+|\theta|} p_A + df_A \wedge (-)^{|\theta|}\psi q^A. \end{aligned} \tag{12}$$

$$\begin{aligned} l + h \equiv t + s &\equiv d(p_A \wedge v^A), \\ h - s \equiv t - l &\equiv d(\psi f_A \wedge q^A). \end{aligned} \tag{13}$$

If differential forms $\{q^A, v^A, p_A, f_A\}$ are the Liouville forms, considered in the next section, then pseudoform Ω (10) is regular and imply equations (6).

The Liouville differential forms

Let M be a manifold and for $p \in M$, T^*_pM be $\mathbb{R}$-space of exterior forms at p. Let $T^kM \xrightarrow{\pi} M$, be a vector bundle of exterior k-forms, $(T^kM)_p \equiv (T^*_pM)^{\wedge k}$, with $(T^*_pM)^{\wedge 0} \equiv \mathbb{R}$.

A differential form $\alpha \in \Lambda^k_M$ determines unique section $\alpha_s \in \Gamma(M, T^kM)$ and $\Lambda_{T^kM} \ni \lambda \longrightarrow \alpha^*_s\lambda \in \Lambda_M$,

$$\begin{array}{ccc} \Lambda_M & \xleftarrow{\alpha_s^*} & \Lambda_{T^kM} \\ \Lambda\uparrow & & \Lambda\uparrow \\ M & \xrightarrow{\alpha_s} & T^kM \end{array}$$

DEFINITION 15. *A differential k-form $\lambda \in \Lambda_{T^kM}$ is said to be the Liouville form if*

$$\alpha^*_s\lambda = \alpha \quad \textit{for every} \quad \alpha \in \Lambda^k_M.$$

The Liouville differential k-form exists, is unique and has a local form

$$\lambda = \frac{1}{k!}\sum \lambda_{\mu_1\ldots\mu_k}\pi^*(dt^{\mu_1}\wedge\ldots\wedge dt^{\mu_k}).$$

The Liouville differential forms of arbitrary degree has been introduced by Tulczyjew in 1979. The Liouville forms are vertical wrt $\theta \equiv \pi^*\mathrm{vol}_M$.

Let E be a vector bundle $E \xrightarrow{\pi} M$ of an exterior forms of different degrees on a manifold M,

$$E \equiv \bigoplus_A \left\{(T^{|q^A|}M)\oplus(T^{|v^A|}M)\oplus(T^{|p_A|}M)\oplus(T^{|f_A|}M)\right\}. \tag{14}$$

Let $\{q^A, v^A, p_A, f_A\}$ be a collection of Liouville's forms on E. Define *dimension* of a form as dimension of a factor module,

$$\dim\alpha \equiv \dim\,\mathrm{Ver}\,\theta - \dim\{(\mathrm{Ver}\,\alpha)\cap\mathrm{Ver}\,\theta\}. \tag{15}$$

One of necessary condition for implication (8) $\Rightarrow$ (6) is

$$\dim\mathrm{Ver} = \sum_A(\dim\,dq^A + \dim\,dv^A + \dim\,dp_A + \dim\,df_A). \tag{16}$$

If λ is the Liouville form on E then according to definition (15-16),

$$\dim(d\lambda) = \binom{\dim\,M}{|\lambda|}. \tag{17}$$

A field theory $\{E\,(14), \pi^*\mathrm{vol}_M, \Omega\,(10)\}$ is regular and imply equations (6). From formula (17) we get the dimensions, $\dim E$, listed in the Table after definition 10. The Liouville forms $\{K, \Gamma, \chi\}$, for simplicity, are not included in the bundle E (14). The Liouville forms $\{K, \Gamma, \chi\}$ in (8) contribute to $\dim E$. In mechanics with (8), $|\theta| = 1$, $\dim E = 1 + 4n + 2n^2 - n$.

Poincaré-Cartan subbundles

Consideration of this section are the same for the case of presymplectic and Poincaré-Cartan subbundles. To be specific we will consider Poincaré-Cartan subbundles only, definition 10 (ii), and for E being a vector bundle of exterior forms (14) with conditions (9).

Let a bundle E be splitted with a fiber-preserving projectors π_P and π_C,

$$\begin{array}{ccccc} P & \xleftarrow{\pi_P} & E = P \oplus C & \xrightarrow{\pi_C} & C \\ \downarrow \pi & & \downarrow \pi & & \downarrow \pi \\ M & = & M & = & M \end{array}$$

Let subbundles P and C be of equal dimensions, dim Ver (16) on E is even. For a Poincaré-Cartan subbundle $\mathcal{P} \hookrightarrow E$ we have $\dim\mathcal{P} = \dim P = \dim C$. Let $\pi_P|\mathcal{P}$ be a fiber-preserving isomorphism. A Poincaré-Cartan subbundle $\mathcal{P}$ will be identified with an injection $\mathcal{P} : P \hookrightarrow \mathcal{P} \subset E$, $\pi_P \circ \mathcal{P} = \mathrm{id}_P$, and $\varphi \equiv \pi_C \circ \mathcal{P} : P \longrightarrow C$ is a fiber-preserving bundle map. If $\theta \equiv \pi^*\mathrm{vol}_M$ then $\mathcal{P}^*\theta \equiv \theta \in \Lambda_P$.

Consider splittings of a bundle E for which Ω have the following form as in decomposition (2-3),

$$\Omega = \pm \sum d\pi_P^*\alpha \wedge \pi_C^*\beta + d\omega. \tag{18}$$

On Poincaré-Cartan subbundle a form $\sum d\pi_P^*\alpha \wedge \pi_C^*\beta$ is exact,

$$\mathcal{P}^*(\sum d\pi_P^*\alpha \wedge \pi_C^*\beta) = \sum d\alpha \wedge \varphi^*\beta \equiv dF \in \Lambda_P.$$

Therefore

$$\varphi^*\beta \equiv \frac{\partial F}{\partial \alpha}.$$

A differential forms $\{\Omega, h, l, s, t\}$ (10-13) are exact on Poincaré-Cartan subbundle $\mathcal{P} \hookrightarrow \Psi \hookrightarrow E$. In particular a hamiltonian H and a lagrangian L are differential forms on *different* Poincaré-Cartan subbundles and are defined as potentials,

$$dH \equiv \mathcal{P}_h^*h, \quad dL \equiv \mathcal{P}_l^*l, \quad dS \equiv \mathcal{P}_s^*s, \quad dT \equiv \mathcal{P}_t^*t. \tag{19}$$

A compositions like $\mathcal{L} \equiv \mathcal{P}_h^{-1} \circ \mathcal{P}_l$ etc, are said to be Legendre's transforms. With help of identities (13) Legendre's transforms allow to calculate, for example, a lagrangian L for a given hamiltonian H,

$$\begin{aligned} dL = \mathcal{P}_l^*l &= (\mathcal{P}_l^* \circ \mathcal{P}_h^{-1*} \circ \mathcal{P}_h^*)\{d(p_A \wedge v^A) - h\} \\ &= \mathcal{L}^*\{d\mathcal{P}_h^*(p_A \wedge v^A) - dH\}. \end{aligned}$$

Therefore, modulo cocycles

$$L = (\mathcal{L}^* \circ \mathcal{P}_h^*)(p_A \wedge v^A) - \mathcal{L}^*H \mod Z.$$

A Poincaré-Cartan forms $\{\alpha,\, d\alpha \equiv \mathcal{P}^*\Omega\}$ can be expressed in terms of L, H, S, and T if we identify the decompositions (11-12) with (18-19).

- $$\varphi_h^* v^A \equiv \frac{\partial H}{\partial p_A}, \qquad \varphi_h^* f_A \equiv \psi^{|\theta|}\frac{\partial H}{\partial q^A},$$

$$\alpha_H \equiv -H + p_A \wedge dq^A \mod Z_{\mathcal{P}},$$

$$d\alpha_H = \left(dp_A - \psi^{|\theta|}\frac{\partial H}{\partial q^A}\right) \wedge \left(dq^A - \frac{\partial H}{\partial p_A}\right). \tag{20}$$

$$\varphi_l^* p_A \equiv \psi^{(1+|\theta|)}\frac{\partial L}{\partial v^A}, \qquad \varphi_l^* f_A \equiv -\psi^{|\theta|}\frac{\partial L}{\partial q^A},$$

$$\alpha_L \equiv L + \frac{\partial L}{\partial v^A} \wedge \psi^{(1+|\theta|)}(dq^A - v^A) \mod Z_{\mathcal{P}},$$

$$d\alpha_L = \left\{d\frac{\partial L}{\partial v^A} - (-)^{|\theta|}\psi\frac{\partial L}{\partial q^A}\right\} \wedge \psi^{(1+|\theta|)}(dq^A - v^A). \tag{21}$$

$$\varphi_s^* v^A \equiv \frac{\partial S}{\partial p_A}, \qquad \varphi_s^* q^A \equiv (-)^{1+|\theta|}\psi\frac{\partial S}{\partial f_A},$$

$$\alpha_S \equiv S + \frac{\partial S}{\partial f_A} \wedge \psi^{(1+|\theta|)}(dp_A - f_A) \mod Z_{\mathcal{P}},$$

$$d\alpha_S = -\left\{d\frac{\partial S}{\partial f_A} - (-)^{|\theta|}\psi\frac{\partial S}{\partial p_A}\right\} \wedge \psi^{(1+|\theta|)}(dp_A - f_A). \tag{22}$$

$$\varphi_t^* p_A \equiv \psi^{(1+|\theta|)}\frac{\partial T}{\partial v^A}, \qquad \varphi_t^* q^A \equiv (-)^{|\theta|}\psi\frac{\partial T}{\partial f_A},$$

$$\alpha_T \equiv T + \left\{d\psi^{1+|\theta|}\frac{\partial T}{\partial v^A} - f_A\right\} \wedge \frac{\partial T}{\partial f_A} - v^A \wedge \frac{\partial T}{\partial v^A} \mod Z_{\mathcal{P}},$$

$$d\alpha_T = \left\{(-)^{1+|\theta|}d\frac{\partial T}{\partial f_A} - \psi v^A\right\} \wedge \left\{d\frac{\partial T}{\partial v^A} - (-\psi)^{1+|\theta|}f_A\right\}. \tag{23}$$

A (f, v)-subbundle (23) in electromagnetism was considered by Thirring (1979, p. 109) and therefore one is tempted to call a differential form T as a *thirringian.*

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HAMILTON'S PRINCIPLE FOR CONSTRAINED SYSTEMS

MAURO BREGOLA and CARLO FERRARIÓ
Dipartimento di Fisica Via Paradiso 12 - 44100 Ferrara - Italy

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Abstract. The Hamilton's principle and the Lagrangian formalism in presence of constraints have been analyzed. The differences between the degenerate and the nonholonomic case are intrinsically characterized.

Key words: degenerate Lagrangians – geometric quantization – nonholonomic systems

1. Introduction

When text books introduce Lagrangian mechanics, they implicitly assume that:
(a) the dynamics is described by a second order vector field which is defined on all TQ (i.e. the initial conditions for the differential equations can be arbitrarily chosen in TQ), TQ being the tangent space of the configuration space Q;
(b) the fiber derivative of the Lagrangian $\mathcal{L}$

$$F\mathcal{L} : TQ \longrightarrow T^*Q$$

(T^*Q being the cotangent bundle) is, at least locally, a diffeomorphism. In such a case $\mathcal{L}$ is said to be *regular* or *standard*.

By using these hypotheses in an essential way one can deduce the Euler-Lagrange equations from a variational principle, develop the Lagrangian formalism (in particular the Noether theorem), and build up the Hamiltonian description of motion. But in a lot of physically meaningful cases, the motion of the system is confined to a submanifold of TQ where *a priori* there is no reason to believe that a Lagrangian description would be possible. As a matter of fact the constraint submanifold does not generally maintain a tangent bundle structure.

Nevertheless, by judiciously transferring the principal intrinsic tools of TQ on those submanifolds, it is possible to preserve a Lagrangian point of view. Hence, under particular conditions, we will verify that suitable Euler-Lagrangian vector fields make the action functional stationary and we will connect symmetries of the Lagrangian with conservation laws. These results will be valid in both cases of degenerate Lagrangian (typically the

Lagrangian density in the gauge theories [1]) and of the systems with constraints introduced from the outside (nonholonomic constraints). In the first case, as is well known, the constraints are implicitly linked with the degeneracy of the Lagrangian. The differences between the two situations will be stressed.

2. Lagrangian mechanics on TQ

It is known [2] that it is possible to define on TQ a (1-1)-type tensor field S (the so-called vertical endomorphism) intrinsically, with the properties:

$$(i)\ \ \mathrm{Im}\, S = \ker S = V(TQ)$$
$$(ii)\ \ N_S(X,Y) = 0 \quad \forall X, Y \in \mathcal{X}(TQ)$$

where V(TQ) represents the set of the vertical vector fields defined on TQ and N_S the so-called Nijenhuis tensor

$$N_S(X,Y) = S^2[X,Y] + [SX,SY] - S[SX,Y] + S[Y,SX].$$

S endows TQ with the structure of an integrable, almost tangent manifold. Moreover it induces a derivation with grading rank 0 and allows to define the *vertical derivative*:

$$d_S := [i_S, d];$$

in particular, the action of d_S on functions defined on TQ is

$$d_S f = (df) \circ S \qquad f \in \mathcal{F}(TQ). \tag{1}$$

Use will be made herein of the well-known Liouville vector field $\Delta \in \mathcal{X}(TQ)$. Indeed a second order vector field Γ_0 will be such that $S(\Gamma_0) = \Delta$.

If the Lagrangian $\mathcal{L}$ of a system is regular, the 2-form $\omega = -dd_S\mathcal{L}$ bestows TQ with a symplectic structure; moreover $E := (L_\Delta - 1)\mathcal{L}$ is the usual Lagrangian energy. Besides, use will be made of the semi-basic 1-form

$$\Lambda = dE - i_\Gamma \omega,$$

Γ being the second order dynamics.

As is known in literature [3], the stationarity condition for the functional

$$\int_{t_1}^{t_2} \mathcal{L} dt$$

gives rise to the condition

$$- \int_{t_1}^{t_2} i_X \Lambda dt + \int_{t_1}^{t_2} \left(i_{(\Delta - S\Gamma)} L_X d\mathcal{L} + i_{[X, \Delta - S\Gamma]} d\mathcal{L} \right) dt$$

$$- \int_{t_1}^{t_2} i_{S[\Gamma, X]} d\mathcal{L} dt + \int_{t_1}^{t_2} L_\Gamma \left(i_{SX} d\mathcal{L} \right) dt = 0. \tag{2}$$

Here $X \in \mathcal{X}(TQ)$ represents the variation vector field. In the absence of any constraint, Γ is a well-defined second order vector field ($S\Gamma - \Delta = 0$), X generates point transformations ($S[X, \Gamma] = 0$), therefore

$$dE - i_\Gamma \omega = 0 \tag{3}$$

are the Euler-Lagrange equations for the problem: the integral curves of Γ are critical curves for the above functional.

In order to complete the description of the regular case, we remember that the vector field

$$X(\Gamma) := X + S[\Gamma, X]$$

generates a Cartan symmetry for the Lagrangian if $\exists F \in \mathcal{F}(TQ)$ s.t.

$$L_{X(\Gamma_0)} \mathcal{L} = L_{\Gamma_0} F \qquad \forall \Gamma_0 : S(\Gamma_0) = \Delta. \tag{4}$$

In this case the Cartan symmetry gives rise to a conservation law and represents a dynamical symmetry for Γ; moreover, the energy is constant on its integral curves.

As proved in [4], this characterization of the symmetries allows the proof of a converse Noether theorem which is completely specular to the direct version and leads to a formulation of the theorem which is completely equivalent to the Hamiltonian approach.

3. Constrained motions

(a) nonholonomic systems

Consider the Lagrange equations (3) in the presence of a velocity-dependent constraint

$$\mathcal{F}(TQ) \ni \Phi = 0;$$

let $Z \in \mathcal{X}(TQ)$ be the Hamiltonian vector field which is the solution of the algebraic equation

$$i_Z \omega = d\Phi;$$

consequently

$$i_{SZ} \omega + d_S \Phi = 0. \tag{5}$$

Moreover, taking (3) and (5) into account, we can write

$$dE - i_{\tilde{\Gamma}}\omega = \lambda d_S \Phi, \tag{6}$$

where $\tilde{\Gamma} = \Gamma + \lambda SZ$ is of course a second order vector field. As clarified in [5] the tangency condition of $\tilde{\Gamma}$ to the constraint submanifold determines the multiplier λ. By inserting (6) into (2) it is easy to prove that the stationarity of the functional is guaranteed by

$$L_{SX}\Phi = 0. \tag{7}$$

This is easy to prove since $\tilde{\Gamma}$ is of second order, $X(\tilde{\Gamma})$ is of course Newtonian and, finally, the condition

$$i_{SX} d\mathcal{L}|_{t_1}^{t_2} = 0$$

remains unchanged with regards to the regular case.

But otherwise (7) is not a condition, it is a consequence of the D'Alembert principle: one can see [6] that the vector fields X satisfying (7) generate virtual displacements which are orthogonal to the constraint force associated with Φ. $\tilde{\Gamma}$ is thus the actual constrained dynamics. Note that the fact, that X is not generally tangent to the constraint submanifold, does not represent an obstruction for the validity of Hamilton's principle. X is only supposed to be a complete lifting of a vector field belonging to $\mathcal{X}(Q)$.

We may now discuss the Noether theorem. Introducing

$$G := i_X d_S \mathcal{L} - F \tag{8}$$

and taking (4) into account, we obtain

$$L_{\tilde{\Gamma}} G = i_{X(\tilde{\Gamma})}\left(dE - i_{\tilde{\Gamma}}\omega\right) = \lambda i_{X(\tilde{\Gamma})} d_S \Phi$$

that is

$$L_{\tilde{\Gamma}} G = 0 \Longleftrightarrow L_{SX}\Phi = 0. \tag{9}$$

Again, the main result of Lagrangian formalism is preserved on condition that infinitesimal displacements remain orthogonal to the constraint force; we have to underline that generally the introduction of these constraints does not allow us to relate Cartan symmetries with the dynamical ones. As a matter of fact it is easy to exhibit very simple counterexamples in which $[X(\tilde{\Gamma}), \tilde{\Gamma}] \neq 0$ and the reason resides essentially in the fact that X can be non tangent to the evolution space. In this case X does not transform integral curves into integral curves of the dynamical vector field.

Finally it is relevant to verify that the condition on SX is crucial for the converse theorem as well. Suppose that a function $G \in \mathcal{F}(TQ)$ remains constant during the constrained time evolution:

$$L_{\tilde{\Gamma}} G = 0$$

hence, if

$$i_{X_G}\omega = dG,$$

we define

$$F := i_{X_G} d_S \mathcal{L} - G.$$

By differentiating this last expression we get

$$L_{X_G} G = i_{X_G} \left(L_{\tilde{\Gamma}} d_S \mathcal{L} - \lambda d_S \Phi \right)$$

from which it follows that

$$L_{X_G} G + i_{S[\tilde{\Gamma}, X_G]} d\mathcal{L} = L_{\tilde{\Gamma}} F - \lambda L_{S X_G} \Phi.$$

Once again

$$L_{X(\Gamma_0)}\mathcal{L} = L_{\Gamma_0} F \quad (\forall \Gamma_0 : S(\Gamma_0) = \Delta) \iff L_{S X_G}\Phi = 0.$$

(b) degenerate systems

We now want to consider the case in which $\mathcal{L}$ is not regular, or equivalently $\omega = -dd_S\mathcal{L}$ is degenerate [7]. In such a case we introduce the vector field set

$$Ker\,\omega = \{K \in \mathcal{X}(TQ) | i_K\omega = 0\}$$

together with

$$V(Ker\,\omega) = Ker\,\omega \cap V(TQ).$$

The motion of the system could eventually be restricted to the *primary constraint* submanifold $M \subset TQ$ by looking for a solution for (3) which should also be of the second order; other constraints can arise if one requires the tangency of the solution to M. At the end of the analysis one obtains a dynamics Γ which may exhibit terms depending linearly on arbitrary functions of time. In fact one obtains a class of equivalence $\{\Gamma\}$ of solutions. Note that in the present discussion we do not make use of the Dirac theory but rather of the Lagrangian constraint approach.

It was proved by Gotay, Nestor and Hinds (1978)[7] that the intrinsic expression for the primary Lagrangian constraints is

$$L_{Z(\Gamma_0)}\mathcal{L} = L_{\Gamma_0} L_{SZ(\Gamma_0)}\mathcal{L} \quad \forall Z : SZ \in V(Ker\,\omega)$$

and Γ satisfies

$$i_\Gamma\omega - dE = \Phi_\mu d_S v^\mu \tag{10}$$

(v^μ being known functions belonging to $\mathcal{F}(TQ)$). Then, if we substitute the last expression in (2) and evaluate all integrals, we conclude that

$$\int_{t_1}^{t_2} L_X \mathcal{L} dt \underset{M}{=} 0$$

because of the tangency of the second order dynamics to M. In this case we have that every $\Gamma \in \{\Gamma\}$ represents the Euler-Lagrange equations on M without conditions on the variations. In fact all the four terms in (2) vanish on M; in particular every element of $\{\Gamma\}$ acts in the same way on $i_{SX} d\mathcal{L}$ since this last term is a $F\mathcal{L}$-projectable function. As a matter of fact a function of $\mathcal{F}(TQ)$ is said $F\mathcal{L}$-projectable if it is constant on the leaves of the foliation generated by $V(Ker\,\omega)$ and, on the other hand

$$\Gamma_1 \sim \Gamma_2 \Longleftrightarrow \Gamma_1 = \Gamma_2 + V(Ker\,\omega).$$

The Noether theorem for degenerate Lagrangians is exhaustively proved in [8]. We only want to remind that in this case all the results obtained in the regular case are preserved on condition that use of an appropriate definition of dynamical symmetry would be made: one has to request that $X(\Gamma)$ be tangent to the final submanifold M_f; moreover a set of r functions $\alpha^j \in \mathcal{F}(TQ)$ must exist such that

$$[X(\Gamma), \Gamma] \underset{M_f}{=} \alpha^j K_j^v, \quad K_j^v \in V(Ker\,\omega)$$

so that $X(\Gamma)$ carries Γ into an equivalent dynamics.
Thus we obtain:

(a) $L_\Gamma G \underset{M}{=} 0$;

(b) $X(\Gamma)$ is a (in the above sense) dynamical symmetry;

(c) $L_{X(\Gamma)} E \underset{M}{=} 0$;

(d) the converse Noether theorem.

4. Conclusions.

We can conclude that the difference between the two kinds of constrained systems resides in the form of the equations of motion: as could have been foreseen, the case in which the equations of motion take the usual form on the constraint submanifold is the one in which the Lagrangian formalism is almost completely preserved. From the physical point of view the last situation is the most important one: one has to remember that this request is necessary for both the methods of geometric quantization (canonical quantization and path integral quantization) [9].

If we denote the identification mapping with

$$j : M \hookrightarrow TQ,$$

we have from (10) that

$$j^*(dE - i_\Gamma \omega) = 0$$

hold true in the case of degenerate Lagrangians. Instead in the nonholonomic case the pullback

$$j^* \lambda d_S \Phi$$

does not generally vanish; on the other hand a second order vector field Γ' does not exist such that

$$j^*(dE - i_{\bar{\Gamma}} \omega - \lambda d_S \Phi) = j^*(dE' - i_{\Gamma'} \omega').$$

This is because of the difficulty in solving the so-called inverse problem [10]. So in the case of constraints *added from the outside* we are in the presence of a total arbitrarity, and consequently a part of the formalism is destroyed: the one concerning the symmetries of the dynamics. But it is important to emphasize that the introduction of the constraints preserves the conservation laws, i.e. the possibility to implement the ordinary procedures of reduction and eventually of the integration of the differential problem [11].

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ON THE 2-COCYCLES OF THE GALILEI LIE ALGEBRA

GERALD HARNETT
Department of Mathematics, Florida Atlantic University, Boca Raton, FL 33431-0991

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Abstract. The 2-cocycles of the Galilei Lie algebra are shown to be in one-to-one correspondence with certain affine maps from Galilean spacetime into the dual of the homogeneous Galilei Lie algebra. The kinematic interpretation of these affine maps is detailed. The symplectic orbits in the space of 2-cocycles are described in a direct coordinate-free manner.

1. Introduction

We develop a coordinate-free description of the 2-cocycles of the Galilei Lie algebra and the orbits of the Galilei group in the space of 2-cocycles. These orbits of course include the physically significant symplectic homogeneous spaces of the group—those corresponding to elementary Galilean systems with nonzero mass. (See [3], [4], [7].) The ideas are elementary and can be applied to other semidirect product groups. The work was motivated in part by a desire to have a coordinate-free account of the geometric quantization of the Galilei group, which is interesting due to the connection between nontrivial 2-cocycles and projective representations [8]. Such an account would be helpful for constructing and interpreting coherent states over homogeneous spaces of the Galilei and Poincaré groups.

2. Affine Transformation Groups

The intrinsic structure of an affine transformation group is given by a certain exact sequence. The induced Lie algebra sequence admits natural splitting maps, labelled by the points of the affine space and having an affine position dependence. The Lie algebra sequence and its dual inherit the equivariance of the original; this facilitates the description of orbits.

Let X be a finite dimensional affine space modelled on a vector space $\mathbf{V}$. If $(Y, \mathbf{W})$ is another affine space, an affine map from X to Y is a map $a\colon X \to Y$, together with a linear map $A\colon \mathbf{V} \to \mathbf{W}$, the *linear part* of a, satisfying

$$a(x + Z) = ax + AZ \qquad (x \in X,\ Z \in \mathbf{V}).$$

In particular, the group $\mathrm{GA}(X, \mathbf{V})$ of affine transformations of X fits into an exact sequence

$$0 \to \mathbf{V} \to \mathrm{GA}(X, \mathbf{V}) \to \mathrm{GL}(\mathbf{V}) \to 1. \tag{1}$$

We call a Lie group G an *affine group* of X if there is an exact sequence of groups

$$0 \to \mathbf{V} \to G \to L \to 1 \tag{2}$$

that maps by homomorphisms (identity on $\mathbf{V}$) into the sequence (1). That is, there is an action of G on X by affine transformations extending the action of $\mathbf{V}$ and a representation $\rho: L \to \mathrm{GL}(\mathbf{V})$ such that the linear part of the action of $a \in G$ is $\rho(A)$, where A is the image of a in L. Operations on vector and affine space elements will be abbreviated; for example, $aZ = AZ = \rho(A)Z$.

The Lie algebra $\mathbf{ga}$ of $\mathrm{GA}(X, \mathbf{V})$ is naturally identified with the vector space of affine maps from X into $\mathbf{V}$. Corresponding to the group sequence (1), we have the sequence

$$0 \to \mathbf{V} \to \mathbf{ga} \to \mathbf{gl}(\mathbf{V}) \to 0. \tag{3}$$

The homomorphism from (2) into (1) induces a homomorphism from the G-equivariant Lie algebra sequence

$$0 \to \mathbf{V} \to \mathbf{g} \to \mathbf{l} \to 0, \tag{4}$$

into (3), involving the infinitesimal representation $\dot{\rho}: \mathbf{l} \to \mathbf{gl}(\mathbf{V})$. A Lie algebra action will be indicated by juxtaposition; thus $KZ = \dot{\rho}(K)Z$ for $K \in \mathbf{l}$ and $Z \in \mathbf{V}$. If $k \in \mathbf{g}$ maps to $K \in \mathbf{l}$ we call K the *linear image* of k. Throughout, the symbols j and k are reserved for elements of $\mathbf{g}$ and their linear images are denoted J and K.

For each $x \in X$ there is the evaluation splitting map $\mathcal{E}_x: \mathbf{g} \to \mathbf{V}$ given by $\mathcal{E}_x k = kx$. The complementary splitting map is

$$\mathcal{F}_x: \mathbf{l} \to \mathbf{g}, \qquad \mathcal{F}_x K = k - kx, \tag{5}$$

where k is any element with linear image K. The image of $\mathcal{F}_x$ is the subalgebra of $\mathbf{g}$ whose elements, as infinitesimal transformations of X, fix x.

From the Lie algebra sequence (4) we get the G-equivariant exact sequence of dual vector spaces

$$0 \to \mathbf{l}^* \to \mathbf{g}^* \to \mathbf{V}^* \to 0. \tag{6}$$

Where the context requires, elements of $\mathbf{l}^*$ will be considered members of $\mathbf{g}^*$. The image in $\mathbf{V}^*$ of an element $\mu \in \mathbf{g}^*$ will be denoted $\bar{\mu}$ and called the *linear image* of μ. We shall need the map $\mathbf{V}^* \otimes \mathbf{V} \to \mathbf{l}^*$, $p \otimes Z \mapsto p \mathbin{\widehat{\otimes}} Z$, given by

$$\langle p \mathbin{\widehat{\otimes}} Z, K \rangle = \langle Kp, Z \rangle = -pKZ \qquad (K \in \mathbf{l}). \tag{7}$$

The evaluation splitting maps give us, for each $x \in X$ and $p \in \mathbf{V}^*$, an element

$$\mathcal{E}_x^* p \in \mathbf{g}^*, \qquad \langle \mathcal{E}_x^* p, k \rangle = \langle p, kx \rangle, \tag{8}$$

whose linear image is p. The associated map from $T^*X = X \times \mathbf{V}^*$ into $\mathbf{g}^*$ is just the standard G-equivariant momentum mapping [1]. The splitting maps $\mathcal{F}_x$ give us for each $\mu \in \mathbf{g}^*$ an affine map $X \to \mathbf{l}^*$, $\mu x = \mathcal{F}_x^* \mu$:

$$\langle \mu x, K \rangle = \langle \mu, \mathcal{F}_x K \rangle \qquad (x \in X,\ K \in \mathbf{l}). \tag{9}$$

In the case of the Euclidean group, μx restricts μ to infinitesimal rotations about the point x; μx is the angular momentum of the kinematical system represented by μ. The position dependence is

$$\mu(x + Z) = \mu x + \bar{\mu} \mathbin{\widehat{\otimes}} Z.$$

For $\mu \in \mathbf{g}^*$ we can now write, for any $x \in X$,

$$\mu = \mu x + \mathcal{E}_x^* \bar{\mu}. \tag{10}$$

If $\dot{\rho}(\mathbf{l})\mathbf{V} = \mathbf{V}$, then the linear image $\bar{\mu}$ is determined by the affine map induced by μ. Thus we have the following.

Proposition. Let $\mathrm{Aff}_\rho(X, \mathbf{l}^*)$ be the space of affine maps from X into $\mathbf{l}^*$ with linear parts of the form $Z \mapsto p \mathbin{\widehat{\otimes}} Z$ for some $p \in \mathbf{V}^*$. There is a G-morphism $\mathbf{g}^* \to \mathrm{Aff}_\rho(X, \mathbf{l}^*)$. If $\dot{\rho}(\mathbf{l})\mathbf{V} = \mathbf{V}$, then this map is an isomorphism.

The condition holds for pseudo-Euclidean affine groups but not for the Galilei group.

We now rapidly describe the coadjoint orbits of an affine group (also described in [4], [6], [5]). The G-equivariance of the sequence (6) implies that a G orbit $\mathcal{O} \subset \mathbf{g}^*$ is a fibration over an L orbit $\bar{\mathcal{O}} \subset \mathbf{V}^*$. Fix $p \in \bar{\mathcal{O}}$ and let $\mathcal{O}_p$ be the fiber in $\mathcal{O}$ over p; that is, $\mathcal{O}_p$ is the intersection of $\mathcal{O}$ with the inverse image of p under the projection $\mathbf{g}^* \to \mathbf{V}^*$. Let G_p be the isotropy subgroup of p for G acting on $\mathbf{V}^*$ via L. The fiber $\mathcal{O}_p$ is itself the total space of a G_p-equivariant fibration, which we now describe.

Let L_p be the isotropy subgroup (little group) of p for L acting on $\mathbf{V}^*$. Consider the dual of the L_p-equivariant linear isotropy sequence $\mathbf{l}_p \to \mathbf{l} \to \mathbf{l}/\mathbf{l}_p$, namely

$$0 \to \mathbf{l}_p^o \to \mathbf{l}^* \to \mathbf{l}_p^* \to 0,$$

where $\mathbf{l}_p^o \approx (\mathbf{l}/\mathbf{l}_p)^*$ is the annihilator of the isotropy algebra $\mathbf{l}_p \subset \mathbf{l}$. By computing $(p \mathbin{\widehat{\otimes}} \mathbf{V})^o = \mathbf{l}_p$ directly from the definitions, one finds that $\mathbf{l}_p^o = p \mathbin{\widehat{\otimes}} \mathbf{V}$. If $\pi_p : \mathbf{l}^* \to \mathbf{l}_p^*$ denotes the natural projection, the map

$$\gamma_p : \mathcal{O}_p \to \mathbf{l}_p^*, \qquad \gamma_p(\mu) = \pi_p(\mu x),$$

(independent of $x \in X$) projects onto a coadjoint orbit of L_p and has affine fibers given by $\mathbf{V} \cdot \mu = \mu + p \hat{\otimes} \mathbf{V}$ for $\mu \in \mathcal{O}_p$. (The coadjoint action of $Z \in \mathbf{V}$ on $\mu \in \mathfrak{g}^*$ is given by $Z \cdot \mu = \mu - \bar{\mu} \hat{\otimes} Z$.)

The coadjoint orbits of G may thus be described as follows.

Proposition. Let $\mathcal{O} \subset \mathfrak{g}^*$ be a G orbit and let $\bar{\mathcal{O}}$ be its linear image in $\mathbf{V}^*$. The linear image map gives $\mathcal{O}$ the structure of a fiber bundle $\mathcal{O} \to \bar{\mathcal{O}}$ with the typical fiber $\mathcal{O}_p$ ($p \in \bar{\mathcal{O}}$) itself being an affine fibration over a little group coadjoint orbit $L_p \cdot \gamma_p(\mu)$ ($\mu \in \mathcal{O}_p$). The action of G maps fibers of each type to fibers of the same type.

3. The Galilei Group

Galilean spacetime is a four-dimensional affine space $(X, \mathbf{V})$ equipped with an affine map $X \to \mathbb{R}$ whose linear part, $\theta: \mathbf{V} \to \mathbb{R}$, has kernel a Euclidean vector space $\mathbf{E} \approx \mathbb{R}^3$. The structure of $\mathbf{V}$ is provided by the exact sequence

$$0 \to \mathbf{E} \to \mathbf{V} \xrightarrow{\theta} \mathbb{R} \to 0.$$

The Galilei group G is the group of affine transformations of $(X, \mathbf{V})$ whose linear parts lie in the group

$$L = \{A \in \mathrm{GL}(\mathbf{V}) \text{ s.t. } A(\mathbf{E}) \subset \mathbf{E},\ A|\mathbf{E} \in SO_{\mathbf{E}}, \text{ and } \theta \circ A = \theta\}.$$

The origin-independent structure of G is given by the exact diagram

$$\begin{array}{ccccccccc} & & \mathbf{E} & & & & \mathbf{E} & & \\ & & \downarrow & & & & \downarrow & & \\ 0 & \to & \mathbf{V} & \to & G & \to & L & \to & 1 \\ & & \downarrow & & & & \downarrow & & \\ & & \mathbb{R} & & & & SO_{\mathbf{E}} & & \end{array} \tag{11}$$

The map $\mathbf{E} \to L$ is given by $v \mapsto T_v$, where $T_v(Z) = Z + \theta(Z)v$; T_v is a shear transformation of $\mathbf{V}$ giving the so-called boost with velocity v. A useful way to view the group L is as the Euclidean affine group of $(V_1, \mathbf{E})$, where

$$V_1 := \{ Z \in \mathbf{V} \mid \theta(Z) = 1 \},$$

the space of unit four-velocities. In this context, the elements of $\mathbf{E}$ are interpreted as relative three-velocities.

4. Galilean 2-Cocycles

The dual of the Lie algebra diagram derived from (11) can be used to analyse the elements and orbits in $\mathfrak{g}^*$. But one knows that the physically interesting symplectic homogeneous spaces lie in the space of 2-cocycles $Z^2(\mathfrak{g})$ [4].

Let $Z^1(\mathbf{l}, \mathbf{V}^*)$ denote the vector space of 1-cocycles of $\mathbf{l}$ with values in $\mathbf{V}^*$; these are the maps $b: \mathbf{l} \to \mathbf{V}^*$ satisfying $b([J,K])Z = b(J)KZ - b(K)JZ$. We need to describe some maps into and out of $Z^1(\mathbf{l}, \mathbf{V}^*)$. Because there are no $\mathbf{l}$-invariants in $\mathbf{V}^* \wedge \mathbf{V}^*$, a 2-cocycle α vanishes when restricted to $\mathbf{V} \times \mathbf{V}$. Thus there is a well-defined G-morphism

$$Z^2(\mathbf{g}) \to Z^1(\mathbf{l}, \mathbf{V}^*), \qquad \alpha \mapsto \bar{\alpha},$$

with

$$\bar{\alpha}(J)Z = \alpha(j, Z) \qquad (Z \in \mathbf{V},\ j \mapsto J \in \mathbf{l}).$$

We call $\bar{\alpha}$ the *linear image* of α. Because elements of $\mathbf{l}$ map $\mathbf{V}$ into $\mathbf{E}$, there is a one-to-one coboundary map

$$\delta: \mathbf{E}^* \to Z^1(\mathbf{l}, \mathbf{V}^*), \qquad (\delta p)(J)Z = -pJZ.$$

Because the bilinear form on $\mathbf{E}$ given by $b(u \otimes \theta)v$ for $u, v \in \mathbf{E}$ is $\mathbf{so}$-invariant (i.e., the natural action of $\mathbf{so}$ annihilates it), it must be a multiple of the inner product. Thus there is a well-defined map

$$Z^1(\mathbf{l}, \mathbf{V}^*) \to \mathbb{R}, \qquad b \mapsto b(u \otimes \theta)u,$$

where u is any unit vector in $\mathbf{E}$. It is appropriate to call the image of b the *mass* of b.

Our first main result is the following.

Theorem 1. There is an exact diagram

$$\begin{array}{ccccccccc} & & \mathbf{so}^* & & & & \mathbf{E}^* & & \\ & & \downarrow & & & & \downarrow & & \\ 0 & \to & \mathbf{l}^* & \xrightarrow{\delta} & Z^2(\mathbf{g}) & \to & Z^1(\mathbf{l}, \mathbf{V}^*) & \to & 0 \\ & & \downarrow & & & & \downarrow & & \\ & & \mathbf{E}^* & & & & \mathbb{R} & & \end{array} \tag{12}$$

Sketch of Proof. The exactness of the right vertical sequence involves the semisimplicity of $\mathbf{so}$. Exactness of the horizontal sequence at $\mathbf{l}^*$ follows from $H^1(\mathbf{l}) = 0$ and $H^2(\mathbf{l}) = 0$. To show that the second horizontal map is onto, and for subsequent use, we introduce the pullback analogous to the $\mathcal{E}_x^*$ of (8). Given $b \in Z^1(\mathbf{l}, \mathbf{V}^*)$, each $x \in X$ determines a 2-cocycle $\mathcal{E}_x^\# b \in Z^2(\mathbf{g})$ defined by

$$(\mathcal{E}_x^\# b)(j,k) := b(J)kx - b(K)jx. \tag{13}$$

For any x, this produces a 2-cocycle whose linear image is b. Q.E.D.

Observe that the well-known result of [2], $H^2(\mathbf{g}) = \mathbb{R}$, follows readily from the theorem.

Next, we associate an affine map to a 2-cocycle $\alpha \in Z^2(\mathbf{g})$. A straightforward calculation shows that, in analogy with (10), for any $x \in X$,

$$\alpha = \mathcal{F}_x^* \alpha + \mathcal{E}_x^{\#} \bar{\alpha},$$

where $\mathcal{F}_x$ is the splitting map of (5) and where $\mathcal{F}_x^*\alpha$, an element of $Z^2(\mathbf{l})$, is included in $Z^2(\mathbf{g})$. We define a map $Z^1(\mathbf{l}, \mathbf{V}^*) \otimes \mathbf{V} \to \mathbf{l}^*$, $b \otimes Z \mapsto b \mathbin{\hat{\otimes}} Z$, by

$$\langle b \mathbin{\hat{\otimes}} Z, K \rangle = b(K)Z.$$

If $b = \delta p$ ($p \in \mathbf{V}^*$), then this yields the $p \mathbin{\hat{\otimes}} Z$ of (7).

Proposition. Each $\alpha \in Z^2(\mathbf{g})$ gives rise to an affine map $X \to \mathbf{l}^*$, $x \mapsto \alpha x$, defined by $\delta(\alpha x) = \mathcal{F}_x^* \alpha$. The position dependence is given by

$$\alpha(x + Z) = \alpha x + \bar{\alpha} \mathbin{\hat{\otimes}} Z.$$

If $\alpha = \delta\mu$ ($\mu \in \mathbf{g}^*$), then the affine map for α is the same as that for μ in (9).

Our second main result is the following.

Theorem 2, part i. Let $\mathrm{Aff}_2(X, \mathbf{l}^*)$ be the space of affine maps from X into $\mathbf{l}^*$ with linear parts of the form $Z \mapsto b \mathbin{\hat{\otimes}} Z$ for some $b \in Z^1(\mathbf{l}, \mathbf{V}^*)$. There is a G-isomorphism

$$Z^2(\mathbf{g}) \xrightarrow{\sim} \mathrm{Aff}_2(X, \mathbf{l}^*).$$

For the kinematic interpretation of 2-cocycles it is useful to add a similar description of elements of the 'four-momentum space' $Z^1(\mathbf{l}, \mathbf{V}^*)$.

Theorem 2, part ii. There is an L-isomorphism of $Z^1(\mathbf{l}, \mathbf{V}^*)$ with the set of affine maps $V_1 \to \mathbf{E}^*$ having linear part of the form $v \mapsto -mv^t$ for some m.

For $b \in Z^1(\mathbf{l}, \mathbf{V}^*)$, the associated affine map $p: V_1 \to \mathbf{E}^*$ is given by

$$\langle pY, u \rangle = -b(u \otimes \theta)Y \quad (u \in \mathbf{E}).$$

The element $u \otimes \theta \in \mathbf{l}$ is the infinitesimal boost by u.

We now give the kinematic interpretation of a typical element $\alpha \in Z^2(\mathbf{g})$ representing an elementary Galilean system (assumed to have nonzero mass). The linear image of α in $Z^1(\mathbf{l}, \mathbf{V}^*)$ is a four-momentum object $\bar{\alpha}$. The image of $\bar{\alpha}$ in $\mathbb{R}$ is the mass m; and $\bar{\alpha}$ gives rise to an affine isomorphism $V_1 \to \mathbf{E}^*$ with linear part determined by m. The value of this affine map at $Y \in V_1$ gives the three-momentum of the system relative to Y. The four-velocity of the system is the argument in $Y_{\bar{\alpha}} \in V_1$ that maps to zero. The 2-cocycle α gives rise to an affine map $X \to \mathbf{l}^*$ with linear part determined by $\bar{\alpha}$. The value of this affine map at a point gives the total four-angular momentum of the system about the point. For each $x \in X$, the linear image of αx in

$\mathbf{E}^*$ is $-m$ times the (transpose of the) Euclidean position vector $\mathbf{r}_x$ of the mass-center world line with respect to x. (The vanishing of this image in $\mathbf{E}^*$ determines the world line.) And αx gives rise to an affine map $V_1 \to \mathfrak{so}^*$ with linear part determined by $m\mathbf{r}_x$. The value of this affine map at $Y \in V_1$ gives the three-angular momentum of the system about x relative to Y. The intrinsic angular momentum of the system is the value of this affine map at the four-velocity $Y_{\bar{\alpha}}$ of α; alternatively, it is αo considered as an element of $\mathfrak{so}^*$, where o is any point on the world line.

For use in the final section, we introduce the cocycles $\sigma_Y \in Z^1(\mathfrak{l}, \mathbf{V}^*)$ defined for $Y \in V_1$ by

$$\sigma_Y(J)Z = (JY, Q_Y Z) \qquad (J \in \mathfrak{l},\ Z \in \mathbf{V}),$$

where

$$Q_Y : \mathbf{V} \to \mathbf{V}, \qquad Q_Y Z := Z - \theta(Z)Y,$$

is the projection onto $\mathbf{E}$ along Y and $(\ ,\ )$ denotes the Euclidean inner product. For any Y, the cocycle $m\sigma_Y$ has mass m. In passing, we note that we can now write down an L-isomorphism $Z^1(\mathfrak{l}, \mathbf{V}^*) \to \mathbf{V}$ (respecting the respective exact sequence structures). It is given, for $Y \in V_1$ and $p \in \mathbf{E}^*$, by

$$b \equiv \delta p + m\sigma_Y \ \mapsto\ Y_b \equiv p^t + mY.$$

If $m \neq 0$, then Y_b is the unique element of $\mathbf{V}$ such that $b \,\widehat{\otimes}\, Y_b = 0$ and $\theta(Y_b) = m$; it is the four-velocity associated to b.

5. Orbits of 2-cocycles

By consideration of the G-equivariant diagram (12), the G orbits in $Z^2(\mathfrak{g})$ (with nonzero mass) can be described in the same manner as coadjoint orbits of affine transformation groups were described. A typical such orbit in $Z^2(\mathfrak{g})$ is seen to be a bundle $\mathcal{O} \to \bar{\mathcal{O}}$ over an L orbit in $Z^1(\mathfrak{l}, \mathbf{V}^*)$, a three-dimensional affine subspace $\bar{\mathcal{O}}$ of $Z^1(\mathfrak{l}, \mathbf{V}^*)$. Choice of a fiducial $Y \in V_1$ determines an isomorphism of $\bar{\mathcal{O}}$ with the space $\mathbf{E}$ of relative three-momenta—the space of the conventional $\mathbf{p}$s of nonrelativistic mechanics. Each fiber of $\mathcal{O}$ is itself a fibration over a sphere (or a point), with three-dimensional affine fibers. The spheres attached to the various points of $\bar{\mathcal{O}}$ may be identified with one coadjoint orbit of fixed radius s in $\mathfrak{so}^*$, the sphere of three-angular momenta of length s. (There is an equivariant map from $\mathcal{O}$ onto this coadjoint orbit.) To realize the orbit $\mathcal{O}$ intrinsically, consider the space of lines in X not parallel to $\mathbf{E}$; this is a bundle over V_1 given by

$$\mathcal{M} := \{\, [x, Y] \mid Y \in V_1,\ x \in X \,\},$$

where $[x, Y]$ denotes the line through x parallel to Y. Let $\mathcal{S}$ denote the unit sphere in $\mathbf{E}$. Recall that a vector $w \in \mathbf{E}$ determines an element $w^\times \in \mathfrak{so}$ given by $w^\times u = w \times u$ (for $u \in \mathbf{E}$); for $\mathbf{e} \in \mathbf{E}$, define $\hat{\mathbf{e}} \in \mathfrak{so}^*$ by $\langle \hat{\mathbf{e}},\, w^\times \rangle = (\mathbf{e}, w)$.

Theorem 3. If $m \neq 0$ and $s \neq 0$, there is a symplectic G-isomorphism

$$\Psi: \mathcal{M} \times \mathcal{S} \to \mathcal{O}, \qquad \Psi([x,Y],\mathbf{e}) = m\mathcal{E}_x^{\#}\sigma_Y + s\delta\hat{\mathbf{e}},$$

where $\mathcal{E}_x^{\#}$ is defined in (13). The map Ψ is a bundle map over $V_1 \to \bar{\mathcal{O}}$, $Y \mapsto m\sigma_Y$.

The existence of such a diffeomorphism is well-known from, e.g., [4] or [7]. The direct coordinate-free expression of the map is new. A trivial modification gives the case $s = 0$.

It remains to describe the relevant symplectic structures. The symplectic structure ω on the orbit $\mathcal{O}$ is given at $\alpha \in \mathcal{O}$ by

$$\omega(j\alpha, k\alpha) = \alpha(j,k),$$

where $j\alpha$ denotes the infinitesimal action of j on α. The manifold $\mathcal{M}$ has the symplectic structure $\omega_{\mathcal{M}}$ determined by symplectic reduction via the obvious map $X \times V_1 \to \mathcal{M}$, where $X \times V_1$ has the presymplectic structure $\omega_{X\times V_1}$, defined as follows. At $z \equiv (x,Y)$,

$$\omega_{X\times V_1}(jz,kz) = (\mathcal{E}_x^{\#}\sigma_Y)(j,k),$$

where $jz = (jx, JY)$ denotes the infinitesimal action of j on z. In coordinates $(t, \mathbf{q}, \mathbf{v})$ for $X \times V_1$, defined with respect to an origin $(o, \hat{Y})$ in $X \times V_1$, such that $x = o + \mathbf{q} + tY$ and $Y = \hat{Y} + \mathbf{v}$, we have $\omega_{X\times V_1} = d\mathbf{v} \wedge d\mathbf{q}$. The pullback of $\omega_{\mathcal{M}}$ by the reduction map is $\omega_{X\times V_1}$. Since the coordinates $(\mathbf{q}, \mathbf{v})$ descend to coordinates on $\mathcal{M}$, we can also write $\omega_{\mathcal{M}} = d\mathbf{v} \wedge d\mathbf{q}$. Let $\omega_{\mathcal{S}}$ denote the area form on the unit sphere in $\mathbf{E}$. Then the symplectic nature of the map Ψ of Theorem 3 is expressed by

$$\Psi^*\omega = m\omega_{\mathcal{M}} - s\omega_{\mathcal{S}},$$

where $\omega_{\mathcal{M}}$ and $\omega_{\mathcal{S}}$ here denote the appropriate pullbacks by projections from $\mathcal{M} \times \mathcal{S}$. The term $-s\omega_{\mathcal{S}}$ corresponds to the symplectic form on the spherical coadjoint orbit of radius s in $\mathbf{so}^*$.

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CONFORMAL COMPACTIFICATIONS OF SPACE-TIME AND MOMENTUM-SPACE

Possible Observable Consequences

PAOLO BUDINICH
Interdisciplinary Laboratory for Natural and Humanistic Sciences and International Centre for Theoretical Physics, Strada Costiera, 11, 34014 Trieste, Italy.
budinich@tsmi19.sissa.it

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Abstract. It is conjectured that space-time and momentum space are both conformally compactified and represented by homogeneous spaces of the conformal group generated from two different subgroups transformed in each other by conformal inversion. It is proposed that this hypothesis may be possibly supported by the recently discovered large scale correlation of galaxies in pencil beam surveys, as far as space-time is concerned, while the hydrogen atom in stationary states might represent a support to the conjecture of momentum-space compactification, connected to the space-time one by conformal inversion. Further possible consequences of the hypothesis are briefly outlined.

1. Introduction

Historically the main, often implicit, axiom of physical sciences is represented by the postulated geometrical properties of space and time where physical phenomena occur. As an example newtonian space-time is constituted by $\mathbb{R}^3$-space where euclidean geometry holds and by an oriented straight line $\mathbb{R}^1$ representing time. This space-time is still appropriate for the description of non-relativistic local motions.

The discovery of electromagnetic phenomena and of Maxwell's equations with their Poincaré covariance induced to the axiomatic introduction of Minkowski space-time apt for the local description of relativistic phenomena. This space-time may also be defined group theoretically as an homogeneous space constituted by the Poincaré group divided by its Lorentz subgroup. In this way the axiomatic role is shifted from space-time to its isometry group.

For the description of global phenomena dealt by cosmology Robertson-Walker space-time is often postulated, this also characterized by the symmetry implied by cosmological principle.

Several motivations [1] (of which the main is Maxwell's equations conformal covariance [2]) suggest the hypothesis that the conformal symmetry is a fundamental one for physics of massless systems. If we adopt it, then space-time should be conceived as the compact homogeneous space $\bar{M}$ diffeomorphic to $(S^3 \times S^1)/Z_2$ generated by the conformal group G as follows:

$$\bar{M} = \frac{G}{H_I} = \frac{S^3 \times S^1}{Z_2} \tag{1.1}$$

where the subgroup $H_I = L \times D \bowtie K$ of G is constituted by L = Lorentz-, D = dilatations- and K = special conformal-transformations. In this case the particular space-time mentioned above should be considered as densely imbedded in $\bar{M}$. As a consequence of the embedding, the flat space (as Minkowski) result conformally flat with conformal factors depending on the modality of the embedding [3].

If space-time is represented by the compact manifold $\bar{M}$ given in (1.1) then one should expect that any field theory in it should be free from infrared divergences, but not from ultraviolet ones [4], an unsatisfactory result for a conformal world where dilatation invariance should hold.

An opposite result, that is regularization of ultraviolet divergences but not of infrared ones, could be expected in case of compactified momentum-space.

A suggestion of momentum space compactification may be found in spinor theory. In fact the E. Cartan definition of a simple spinor Ψ associated with the pseudoeuclidean space $V = \mathbb{R}^{m+1,m-1}$ is represented by [5]:

$$p_a \gamma^a \Psi = 0 \quad a = 1, 2, \ldots, 2m \tag{1.2}$$

where p_a are the components of $p \in V$ and γ^a are the generators of the Clifford algebra $Cl(m+1, m-1)$. Notoriously for $\Psi \neq 0$ we have $p_a p^a = 0$, therefore $p \in V$ lies in the projective light cone $\bar{P}$ of V which is compact. In fact:

$$\bar{P} = \frac{S^m \times S^{m-2}}{Z_2} \tag{1.3}$$

The fact that $\bar{P}$ is a good candidate for compactified momentum space derives from the observation that from (1.2) one may easily get several of the fundamental equations of physics like, for $m = 2$, Weyl equation for massless neutrinos:

$$p_\mu \gamma^\mu (1 \pm \gamma_5) \Psi_\pm = 0, \quad \mu = 1, 2, 3, 4$$

and Maxwell equations:

$$p_\mu F_\pm{}^{\mu\nu} = 0, \tag{1.4}$$

and, for $m = 3$, the twistor equations

$$p_a\Gamma^a(1\pm\Gamma_7)\Pi_\pm = 0 \quad a = 1,2,3,4,5,6 \tag{1.5}$$

and so on [6]. All of them in compactified momentum space represented by eq.(1.3)..

Going back to compactified space-time $\bar{M}$ given by eq.(1.1), one may transform it with conformal inversion I, one of the global transformations of the conformal group. The result is:

$$I\bar{M}I^{-1} = \bar{P} = \frac{G}{H_{II}} = \frac{S^3\times S^1}{Z_2} \tag{1.6}$$

where $H_{II} = L \times D \bowtie T$ obtained from H_I above by substituting special conformal transformations K with Poincar'e translations T.

Observe that $\bar{P}$ may be also considered as a compactification of $P = \mathbb{R}^{3,1}$, however if in space-time $M = \mathbb{R}^{3,1}$ the coordinates of a point x have the dimension of a length in $P = \mathbb{R}^{3,1}$, the coordinates of a point p have the dimension of the inverse of a length, since: $Ix_\mu = \pm x_\mu/x^2$.

The action of G in $\bar{M}$ unambiguously defines the action of G in $\bar{P}$, and in conformally flat momentum space it might be identified under certain conditions with the standard one [7].

Adopting the suggestion of spinor geometry, we will then adopt the hypothesis that $\bar{P}$ represents compactified momentum space. Furthermore, if we admit the validity of global conformal covariance in nature, since $\bar{P}$ is obtained from $\bar{M}$ as a consequence of conformal inversion: one of the transformations of G, then we will suppose that both space-time $M = \mathbb{R}^{3,1}$ and momentum space $P = \mathbb{R}^{3,1}$ are simultaneously compactified in $\bar{M}$ given by and $\bar{P}$ given by (1.2) respectively, connected by I, conformal inversion.

As a consequence then in a fully conformal theory both infrared and ultraviolet divergences should both be absent, as expected in a conformal world.

It is well known that if one adopts $\bar{M}$ given in (1.1) as space-time, then its dual momentum space may be conceived as an infinite lattice, whose points are labelled by the indices of spherical harmonics [4]. If momentum space is also compactified, then this lattice will have to be mapped in $\bar{P}$ and result therefore finite. From $\bar{P}$ in turn a finite lattice will result in $\bar{M}$. It could be expected that on these two finite lattices, labelled by the quantum number of the spherical functions on $S^3\times S^1$, Hopf groups may act[1]. This will be further analyzed elsewhere.

In the following we will try to explore if our hypothesis of simultaneous compactification of space-time and momentum space, correlated by eq.(1.6) may find support from some observable phenomena.

[1] This possibility was suggested to me by S.Majid, to whom I am grateful.

2. Observable Consequences

2.1. Space-compactification and the Structure of the Universe

In cosmological applications compactified space-time $\bar{M}$ given in eq.(1.1) is often substituted by [4]:

$$M_{R.W} = S^3 \times I\!R^1 \tag{2.1}$$

where $I\!R^1$ is interpreted as infinite covering of S^1. It is one of the possible compact-space realizations of Robertson-Walker space-time with metric:

$$ds^2 = -dt^2 + R^2(t)[d\chi^2 + \sin^2\chi(d\theta^2 - \sin^2\theta d\phi^2)], \tag{2.2}$$

which may be also obtained by imposing the cosmological principle stating: "the homogeneity and isotropy of space in the universe"[8].

Several cosmological models consider the universe evolution in Robertson-Walker space-time. In a previous paper [9] we took as an example the inflationary model which considers a scalar field ϕ: the "inflaton field", for which the equation of motion $M_{R.W.}$ is:

$$\{R^{-3}\frac{\partial}{\partial t}R^3\frac{\partial}{\partial t} + \frac{1}{R^2}[\Delta(S^3) - 1]\}\phi = 0 \tag{2.3}$$

where $\Delta(S^3)$ is the Laplace-Beltrami operator in S^3, with elementary solutions:

$$\phi_{n\ell m} = f_n(t)Y_{n\ell m}(\chi, \theta, \phi) \tag{2.4}$$

and the spherical functions $Y_{n\ell m}(\chi, \theta, \phi)$ are defined by

$$\Delta(S^3)Y_{n\ell m} = -n(n+2)Y_{n\ell m}. \tag{2.5}$$

Then the component T_{00} of the energy momentum tensor $T_{\mu\nu}$ interpreted as energy-density $\aleph$ has, for eigenmode, the form:

$$\aleph_{n\ell m} = K(t)[Y_{n\ell m}(\chi, \theta, \phi)]^2, \tag{2.6}$$

which represents then a possible eigenvibration of a closed universe compatible with the cosmological principle as already anticipated by E. Schroedinger [10].

Recent observations up to 1400 Mpc in the North-South galactic poles directions have revealed striking correlations of galaxies showing 10 peaks spaced by 128 Mpc [11]. They may be described by eq. (2.6) by identifying energy density with matter density. Assuming the value 3000 Mpc for the inverse $H^{-1} = R\pi/2$ of the Hubble constant, the observations may be rather well represented with just one eigenmode $Y_{n,0,0}$, for which

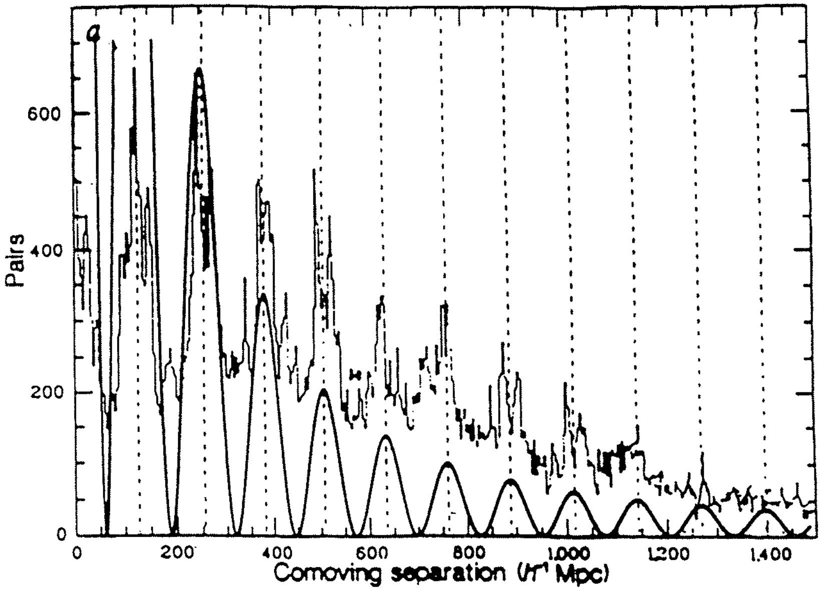

Fig. 1. Observational result on large-scale distribution of galaxies in the direction of south and north galactic poles reproduced from ref. [11], fig. 2a) (broken line). The continuous curve represents the theoretical energy density $\rho(x, x_0)$ given by eq.(2.8). It is normalized to the third peak of the observational data.

$$\aleph_n := \aleph_{n,00}(\chi, t) = K(t)\frac{\sin^2(n+1)\chi}{\sin^2\chi} \tag{2.7}$$

and the unknown time-dependent factor may be eliminated by considering the relative density. For $n = 46$ one has:

$$\rho(\chi_0\chi) = \frac{\aleph_{46}(\chi, t)}{\aleph_{46}(\chi_0 t)} = \frac{\sin^2 47\chi \sin^2\chi_0}{\sin^2\chi \sin^2 47\chi_0}. \tag{2.8}$$

The result is reported from reference [9] in fig. 1.

More observations are being performed confirming the previous ones with some variations in other directions [12]. They seem to be rather well reproducible with the single eigenmode $Y_{45,1,0}$ as it will be reported in a subsequent paper.

Should the present trend be confirmed from further observations and computations, one might conclude that they may constitute a confirmation of the role of $\bar{M}$ in the universe (or at least of the S^3 part of it).

2.2. Momentum-space Compactification and the H-atom

In the frame of the hypothesis proposed in the introduction the compactification $\bar{M}$ of space-time should imply the compactification $\bar{P}$ of momentum space and the two should be correlated by conformal inversion I, as shown in eq.(1.6).

It is well known that I maps every point x_μ of Minkowski space-time $\mathbb{R}^{3,1}$ in x_μ/x^2, say:

$$I : x_\mu \rightarrow \frac{x_\mu}{x^2}$$

and therefore for space-like x_μ, every point inside a sphere S^2 of radius one centered in the origin is mapped by I to a point outside it; one could call "small" and "large" the space inside and outside S^2 respectively however these words have no meaning in a conformal world. In order to give them meaning one should give a dimensional radius to S^2 but then conformal covariance would be broken. The situation is different with the dimensionless scalar product $x_\mu k^\mu$ where k^μ is a point of the wave-number space dual to x_μ; then, in fact:

$$I : x_\mu k^\mu \rightarrow \frac{x_\mu k^\mu}{x^2 k^2},$$

and if one substitutes k_μ with momentum $p_\mu = \hbar k_\mu$, then S^2 has radius $\hbar^2$ and the words "large" and "small" refer to classical and quantum system. Therefore if we consider the universe considered in section 2.1 as a "large" classical system we should, by conformal inversion, obtain a system in

$$P_{R.W.} = S^3 \times \mathbb{R} \tag{2.9}$$

where S^3 represents compactified momentum space (3-dimensional) and R energy and search for quantum systems whose equation of motion is represented by eq.(2.5) in momentum space. Such systems indeed exist and they represent the most common element in the universe: the H-atom.

In fact let us represent the sphere S^3 in (2.9) by:

$$S^3 : \pi_1^2 + \pi_2^2 + \pi_3^2 + \pi_5^2 = 1,$$

then, as shown by V.Fock [13] eq.(2.5) on S^3 is equivalent to the integral equation

$$\phi(\pi) = \frac{\lambda}{2\pi^2} \int \frac{\phi(\pi')}{(\pi - \pi')^2} d\Omega'$$

where $\pi = \{\pi_1\pi_2\pi_3\pi_5\}$ is a point on S^3 and which, for $\lambda = n+1$ is satisfied by the spherical harmonics $\phi = Y_{n\ell m}$. If we stereographically project S^3 on $\mathbb{R}^3$-momentum space through

$$p_j = \frac{p_0\pi_j}{1+\pi_5}, \qquad j = 1,2,3,$$

where $p_0 = mc$ is a unit of momentum (m is the mass of the electron) we easily obtain, after setting

$$n+1 = \lambda = \frac{e^2}{\hbar c}\sqrt{\frac{mc^2}{-2E}}, \qquad n = 0,1,2...$$

and

$$\Psi_{n\ell m}(\mathbf{p}) = K(p,p_0)Y_{n\ell m}(\chi,\theta,\phi),$$

with $K(p,p_0)$ a normalization factor which renders $\Psi_{n\ell m}$ orthonormal in $\mathbb{R}^3$, the Schrödinger equation of the H-atom in momentum space:

$$\frac{1}{2m}p^2\Psi_{n\ell m}(\mathbf{p}) + \frac{e^2}{2\pi\hbar^2}\int\frac{\Psi_{n\ell m}(\mathbf{p}')}{|\mathbf{p}-\mathbf{p}'|^2}d^3p = E_n\Psi_{n\ell m}$$

where

$$E_n = \frac{me^4}{2\hbar(n+1)^2}.$$

It may be shown that the introduction of p_0 is equivalent to giving a radius p_0 to S^3.

It is known that the whole spectrum of the H-atom may be obtained by acting on $\Psi_{n\ell m}$ with the algebra of SO(4,2) realized in momentum space [14].

Therefore the eigenfunction $Y_{n\ell m}(\chi,\theta,\phi)$, possible candidates for the interpretation of the large-scale distribution of matter in the universe are proportional to the Fourier transforms of the H-atom eigenfunctions in stationary states.

Obviously this correspondence is not unique. In fact there might be more systems both in space-time and, correspondingly, in momentum space, whose geometrical structure (and dynamics) is determined by that of $\bar{M}$ and correspondingly of $\bar{P}$. One of them is represented by planetary systems. But more may exist. Each might be characterized by the possible radius of S^3 like the two considered here (R for the universe and $p_0 = mc$ for the H-atom).

3. Conclusion

Should the hypothesis formulated in this paper be confirmed by further observations and computations then one might draw from it several further interesting consequences. One of them could be that if the eigenvibrations $Y_{n\ell m}(\chi, \theta\phi)$ are apt to represent the observed large scale structure of the universe, then they would mean that the cosmological principle is broken, and not just locally by matter condensations as we well know, but also by the large scale structure of space represented in our particular model by the eigenvibrations of the inflaton field ϕ.

It would be an example of spontaneous symmetry breaking, of the group $SO(4)$ in this case, which is instead the isometry group of S^3 in $M_{R.W.}$ given by (2.1), where the equations of motion (2.3) and (2.5) are formulated. One should then distinguish between the "homogeneous space" on one side, which is determined by the group of symmetry, in this case $SO(4)$, dictated by the symmetry implied by the Cosmological principle, where the equations of motions are written and the "physical space" on the other side obtained from the solutions of equations of motion written in that homogeneous space, which then may break spontaneously the postulated symmetry (in a similar way as the Kepler orbits spontaneously break the $SO(3)$ symmetry of the equations of motions). The cosmological principle then applies to the "homogeneous space" but not to the "physical one". Furthermore, while the "homogeneous space" is finite and continuous for the transitive action of the group, the "physical space" might be both finite and discrete.

The concept of a finite space is not new: it is contemplated in the cosmology of a closed universe where space is represented by S^3 with radius R equal $\sim 10^{10}$ light years, say. In such a universe the concept of distance has an upper limit: πR beyond which the mere concept of distance looses its meaning. Such a closed universe implies the existence of a minimal momentum $\sim \hbar R^{-1}$, below which the mere concept of momentum is meaningless. This, by itself originates the conception of physical momentum space as an infinite lattice.

We have seen that spinor geometry and conformal inversion induces to conjecture that homogeneous momentum space should also be compact, represented by a sphere S^3 of radius M, say, But then the above lattice will also have to be finite; not only but the existence of a maximal momentum $\sim \pi M$ will imply the existence of a minimal distance $\sim \hbar M^{-1}$ in physical ordinary space, below which the mere concept of distance is meaningless, and the physical ordinary space will also have to be represented by a finite lattice, dual to the one of physical momentum space.

Concluding, we may affirm that the postulated conformal symmetry axiomatically defines the "homogeneous, compact space" where the postulated group acts continuously and transitively. The "physical space" instead may not only spontaneously break that symmetry, but also might naturally result discrete and finite, on which then the mere concepts of infinity and infinitesimal are absent.

It is to be underlined that this last property derives not only from the identification of the "physical space" as originated from some functional of the solutions of some field equation on the compact "homogeneous space", but also from any single valued function on it. This should then in principle be applicable also to general relativity where space-time, conceived as a field (obviously not scalar), should, in our language, be classified as a "physical space" to which all the considerations exposed above (after extending them to tensor fields) should apply.

Observe that this possible group-theoretical genesis of finite, dual lattices in ordinary and momentum "physical spaces" differs both from the one pursued, after the introduction of a fundamental length, in non local field theories some decades ago, and from the one adopted for the numerical solution of Q.C.D. field equations.

These ideas will be further analyzed elsewhere.

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TORSION RELATIONS IN FINSLER SPACETIME TANGENT BUNDLE

HOWARD E. BRANDT
Army Research Laboratory
2800 Powder Mill Road
Adelphi, Maryland 20783-1197
USA.

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Abstract. A set of algebraic relations involving the bundle torsion, gauge curvature field, and four-velocity in the Finsler-spacetime tangent bundle is presented that maintains (1) compatibility with Cartan's theory of Finsler space, (2) the almost complex structure, and (3) the vanishing of the covariant derivative of the almost complex structure. This avoids the much more restrictive condition of vanishing gauge curvature field. A simple solution to the torsion relations is also obtained.

1. Introduction

It was demonstrated recently that the spacetime tangent bundle of a Finsler spacetime [1, 2] is almost complex, and also Kählerian [1, 3] and complex [4] with vanishing covariant derivative of the almost complex structure, provided that the gauge curvature field is vanishing. A vanishing gauge curvature field is equivalent to the condition that the four-velocity tangent-space coordinate be a parallel vector field. The vanishing of the gauge curvature field was also shown to be a sufficient condition for the bundle connection to have a form consistent with Cartan's theory of Finsler space [1, 2]. However, through the introduction of bundle torsion satisfying prescribed conditions, the Finsler-spacetime tangent bundle can be made to remain consistent with Cartan's theory of Finsler space, and remain almost complex with a vanishing covariant derivative of the almost complex structure, without the need to impose the relatively restrictive condition of vanishing gauge curvature field [5]. However, a nonvanishing gauge curvature field precludes that the bundle be complex [5]. A number of implied relations involving the torsion, gauge curvature field, and four-velocity can be demonstrated.

In the present work, we first review the basis for the torsion relations and then obtain a simple solution, in which the only nonvanishing component of the torsion is in the fiber-base-base sector of the bundle, and is given by the negative of the gauge curvature field.

2. Finsler-Spacetime tangent bundle with torsion.

The components of the bundle connection ${}^{(8)}\Gamma^M{}_{AB}$ of the Finsler-spacetime tangent bundle, including bundle torsion, and written in an anholonomic basis adapted to the spacetime connection, are given by [5]

$$ {}^{(8)}\Gamma^\mu{}_{\alpha\beta} = \overline{\left\{ {}^{\mu}{}_{\alpha\beta} \right\}} + {}^{(8)}K^\mu{}_{\alpha\beta}, \tag{1} $$

$$ {}^{(8)}\Gamma^\mu{}_{\alpha b} = \Pi^\mu{}_{\alpha b} + \frac{1}{2} F_{b\alpha}{}^\mu + {}^{(8)}K^\mu{}_{\alpha b}, \tag{2} $$

$$ {}^{(8)}\Gamma^\mu{}_{b\alpha} = \Pi^\mu{}_{\alpha b} + \frac{1}{2} F_{b\alpha}{}^\mu + {}^{(8)}K^\mu{}_{b\alpha}, \tag{3} $$

$$ {}^{(8)}\Gamma^\mu{}_{ab} = \rho_0 v^\lambda \frac{\overline{D}}{Dx^\lambda} \Pi_{ab}{}^\mu + {}^{(8)}K^\mu{}_{ab}, \tag{4} $$

$$ {}^{(8)}\Gamma^m{}_{\alpha\beta} = -\Pi_{\alpha\beta}{}^m + \frac{1}{2} F^m{}_{\alpha\beta} + {}^{(8)}K^m{}_{\alpha\beta}, \tag{5} $$

$$ {}^{(8)}\Gamma^m{}_{\alpha b} = -\rho_0 v^\lambda \frac{\overline{D}}{Dx^\lambda} \Pi_b{}^m{}_\alpha + {}^{(8)}K^m{}_{\alpha b}, \tag{6} $$

$$ {}^{(8)}\Gamma^m{}_{b\alpha} = \overline{\left\{ {}^{m}{}_{b\alpha} \right\}} + {}^{(8)}K^m{}_{b\alpha}, \tag{7} $$

$$ {}^{(8)}\Gamma^m{}_{ab} = \Pi^m{}_{ab} + {}^{(8)}K^m{}_{ab}. \tag{8} $$

Here recall that a generic point in the bundle manifold has coordinates $\{\chi^M; M = 0, 1, ..., 7\} = \{\chi^\mu, \chi^m; \mu = 0, 1, 2, 3; m = 4, 5, 6, 7\} \equiv \{\chi^\mu, \rho_0 v^\mu;$ $\mu = 0, 1, 2, 3\}$, where χ^μ and v^μ are the spacetime and four-velocity coordinates, respectively. Greek indices refer to spacetime and range from 0 to 3; lower case Latin indices refer to four-velocity space and range from 4 to 7; and upper case Latin indices refer to a point in the bundle and range from 0 to 7. Any lower case Latin index n appearing in a canonical spacetime tensor or connection is defined to be $n-4$ implicitly. The length ρ_0 is of the order of the Planck length [6]. Also in the above equations, there appears the spacetime connection

$$ \Gamma^\mu{}_{\alpha\beta} = \overline{\left\{ {}^{\mu}{}_{\alpha\beta} \right\}} = \left\{ {}^{\mu}{}_{\alpha\beta} \right\} - A^\lambda{}_\alpha \Pi_{\lambda\beta}{}^\mu - A^\lambda{}_\beta \Pi_{\lambda\alpha}{}^\mu - A^{\lambda\mu} \Pi_{\lambda\alpha\beta}, \tag{9} $$

in which the gauge potential is given by

$$A^{\mu}{}_{\nu} = \rho_0 v^{\lambda}\Gamma^{\mu}{}_{\lambda\nu} = \rho_0 v^{\lambda}\left\{ {\mu \atop \lambda\nu} \right\} - \rho_0^2 v^{\alpha} v^{\beta} \Pi^{\mu}{}_{\nu\lambda}\left\{ {\lambda \atop \alpha\beta} \right\}, \tag{10}$$

where $\left\{ {\mu \atop \alpha\beta} \right\}$ is the ordinary Christoffel symbol

$$\left\{ {\mu \atop \alpha\beta} \right\} = \frac{1}{2} g^{\mu\nu}\left(\frac{\partial}{\partial x^{\beta}} g_{\nu\alpha} + \frac{\partial}{\partial x^{\alpha}} g_{\nu\beta} - \frac{\partial}{\partial x^{\nu}} g_{\alpha\beta}\right), \tag{11}$$

and $g_{\mu\nu}$ is the spacetime metric tensor. Also, the Christoffel symbol of four-velocity space is given by

$$\Pi^{\mu}{}_{\alpha\beta} = \frac{1}{2}\rho_0^{-1} g^{\mu\lambda}\frac{\partial}{\partial v^{\lambda}} g_{\alpha\beta}. \tag{12}$$

Also in the above, ${}^{(8)}K^{M}{}_{AB}$ is the bundle contorsion

$${}^{(8)}K^{M}{}_{AB} = \frac{1}{2}\left(G^{ML}G_{AD}{}^{(8)}\bar{T}^{D}{}_{BL} + G^{ML}G_{BD}{}^{(8)}\bar{T}^{D}{}_{AL} + {}^{(8)}\bar{T}^{M}{}_{AB}\right), \tag{13}$$

where ${}^{(8)}\bar{T}^{M}{}_{AB}$ is the bundle torsion, and the bundle metric is given by

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & g_{mn} \end{pmatrix} \tag{14}$$

in the adapted anholonomic basis. Also in the above, the gauge curvature field is given by

$$F^{\mu}{}_{\alpha\beta} = \rho_0 v^{\lambda}\bar{R}^{\mu}{}_{\lambda\alpha\beta}, \tag{15}$$

where

$$\bar{R}^{\mu}{}_{\lambda\alpha\beta} = \Gamma^{\mu}{}_{\lambda\beta,\alpha} - \Gamma^{\mu}{}_{\lambda\alpha,\beta} + \Gamma^{\mu}{}_{\gamma\alpha}\Gamma^{\gamma}{}_{\lambda\beta} - \Gamma^{\mu}{}_{\gamma\beta}\Gamma^{\gamma}{}_{\lambda\alpha} \tag{16}$$

is the spacetime Riemann curvature tensor, written in the adapted basis. Here, the comma followed by a lower case Greek index denotes the operator ${}_{,\nu} \equiv \partial/\partial x^{\nu} - \rho_0^{-1}A^{\beta}{}_{\nu}\partial/\partial v^{\beta}$, corresponding to the adapted basis. Also in the above $\bar{D}/Dx^{\lambda}$ denotes the ordinary spacetime convariant derivative with the spacetime connection Eq. (9). The anholonomic basis vectors are defined by

$$\{E_M\} \equiv \{E_{\mu}, E_m\} \equiv \left\{\frac{\partial}{\partial x^{\mu}} - \rho_0^{-1}A^{\beta}{}_{\mu}\frac{\partial}{\partial v^{\beta}}, \rho_0^{-1}\frac{\partial}{\partial v^{\mu}}\right\}. \tag{17}$$

The associated structure coefficients $C_{AB}{}^{M}$ are defined by the commutator

$$[E_A, E_B] = C_{AB}{}^{M}E_M, \tag{18}$$

and the only nonvanishing components are

$$C_{\alpha\beta}{}^{m} = -F^{m}{}_{\alpha\beta}, \tag{19}$$

$$C_{\alpha b}{}^{m} = -C_{b\alpha}{}^{m} = \phi^{m}{}_{\alpha b}, \tag{20}$$

where

$$\phi^{\mu}{}_{\alpha\beta} = \rho_0^{-1} \frac{\partial}{\partial v^{\beta}} A^{\mu}{}_{\alpha}. \tag{21}$$

In Cartan's theory of Finsler geometry, involving the base manifold only, the connection coefficients are those corresponding to Eqs. (1) and (2). Those of Eq. (1) are identical to one set of connection coefficients appearing in Cartan's theory, provided

$$^{(8)}K^{\mu}{}_{\alpha\beta} = 0. \tag{22}$$

Those appearing in Eq. (2) are identical to the remaining set of connection coefficients of Cartan's theory, only if

$$^{(8)}K^{\mu}{}_{\alpha b} = -\frac{1}{2} F_{b\alpha}{}^{\mu}. \tag{23}$$

If the bundle torsion is not present, then the contorsion is vanishing, and Eq. (23) then requires that the gauge curvature field be vanishing, but a nonvanishing torsion circumvents the latter more severe restriction. From Eqs. (23) and (13) and the antisymmetry of the torsion, it follows that

$$^{(8)}\bar{T}^{\mu}{}_{b\alpha} = -{}^{(8)}\bar{T}^{\mu}{}_{\alpha b} = \frac{1}{2} F_{b\alpha}{}^{\mu} + {}^{(8)}K^{\mu}{}_{b\alpha}. \tag{24}$$

Next define the antisymmetric part of the bundle connection by

$$^{(8)}\underset{V}{\Gamma^{M}{}_{AB}} = -{}^{(8)}\underset{V}{\Gamma^{M}{}_{BA}} \equiv \frac{1}{2} {}^{(8)}\Gamma^{M}{}_{[AB]} = \frac{1}{2}\left(-C_{AB}{}^{M} + {}^{(8)}\bar{T}^{M}{}_{AB}\right). \tag{25}$$

Throughout, we employ the notation $T^{\cdot\cdot}_{\cdot\cdot[\mu..\nu]..} \equiv T^{\cdot\cdot}_{\cdot\cdot\mu..\nu..} - T^{\cdot\cdot}_{\cdot\cdot\nu..\mu..}$. According to Eqs. (13) and (22) and the antisymmetry of the torsion, one also has

$$^{(8)}\bar{T}^{\mu}{}_{\alpha\beta} = {}^{(8)}K^{\mu}{}_{[\alpha\beta]} = 0. \tag{26}$$

Then using Eqs. (25), (18), and (26), we obtain

$$^{(8)}\underset{V}{\Gamma^{\mu}{}_{\alpha\beta}} = 0. \tag{27}$$

Next, if we use Eqs. (25), (18), and (24), we deduce that

$$\underset{V}{{}^{(8)}\Gamma^{\mu}{}_{\alpha b}} = \frac{1}{2}{}^{(8)}\bar{T}^{\mu}{}_{\alpha b} = -\frac{1}{4}F_{b\alpha}{}^{\mu} - \frac{1}{2}{}^{(8)}K^{\mu}{}_{b\alpha}, \tag{28}$$

and

$$\underset{V}{{}^{(8)}\Gamma^{\mu}{}_{b\alpha}} = \frac{1}{2}{}^{(8)}\bar{T}^{\mu}{}_{b\alpha} = \frac{1}{4}F_{b\alpha}{}^{\mu} + \frac{1}{2}{}^{(8)}K^{\mu}{}_{b\alpha}. \tag{29}$$

Also, according to Eqs. (25) and (18), one has

$$\underset{V}{{}^{(8)}\Gamma^{\mu}{}_{ab}} = \frac{1}{2}\bar{T}^{\mu}{}_{ab}. \tag{30}$$

Only the components of the antisymmetric part of the connection Eqs. (27)-(30) are needed explicitly for the considerations that follow.

3. Almost complex structure

The Finsler-spacetime tangent bundle is almost complex, and in the anholonomic basis adapted to the spacetime connection, the almost complex structure is given by [1, 3]

$$J_{AB} = \begin{pmatrix} 0 & -g_{\alpha b} \\ g_{a\beta} & 0 \end{pmatrix} \tag{31}$$

in the absence of torsion. In the presence of bundle torsion, the bundle connection has an antisymmetric part, and the almost complex structure becomes [5]

$$J_{AB} = \begin{bmatrix} 2\rho_0 \underset{V}{{}^{(8)}\Gamma^{\mu}{}_{\alpha\beta}} v_\mu & -g_{\alpha b} + 2\rho_0 \underset{V}{{}^{(8)}\Gamma^{\mu}{}_{\alpha b}} v_\mu \\ g_{a\beta} + 2\rho_0 \underset{V}{{}^{(8)}\Gamma^{\mu}{}_{a\beta}} v_\mu & 2\rho_0 \underset{V}{{}^{(8)}\Gamma^{\mu}{}_{ab}} v_\mu \end{bmatrix}. \tag{32}$$

If we use Eqs. (27)-(30) in Eq. (32), and compare with Eq. (31), we conclude that the almost complex structure (Eq. (31)) is preserved in the presence of torsion, if the following conditions are satisfied:

$${}^{(8)}K^{\mu}{}_{b\alpha} v_\mu = -\frac{1}{2} v^\mu F_{b\alpha\mu} \tag{33}$$

and

$${}^{(8)}\bar{T}^{\mu}{}_{ab} v_\mu = 0. \tag{34}$$

Next, if one expands the convariant derivative of the almost complex structure $\nabla_E J_A{}^B$ by using Eqs. (31), (14), (1)-(8), (22) and (23), together with the corresponding results of [3] one concludes that

$$\nabla_E J_A{}^B = 0, \tag{35}$$

provided that the following relations involving the bundle torsion are satisfied (including the other relations obtained above):

$$ {}^{(8)}K^{\mu}{}_{\delta\varepsilon} = 0, \quad {}^{(8)}\bar{T}^{\mu}{}_{\delta\varepsilon} = 0, \quad {}^{(8)}K^{\mu}{}_{\delta e} = -\frac{1}{2}F_{e\delta}{}^{\mu}, \tag{36 - 38} $$

$$ {}^{(8)}\bar{T}^{\mu}{}_{\delta e} = -\frac{1}{2}F_{e\delta}{}^{\mu} - {}^{(8)}K^{\mu}{}_{e\delta}, \quad {}^{(8)}K^{\mu}{}_{d\varepsilon}P^{d}_{b\;\mu a} = \frac{1}{2}F_{[ab]\varepsilon}, \tag{39,40} $$

$$ {}^{(8)}K^{\mu}{}_{d\varepsilon}v_{\mu} = -\frac{1}{2}v^{\mu}F_{d\varepsilon\mu}, \quad {}^{(8)}\bar{T}^{\mu}{}_{d\varepsilon} = \frac{1}{2}F_{d\varepsilon}{}^{\mu} + {}^{(8)}K^{\mu}{}_{d\varepsilon}, \tag{41,42} $$

$$ {}^{(8)}K^{\mu}{}_{de}P^{db}{}_{\mu a} = 0, \quad {}^{(8)}\bar{T}^{\mu}{}_{de}v_{\mu} = 0, \quad {}^{(8)}K^{m}{}_{\delta\varepsilon}P^{\delta}_{\beta\; m\alpha} = \frac{1}{2}F_{[\alpha\beta]\varepsilon}, \tag{43 - 45} $$

$$ {}^{(8)}K^{m}{}_{\delta e}P^{\beta\delta}{}_{m\alpha} = 0, \quad {}^{(8)}K^{m}{}_{d\varepsilon} = 0, \quad {}^{(8)}K^{m}{}_{de} = 0, \tag{46 - 48} $$

where

$$ P^{\beta\delta}{}_{\mu\alpha} = \delta^{\beta}{}_{\mu}\delta^{\delta}{}_{\alpha} - g^{\beta\delta}g_{\mu\alpha}. \tag{49} $$

In summary, Eqs. (36)-(39) and (42) insure compatibility of the bundle connection with Cartan's theory of Finsler space; Eqs. (41) and (44) insure that the almost complex structure is maintained; and Eqs. (40), (43), and (45)-(48) insure that the covariant derivative of the almost complex structure is vanishing.

By means of the following identity [5],

$$ v^{\lambda}F_{[\alpha\beta]\lambda} = 0, \tag{50} $$

together with Eqs. (39)-(42) and (45), the following additional torsion relations can also be demonstrated [5]:

$$ {}^{(8)}\bar{T}^{\mu}{}_{\delta e}v_{\mu} = 0, \quad {}^{(8)}\bar{T}^{\mu}{}_{d\varepsilon}v_{\mu} = 0, \tag{51,52} $$

$$ {}^{(8)}\bar{T}^{\mu}{}_{\delta e}P^{e}_{b\;\mu a} = 0, \quad {}^{(8)}\bar{T}^{\mu}{}_{d\varepsilon}P^{d}_{b\;\mu a} = 0, \tag{53,54} $$

$$ {}^{(8)}K^{\mu}{}_{de}v^{\varepsilon}P^{d}_{b\;\mu a} = 0, \quad {}^{(8)}K^{m}{}_{\delta\varepsilon}v^{\varepsilon}P^{\delta}_{\beta\; m\alpha} = 0. \tag{55,56} $$

4. Simple solution to torsion relations

A general solution to the torsion relations, Eqs. (36)-(48) and (51)-(56), expressing the components of the bundle torsion explicitly in terms of the gauge curvature field and four-velocity, will be addressed elsewhere. Here we instead seek a simple particular solution.

Begin by considering Eq. (45) with the following ansatz

$$^{(8)}K^{m}{}_{\delta\varepsilon} = \kappa F^{m}{}_{\delta\varepsilon}, \tag{57}$$

as part of a possible self-consistent solution, where κ is a constant. If we substitute Eq. (57) in Eq. (45), it follows, that κ = -1/2, and therefore

$$^{(8)}K^{m}{}_{\delta\varepsilon} = -\frac{1}{2}F^{m}{}_{\delta\varepsilon}. \tag{58}$$

Also, Eq. (41) immediately suggests

$$^{(8)}K^{\mu}{}_{d\varepsilon} = -\frac{1}{2}F_{d\varepsilon}{}^{\mu}. \tag{59}$$

Furthermore, in accordance with Eqs. (43) and (46), we can make the simple ansatz

$$^{(8)}K^{\mu}{}_{de} = 0, \quad ^{(8)}K^{m}{}_{\delta e} = 0. \tag{60,61}$$

Thus, the only nonvanishing components of the contorsion are given by Eqs. (38), (58), and (59), which are assembled here:

$$^{(8)}K^{\mu}{}_{d\varepsilon} = ^{(8)}K^{\mu}{}_{\varepsilon d} = -\frac{1}{2}F_{d\varepsilon}{}^{\mu} \tag{62}$$

and

$$^{(8)}K^{m}{}_{\delta\varepsilon} = -\frac{1}{2}F^{m}{}_{\delta\varepsilon}. \tag{63}$$

All other components of the bundle contorsion are taken to be vanishing (Eqs. (36), (60), (61), (47), and (48)), namely,

$$^{(8)}K^{\mu}{}_{\delta\varepsilon} = ^{(8)}K^{\mu}{}_{de} = ^{(8)}K^{m}{}_{\delta e} = ^{(8)}K^{m}{}_{d\varepsilon} = ^{(8)}K^{m}{}_{de} = 0. \tag{64}$$

Next we can substitute Eq. (62) in Eq. (39) and obtain

$$^{(8)}\bar{T}^{\mu}{}_{\delta e} = 0. \tag{65}$$

Also, if we substitute Eq. (62) in Eq. (42), we get

$$^{(8)}\bar{T}^{\mu}{}_{d\varepsilon} = 0. \tag{66}$$

Furthermore, in accordance with Eq. (44), we also make the simple ansatz

$$ {}^{(8)}\bar{T}^{\mu}{}_{de} = 0. \tag{67} $$

Next, if we substitute Eqs. (64) and (67) in the expression for ${}^{(8)}K^{m}{}_{\alpha b}$ given by Eq. (13), we obtain

$$ (g^{ml} g_{bd} + \delta^{m}{}_{d} \delta^{l}_{b})\, {}^{(8)}\bar{T}^{d}{}_{\alpha l} = 0. \tag{68} $$

Equation (68) suggests the simple ansatz

$$ {}^{(8)}\bar{T}^{d}{}_{\alpha l} = 0. \tag{69} $$

Next, if we substitute Eqs. (64), (69), and (67) in the expression for ${}^{(8)}K^{m}{}_{b\alpha}$ given by Eq. (13), we obtain directly,

$$ {}^{(8)}\bar{T}^{m}{}_{b\alpha} = 0. \tag{70} $$

Furthermore, if we substitute Eq. (64) in the expression for ${}^{(8)}K^{m}{}_{ab}$ given by Eq. (13), we obtain

$$ \left(g^{ml} g_{ad} \delta^{n}{}_{b} + g^{ml} g_{bd} \delta^{n}{}_{a} + \delta^{m}{}_{d} \delta^{n}{}_{a} \delta^{l}_{b}\right)^{(8)} \bar{T}^{d}{}_{nl} = 0. \tag{71} $$

Equation (71) suggests the simple ansatz

$$ {}^{(8)}\bar{T}^{d}{}_{nl} = 0. \tag{72} $$

Finally, if we substitute Eqs. (65) and (63) in the expression for ${}^{(8)}K^{m}{}_{\alpha\beta}$ given by Eq. (13), we get

$$ {}^{(8)}\bar{T}^{m}{}_{\alpha\beta} = -F^{m}{}_{\alpha\beta}. \tag{73} $$

In summary, the only nonvanishing component of the torsion is in the fiber-base-base sector, and is given by Eq. (73). All other components of the bundle torsion are vanishing (Eqs. (37), (65)-(67), (69), (70), and (72)), namely,

$$ {}^{(8)}\bar{T}^{\mu}{}_{\delta\varepsilon} = {}^{(8)}\bar{T}^{\mu}{}_{\delta e} = {}^{(8)}\bar{T}^{\mu}{}_{d\varepsilon} = {}^{(8)}\bar{T}^{\mu}{}_{de} = {}^{(8)}\bar{T}^{m}{}_{\delta e} = {}^{(8)}\bar{T}^{m}{}_{d\varepsilon} = {}^{(8)}\bar{T}^{m}{}_{de} = 0. \tag{74} $$

Equations (36), (37), (43), (44), (46)-(48), and (51)-(54) are trivially satisfied by Eqs. (74) and (64). Equations (39) and (42) are satisfied by Eqs. (74) and (62). Equation (40) is satisfied by Eq. (62) together with Eq. (49). Equation (45) is satisfied by Eq. (63) together with Eq. (49). Equation (55) is satisfied by Eq. (62) together with Eq. (50). Equations (38) and (41) are satisfied by Eq. (62). And finally, Eq. (56) is satisfied by Eq. (63) together with Eq. (50). Thus, all of the torsion relations are satisfied by the simple solution given by Eqs. (73) and (74).

5. Conclusion

The Finsler-spacetime tangent bundle with bundle torsion is compatible with Cartan's theory of Finsler space, and is almost complex with a vanishing covariant derivative of the almost complex structure, provided that the torsion satisfies the relations given by Eqs. (36)-(48) and (51)-(56). A simple particular solution to these torsion relations is given by Eqs. (73) and (74), in which the only nonvanishing component of the torsion is in the fiber-base-base sector of the bundle, and is given by the negative of the gauge curvature field.

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CHAPTER V

CLIFFORD ALGEBRAS AND SPINORS

CLIFFORD ALGEBRA AND THE CONSTRUCTION OF A THEORY OF ELEMENTARY PARTICLE FIELDS

JAIME KELLER
División de Ciencias Básicas, F.Q.
and Quantum Theory of Matter Project
Facultad de Estudios Superiores-Cuautitlán,
Universidad Nacional Autónoma de México
A. Postal 70-528, 04510 México, D. F., MEXICO
keller@redvax.unam.mx1

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Abstract. We present a review of the development of a theory of elementary particle fields. Instead of a mathematical model based in a mathematical group, we show that we can actually develop a theory which, as a consequence, points to a mathematical structure. Clifford algebra is used as the basic tool.

We show that an extended representation of the Multivector Clifford algebra allows, first, a series of factorizations of the Laplacian operator, and, second, generates 3 families of elementary particles with the experimentally observed lepton and quark content for each family and the experimentally observed electroweak color interactions and other related properties. The factorizations $\nabla^2 = (\Gamma^{\mu}_{(f)}\partial^{(d)}_{\mu})^*(\Gamma^{\nu}_{(f)}\partial^{(d)}_{\mu})$ and the related Dirac-like equations

$$\Gamma^{\mu}_{(f)}\partial^{(d)}_{\mu}\psi_{(d,f)} = 0$$

are studied, its symmetries given. The $\Gamma^{\mu}_{(f)}$ generate the 3 families, the $\partial^{(d)}_{\mu}$ generate the observed lepton and quark content of the families.

In contrast to the usual approach to the standard model the properties for the different fields of the model are consequences of the relative properties of the equations, among themselves and in relation to spacetime, and therefore, they do not need to be postulates of the theory.

1. Introduction

In the years 1980-1983 it became apparent that besides the accepted SU(3)⊗SU(2)⊗U(1) structures of the elementary fields corresponding to a family of elementary particles, there were 3 possible families, and perhaps more, each one repeating the group structure of the fundamental family. All the experimental analysis, in the decade elapsed since that time, confirms that scheme. The construction of the basic field as composite of other, more fundamental fields, pointed to the need of combining the gauge, or interaction, fields with the study of the basic fields and moreover to the need of incorporating basic concepts like "confinement" and "colorless composites" together with the consequences of the basic group. Of paramount importance is violation of parity in weak interactions, the massless character of neutrinos and their associated left (right) handedness.

For us all the phenomenological concepts and models fit together in such a way that, if an appropriate mathematical framework is used, we could develop a theory of elementary particles and their interaction fields which should then be the foundation of the set of phenomenological laws.

Here we show that this task is now possible and that a useful mathematical tool which reduces the need for additional, or ad hoc definitions, is the Clifford algebra approach to the mathematical formulation of spacetime and the basic fields.

In fact a usual approach in mathematical physics is to use the concept of spacetime as a frame of reference for the description of the matter and their interaction fields. Spacetime, having a multivector structure and containing a spinor (and dual spinor) space, not only describes our perception of the physical nature but is also a powerful mathematical tool. Adopting spacetime as a basic frame of reference for physical phenomena should imply that its structure and symmetries corresponds to the observed characteristics of the matter and interaction fields. If a contradiction or insuficience were found a wider reference frame should then be constructed and used, but this does not seem to be the actual case.

We have several motivations for the analysis presented here which follow from studies we have performed in the last 14 years [Keller 1991]:

1. Given spacetime and its multivector Clifford algebra, $C\ell_{1,3}$ or its complexification $C\ell_{0,5}$, we can ask: which fields may exist in it obeying the Klein-Gordon wave equation $\Delta\psi = -a^2\psi$, with: $(a^2 \geq 0)$? Introducing the fields from first principles and guiding our analysis of thore fields (to make connection with experiment) from the accepted form of the rtandard phenomenology.
2. In the standard model, if we consider the fields that may exist in spacetime according to 1): do we need to add isospace to spacetime? After all the natural tangent space T_M to spacetime $R^{1(3}$ contains 16 elements and the T_M to the complex spacetime $C\ell_{0,5}$ contains 32 elements.

The elements γ^A of $R_{1,3}$ are the dimentionless Grassmann numbers

$$\begin{gathered}
\mathbf{1},\ \gamma_\mu,\ \gamma_\mu\gamma_\nu = g_{\mu\nu} + \gamma_{\mu\nu},\ g_{\mu\nu} = diag(\mathbf{1},-\mathbf{1},-\mathbf{1},-\mathbf{1}),\\
\gamma_{\mu\nu} = -\gamma_{\nu\mu}, \gamma_\rho\gamma_{\mu\nu} = g_{\rho\mu}\gamma_\nu - g_{\rho\nu}\gamma_\mu + \gamma_{\rho\mu\nu}\\
\text{and}\quad \gamma_\lambda\gamma_{\mu\nu\rho} = g_{\lambda\mu}\gamma_{\nu\rho} - g_{\lambda\nu}\gamma_{\mu\rho} + g_{\lambda\rho}\gamma_{\mu\nu} + \gamma_5 \quad \text{or} \quad \gamma_5 = \gamma_{0123},\\
\text{all } \{\mu,\nu,\lambda,\rho\} = 0,1,2,3.
\end{gathered}$$

The complexification of $C\ell_{1,3} \to C\ell_{0,5}$ can be denoted by $\{\gamma^A + i\gamma^A;\quad \gamma^A \subset C\ell_{1,3}\}$. All multivectors act as operators among themselves and on the ψ's describing the matter and interaction fields, the definitions are such that $\gamma^0, i\gamma^{12}$ and $i\gamma^5$ are hermitian. Multivectors are defined from the vectors γ^μ through their Grassmann outer product $\gamma^{\mu\nu\cdots} = \gamma^\mu \wedge \gamma^\nu \wedge \ldots$ (See Chisholm and Common (1986) or Micali, Boudet and Helmstetter (1992)).

We have discussed elsewhere the use of multivectors as generators of Lie groups, see for example [Keller and Rodríguez-Romo 1990, 1991a] where we analyse the construction within $C\ell_{0,5}$ of frequently used groups as for example S0(2,3), SU(3) or SU(2). Also the integration of spinors and multivectors into a geometric superalgebra [Keller and Rodríguez 1992].

Here we show that the basic phenomenology, and the essential lefthandedness of the neutrino, can all be combined in a generalization of the Dirac equation and the postulate that all physical possibilites implied should be included.

2. Chiral symmetry in spacetime

We assume that a local observer describes spacetime by an orthonormal tetrad a) $(\gamma^0)^2 = -(\gamma^1)^2 = -(\gamma^2)^2 = -(\gamma^3)^2 = 1$. In this frame b) $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$ is both the duality transform operator and the pseudoscalar $(\gamma^5)^2 = -1$. It is important that if another observer uses a different coordinate system, related by a Lorentz transformation L, the fundamental properties $(i\gamma^5)^2 = 1$ and $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$ are also preserved, together with a).

The handedness operator $H = i\gamma^5$ can be used to construct the chirality projectors P_R and P_L:

$$P_R + P_L = 1,\ P_R P_R = P_R;\ P_L P_L = P_L,\ P_R P_L = P_L P_R = 0,$$

where $P_R = \frac{1}{2}(1 + i\gamma^5)$, $P_L = \frac{1}{2}(1 - i\gamma^5)$ or, as discussed here below,

$$P_{R,L} = \frac{1}{2}(1 \pm H).$$

If a coordinate transformation $\gamma^5 \to (\gamma^5)'$ is allowed where a), and consequently b), is not preserved (that is if the determinant ς of the transformation is not $\varsigma = +1$) then $H \neq i(\gamma^5)'$ showing that a chirality operator $H = i(\gamma^5)'/\varsigma$,with $H^2 = 1$ in all frames has to be used.

H is in fact an invariant dimentionless quantity, it obeys $H^2 = 1$ in all frames of reference. Even if the handedness of the frame F' is changed relative to frame F' because $(\gamma^5)' = \varsigma\gamma^5$. Given that $g' = \varsigma^2 g$ and then the effect of the sign of ς is lost, we cannot define H in terms of $\sqrt{|g|}$, we have to define it in relation to the "handedness (F)" of a given frame F and then the use of ς ensures that in a change to F' we obtain handedness(F') = sign(ς)• handedness(F). In terms of this, relative, handedness definition we could write

$$H = \text{handedness}\,(F) i\gamma^5/\sqrt{|g|} \qquad (A1)$$

which is equivalent to our definition $H = i\gamma^5$ when and only when the conditions mentioned in the text are satisfied. γ^5 and $\sqrt{|g|}$ could both, simultaneously, be considered to have (length)^4 dimention and H still would be a dimentionless quantity.

Here we will assume $H = i\gamma^5$, because of the restriction a) and the assumption that we have selected a "right" handeded frame of reference. The P_R and P_L can better be considered numbers of a new mathematical field, with basis **1** and **H**, in an hypercomplexification of the Clifford algebra. $\mathbf{H}(= H)$ is coordinate invariant.

3. Chiral symmetry theory of elementary particles

Using spinors, vectors and multivectors [Fock and Ivanenko (1929), see also Keller 1991, Keller and Rodríguez-Romo 1991b, Hestenes 1966, Casanova 1976, Keller and Viniegra 1992, Keller and Rodríguez 1992] we will now construct a theory for lepton and quark fields using the possible multivector generalization of the Dirac factorization of the Laplacian (d'Alembert operator $\nabla^2 = \partial^\mu\partial_\mu$). We start, as a guiding concept, by considering the Klein-Gordon equation operator and its factorization

$$(\partial^\mu\partial_\mu + m^2) \quad = (D^\dagger + mi)(D - mi) \qquad (1)$$

which requires that

$$-D^\dagger m + mD = 0 \qquad \text{and} \qquad D^\dagger D = \partial^\mu\partial_\mu = \nabla^2 \qquad (2)$$

we can have then a set of choices, either

1. any value of m and $D^\dagger = D$ (the standard Dirac operator D_0), or
2. for the case where m = 0 the possibility $D^\dagger \neq D$ also becomes acceptable. Here we will use the field generated by **1** and **H**.

In multivector algebra the Dirac operator is the standard vector operator (using the vectors γ^μ) $D \to D_0 = \gamma^\mu\partial_\mu$. (Sometimes $D \to \gamma^0 D_0 = \gamma^{0\mu}\partial_\mu$ is used).

The basic requirement $D^\dagger D = DD^\dagger = \partial^\mu\partial_\mu$ limits the choices of D, it can be taken to be written in the Lorentz invariant form

$$D_{(d,f)} = \Gamma^\mu_{(f)}\partial^{(d)}_\mu, \quad \text{also} \quad D_{(d,f)}\psi_{(d,f)} = 0, \qquad (3)$$

in order to show the relation to the Dirac's original factorization in the simplest possible form. Here the $\Gamma^{\mu}_{(f)}$ are operators on the ψ which can be represented by generalized Dirac γ^{μ} matrices, see below. The limitation is so strong that the only possible choice is where the multivector $i\gamma^5$ (or the invariant $\mathbf{H}$), which has the same action on all γ^{μ}, that is $i\gamma^5\gamma^{\mu} = -\gamma^{\mu}i\gamma^5$, is used see [Keller 1982, 1984, 1986 and 1991, pag. 158 and following], this is particularly interesting because chirality comes naturally into the theory.

We construct the following (Lorentz invariant but coordinate system dependent) operator

$$\partial^{(d)}_{\mu} = \left\{ \mathbf{1}\cos\left(n + t^d_{\mu}\right)\frac{\pi}{2} + \mathbf{H}\sin\left(n + t^d_{\mu}\right)\frac{\pi}{2} \right\} \partial_{\mu} \tag{4}$$

condition (2) requires n and t^d_{μ} integer and it results in the simplest multivectors. Here, to take the electron as reference we use $n = -1$.

With this choice of presentation we can have the "diagonal" structure:

$$\partial^{(d)}_{\mu} = \begin{cases} \partial_{\mu} & if \;\; n + t^d_{\mu} \text{ are even} \\ i\gamma^5\partial_{\mu} & if \;\; n + t^d_{\mu} \text{ are odd} \end{cases} \tag{5}$$

The standard $\gamma^{\mu} = \Gamma^{\mu}_{(1)}$ matrices which correspond to an irreducible representation of $C\ell_{1,3}$ are found to be useful to write the wave equations of the first or fundamental family $(e^-_R, e^-_L, \nu_L, \{u_L, d_L; color\})$ of elementary particles. The electron requires a combination of two fields $e^- = (e^-_R, e^-_L)$ for the standard phenomenology of electroweak-color interactions.

The study of families other than the electron family suggested that, a more general, non reducible representations of $C\ell_{1,3}$, could in fact be needed. They are collectively denoted by $\Gamma^{\mu}_{(f)}$. In Clifford algebra their Lorentz transformations $\Gamma^{\mu}_f \to (\Gamma^{\mu}_f)'$ do not change the $\partial^{(d)}$. From our basic postulates the Γ^{μ} can all be written as exterior products of the $\gamma^{\mu}, \gamma^5, i\gamma^5$ and $\mathbf{1}$, . A fundamental representation would be for example [see Królikowski 1990]

$$\Gamma^{\mu}_f = \gamma^{\mu} \otimes (\mathbf{1} \otimes \mathbf{1} \otimes \ldots)_{2(f-1) products} \tag{6}$$

other equivalent, but different, representations, being also possible. We call these representations of the Clifford algebra "capital representation" [see Keller 1993]. The corresponding spinors would then be the, totally antisymmetric, exterior products

$$\psi_{(f)} = \psi(x) \wedge (\psi_1 \wedge \psi_2 \wedge \ldots)_{2(f-1) products}.$$

For a local theory (assumed here) the first factor $\psi(x)$ is the only one that carries spacetime position dependence.

Then the ψ_i are $2(f-1)$ non null, normalized constant Dirac spinors which correspond to extra, mathematical, internal degrees of freedom of the diracon fields. For the fundamental $s = 1/2$ fields their spin should add to zero (f odd integer). The total antisymmetry of $\psi_{(f)}$ limits the value of f to $f = 1, 2, 3$ otherwise the exterior product is null.

The degeneracy n_f of the representations of the $\Gamma^{\mu}_{(f)}$ gives statistical weight to each family: $n_1 = 1$, $n_2 = 4$ and $n_3 = 24$. This will result in factors for the terms of the mass matrix.

The elementary fields thus described are mathematically composite, but still elementary in the sense that they cannot be decomposed experimentally into their components. No size of the particle is required by the theory, they are representations of the basic elementary fermion equations, no spacetime structure is involved, there is only the mathematical complexity of the wave function. Each family has an internal relationship identical to the fundamental family $f = 1$ and the same $SU(3)_{color} \otimes SU(2) \otimes U(1)$ symmetry. No additional gauge interaction field is needed to relate the different families. They are algebraic families of otherwise structureless leptons and quarks. The algebra of the $\Gamma^{\mu}_{(f)}$ has been developed and studied by Królikowski (1990), as well as the consequences for the phenomenology of the elementary particle families.

4. Chiral geometry theory of charge isospin and color

For the quarklike diracons, an introductory analysis to study the consequences of (3), we use a reference frame F in such a way that a local reference direction is defined to be $\gamma_\ell = (\gamma_1 + \gamma_2 + \gamma_3)\sqrt{3}$ and the notation $\gamma^D_\mu \equiv i\gamma_5\gamma_\mu$ is used. Such that we can explicitly exhibit the vector-(imaginary) axial vector momentum admixture and show that it is a constant (independent of the "color" of the diracon field).

Let us write in detail the "energy momentum multivector" $\mathbf{p}$ of every diracon field d, including the different "colors" red (r), blue (b), or green (g) of the quarks, according to Table I, ($\mathbf{p}_{dir} = p^0\gamma_0 + p^\ell\gamma^{dir}_\ell$):

$$
\begin{array}{lll}
\text{electron } e: & \mathbf{p}_e & = p^0\gamma_0 + p^\ell(\gamma_1 + \gamma_2 + \gamma_3)/\sqrt{3} \\
& \mathbf{p}^r_{\bar{u}} & = p^0\gamma_0 + p^\ell(\gamma_1^D + \gamma_2 + \gamma_3)/\sqrt{3} \\
\text{quark } \bar{u}: & \mathbf{p}^b_{\bar{u}} & = p^0\gamma_0 + p^\ell(\gamma_1 + \gamma_2^D + \gamma_3)/\sqrt{3} \\
& \mathbf{p}^g_{\bar{u}} & = p^0\gamma_0 + p^\ell(\gamma_1 + \gamma_2 + \gamma_3^D)/\sqrt{3} \\
& \mathbf{p}^r_d & = p^0\gamma_0 + p^\ell(\gamma_1 + \gamma_2^D + \gamma_3^D)/\sqrt{3} \\
\text{quark } d: & \mathbf{p}^b_d & = p^0\gamma_0 + p^\ell(\gamma_1^D + \gamma_2 + \gamma_3^D)/\sqrt{3} \\
& \mathbf{p}^g_d & = p^0\gamma_0 + p^\ell(\gamma_1^D + \gamma_2^D + \gamma_3)/\sqrt{3} \\
\text{neutrino } \nu: & \mathbf{p}_\nu & = p^0\gamma_0 + p^\ell(\gamma_1^D + \gamma_2^D + \gamma_3^D)/\sqrt{3}
\end{array}
\qquad (7)
$$

Here p^ℓ is the three-momentum and p^0 is the energy. We can see that the energy-momentum vectors are all in different phases of the $p_\mu \to p_\mu^D$ rotations, with none, one, two or three vector rotations.

Let us now consider a gauge energy-momentum vector field $A^\mu\gamma_\mu$, in the Coulomb gauge $A^0 = 0$, added to the diracon fields with coupling constant proportional to Q_e, modifying the *vector* part of the momentum, with the energy-momentum components given in the same proportion to the time part and to the spatial parts (calling $\gamma_\perp$ a vector perpendicular to the direction of motion γ_ℓ). For the electron

$$\mathbf{p}' = p^0\gamma_0 + (p^\ell + Q_e A^\ell)\gamma_\ell + Q_e A^\perp \gamma_\perp \qquad (8)$$

has components

$$\text{timelike } \gamma_0 \cdot p = p^0, \text{ spacelike parallel } \gamma_\ell \cdot \mathbf{p}' = p^\ell + Q_e A^\ell,$$

$$\text{spacelike perpendicular } \gamma_\perp \cdot \mathbf{p}' = Q_e A^\perp. \qquad (9)$$

All of them are scalar quantities.

However, for a $\bar{u}$ quark (taking, for example, a red quark, the result being invariant with respect to color),

$$\gamma_\ell \cdot \gamma_\ell^{\bar{u}} = \frac{1}{\sqrt{3}}(\gamma_1 + \gamma_2 + \gamma_3) \cdot \frac{1}{\sqrt{3}}(\gamma_1^D + \gamma_2 + \gamma_3) = \frac{2}{3} + \frac{1}{3} i\gamma_5 \qquad (10)$$

the scalar components will be affected by a factor of $\frac{2}{3}$, and following the same procedure for a down quark, the scalar components will be affected by a factor of $\frac{1}{3}$, and for a neutrino the scalar components will be affected by a factor 0.

Then if we make the obvious definition that the scalar part of the gauge field, treated on an equal basis for the electrons and for the quarks of the neutrino, is to be considered as gauged by the electromagnetic field A, the "electric charges" have to be Q_e, $\frac{2}{3}Q_e$, $\frac{1}{3}Q_e$, and 0, respectively. The pseudoscalar (proportional to $i\gamma_5$) parts are to be treated on a different basis, and will be shown to correspond to the weak and color interactions.

In the full Lagrangian, introduced and discussed in [Keller 1991], a first term equivalent to the standard Dirac matter-field Lagrangian

$$\mathcal{L}_m = i\bar{\psi}\gamma^\mu D_\mu \psi \quad (\text{here } \partial_\mu \to D_\mu \text{after gauging}) \tag{11}$$

is to be replaced by the corresponding expression for diracons:

$$\mathcal{L}_d = i\bar{\psi}\gamma_d^\mu D_\mu \psi \tag{12}$$

It is in this term of the Lagrangian where we have to introduce an electromagnetic gauging with a coefficient e for the electron field, $2e/3$ for the (anti) up-quark field, $e/3$ for the down-quark field, and 0 for the neutrino field. Then in the gauge theory we are constructing, the charges for the $U(1)$ part of the gauge fields are the (postulated usually) integer, fractional, or zero values of the standard theory. In general our method will allow us to *develop* a gauge theory instead of postulating it as in the standard approaches. In this form we are showing the physical origin of the various couplings of the gauge fields, and the role played by $i\gamma_5$ in it, as a part of the symmetry-constrained Dirac particle theory.

For this purpose the A field discussed above will have to be enlarged and split into contributions, usually called B and W^3 in the literature, and new "charges" T^3 and Y are introduced with the standard notation

$$Q = T^3 + Y/2 \tag{13}$$

but the assignment of T^3 and Y to give our values of Q will be straightforward and its physical origin clear.

It is convenient to start with a rearrangement of the set of diracon fields in groups which will show an explicit $SU(2) \times SU(3) \subset \text{spin}(8)$ symmetry as shown in Table I on page 387.

To start, we explore the $SU(2)$ relations; for each given family we can see that the addition of a set of symmetry coefficients $\{W^-\} = (0, -1, -1, -1)$, modulus -2, to the first row produces the last row and their addition to any one of the first group of three up-quark fields produces one of the group of three down-quark fields. That is: the same chiral phase change that takes the neutrino field into a left electron field will change an up quark into a down quark. The reverse process proceeds in the corresponding way. The "neutral" interaction will arise from a change in the phase of one of the partner fields canceling that of the change of the other.

In the language of bilinear spinor operators, creation-anihilation, we could write all these processes in terms of spinors: if $\{\chi_\nu, \chi_u, \chi_d, \chi_e\} = \chi_a$ represent the neutrino, up-quark, down-quark, and electron fields, respectively, and their respective dual fields are $\{\chi_\nu^\dagger, \chi_u^\dagger, \chi_d^\dagger, \chi_e^\dagger\} = \chi_a^\dagger$, with the orthogonality condition $\chi_a^\dagger \psi_b = \delta_{ab}$, then the processes above can be described by

$$\hat{W}^- = w^-(\chi_e \chi_\nu^\dagger + \chi_d \chi_u^\dagger) \tag{14}$$

$$\hat{W}^+ = w^+(\chi_\nu \chi_e^\dagger + \chi_u \chi_d^\dagger) \tag{15}$$

and the neutral interaction (to be combined with the electromagnetic) is

$$\hat{W}^3 = w^3 \tfrac{1}{2}(\hat{W}^+ \hat{W}^- - \hat{W}^- \hat{W}^+) \tag{16}$$

provided that, in order to account for the spin $h/2\pi$ of the gauge fields, in all cases the spins of each spinor operator of the product are opposite, i.e., that the spinor of the electron field created is opposite to that of the neutrino field annihilated, etc. Then these processes correspond to vector interactions with total spin one, equal to the change in spin of the field during the interaction.

What we will show below is the correspondence between the interaction fields and each product of an interaction operator, written here in a formal way. We should add at this stage that, besides spin, energy-momentum is being exchanged during the interaction; for example, a photon interacting with an electron, with energy-momentum exchange q, could be written

$$\hat{A} = \sum_p \hat{\chi}_{e(p+q, \mp s \pm 1)} \hat{\chi}_e^\dagger(p, \mp s) \tag{17}$$

stating that the electromagnetic interaction annihilates an electron of momentum p and spin component s and creates an electron of momentum $p+q$ and of opposite spin.

The *color* interaction will change one of the spacelike t_i^d indexes of the quarks from the value 1 to 0 and produce a value 1 for one of the other indexes (which was zero previously), or change the axial vector momentum of two of those indexes simultaneously to a total of the eight operations $\{1 \to 2, 1 \to 3, 2 \to 3, 2 \to 1, 3 \to 1, 3 \to 2, 11 \to 22, 22 \to 33\}$, corresponding to the $SU(3)$ color symmetry; we can also write these results in a formal operator way if we add a color subindex to the quark fields; then

$$\hat{G}_{ab} = \hat{\chi}_{q,a} \hat{\chi}_{q,b}^\dagger \tag{18}$$

will correspond to a gluonic interaction changing color b into color a.

All these interactions in our diracon fields and in our chiral phase language correspond to a change in the free particle wave function

$$\psi_d = u \exp(p^d \cdot x + \phi_d^0) = u \exp(\phi_d) \tag{19}$$

with u a spinor and the de Broglie phases ϕ_d being the sum of the scalar and the pseudoscalar parts of the products of the vector x with the momenta given by equations (12). The de Broglie phases are gauged by the ϕ_d^0 which also contain scalar and pseudoscalar parts. For the leptons the de Broglie phases are

$$\phi_{\text{electron}} = p^\mu x_\mu + \phi_e^0 , \tag{20}$$

$$\phi_{\text{neutrino}} = p^0 x_0 + i\gamma_5 p^k x_k + \phi_\nu^0, \qquad k = 1,2,3 \tag{21}$$

The spinor u for the electron can be left- or right-handed, whereas for the neutrino, in order to satisfy equations (2-3), only the left-handed field is possible.

In order to preserve rotational symmetry, for each one of the quarks we need to show explicitly the gauge phase $\phi_{q,a}^0$ ensuring that the overall de Broglie phase is space-symmetric. This requires a complicated vector notation. If a space index is k (with values 1,2,3), a references space index is $r = 1,2,3$, and a color index is a or b (with values r, b, g), we have a set of three multivectors [vector $+i$ axial vector, $i = (-1)^{1/2}$],

$$e_k^a = c_k^{ar}\gamma_r; \qquad c_k^{ar} = \cos\omega_{rk}[\cos(\pi t_r^a/2) + i\gamma_5 \sin(\pi t_r^a/2)] \tag{22}$$

for each color a of a given quark, direction k in space, and quantum number t^a in Table I, for reference space direction r, this reference space direction at an angle ω_{rk} with the observer's space coordinates k. This is a more general notation than that of equation (7), where, for simplicity, the particle was taken to move in a direction with all $\cos\omega_{rv} = 1/\sqrt{3}$. The c_k^{ar} are then the sum of a scalar and (i times) a pseudoscalar.

For the purpose of our formalism we need a duality-symmetric set of coefficients b_k^{ar} such that $c_k^{ar} + b_k^{ar} = \cos\omega_{rk}$, the ordinary cosine directors (no axial vector mixing).

In terms of the multivectors (22) the de Broglie phases for the quarks are

$$\text{up quark} \qquad \phi_{u,a} = p^0 x_0 + c_k^{ar} p^k x_r + b_k^{ar}\phi^k x_r + \phi_{u,a}^0 \tag{23}$$

$$\text{down quark} \qquad \phi_{d,b} = p^0 x_0 + c_k^{br} p^k x_r + b_k^{br}\phi^k x_r + \phi_{d,b}^0 \tag{24}$$

The constants c_k^{ar} are different for up quarks and for down quarks, corresponding to the t_d^a quantum numbers.

Now, the phase angles ϕ_d^0 can either change the scalar-pseudoscalar structure of the de Broglie phases or leave them with the same structure. In the first case we have a change of the particle's nature (the resulting wave function will obey a different wave equation), and in the second case we have a type-conserving interaction. For this purpose we construct a Lagrangian which is invariant to the changes of the phase structure of the different

TABLE I. Allowed Sets of Symmetry-Constrained Quantum Numbers $\{t'^d_\mu \equiv t^d_\mu + n\}$ for Chiral Fields Corresponding to the Electron Family. Satisfying the Generalized Dirac Equation $D_{(f,d)}\psi_{(f,d)} = 0$. The quantum numbers n, t^d_μ, and operator $D_{(f,d)}$ are defined in equations (3)-(4) in the text. They correspond to the choice of e^- as reference. The charges are given by the average value $(t'_1 + t'_2 + t'_3)/3t'_0$ as described by the explanation of (72) in [Keller 1991]. The isospin pairs are connected by a change in the t'_μ such that $|t^{d'}_\mu - t^d_\mu| = (2,1,1,1)$ mod 2, and the color triplets by a change in the t'_μ such that $t^d_\mu - t^{d'}_\mu = t^{d'}_\nu - t^d_\nu$.

t'_0	t'_1	t'_2	t'_3	Q	2I	Color	Name
-1	-1	-1	-1	-1	-1	-	electron
1	0	1	1	$+\frac{2}{3}$	1	r	
1	1	0	1	$+\frac{2}{3}$	1	b	up quark
1	1	1	0	$+\frac{2}{3}$	1	g	
-1	-1	0	0	$-\frac{1}{3}$	0	r	
-1	0	-1	0	$-\frac{1}{3}$	0	b	down quark
-1	0	0	-1	$-\frac{1}{3}$	0	g	
1	0	0	0	0	0	-	neutrino

$\phi_d = \rho^u_d\chi_\mu + \phi^0_d$ shown above. We have done this in [Keller 1991] using matrix notation for isospin to conform to the usual expression of the standard theory.

Here we should remember that the idempotents $\frac{1}{2}(1 \pm i\gamma^5)$ correspond to the operators selecting handedness (or chirality) in spacetime algebra. The set of t^d_μ are then restricted forms of handling the chiral symmetry of the different fields. The relative chiral symmetries of the fields are the relevant quantities. The properties are <u>relative</u> properties, only the relations are meaningful not the actual components which are frame of reference dependent (or even coordinate dependent if general transformations are allowed). The group of these relations (see Table I) is the mathematical structure of physical interest. It is a $SU(2) \otimes SU(3)_c$ structure for each f. The $U(1)$ additional symmetry is related to the standard gauge freedom of the wave function.

In Table I Q = charge and I = isospin. Color and name refer to standard nomenclature. See [Close 1979 , Field 1979, Okun 1982, or Halzen and Martin 1984].

The basic equations for the set of spinor fields being (with $D_{(d,f)}$ explicitly defined above)

$$D_{(d,f)}\psi_{(d,f)} = 0 \ , \qquad \psi_{(d,f)} = D^{\dagger}_{(d,f)}\Phi \quad \text{and} \quad \partial^{\mu}\partial_{\mu}\Phi = 0 \qquad (25)$$

where the subindex d stands for symmetry constrained Dirac fields (Diracons), it is given the values $(\text{electron})_{left}$, electron_{right}, $u_r, u_b, u_g, d_r, d_b, d_g$ and ν, for the first family, to conform to standard phenomenology and the subindex f refers to the family number.

We have shown [Keller 1991] that they constitute a set with all the known properties of each elementary particle's family, the fields they represent can be:

— massless or massive in the particular case of $e_L + e_R$

— charged (integer or fractional).

and it is discussed in [Keller 1991, pages 158 and following], that the collection of the fields constructed with (5(and (6) have weak charge and color, and in general the characteristics usually postulated on phenomenological basis, like composites being colorless, confinement, etc. these being immediate consequences of the defining equations.

Because of the appearance, or not, of the $i\gamma^5$ factors in (5), the fields have definite chiral properties. Only one type (for each family) of field in the theory may have simultaneously both chiralities and therefore can be, as a free field, massive, charged (reference charge ± 1) and weak charged: this is, for the first family, identified as the electron field.

We should stress, again, that in Table I properties are not assigned they are relative and are properties of the gauged Lagrangian. See [Keller 1991 pages 161 and following] for a full discussion of this point.

— The resulting theory is a **chiral geometry theory of charge, isospin and color.**

The theory has a Lagrangian formulation that reproduces all aspects of the standard theory. Higgs particles have not, in its first approximation (see below) the same motivation as in the standard theory. Confinement results, within the theory, from the requirement that the Lorentz symmetry should not be broken even at local level. The same requirement gives rise to the colorless condition for hadrons, the new feature is that hadrons should be both globally and locally colorless. Fractional charges are also a natural consequence of the gauging properties of the Lagrangian.

Mass results from vector and axial vector gauging, this procedure conserves the succesfull role of the Higgs field in the standard theory, weak bosons acquiring the same mass.

The theory shows the reason for chirality being a basic property of nature as shown by the set of elementary particles. This can be clearly seen with the gauging of the diracons equations

$$D_{(d,f)} = \Gamma^{\mu}_{(f)}[\partial^{(d)}_{\mu} - \mathbf{I}\frac{e^{\cdot}}{\hbar}A^{\mu}_{(d,f)}(x)] \tag{26}$$

the gauging fields having the multivector composition

$$\begin{aligned} A^{\mu}_{d}(x) = A^{\mu}_{d,\,\text{scalar (electromagnetic)}} + i\gamma^5\, A^{\mu}_{d,\text{pseudoscalar (weak, color)}} \\ + A^{\mu}_{\alpha\beta,\text{tensor(gravity)}}\gamma^{\alpha\beta} + \gamma^{\alpha}\, A^{\mu}_{\alpha,\,poincaré} + \gamma^5\gamma^{\beta}\, A^{\mu}_{\beta,\,poincaré} \end{aligned} \tag{27}$$

That is, the gauging has electromagnetic, weak, color and gravity parts. Then the wave function becomes upon gauging (φ a reference spinor).

$$\psi_d(x) = B\exp\{\mathbf{I}(p^{\mu}_{d}x_{\mu} + \phi_d(x))\}\varphi \tag{28}$$

with the phase factor being a multivector

$$\phi_d(x) = \phi_{d,\text{scalar}}(x)\mathbf{1} + \phi_{d,\text{pseudoscalar}}(x)i\gamma^5 + \phi_{d,\alpha\beta}(x)\gamma^{\alpha\beta} + \phi_{d,\,poincaré} \tag{29}$$

the particular, relative, combinations for the phase factor of the $i\gamma^5$ terms generate isospin and color and the $\gamma^{\alpha\beta}$ generate the local Lorentz transformations which are a consequence of gravity. To get a more common formulation of the theory we take first $\mathbf{I} = \gamma^5$ and second replace it by its eigenvalues $\pm$ i. The usefullness of γ^5 stems from the fact that it commutes with $\mathbf{1}$, $\gamma^{\alpha\beta}$ and $i\gamma^5$, (or H = handededness $(F)i\gamma^5/\sqrt{|g|}$). The symmetries of $\phi_{d,\text{scalar}}(x) + \phi_{d,\text{pseudoscalar}}(x)\, i\gamma^5$ generate the well known $SU(3)_c \otimes [SU(2)\otimes U(1)]_{ew}$ standard theory. The mass matrix for the $f > 1$ families of elementary particles has a very interesting form in its first approximation:

$$\hat{m}_{(f,d)} = N_f m_d(\,5.75 + \text{effect of nondiagonal terms}) \tag{30}$$

with $N_f = n_f c_f^2$ and $m_d = m_o(n_c)_d Q_d^4$, where n_f is the degeneracy of the family's wave function, $c_f = 2f - 1$ the number of spinors in the outer product of ψ, m_o the electron mass, n_c the number of color degrees of freedom: 1 (for ν and e^-) and 3 (for the quarks) and Q_d the charge of the lepton or quark field. Then the masses are all, in a first approximation, proportional to the electron mass. The factor Q_d^4 suggests that the mass matrix is directly related to the electromagnetic, gauge, field as of a self interaction origin. The creation of a pair of elementary particles at a given point, and its subsequent separation, involves the creation of their gauge fields, Q_d^2 is the factor for the energy required to create the particle's electromagnetic field, an inseparable field from the concept of the existence of the particle, whereas Q_d^4 should correspond to self interaction.

The phase factor (29) may contain additional terms in the vector + axial vector part of the Clifford algebra. In particular the possibility of a vector contribution $(m/4)\gamma^\mu\chi_\mu$ will result in the term called the "frame field" by Chisholm and Farwell (1992) generating the mass of the matter field.

5. The basic set of equations

It is interesting that the fundamentals of the theory can be summarized in the set of equations (25 and 26) labeled by (f, d) which should be treated together and with the corresponding equations for the gauge fields.

The comparison of the matter fields to see their relative properties is mathematically a spin (8) $\oplus$ spin (1) model for each family of elementary particles. This substantiates the work of Chisholm and Farwell as a further evidence that we have presented here a **theory of elementary particles**.

6. Conclusions

In the theory we have presented here the physical properties are now a constitutive part of the wave equations. The relative properties are clearly shown [Keller 1991] when supermatrices describe a collection of fields. Off diagonal terms couple them among themselves.

We have seen that spacetime and its T_M (complex) allows enough degrees of freedom to construct a theory of elementary particles and their interactions. Specially important is that all known interactions are properly described. No additional isospin space is therefore needed.

Nucleons like proton or neutron and mesons are, within this theory, composite fields but elementary particles. In fact these composite "elementary" particles cannot, even if enough energy is available, be split into smaller components; the requirement of rotational invariance forces the "colorless" combination of quarks, even to the smallest possible experimental probe size.

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MULTIPLICATION OF SPINORS AND THE DIRAC OPERATOR

FRANK FLAHERTY
Department of Mathematics
Oregon State University
Corvallis, OR, 97331-4605, USA

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Abstract. Multiplication of spinors leads to triality (weak triality) and then to Lie (Jordan) superalgebras, which form the basis of our understanding of the exceptional geometries. Together with the corresponding Dirac operator, these algebras shine a light on the geometry of manifolds and can also be used to describe particles.

On the algebraic side, this talk is devoted to versions of triality in low dimensions (due to E. Cartan and N. Jacobson) as well as the structure group of the exceptional Jordan algebra. Using the idea of genralized multiplication on the geometric or analytic side, a variety of Dirac operators can be investigated. With some mild curvature conditions harmonic sections are parallel, (The vanishing theorem). Moreover the vanishing theorem gives one information on the index of the Dirac operator, which also has ramifications in particle physics.

1. Introduction

Spinors admit a variety of multplications. Although spinors form a module over the Clifford algebra, the operation of the tensor product is the most obvious internal product rule. Moreover representation theory produces a multiplication of spinors by decomposing the tensor product. These products are sensitive to the parity of the dimension of the vector space. In low dimensions there are several fortuitous coincidences that allow the construction of certain non-associative algebras. The path followed will develop the triality algebra of Chevalley and then sketch the exceptional algebra of A. A. Albert. Along the way octonians arise from the triality algbera.Finally, a Dirac operator is introduced and the existence of harmonic sections is examined.

The talk is organized around a number of examples.

Warning If no specific mention is made, the underlying ground field is the complex numbers. If confused, complexify!

2. Example 1

Let V be a vector space of dimension eight with a quadratic form q. Denoting the Clifford algebra of V by

$$CL(V)$$

recall that $CL(V)$ is obtained from the full tensor algebra by dividing by the two-sided ideal generated by elements of the form

$$v \otimes v - q(v) \quad v \in V$$

The fundamental relation in $CL(V)$ is then

$$v^2 = q(v)$$

There are several important observations to be made.

(1) From general considerations the space of spinors S is defined to be an irreducible representation of $CL(V)$ and as such has the property that

$$CL(V) = \mathrm{End}(S)$$

(2) There is no unique way to choose S. For example, in this lecture, you may think of polarizing V:

$$V = V' \oplus V''$$

in which the spaces V', V'' are maximal null spaces and choosing S to be the left ideal

$$S = \wedge(V')\varphi$$

where φ is a product of basis elements in V''

(3) The spinor representation in this case is just the regular representation arising from left multiplication.

(4) The reduced representation of the even Clifford algebra $CL(V)_0$ yields the decomposition

$$S = S_0 \oplus S_1$$

into even and odd spinors. Since the dimension of S is sixteen, the even and odd spinors each have dimension eight.

Fundamental bilinear for spinors. For $u, v \in S$ define the bilinear form β on S as follows:

$$\beta(u,v)\psi = \text{ homogeneous component of deg four of } u^t v$$

where ψ is the product of a basis of V' and $u \to u^t$ is the transpose anti-automorphism.

(1) β is non-degenerate.

(2)

$$\beta(xu, xv) = q(x)\beta(u, v) \quad x \in V \quad u, v \in S$$

$$\beta(su, sv) = N(s)\beta(u, v) \quad s \in \text{Clifford group}$$

where $N(s) = u^t u$ is the spinorial norm defined on the Clifford group.

(3)

$$\beta(xu, v) = \beta(u, xv)$$

(4) β is symmetric. (That V is eight dimensional enters here)

The Chevalley triality algebra. Consider the vector space

$$\mathcal{A} = V \oplus S = V \oplus S_0 \oplus S_1$$

provided with the quadratic form which is the direct sum of β and q. The space $\mathcal{A}$ has dimension twenty-four and carries a cubic form.

$$(x, u, v) \to \beta(xu, v)$$

which gives rise to a symmetric trilinear form Φ on $\mathcal{A}$ via the process of polarization. Using the quadratic form on $\mathcal{A}$ one obtains a product:

$$\Phi(A, B, C) = (A \circ B, C)$$

This is the same way that the cross-product is introduced in the calculus of $\mathbb{R}^3$.

Theorem. The vector space $\mathcal{A}$ with the $\circ$-product forms a commutative, non-associative algebra satisfying:

(1)

$$A \circ B = 0 \quad if\, A,\, B \quad \in \text{same summand}$$

(2)

$$S_0 \circ S_1 \subset V \quad \text{and cyclically}$$

(3)

$$x \circ u = xu \quad x \in V, \quad u \in S \quad \text{Clifford multiplication}$$

$$x \circ (x \circ u) = q(x)u,$$

$$\beta(x \circ u, y \circ u) = q(x, y)\beta(u, v).$$

Remark. The form $q(x, y)$ is defined as the bilinear form associated with the quadratic form in the following manner

$$(x, y) \to q(x, y) = q(x + y) - q(x) - q(y).$$

Note that a factor of two has been suppressed in the definition of $q(x, y)$, following the convention of Chevalley.

This example ends with a discussion of the triality principle of É. Cartan.

The Clifford group acts on the algebra $\mathcal{A}$ in the following way:

$$s(x,u,v) = (\chi(s)x, \rho(s)u, \rho(s)v)$$

in which ρ is the spinor representation and χ is the vector representation which actually defines the Clifford group, Γ; indeed

$$\Gamma = \{s \in CL^{\times} \quad \text{such that} \quad sxs^{-1} \in V \quad \text{for any} \quad x \in V\}$$

and χ is then defined by

$$\chi(s)x = sxs^{-1} \quad \text{for } x \in V.$$

Note that the action of s on $\mathcal{A}$ is an automorphism. Now choose some special automorphisms by first fixing a base point in $S_{-1} \stackrel{def}{=} V$; call the base point u_{-1} and make sure that

$$q(u_{-1}) = 1.$$

To insure that spinorial norms play no role, restrict attention to the group $Spin(8)$ which is the set of even elements in Γ whose spinorial norm satisfies

$$N(s)^2 = 1.$$

In other words, Γ is the conformal spin group.
Triality. First define a map

$$\iota(u_{-1}) = (B(u_{-1}), \rho(u_{-1}), \rho(u_{-1}))$$

where $B(u_{-1})$ is minus the mirror through u_{-1}. Now define the triality map T as

$$T = \iota(u_{-1})\iota(u_0)$$

where u_0 is a unit even spinor.
Principle of triality. The triality mapping T is an automorphism of the algebra $\mathcal{A}$ which has order three and which has the property that

$$T\, S_i \to S_{i+1} \quad (\text{modulo}\, 3)$$

Remark. All of the automorphisms of $\mathcal{A}$ that fix one of the summands can now be described using triality. More succinctly, every automorphism of $\mathcal{A}$ which fixes a summand is up to triality an element of the Clifford group Γ.
This ends the first example.

3. Example 2

This example is a corollary to example 1. Triality helps to define a restricted type of product on the vector space V, making V into a composition algebra. Of course this is the algebra of octonians or Cayley numbers and is really a trivial application of triality. So for $x, y \in V$

$$x \cdot y = (x \circ u_1) \circ (y \circ u_0) = T(\overline{x}) \circ T^2(\overline{y})$$

where $u_1 = u_{-1} \circ u_0$ and the bar indicates mirror reflection through the vector u_{-1}. The bar operation so defined is the conjuagation operator of the octonians.

One may ask the purpose of introducing the octonians this way but the proof of all the properties of the octonians now become transparent. Moreover, one does not have to refer to a particular choice of basis for anything but computation.

4. Example 3

The group $Spin(9)$ has a 25 dimensional representation on the direct sum of a vector space

$$W = V^8 \oplus E_0^1$$

and the spinor space S just as the Clifford group acts on $\mathcal{A}$, that is, componentwise. Define the space $\mathcal{B}$ as

$$\mathcal{B} = W \oplus S(9)$$

Note that $S(9)$ is 16 dimensional and that restricting to $Spin(8)$

$$S(9) = S(8)_0 \oplus S(8)_1$$

From representation theory $S(9) \otimes S(9)$ contains W as a summand. As a result $\mathcal{B}$ forms a kind of loop, in the sense of binary systems. In less formal terms, a spin invariant product can be introduced on S with values in W in the following manner:

$$u \cdot v = u_0 \circ v_1 + u_1 \circ v_0 + (\beta(u_0) - \beta(u_1))\varepsilon$$

in which ε is a basis element for E_0 and β denotes the quadratic form associated with the fundamental bilinear on $S(8)$.

5. Example 4

Next consider the direct sum

$$\mathcal{A} \oplus \mathcal{P}$$

in which $\mathcal{A}$ is the triality algebra and $\mathcal{P}$ is the plane in $\mathbb{R}^3$ defined by the equation:

$$x_{-1} + x_0 + x_1 = 0.$$

The plane $\mathcal{P}$ is the reduced representation of the permutation group on three letters, $\mathcal{S}_3$.

Denote this new space by $\mathcal{J}'$, that is,

$$\mathcal{J}' = \mathcal{A} \oplus \mathcal{P}.$$

$\mathcal{J}'$ is called the restricted Albert algebra and is 26 dimensional.

Construction. The space $\mathcal{J}'$ carries the structure of an algebra.

(1) You already know the multiplication on $\mathcal{A}$

(2) On $\mathcal{P} \otimes S_j \otimes S_j$, or any permutation thereof, map

$$(x_j, u_j, v_j) \to x_j \beta(u_j, v_j)$$

(3) On $(\mathcal{P})^{\otimes 3}$ take

$$x \otimes y \otimes z \to \mathcal{S}(x_j y_j z_j)$$

Remarks.

(1) Some account of signs must be taken.

(2) $\mathcal{J}'$ can be represented by order three matrices with real diagonal entries, trace zero, and symmetric octonionic entries, otherwise.

This matrix representation appears as:

$$\begin{pmatrix} x_1 & a_1 & a_0 \\ a_1 & x_0 & a_{-1} \\ a_0 & a_{-1} & x_{-1} \end{pmatrix}$$

where $x_j \in \mathbb{R}$, $\Sigma x_j = 0$ and $a_j \in S_j$. Note that the resulting algebra is not associative but after a change of multiplication and extension, $\mathcal{J}'$ does satisfy a weaker relation, namely, power associativity, which flows from the fact that the larger algebra, discussed in the next example is a Jordan algebra.

6. Example 5

The full Albert algebra is obtained by adding a unit element to the restricted algebra. In terms of the matrix representation above the identity matrix is thrown in. In more precise terms let

$$\mathcal{J} = \mathbb{R}^3 \oplus \mathcal{A}$$

and extend the multiplication to $\mathcal{J}$ by requiring

$$(\varepsilon_i, \varepsilon_j, \varepsilon_k) \to 1 \quad for \quad i = j = k$$

and zero otherwise, with ε_j a basis for $\mathbb{R}^3$.

Going back to the octonians and recalling that

$$tr(x) \stackrel{def}{=} q(x, u_{-1})$$

you can easily see that

$$tr(x \cdot y) = q(\overline{x}, y)$$

and the elements of $\mathcal{J}$ can be represented as

$$\begin{pmatrix} x_1 & \overline{a_1} & a_0 \\ a_1 & x_0 & \overline{a_{-1}} \\ \overline{a_0} & a_{-1} & x_{-1} \end{pmatrix}$$

where the multiplication between matrices is defined in terms of octonianic multiplication. Multiplication on S_0, S_1 is carried over from S_{-1} by means of triality. The Jordan product is defined then in the usual manner

$$2A * B = A \cdot B + B \cdot A$$

where $A \cdot B$ is octonianic matrix multiplication.

7. Differentiating spinors

Start with a spin manifold M and use a local frame field e_j to define the Dirac operator on spinor fields, locally by

$$D = \Sigma e_j D_j$$

The assumption that M is a spin manifold implies that the Leibniz rule applies for the Clifford multiplication on the spinors.

Note that the principal symbol of the Dirac operator at a contangent vector v is given by:

$$\text{symb}(D) \cdot v = L_v$$

where L indicates the operation of left Clifford multiplication.

For sections, X, of the bundle of restricted Albert algebras one may also define a Dirac operator:

$$DX = \Sigma e_j D_j X$$

with the principal symbol for the operator square:

$$\text{symb}(D^2) = q(v)(\mathbf{1} - M_v)$$

where as usual M_v is the mirror through v.

Sideremark Using the bimultiplication on the octonians there is a second order differential operator whose principal symbol is bimultiplication:

$$\mathrm{symb}(\delta) \cdot v = B_v.$$

It then follows from the Moufang identities that

$$B_v = L_v R_v = R_v L_v$$

and so δ is similar to the Laplace operator in the case of a Kähler manifold. Indeed, since

$$B_v = -q(v) M_v$$

it is clear that δ is an elliptic operator. In local co-ordinates

$$\delta(f) = \Sigma(e_j D_j f) D_i e_i$$

where the sum is over i, j and multiplication is in the octonians.

Harmonic sections. A cross section X of the bundle of restricted Albert algebras is called harmonic iff

$$DX = 0.$$

Vanishing theorem. Let M be a compact manifold supporting a bundle $\mathcal{J}'$ of restricted Albert algebras. If suitable curvature functions are non-negative then every harmonic section is parallel. Moreover, if the curvature condition is strict at a point then there are no non-trivial harmonic sections.

Proof: This is a Lichnerowicz-Hopf type argument.

(1) One proves an identity involving the Laplacian

(2) One looks at the remaining terms which are dominated by a quadratic curvature form.

Remark. The curvature calculations are forthcoming in the Ph. D. thesis of Troy Warwick of Oregon State University.

8. Questions and discussion

Two questions posed at the lecture were:

(1) Do harmonic sections have anything to do with string theory?

(2) In dimension eight, points, simple even spinors, and simple odd spinors are all on an equal footing. Does this mean anything physically or does this mean anything for strings?

The discussion indicated that the triality algebra may lead to a low dimensional or toy model conformal field theory (L. Dolan, UNC). Werner Nahm (Bonn) commented that the numbers 24 and 26 had already entered physics and that an earlier article by Dyson on missed opportunities pointed out the occurendce of 24 as the power to which the eta-function must be raise to obtain Ramanujan's τ-function. Strings themselves can be looked upon as mappings of circles into a space so perhaps the ideas in this note can be recast in that light (A. Voronov, Princeton). Finally, P. Budinich (SISSA, Trieste) noted that the real compactified Minkowski space requires eight dimensions to find its place and that perhaps the eight dimensional theory as presented here is the next logical step in Penrose's twistor program.

Due to lack of time the discussion of nine dimensional triality was deleted. That discussion will appear in the expanded version of this note.

9. References

The main reference for this note is Chevalley's book on the algebraic theory of spinors from 1954. In addition, the article by J. F. Adams on spin and triality from the Nuffield conference on superspaces and supergravity (Hawking and Rocek, 1981) proved invaluable in the preparation of this article.

SOLUTIONS OF THE YANG-MILLS EQUATIONS AND A CLIFFORD ALGEBRAS

TATIANA A. IVANOVA and A. D. POPOV
Laboratory of Theoretical Physics,
Joint Institute for Nuclear Research (Dubna),
Head Post Office, P.O. Box 79, Moscow 101 100, Russia

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Abstract. We consider the Yang-Mills and the Klein-Gordon equations in the external Yang-Mills fields in the spaces $\mathbb{R}^n$. Using the generators of the Clifford algebra, we construct the ansätze for the Yang-Mills potentials and for the scalar field. New classes of solutions of the Klein-Gordon and Yang-Mills equations in the spaces $\mathbb{R}^n$ with $n \geq 4$ are described.

1. Introduction

We will show that the Clifford algebras may be used in constructing the solutions of the Yang-Mills (YM) equations in $\mathbb{R}^n$. Our goal is to find some solutions of the equations for a pure classical YM theory in the Euclidean space $\mathbb{R}^n$ with the metric δ_{ab}, $a, b, ... = 1, \ldots, n$. Let A_a be the YM potentials with values in the semisimple Lie algebra $\mathcal{G}$ of the Lie group G and $F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b]$ be the curvature tensor for A_a.

The YM equations for the gauge potentials A_a have the form

$$\partial_a F_{ab} + [A_a, F_{ab}] = 0. \tag{1.1}$$

The Einstein summation convention is used throughout, if not stated otherwise.

Some solutions of Eqs.(1.1) in the spaces $\mathbb{R}^7, \mathbb{R}^8$ and $\mathbb{R}^{4k}$ were obtained in [1, 2, 3, 4, 5, 6] (see also [7]). In what follows we shall show that it is possible to obtain other classes of solutions of the YM equations in the spaces of dimension $n \geq 4$ using the properties of Clifford algebras.

2. Ansatz for Gauge Potentials

Let us suppose that in the space $\mathbb{R}^n$ with metric δ_{ab} there are q constant tensors $J^1_{ab}, \ldots, J^q_{ab}$ that are antisymmetric in indices a and b and obey the relations

$$J^\alpha_{ac} J^\beta_{bc} = \delta^{\alpha\beta}\delta_{ab} + \Sigma^{\alpha\beta}_{ab}, \tag{2.1}$$

where $\Sigma_{ab}^{\alpha\beta}$ are some constant antisymmetric in a and b tensors, $\alpha, \beta, \ldots = 1, \ldots, q$. From (2.1) it follows that

$$J^{\alpha}_{ac}J^{\beta}_{cb} + J^{\beta}_{ac}J^{\alpha}_{cb} = -2\delta^{\alpha\beta}\delta_{ab},$$

i.e., $J^{\alpha} = (J^{\alpha}_{ab})$ give a real matrix representation for the generators J^{α} of the Clifford algebra for the space $\mathbb{R}^q$ with the metric $g_{\alpha\beta} = -\delta_{\alpha\beta}$.

We shall look for solutions of the YM equations (1.1) in the form

$$A_a = -J^{\alpha}_{ac}T_{\alpha}(\varphi)\partial_c\varphi, \tag{2.2}$$

where the real antisymmetric tensors J^{α}_{ab} satisfy (2.1); φ is an arbitrary function of coordinates $x_a \in \mathbb{R}^n$; $T_1, \ldots, T_q$ depend only on φ, take values in the Lie algebra $\mathcal{G}$ and satisfy the Rouhani-Ward (RW) equations (see [4, 5, 6, 7, 8, 9]):

$$f_{\alpha\beta\gamma}\dot{T}_{\gamma} + [T_{\alpha}, T_{\beta}] = 0. \tag{2.3}$$

Here $f_{\alpha\beta\gamma}$ is some totally antisymmetric three-index tensor in $\mathbb{R}^q$ satisfying $f_{\alpha\gamma\delta}f_{\beta\gamma\delta} = 2\delta_{\alpha\beta}$ and $\dot{T}_{\gamma} \equiv dT_{\gamma}/d\varphi$. If q coincides with the dimension of the simple compact Lie algebra $\mathcal{H}$, then as $f_{\alpha\beta\gamma}$ one may take the structure constants of $\mathcal{H}$.

It may be shown that after substituting (2.2) into (1.1) and using the identities (2.1), the YM equations are reduced to the following system of linear equations:

$$f^{\alpha}_{\beta\gamma}\Sigma^{\beta\gamma}_{ac}\partial_c\partial_b\varphi - 2J^{\alpha}_{bc}\partial_c\partial_a\varphi + 2f^{\alpha}_{\beta\gamma}J^{\beta}_{ac}J^{\gamma}_{be}\partial_c\partial_e\varphi + J^{\alpha}_{ab}\Box\varphi = 0, \tag{2.4}$$

where $\Box \equiv \partial_c\partial_c$.

PROPOSITION: If tensors J^{α}_{ab} satisfy the relations (2.1) and $q = \dim\mathcal{H}$, then to each solution of system $\{(2.3), (2.4)\}$ one may correspond the solution (2.2) of the YM equations (1.1) for gauge fields A_a of an arbitrary semisimple Lie group G in the Euclidean space $\mathbb{R}^n$.

3. Explicit Form of Tensors J^{α}_{ab}

To find solutions of Eqs. (2.4), one should give the concrete expressions to the tensors J^{α}_{ab} and $\Sigma^{\alpha\beta}_{ab}$. The theory of Clifford algebras gives the examples of such tensors.

Let us denote by $Cl(0,q)$ the Clifford algebra for the space $\mathbb{R}^q$ with the metric $g_{\alpha\beta} = -\delta_{\alpha\beta}$, $\alpha, \beta, \ldots = 1, \ldots, q$. It has been known for a long time that the algebra $Cl(0,q)$ can be realized in terms of matrices. In particular, $Cl(0,6) \cong M(8, \mathbb{R})$ and $Cl(0,8) \cong M(16, \mathbb{R})$(see, e.g., [10]), where through

$M(s, \mathbb{R})$ the full $s \times s$ matrix algebra over $\mathbb{R}$ is denoted. Let us give some examples of tensors J^{α}_{ab} .
Example 1: Consider the algebra $Cl(0,2)$ with generators γ^1 and γ^2. It is well-known [10] that $Cl(0,2)$ is isomorphic to the algebra of quaternions **H**, and elements $\gamma^1, \gamma^2, \gamma^3 \equiv \gamma^1\gamma^2$ can be realized in terms of real antisymmetric 4×4 matrices η^1, η^2, η^3 with components: $\eta^{\alpha}_{\beta\gamma} = \epsilon^{\alpha}_{\beta\gamma}$, $\eta^{\alpha}_{\mu 4} = -\eta^{\alpha}_{4\mu} = \delta^{\alpha}_{\mu}$, where $\epsilon_{\alpha\beta\gamma}$ are structure constants of $SU(2)$, $\alpha, \beta, \gamma, \delta = 1,2,3$; $\mu, \nu, \ldots = 1, \ldots, 4$. Tensors $\eta^1_{\mu\nu}, \eta^2_{\mu\nu}$ and $\eta^3_{\mu\nu}$ coincide with the well-known 't Hooft tensors that obey the relations (2.1) with $\Sigma^{\alpha\beta}_{\mu\nu} = \epsilon^{\alpha\beta\gamma}\eta^{\gamma}_{\mu\nu}$.

Now, let us introduce the tensors

$$J^{\alpha}_{(\mu i)(\nu j)} = \delta_{ij}\eta^{\alpha}_{\mu\nu} \tag{3.1}$$

with the double indices $(\mu i), (\nu j), \ldots$, where $i, j, \ldots = 1, \ldots, p$. If we denote the double indices by $a, b, \ldots = 1, \ldots, 4p$, then it is not difficult to verify that the tensors J^{α}_{ab} will satisfy the relations (2.1) with $\Sigma^{\alpha\beta}_{ab} = \epsilon^{\alpha\beta\gamma}J^{\gamma}_{ab}$. Thus, in the spaces $\mathbb{R}^{4p}$ one may always introduce three tensors J^{α}_{ab} satisfying (2.1).
Example 2: Let us consider the algebra $Cl(0,6)$ with generators $\gamma^1, \ldots, \gamma^6$ and also introduce $\gamma^7 \equiv \gamma^1\gamma^2\gamma^3\gamma^4\gamma^5\gamma^6$. It is known [10] that γ^{α} ($\alpha = 1, \ldots, 7$) can be realized in terms of real antisymmetric 8×8 matrices. The components $\gamma^{\alpha}_{\mu\nu}$ ($\mu, \nu, \ldots = 1, \ldots, 8$) of these matrices satisfy the relations (2.1) with $\Sigma^{\alpha\beta}_{\mu\nu} = \frac{1}{2}\gamma^{[\alpha}_{\mu\lambda}\gamma^{\beta]}_{\nu\lambda} \equiv \frac{1}{2}(\gamma^{\alpha}_{\mu\lambda}\gamma^{\beta}_{\nu\lambda} - \gamma^{\beta}_{\mu\lambda}\gamma^{\alpha}_{\nu\lambda})$.

Now we introduce the tensors

$$J^{\alpha}_{(\mu i)(\nu j)} = \delta_{ij}\gamma^{\alpha}_{\mu\nu}, \tag{3.2}$$

where $\mu, \nu, \ldots = 1, \ldots, 8$; $i, j, \ldots = 1, \ldots, p$. Numbering the components of these tensors by the indices $a, b, \ldots = 1, \ldots, 8p$, in the space $\mathbb{R}^{8p}$ we obtain seven tensors J^{α}_{ab} satisfying (2.1) with $\Sigma^{\alpha\beta}_{ab} = \frac{1}{2}J^{[\alpha}_{ac}J^{\beta]}_{bc}$. It is clear that for ansatz (2.2) one can choose not all seven tensors but only q of them with $4 \leq q \leq 7$.
Example 3: Let us consider the algebra $Cl(0,8)$ with generators γ^{α}, $\alpha, \beta, \ldots = 1, \ldots, 8$. It is known [10] that γ^{α} can be realized in terms of real antisymmetric 16×16 matrices. The components $\gamma^{\alpha}_{\mu\nu}$ ($\mu, \nu, \ldots = 1, \ldots, 16$) of these matrices satisfy (3.1) with $\Sigma^{\alpha\beta}_{\mu\nu} = \frac{1}{2}\gamma^{[\alpha}_{\mu\lambda}\gamma^{\beta]}_{\nu\lambda}$. Let us also introduce the tensors $J^{\alpha}_{(\mu i)(\nu j)}$ defined by (3.2) but with $\mu, \nu, \ldots = 1, \ldots, 16$; $i, j, \ldots = 1, \ldots, p$. Numbering the components of these tensors by the indices $a, b, \ldots = 1, \ldots, 16p$, we obtain eight tensors J^{α}_{ab}. In the space $\mathbb{R}^{16p}$ all these tensors satisfy the relations (2.1) with $\Sigma^{\alpha\beta}_{ab} = \frac{1}{2}J^{[\alpha}_{ac}J^{\beta]}_{bc}$ and can be used in constructing of the ansatz (2.2).

And finally, we point out that in the spaces $\mathbb{R}^n$ one may introduce q tensors J^{α}_{ab} satisfying (2.1) in the following cases:

$$n = p2^{2+4m} \Rightarrow 1 + 8m \le q \le 3 + 8m, \tag{3.3a}$$

$$n = p2^{3+4m} \Rightarrow 4 + 8m \le q \le 7 + 8m, \tag{3.3b}$$

$$n = p2^{4+4m} \Rightarrow \qquad q = 8 + 8m, \tag{3.3c}$$

where $m = 0, 1, 2, \ldots;\ p = 1, 2, \ldots$. Proof may be obtained with the help of formula [10]:

$$Cl(0, s + 8m) = Cl(0, s) \otimes Cl(0, 8) \underbrace{\otimes \ldots \otimes}_{m\ times} Cl(0, 8), \tag{3.4}$$

where $1 \le s \le 8$. Using the recurrence relations given in [10], one can easily obtain the explicit form of tensors $J^1_{ab}, \ldots, J^q_{ab}$ in the spaces of dimension n indicated in (3.3).

4. Constructing of Solutions for the Scalar Field Equations

Substituting the explicit form of J^α_{ab} into Eqs.(2.4), one may try to solve (2.4). Solutions exist. Rather then make an exhaustive study of all the possibilities we shall restrict ourselves to the case of $n = 4p$ and $q = 3$.

So, let us substitute (3.1) into Eqs.(2.4) where $\epsilon_{\alpha\beta\gamma}$ are taken instead of $f_{\alpha\beta\gamma}$ and $\Sigma^{\alpha\beta}_{ab} = \epsilon^{\alpha\beta\gamma} J^\gamma_{ab}$. We use the following identities for $\eta^\alpha_{\mu\nu}$ [11]:

$$\eta^\alpha_{\mu\lambda}\eta^\beta_{\nu\lambda} = \delta^{\alpha\beta}\delta_{\mu\nu} + \epsilon^{\alpha\beta\gamma}\eta^\gamma_{\mu\nu}, \tag{4.1a}$$

$$\epsilon^\alpha_{\beta\gamma}\eta^\beta_{\mu\lambda}\eta^\gamma_{\nu\sigma} = \delta_{\mu\nu}\eta^\alpha_{\lambda\sigma} - \delta_{\mu\sigma}\eta^\alpha_{\lambda\nu} - \delta_{\lambda\nu}\eta^\alpha_{\mu\sigma} + \delta_{\lambda\sigma}\eta^\alpha_{\mu\nu}, \tag{4.1b}$$

and obtain the equations:

$$\begin{aligned} 2\eta^\alpha_{\mu\lambda}(\partial_{\lambda i}\partial_{\nu j}\varphi - \partial_{\lambda j}\partial_{\nu i}\varphi) - 2\eta^\alpha_{\nu\lambda}(\partial_{\lambda j}\partial_{\mu i}\varphi - \partial_{\lambda i}\partial_{\mu j}\varphi) \\ +\delta_{\mu\nu}\eta^\alpha_{\lambda\sigma}(\partial_{\lambda i}\partial_{\sigma j}\varphi - \partial_{\lambda j}\partial_{\sigma i}\varphi) + \\ +\eta^\alpha_{\mu\nu}(2\partial_{\lambda i}\partial_{\lambda j}\varphi + \delta_{ij}\Box\varphi) = 0, \end{aligned} \tag{4.2}$$

where $\partial_{\lambda i} \equiv \partial/\partial x^{\lambda i}$. It is clear that Eqs.(4.2) are satisfied if φ obeys the following equations

$$\partial_{\mu i}\partial_{\nu j}\varphi = \partial_{\mu j}\partial_{\nu i}\varphi, \quad \partial_{\lambda i}\partial_{\lambda j}\varphi = 0, \tag{4.3}$$

where $\mu, \nu, \ldots = 1, \ldots, 4;\ i, j, \ldots = 1, \ldots, p$. Equations (4.3) are simpler then Eqs.(2.4) and appear in study of the hyper-Kähler manifolds of dimension $4p$ (see [12]). In principle, for Eqs.(4.3) one may write a general solution (see [12]), but we shall not do this here. As an example, we write out one of the particular solutions of Eqs.(4.3) (and Eqs.(4.2)):

$$\varphi = 1 + \sum_{I=1}^{N} \frac{B_I^2}{(X_\mu - C_\mu^I)(X_\mu - C_\mu^I)}, \tag{4.4}$$

where $X_\mu = x_{\mu i}p_i$, p_i=const, N is any integer number, B_I and C_μ^I are arbitrary constants. For a special case of the space $\mathbb{R}^8$ and group $G = SU(2)$ the solution of this type was obtained by Ward [1].

Equations (2.3) with $q = 3$ and $\mathcal{H} = su(2)$ coincide with the well-known Nahm equations (see [8, 9] and [13, 14]). These equations appeared in constructing the solutions of the YM equations in $\mathbb{R}^4$ [14, 15, 16] and of the model of chiral fields in $\mathbb{R}^2$ [17]. Nahm's equations have a Lax-type representation with a spectral parameter, and in terms of theta functions one can write a general solution of Nahm's equations for any semisimple Lie algebra $\mathcal{G}$ (see [13] and [9]). The explicit form of particular solutions of Nahm's equations may be found in [15] and [16]. We shall not write it here.

5. Solutions of the Massless Klein-Gordon Equation

In $\mathbb{R}^n$ let us consider the massless scalar field χ with values in the adjoint representation of the Lie algebra $\mathcal{G}$. The Klein-Gordon equation for χ in the external field A_a has the form

$$\Big(\partial_a + [A_a,\ ]\Big)\Big(\partial_a + [A_a,\ ]\Big)\chi = 0, \tag{5.1}$$

where $a, ... = 1, ..., n$.

Now, substitute the ansatz (2.2) for A_a into (5.1). Suppose that $T_\alpha(\varphi)$ and φ obey the equations (2.3) and (2.4). For χ let us consider the following ansatz:

$$\chi = \chi_\alpha T_\alpha(\varphi), \qquad \chi_\alpha = const. \tag{5.2}$$

In this case, the Klein-Gordon equation (5.1) is reduced to the following equation:

$$\chi_\alpha \dot{T}_\alpha \Box\varphi + \chi_\alpha \partial_c\varphi\partial_c\varphi\Big\{\ddot{T}_\alpha - [T_\beta, [T_\alpha, T_\beta]]\Big\} = 0. \tag{5.3}$$

Here we have used the identities (2.1); $\ddot{T}_\alpha \equiv d^2T_\alpha/d\varphi^2$.

Thus, if $T_\alpha(\varphi)$ satisfy the equations

$$\ddot{T}_\alpha - \Big[T_\beta, [T_\alpha, T_\beta]\Big] = 0. \tag{5.4}$$

and φ satisfies the Laplace equation

$$\Box\varphi = 0, \tag{5.5}$$

then the ansatz (5.2) gives the solution of the massless Klein-Gordon equation (5.1).

It is easy to see that each solution of the RW equations (2.3) satisfies Eqs. (5.4). Indeed, if one multiplies Eqs. (5.4) by $f_{\alpha\beta\delta}$ and differentiates these equations once more, then obtains

$$\ddot{T}_\alpha = -f_{\alpha\beta\gamma}[T_\beta, \dot{T}_\gamma].$$

At the same time, from Eqs.(2.3) it follows that

$$\Big[T_\beta, [T_\alpha, T_\beta]\Big] = -f_{\alpha\beta\gamma}[T_\beta, \dot{T}_\gamma].$$

Therefore, if T_α satisfy Eqs.(2.3), then T_α satisfy Eqs.(5.4). Remind that the function φ must satisfy Eqs.(2.4). Comparing Eqs.(2.4) with Eq.(5.5), we obtain the following system of equations:

$$f^\alpha_{\beta\gamma}\Sigma^{\beta\gamma}_{ac}\partial_c\partial_b\varphi - 2J^\alpha_{bc}\partial_c\partial_a\varphi + 2f^\alpha_{\beta\gamma}J^\beta_{ac}J^\gamma_{be}\partial_c\partial_e\varphi = 0, \qquad (5.6a)$$

$$\Box\varphi = 0. \qquad (5.6b)$$

Equations (5.6) have solutions. Some of them have been written out in Section IV (see also [6, 7]).

6. Conclusion

An example for $n = 4p$ and $q = 3$ shows that Eqs.(2.4) may have not only solution linear on coordinates x^a , but also more complicated solutions. It is interesting to study Eqs. (2.4) in the spaces $\mathbb{R}^n$ with q tensors J^α_{ab} and $n > 4p$ from (3.3) in the case when q coincides with the dimension of some simple Lie algebra $\mathcal{H}$. In this case, as $f_{\alpha\beta\gamma}$ in Eqs.(2.3) one may take structure constants of $\mathcal{H}$.

We have considered the case of Example 1 when $n = 4p$, $q = 3$ and $\mathcal{H} = su(2)$. If one takes eight tensors J^α_{ab} in $\mathbb{R}^{16p}$ from Example 3, then as $f_{\alpha\beta\gamma}$ one may choose the structure constants of the Lie algebra $su(3)$. In particular, from (3.3c) it follows that in spaces of dimension $n = 4096p$ one may introduce 24 tensors J^α_{ab} satisfying the relations (2.1), and as $f_{\alpha\beta\gamma}$ one may take the structure constants of the Lie algebra $su(5)$. All these cases need a special investigation.

Thus, we have shown that in constructing the solutions of the Yang-Mills equations in the spaces of dimension greater than four the technique of Clifford algebras plays an importent role. It permits one to reduce these equations to more simple system $\{(2.3), (2.4)\}$. Our results show strong evidence for detailed study of the integrability of the Rouhani-Ward equations (2.3) and Eqs.(2.4) for scalar field φ.

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TWO TYPES OF ANALYSIS ASSOCIATED TO THE NOTION OF HURWITZ PAIRS

ENRIQUE RAMíREZ DE ARELLANO†, MICHAEL V. SHAPIRO‡ and NIKOLAI L. VASILEVSKI†*
†Departamento de Matemáticas
Centro de Investigación y de Estudios Avanzados del IPN
Apartado Postal 14–740, México, D.F. 07000 – México
‡ Departamento de Matemáticas, ESFM del IPN
México, D.F. 07000 – México

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Introduction

The Hurwitz problem stated by him in 1898 [1] and its further development [2] were the motivation to introduce and to study the so–called Hurwitz pairs (see [9, 10] for an extensive literature).

In our works [6, 7] the precise relation between Hurwitz pairs and Clifford algebras has been established. Two canonical algorithms have been described for constructing an irreducible representation of a certain Clifford algebra for a given Hurwitz pair and, conversely, for constructing all possible Hurwitz pairs for given Clifford algebra and its irreducible representation.

All this has been inspired by our wish to develop the so–called Hurwitz analysis initiated in [3, 4].

The present article gives our vision of the situation, that is, which analytic theories can be related to a given Hurwitz pair. Our previous studies [6, 7], see also Section 1) show that a Hurwitz pair generates two types of multiplication which are essentially different, in general. In accordance with this fact, in Section 2 we introduce two types of generalized Cauchy–Riemann operators and show that all main formulas for the functions from their kernels can be obtained in the traditional way.

We stop our considerations at the place where it is perfectly clear how to develop both corresponding theories. We are going to explain how they are connected with the Clifford analysis elsewhere.

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1. Algebraic fundamentals of Hurwitz pairs

1.1 Let S be a $(p+1)$–dimensional real vector space with basis $\{\varepsilon_\alpha\}, \alpha \in \{0\} \cup N_p, N_p := \{1, 2, \ldots, p\}$, and let the R–bilinear form

$$(\cdot,\cdot)_S : S \times S \to R$$

be defined by the following metric matrix

$$\eta := [\eta_{\alpha\beta}] := [(\varepsilon_\alpha, \varepsilon_\beta)_S] = \text{diag}\ (\underbrace{1,\ldots,1}_{r+1}; \underbrace{-1,\ldots,-1}_{s}),$$

where $p = r + s$.

Introduce also the n–dimensional real vector space V with the basis $\{e_j\}, j \in N_n$, provided with an R–bilinear form

$$(\cdot,\cdot)_V : V \times V \to R,$$

which is defined by the following nonsingular metric matrix

$$\kappa := [\kappa_{kj}] := [(e_k, e_j)_V].$$

We assume also that the form $(\cdot,\cdot)_V$ in V is either symmetric: $\kappa = \kappa^t$, or antisymmetric: $\kappa = -\kappa^t$, where "t" means transposition.

Let

$$\circ : S \times V \to V$$

be an R–bilinear mapping. We call it (see, for instance [5]) a Hurwitz multiplication (of elements from V by elements of S on the left–hand side) if the following axioms are fulfilled:

H.1 for all $\{f, g\} \subset V$ and all $a \in S$

$$(a, a)_S (f, g)_V = (a \circ f, a \circ g)_V;$$

H.2 there exists the unit element $\varepsilon \in S$ with respect to the mapping "$\circ$", i.e. for all $f \in V$

$$\varepsilon \circ f = f;$$

H.3 the mapping "$\circ$" does not leave invariant any proper subspace of V. The set $(S, V, \circ)$ is called a Hurwitz pair.

1.2 Introduce the R–linear isomorphisms

$$\nu_S : S \to R^{p+1}$$

and

$$\nu_V : V \to R^n$$

by the rules

$$\nu_S : a = \sum_{\alpha=0}^{p} a_\alpha \varepsilon_\alpha \longmapsto \tilde{a} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_p \end{pmatrix} \in R^{p+1}$$

and

$$\nu_V : f = \sum_{j=1}^{n} f_j e_j \longmapsto \tilde{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \in R^n.$$

For each basis elements $\varepsilon_\alpha \in S$ and $e_j \in V$ we have $\varepsilon_\alpha \circ \varepsilon_j \in V$. Thus for some real constants $c_{\alpha j}^k$:

$$\varepsilon_\alpha \circ e_j = \sum_{k=1}^{n} c_{\alpha j}^k e_k,$$

and for every $f = \sum\limits_{j=1}^{n} f_j e_j \in V$:

$$\varepsilon_\alpha \circ f = \sum_{k=1}^{n} (\sum_{j=1}^{n} c_{\alpha j}^k f_j) e_k.$$

Applying the isomorphism ν_V to the both sides of this equality we obtain

$$\nu_V(\varepsilon_\alpha \circ f) = C_\alpha \cdot \nu_V(f),$$

where $C_\alpha := [c_{\alpha j}^k]_{j,k=1}^n$.

Thus, each element $\varepsilon_\alpha \in S$ determines uniquely the matrix C_α (and vice versa), and the following diagram

$$\begin{array}{ccc} V & \xrightarrow{m(\varepsilon_\alpha)} & V \\ {\scriptstyle \nu_V}\Big\downarrow & & \Big\downarrow{\scriptstyle \nu_V} \\ R^n & \xrightarrow{m(C_\alpha)} & R^n \end{array}$$

is commutative. Here $m(\varepsilon_\alpha) : f \mapsto \varepsilon_\alpha \circ f$ and $m(C_\alpha) : \tilde{f} \mapsto C_\alpha \cdot \tilde{f}$.

1.3 Without loss of generality we may assume that the unit element ($\equiv$ identity $\varepsilon \in S$) coincides with ε_0.

Under this assumption we have (see [6]): for each $\{\alpha, \beta\} \subset N_p$

$$C_\alpha \cdot C_\beta + C_\beta \cdot C_\alpha = -2\eta_{\alpha\beta} I_n.$$

1.4 Consider a fixed Hurwitz pair $(S, V, \circ)$ with the unit element $\varepsilon = \varepsilon_0$. Each element $a \in S$ generates in a natural way the operator of "Hurwitz multiplication by a" acting on V by the rule

$$f \in V \longmapsto a \circ f \in V.$$

Denote this operator by $m(a)$. It is clear that for $a = \sum_{\alpha=0}^{p} a_\alpha \varepsilon_\alpha$

$$m(a) = \sum_{\alpha=0}^{p} a_\alpha m(\varepsilon_\alpha),$$

where $m(\varepsilon_0) = I$, the identity operator on V.

The mapping

$$\mu : a \longmapsto \mu(a) \in \mathrm{Hom}\,(V, V)$$

gives a linear isomorphism between the space S and some linear subspace of $\mathrm{Hom}(V, V)$.

Denote by $Al(S, V, \circ)$ the algebra generated by all operators $m(a)$ acting on V. We have obviously

$$\mu(S) \subset Alg(S, V, \circ) \subset \mathrm{Hom}\,(V, V) \cong R(n),$$

and in general both inclusions are proper.

All above said can be found in [6, 7], but to construct the corresponding function theory we need to complement the algebraic part of those works with some new results.

1.5 The algebra $Alg(S, V, \circ)$ allows us to introduce a multiplication on elements of S in such a way that the algebra $\widetilde{S}$ generated by this multiplication becomes isomorphic to $Al(S, V, \circ)$. The corresponding isomorphism will be an extension of the linear mapping μ from S onto the algebra $\widetilde{S}$.

Let us describe this procedure.

Denote by $\check{S}$ the *free* algebra generated by the elements of S and let

$$\check{\mu} : \check{S} \longrightarrow Alg(S, V, \circ)$$

be the real algebra homomorphism which is generated by the following mapping of the generators $a \in S$ of the algebra $\check{S}$:

$$\check{\mu} : a \in S \longmapsto \check{\mu}(a) := \mu(a) = m(a) \in \mathrm{Hom}\,(V, V),$$

i.e. the mapping $\check{\mu}$ is an extension (up to a real algebra homomorphism) of the mapping

$$\mu : S \longrightarrow Alg(S, V, \circ).$$

Now introduce the algebra $\widetilde{S} := \check{S}/ker\check{\mu}$ and the mapping

$$\widetilde{\mu} : \widetilde{S} \longrightarrow Alg(S, V, \circ) \tag{1.1}$$

which is defined by the rule

$$\widetilde{\mu} : [\check{a}] \longmapsto \check{\mu}(\check{a}),$$

where $[\check{a}] := \check{a} + ker\check{\mu}$ for $\check{a} \in \check{S}$.

We will denote by "$*$" the multiplication symbol in $\widetilde{S}$.

The described procedure provides a natural imbedding of S into $\widetilde{S}$. Identifying, as usual, S and its image under this imbedding, we can say now that S is a subset (and a linear subspace) of the algebra $\widetilde{S}$. It is clear that

$$\widetilde{\mu}|S = \mu.$$

The mapping $\widetilde{\mu}$ gives now a real algebra isomorphism of $\widetilde{S}$ onto $Alg(S, V, \circ) \subset \mathrm{Hom}(V, V)$, and thus also a representation of the algebra $\widetilde{S}$ on the space V.

Moreover, we have a well-defined extension (from $S \times V$ onto $\widetilde{S} \times V$) of the Hurwitz multiplication, also denoted by "$\circ$", as follows:

$$\text{for each} \qquad s \in \widetilde{S} \quad \text{and} \quad f \in V$$

$$s \circ f := \widetilde{\mu}(s)(f) \in V.$$

1.6 Remark. For any elements $s_1, s_2 \in \widetilde{S}$ and $f \in V$ the following "associativity law" is true

$$(s_1 * s_2) \circ f = s_1 \circ (s_2 \circ f),$$

and thus we can write $s_1 * s_2 \circ f := s_1 \circ s_2 \circ f := (s_1 * s_2) \circ f = s_1 \circ (s_2 \circ f)$.

2. Conceptions of Hurwitz analyses

2.1 Given a Hurwitz pair $(S, V, \circ)$ (with no restrictions on η), denote by l any integer with the condition $2 \le l \le p$, and let Ω be a domain in R^{l+1}. For any set of vectors $(\psi^0, \psi^1, \ldots, \psi^l) =: \psi \in S^{l+1}$ we can write the formal expression

$$\sum_{\alpha=0}^{l} \psi^\alpha \circ \frac{\partial}{\partial x_\alpha} \tag{2.1}$$

where $\frac{\partial}{\partial x_\alpha}$ denotes the operation of the usual partial derivation of a given ($\widetilde{S}$–valued or V–valued) function defined in Ω. Depending on the type of function the expression (2.1) allows to introduce two kinds of operators.

2.2 For an arbitrary function $f \in C^1(\Omega, V)$ define the operator $\mathcal{D}_\psi$ by the rule

$$\mathcal{D}_\psi[f] := \sum_{\alpha=0}^{l} \psi^\alpha \circ \frac{\partial}{\partial x_\alpha}[f] := \sum_{\alpha=0}^{l} \psi^\alpha \circ \frac{\partial f}{\partial x_\alpha}. \tag{2.2}$$

The operator $\mathcal{D}_\psi$ becomes later the Cauchy–Riemann operator in the analysis of V–valued functions. It is necessary to emphasize strongly that just here we see the consequence of the Hurwitz multiplication asymmetry: in contrast with the usual hyperholomorphic settings we can introduce the left operator only, not the left and the right ones (compare with what will be done below).

2.3 Consider an arbitrary $g \in C^1(\Omega, \widetilde{S})$. Introduce the left ${}^\psi D$ and the right D^ψ analogs of the operator $\mathcal{D}_\psi$ by the rules:

$${}^\psi D[g] := \sum_{\alpha=0}^{l} \psi^\alpha * \frac{\partial}{\partial x_\alpha}[g] =: \sum_{\alpha=0}^{l} \psi^\alpha * \frac{\partial g}{\partial x_\alpha} \tag{2.3}$$

and

$$D^\psi[g] := \sum_{\alpha=0}^{l} \frac{\partial}{\partial x_\alpha} * m(\psi^\alpha)[g] := \sum_{\alpha=0}^{l} \frac{\partial g}{\partial x_\alpha} * \psi^\alpha, \tag{2.4}$$

where m is the map defined in 1.6.

2.4 Using the notion of the "natural conjugation" on S we introduce the "conjugate" operators $\mathcal{D}_\psi^\#$, ${}^\psi D^\#$ and $D^{\psi,\#}$:

$$\begin{aligned} \mathcal{D}_\psi^\#[f] &:= \sum_{\alpha=0}^{l} \psi^{\alpha\#} \circ \frac{\partial}{\partial x_\alpha}[f], \\ {}^\psi D^\#[g] &:= \sum_{\alpha=0}^{l} \psi^{\alpha\#} * \frac{\partial}{\partial x_\alpha}[g], \\ D^{\psi\#}[g] &:= \sum_{\alpha=0}^{l} \frac{\partial}{\partial x_\alpha}[g] * \psi^{\alpha\#}, \end{aligned} \tag{2.5}$$

where "#" is a linear mapping on S defined on the basis elements ε_k by the rule

$$\varepsilon_k^\# = -\varepsilon_k, \; k \in N_p$$

2.5 Remark. To develop the corresponding function theory it is necessary to be able to multiply the above defined operators.

The following peculiarities arise from the asymmetry of the Hurwitz multiplication. We have sets of S–valued functions, of V–valued functions, of $\widetilde{S}$–valued operators (that is, differential operators with coefficients from $\widetilde{S}$), etc. We can "multiply" V-valued functions on the *left*-hand side by $\widetilde{S}$-valued operators obtaining in V-valued functions: ${}^{\psi}D \circ f := {}^{\psi}D[f]$.

We can "multiply" various $\widetilde{S}$-valued operators on both sides resulting in $\widetilde{S}$-valued operators: ${}^{\psi}D * {}^{\psi}D^{\#} := {}^{\psi}D \cdot {}^{\psi}D^{\#}$ with the "$\cdot$" denoting the usual operator product. We should take into account that, according to the definition of ${}^{\psi}D$ and ${}^{\psi}D^{\#}$, the result of the multiplication is an operator acting on $\widetilde{S}$-valued (not on V-valued) functions.

Finally we can "multiply", in the sense of the Hurwitz multiplication, operators of the type of $\mathcal{D}_{\psi}$:

$$\mathcal{D}_{\psi} \circ \mathcal{D}_{\psi} : f \mapsto \mathcal{D}_{\psi} \circ (\mathcal{D}_{\psi} \circ f) = \mathcal{D}_{\psi}\left[\mathcal{D}_{\psi}[f]\right].$$

The operator $\mathcal{D}_{\psi}\circ\mathcal{D}_{\psi}$ acts on V-valued functions and in this sense $\mathcal{D}_{\psi} * \mathcal{D}_{\psi}^{\#} \neq {}^{\psi}D * {}^{\psi}D^{\#}$.

2.6 For a fixed set ψ introduce the differential l-form

$$\sigma_{\psi,x}^{(l)} := \sum_{\alpha=0}^{l} (-1)^{\alpha} \cdot \psi^{\alpha} dx_{[\alpha]} \tag{2.6}$$

where $dx_{[\alpha]}$ is the differential l-form $dx := dx_0 \wedge \ldots \wedge dx_l$ with dx_{α} omitted. The operator of exterior differentiation d acts on such $\widetilde{S}$-valued differential forms as a $\widetilde{S}$-linear mapping. Then, if $g \in \mathrm{C}^1(\Omega, \widetilde{S}), f \in \mathrm{C}^1(\Omega, V)$, an easy calculation gives:

$$d\left(g * \sigma_{\psi,x}^{(l)} \circ f\right) = \left(D^{\psi}[g] \circ f + g * \mathcal{D}_{\psi}[f]\right) dx. \tag{2.7}$$

And analogously, for $g, h \subset \mathrm{C}^1(\Omega, \widetilde{S})$

$$d\left(g * \sigma_{\psi,x}^{(l)} * h\right) = \left(D^{\psi}[g] * h + g * {}^{\psi}D[h]\right) dx. \tag{2.8}$$

2.7 If now we assume that Ω is a bounded domain with smooth enough boundary $\Gamma = \partial\Omega$, then application of the Stokes formula results immediately in the following equalities:

$$\int_{\Gamma} g * \sigma_{\psi,x}^{(l)} \circ f = \int_{\Omega} \left(D^{\psi}[g] \circ f + g * \mathcal{D}_{\psi}[f]\right) dx, \tag{2.9}$$

$$\int_{\Gamma} g * \sigma_{\psi,x}^{(l)} * h = \int_{\Omega} \left(D^{\psi}[g] * h + g * {}^{\psi}D[h]\right) dx. \tag{2.10}$$

2.8 Up to now we assumed no restrictions on η. But if we want to have a good function theory, we should limit ourselves to the cases where $\eta = I_{p+1}$ or $\eta = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -I_p \end{smallmatrix}\right)$. Let one of these conditions be fulfilled. Denote by

$\Delta_{l+1}(\widetilde{S})$ and $\Delta_{l+1}(V)$ the usual $(l+1)$–dimensional Laplace operator acting on $C^2(\Omega, \widetilde{S})$ and $C^2(\Omega, V)$, respectively.

Then

$$ {}^{\psi}D^{\#} * {}^{\psi}D = {}^{\psi}D * {}^{\psi}D^{\#} = D^{\psi\#} * D^{\psi} = D^{\psi} * D^{\psi\#} = \Delta_{l+1}(\widetilde{S}), \quad (2.11) $$

$$ \mathcal{D}_{\psi} * \mathcal{D}_{\psi}^{\#} = \mathcal{D}_{\psi}^{\#} * \mathcal{D}_{\psi} = \Delta_{l+1}(V). \quad (2.12) $$

It is easy to describe all ψ's with the properties (2.11)–(2.12).

2.9 It is well–know that most part of the usual one–dimensional complex analysis (i.e. the theory of holomorphic functions of one complex variable) can be constructed starting from only two facts: a) factorization of the Laplace operator by the conjugate Cauchy–Riemann operators and b) the Green's (or 2–dimensional Stokes) formula. Some multidimensional generalizations, such as the quaternionic and the Clifford analysis, are based on these two facts (one can find the detailed substantiation of this point of view in [8], for example).

Formulas (2.9) and (2.11), as well as (2.10) and (2.12) show that we can develop the corresponding theories for V–valued and $\widetilde{S}$–valued functions in the same way. We will show the initial part of this procedure just to illustrate the idea.

We shall use the notations

$$ \ker\ \mathcal{D}_{\psi} =: \mathcal{N}_{\psi}(\Omega, V);\ \ker\ {}^{\psi}D =: {}^{\psi}\mathcal{M}(\Omega, \widetilde{S});\ \ker\ D^{\psi} =: \mathcal{M}^{\psi}(\Omega, \widetilde{S}), \quad (2.13) $$

and call the elements of these sets V–valued and $\widetilde{S}$–valued hyperholomorphic functions, respectively (in the latter case adding sometimes the word "left" of "right").

2.10 Let θ_{l+1} denote the fundamental solution of Δ_{l+1} in R^{l+1}, i.e. $\Delta_{l+1}(\theta_{l+1}) = \delta$,

$$ \theta_{l+1} : x \in R^{l+1} \setminus \{0\} \longrightarrow \frac{1}{(1-l)|S^l|} \cdot |x|^{1-l}, \quad (2.14) $$

where $|S^l|$ is the area of the unit sphere in R^{l+1}. We cannot identify θ_{l+1} in a natural way with a V–valued function, but we can identify it with the $\widetilde{S}$–valued function $\theta_{l+1} \cdot \varepsilon_0$. Hence we can introduce the function

$$ \mathcal{K}_{\psi}(x) := {}^{\psi}\mathcal{D}^{\#}[\theta_{l+1}](x) = D^{\psi\#}[\theta_{l+1}](x) = \quad (2.15) $$
$$ = \frac{1}{|S^l| \cdot |x|^{l+1}} \cdot \sum_{\alpha=0}^{l} \psi^{\alpha\#} \cdot x_{\alpha}, $$

which will play the role of the Cauchy kernel for both theories. It has the following important properties:

a) $\mathcal{K}_\psi \in C^\infty(R^{l+1} \setminus \{0\}, \widetilde{S})$
b) $\mathcal{K}_\psi \in {}^\psi\mathcal{M}(R^{l+1} \setminus \{0\}, \widetilde{S}) \cap \mathcal{M}^\psi(R^{l+1} \setminus \{0\}, \widetilde{S})$,
c) Let $y \in R^{l+1}$, $\mathcal{K}_{\psi,y}(x) := \mathcal{K}_\psi(y-x)$; then $\mathcal{K}_{\psi,y} \in {}^\psi\mathcal{M}(R^{l+1} \setminus \{y\}) \cap \mathcal{M}^\psi(R^{l+1} \setminus \{y\})$.

2.11 Theorem (Borel–Pompeiu formula). *Let* $f \in C^1(\overline{\Omega}, V)$, $g \in C^1(\overline{\Omega}, \widetilde{S})$, *then for* $\forall x \in \Omega$,

$$f(x) = \int_\Gamma \mathcal{K}_\psi(\tau - x) * \sigma^{(l)}_{\psi,\tau} \circ f(\tau) - \int_\Omega \mathcal{K}_\psi(\tau - x) * \mathcal{D}_\psi[f](\tau)d\tau \quad (2.16)$$

$$g(x) = \int_\Gamma \mathcal{K}_\psi(\tau - x) * \sigma^{(l)}_{\psi,\tau} * g(\tau) - \int_\Omega \mathcal{K}_\psi(\tau - x) * {}^\psi D[g](\tau)d\tau \quad (2.17)$$

$$= \int_\Gamma g(\tau) * \sigma^{(l)}_{\psi,\tau} * \mathcal{K}_\psi(\tau - x) - \int_\Omega D^\psi[g] * \mathcal{K}_\psi(\tau - x)d\tau. \quad (2.18)$$

Proof. Cut out a small ball centered in x, apply (2.9) and (2.10) to the rest of Ω; substitute $\mathcal{K}_\psi$ instead of g or f. Standard routine calculations give the answer.

2.12 Theorem (Cauchy integral theorem). *Let* $f \in \mathcal{N}_\psi(\overline{\Omega}, V)$, $g \in \mathcal{M}^\psi(\overline{\Omega}, \widetilde{S})$, $h \in {}^\psi\mathcal{M}(\overline{\Omega}, \widetilde{S})$, *then*

$$\int_\Gamma g * \sigma^{(l)}_{\psi,\tau} \circ f = 0,$$

$$\int_\Gamma g * \sigma^{(l)}_{\psi,\tau} * h = 0. \quad (2.19)$$

Proof. Directly from (2.9)–(2.10).

2.13 Theorem (Cauchy integral formula). *Let* $f \in \mathcal{N}_\psi(\overline{\Omega}, V)$, $g \in \mathcal{M}^\psi(\overline{\Omega}, \widetilde{S})$, $h \in {}^\psi\mathcal{M}(\overline{\Omega}, \widetilde{S})$, *then for* $\forall x \in \Omega$

$$f(x) = \int_\Gamma \mathcal{K}_\psi(\tau - x) * \sigma^{(l)}_{\psi,\tau} \circ f(\tau),$$

$$h(x) = \int_\Gamma \mathcal{K}_\psi(\tau - x) * \sigma^{(l)}_{\psi,\tau} * h(\tau),$$

$$g(x) = \int_\Gamma g(\tau) * \sigma^{(l)}_{\psi,\tau} * \mathcal{K}_\psi(\tau - x).$$

Proof. Directly from Theorem 2.11.

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CLIFFORD TENSOR CALCULUS

FRANK SOMMEN*
Department of Mathematical Analysis, University of Gent
Galglaan 2, B-9000 Gent/Belgium

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Abstract. In this paper we give a description of some basic operations on Clifford tensors involving symmetrization and alternation. We also define monogenic tensors and establish the so called monogenic decomposition of tensors.

Key words: Clifford analysis, tensors, group representations

1. Introduction

Clifford algebras arise in many areas of mathematics and mathematical physics, especially in connection with the Dirac operator. Clifford analysis studies function theory of Clifford algebra-valued operators, such as the Dirac operator acting on functions with values in spinor spaces or Clifford algebras. An introduction to this field of research may be found in the recent books [3] and [5], while a number of related topics was treated in [6] and other works. Clifford algebras do contain the representation spaces of the basic irreducible representations of the spin group $Spin(m)$ as subspaces, namely the spaces $\mathbf{R}_m^k$ of k-vectors and the spinor spaces, which may be realized as minimal left ideals of the Clifford algebra. This motivates the consideration of Clifford algebra-valued functions. But all the other irreducible representations of $Spin(m)$ are not realized on subspaces of the Clifford algebra and so, other analytic tools are needed to represent them. From an abstract point of view one can construct all irreducible representations of $Spin(m)$ out of the basic ones by Cartan composition. But then the theory of the Dirac operator acting on functions with values in representation spaces remains rather abstract (see [12] for a definition of the Dirac operator). Both from a theoretical and from a computational point of view it seems better to work with "simple analytic tools" such as polynomials $P(\underline{u})$, $\underline{u} = \sum u_j e_j$ being a vector variable and differential operators $P(\partial_{\underline{u}})$, $\partial_{\underline{u}} = \sum \partial_{u_j} e_j$ being a Clifford Dirac operator, also called vector derivative. In [10] e.g. we considered spin-invariant differential operators acting on functions with values in special spaces of so called spherical monogenic polynomials (Clifford polynomials $P(\underline{u})$ satisfying $\partial_{\underline{u}} P(\underline{u}) = 0$). Similar operators acting on functions with values in sphe-

* Senior Research Associate, NFWO, Belgium

rical harmonics of a vector or matrix variable were considered in [5] and [13]. But these are just examples of function theory with values in representation spaces of $Spin(m)$ and none of these examples contains all representation spaces. However, if instead of one vector variable $\underline{u}$ and vector derivative $\partial_{\underline{u}}$ we consider polynomials $P(\underline{u}_1, \underline{u}_2, \ldots)$ depending on several vector variables and the corresponding vector derivatives $\partial_{\underline{u}_1}, \partial_{\underline{u}_2}, \ldots$ we have a rich enough calculus to represent all irreducible representations of $Spin(m)$ analytically. In fact it is possible to realise all irreducible representations of $Spin(m)$ on the space of Clifford tensors (multilinear Clifford polynomials $P(\underline{u}_1, \ldots, \underline{u}_k)$ of several vector variables) on which only two basic $Spin(m)$-representations are defined, one corresponding to representations with integer weight and the other to representations with half integer weight. In a previous paper [8] we studied all the $Spin(m)$- and $Gl(m)$-invariant operators acting on Clifford tensors (see also our related papers [4] and [9]). In the present paper we first give a description of how tensors are related to polynomials of several vector variables and study the basic symmetrization and alternation operators, including Young tables and Young symmetry operators (see also [6], [14]). The second section deals with so called "monogenic tensors", i.e. multilinear functions $P(\underline{u}_1, \ldots, \underline{u}_k)$ satisfying the equations $\partial_{\underline{u}_j} P(\underline{u}_1, \ldots, \underline{u}_k) = 0$ as well as the monogenic decomposition of tensors (decomposition into monogenic pieces).

Both the Young symmetry operators and the operators of monogenic decomposition are essential for the decomposition of Clifford tensors into irreducible pieces.

2. Clifford polynomials and tensors

Let $\{e_1, \ldots, e_m\}$ be the standard basis of $\mathbf{R}^m$, then $\mathbf{R}_m$ denotes the Clifford algebra determined by the relations

$$e_i e_j + e_j e_i = -2\delta_{ij}.$$

A vector variable $\underline{u} \in \mathbf{R}^m$ is just a variable Clifford vector, also expressed as the polynomial $\underline{u} = \sum u_j e_j$. The Dirac operator or vector derivative $\partial_{\underline{u}}$ is then given by $\partial_{\underline{u}} = \sum \partial_{u_j} e_j$.

A Clifford tensor F on $\mathbf{R}^m$ is a multilinear map $F : (\underline{u}_1, \ldots, \underline{u}_k) \rightarrow F(\underline{u}_1, \ldots, \underline{u}_k)$ from $\mathbf{R}^m$ into the Clifford algebra (see also [6]). We think of it as a multilinear polynomial $F(\underline{u}_1, \ldots, \underline{u}_k)$ depending on k vector variables.

The tensor product FG of two tensors is then represented by the polynomial product

$$FG(\underline{u}_1, \ldots, \underline{u}_{k+l}) = F(\underline{u}_1, \ldots, \underline{u}_k)\, G(\underline{u}_{k+1}, \ldots, \underline{u}_{k+l}).$$

T_k denotes the space of Clifford k-tensors. T denotes their algebra. At present one might consider tensors merely as multilinear polynomials of a sequence

$\underline{u}_j$ of vector variables. But the tensor product requires a shift of indices, so the tensor product is not merely a polynomial product. To be more precise, let $\mathcal{P}\{\underline{u}\}$ denote the algebra of polynomials of an infinite sequence $\{\underline{u}_1, \underline{u}_2, \ldots\}$ of vector variables and let $Lin\{\underline{u}\}$ be the subspace of multilinear polynomials. Then for any $a_1 < \ldots < a_k$ and multilinear polynomial $P(\underline{u}_{a_1}, \ldots, \underline{u}_{a_k}) \in Lin\{\underline{u}\}$ we consider the tensor $t(P)$ with "standard polynomial representation" $P(\underline{u}_1, \ldots, \underline{u}_k)$. Hence T is a quotient of $Lin\{\underline{u}\}$.

Next, let $\mathbf{R}^{m'}$ be the "dual of $\mathbf{R}^m$", i.e. the space of linear functions $\underline{x} \to \langle \underline{u}', \underline{x} \rangle$, $\underline{x} \in \mathbf{R}^m$. Then we may think of $\mathbf{R}^{m'}$ as another copy of $\mathbf{R}^m$ and represent $\underline{x} \to \langle \underline{u}', \underline{x} \rangle$ by the Clifford vector $\underline{u}'$. A dual l-tensor is a l-linear map from $\mathbf{R}^{m'}$ into $\mathbf{R}^m$. Hence if T' (resp. T_l') denotes the space of dual tensors (resp. of degree l), then $G \in T_l'$ is represented by a multilinear polynomial $G(\underline{u}_1', \ldots, \underline{u}_l') \in Lin\{\underline{u}'\} \subset \mathcal{P}\{\underline{u}'\}$.

Tensors and dual tensors can only be distinguished by their transformation properties under the group $Gl(m)$. The action of $Gl(m)$ on Clifford tensors was studied in our previous paper [8]. One can also consider the spaces $T_k \otimes T_{l'}$ of tensors of mixed type represented by multilinear polynomials of the form

$$F(\underline{u}_1, \ldots, \underline{u}_k; \underline{u}_1', \ldots, \underline{u}_l') \in Lin\{\underline{u}, \underline{u}'\} \subset \mathcal{P}\{\underline{u}, \underline{u}'\}$$

where $Lin\{\underline{u}, \underline{u}'\}$ is the subspace of multilinear polynomials of the space $\mathcal{P}\{\underline{u}, \underline{u}'\}$ of polynomials in the sequences $\{\underline{u}_1, \ldots; \underline{u}_1', \ldots\}$ of vector variables. Note that the tensor product FG of $F \in T_k$ and $G \in T_l'$ is given by

$$FG: (\underline{u}_1, \ldots, \underline{u}_k; \underline{u}_1', \ldots, \underline{u}_l') \to F(\underline{u}_1, \ldots, \underline{u}_k) G(\underline{u}_1', \ldots, \underline{u}_l')$$

so there is no index shifting here.

Next let $F \in T_k$ and $G \in T_l'$ with $l \leq k$. Then the tensor contraction $G \cdot F$ is the $(k-l)$-tensor given by

$$G \cdot F(\underline{u}_1, \ldots, \underline{u}_{k-l}) = G(\partial_{\underline{v}_1}, \ldots, \partial_{\underline{v}_l}) F(\underline{v}_1, \ldots, \underline{v}_l, \underline{u}_1, \ldots, \underline{u}_{k-l}).$$

For $l = k$ we can consider the bilinear form $\langle G, F \rangle = [G \cdot F]_0$, where for $a \in \mathbf{R}_m$, $[a]_0$ denotes the scalar part of a. This bilinear form can be seen to be $Gl(m)$-invariant and determines the duality between T_k and T_k'.

In this paper we are more interested in the action of $Spin(m)$ on the spaces T_k of Clifford tensors and decompositions of T_k in invariant subspaces. Recall that

$$Spin(m) = \{s = \underline{\omega}_1 \cdots \underline{\omega}_{2k} : \quad \underline{\omega}_j \in S^{m-1}\}.$$

Moreover, consider on $\mathbf{R}_m$ the main involution $a \to \tilde{a}$ and the main anti-involution $a \to \bar{a}$ determined by

$$\widetilde{(ab)} = \tilde{a}\tilde{b}, \quad \overline{ab} = \bar{b}\bar{a}, \quad \tilde{\underline{v}} = \bar{\underline{v}} = -\underline{v}, \quad \underline{v} \in \mathbf{R}^m.$$

Then for $s \in Spin(m)$, the map $h(s) : \underline{x} \to s\underline{x}\bar{s}$ is a rotation and the map $s \to h(s)$ is the double covering of $Spin(m)$ on $SO(m)$. The map $h(s)$ can be extended to the whole Clifford algebra $\mathbf{R}_m$. Moreover, let $\mathbf{R}_m^k$ be the space of k- vectors and denote by $[a]_k$ the projection of $a \in \mathbf{R}_m$ on $\mathbf{R}_m^k$. Then $[\cdot]_k$ commutes with $h(s)$ and so $\mathbf{R}_m^k$ are invariant subspaces under this representation of $Spin(m)$, which for $k < [m/2]$ are irreducible. The "Hodge star map" $a \to e_{1\dots m}a$ determines an isomorphism between $\mathbf{R}_m^k$ and $\mathbf{R}_m^{m-k}$ which commutes with $h(s)$. Hence $Spin(m)$ acts in the same way on $\mathbf{R}_m^{m-k}$ as on $\mathbf{R}_m^k$. For $m = 2p$, $\mathbf{R}_m^p$ splits into two inequivalent irreducible subspaces, namely the eigenspaces of the Hodge star map.

Of course the standard spin representation is simply given by $l(s)a = sa$, $a \in \mathbf{R}_m$ and under this representation $\mathbf{R}_m$ splits into so called spinor spaces which are irreducible and representable as minimal left ideals of Clifford algebras (see [1], [3], [7]). These are the basic irreducible representations of $Spin(m)$ from which all others may be obtained by Cartan composition. However, this procedure cannot be carried out in the Clifford algebra setting. For functions $f : \mathbf{R}_m \to \mathbf{R}_m$, we consider the representations

$$L(s)f(a) = sf(\bar{s}as), \quad H(s)f(a) = sf(\bar{s}as)\bar{s}.$$

One can of course extend the definition of L and H to functions $f(a_1, \dots, a_k)$ of several Clifford variables and in particular to polynomials $P(\underline{u}_1, \dots, \underline{u}_k)$ of several vector variables. From this follow the representations of L and H on the spaces T_k (resp. T_l', $T_k \otimes T_l'$) of Clifford tensors:

$$L(s)F(\underline{u}_1, \dots, \underline{u}_k) = sf(\bar{s}\underline{u}_1 s, \dots, \bar{s}\underline{u}_k s),$$

$$H(s)F(\underline{u}_1, \dots, \underline{u}_k) = sf(\bar{s}\underline{u}_1 s, \dots, \bar{s}\underline{u}_k s)\bar{s}.$$

A natural inner product on T_k is defined as follows. Let $F \in T_k$, then we consider $F^* \in T_k'$ given by

$$F^* : (\underline{u}_1', \dots, \underline{u}_k') \to \bar{F}(\underline{u}_1', \dots, \underline{u}_k').$$

The inner product on T_k is then given by

$$(G, F) = [G^* \cdot F]_0 = [\overline{G}(\partial_{\underline{v}_1}, \dots, \partial_{\underline{v}_k}) F(\underline{v}_1, \dots, \underline{v}_k)]_0.$$

It is clear that this inner product is invariant under both $L(s)$ and $H(s)$, $s \in Spin(m)$.

Apart from the groups $Gl(m)$ and $Spin(m)$ we also consider the action of the permutation group $Sym(k)$ of $\{1, \dots, k\}$ on the space T_k (resp. T_k') of Clifford tensors. Let $\pi \in Sym(k)$, then we put

$$\pi(F)(\underline{u}_1, \dots, \underline{u}_k) = F(\underline{u}_{\pi(1)}, \dots, \underline{u}_{\pi(k)}).$$

In this way, we have a representation of $Sym(k)$ on tensors, which allows us to think of $\pi \in Sym(k)$ as an operator and to consider the operator algebra $\mathcal{A}(k)$ generated by $Sym(k)$, which is in fact the algebra of the group $Sym(k)$ (see also [14]). Note that the elements of $\mathcal{A}(k)$ are $Gl(m)$-invariant, i.e. they commute with the action of $Gl(m)$ on Clifford tensors. In the representation theory of both groups $Sym(k)$ and $Gl(m)$ on tensors, a crucial role is played by $\mathcal{A}(k)$ and in particular by operators of symmetrization and alternation which are natural projection operators belonging to $\mathcal{A}(k)$. The simplest examples are the operators Sym and Alt given by

$$Sym\ F = \frac{1}{k!}\sum \pi(F), \quad Alt\ F = \frac{1}{k!}\sum sgn(\pi)\,\pi(F),$$

leading to the subspaces S_k and L_k of T_k of symmetric resp. alternating k-tensors. Similarly, one can define Sym and Alt on T'_k, leading to subspaces S'_k and L'_k. The space S_k is naturally isomorphic to $\mathcal{P}_k\{\underline{u}\}$ of homogeneous polynomials of degree k in one vector variable $\underline{u}$ and the isomorphism is the restriction of S_k to the so called "polynomial projection" on T_k, given by

$$P:\ F(\underline{u}_1,\ldots,\underline{u}_k) \to P(F)(\underline{u}) = F(\underline{u},\ldots,\underline{u}).$$

Let $Q:\ \mathcal{P}_k\{\underline{u}\} \to S_k$ be the right inverse of P, then the operator Sym is clearly given by $Sym = Q\,P$. Let $F, G \in S_k$, then the symmetric tensor product $F \circ G$ is given by $F \circ G = Sym(FG)$. It is immediately clear that $P(F \circ G) = P(F)P(G)$. Next let $\underline{u} \in \mathbf{R}^m$; then $\underline{u}$ determines an element $T'_1 = S'_1$, namely $\langle \underline{u}, \cdot\rangle :\ \underline{v} \to \langle \underline{u}, \underline{v}\rangle$ and the contraction $\langle \underline{u}, \cdot\rangle \cdot F$ is given by $\langle \underline{u}, \partial_{\underline{v}}\rangle F(\underline{v}, \underline{u}_2, \ldots, \underline{u}_{k-1})$ so that

$$P(\langle \underline{u}, \cdot\rangle \cdot\ F)(\underline{v}) = \frac{1}{k}\langle \underline{u}, \partial_{\underline{v}}\rangle P(F)(\underline{v}).$$

More generally, the space S'_k of symmetric dual tensors is the dual space of S_k and the duality is determined by the contraction

$$G \cdot F = \frac{1}{k!}P(G)(\partial_{\underline{u}})P(F)(\underline{u}), \quad F \in S_k, \quad G \in S'_k.$$

Note that the inner product on T_k restricted to S_k is given by

$$(G, F) = [G^* \cdot\ F]_0 = \frac{1}{k!}[\overline{P(G)}(\partial_{\underline{u}})P(F)(\underline{u})]_0 = (P(G), P(F)),$$

where for $P, P' \in \mathcal{P}_k\{\underline{u}\}$, (P, P') denotes the Fischer inner product $\frac{1}{k!}[\bar{P}(\partial_{\underline{u}})\ P'(\underline{u})]_0$ (see also [3], [10]). Next consider the spaces L_k of alternating k-tensors and their duals L'_k; then the definitions introduced for the spaces S_k have an "alternative analogue". Instead of the symmetric tensor product we can introduce the "cup product" of $F \in L_k$ and $G \in L_l$ by $F \wedge G = Alt\, F\, G$ which turns the direct sum $L = L_0 + L_1 + \ldots + L_m$ into the so called algebra of Clifford forms on $\mathbf{R}^m$ (do not confuse the cup product with the wedge product of Clifford numbers). Next let again $\underline{u} \in \mathbf{R}^m$ and consider the corresponding element $\langle \underline{u}, \cdot \rangle \in T_1' = L_1'$; then we can define the contraction $\langle \underline{u}, \cdot \rangle \rfloor F$ of a k-form F with a "vector" $\langle \underline{u}, \cdot \rangle$by $\langle \underline{u}, \cdot \rangle \rfloor F = k \langle \underline{u}, \cdot \rangle \cdot F$. This notion of contraction is the one usually introduced for differential forms. Let $\underline{v} = \langle \underline{u}, \cdot \rangle$, then we have that for $F \in L_k$ and $G \in L_l$,

$$\underline{v} \rfloor (F \wedge G) = \underline{v} \rfloor F \wedge G + (-1)^k F \wedge \underline{v} \rfloor G,$$

which is an anticommutative analogue of the usual law of differentiation of a product. More generally one can consider the duals L_k' of the spaces L_k for which one can also introduce the cup product, and the contraction $G \rfloor F$ is given by $\sum e_A G_A \rfloor F$, where G_A are the scalar components of G and where for scalar valued elements $G_1 \in L_{k_1}'$ and $G_2 \in L_{k_2}'$ we put $(G_1 \wedge G_2) \rfloor = G_2 \rfloor G_1 \rfloor$. Note that like for the Fischer inner product for $F, G \in L_k$, $G^* \in L_k'$ and

$$(G, F) = [G^* \cdot F]_0 = \frac{1}{k!} [G^* \rfloor F]_0.$$

Forms can be represented as linear functions on multivector space (see also [6]). Indeed, note that any linear function $F(\underline{u}_1 \wedge \ldots \wedge \underline{u}_k)$ defined on the space $\mathbf{R}_m^k$ of k-vectors is interpretable as an element of L_k. Moreover, as the dimensions are the same, every element of L_k can uniquely be represented by such a linear function. One may of course also use the standard differential form notation, whereby one makes use of "differential form variables" like $d\underline{x} = \sum dx_j e_j$, where dx_j are anti-commuting (see e.g. [2], [3]). But in the context of this paper, the polynomial Frepresentation of tensors will be quite sufficient.

In tensor analysis we need more general types of symmetrization and alternation operators. Let A be a subset of $\{1, \ldots, k\}$; then we put

$$Sym_A\, F = \frac{1}{|A|!} \sum \pi(F), \quad Alt_A\, F = \frac{1}{|A|!} \sum sgn(\pi) \pi(F),$$

where the sum runs over all permutations π of the set A. Let $k_1, \ldots, k_n$ be numbers such that $k = \sum k_j$ and let $A_1 = \{1, \ldots, k_1\}$, $A_2 = \{k_1 + 1, \ldots, k_1 + k_2\}$, ... ,
$A_n = \{k - k_n + 1, \ldots, k\}$; then we can consider the operators

$$Sym_{k_1 \ldots k_n} = Sym_{A_1} \ldots Sym_{A_n}, \quad Alt_{k_1 \ldots k_n} = Alt_{A_1} \ldots Alt_{A_n}.$$

Moreover, let $S_{k_1 \ldots k_n}$ resp. $S_{k_1 \ldots k_n}'$ be the subspace of T_k resp. T_k' of elements of the form $Sym_{k_1 \ldots k_n} F$, $F \in T_k$ resp. T_k' and let $\mathcal{P}_{k_1 \ldots k_n}\{\underline{u}\}$ be the space of polynomials of the form $P(\underline{u}_1, \ldots, \underline{u}_n)$ that are homogeneous of degree k_1 in $\underline{u}_1$, k_2 in $\underline{u}_2$ etc. Then we can consider the canonical polynomial projection

$$P: F(\underline{u}_1, \ldots, \underline{u}_k) \to P(F)(\underline{u}_1, \ldots, \underline{u}_n) = F(\underline{u}_1, \ldots, \underline{u}_1, \ldots, \underline{u}_n, \ldots, \underline{u}_n)$$

of T_k on $\mathcal{P}_{k_1 \ldots k_n}\{\underline{u}\}$ which is an isomorphism on $S_{k_1 \ldots k_n}$. Moreover, for $G \in S'_{k_1 \ldots k_n}$ we have that

$$\begin{aligned} G \cdot F &= G(\partial_{\underline{u}_1}, \ldots, \partial_{\underline{u}_k}) F(\underline{u}_1, \ldots, \underline{u}_k) \\ &= \frac{1}{k_1!} \cdots \frac{1}{k_n!} P(G)(\partial_{\underline{u}_1}, \ldots, \partial_{\underline{u}_n}) P(F)(\underline{u}_1, \ldots, \underline{u}_n) \end{aligned}$$

and in particular for $F, G \in S_{k_1 \ldots k_n}$,

$$\begin{aligned} (G, F) &= (P(G), P(F)) \\ &= \frac{1}{k_1!} \cdots \frac{1}{k_n!} [\overline{P(G)}(\partial_{\underline{u}_1}, \ldots, \partial_{\underline{u}_n}) P(F)(\underline{u}_1, \ldots, \underline{u}_n)]_0. \end{aligned}$$

Similarly, let $L_{k_1 \ldots k_n}$ resp. $L'_{k_1 \ldots k_n}$ be the subspace of T_k resp. T'_k of elements of the form $Alt_{k_1 \ldots k_n} F$, $F \in T_k$ resp. T'_k; then $F \in L_{k_1 \ldots k_n}$ may be represented by a multilinear polynomial of the form $F(\underline{u}_1 \wedge \ldots \wedge \underline{u}_{k_1}, \ldots, \underline{u}_{k-k_n+1} \wedge \ldots \wedge \underline{u}_k)$ and the contraction operator $G\cdot$ corresponding to $G \in L'_{k_1 \ldots k_n}$ may be thought of as a differential operator of the form $G(\partial_{\underline{u}_1} \wedge \ldots \wedge \partial_{\underline{u}_{k_1}}, \ldots, \partial_{\underline{u}_{k-k_n+1}} \wedge \ldots \wedge \partial_{\underline{u}_k})$. Finally we consider so called Young symmetry operators and Young tables. For most of the results mentioned here we refer to [14]. Let again $k_1, \ldots, k_n$ be numbers such that $k = k_1 + \ldots + k_n$ and ordered in such a way that $k_1 \geq \ldots \geq k_n$. Then the tuple $K = (k_1, \ldots, k_n)$ constitutes a so called Young table. Let $A_1 = \{1, \ldots, k_1\}$, $A_2 = \{k_1 + 1, \ldots, k_1 + k_2\}$, ..., $A_n = \{k - k_n + 1, \ldots, k\}$ as before; then we consider for $n' = |A_1|$ the "dual sets" $A'_1, \ldots, A'_{n'}$, where A'_j consists of the elements on the j-th position in the sets $A_1, \ldots, A_n$ (which are ordered lexicographically). The basic Young symmetry operator $Y_{k_1 \ldots k_n}$ is then given by

$$Y_{k_1 \ldots k_n} = Sym_{A'_1} \ldots Sym_{A'_n} Alt_{A_1} \ldots Alt_{A_n}.$$

Of course one could start with arbitrary ordered sets $A_1, \ldots, A_n$ with cardinality $k_1, \ldots, k_n$ and form the corresponding Young symmetry operator. Such symmetry operator would still correspond to the same Young table and it is readily seen that any other Young symmetry operator corresponding to the same Young table has the form $Y = \pi Y_{k_1 \ldots k_n} \pi^{-1}$, $\pi \in Sym(k)$. This suggest to consider the sum

$$C_{k_1 \ldots k_n} = \sum_{\pi} \pi Y_{k_1 \ldots k_n} \pi^{-1}$$

which is a canonically defined operator only depending on the Young table K itself. These operators $C_K = C_{k_1 \ldots k_n}$ satisfy some interesting properties, the proof of which can be found in [14].

(1) the operators C_K are permutation invariant, i.e. $\pi C_K = C_K \pi$, $\pi \in Sym(k)$.
(2) if K and K' are different tables, then $C_K C_{K'} = C_{K'} C_K = 0$.
(3) $1 = \sum l_K C_K$, for suitable positive numbers l_K.
The Young operators $Y_{k_1 \dots k_n}$ themselves do not satisfy these nice properties and are not even self-adjoint w.r.t. the Fischer inner product. As the operators Sym and Alt are orthogonal projectors, the adjoint operator to $Y_{k_1 \dots k_n}$ is given by

$$Y'_{k_1 \dots k_n} = Alt_{A_1} \dots Alt_{A_n} Sym_{A'_1} \dots Sym_{A'_{n'}}.$$

Yet we have that
(4) the operators C_K are self-adjoint, i.e. $C'_K = C_K$.
Note that the operators $\chi_K = l_K C_K$ form a system of mutually orthogonal projection operators, one for each Young table, such that $1 = \sum \chi_K$. Moreover, each of the operators χ_K is permutation invariant and hence commutes with each Sym_A or Alt_A.
We finish this section by giving a Clifford analytic representation of $Y_{k_1 \dots k_n} F$, $F \in T_k$. First note that a tensor of the form $Alt_{A_1} \dots Alt_{A_n} F(\underline{u}_1, \dots, \underline{u}_k)$ may always be written into the form $G(\underline{u}_1 \wedge \dots \wedge \underline{u}_{k_1}, \underline{u}_{k_1+1} \wedge \dots \wedge \underline{u}_{k_1+k_2}, \dots, \underline{u}_{k-k_n+1} \wedge \dots \wedge \underline{u}_k)$. Moreover, the further symmetrization $Sym_{A'_1} \dots Sym_{A'_{n'}} G$ of this is determined by the polynomial projection obtained by equating $\underline{u}_1 = \underline{u}_{k_1+1} = \dots = \underline{u}_{k-k_n+1}, \dots, \underline{u}_2 = \underline{u}_{k_1+2} = \dots = \underline{u}_{k-k_n+2}$ etc. , i.e. by the polynomial $G(\underline{u}_1 \wedge \dots \wedge \underline{u}_{k_1}, \underline{u}_1 \wedge \dots \wedge \underline{u}_{k_2}, \dots, \underline{u}_1 \wedge \dots \wedge \underline{u}_{k_n})$. It is a very good costum to rewrite this polynomial as a polynomial of the form $G(\underline{u}_1, \underline{u}_1 \wedge \underline{u}_2, \dots, \underline{u}_1 \wedge \dots \wedge \underline{u}_m)$, of degree $(t_1, \dots, t_m)$ in the simplicial variables $\underline{u}_1, \underline{u}_1 \wedge \underline{u}_2, \dots, \underline{u}_1 \wedge \dots \wedge \underline{u}_m$, where t_j is the occurence of j in the Young table $(k_1, \dots, k_n)$. Unfortunately, this representation depends on the operator $Y_{k_1 \dots k_n}$ and cannot be used for $C_K F$.

3. Monogenic decomposition of tensors

Similar to spherical monogenics and monogenic or "primitive" forms (see [3]) one can introduce in general so called "monogenic tensors" as follows.

DEFINITION 3.1. *A tensor F is called monogenic k-tensor if its polynomial representation $F(\underline{u}_1, \dots, \underline{u}_k)$ is monogenic in each of the variables $\underline{u}_j$, that is if*
$\partial_{\underline{u}_j} F(\underline{u}_1, \dots, \underline{u}_k) = 0$, $j = 1, \dots, k$. *By MT_k we denote the space of monogenic k-tensors.*

The main motivation for monogenic tensors is the so called Fischer decomposition, also called the monogenic decomposition of tensors.

THEOREM 3.1. *Every $F \in T_k$ admits a canonical decomposition of the form $F = M(F) + M^{\perp}(F)$, where $M(F)$ is spherical monogenic and $M^{\perp}(F)$ has the form*

$$M^{\perp}(F)(\underline{u}_1, \ldots, \underline{u}_k) = \underline{u}_1 G_1(\underline{u}_2, \ldots, \underline{u}_k) + \ldots + \underline{u}_k G_k(\underline{u}_1, \ldots, \underline{u}_{k-1}).$$

Moreover, the space of spherical monogenic k-tensors is the orthogonal complement of the space of tensors of the above form, hence denoted by $MT_k^{\perp}$.

Proof. Let $G = \underline{u}_j G_j(\underline{u}_1, \ldots, \underline{u}_{j-1}, \underline{u}_{j+1}, \ldots, \underline{u}_k)$; then for $F \in T_k$,

$$(G, F) = -[\overline{G_j}(\partial_{\underline{u}_1}, \ldots, \partial_{\underline{u}_{j-1}}, \partial_{\underline{u}_{j+1}}, \ldots, \partial_{\underline{u}_k})\partial_{\underline{u}_j} F]_0 = -(G_j, \partial_{\underline{u}_j} F).$$

Hence the condition $(G, F) = 0$ for all $G_j \in T_{k-1}$ clearly means that $\partial_{\underline{u}_j} F = 0$. This is true for all j, if and only if $F \in MT_k$. □

The orthogonal projection operator $M : T_k \to MT_k$ is called the monogenic projection operator. It is clear that the monogenic projection is uniquely determined. The question is whether also the polynomials G_j are uniquely determined. We have the following

LEMMA 3.1. *Let $m > 2$ and G_i and G_j be multilinear of degree $(k-1)$ in their variables and let for $i \neq j$*

$$\underline{u}_i G_i(\underline{u}_1, \ldots, \underline{u}_{i-1}, \underline{u}_{i+1}, \ldots, \underline{u}_k) = \underline{u}_j G_j(\underline{u}_1, \ldots, \underline{u}_{j-1}, \underline{u}_{j+1}, \ldots, \underline{u}_k),$$

then $G_i = G_j = 0$.

Proof. It suffices to prove this for $k = 2$. Indeed, one can derive the above equation w.r.t. all variables $u_{l,n}$ where n is different from i and j. If the lemma holds for $k = 2$ it then follows that each of these derivatives of G_i and G_j vanish. But they determine G_i and G_j so that also $G_i = G_j = 0$. Now let $\underline{u}_1 G_1(\underline{u}_2) = \underline{u}_2 G_2(\underline{u}_1)$, then symmetrization $\underline{u} = \underline{u}_1 = \underline{u}_2$ leads to $G_1(\underline{u}) = G_2(\underline{u}) = G(\underline{u})$. Next, $\underline{u}_1 G(\underline{u}_2) = \underline{u}_2 G(\underline{u}_1)$ means that $e_i G_j = e_j G_i$, $G_j = G(e_j)$ or, after multiplication with e_{ij}, $e_j G_j = -e_i G_i$. For $m > 2$ this only has the nullsolution. For $m = 2$ one could have that $G_1 = 1$, $G_2 = -e_{12}$ so that $G(\underline{u}_1) = u_{11} - e_{12} u_{12} = -e_1 \underline{u}_1$. Note that $\underline{u}_1 e_1 \underline{u}_2 = \underline{u}_2 e_1 \underline{u}_1$! □

Hence in case $m > 2$ we can apply the Fischer decomposition recursively, leading to the following

THEOREM 3.2. *For $m > 2$, every $F \in T_k$ has a unique decomposition of the form*

$$F = \sum \underline{u}_{a_1} \ldots \underline{u}_{a_l} M_{a_1 \ldots a_l}(F),$$

where the sum is taken over all ordered subsets $(a_1 \ldots a_l)$ of the set $\{1, \ldots, k\}$ and $M_{a_1 \ldots a_l}(F)$ is a monogenic $(k-l)$-tensor in the remaining variables.

Although unique, the above Fischer decomposition is far from being orthogonal. Already tensors of the form $\underline{u}_1 F(\underline{u}_2)$ need not be orthogonal to tensors of the form $\underline{u}_2 G(\underline{u}_1)$. For example, if $M_1(\underline{u})$ and $M_2(\underline{u})$ are linear monogenic, then

$$(\underline{u}_1 M_1(\underline{u}_2), \underline{u}_2 M_2(\underline{u}_1) = (M_1(\underline{u}_2), M_2(\underline{u}_2)),$$

which vanishes only seldom. Hence, next question is whether certain parts of the decomposition are orthogonal to others. And to verify this we have to compute expressions of the form

$$\partial_{\underline{u}_j} \underline{u}_{a_1} \dots \underline{u}_{a_l} M_{a_1 \dots a_l}(\underline{u}_{b_1}, \dots, \underline{u}_{b_{k-l}}),$$

where $\{1, \dots, k\} = \{a_1, \dots, a_l, b_1, \dots, b_{k-l}\}$. If j is one of the elements $\{a_1, \dots, a_l\}$ we use the relations $\underline{u}_k \underline{u}_j = -\underline{u}_j \underline{u}_k - \langle \underline{u}_k, \underline{u}_j \rangle$ together with $\partial_{\underline{u}_j} \underline{u}_j = -m$ and $\partial_{\underline{u}_j} \langle \underline{u}_k, \underline{u}_j \rangle = \underline{u}_k$ at many places to see that the above expression has the form

$$U_{a_1 \dots \not{j} \dots a_l} M_{a_1 \dots a_l}(\underline{u}_{b_1}, \dots, \underline{u}_{b_{k-l}})$$

where $U_{a_1 \dots \not{j} \dots a_l}$ is a sum of products of $\underline{u}_{a_1} \dots \underline{u}_{a_l}$, with $\underline{u}_j$ not included. Secondly, if j belongs to the set $\{b_1, \dots, b_{k-l}\}$, then we make use of the relations $\partial_{\underline{u}_j} \underline{u}_k = -\underline{u}_k \partial_{\underline{u}_j} - \langle \underline{u}_k, \partial_{\underline{u}_j} \rangle$ together with $\partial_{\underline{u}_j} M_{a_1 \dots a_l} = 0$ and $\langle \underline{u}_k, \partial_{\underline{u}_j} \rangle M_{a_1 \dots a_l}(\underline{u}_{b_1}, \dots, \underline{u}_{b_{k-l}})$
$= M_{a_1 \dots a_l}(\underline{u}_{b_1}, \dots, \underline{u}_{b_{k-l}})|_{\underline{u}_j \to \underline{u}_k}$ to see that the above expression is a sum of terms of the form

$$\underline{u}_{a'_1} \dots \underline{u}_{a'_{l-1}} M_{a'_1 \dots a'_{l-1}}(\underline{u}_{b'_1}, \dots, \underline{u}_{b'_{k-l+1}})$$

where $\{a'_1 \dots a'_{l-1}\}$ is a subset of $\{a_1 \dots a_l\}$. This leads to

THEOREM 3.3. *Let $m > 2$; then every k-tensor $F \in T_k$ admits a unique orthogonal decomposition of the form*

$$F = \sum_{l=0}^{k} M_l(F),$$

where $M_0(F) = M(F)$ is the monogenic projection of F and where $M_l(F) = \sum \underline{u}_{a_1} \dots \underline{u}_{a_l} M_{a_1 \dots a_l}(F)$ is called the l-monogenic projection of F.

Proof. Let $l' > l$; then we have that the Fischer inner product

$$(\underline{u}_{a'_1} \dots \underline{u}_{a'_{l'}} M_{a'_1 \dots a'_{l'}}(F), \underline{u}_{a_1} \dots \underline{u}_{a_l} M_{a_1 \dots a_l}(F))$$

vanishes for $l = 0$ while for $l > 0$ it equals

$$-(\underline{u}_{a'_2} \dots \underline{u}_{a'_{l'}} M_{a'_1 \dots a'_{l'}}(F), \partial_{\underline{u}_{a'_1}} \underline{u}_{a_1} \dots \underline{u}_{a_l} M_{a_1 \dots a_l}(F)).$$

But in view of the above computations, $\partial_{\underline{u}_{a'_1}} \underline{u}_{a_1} \cdots \underline{u}_{a_l} M_{a_1 \ldots a_l}(F)$ is a sum of terms of the form

$$\underline{u}_{a''_1} \cdots \underline{u}_{a''_{l-1}} M_{a''_1 \ldots a''_{l-1}}.$$

Hence, by induction on l, the above inner product vanishes. □

Whereas the space $M(T_k)$ was also the space of all tensors F satisfying the first order equations $\partial_{\underline{u}_j} F = 0$, we have more in general

LEMMA 3.2. *The space of $F \in T_k$ satisfying the system $\partial_{\underline{u}_{a_1}} \cdots \partial_{\underline{u}_{a_{l+1}}} F = 0$, where each $(a_1, \ldots, a_{l+1})$ is an ordered subset of $\{1, \ldots, k\}$, is the direct sum $M(T_k) + \ldots + M_l(T_k)$.*

Proof. Both spaces are the orthogonal complement of the space of tensors of the form $\sum \underline{u}_{a_1} \cdots \underline{u}_{a_{l+1}} G_{a_1, \ldots, a_{l+1}}(\underline{u}_{b_1}, \ldots, \underline{u}_{a_{k-l-1}})$. □

Important is also the interaction between monogenic decomposition and symmetrization and alternation operators.

LEMMA 3.3. *Let $\pi \in Sym(k)$, then π commutes with the orthogonal monogenic decomposition, that is, $\pi(M_l(F)) = M_l(\pi(F))$, $F \in T_k$.*

Proof. It is immediately clear that the spaces of tensors of the form $\sum \underline{u}_{a_1} \cdots \underline{u}_{a_l} M_{a_1 \ldots a_l}(F)$ are permutation invariant, while the operators M_k are the orthogonal projectors onto these spaces. □

COROLLARY 3.1. *The operators Sym_A and Alt_A commute with the monogenic projectors M_l. The same is true for the Young operators $l_K C_K$. Thus every k-tensor $F \in T_k$ has a unique orthogonal decomposition of the form*

$$F = \sum_{l,K} l_K C_K M_l(F),$$

which refines both the monogenic and the Young decompositions.

It is this decomposition which plays a fundamental role in the decomposition of tensors into irreducible pieces under $SO(m)$. We hence call it the Fischer decomposition. Next let us apply this to the decomposition of polynomials. First, let $P(\underline{u}_1, \ldots, \underline{u}_n) \in \mathcal{P}_{k_1 \ldots k_n}\{\underline{u}\}$ be a polynomial in vector variables $\underline{u}_1, \ldots, \underline{u}_n$ and homogeneous of degrees k_1 up to k_n. If $k = k_1 + \ldots k_n$, this space is the polynomial projection of $Sym_{k_1 \ldots k_n} T_k$. From theorems 2 and 3 we readily obtain (see also [11])

THEOREM 3.4. *Every $P \in \mathcal{P}_{k_1 \ldots k_n}\{\underline{u}\}$ admits a unique decomposition of the form*

$$P(\underline{u}_1, \ldots, \underline{u}_n) = \sum \prod_{i<j} \langle \underline{u}_i, \underline{u}_j \rangle^{s_{ij}} \prod_i \underline{u}_i^{s_i} M_{s_{ij}, s_i}(P),$$

where $M_{s_{ij},s_i}(P)$ are homogeneous polynomials satisfying the monogenicity conditions $\partial_{\underline{u}_j} f(\underline{u}_1, \ldots, \underline{u}_n) = 0$, $j = 1, \ldots, n$.

Proof. Let $F(\underline{v}_1, \ldots, \underline{v}_k) \in Sym_{k_1 \ldots k_n} T_k$ be the symmetrical tensor whose polynomial projection is P; then we can apply theorem 2 to F and apply the polynomial projection $\underline{u}_1 = \underline{v}_1 = \ldots = \underline{v}_{k_1}, \ldots, \underline{u}_n = \underline{v}_{k-k_n+1} = \ldots = \underline{v}_k$ on the result. □

THEOREM 3.5. *Every $P \in \mathcal{P}_{k_1 \ldots k_n}\{\underline{u}\}$ admits a unique orthogonal decomposition of the form $P = \sum M_l(P)$, where $M_l(P)$ consists of terms of the form $\underline{u}_{a_1} \ldots \underline{u}_{a_k} f(\underline{u}_1, \ldots, \underline{u}_n)$, $a_j \in \{1, \ldots, n\}$ with $\partial_{\underline{u}_j} f(\underline{u}_1, \ldots, \underline{u}_n) = 0$, $j = 1, \ldots, n$.*

Proof. Let again F be the symmetric tensor with P as polynomial projection, then due to lemma 3, $Sym_{k_1 \ldots k_n} M_l(F) = M_l(F)$, which is a symmetric tensor with polynomial projection denoted by $P(M_l(F)) = M_l(P)$ belonging to $\mathcal{P}_{k_1 \ldots k_n}\{\underline{u}\}$. $M_l(P)$ is clearly determined by the above stated properties and the polynomial projection preserves orthogonality. □

Interesting is of course the fact that

LEMMA 3.4. *The space of polynomial solutions $P \in \mathcal{P}_{k_1 \ldots k_n}\{\underline{u}\}$ of the system of equations $\partial_{\underline{u}_{a_1}} \ldots \partial_{\underline{u}_{a_{l+1}}} P = 0$, $a_j \in \{1, \ldots, n\}$ is the direct sum $M(E) + M_1(E) + \ldots + M_l(E)$, $E = \mathcal{P}_{k_1 \ldots k_n}\{\underline{u}\}$.*

This lemma leads to the analogue of the notion of k-monogenicity for functions of several vector variables. Next, let us consider forms. Let $F = Alt_{k_1 \ldots k_n} F$, then we know that F can be represented as a multilinear function of the form $F(\underline{u}_1 \wedge \ldots \wedge \underline{u}_{k_1}, \ldots, \underline{u}_{k-k_n+1} \wedge \ldots \wedge \underline{u}_k)$. On the other hand, due to lemma 3, we know that $M_l(F) = Alt_{k_1 \ldots k_n} M_l(F)$ is again a differential form of the same type, where $M_l(F)$ is as in theorem 3.
Finally note that if P is a Young-type polynomial of the form $P(\underline{u}_1, \underline{u}_1 \wedge \underline{u}_2, \ldots, \underline{u}_1 \wedge \ldots \wedge \underline{u}_m)$ of degree t_1 in $\underline{u}_1, \ldots, t_m$ in $\underline{u}_1 \wedge \ldots \wedge \underline{u}_m$ coming from the polynomial projection of a Young-type tensor F, then from lemma 3 we again know that $M_l(F) = Y_{k_1 \ldots k_n} M_l(F)$, so that $M_l(F)$ projects to a polynomial $M_l(P)(\underline{u}_1, \underline{u}_1 \wedge \underline{u}_2, \ldots, \underline{u}_1 \wedge \ldots \wedge \underline{u}_m)$ of the same form, where at the same time $M_l(P)$ still has the form stated in theorem 5. $M_l(P)$ is hence also characterized by

(i) $M_l(P)$ has the form $M_l(P)(\underline{u}_1, \underline{u}_1 \wedge \underline{u}_2, \ldots, \underline{u}_1 \wedge \ldots \wedge \underline{u}_m)$
(ii) $\partial_{\underline{u}_{a_1}} \ldots \partial_{\underline{u}_{a_{l+1}}} M_l(P)(\underline{u}_1, \underline{u}_1 \wedge \underline{u}_2, \ldots, \underline{u}_1 \wedge \ldots \wedge \underline{u}_m) = 0$
(iii) $M_l(P)$ is Fischer orthogonal to every $M_{l-1}(P)$

The study of functions satisfying conditions (i) and (ii) forms a very interesting topic in analysis. Of particular interest are the homogeneous solutions because they lead to irreducible representations of $Spin(m)$. So far we only have obtained a description of the monogenic decomposition of polynomials, not the complete Fischer decomposition, which also involves the operators C_K. This however seems to require the translation of polynomials into tensors followed by the Fischer decomposition of tensors.

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A CLIFFORD DYADIC SUPERFIELD FROM BILATERAL INTERACTIONS OF GEOMETRIC MULTISPIN DIRAC THEORY

WILLIAM M. PEZZAGLIA JR.
Department of Physics, Santa Clara University
Santa Clara, CA 95053, U.S.A., wpezzaglia@scuacc.scu.edu

and

ALFRED W. DIFFER
Department of Physics, American River College
Sacramento, CA 95841, U.S.A.

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Abstract. Multivector quantum mechanics utilizes wavefunctions which are Clifford aggregates (e.g. sum of scalar, vector, bivector). This is equivalent to multispinors constructed of Dirac matrices, with the representation independent form of the generators geometrically interpreted as the basis vectors of spacetime. Multiple generations of particles appear as left ideals of the algebra, coupled only by now-allowed right-side applied (dextral) operations. A generalized bilateral (two-sided operation) coupling is proposed which includes the above mentioned dextrad field, and the spin-gauge interaction as particular cases. This leads to a new principle of *poly-dimensional covariance*, in which physical laws are invariant under the reshuffling of coordinate geometry. Such a multigeometric superfield equation is proposed, which is sourced by a bilateral current. In order to express the superfield in representation and coordinate free form, we introduce Eddington E-F *double-frame* numbers. Symmetric tensors can now be represented as 4D "dyads", which actually are elements of a global 8D Clifford algebra. As a restricted example, the dyadic field created by the Greider-Ross multivector current (of a Dirac electron) describes both electromagnetic and Morris-Greider gravitational interactions.

Key words: spin-gauge, multivector, clifford, dyadic

1. Introduction

Multivector physics is a grand scheme in which we attempt to describe all basic physical structure and phenomena by a single geometrically interpretable Algebra. A conservative approach recognizes the Dirac algebra as belonging to a Clifford manifold having both spin and coordinate aspects. The *spin gauge theory* approach to grand unification makes use of a spin Clifford algebra which necessarily commutes with coordinate geometry. We propose a direct projection from this abstract space into concrete coordinate geometric algebra. Ultimately we eliminate spin space entirely by using Clifford aggregates of coordinate geometry to replace 'spinors'. Spin gauge theory, an artifact of spin geometry therefore vanishes. However, we gain in having multiple generations of particles appear which are coupled by new dextrad (right-sided multiplication) gauge transformations.

To accommodate all the known couplings we must somehow recover the spin-gauge formalism. This requires transformations which literally reshuffle the geometry, i.e. the basis vectors for one observer might be the trivectors for another observer. This leads us to propose that the general physical laws are invariant under these transformations, a new principle called *poly-dimensional covariance.* We postulate a single multigeometric superfield equation, which will require two commuting coordinate Clifford algebras, analogous to Eddington's E-F 'double frame" numbers [7]. This *dyadic* Clifford algebra can be reinterpreted as a single 8D multigeometric space. Multivector Dirac theory expressed in this full algebra potentially has enough degrees of freedom to represent all the fermions of the standard model.

2. Geometric Algebras and Multi-Spinors

We present at first the standard view that abstract entities (e.g. spinors) exist outside of the realm of concrete coordinate geometry. Dirac algebra belongs to a Clifford manifold which has both spin and coordinate features. We propose a direct projection between spin space and coordinate geometry in eq. (2) below.

2.1. Spacetime and the Majorana Algebra

Factoring the second-order *meta-harmonic* Klein-Gordon equation to the first order *meta-monogenic* Dirac form requires four mutually anticommuting algebraic elements $\{\gamma^1, \gamma^2, \gamma^3, \gamma^4\}$,

$$(\Box^2 - m^2)\Phi(x) = (\Box - m)(\Box + m)\Phi(x), \qquad (1a)$$

$$\Psi(x) = (\Box + m)\Phi(x) = (\gamma^\mu \nabla_\mu + m)\Phi(x), \qquad (1b)$$

$$(\gamma^\mu \nabla_\mu - m)\Psi(x) = 0, \qquad (1c)$$

where $\nabla_\mu = \partial_\mu$ in flat spacetime. Requiring the formulation to be Lorentz covariant imposes the defining condition of a Clifford algebra, $\frac{1}{2}\{\gamma_\mu, \gamma_\nu\} = g_{\mu\nu} = \mathbf{e}_\mu \bullet \mathbf{e}_\nu$, where $\mathbf{e}_\mu$ are the coordinate basis vectors. If the use of the abstract i is excluded, the above factorization of eq. (1a) only works in the metric signature of $(+++-)$. The lowest order matrix representation of the $\{\gamma_\mu\}$ is $\mathbb{R}(4)$, i.e. 4 by 4 real (i.e. no commuting i) matrices, commonly known as the (16 dimensional) *Majorana algebra.* The explicit matrix form of the algebra generator $\gamma_\mu{}^{\alpha\beta}$ can be determined from the Riemann space metric $g_{\mu\nu}$ up to a similarity *spin transformation*: $\gamma'_\mu = S\gamma_\mu S^{-1}$.

2.2. Spin Space

The solution of the Dirac eq. (1c) is usually taken to be a four component column *bispinor* Ψ^α, belonging to the left linear space for which the endomorphism algebra is the Majorana matrices. This *spin space* is transcendental, i.e. the postulates of quantum mechanics ordain that some attributes (e.g. quantum phase) of the wavefunction cannot be directly observed. The *principle of representation invariance* states that tangible results should be invariant under a spin transformation: $\Psi^{\alpha\prime} = S^\alpha{}_\beta \Psi^\beta$. It should therefore be possible to express the theory in a form which eliminates any reference to a particular representation without sacrificing any "physics". To this end we introduce the *spinor basis* ξ_α as carriers for the representation. A spin transformation can now be interpreted as a passive change in spinor basis, which leaves the *spin vector* $\Psi = \Psi^\alpha \xi_\alpha$ unchanged.

The *dual* spinor basis $\bar{\xi}_\alpha$ is defined such that $\bar{\xi}_\alpha \xi_\beta = \eta_{\alpha\beta}$, where the spin metric $\eta_{\alpha\beta}$ has the diagonal signature $(++--)$ in the standard matrix representation. We propose to interpret the representation independent form,

$$\mathbf{e}_\mu = \xi_\alpha \gamma_\mu{}^{\alpha\beta} \bar{\xi}_\beta, \tag{2a}$$

as the "observable" basis vector of coordinate space. Mathematically this can be viewed as a map or projection from the Clifford manifold to the coordinate manifold. Hence we get a Dirac equation completely independent of spin basis or matrix representation: $(\Box - m)\Psi = 0$, where $\Psi = \Psi^\alpha \xi_\alpha$ and $\Box = \mathbf{e}^\mu \partial_\mu$ is now the coordinate gradient.

2.3. Geometric interpretations of Gauge Algebras

There is a long standing tradition which views i as only "existing" in spin space, as the internal $\mathbf{U}(1)$ generator of unobservable quantum phase. Factors of i are included as needed to make operators Hermitian (e.g. γ_4) so that expectation values will never contain a non-observable "imaginary" number. The usual Dirac matrices are the complexified Majorana algebra: $\mathbb{C}(4) = \mathbb{C} \otimes \mathbb{R}(4)$. This can be geometrically reinterpreted as a 5D geometric (anti de-Sitter) space, where the unit pseudoscalar (5-volume) plays the role of the $i = \gamma^1\gamma^2\gamma^3\gamma^4\gamma^5$, only if the fifth basis vector has positive signature. The obvious question would be the physical interpretation of the new fifth dimension, and the identification of its associated coordinate variable and conjugate momenta (mass?). We will address this question briefly below.

To represent an isospin doublet of bispinors (e.g. u & d quark) requires a commuting isospin Pauli $\{\sigma_j\}$ algebra. The wavefunction can be expressed as a matrix set of components $\Psi^{\alpha\kappa}$ contracted on a product basis $\xi_\alpha \lambda_\kappa$. There are only two elements to the isospinor basis $\{\lambda_1, \lambda_2\}$ which necessarily commute with the spinor basis ξ_α. The direct product of a commuting

Majorana "spin" algebra and a Pauli "isospin" algebra can be reinterpreted as a 7D geometric algebra with metric signature $(+++----)$. The column spinor for which the endomorphsim algebra is $\mathbb{C}(8) = \mathbb{C}(2) \otimes \mathbb{R}(4)$ would now have 8 components.

To represent two observers in real spacetime requires a pair of coordinates, each with their own Clifford algebras [5]. The direct product of these two commuting algebras can be geometrically reinterpreted as a 8D space with a *mother algebra* [14] $\mathbb{R}(16) = \mathbb{R}(4) \otimes \mathbb{R}(4)$. This encompases all the above algebras, where the 'second frame' $\mathbb{R}(4)$ algebra commutes with that of the 'first frame'. Hence the 'second' algebra is the 'internal' gauge algebra for the 'first' frame observer and visa versa.

3. Spin Covariant Dirac Theory

The *special theory of relativity* requires the Dirac equation to have the same form under Lorentz transformations: $dx^\mu = a^\mu{}_\nu dx^\nu$. It is usually argued [10] that the generators γ^μ are invariant scalars, i.e. the same for all observers, at the cost of forcing the bispinor wavefunction to obey a compensating spin transformation: $\psi^{\alpha\prime} = S^\alpha{}_\beta \psi^\beta$, where $S^{-1}\gamma^\mu S = a^\mu{}_\nu \gamma^\nu$.

3.1. Coordinate Covariant Dirac Theory

The *general principle of covariance* will require the spin transformation to be local (different at each point in spacetime). This introduces a *spin connection* $\boldsymbol{\Omega}_\mu$ to the derivative ∇_μ of the Dirac eq. (1c),

$$\nabla_\mu = \partial_\mu + \boldsymbol{\Omega}_\mu, \tag{3a}$$

$$\partial_\mu \xi_\alpha = \boldsymbol{\Omega}_\mu \xi_\alpha = \xi_\beta \Omega_\mu{}^\beta{}_\alpha, \tag{3b}$$

$$\boldsymbol{\Omega}_\mu = \Omega^{(j)}_\mu \, \mathbf{E}_{(j)} = \Omega^{(j)}_\mu \, \boldsymbol{\Gamma}_{(j)}^{\beta\alpha} \xi_\beta \bar{\xi}_\alpha = \Omega_\mu{}^{\beta\alpha} \, \xi_\beta \bar{\xi}_\alpha, \tag{3c}$$

$$\boldsymbol{\Omega}'_\mu = S \, \boldsymbol{\Omega}_\mu S^{-1} + S \partial_\mu S^{-1}. \tag{3d}$$

One of the 16 basis elements $\mathbf{E}_{(j)}$ of the geometric Clifford algebra is given by the generalization of eq. (2a),

$$\mathbf{E}_{(j)} = \Gamma_{(j)}{}^{\alpha\beta} \, \xi_\alpha \bar{\xi}_\beta, \tag{2b}$$

where $\Gamma_{(j)}$ is the corresponding basis element of the Dirac matrix algebra. Under the general coordinate transformations required by the *equivalence principle*, one must replace $\gamma^\mu \to \gamma^a h_a{}^\mu(x)$ where the *tetrad* (vierbein) field $h_a{}^\mu(x)$ transforms as a vector. This is equivalent to introducing position dependent $\gamma^\mu(x)$ which transform like basis vectors.

For this reason and others, we adopt the "nontraditional" view that both $\mathbf{e}^\mu$ and γ^μ of eq. (2a) transform as vectors, while ξ_β and $\bar{\xi}_\alpha$ transform as coordinate scalars [9]. With this definition of constant spin basis, the spin connection is everywhere zero, hence the generally covariant Dirac equation is simply eq. (1c) with Dirac matrices which are a function of position. However, when the coordinate space is curved, one cannot have the spin connection vanish everywhere. The geometric definition of eq. (2a) forces the following relations:

$$\partial_\nu \gamma_\mu = C_{\nu\mu}{}^{\omega}\, \gamma_\omega - \mathbf{\Omega}^{(j)}_\nu [\mathbf{\Gamma}_{(j)}, \gamma_\mu], \tag{4a}$$

$$[\mathbf{K}_{\omega\sigma}, \mathbf{e}_\mu] = R_{\omega\sigma\mu}{}^{\nu} \mathbf{e}_\nu, \tag{4b}$$

$$\mathbf{K}_{\omega\sigma} = K^{(j)}_{\omega\sigma}\, \mathbf{E}_{(j)} = [\nabla_\omega, \nabla_\sigma], \tag{4c}$$

The coordinate connection coefficient $C_{\nu\mu}{}^{\omega}$ (Christoffel symbol) is directly related to the spin connection by eq. (4a). Restricting our discussion to real spacetime algebra (no commuting i), the *spin curvature* $\mathbf{K}_{\sigma\omega}$ is forced by eq. (4b) to be a bivector. Clearly it must be nonzero if the coordinate space is curved, i.e. described by the Riemann curvature tensor: $R_{\omega\sigma\mu}{}^{\nu} \mathbf{e}_\nu = [\partial_\omega, \partial_\sigma]\mathbf{e}_\mu$. It follows from eq. (4c) that the spin connection (which appears in the spin covariant derivative ∇_σ) must have a nontrivial bivector part, commonly called the *Fock-Ivanenko coefficient* [9].

3.2. Spin Gauge Theory

The principle of *local matrix representation invariance* or equivalently a principle of *spin basis covariance* is invoked to induce via minimal coupling a non-trivial spin connection [4]. This is a gauge theory where the generators $\mathbf{\Gamma}_{(j)}$ of the general spin transformation are usually restricted to be Dirac bar-negative in order to preserve the spin norm $\bar{\Psi}\Psi$ (i.e. the spin metric $\bar{\xi}_\alpha \xi_\beta = \eta_{\alpha\beta}$ is invariant). The standard (5D) Dirac algebra which has the bar negative pseudoscalar i, would contain the 16 element group structure $U(2,2)$. Electromagnetism is associated with i, which by itself would force the space curvature to be zero. It is tempting to interpret the 10 bivectors (of 5D) with group structure $SO(4,1)$ as the gauge fields which cause gravitational curvature through eq. (4b).

Grand unification is approached by Chisholm and Farwell [1] by resorting to higher dimensions (e.g. 11D) to introduce more fields. They only consider spin transformations of the form: $\gamma^\mu = \gamma^a h_a{}^\mu(x)$, generated by bivectors or the pseudoscalar i. They avoid those bivectors which would rotate spacetime into a higher dimension (e.g. $\gamma^5\gamma^1$). The remaining bivectors which operate on spacetime form the 6 element Lorentz group $SL(2,\mathbb{C})$, potentially insufficient to accommodate a full description of gravitation.

3.3. Local Automorphism Invariance

Alternatively, the entire automorphism group $U(2,2)$ of the Dirac algebras is allowed by Crawford [2]. Previously, the non-bivector generators were excluded by equations (4a) & (4b). These constraints are relaxed because Crawford does not require the geometric interpretation of eq. (2ab). This allows him to consider generalized spin transformations of the form: $\mathbf{\Gamma}^{(j)} = \mathbf{\Gamma}^a \Delta_a{}^{(j)}(x)$, where $\mathbf{\Gamma}^{(j)}$ is a basis element of the full "spin" (Dirac) Clifford algebra. The *drehbein* fields $\mathbf{\Delta}_a{}^{(j)}(\mathrm{x})$ ("spin-legs") reshuffle multivector rank in the Clifford spin manifold (e.g. vector $\leftrightarrow$ bivector) without doing the same to the "observable" coordinate geometry.

A Lagrangian formulation can show that the field equation is,

$$\mathbf{K}_{\mu\nu}{}^{;\mu} + [\mathbf{\Omega}^{\mu}, \mathbf{K}_{\mu\nu}] = \mathbf{j}_{\mu} = j_{\mu}^{(i)}\, \mathbf{\Gamma}_{(i)}, \tag{5a}$$

$$j_{\mu}^{(i)} = \frac{1}{2}\bar{\Psi}\{\mathbf{\Gamma}^{(i)}, \gamma_{\mu}\}\Psi = \frac{1}{8}Tr(\Psi\bar{\Psi}\{\mathbf{\Gamma}^{(i)}, \gamma_{\mu}\}), \tag{5b}$$

The current $\mathbf{j}_{\mu}$ is similar to the spin gauge connection $\mathbf{\Omega}_{\mu}$ in being a coordinate vector while also a Clifford aggregate over the spin algebra $\mathbf{\Gamma}^{(i)}$. The spin curvature $\mathbf{K}$ can be geometrically interpreted as a dyad of a *coordinate* geometric bivector and a *spin algebra* Clifford aggregate.

$$\mathbf{K} = K^{\mu\nu}_{(i)}\ \mathbf{e}_{\mu} \wedge \mathbf{e}_{\mu}\ \mathbf{\Gamma}^{(i)}. \tag{5c}$$

Elements of the coordinate geometry $\mathbf{E}_{(i)}$ commute with the spin algebra $\mathbf{\Gamma}^{(i)}$ because Crawford does not postulate the geometric connection of eq. (2ab). Note that the bivector part of the spin curvature is no longer constrained by eq. (4b) to be related to the space curvature.

4. Multivector Gauge Theory

The basic difference from standard theory is the replacement of column spinors by algebraic wavefunctions, i.e. Clifford aggregates of Dirac matrices [6]. Most authors only consider restricted combinations called *minimal ideals*, which have the same degrees of freedom as a single column spinor. In our approach, the form of the multivector wavefunction is unrestricted, having the same number of degrees of freedom as the elements of the Clifford group. The complete solution can be interpreted as a geometric multispinor: $\Psi = \Psi^{(i)}\mathbf{E}_{(i)} = \Psi^{\alpha\beta}\xi_{\alpha}\lambda_{\beta}$. Here the ξ_{α} is no longer a basis spinor, but an element of a left ideal, hence eq. (3b) is no longer valid. The isospin element is part of the same algebra: $\lambda_{\beta} = \bar{\xi}_{\beta}$, which does not commute with ξ_{α} whereas it did in standard formulation. In 4D spacetime algebra (no commuting i) the geometric multispinor has been shown [11] to be an isospin doublet of

Dirac bispinors, where the role of i is played by right-side applied (i.e. *dextrad multiplication*) time basis vector $\mathbf{e}_4$. In 5D (standard Dirac algebra) one has enough degrees of freedom to represent four quarks (i.e. u,d,s,c), where the (u,d) and (s,c) isospin doublets are uncoupled.

4.1. Dextral Gauge Theory

The generally covariant multivector Dirac equation $(\mathbf{e}^\mu\partial_\mu - m)\Psi^{(i)}\mathbf{E}_{(i)} = 0$, where $\mathbf{e}_\mu(x)$ are the local coordinate basis vectors, is manifestly matrix representation independent. We have in fact completely eliminated spin space, specifically spin basis ξ_α and spin algebra $\gamma_\mu{}^{\alpha\beta}$ in favor of there being only the geometrically interpretable coordinate Clifford algebra $\mathbf{E}_{(i)}$. Hence, spin gauge theory, an artifact of spin space, is now inaccessible!

The multiple particle generations in the multivector wavefunction can be coupled by now-allowed right-side applied *dextral gauge transformation* [3]. The new gauge fields enter as a *dextrad connection*: $\mathbf{D}_\mu = D_\mu{}^{(i)}\mathbf{E}_{(i)}$,

$$\nabla_\mu(\Psi) = \partial_\mu\Psi + \Psi\mathbf{D}_\mu, \tag{6a}$$

coupling to the multivector parts of Greider's current [6],

$$j_\mu^{(i)} = Tr(\mathbf{E}^{(i)}\bar{\Psi}\mathbf{e}_\mu\Psi) = Tr(\Psi\mathbf{E}^{(i)}\bar{\Psi}\mathbf{e}_\mu). \tag{6b}$$

A Lagrangian formulation [11] will require the geometric generators $\mathbf{E}_{(i)}$ of the dextrad connection$\mathbf{D}_\mu$ to be bar negative. In 4D spacetime, the subset which is also unitary generates the electroweak group: $U(1)\otimes SU(2)$, where isospin rotations are generated by spacelike bivectors and the role of i played by right-sided (dextrad) multiplication of the time basis element $\mathbf{e}_4$.

4.2. Poly-Dimensional Covariance

The spin gauge formalism can be recovered by proposing that the automorphism transformations operate on the very real, concretely observable spacetime coordinate Clifford algebra: $\mathbf{E}'_{(i)} = \mathbf{E}_{(j)}\Delta^{(j)}{}_{(i)}(x)$. The *geobein* fields $\Delta(x)$ ("geometry-legs") are completely analogous to Crawford's *drehbeins* [2] except that now we are reshuffling observable geometry. We are tautologically committed to propose a new principle of *local poly-dimensional covariance*. By this we mean that the basis vectors of a coordinate frame displaced from the origin may be "rotated" in dimension, e.g. be a multivector that is part vector plus part bivector relative to the reference geometry.

The generalized *poly-dimensional connection* $\Lambda_{(i)}{}^{(j)}$ is defined,

$$\Box\mathbf{E}_{(i)} = \Lambda_{(i)}{}^{(j)}\mathbf{E}_{(j)}. \tag{7a}$$

The right side of this equation is recognized as a linear transformation on the full Clifford algebra $\mathbb{R}(4)$. In general $\Lambda_{(i)}{}^{(j)}$ belongs to the endomorphism algebra End $\mathbb{R}(4) \cong \mathbb{R}(4) \otimes \mathbb{R}(4)$, hence it is an element of the *mother* algebra $\mathbb{R}(16)$ [14]. This leads to a new generalized *poly-dimensional covariant Dirac equation*,

$$[(\Box - m)\Psi^{(j)} + \Psi^{(i)}\Lambda_{(i)}{}^{(j)}]\mathbf{E}_{(j)} = 0, \tag{7b}$$

where the coordinate gradient in eq. (7b) is understood now NOT to operate on $\mathbf{E}_{(j)}$. This is not a particularly useful form, as it is expressed in terms of the *multivector* basis $\mathbf{E}_{(j)}$ instead of an *ideal* basis which would more closely resemble standard spinor form. The main annoying feature is that each multivector piece of the wavefunction couples to a different connection coefficient. Further, the poly-dimensional connection cannot itself be expressed as a multivector within the $\mathbb{R}(4)$ spacetime algebra.

Alternatively, the linear transformation can be written entirely within the smaller original $\mathbb{R}(4)$ spacetime algebra using two-sided multiplication [13]. We re-express eq. (7a) in terms of a new *bilateral connection* $\Omega^{(jk)}$,

$$\Box\mathbf{E}_{(i)} = \Omega^{(jk)}\mathbf{E}_{(j)}\mathbf{E}_{(i)}\mathbf{E}_{(k)}. \tag{8a}$$

The advantage of eq. (8a) over eq. (7a) is that the connection is now completely form independent of the operand element $\mathbf{E}_{(i)}$. This allows us to rewrite the interaction term of the Dirac equation in terms of the full multivector wavefunction Ψ instead of having to consider each multivector component $\psi^{(j)}$ separately as was done in eq. (7b). The resulting Dirac equation has the *bilateral interaction term* which was proposed earlier to empirically fit known mesonic couplings [12],

$$(\Box - m)\Psi = -\mathbf{E}_{(i)}\Psi\mathbf{E}_{(j)}\Omega^{(ij)}, \tag{8b}$$

where again it is understood that the gradient does not operate on the multivector basis (as that has already been included on the right side of the equation). From a multivector Lagrangian formulation [12] it can be shown that the gauge connection $\Omega^{(ij)}$ of eq. (8a) couples to the *bilateral current*,

$$j^{(ij)} = \frac{1}{4}Tr(\bar{\Psi}\mathbf{E}^{(i)}\Psi\mathbf{E}^{(j)}) = \frac{1}{4}Tr(\Psi\mathbf{E}^{(j)}\bar{\Psi}\mathbf{E}^{(i)}), \tag{8c}$$

where $\mathbf{E}^{(i)}$ and $\mathbf{E}^{(j)}$ must both be bar-positive or both bar-negative. The dextral interactions of eq. (6a) are the special case where the *sinistrad* [12] (left-side applied) interaction element of eq. (8b) is the set of basis vectors: $\mathbf{E}_{(i)} = \mathbf{e}_\mu$. When the *dextrad* (right-side applied) element $\mathbf{E}_{(j)}$ of eq. (8b) is either 1 or i, the interactions are of the same form proposed by Crawford [2].

4.3. Multivector Field Theory

In order to have a fully geometric description of symmetric tensors, Greider [8] introduced a second commuting Clifford algebra $\mathbf{F}_{(k)}$ in analogy with Eddington's E-F *double-frame* numbers [7]. A product of two elements $\mathbf{E}_{(j)}\mathbf{F}_{(k)}$ is a *geometric dyad* which is an element of a global 8D *mother* [14] algebra: $\mathbb{R}(16) = \mathbb{R}(4) \otimes \mathbb{R}(4)$. Potentially this allows us to write a single superfield equation which is completely coordinate and poly-dimensional covariant in form. In the particular case of dextrad connection of eq. (6a), the superfield equation can be written in a sourced *monogenic* form,

$$\Box\mathcal{F} = \mathcal{J}, \tag{9a}$$

where $\mathcal{J} = j^{\mu(j)}\mathbf{f}_\mu\mathbf{E}_{(j)}$ is the vector-multivector supercurrent made from eq. (6b). The coordinate derivative is in the $\mathbf{F}_{(j)}$ algebra vector basis: $\Box = \mathbf{f}^\mu\partial_\mu$. The superfield $\mathcal{F} = F^{\mu\nu(j)}\mathbf{f}_\mu \wedge \mathbf{f}_\nu\mathbf{E}_{(j)}$ is a bivector in the "first-frame *coordinate* algebra" $\mathbf{F}_{(k)}$, while a Clifford aggregate in the "second-frame *charge* algebra" $\mathbf{E}_{(j)}$.

The Morris-Greider [8] theory of gravitation was based upon the particular case where $\mathbf{E}_{(j)}$ is limited to be a vector, the supercurrent then being a vector-vector dyad. The case where $\mathbf{E}_{(j)}$ is a trivector was explored by Differ [5]. It appears that the spin-gauge field eq. (5a) can also be written in this general form, where the commutator term is built into the equation if assumptions are made about the generalized connection coefficient of eq. (7a). The field equation for the general bilateral interaction of eq. (8b) has yet to be fully formulated.

5. Summary

Our development was based upon an underlying theme of using only algebra that is based on concrete spacetime geometry. This has led us to eliminate spin space, the principle of local spin covariance and ultimately spin gauge theory. In its place we propose the more grand scheme of *local polydimensional covariance.* While the results are promising for Dirac, gauge and classical field theory, it remains to be seen if its domain can be extended to classical mechanics. Further, the interpretation of the 8D geometry needed is not completely clear, although it appears to be connected with the classical symmetric tensor objects of 4D which are needed for formulations of gravitation.

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DIVISION ALGEBRAS, (1,9)-SPACE-TIME, MATTER-ANTIMATTER MIXING

GEOFFREY DIXON
Department of Physics, Brandeis University,
Waltham, MA 02254,
and Department of Mathematics,
University of Massachusetts, Boston, MA 02125
dixon@binah.cc.brandeis.edu

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Abstract. The tensor product of the division algebras, which is a kernel for the structure of the Standard Model, is also a root for the Clifford algebra of (1,9)-space-time. A conventional Dirac Lagrangian, employing the (1,9)-Dirac operator acting on the Standard Model hyperfield, gives rise to matter into antimatter transitions not mediated by any gauge field. These transitions are eliminated by restricting the dependencies of the components of the hyperfield on the extra six dimensions, which appear in this context as a complex triple.

This article is an extension of my work on applying the tensor product of the division algebras to the lepto-quark Standard Model [1-4] and beyond. Although it is selfcontained, many results derived previously are not rederived here.

Applications of the division algebras to particle physics [5-10] are not new, nor are all the same. This application, to the best of my knowledge, while owing a debt to the work of Gürsey and Günaydin, is the only one of its kind. Like all applications of these algebras, however, it is motivated by the attractive notion that the special structures of mathematics play a role in the design of reality. Most theorists share a faith - or at least a hope - of this sort; here it has been allowed to become a guiding principle.

In this article I present the first radical extension of my ideas beyond the Standard Model and its foundation. Because it combines the Standard Model with (1,9)-space-time ($\mathbf{R}^{1,9}$), it may well prove a step toward the development of a connection to, and a narrowing of, string theory, the initial euphoria to which has - in the fashion of GUTs and SUSY - succumbed to the curse of multiple realities.

The nontrivial real division algebras with unity are the complexes, $\mathbf{C}$, quaternions, $\mathbf{Q}$, and octonions, $\mathbf{O}$. They are 2-, 4-, and 8-dimensional. Multiplication tables for $\mathbf{Q}$ and $\mathbf{O}$ are constructable from the following elegant rules:

Division Algebra	Q	O	
Imaginary Units	$q_i, i = 1,2,3,$	$e_a, a = 1,...,7,$	
Anti-commutators	$q_iq_j + q_jq_i = 2\delta_{ij},$	$e_ae_b + e_be_a = 2\delta_{ab},$	(1)
Cyclic Rules	$q_iq_{i+1} = q_{i-1} = q_{i+2},$	$e_ae_{a+1} = e_{a-2} = e_{a+5},$	
Index Doubling	$q_iq_j = q_k \Longrightarrow$ $q_{(2i)}q_{(2j)} = -q_{(2k)},$	$e_ae_b = e_c \Longrightarrow$ $e_{(2a)}e_{(2b)} = e_{(2c)},$	

where **Q**-indices run from 1 to 3, modulo 3, and **O**-indices run from 1 to 7, modulo 7.

C⊗**Q** is spanned by the 8 elements $\{1, i, q_j, iq_j\}$. It is isomorphic to the Pauli algebra, **C**(2), which is the Clifford algebra of $\mathbf{R}^{3,0}$ space. Represented by **C**(2), the spinors of that Clifford algebra are 2×1 over **C**, the so-called Pauli or Weyl spinors. The spinor space of **C**⊗**Q**, however, is 1×1 over **C**⊗**Q**, hence is **C**⊗**Q** itself. In this case, to distinguish the Clifford algebra from its spinor space, we denote the former $\mathbf{C}_L \otimes \mathbf{Q}_L$, the subscript indicating action from the left on the spinor space, which we denote **C**⊗**Q**.

C⊗**Q** is twice as large as it needs to be. It is the direct sum of two 2-dimensional (over **C**) Weyl spinor spaces unmixed by $\mathbf{C}_L \otimes \mathbf{Q}_L$ (just as $\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$ in **C**(2) is the direct sum of the Weyl spinor spaces $\begin{bmatrix} x_1 & 0 \\ x_2 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & y_1 \\ 0 & y_2 \end{bmatrix}$). If $\mathbf{x} \in \mathbf{Q}$ satisfies $\mathbf{x}^2 = -1$, then multiplication from the right on **C**⊗**Q** by the idempotents $\frac{1}{2}(1 \pm i\mathbf{x})$ projects two such Weyl spinor spaces (just as multiplication from the right by the idempotents $\frac{1}{2}(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix})$ on $\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$ projects the **C**(2) Weyl spinor spaces above). $\mathbf{Q}_R$, which acts from the right on **C**⊗**Q**, mixes these two independent spinor spaces. $\mathbf{Q}_R$ commutes with $\mathbf{C}_L \otimes \mathbf{Q}_L$, so it is an "internal" algebra, where the Clifford (geometric) algebra is "external". The elements of unit length of $\mathbf{Q}_R$ form the group $SU(2)$, which in previous work along these lines was manifested as the isospin gauge symmetry [1].

The octonion algebra is generally considered ill-suited to Clifford algebra theory because **O** is nonassociative, and Clifford algebras are associative.

This problem disappears once we identify **O** as the spinor space of $\mathbf{O}_L$, the adjoint algebra of actions of **O** on itself from the left. $\mathbf{O}_L$ *is* associative. $\mathbf{O}_L$ is linear in actions of the form

$$e_{Lab...c}[x] = e_a(e_b(...(e_c x)...)), \tag{2}$$

$x \in \mathbf{O}$. For example, although $e_1 e_2 = e_6$,

$$e_{L12}[x] = e_1(e_2 x) \neq e_6 x = e_{L6}[x]$$

in general; and although $e_1(e_2 e_4) = e_7$,

$$e_{L124}[x] = e_1(e_2(e_4 x)) \neq e_7 x = e_{L7}[x]$$

in general. These are consequences of nonassociativity. The elements $e_{Lab...c}$ satisfy

$$\begin{aligned} e_{Labcc...d} &= -e_{Lab...d}, \\ e_{Lab...c} &= \pm e_{Lpq...r}, \end{aligned} \tag{3}$$

pq...r an even-odd permutation of ab...c, and

$$e_{Lab...c} e_{Ldf...g} = e_{Lab...cdf...g}. \tag{4}$$

It is also not difficult to prove that $e_{L7654321}[x] = x$ for all x in **O**. Therefore, for example, using (4) and (5) one can easily prove

$$e_{L4567} = e_{L4567} e_{L7654321} = e_{L321}. \tag{5}$$

That is, any element of $\mathbf{O}_L$ with four or mcre indices can be reduced to an element with three indices or less. So a complete basis for $\mathbf{O}_L$ consists of the elements

$$1,\ e_{La},\ e_{Lab},\ e_{Labc}. \tag{6}$$

Therefore $\mathbf{O}_L$ is 1+7+21+35=64-dimensional, and $\mathbf{O}_L \simeq \mathbf{R}(8)$. The embedding of parentheses in the definition (2), implying (4), trivially implies $\mathbf{O}_L$ is associative.

$\mathbf{O}_L$ is isomorphic to the Clifford algebra of the space $\mathbf{R}^{0,6}$, the spinor space of which is 8-dimensional over **R**. In this case the spinor space is **O** itself, the object space of $\mathbf{O}_L$. It is significant that the dimensionality of **O** is correct in this case. This is tied to the remarkable fact that the algebra $\mathbf{O}_R$ of right adjoint actions of **O** on itself is the *same* algebra as $\mathbf{O}_L$. Every action in $\mathbf{O}_R$ can be written as an action in $\mathbf{O}_L$.

A 1-vector basis for $\mathbf{O}_L$, playing the role of the Clifford algebra of $\mathbf{R}^{0,6}$, is $\{e_{Lp}, p = 1,...,6\}$. The resulting 2-vector basis is then $\{e_{Lpq}, p, q = 1,...,6, p \neq q\}$. This subspace is 15-dimensional, closes under the commutator product, and is in that case isomorphic to $so(6)$. The intersection of this Lie algebra with the Lie algebra of the automorphism group of $\mathbf{O}$, G_2, is $su(3)$, with a basis

$$su(3) \rightarrow \{e_{Lpq} - e_{Lrs}, p, q, r, s \text{ distinct, and from 1 to 6}\}. \tag{7}$$

The group $SU(3)$ generated by these elements arises as the color gauge group in applications [1] (note that $SU(3)$ is the stability group of e_7, hence the index doubling automorphism of $\mathbf{O}$ is an $SU(3)$ rotation).

Finally we let $\mathbf{C} \otimes \mathbf{Q} \otimes \mathbf{O}$ play the role of spinor space to $\mathbf{C}_L \otimes \mathbf{Q}_L \otimes \mathbf{O}_L$, which is isomorphic to $\mathbf{C}(16)$, hence isomorphic to the Clifford algebra of the space $\mathbf{R}^{0,9}$. With respect to the gauge symmetry $SU(2) \times SU(3)$ outlined above, which expands to $U(2) \times U(3)$ (in [1] this symmetry is derived; it is associated with the inner product on $\mathbf{C} \otimes \mathbf{Q} \otimes \mathbf{O}$, specifically with a set of projection operators (associative idempotents) from which the inner product is constructed), the spinor space $\mathbf{C} \otimes \mathbf{Q} \otimes \mathbf{O}$ transforms exactly like the direct sum of a family and antifamily of lepto-quark Weyl spinors. That is, various algebraic bits of the spinor space are identifiable by their $U(2) \times U(3)$ transformation properties as being quark or lepton. Quantum numbers for the (family) spinors can be manifested in two ways, one corresponding to righthanded particles, one to lefthanded. They can be simultaneously incorporated by expanding $\mathbf{C}_L \otimes \mathbf{Q}_L \otimes \mathbf{O}_L$ to $\mathbf{C}_L \otimes \mathbf{Q}_L \otimes \mathbf{O}_L(2)$ (2×2 over $\mathbf{C}_L \otimes \mathbf{Q}_L \otimes \mathbf{O}_L$), the "Dirac" algebra for $\mathbf{R}^{1,9}$ space-time (just as $\mathbf{C}_L \otimes \mathbf{Q}_L(2)$, isomorphic to $\mathbf{C}(4)$, is the Dirac algebra for $\mathbf{R}^{1,3}$). The spinor space in this case is 2×1 over $\mathbf{C} \otimes \mathbf{Q} \otimes \mathbf{O}$.

Let Ψ be such a spinor, and give it a functional dependence on $\mathbf{R}^{1,9}$ space-time. Let

$$\rho_{\pm} = (1 \pm ie_7)/2, \tag{8}$$

a $U(3)$ invariant component of the projection operator set mentioned above. Then $\rho_+\Psi$ is the matter half of Ψ, and $\rho_-\Psi$ the antimatter half. $\rho_+\Psi\rho_+$ is an $SU(2)$ lepton doublet, and $\rho_+\Psi\rho_-$ is a quark $SU(2)$ doublet, $SU(3)$ triplet (reverse signs for antimatter).

Define in $\mathbf{R}(2)$:

$$\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

A 1-vector basis for the Clifford algebra of $R^{1,9}$ consists of the elements:

$$\gamma_0 = \beta,\ \ \gamma_j = q_j e_{L7}\omega, j = 1,2,3,\ \ \gamma_h = ie_{h-3}\omega, h = 4,...,9. \tag{9}$$

These satisfy:

$$\gamma_h\gamma_l + \gamma_l\gamma_h = 2\eta_{hl}\epsilon,$$

η_{hl} diagonal $(1(+), 9(-))$. In particular note that the set $\{\gamma_h, h = 4,...,9\}$ are not $SU(3)$ invariant, from which we infer that the extra 6 space dimensions carry $SU(3)$ charges.

The (1,9)-Dirac operator is $\partial\!\!\!/_{1,9} = \gamma_f \partial^f, f = 0,1,...,9$, and I define $\partial\!\!\!/_{1,3} = \gamma_\mu \partial^\mu, \mu = 0,1,2,3$, $\partial\!\!\!/_{0,6} = \partial\!\!\!/_{1,9} - \partial\!\!\!/_{1,3}$. Define

$$\rho_{L\pm} = (1 \pm ie_{L7})/2 \tag{10}$$

(the left adjoint version of $\rho_\pm$). Using these adjoint idempotents we can decompose $\partial\!\!\!/_{1,9}$ into its (1,3)- and (0,6)-Dirac operator parts, one of each for both matter and antimatter:

$$\begin{aligned}\partial\!\!\!/_{1,9} &= \rho_{L+}\partial\!\!\!/_{1,9}\rho_{L+} + \rho_{L-}\partial\!\!\!/_{1,9}\rho_{L-} + \rho_{L+}\partial\!\!\!/_{1,9}\rho_{L-} + \rho_{L-}\partial\!\!\!/_{1,9}\rho_{L+} \\ &= \rho_{L+}\partial\!\!\!/_{1,3}\rho_{L+} + \rho_{L-}\partial\!\!\!/_{1,3}\rho_{L-} + \rho_{L+}\partial\!\!\!/_{0,6}\rho_{L-} + \rho_{L-}\partial\!\!\!/_{0,6}\rho_{L+}, \\ &= \partial\!\!\!/_{1,3}\rho_{L+} + \partial\!\!\!/_{1,3}\rho_{L-} + \partial\!\!\!/_{0,6}\rho_{L-} + \partial\!\!\!/_{0,6}\rho_{L+}\end{aligned} \tag{11}$$

(note that $\partial\!\!\!/_{1,3}\rho_{L\pm}$ are the matter/antimatter Dirac operators for (1,3)-space-time, and that because $e_{L7}\rho_{L\pm} = \mp i\rho_{L\pm}$, the partials of the latter are space-reflected relative to the former). Therefore,

$$\begin{aligned}\partial\!\!\!/_{1,9}\Psi &= (\partial\!\!\!/_{1,3}\rho_{L+} + \partial\!\!\!/_{1,3}\rho_{L-} + \partial\!\!\!/_{0,6}\rho_{L-} + \partial\!\!\!/_{0,6}\rho_{L+})\Psi \\ &= \partial\!\!\!/_{1,3}(\rho_+\Psi) + \partial\!\!\!/_{1,3}(\rho_-\Psi) + \partial\!\!\!/_{0,6}(\rho_-\Psi) + \partial\!\!\!/_{0,6}(\rho_+\Psi).\end{aligned} \tag{12}$$

To form a Lagrangian for the field we use the inner product of $\mathbf{C}\otimes\mathbf{Q}\otimes\mathbf{O}$ [1]:

$$\begin{aligned}\mathcal{L} &= <\Psi, \partial\!\!\!/_{1,9}\Psi> \\ &= <\rho_+\Psi + \rho_-\Psi, \partial\!\!\!/_{1,3}(\rho_+\Psi) + \partial\!\!\!/_{1,3}(\rho_-\Psi) + \partial\!\!\!/_{0,6}(\rho_-\Psi) + \partial\!\!\!/_{0,6}(\rho_+\Psi)> \\ &= <\rho_+\Psi, \partial\!\!\!/_{1,3}(\rho_+\Psi)> + <\rho_-\Psi, \partial\!\!\!/_{1,3}(\rho_-\Psi)> \\ &+ <\rho_+\Psi, \partial\!\!\!/_{0,6}(\rho_-\Psi)> + <\rho_-\Psi, \partial\!\!\!/_{0,6}(\rho_+\Psi)>\end{aligned} \tag{13}$$

(the last equality arising from the algebra of the inner product). The first two terms after the last equality in (13), $<\rho_\pm\Psi, \partial\!\!\!/_{1,3}(\rho_\pm\Psi)>$, are ordinary. One can obtain a list of viable particle transitions from such Lagrangians, as each Weyl component of Ψ has an obvious particle identification. For example, after gauging $U(2)\times U(3)$, algebraic combinations of spinor and gauge fields that survive the inner product correspond to viable transitions, and these are just those that are built into the Standard Model on phenomenological grounds, the major difference being the presence of noninteracting righthanded neutrino terms (this aspect won't be developed further here; see [1], [11]). These first two terms connect matter/antimatter to matter/antimatter ($\rho_\pm\Psi=$ matter/antimatter), hence they are in that sense conventional. They conserve lepton and baryon numbers.

The last two terms of (13), $<\rho_\mp\Psi, \partial\!\!\!/_{0,6}(\rho_\pm\Psi)>$, are a problem, even without gauge fields, for they imply matter/antimatter ($\rho_\pm\Psi$) into antimatter/matter ($\rho_\mp\Psi$) transitions, mediated algebraically by $\partial\!\!\!/_{0,6}$. As such transitions are unobserved, the rest of the article will be devoted to getting rid of the last two terms of (13).

The 2-vector basis for the Clifford algebra of $\mathbf{R}^{1,9}$, derived from the 1-vectors in (8), is

$$q_j\epsilon,\ q_je_{L7}\alpha,\ ie_{Lp}\alpha,\ iq_je_{Lp7}\epsilon,\ e_{Lpq}\epsilon, \tag{14}$$

j=1,2,3, p,q∈ {1,...,6}. This 45-dimensional subspace closes under the commutator product and is in that case isomorphic to $so(1,9)$. The first six elements, $\{q_j\epsilon,\ q_je_{L7}\alpha\}$, form a basis for $so(1,3)$, the last fifteen, $\{e_{Lpq}\epsilon\}$, a basis for $so(6)$. This is the same $so(6)$ we saw earlier, and it contains color $su(3)$ (see (7)). That is, the space $\mathbf{R}^{0,6}$, hence $\partial\!\!\!/_{0,6}$, carry color charges (one consequence of these charges: in none of the unwanted transitions implied by (13) can a particle make a transition to its own antiparticle; hence, for example, quarks may mix with anitleptons, violating baryon and lepton number conservation).

Consider the element $\partial\!\!\!/_{0,6}(\rho_+\Psi)$ which appears in the last term of (13). Because

$$\rho_\pm e_7 = \mp i\rho_\pm,\ \rho_\pm e_5 = \mp i\rho_\pm e_1,\ \rho_\pm e_3 = \mp i\rho_\pm e_2,\ \rho_\pm e_6 = \mp i\rho_\pm e_4, \tag{15}$$

$\rho_+\Psi$ may be decomposed into

$$\rho_+\Psi = \rho_+[\Psi_+^0 + \Psi_+^1 e_1 + \Psi_+^2 e_2 + \Psi_+^4 e_4], \tag{16}$$

where the Ψ_+^m, m=0,1,2,4, are 2×1 over $\mathbf{C}\otimes\mathbf{Q}$. These four fields can be designated lepton, red-, green-, and blue-quark.

Now consider $\partial\!\!\!/_{0,6}(\rho_+\Psi)$, and in particular, for example, the term (sum p=1,...,6)

$$\begin{aligned}
\partial\!\!\!/_{0,6}(\rho_+\Psi_+^1 e_1) &= i\omega e_p\partial^{p+3}[\rho_+\Psi_+^1 e_1]\\
&= i\omega(\rho_- e_1\partial^4 + \rho_+e_2\partial^5 + \rho_+e_3\partial^6 + \rho_+e_4\partial^7 + \rho_-e_5\partial^8 + \rho_+e_6\partial^9)[\Psi_+^1 e_1]\\
&= i\omega(\rho_-e_1(\partial^4 + i\partial^8) + \rho_+e_2(\partial^5 - i\partial^6) + \rho_+e_4(\partial^7 - i\partial^9))[\Psi_+^1 e_1]\\
&= i\omega(e_1(\partial^4 + i\partial^8) + e_2(\partial^5 - i\partial^6) + e_4(\partial^7 - i\partial^9))[\rho_+\Psi_+^1 e_1]\\
&\equiv \partial\!\!\!/_{6+--}[\rho_+\Psi_+^1 e_1]
\end{aligned} \tag{17}$$

(in the second line the nonassociativity of **O** plays a part in altering the sign subscripts of $\rho_\pm$; in general nonassociativity plays an essential role in keeping the mathematics consistent with phenomenology). $\partial\!\!\!/_{6+--}$ (generalized below) is defined in the penultimate line. In like manner one can demonstrate that

$$\begin{aligned}
\partial\!\!\!/_6\rho_+\Psi_+^0 &= \partial\!\!\!/_{6+++}\rho_+\Psi_+^0,\\
\partial\!\!\!/_6(\rho_+\Psi_+^2 e_2) &= \partial\!\!\!/_{6-+-}(\rho_+\Psi_+^2 e_2)\\
\partial\!\!\!/_6(\rho_+\Psi_+^4 e_4) &= \partial\!\!\!/_{6--+}(\rho_+\Psi_+^4 e_4)
\end{aligned} \tag{18}$$

(no parentheses are needed in the first of these equations (lepton term), for nonassociativity only becomes an issue on the quark terms). For any real variables x and y, and differentiable f: $(\partial_x + i\partial_y)f(x+iy) = 0$. Therefore, ignoring $\mathbf{R}^{1,3}$ coordinates, if

$$\begin{aligned}
\Psi_+^0 &= \Psi_+^0(x_4 + ix_8, x_5 + ix_6, x_7 + ix_9),\\
\Psi_+^1 &= \Psi_+^1(x_4 + ix_8, x_5 - ix_6, x_7 - ix_9),\\
\Psi_+^2 &= \Psi_+^2(x_4 - ix_8, x_5 + ix_6, x_7 - ix_9),\\
\Psi_+^4 &= \Psi_+^4(x_4 - ix_8, x_5 - ix_6, x_7 + ix_9),
\end{aligned} \tag{19}$$

then

$$\partial\!\!\!/_{0,6}(\rho_+\Psi) = 0 \tag{20}$$

identically.

The antimatter fields of $\rho_-\Psi$ would have functional dependencies conjugate to those above. Any fluctuation from these would give rise to so far unobserved matter-antimatter mixing.

Under $U(3)$ the lepton term Ψ_+^0 is supposed invariant, but its 3 complex coordinates in (19) are not. In making $U(3)$ a local gauge symmetry, dependent upon $\mathbf{R}^{1,3}$ coordinates, the complex coordinates of Ψ_+^0 also acquire a functional dependence on $\mathbf{R}^{1,3}$. The orbit of $U(3)$ is S^5, the 5-sphere. Because Ψ_+^0 is dependent on 3 complex coordinates, and not 6 real, this precludes a variation of Ψ_+^0 by even so much as a phase factor under $U(3)$. It would seem then that the colorless lepton term Ψ_+^0 must be independent entirely of the color-carrying coordinates of $\mathbf{R}^{0,6}$.

The complex triple associated with Ψ_+^1 in (19) has a more complicated SU(3) transformation, further complicated by the fact that Ψ_+^1 is itself simultaneously transformed. However, Ψ_+^1 is invariant under the action of the SU(2) subgroup of SU(3) that leaves e_1 and e_5 invariant. Following the same reasoning used above we now conclude that Ψ_+^1 must be independent, not of all of $\mathbf{R}^{0,6}$ as was Ψ_+^0, but of x_r, r=5,6,7,9.

In general we may now conclude, inorder to preserve (20), that

$$\begin{aligned}
\Psi_+^0 &= \Psi_+^0(x_\mu, ..., ..., ...),\\
\Psi_+^1 &= \Psi_+^1(x_\mu, x_4 + ix_8, ..., ...),\\
\Psi_+^2 &= \Psi_+^2(x_\mu, ..., x_5 + ix_6, ...),\\
\Psi_+^4 &= \Psi_+^4(x_\mu, ..., ..., x_7 + ix_9),
\end{aligned} \tag{21}$$

where (,...) indicates independence of the complex coordinate in that slot, and x_μ denote the coordinates of $\mathbf{R}^{1,3}$.

Does any of this have anything to do with string theory? I confess myself not a string theorist, so I can not supply a definitive answer to that question. String theory uses $\mathbf{R}^{1,9}$, and it deals with the extra 6 dimensions by balling them up into a complex 3-manifold too small to be observed. My route to $\mathbf{R}^{1,9}$ is certainly different, but in requiring (20) the space $\mathbf{R}^{0,6}$ is forced to appear in the guise of a complex 3-space. It has not yet been investigated if some specific compactification is required of the model, much less if there is an associated $SU(3)$ holonomy group [12]. As to its unobservability, everything in this model (specifically quarks and $\mathbf{R}^{0,6}$) associated with the octonion units $e_p, p = 1, ..., 6$ (also associated with nonassociativity) is unobserved. There may be some nice algebraic/quantum mechanical explanation for this, but even so one finds such subtlety is generally manifested by more prosaic explanations as well, like infrared slavery, and, presumably, compactification.

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DERIVATIONS OF ENDOMORPHISMS

GARRET SOBCZYK
Universidad de las Americas, Departmento de Fisico-Matematicas
Apartado Postal #100, Santa Catarina Mártir
72820 Puebla, Pue., México
sobczyk@udlapvms.pue.udlap.mx

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Abstract. Let $End(\mathbb{R}_n)$ be the algebra of endomorphisms on the real 2^n-dimensional Clifford geometric algebra $\mathbb{R}_n$ of the n-dimensional Euclidean space $\mathbb{R}^n$. In this work we study the *structure* induced by a family of derivations on the algebra $End(\mathbb{R}_n)$. The shape and curvature bivectors of a projection with respect to a given structure are defined. The concept of a structure allows us to lift the local properties of a vector manifold to the algebra $End(\mathbb{R}_n)$.

1. Algebraic Framework

Let $End(\mathbb{R}_n)$ be the algebra of endomorphisms on the real 2^n-dimensional Clifford geometric algebra $\mathbb{R}_n$ of the n-dimensional Euclidean space $\mathbb{R}^n$. As has been shown in [1, p.3656], the algebra $End(\mathbb{R}_n)$ is itself isomorphic to a Clifford geometric algebra, namely the 2^{2n}-dimensional geometric algebra $\mathbb{R}_{n,n}$, and also to the algebra of $2^n \times 2^n$ real matrices. Although the matrix algebra formalism is fully isomorphic to the corresponding geometric algebra, the great advantage of latter is its comprehensive geometric significance. Geometric algebra can also be nicely formulated in an infinite dimensional setting, but we will not consider this here.

A familiar subalgebra of $End(\mathbb{R}_n)$ is $End(\mathbb{R}^n)$ the algebra of all linear operators on the Euclidean space $\mathbb{R}^n$. Let $f \in End(\mathbb{R}^n)$. The characteristic equation of f is $det(\lambda - f) = 0$, and in general can have both real and complex eigenvalues as roots. If the minimal polynomial of f is known, then f can be put in the *eigenprojector* form:

$$f = \sum_{j=1}^{r} \lambda_j p_j + q_j,$$

where the $p_j, q_j \in End(\mathbb{R}^n)$ are respectively idempotents and nilpotents, [2], [3].

One disturbing question when reformulating linear algebra in terms of geometric algebra [4], has been the lack of a suitable geometric interpretation of the complex eigenvalues of f in the geometric algebra $\mathbb{R}_n$. The difficulty is in the interpretation of the "imaginary" unit i.[1]

In $\mathbb{R}_n$ each unit bivector has square -1, but bivectors do not algebraically commute with all the other elements of $\mathbb{R}_n$. In $\mathbb{R}_3$ the unit pseudoscalar has square -1 and commutes with all other elements, and therefore is an attractive choice in this algebra, as well in the similar algebras $\mathbb{R}_{4k+3}$ for $k = 1, 2, \ldots$. We shall use this as the guiding principle for the selection of the geometric interpretation of the complex eigenvalues of $f \in End(\mathbb{R}^n)$: We are looking for an encompassing geometric algebra containing $\mathbb{R}_n$ whose pseudoscalar element commutes with all other elements and has square -1.

We have noted that $End(\mathbb{R}_n) \cong \mathbb{R}_{n,n}$, so it is natural to look in this larger geometric algebra which contains $\mathbb{R}_n$. The unit pseudoscalar of $\mathbb{R}_{n,n}$ commutes with all elements of $\mathbb{R}_{n,n}$, but has square -1 only when n is odd. This situation is at least an improvement over the previous values of $n = 4k + 3$ in the case of the algebras $\mathbb{R}_n$.

The solution to this search for the elusive i is to note the isomorphism

$$\mathbb{R}_{n,n}(i) \cong \mathbb{R}_{n,n+1}$$

between the formal complexification $\mathbb{R}_{n,n}(i)$ of $\mathbb{R}_{n,n}$ and the geometric algebra $\mathbb{R}_{n,n+1}$ of the signature $(p, q) = (n, n + 1)$. The pseudoscalar of this algebra has all of the desired algebraic properties for all $n = 0, 1, 2, \ldots$.

Let $\mathbb{C}^n \equiv \mathbb{R}^n(i)$ denote the formal complexification of the Euclidean n-space $\mathbb{R}^n$, and $\mathbb{C}_n \equiv \mathbb{R}_n(i)$ the corresponding complexification of the real geometric algebra $\mathbb{R}_n$. Complex vectors $c \in \mathbb{C}^n$ are of the form $c = a + ib$ where $a, b \in \mathbb{R}^n$. The algebra

$$End(\mathbb{C}_n) = End^+(\mathbb{C}_n) \oplus End^-(\mathbb{C}_n) \cong \mathbb{R}_{n,n+1},$$

is the direct sum of all complex linear operators $End^+(\mathbb{C}_n)$ and complex antilinear (sesquilinear) operators $End^-(\mathbb{C}_n)$, and is isomorphic to $\mathbb{R}_{n,n+1}$. Complex linear and antilinear operators have been studied for $n = 3, 4$ in [6], [7]. The remainder of this paper is concerned with the algebra of complex linear operators $End^+(\mathbb{C}_n)$, the special case of the real operators in $End(\mathbb{R}_n)$ being naturally included.

2. Structures

Each element $C \in \mathbb{C}_n$ can be written in the form $C = A + iB$ for $A, B \in \mathbb{R}_n$. We can also decompose $C \in \mathbb{C}_n$ into the sum

$$C = \sum_{j=0}^{n} <C>_j,$$

[1] See [5] for a related discussion of this point.

of complex homogeneous k-vector parts $< C >_k$. For what follows we will need an orthonormal basis of complex 1-vectors $\{e_1, \ldots, e_n\}$ which satisfy the usual relations $e_j \cdot e_k = \delta_{jk}$ for $j, k = 1, 2, \ldots, n$. In general, the *complex linear* inner product between two complex 1-vectors in $\mathcal{C}_n$ is a complex scalar.

One of the most fundamental identities for the geometric product of a complex 1-vector a and a complex k-vector A for $k \geq 1$, is

$$aA = a \cdot A + a \wedge A$$

where $a \cdot A$ is a complex $(k-1)$-vector, and $a \wedge A$ is a complex $(k+1)$-vector.

Let $f, g \in End^+(\mathcal{C}_n)$. Then $f + g, fg \in End^+(\mathcal{C}_n)$ is respectively the sum and product (composition) of these elements in the algebra $End^+(\mathcal{C}_n)$. For complex elements $A, B \in \mathcal{C}_n$, the mixed sums and products such as

$$f(A)^2 g(B) + fg(AB) \in \mathcal{C}_n$$

are also well defined in terms of the operations of addition and multiplication in both $End^+(\mathcal{C}_n)$ and in $\mathcal{C}_n$ itself.

Let $f, g \in End^+(\mathcal{C}_n)$, $a \in \mathcal{C}^n$, and $A, B \in \mathcal{C}_n$.

DEFINITION 1. *A structure is a bilinear mapping* $\varphi : \mathcal{C}^n \times End^+(\mathcal{C}_n) \to End^+(\mathcal{C}_n)$, *where* $\varphi(a, f) \equiv f_a \in End(\mathcal{C}_n)$, *that satisfies the following properties*

1. *If* $id \in End(\mathcal{C}_n)$ *is the identity operator, then* $id_a = 0$.
2. $[f + g]_a = f_a + g_a$,
3. $[fg]_a = f_a g + f g_a$.
4. $[f(A)g(B)]_a \equiv f_a(A)g(B) + f(A)g_a(B)$.

There are many different ways of extending the operators in $End^+(\mathcal{C}^n)$ to operators in $End^+(\mathcal{C}_n)$. We mention here the "outermorphism" rule [4, p. 67], and the "derivation" rule [8, p.107-110]. Given $f \in End^+(\mathcal{C}^n)$, f is extended to an outermorphism $\underline{f} \in End^+(\mathcal{C}_n)$ by defining

$$\underline{f}(\alpha + c_1 \wedge c_2 \wedge \ldots \wedge c_r) = \alpha + f(c_1) \wedge \ldots \wedge f(c_r)$$

for each complex scalar $\alpha \in \mathcal{C}$, and each r-vector $c_1 \wedge c_2 \wedge \ldots \wedge c_r$. The definition of $\underline{f}$ on all of $\mathcal{C}_n$ is completed by enforcing complex linearity. The determinant of $f \in End(\mathcal{C}^n)$ can be nicely defined in terms of its outermorphism $\underline{f}$,

$$det(f) \equiv (e_1 \wedge \cdots \wedge e_n)^{-1} \underline{f}(e_1 \wedge \ldots \wedge e_n).$$

Given $f \in End^{+}(\mathbb{C}^n)$, f is extended to a derivation $\dot{f} \in End^{+}(\mathbb{C}_n)$ by defining

$$\dot{f}\,(\alpha + c_1 \wedge c_2 \wedge \ldots \wedge c_r) = \sum_{j=1}^{r} c_1 \wedge \ldots \wedge c_{j-1} \wedge \dot{f}\,(c_j) \wedge c_{j+1} \wedge \ldots \wedge c_r$$

for each complex scalar $\alpha \in \mathbb{C}$, and each r-vector $c_1 \wedge c_2 \wedge \ldots \wedge c_r$. The extension of $\dot{f}$ on all of $\mathbb{C}_n$ is completed by enforcing complex linearity.

3. Shape and Curvature

Let φ be a structure on $End^{+}(\mathbb{C}_n)$, and P be a projection in $End^{+}(\mathbb{C}_n)$ satisfying $P^2 = P$. A projection is automatically an outermorphism, and therefore satisfies the property $P(A \wedge B) = P(A) \wedge P(B)$. Using definition 1, we get the basic identity

$$P_a(A \wedge B) = P_a(A) \wedge P(B) + P(A) \wedge P_a(B).$$

This identify is useful in proving properties about the *shape and curvature bivectors* whose definitions are given below.

DEFINITION 2. *The shape bivector* $S : \mathbb{C}^n \rightarrow <\mathbb{C}_n>_2$ *of the projection* $P \in End^{+}(\mathbb{C}_n)$ *is given by*

$$S(a) \equiv \sum_{j=1}^{n} e_j \wedge P_a(e_j).$$

Of course, the shape bivector of the projection P is defined with respect to the structure φ.

We can also define a *curvature* bivector of the projection P with respect to the structure φ.

DEFINITION 3. *The curvature bivector* $R : <\mathbb{C}_n>_2 \rightarrow <\mathbb{C}_n>_2$ *of the projection* P *is given by*

$$R(a \wedge b) \equiv \sum_{j=1}^{n} e_j \wedge P_a P_b(e_j) = P_a(S(b)).$$

4. Discussion

Many other objects from differential geometry can be defined in terms of a structure φ on $End^{+}(\mathbb{C}_n)$, such as k-forms, k-fields and the bracket of fields[2]. The main idea of this paper is to lift the local structure of these objects to the very rich $End^+(\mathbb{C}_n)$ by utilizing the concept of a structure given in definition 1. Many of these ideas have been developed in the context of a *vector manifold* in [4, Chapter 4].

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CLIFFORD ALGEBRAS AND SPINORS IN BRAIDED GEOMETRY

MIĆO ĐURĐEVIĆ
Instituto de Matematicas, UNAM
Area de la Investigacion Cientifica, Circuito Exterior
Ciudad Universitaria, 04510 México DF, México
miko@unamvm1.dgsca.unam.mx
and
Centro de Investigaciones Teoricas
Facultad de Estudios Superiores Cuautitlan, UNAM
54700 Cuautitlan Izcalli, México

and

ZBIGNIEW OZIEWICZ*
Centro de Investigaciones Teoricas
Facultad de Estudios Superiores Cuautitlan, UNAM
54700 Cuautitlan Izcalli, México
oziewicz@redvax1.dgsca.unam.mx

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Abstract. The paper provides an introduction to Clifford algebras and spinors for an arbitrary braid. Braided Clifford algebras are defined as Chevalley-Kähler deformations of braided exterior algebras (the Woronowicz algebras). Spinor representations are introduced, following classical Cartan's approach.

1. Introduction

The aim of this contribution is to present an incorporation of classical theory of Clifford algebras and spinors into a braided framework, starting from Chevalley-Kähler interpretation of Clifford algebras, as deformations of exterior algebras. A detailed exposition is given in (Đurđević and Oziewicz 1994).

Woronowicz introduced in 1989 a braided exterior algebras (the Woronowicz algebras). In the next section the main properties of the Woronowicz algebras are collected. Section 3 deals with inner products. Section 4 is devoted to the construction and general analysis of braided Clifford algebras. The construction conceptually follows classical Chevalley approach. We shall introduce a new product in the exterior algebra space. This prod-

* On leave of absence from University of Wroclaw, Poland. Research of the second author partially suported by State Committee for Scientific Research, Poland, KBN grant # 2 P302 023 07.

uct is expressible in terms of the exterior product, and various "relative" contractions, which are constructed from the corresponding scalar product on the vector space. In such a way the Clifford algebra becomes a Chevalley's deformation of the exterior algebra. We shall also introduce an analog of the Crumeyrolle map (Crumeyrolle 1990) connecting Clifford and exterior ideals in the tensor algebra. This allows to define braided Clifford algebras as a quantization of the Woronowicz exterior algebras.

In Section 5 we study counterparts of algebraic spinors, conceptually following Cartan's geometrical approach, and varying and generalizing a construction given by Bautista *at al.* (1994). Spinors are defined as elements of braided exterior algebras over certain isotropic subspaces of the initial vector space. The spinor space is a left Clifford module. We shall prove that (under certain assumptions concerning the braid) the spinor representation is irreducible and faithful, as in the classical theory.

2. The Woronowicz algebras

Woronowicz introduced in 1989 exterior algebras for an arbitrary braid operator. In this section we collect the main properties of the Woronowicz algebras. Let W be a (complex) finite-dimensional vector space, and let $\sigma: W \otimes W \to W \otimes W$ be a bijective map satisfying the braid equation

$$(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id}) = (\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma). \tag{1}$$

Let $A: W^{\otimes} \to W^{\otimes}$ be the total antisymetrizer map. Its components $A_n: W^{\otimes n} \to W^{\otimes n}$ are given by

$$A_n = \sum_{\pi \in S_n} (-1)^{\pi} \sigma_{\pi}$$

where $\sigma_{\pi}: W^{\otimes n} \to W^{\otimes n}$ are maps obtained by replacing transpositions figuring in a minimal decomposition of π by the corresponding σ-twists. The following identities hold

$$A_{n+k} = (A_n \otimes A_k) A_{nk} \tag{2}$$
$$A_{n+k} = B_{nk}(A_n \otimes A_k) \tag{3}$$

where

$$A_{nk} = \sum_{\pi \in S_{nk}} (-1)^{\pi} \sigma_{\pi^{-1}}$$
$$B_{nk} = \sum_{\pi \in S_{nk}} (-1)^{\pi} \sigma_{\pi}$$

and $S_{nk} \subseteq S_{n+k}$ is the set of permutations preserving the order of sets $\{1, \ldots, n\}$ and $\{n+1, \ldots, n+k\}$.

The Woronowicz algebra (braided exterior algebra) $W^\wedge$ is the factoralgebra of the tensor algebra $W^\otimes$ modulo the ideal $\ker A$ (Woronowicz 1989).

The algebra $W^\wedge$ can be naturally realized as a subspace $\mathrm{im} A$ in $W^\otimes$. This realization is given by

$$\Big[\psi + \ker A\Big] \leftrightarrow A\psi. \tag{4}$$

In terms of the above identification the exterior product is given by

$$\psi \wedge \varphi = B_{nk}(\psi \otimes \varphi), \quad \text{for } \psi \in W^{\wedge n} \text{ and } \varphi \in W^{\wedge k}. \tag{5}$$

In general $\ker A$ and $\mathrm{im} A$ are not mutually complementary subspaces and $W^\wedge$ is generally not a quadratic algebra, although in the case when σ is a Hecke braiding $W^\wedge$ is quadratic and A_n are projectors, up to scalar factors (modulo some singular cases, giving $A_n^2 = 0$). Moreover, it is possible to construct examples with the trivial second-order constraint $\ker A_2$, and non-trivial higher-order constraints.

3. Inner Products

For each $f \in W^*$ and $\xi \in W^{\wedge n}$, let $f \sqcup \xi \in W^{\wedge n-1}$ be an element given by

$$f \sqcup \xi = (f \otimes \mathrm{id}^{n-1})(\xi). \tag{6}$$

In the above formula, it is assumed that $W^\wedge$ is embedded in $W^\otimes$ (as described in the previous section). The fact that $f \sqcup \xi$ belongs to $W^\wedge$ easily follows from (2).

In such a way we have constructed a map $\sqcup : W^* \otimes W^\wedge \to W^\wedge$ (a counterpart of the standard contraction operation). For each $f \in W^*$ we shall denote by $\sqcup_f : W^\wedge \to W^\wedge$ the corresponding contraction map.

We will assume that σ is naturally extended to a braiding on $W^\wedge \otimes W^\wedge$, by requiring

$$\sigma(m \otimes \mathrm{id}) = (\mathrm{id} \otimes m)(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma) \tag{7}$$
$$\sigma(\mathrm{id} \otimes m) = (m \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id}), \tag{8}$$

where $m : W^\wedge \otimes W^\wedge \to W^\wedge$ is the product map.

LEMMA 1. *The following braided Leibniz rule holds*

$$\sqcup_f(\xi\eta) = \sqcup_f(\xi)\eta + (-1)^{\partial\xi} m\sigma^{-1}(\sqcup_f \otimes \mathrm{id})\sigma(\xi \otimes \eta). \ \square$$

4. Braided Clifford Algebras

Let $F : W \otimes W \to \mathbf{C}$ be a scalar product on a space W. Let $\ell^F : W \to W^*$ be a map given by

$$[\ell^F(x)](y) = F(x, y).$$

Let $\iota^F: W \times W^\wedge \to W^\wedge$ be a contraction map given by

$$\iota_x^F \xi = \ell^F(x) \sqcup \xi.$$

In what follows it will be assumed that F and σ are mutually related such that the following "functoriality property" holds

$$(F \otimes \mathrm{id})(\mathrm{id} \otimes \sigma) = (\mathrm{id} \otimes F)(\sigma \otimes \mathrm{id}). \tag{9}$$

Then the contraction operator ι^F satisfies the following braided variant of the Leibniz rule

$$\iota_x^F(\vartheta\eta) = \iota_x^F(\vartheta)\eta + (-1)^{\partial\vartheta} \sum_k \vartheta_k \iota_{x_k}^F(\eta)$$

where $\sum_k \vartheta_k \otimes x_k = \sigma(x \otimes \vartheta)$.

We can trivially extend the introduced contraction operator, to the map of the form $\iota^F: W^\otimes \times W^\wedge \to W^\wedge$ such that

$$\iota_{u\otimes v}^F = \iota_u^F \iota_v^F$$

for each $u, v \in W^\otimes$.

LEMMA 2. *If* $u \in \ker A$ *then* $\iota_u^F = 0$. □

Therefore, we can pass from $W^\otimes$ to $W^\wedge$ in the first argument of ι^F. In such a way we obtain a contraction map of the form $\iota^F: W^\wedge \times W^\wedge \to W^\wedge$ (we use the same symbol for different contraction maps, because the domain is clear from the context).

Let us define "relative" contraction operators $\langle\,\rangle_k: W^\wedge \times W^\wedge \to W^\wedge$ as follows

$$\langle \zeta, \xi \rangle_k = \sum_j \psi_j \wedge (\iota_{\varphi_j}^F \xi)$$

Here, it is assumed that $\zeta \in W^{\wedge n}$ and $\sum_j \psi_j \wedge \varphi_j = [A_{n-k\,k}\tilde{\zeta}]^\wedge$, where $\tilde{\zeta} \in W^{\otimes n}$ satisfies $[\tilde{\zeta}]^\wedge = \zeta$, and $\varphi_j \in W^{\wedge k}$. Consistency of this definition follows from (2). If $n < k$ we define $\langle\,\rangle_k = 0$.

Now, we can define a *new product* on $W^\wedge$, in the spirit of classical Chevalley's construction. This product is defined by the following formula

$$\psi \vee \varphi = \psi \wedge \varphi + \sum_{k \geq 1} \langle \psi, \varphi \rangle_k.$$

In particular for $x \in W$,

$$x \vee \psi = x \wedge \psi + \iota_x^F(\psi).$$

THEOREM 3. *Endowed with* $\vee$, *the space* $W^\wedge$ *becomes a unital associative algebra, with the unity* $1 \in W^\wedge$. □

Definition. The algebra $c\ell(W) = (W^\wedge, \vee)$ is called *the braided Clifford algebra* (associated to $\{\sigma, F\}$).

The constructed algebra can be understood as a deformation of the exterior algebra $W^\wedge$. The graded algebra associated to the filtered algebra $c\ell(W)$ naturally coincides with $W^\wedge$. The exterior algebra $W^\wedge$ is in fact a special case of the constructed Clifford algebra, when $F = 0$.

The algebra $c\ell(W)$ can be viewed as a factoralgebra $c\ell(W) = W^\otimes / J_F$, where J_F is the kernel of the canonical epimorphism $j_F\colon W^\otimes \to c\ell(W)$ extending the identity map on W. Now, we shall describe this ideal in an independent way, using a generalization of the construction presented by Crumeyrolle (1990).

A linear map $\lambda_F\colon W^\otimes \to W^\otimes$ defined by

$$\lambda_F(1) = 1 \qquad \lambda_F(x \otimes \vartheta) = x \otimes \lambda_F(\vartheta) + \iota_x^F \lambda_F(\vartheta), \quad x \in W, \quad \vartheta \in W^\otimes$$

is said to be the Crumeyrolle map. In the above formula, ι_x^F is considered as a braided antiderivation on $W^\otimes$. The Crumeyrolle map λ_F is bijective. Let $\vee$ be a new product in $W^\otimes$, given by

$$\vartheta \vee \eta = \lambda_F(\lambda_F^{-1}(\vartheta) \otimes \lambda_F^{-1}(\eta)).$$

By construction the space $\ker A$ is a left ideal in $W^\otimes$, relative to this new product. Condition (9) ensures that $\ker A$ is also a right $\vee$-ideal.

THEOREM 4. *We have* $(W^\otimes, \vee)/\ker A = c\ell(W)$. □

In other words, the factorization map $[\,]^\wedge\colon W^\otimes \to W^\wedge$ is also a homomorphism of corresponding deformed algebras. The map λ_F is a braided counterpart of the map introduced by Crumeyrolle (1990).

LEMMA 5. *We have* $\lambda_F^{-1}[\ker A] = J_F$. □

5. The Spinor Representation

This section is devoted to a braided generalization of classical Cartan theory of spinors (Cartan 1938). Let us assume that the space W is splitted into a direct sum

$$W = W_1 \oplus W_2$$

where W_1, W_2 are F-isotropic subspaces. Furthermore, let us assume that this decomposition is compatible with the braiding σ in the following way

$$\sigma(W_i \otimes W_j) = W_j \otimes W_i \tag{10}$$

$$\sigma^2 \Big| \Big\{(W_1 \otimes W_2) \oplus (W_2 \otimes W_1)\Big\} = \mathrm{id}. \tag{11}$$

Finally, it will be assumed that $(F|W_1 \otimes W_2) = 0$ and that $F|W_2 \otimes W_1$ is *nondegenerate*. In this case, $W_2 = W_1^*$, in a natural manner. The duality is given by $F(f, x) = f(x)$, for $f \in W_2$ and $x \in W_1$.

Exterior algebras $W_1^\wedge$ and $W_2^\wedge$ understandable as subalgebras of $c\ell(W)$, in a natural manner.

LEMMA 6. *The map* $\mu: W_1^\wedge \otimes W_2^\wedge \to c\ell(W)$ *defined by*

$$\mu(u \otimes v) = uv$$

is bijective. □

The corresponding "spinor space" can be defined as follows. Let us consider the space $\mathcal{K} = W_2^\wedge$, and let $\kappa: \mathcal{K} \to \mathbf{C}$ be a natural character, specified by $\kappa(1) = 1$ and $\kappa(W_2) = \{0\}$. This gives a left $\mathcal{K}$-module structure on the number field $\mathbf{C}$. On the other hand, $c\ell(W)$ is a right $\mathcal{K}$-module, in a natural manner. Let $\mathcal{S}$ be a left $c\ell(W)$-module, given by

$$\mathcal{S} = c\ell(W) \otimes_{\mathcal{K}} \mathbf{C}.$$

According to Lemma 6, the space $\mathcal{S}$ is naturally identificable with the exterior algebra $W_1^\wedge$. In terms of this identification, we have

$$x\xi = x_1 \wedge \xi + x_2 \sqcup \xi,$$

for each $x \in W$, where $x = x_1 + x_2$ and $x_i \in W_i$. In other words, a complete analogy with the classical Cartan formalism holds.

THEOREM 7. *The algebra* $c\ell(W)$ *acts on* $\mathcal{S}$ *faithfully and irreducibly.* □

The module $\mathcal{S}$ is completely characterized by the existence of a cyclic vector (the unit element $1_{\mathcal{S}}$), killed by the space W_2.

In other words let $\mathcal{V}$ be an arbitrary (left) $c\ell(W)$-module, possesing a vector v satisfying $\{W_2\}v = \{0\}$. Then there exists the unique module map $\varrho: \mathcal{S} \to \mathcal{V}$ satisfying $\varrho(1_{\mathcal{S}}) = v$. The map ϱ is injective (because of the simplicity of $\mathcal{S}$). In particular, if v is cyclic then ϱ is a module isomorphism.

6. Concluding Remarks

If the braid operator σ is such that $\ker A$ is quadratic, then the ideal J_F is generated by elements of the form

$$Q = \psi - F(\psi)1 \otimes 1 \tag{12}$$

where $\psi \in W^{\otimes 2}$ is σ-invariant. This covers Clifford algebras based on Hecke braidings, and in particular includes classical Weyl algebras (Oziewicz 1994).

Quantum Clifford algebras and spinors (for a Hecke braiding) introduced and analyzed by (Bautista *at al.*, 1994) can be included in the theory presented here. Clifford algebras introduced in the mentioned paper are based on Hecke braidings $\tau: V \otimes V \to V \otimes V$ (where V is a finite-dimensional vector space) admitting extensions to all possible braidings between V and V^*, so that the contraction map is functorial, in the standard sense. Then $W = V \oplus V^*$ and the corresponding scalar product F and the braiding σ are expressible in terms of the extended braiding τ and the contraction map.

The construction of the Crumeyrolle map λ_F works for an arbitrary F and in particular, it is independent of functoriality-type assumptions (9). For a possibility to define braided Clifford algebras as deformations of braided exterior algebras, it is sufficient to assume that $\ker A$ is also a right-ideal in $(W^{\otimes}, \vee)$. This assumption is weaker then (9). However, if (9) does not hold, then the symmetry between left and right is broken.

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CHAPTER VI

DIFFERENTIAL EQUATIONS & GEOMETRY

DIFFERENCE SCHRÖDINGER OPERATORS WITH THE FIXED SYMMETRY PROPERTIES

VYACHESLAV SPIRIDONOV*, LUC VINET
Centre de Recherches Mathématiques, C.P. 6128-A, Montréal, Québec, H3C 3J7, Canada

and

ALEXEI ZHEDANOV
Physics Department, Donetsk University, Donetsk, 340055 Ukraine

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Abstract. Using the factorization method we construct finite-difference Schrödinger operators (Jacobi matrices) whose discrete spectra are composed from independent arithmetic, or geometric series. Such systems originate from the periodic, or q-periodic closure of a chain of corresponding Darboux transformations. The Charlier, Krawtchouk, Meixner orthogonal polynomials, their q-analogs, and some other classical polynomials appear as the simplest examples for $N = 1$ and $N = 2$ (N is the period of closure). A natural generalization involves discrete versions of the Painlevé transcendents.

Spectral problems of the Sturm-Liouville type have many applications in physics. Quantum mechanics and the theory of solitons are essentially based on the spectral analysis of Schrödinger and Dirac operators. The one-dimensional finite-difference Schrödinger equation

$$L\psi(x) \equiv a(x+1)\psi(x+1) + a(x)\psi(x-1) + b(x)\psi(x) = \lambda\psi(x), \qquad (1)$$

which is the main object of investigation in this paper, may be interpreted either as an equation determining harmonic oscillation frequencies of a non-homogeneous discrete string, or as an energy eigenvalue problem for a particle moving along some non-uniform lattice (tight binding model). Alternatively, equation (1) may be considered as an auxiliary spectral problem helping to integrate the Toda chain equations of motion. The latter system is known to be integrable through the inverse scattering method in the periodic case $a(x+m) = a(x)$, $b(x+m) = b(x)$, or for fast decreasing potentials $a(x) \to 1$, $b(x) \to 0$, $x \to \pm\infty$. The operator L is tridiagonal and said to be in Jacobi form. In the following we do not assume any particular physical interpretation of (1) due to the universal character of this equation.

* On leave of absence from the Institute for Nuclear Research, Moscow, Russia.

Replacing $x \pm 1$ in (1) by $x \pm h$ and taking the zero lattice spacing limit $h \to 0$ one gets a continuous Sturm-Liouville problem:

$$h^2(a_0(x)\psi'(x))' + (a(x+h) + a(x) + b(x) - \lambda)\psi(x) + O(h^3) = 0,$$

where prime denotes the derivative d/dx and $a_0(x)$ is the leading asymptotic term of $a(x), a(x) = a_0(x)(1 + O(h))$. If $a_0 = constant$ and the asymptotic expansions of $a(x)$ and $b(x)$ properly match then one obtains in the continuous limit the standard Schrödinger equation: $-\psi''(x) + u(x)\psi(x) = \tilde{\lambda}\psi(x)$, which was studied in great detail. In particular, the special technique based on the factorization of Hamiltonians was developed in order to simplify the solution of such spectral problems [4]. In the theory of solitons it is known as the dressing method. It has proven to be very powerful and it provides a guide to the classification of exactly solvable potentials. This method has been used recently to describe one-dimensional potentials with discrete spectra composed from N arithmetic, or geometric series [18, 13, 14, 15, 16]. The first class of potentials is related to the Painlevé nonlinear ordinary differential equations [18], the second one is connected with infinite-soliton solutions of the Korteweg-de Vries equation [13, 15], quantum algebras [14, 15], and q-deformed Painlevé transcendents [16]. In this paper we construct difference Schrödinger operators obeying analogous properties with the help of a discretized version of the same technique. The simplest systems that are found are related to the Charlier, Krawtchouk, Meixner orthogonal polynomials of a discrete variable, or to their q-analogs. The Stieltjes-Wigert and continuous q-Hermite polynomials are also incorporated into the scheme. More complicated systems are related to discrete versions of the Painlevé transcendents. Note that the factorization of finite-difference equations has been considered in [9, 10, 1, 2, 3]. Our approach differs in that we are not describing the symmetries of known systems but determining rather, whole classes of systems with fixed symmetry properties.

The equation (1) needs to be supplemented with boundary conditions. Let Γ_c be the coordinate lattice, i.e. the set of discrete points x on which $a(x)$, $b(x)$ are defined, and Γ_s the spectral parameter lattice, i.e. the set of indices of λ in the eigenvalue problem $L\psi_n = \lambda_n\psi_n$, $n \in \Gamma_s$. We shall deal with systems for which Γ_s consists in a number of discrete points and, possibly, a continuous part extending from some point to infinity. The lattice Γ_c may lie on a finite interval, or half line, or cover the whole real line. In the first two cases it is convenient to take $x = 0$ as the left edge of Γ_c. For finite Γ_c, the standard boundary conditions are

$$a(0)\psi(-1) = 0, \quad \psi(0) \neq 0, \quad a(x_{max})\psi(x_{max}) = 0.$$

When Γ_c extends from 0 to infinity, the boundary condition at zero is the same and in addition $\psi(x)$ is required to be bounded. Note that if the edges of Γ_c are determined by two successive zeros of $a(x)$ then one only needs $\psi(x)$ to be finite at those points. If Γ_c covers the whole line then there is only the requirement that $\psi(x)$ be bounded.

When Γ_c has a boundary, the equation (1) provides a recurrence relation that defines $\psi_\lambda(x)$ as a set of orthogonal polynomials of order x in the argument λ. For some problems, the opposite situation can also take place: $\psi_n(x) = p_n(z(x))\psi_0(x)$, $n \in \Gamma_s$, where $p_n(z)$ are orthogonal polynomials of the argument $z(x)$ and order n. The latter interpretation of the formulas is similar to the quantum mechanical one. For problems with purely discrete spectra, when $\psi(x) \in l_2(\Gamma_c)$, the orthogonality and completeness relations look as follows:

$$\sum_{x\in\Gamma_c} \psi_n^*(x)\psi_m(x) = \delta_{nm}, \qquad \sum_{n\in\Gamma_s} \psi_n^*(x)\psi_n(y) = \delta_{xy}, \tag{2}$$

where $\delta_{xy} = 0$ if $x \neq y$ and 1 otherwise. If $\Gamma_c = \Gamma_s$ and x and λ enter in a symmetric fashion we get a so-called self-dual system.

Let us now take a set of equations, all of the form (1):

$$L_j\psi^{(j)}(x) = \lambda\psi^{(j)}(x), \qquad j \in \mathbf{Z}, \tag{3}$$

where

$$L_j = a_j(x+1)T^+ + a_j(x)T^- + b_j(x), \qquad T^\pm\psi(x) = \psi(x \pm 1). \tag{4}$$

The coordinate x is assumed to be real, so that the operators L_j are formally hermitian.

Consider the factorization of (4)

$$L_j = A_j^+ A_j^- + \lambda_j, \tag{5}$$

where

$$A_j^+ = p_j(x)T^- + f_j(x), \qquad A_j^- = p_j(x+1)T^+ + f_j(x). \tag{6}$$

These operators are assumed to be hermitian conjugates one of the other, $(A_j^\pm)^\dagger = A_j^\mp$. From (4)-(6), one finds the relation between the "potentials" $a_j(x)$, $b_j(x)$ and the "superpotentials" $p_j(x), f_j(x)$:

$$a_j(x) = p_j(x)f_j(x-1), \qquad b_j(x) = p_j^2(x) + f_j^2(x) + \lambda_j. \tag{7}$$

It is well known that the discrete spectra of the operators A^+A^- and A^-A^+ can differ only by the lowest eigenvalue when $A^\pm$ are first order differential or difference operators. This observation plays a crucial role in the factorization method since it allows to construct new solvable spectral problems from known ones. We impose the following condition:

$$A_j^- A_j^+ + \lambda_j = (-1)^{\sigma_j}(A_{j+1}^+ A_{j+1}^- + \lambda_{j+1}), \tag{8}$$

relating L_j and L_{j+1}. The eigenfunctions of the Hamiltonians L_j are now said to be related by Darboux transformations. The chain of equations (8) differs from the one used in [4, 18, 13, 14, 15, 16] by the presence of the sign-factors $(-1)^{\sigma_j}$. (They are absent in the continuous case because of the special structure of $A_j^{\pm}$-operators). These signs can not be removed by a renormalization of the variables with index $j+1$ and they represent a unique feature of the above systems.

The algebraic relations arising from the chain (8) are of prime importance because they are independent of any particular realization. Let us introduce the operators M_j:

$$M_j^+ = A_j^+ A_{j+1}^+ \dots A_{j+N-1}^+, \qquad M_j^- = (M_j^+)^\dagger, \tag{9}$$

where N is some positive integer. One can check easily the identities

$$\begin{aligned} L_j M_j^+ &= (-1)^{s_j} M_j^+ L_{j+N}, \qquad s_j = \sum_{k=0}^{N-1} \sigma_{j+k} \\ M_j^- L_j &= (-1)^{s_j} L_{j+N} M_j^-. \end{aligned} \tag{10}$$

These equations show that the operators $M_j^{\pm}$ map the eigenfunctions of the operators L_j and L_{j+N} onto each other. The structure relations are completed by

$$\begin{aligned} M_j^+ M_j^- &= \prod_{k=0}^{N-1} ((-1)^{s_j - s_{jk}} L_j - \lambda_{j+k}), \qquad s_{jk} = \sum_{l=k}^{N-1} \sigma_{j+l}, \\ M_j^- M_j^+ &= \prod_{k=0}^{N-1} ((-1)^{s_{jk}} L_{j+N} - \lambda_{j+k}). \end{aligned} \tag{11}$$

Now suppose a closure condition (see below) relating L_j and L_{j+N} is imposed. From (10) and (11) we see that L_j and $M_j^{\pm}$ would then generate a polynomial algebra generalizing $sl(2)$ or its q-analog. The role of polynomial algebras as dynamical symmetry algebras is discussed in a different context in [7].

Substituting (6) in (8), we derive the following two-dimensional discrete dressing chain ($\mu_j \equiv (-1)^{\sigma_j} \lambda_{j+1} - \lambda_j$):

$$p_j(x) f_j(x) = (-1)^{\sigma_j} p_{j+1}(x) f_{j+1}(x-1), \tag{12}$$

$$p_j^2(x+1) + f_j^2(x) = (-1)^{\sigma_j} (p_{j+1}^2(x) + f_{j+1}^2(x)) + \mu_j. \tag{13}$$

A system of equations analogous to the one presented above was used in [9, 10] to provide a Lie-algebraic interpretation of some solvable finite-difference equations. Let us stress that our approach is more general because we do not restrict ourselves to simple Lie-algebras, or their q-analogs but allow rather, for generalizations in the form of polynomial algebras of arbitrary order.

The basic idea for obtaining Schrödinger operators with linear and exponential spectra is the following. As already mentioned, the operators $M_j^{\pm}$ intertwine L_j and L_{j+N}, hence if these Hamiltonians are related to each other as follows [15, 16]:

$$L_{j+N} = qUL_jU^{-1} + \lambda_{j+N} - q\lambda_j, \tag{14}$$

with q an arbitrary positive parameter and U a unitary operator, the combinations

$$B_j^+ \equiv M_j^+U, \qquad B_j^- \equiv U^{-1}M_j^-$$

become symmetry generators for the Hamiltonian L_j. The operator U plays a very important role in our considerations because it allows to generate an infinite amount of systems with given spectral properties.

Setting $B^{\pm} \equiv B_1^{\pm}$, $\omega \equiv \lambda_{N+1} - q\lambda_1$, $H \equiv L_1, s_1 = \sum_{l=1}^N \sigma_l = 0$ and substituting (14) in (10) and (11) we get the following dynamical symmetry algebra:

$$HB^+ - qB^+H = \omega B^+, \qquad B^-H - qHB^- = \omega B^-, \tag{15}$$

$$B^+B^- = (-1)^S \prod_{k=1}^N (H - \tau_k), \tag{16}$$

$$B^-B^+ = (-1)^S \prod_{k=1}^N (qH + \omega - \tau_k), \tag{17}$$

where

$$\tau_k = (-1)^{\sum_{l=k}^N \sigma_l}\lambda_k, \qquad S = \sum_{k=1}^N \sum_{l=k}^N \sigma_l.$$

The relations (15)-(17) clearly define a spectrum generating algebra. This algebra generalizes the one obtained in [16] owing to the sign factors that appear when $\sigma_j \neq 0$. Note that for $N = 1$, we recover the q-oscillator algebra, while for $N = 2$, we get the q-analogs of the $su(1,1)$ and $su(2)$ algebras for $S = 0$ and $S = 1$ respectively.

Suppose that $\tau_k < \tau_{k+1}$, $k = 1, \ldots, N$, then the equation $B^-\psi_0^{(k)} = 0$ defines a set of N "vacuum" states with energies equal to τ_k. This is of course a formal conclusion because one needs to check that all these states satisfy the boundary conditions. Acting with the "creation" operator B^+ upon these vacua, $\psi_m^{(k)} = (B^+)^m\psi_0^{(k)}$, one generates all physical bound states. For $q = 1$, the spectrum of H will consist of N independent arithmetic (equidistant) series with ω the step between two successive members of one series, for $0 < q < 1$, the spectrum has a discrete part composed from N geometric series with accumulation point $\lambda_1 - \omega/(1-q)$ and continuous part starting at that point and going up to infinity, for $q>1$, the spectrum is purely discrete and grows exponentially. Note that the latter possibility is excluded in the case of differential Schrödinger operators [14].

The unitary operator U may be taken as a general element of the unitary transformations of the line, here we will choose it to be the shift operator $U\equiv T_\delta^+$, $T_\delta^\pm\psi(x)=\psi(x\pm\delta)$, where δ is an arbitrary real parameter. The closure equation (14) is then equivalent to the following conditions:

$$\begin{aligned} p_{j+N}(x) &= \sqrt{q}\,p_j(x+\delta), \qquad f_{j+N}(x)=\sqrt{q}f_j(x+\delta),\\ \mu_{j+N} &= q\,\mu_j, \qquad \sigma_{j+N}=\sigma_j. \end{aligned} \tag{18}$$

Requiring δ to be commensurable with the original lattice size "1", i.e. δ to be a rational number, we get a system of difference equations to which standard integration techniques apply.

We have analyzed the integrability of the discrete dressing chain equations under the closure (18) for the simplest choices of N and δ. The results are presented below.

For $N=2$ we have found only one value of δ, namely $\delta=1$, for which the system is integrable in terms of elementary functions. The corresponding equations explicitly read:

$$\begin{aligned} p_1(x)f_1(x) &= \pm\,p_2(x)f_2(x-1),\\ p_2(x)f_2(x) &= \pm\,q\,p_1(x+1)f_1(x), \end{aligned} \tag{19}$$

$$\begin{aligned} p_1^2(x+1)+f_1^2(x) &= \pm\,(p_2^2(x)+f_2^2(x))+\mu_1,\\ p_2^2(x+1)+f_2^2(x) &= \pm\,q\,(p_1^2(x+1)+f_1^2(x+1))+\mu_2. \end{aligned} \tag{20}$$

Letting $F_j\equiv f_j^2, P_j\equiv p_j^2$, we have the solution:

$$P_1(x)=\frac{\mu_1\pm\mu_2+\gamma^2(\mu_2\pm\mu_1 q)q^{2x-1}+cq^x}{(1-q)(1\mp\gamma^2q^{2x-1})(1\mp\gamma^2q^{2x})}, \tag{21}$$

$$F_1(x)=\gamma^2q^{2x}\,\frac{\mu_2\pm\mu_1 q+\gamma^2(\mu_1\pm\mu_2)q^{2x+1}\pm cq^x}{(1-q)(1\mp\gamma^2q^{2x+1})(1\mp\gamma^2q^{2x})}, \tag{22}$$

$$F_2(x)=\gamma^2q^{2(x+1)}P_1(x+1), \qquad P_2(x)=\gamma^{-2}q^{-2x}F_1(x), \tag{23}$$

where γ^2 and c are two integration constants. In fact, γ^2 and c can be arbitrary periodic functions with period 1 but we shall not consider further such a possibility. For generic values of the parameters $\mu_{1,2},\gamma^2$ in (21)-(23) the constant c is determined from the requirement $a(0)=0$, implying that Γ_c lies either on the half line or on a finite interval. When the upper signs are taken, the wavefunction $\psi(x)$ is seen to involve q-analogs of the Meixner polynomials and the spectrum generating algebra is a q-analog of $su(1,1)$ (these results were also derived in [19] with the help of a different method).

In the case when the lower signs are taken, we have $su_q(2)$ as symmetry algebra and the formulae (21)-(23) yield Stanton's q-analogs of the Krawtchouk polynomials which can be expressed in terms of the ${}_3\varphi_2$ basic hypergeometric series [12]. In the limit $q \to 1$, we recover the classical Meixner and Krawtchouk polynomials of a discrete variable. An algebraic interpretation and some physical applications of these polynomials can be found in [8, 5].

A particular subcase of (21)-(23) with the upper signs, corresponds to the q-oscillator algebra. Indeed, substituting $q \to q^2$, $\mu_2 \to q\mu_1$, we get the solution for the $N = 1$, $\delta = 1/2$ system:

$$\begin{aligned} P_1(x) &= \frac{\mu_1(1+\gamma^2 q^{4x-1}) + cq^{2x}(1+q)^{-1}}{(1-q)(1-\gamma^2 q^{4x-2})(1-\gamma^2 q^{4x})}, \\ F_1(x) &= \gamma^2 q^{4x+1} P_1(x+\tfrac{1}{2}). \end{aligned} \tag{24}$$

Now there exists a special choice of the constant c, namely $c = -\mu_1\gamma(1+q)^2/q$, such that the singularities in the denominators of $P_1(x)$ and $F_1(x)$ cancel with the zeros of the numerators. The potentials $a(x)$ and $b(x)$ are then defined on the whole line (i.e. $\Gamma_c = \mathbf{Z}$) and lead to the so-called continuous q-Hermite polynomials whose relation to the q-oscillator algebra was discussed recently in [3].

The representation theory of the symmetry algebra (15)-(17) will characterize the Hilbert space of wave functions provided the operators $H, B^{\pm}$ are well defined on $l_2(\Gamma_c)$. Let us discuss this on the example of the ordinary Meixner polynomials, i.e. the case $N = 2$, $q = 1$, $\delta = 1$, for which $\Gamma_c = \mathbf{N}$. Formally, the equation $B^-\psi^{(k)} = 0$ has two independent solutions corresponding to the two roots ν of $C = \nu(\nu - 1)$, where C is some fixed eigenvalue of the $su(1,1)$ Casimir operator. Namely,

$$\psi^{(1)}(x) \propto \alpha^x \sqrt{\frac{\Gamma(2\nu + x)}{\Gamma(x+1)}}, \qquad \psi^{(2)}(x) \propto \alpha^x \sqrt{\frac{\Gamma(x+1)}{\Gamma(2\nu + x)}}, \tag{25}$$

where α and ν are combinations of the parameters entering in (21)-(23). The Hamiltonian H is self-adjoint for the boundary condition: $\psi(-1) = g\psi(0)$, where g is some real constant. However it is easy to check that the operator B^+ is conjugate to B^- only on states for which $a(0)\psi(-1) = 0$ (it is assumed that $\psi(0)$ is always finite). The first state in (25) satisfies this condition, but for the second one we have $\lim_{\epsilon\to 0} a(\epsilon)\psi^{(2)}(\epsilon - 1) \neq 0$ and the function $\psi^{(2)}$ should thus be discarded. So we conclude that although we have a $N = 2$ closure, the physical spectrum consists only of one arithmetic series. This result should be contrasted with the situation that prevails for a continuous oscillator with a $1/x^2$ singular potential which also possesses a $su(1,1)$ dynamical symmetry algebra but where a range of parameters exists when both spectral series are physical.

The most simple system emerges for $N = 1$, $\delta = 0$:

$$F_1(x) = \gamma^2 q^{2x}, \qquad P_1(x) = (1-q)^{-1}(\mu_1 + \gamma^2(1-q)q^{2x-1} + cq^x). \quad (26)$$

Here one has two possibilities. When $P_1(x)$ does not have zeros as a function of the continuous argument x, then $\Gamma_c = \mathbf{Z}$, otherwise the parameter c is fixed by the requirement $P_1(0) = 0$ and then $\Gamma_c = \mathbf{N}$. The latter case corresponds to the q-Charlier polynomials.

The $N = 2$, $\delta = 0$ system of equations admits only one integral for $q \neq 1$. When $q = 1$, a second integral is found and we have:

$$F_2(x) = \frac{\gamma^2}{F_1(x)}, \qquad P_2(x) = \mu_\pm x + c \mp P_1(x),$$

$$P_1(x) = \frac{\gamma^2(\mu_\pm x + c)}{F_1(x)F_1(x-1) \pm \gamma^2},$$

and the following equation:

$$\frac{\gamma^2(\mu_\pm x + c)}{F_1(x)F_1(x-1) \pm \gamma^2} + \frac{\gamma^2(\mu_\pm (x+1) + c)}{F_1(x)F_1(x+1) \pm \gamma^2} = \mu_1 \pm (\mu_\pm x + c + \frac{\gamma^2}{F_1(x)}) - F_1(x), \quad (27)$$

where $\mu_\pm = \mu_2 \pm \mu_1$ and γ^2 and c are constants of integration. It has not proven possible to integrate (27) further. However, if one sets $\mu_\pm = 0$, the order of (27) can then be lowered:

$$I = \frac{\gamma^2 c}{(F_1(x)F_1(x+1) \pm \gamma^2)^2} + \frac{F_1(x) + F_1(x+1) - \mu_1 \mp c}{F_1(x)F_1(x+1) \pm \gamma^2} = const.$$

The general solution of this equation can be written in terms of elliptic functions. Note that for $\mu_\pm = 0$, the operators $B^\pm$ commute with the Hamiltonian, i.e. they are integrals of motion.

An interesting system emerges for $N = 1$, $\delta = 1/3$:

$$P_1(y) = \gamma^2 q^{-2y} F_1(y-1)F_1(y-2), \quad (28)$$

$$\gamma^2 q^{-2(y+3)}(F_1(y+2)F_1(y+1) - q^2 F_1(y)F_1(y-1)) + F_1(y) - qF_1(y+1) = \mu_1,$$

where $y = 3x$. The latter equation is not integrable, but when $q = 1$ it admits one additional integral that leads to

$$\gamma^2 F(y)(F(y-1) + F(y+1) - \gamma^{-2}) = \mu y + c, \quad (29)$$

where $F \equiv F_1$, $\mu \equiv \mu_1$. This equation is very close to the discrete Painlevé-I (PI) transcendent considered in [6] and in the continuous limit $h \to 0$,

$$F(y) = \frac{1 - 3h^2 u(\xi)}{4\gamma^2}, \quad \xi = \frac{h}{\mu}(\mu y + c + \frac{1}{8\gamma^2}), \qquad \mu \to -\frac{3h^5}{16\gamma^2}, \quad (30)$$

it goes to the standard PI ordinary differential equation: $d^2u(\xi)/d\xi^2 = 6u^2(\xi) + \xi$.

For $N = 1$, $\delta = 2$, one has

$$F_1(x) = \gamma^2\beta(x)\beta(x+1), \qquad P_1(x) = q^{-x}\beta^{-1}(x), \tag{31}$$

where $\beta(x)$ is defined by the following equation ($\mu \equiv \mu_1 > 0$):

$$q^{-x}\left(\frac{1}{\beta(x)} - \frac{1}{\beta(x+1)}\right) + \gamma^2 q(\beta(x-1)\beta(x) - q\beta(x+1)\beta(x+2)) = \mu. \tag{32}$$

We were only able to find one elementary solution of this equation, namely, $\beta^2 = \mu/\gamma^2 q(1-q) = constant$. It happens to lead to the Stieltjes-Wigert polynomials which were shown in [2] to be related to the q-oscillator algebra. Note that this solution disappears when $q \to 1$ if we keep the lattice size finite. Indeed, for $q = 1$, the order of the equation (32) can be lowered:

$$\beta(x-1) + \beta(x+1) = -\frac{1 + (\mu x + c)\beta(x)}{\gamma^2\beta^2(x)}, \tag{33}$$

where c is an integration constant. This is a special case of the discrete PII equation considered in [11] and it does not admit constant solutions. Since $\beta(x) > 0$, the lattice Γ_c has to be semi-infinite, extending from $-\infty$ to some point. If $\mu = 0$, then (33) is integrable in terms of elliptic functions.

It is natural to expect that for the higher orders of periodic closure N, the discrete versions of the higher Painlevé equations (and their q-analogs) are emerging. E.g. the $N = 3$, $\delta = 1$, $\sigma_j = 0$ system reduces under the special choice of parameters to the $N = 1$, $\delta = 1/3$ case, i.e. to the discrete PI, and this looks similar to the coalescence procedure.

To sum up, we briefly outlined a symmetry approach to the linear second order finite-difference equation (1) and derived explicitly a class of Schrödinger operators with one series of equidistant, or exponential discrete eigenvalues. They arose from the (q-)periodic closures of a chain of Darboux transformations. For the simplest periods of closure $N = 1, 2$, we encountered many classical orthogonal polynomials of a discrete variable. Although we did not succeed in finding Hamiltonians with at least two independent series in physical spectrum, we do not see any fundamental reason forbidding this. An interesting fact consists in the appearance of the discrete Painlevé equations (and their "q-analogs") within the context of spectral problems on lattices. This observation is parallel to the one made in [18, 16] for the continuous Schrödinger equation although a precise correspondence has not been established yet. Note that the discrete versions of PI found in this work differ from those considered in [6, 11] and that q-deformation of the Painlevé maps (i.e. transition from $q = 1$ to $q \neq 1$) raises their order by one (in the continuous case such a procedure leads to the differential-difference nonlinear equations).

There are many interesting aspects of the equation (1) that were not discussed here. One of them is related to the supersymmetric structures underlying the chain (8). Another problem is the Hamiltonian formulation of the derived mappings and the general analysis of their integrability properties (for a review see, e.g., [17]). The factorization (dressing) method may be formulated for the difference equations when the operators $T^{\pm}$ effect shifts along the imaginary axis, $T^{\pm}\psi(x) = \psi(x \pm i)$. This opens a new direction for further generalizations of solvable difference Schrödinger equations with the simple symmetry algebras described in the present Letter. All these questions and the problem of classification of solutions of the discrete dressing chain are worth of separate consideration.

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SUPERSYMMETRY AND TOPOLOGICAL EFFECTS IN QUANTUM MECHANICAL SYSTEMS

ALLINA A. SUZKO
Joint Institute for Nuclear Research, Dubna
Head Post Office, Box 79, Moscow, Russia
and
Institute of Radiation Physical and Chemical Problems
Byelorussian Academy of Sciences, Minsk

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Abstract. The problems of the supersymmetric extension for the gauge equations connected with introducing and changing gauge scalar and vector potentials and geometric phases are investigated. A direct generalization of the supersymmetric quantum mechanics by Witten for gauge equations in two-dimensional space with additional scalar potentials is given. The influence of scalar potentials on the degeneracy of the ground state, and as a result on topological effects, is studied.

1. Supersymmetry of gauge equations.

In the adiabatic approach to nonrelativistic quantum mechanics the systems of gauge equations appear and, as a result, there are opportunities for prediction and explanations of Aharonov–Bom, Hall and geometric phases phenomena. Some topological effects take place in the presence of the supersymmetry.

Induced gauge potentials appear naturally in the description of quantum-mechanical systems dependent upon slowly varying external parameters and upon fast varying intrinsic ones. This occurs in many real systems, there are fast and slow degrees of freedom and one should estimate the effect of the slow dynamic on the behaviour of the fast ones and vice versa. In this case the total Hamiltonian H is decomposed into

$$H = H^s \otimes I + H^f, \tag{1}$$

where $H^f(\mathbf{R})$ is the parametric family of the "fast" Hamiltonians depending on slow variables $\mathbf{R}$. The searched wave function of H is expressed as a sum over the eigenfunctions $\Phi_n(\mathbf{R};\mathbf{r})$ of the instanteneous Hamiltonian H^f for each fixed value of the slow variable $\mathbf{R}$

$$|\Psi(\mathbf{R},\mathbf{r})>=|n><n|\Psi>=\sum_n \int \Phi_n(\mathbf{R};\mathbf{r})F_n(\mathbf{R}). \tag{2}$$

We use the orthonormalization $< n \mid m >= \delta_{nm}$ and completeness $\mid n >< n \mid= 1\delta(\mathbf{r}-\mathbf{r}')$ of the eigenstates $|\Phi_n(\mathbf{R};\mathbf{r})>$ of the Hamiltonian H^f at fixed $\mathbf{R}$ and get the "slow" system of equations of the gauge type (3) for the expansion coefficients $\{F_n\} = F$

$$-1/2[\nabla \otimes \mathbf{I} - i\mathbf{A}(\mathbf{R})]^2 F(\mathbf{R}) + \mathcal{E}(\mathbf{R})F(\mathbf{R}) = EF(\mathbf{R}). \tag{3}$$

Here

$$A_{nm}(\mathbf{R}) =< \Phi_n(\mathbf{R}) \mid i\nabla_{\mathbf{R}} \mid \Phi_m(\mathbf{R}) > \qquad \text{and} \tag{4}$$

$$U_{nm}(\mathbf{R}) =< \Phi_n(\mathbf{R}) \mid H^f(\mathbf{R}) \mid \Phi_m(\mathbf{R}) >= \mathcal{E}_n(\mathbf{R})\delta_{nm} \tag{5}$$

act as operator-values the vector and scalar matrix components of the gauge field, F is a column-vector of dimension $M, \mathbf{I}$ is the unit matrix. In the one-state approximation (Born-Oppengeimer approximation) $F(\mathbf{R})$ becomes a scalar wave function and Eq.(3) is an ordinary gauge equation.

As is well known from vector analysis, if the curl of the vector potential vanishes at all $\mathbf{R}$, $\mathbf{B} = \nabla \times \mathbf{A} = 0$, we can eliminate the gauge vector potential by a phase transformation. In the adiabatic representation, vanishing of the matrix tensor $R_{\mu\nu} = \partial_\mu A^\nu - \partial_\nu A^\mu - ig[A^\nu, A^\mu]$ is equivalent to this requirement, because we have non-Abelian gauge fields. Note, more interesting effects take place when $R_{\mu\nu} \neq 0$. These phenomena are connected with Berry's opening of geometric phases in simple quantum systems [1]. Berry demostrated the existence of magnetic monopole fields in dynamical systems, which arise naturally in a gauge theory framework [2]

$$\delta = \frac{1}{2} i \oint_C \mathbf{A} \cdot d\mathbf{R} = i \int\int_S \mathbf{B} \cdot d\mathbf{S} \neq 0. \tag{6}$$

It takes place when we have the supersymmetry, then $\int\int_S \mathbf{B} \cdot d\mathbf{S} \neq 0$, or when level crossing or quasicrossing take place between two terms, then the vector potential $\mathbf{A}(\mathbf{R})$ is singular and closed loop gives a non-zero result. Then the S–matrix of geometric phase factors [3], [4,5] is defined as follows

$$S_{nm} = \exp Im(\oint \mathbf{A}_{nm}(\mathbf{R}) \cdot d\mathbf{R}) = \tag{7}$$

$$= \exp \pi i \sum_{j\neq n,m}^{N} Res \frac{< \Phi_n|\nabla_{\mathbf{R}} H^f(\mathbf{R})|\Phi_j >}{\mathcal{E}_n(\mathbf{R}) - \mathcal{E}_j(\mathbf{R})} \times \frac{< \Phi_j|\nabla_{\mathbf{R}} H^f(\mathbf{R})|\Phi_m >}{\mathcal{E}_j(\mathbf{R}) - \mathcal{E}_m(\mathbf{R})}.$$

It is easy to see, in this case there is an opportunity for three terms to cross at one point. Then, at $n = m$ we obtain the Berry relations for geometric phases.

Let us generalize the approach has been considered by Aharonov and Casher [6]. They studied the problem of a spin–1/2 charged particle moving in a plane under the influence of a perpendicular magnetic field. Aharonov and Casher showed the total magnetic flux is

$$\int\int \mathbf{B}(x,y)dxdy \equiv \Phi = 2\pi(N+\varepsilon), \quad 0<\varepsilon<1.$$

They proved the following two theorems:
1. *If* $(N+\varepsilon) > 1$ *the Pauli Hamiltonian has exactly* $N-1$ *zero–energy normalizable eigenstates whose spin has the same sign as the flux.*
2. *All nonzero energy eigen–states are degenerate with respect to spin flip.*
After that there are opportunities for the Hall quantum effects and fractial statistic.

In fact, their proof is based on the supersymmety of the Pauli Hamiltonian $(\hbar = c = m = 1)$

$$H^P = 1/2[-i\nabla - e\mathbf{A}]^2 - \frac{e}{2}\boldsymbol{\sigma}\cdot\mathbf{B}. \tag{8}$$

It can be written as the square of the Dirac Hamiltonian

$$H^P = 1/2(H^D)^2; \qquad H^D = (\boldsymbol{\sigma}\cdot\boldsymbol{\pi}) = \boldsymbol{\sigma}\cdot(\mathbf{p}-e\mathbf{A}) \tag{9}$$

and H^D is supercharge. In adiabatic representation we have the eq·1ations (3) of the gauge type. But conditions allowing supersymmetry for (8) are special and strongly limited: the vector and scalar components of the gauge field ought to satisfy the principle of minimal coupling $\mathbf{B} = \nabla\times\mathbf{A} = V$. That is why it is natural desire to widen opportunities of the approach by introducing an additional scalar potential but also conserving supersymmetry. Is this possible and will the additional scalar potential influence the geometric phases and topological effects? As it turns out, the additional scalar potential, conserving supersymmetry, affects the degeneracy of the ground state and may lead to its increase and vice versa to its decrease up to a vanishing of the degenaracy and so on to topological effects. An action of such scalar potential is analogous to an action of the field strength tensor $\mathbf{B}$.

Now, we try to construct a model of a SUSY quantum mechanics in two-dimensional space with an additional scalar potential $W(x,y)$ with respect to the classic problem (8)–(9). First, we define the hermitian supercharges $Q_i \equiv Q_i(x,y)$ $(i=1,2)$, $Q_2 = i\sigma_3 Q_1$

$$\begin{aligned} Q_1 &= 1/2[\sigma_1(\pi_x - \partial_y W(x,y)) + \sigma_2(\pi_y + \partial_x W(x,y))], \\ Q_2 &= 1/2[\sigma_2(\pi_x - \partial_y W(x,y)) - \sigma_1(\pi_y + \partial_x W(x,y))] \end{aligned} \tag{10}$$

where $i\pi_\mu = D_\mu = \partial_\mu - iA_\mu$. It is a generalization of the definition of supercharges introduced by Witten [7] for the description of a particle moving in a line in one-dimensional space

$$Q_1 = 1/2[\sigma_1 p_x + \sigma_2 W(x)], \quad Q_2 = 1/2[\sigma_2 p_x - \sigma_1 W(x)]. \tag{11}$$

Now, as usual, we introduce the non-hermitian supercharges

$$\bar{\mathbf{Q}}^+ = \frac{1}{\sqrt{2}}[-Q_2 + iQ_1], \qquad \bar{\mathbf{Q}}^- = \frac{1}{\sqrt{2}}[-Q_2 - iQ_1]. \tag{12}$$

They are represented as block two-by-two matrices

$$\bar{\mathbf{Q}}^+ = \frac{1}{\sqrt{2}}\tau_+[(i\pi_x + \partial_x W) + (\pi_y - i\partial_y W)],$$
$$\bar{\mathbf{Q}}^- = \frac{1}{\sqrt{2}}\tau_-[(-i\pi_x + \partial_x W) + (\pi_y + i\partial_y W)]. \tag{13}$$

They can be written as

$$\bar{\mathbf{Q}}^+ = \frac{1}{\sqrt{2}}\tau_+[Q_x^+ - iQ_y^+], \;\; \bar{\mathbf{Q}}^- = \frac{1}{\sqrt{2}}\tau_-[Q_x^- + iQ_y^-] \tag{14}$$

or

$$\bar{\mathbf{Q}}^+ = \frac{1}{\sqrt{2}}\tau_+[\Pi_x^+ - i\Pi_y^+], \;\; \bar{\mathbf{Q}}^- = \frac{1}{\sqrt{2}}\tau_-[\Pi_x^- + i\Pi_y^-] \tag{15}$$

where coordinate components of $Q_\mu^\pm$ are defined as [8]

$$Q_\mu^\pm = \pm D_\mu + \partial_\mu W, \tag{16}$$

and corresponding to (10) we define $\Pi_\mu^\pm$ another way as

$$\Pi_x^\pm = \pm D_x - i\partial_y W, \;\; \Pi_y^\pm = \pm D_y + i\partial_x W. \tag{17}$$

Here $\tau_\pm = \frac{1}{2}(\sigma_1 \pm \sigma_2)$, $\sigma_\mu(\mu = 1, 2, 3)$ are Pauli spin matrices. It is easy to examine that the supercharges Q_i (10) and $\bar{\mathbf{Q}}^\pm$ (14) or (15) satisfy the set of relations of the Witten supersymmetric quantum mechanics

$$\{Q_i, Q_j\} = \delta_{ij} H^s \text{ and } [H, Q_i] = 0, \tag{18}$$

and

$$\bar{\mathbf{Q}}^{+^2} = \bar{\mathbf{Q}}^{-^2} = [H^s, \bar{\mathbf{Q}}^\pm] = 0. \tag{19}$$

Using (14) or (15) we construct the supersymmetric Hamiltonian

$$H^s = 1/2\{\bar{\mathbf{Q}}^+, \bar{\mathbf{Q}}^-\} =$$

$$\frac{1}{2}\begin{pmatrix} (\mathbf{Q}^+ \cdot \mathbf{Q}^-) + i(\mathbf{Q}^+ \times \mathbf{Q}^-) & 0 \\ 0 & (\mathbf{Q}^- \cdot \mathbf{Q}^+) - i(\mathbf{Q}^- \times \mathbf{Q}^+) \end{pmatrix}. \tag{20}$$

It is evident, that similar equations are obtained with operators $\mathbf{\Pi}^\pm$ instead of $\mathbf{Q}^\pm$.

In the general case for a Dirac spin -1/2 particle in the extra magnetic field, four-component wave functions have to be introduced. We can make a generalization of relations (8)–(20) in two ways.

One of them follows the logic of choosing the supercharges Q_1 and Q_2 (11): in the definition of supercharge Q_1, σ–matrices are replaced by γ–matrices. Then $\bar{\mathbf{\Pi}}^+$ and $\bar{\mathbf{\Pi}}^-$ (13) are defined as:

$$\bar{\mathbf{\Pi}}^+ = \frac{1}{\sqrt{2}}\tau_+(\sigma\cdot\mathbf{\Pi}^+) = \frac{1}{\sqrt{2}}\tau_+\sum_\mu \sigma_\mu \Pi_\mu^+ ;$$
$$\bar{\mathbf{\Pi}}^- = \frac{1}{\sqrt{2}}\tau_-(\sigma\cdot\mathbf{\Pi}^-) = \frac{1}{\sqrt{2}}\tau_-\sum_\mu \sigma_\mu \Pi_\mu^- . \tag{21}$$

Another way [8] corresponds to direct use of definition (16)

$$\bar{\mathbf{Q}}^+ = \frac{1}{\sqrt{2}}\tau_+(\sigma\cdot\mathbf{Q}^+) = \frac{1}{\sqrt{2}}\tau_+\sum_\mu \sigma_\mu Q_\mu^+ ;$$
$$\bar{\mathbf{Q}}^- = \frac{1}{\sqrt{2}}\tau_-(\sigma\cdot\mathbf{Q}^-) = \frac{1}{\sqrt{2}}\tau_-\sum_\mu \sigma_\mu Q_\mu^- , \tag{22}$$

where $\bar{\mathbf{\Pi}}^\pm$ and $\bar{\mathbf{Q}}^\pm$ are represented as block two-by-two matrices combined from the two-by-two matrices generators of the supersymmetry $\mathbf{\Pi}^\pm$ and $\mathbf{Q}^\pm$. Then instead of the supersymmetric Hamiltonian (20) in the first case we have

$$H^s = \frac{1}{2}\begin{pmatrix} (\mathbf{\Pi}^+\cdot\mathbf{\Pi}^-) + i\sigma_3(\mathbf{\Pi}^+\times\mathbf{\Pi}^-) & 0 \\ 0 & (\mathbf{\Pi}^-\cdot\mathbf{\Pi}^+) + i\sigma_3(\mathbf{\Pi}^-\times\mathbf{\Pi}^+) \end{pmatrix} \tag{23}$$

and in the second case

$$H^s = \frac{1}{2}\begin{pmatrix} (\mathbf{Q}^+\cdot\mathbf{Q}^-) + i\sigma(\mathbf{Q}^+\times\mathbf{Q}^-) & 0 \\ 0 & (\mathbf{Q}^-\cdot\mathbf{Q}^+) + i\sigma(\mathbf{Q}^-\times\mathbf{Q}^+) \end{pmatrix}. \tag{24}$$

In addition to the terms of $H^+ = \frac{1}{2}Q^+Q^-$ and $H^- = \frac{1}{2}Q^-Q^+$ of the one-dimensional supersymmetric Hamiltonian in (23) and (24) the field strength tensor

$$F_{\mu\nu} = \partial_\mu A^\nu - \partial_\nu A^\mu = \epsilon^{\nu\mu k}B^k \tag{25}$$

arises, associated with a magnetic field, terms $(\boldsymbol{\pi}\cdot\nabla W)$ and in (24) terms $\sigma(\nabla W\times\boldsymbol{\pi})$ similar to spin-orbital couplings $(2/R)(dW/dR)(\sigma\mathbf{L})$ for the central fields. In our case $\mathbf{L}$ is a generalized orbital momentum, defined as $\mathbf{L} = \mathbf{r}\times\boldsymbol{\pi}$. Note, that only in the one-dimensional case is the supersymmetry completely defined scalar potential. At $W(x,y) = 0$ two supersymmetric partners H^+ and H^- coincide. Another situation takes place in many-dimensional space ($N \geq 2$). Here even at $W = 0$ there is a supersymmetry defined by the gauge vector potential A (precisely by the field strength tensor). In this case vector and scalar components of the gauge field are minimally coupled.

Do the obtained relations (23), (24) satisfy the principal of minimal coupling of the gauge fields ? If the gauge vector potential in (16), (23) and in (16), (24) is replaced by

$$A_x \rightarrow A_x + \partial_y W; \quad A_y \rightarrow A_y - \partial_x W;$$

$$A_\mu \rightarrow A_\mu + \partial_\mu W,$$

respectively, then the tensor (25) F_{xy} will be represented as

$$F_{xy} = \partial_x(A_y - \partial_x W) - \partial_y(A_x + \partial_y W) = B_{xy} \tag{26}$$

$$F_{xy} = \partial_x(A_y + \partial_y W) - \partial_y(A_x + \partial_x W) \tag{27}$$

and H^s (23), (24) with a noncentral field is rewritten in the form of the Pauli equation H^p with the following exchanges

$$\boldsymbol{\pi}^+ \rightarrow \bar{\boldsymbol{\Pi}}^+, \quad \boldsymbol{\pi}^- \rightarrow \bar{\boldsymbol{\Pi}}^-;$$

$$\boldsymbol{\pi}^+ \rightarrow \mathbf{Q}^+, \quad \boldsymbol{\pi}^- \rightarrow \mathbf{Q}^-.$$

Note in the first case this is not a gauge transformation, while in the second case it is a gauge transformation.

At the application of the first approach for 1/2 -spin charged particles the so-called spin-flip effect takes place with the simultaneously changing coordinate dependence of wave functions, when the generators $(\boldsymbol{\sigma} \cdot \boldsymbol{\Pi}^+)$ and $(\boldsymbol{\sigma} \cdot \boldsymbol{\Pi}^-)$ turn into superpartner states of each other

$$\chi_+ = (\boldsymbol{\sigma} \cdot \boldsymbol{\Pi}^+)\chi_- \text{ and } \chi_- = (\boldsymbol{\sigma} \cdot \boldsymbol{\Pi}^-)\chi_+.$$

This is the second result of Aharonov and Casher obtained here for the case whith an additional scalar potential. If $\chi_{-\sigma_3}$ are eigenstates of the H^- Hamiltonian (23) then the eigenstates of H^+ are

$$\chi_{+\sigma_3} = i[(\sigma_1(\pi_x - \partial_y W) + \sigma_2(\pi_y + \partial_x W)]\chi_{-\sigma_3}. \tag{28}$$

A zero eigenstate $\chi_o = \chi_{-\sigma_3}$ of H^- Hamiltonian must be annihilated by $(\boldsymbol{\sigma} \cdot \boldsymbol{\Pi}^+)$

$$(\boldsymbol{\sigma} \cdot \boldsymbol{\Pi}^+)\chi_o = 0.$$

This relation after multiplication from the left by σ_2 is rewritten as

$$\{\sigma_3[\partial_x - i(A_x + \partial_y W)] + (i\partial_y + A_y - \partial_x W)\}\chi_o = 0. \tag{29}$$

One can easily see that in the adiabatic representation for (3) we arrive at the well-known situation stated by Aharonov and Casher [6] in the case of zero scalar potential W. The ground state of the Hamiltonian H (23) is degenerated. The number of zero energy states is governed by the Atiah-Singer index theorem

$$\chi_o(x,y) = f(x,y)P\exp\{-\sigma_3\int\int B(x,y)dx\,dy\}, \tag{30}$$

where B is defined by (26)

$$B_{xy} = F_{xy} - (\partial_x + i\sigma_3\partial_y)(\partial_x W(x,y) - i\sigma_3\partial_y W(x,y))$$

and the function $f(x,y)$ satisfies

$$(\partial_x + i\sigma_3\partial_y)f(x,y) = 0.$$

Hence in the presence of $W(x,y)$, the function $f(x,y)$ is an entire function of $(x + i\sigma_3 y)$ as before with $W = 0$. As a result we have a degeneracy of the ground state modes which are defined by

$$\chi_{o_j} = (x + i\sigma_3 y)^j \exp\{-\sigma_3\int\int B_{xy}(x,y)dx\,dy\}, \quad (j = 1,\ldots,N-1). \tag{31}$$

The number N defines the degenerate multiplicity of zero-energy states being governed by the Atiah-Singer index theorem. It relates to the number of zero modes of a particle moving in a external gauge field with the topological number N defined as the surface integral from the tensor B_{xy}

$$\int\int B(x,y)dx\,dy = 2\pi(N+\epsilon)\ , \quad 0 < \epsilon < 1. \tag{32}$$

If $\epsilon = 0$, the flux defined by relations (32) and (26) is quantized and one can speak about the Hall effect and nonstandard statistics in nonrelativistic systems (for example three-body ones). It is now trivial to see from (30) and (26) that the presence of a scalar potential can lead to an increase and vice versa to a decrease and even to the cancellation of a positive integer N, and as result, removing the degeneracy of the ground state, i.e. to lose conditions for topological effects.

But the fact of the presence or the absence of the degeneracy due to the supersymmetry does not eliminate other geometric phases related to the transfers between levels (see Eq.(7)) referred to as the nonadiabatic Aharonov–Anandan phases [3].

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COHOMOLOGY AND SPECTRAL SEQUENCES IN GAUGE THEORY

M. A. AGUILAR
Instituto de Matemáticas, Universidad Nacional Autónoma de México, Cd. Universitaria, 04510 México D.F.

J. M. LÓPEZ-ROMERO
Departamento de Física, Centro de Investigación y Estudios Avanzados del IPN, Apdo. Postal 14-740, 07000 México D.F.

and

MIGUEL SOCOLOVSKY
Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Circuito Exterior C.U., A.P. 70543, 04510 México, D.F.

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Abstract. We define a double Chevalley-Eilenberg complex associated with the classical Becchi-Rouet-Stora-Tyutin symmetry of a gauge theory in the context of differential geometry, and describe the use of spectral sequences for the calculation of the corresponding cohomology groups.

Key words: gauge theory, spectral sequences, Lie algebra cohomology

1. Introduction

As is well know several years ago Bonora and Cotta-Ramusino [1] introduced a geometrical picture of the BRST [2] symmetry of a gauge theory, based on the concept of Lie algebra cohomology first introduced in the finite dimensional case by Chevalley and Eilenberg (CE) [3]. They considered the principal fiber bundle ξ where the gauge theory in question is defined and they generalized to the infinite dimensional case the CE theory. The generator of the symmetry could be identified with the coboundary operator associated with the representation of the Lie algebra of the gauge group with coefficients in the zero-forms on the space of connections; thus the symmetry (at least its generator) is not a property of a particular gauge theory but of the principal fiber bundle (which is a global geometrical concept) where the fields are defined. The local version of this cohomology was also shown in reference [1] to be related to the quantum anomalies of the theory; this point was studied in more detail in reference [4] and more recently by Dixon [5] and Dubois-Viollete *et al* [6].

In this note we extend the original BRST complex of a principal fiber bundle to a doble complex which in turn induces a total cochain complex: the *total BRST complex of a principal fiber bundle.* Double complexes appear when one considers representations with coefficients in forms of arbitrary order and uses the commutativity of the exterior and Lie derivatives. Though we have not made an explicit calculation of the relevant cohomology groups for any particular case, we present general arguments that show how spectral sequences [9] can be used to compute these groups. We think that the investigation of the meaning of the additional cohomology groups has both physical and geometrical interest, and leave it as a subject for further research.

2. Connections on principal bundles

2.1. PRINCIPAL BUNDLES AND CONNECTIONS

The starting point is a smooth (C^∞) principal fiber bundle principal fiber bundle $\xi : G \to P \overset{\pi}{\to} B$ and the space of connections on it, $\mathcal{C}(\xi)$.

The *gauge group of* ξ, $\mathcal{G}(\xi)$, is the group of vertical automorphisms of ξ *i.e.* the set of C^∞ functions $P \overset{f}{\to} P$ such that the following diagram commutes:

$$\begin{array}{ccccc} P \times G & & \overset{f \times id}{\longrightarrow} & & P \times G \\ \psi \downarrow & & & & \downarrow \psi \\ P & & \overset{f}{\longrightarrow} & & P \\ & \pi \searrow & & \swarrow \pi & \\ & & B & & \end{array}$$

where ψ is the right free action of G on P. $\mathcal{G}(\xi)$ acts on $\mathcal{C}(\xi)$ through pull-backs *i.e.* if $\omega \in \Gamma(T^*P \otimes \mathbf{g})$ ($\mathbf{g} = LieG$) is a connection on ξ then $\omega' = f^*\omega$ gives the *gauge transformed* connection. This is the action that induces the BRST cohomology as we shall see below. We emphasize that gauge transformations are global and that it is only when one restricts to local trivializations and considers the pull-backs $A_U = \sigma_U^* \omega_U$ with local sections σ_U on open subsets U of the base space that one gets the familiar local gauge transformations used in physics. One has the quotient (moduli) space $\eta := \mathcal{C}/\mathcal{G}$, however the projection $\mathcal{C} \to \eta$ is not a principal fiber bundle and typically one restricts $\mathcal{G}$ to $\bar{\mathcal{G}} := \mathcal{G}/z$, where z is the center of $\mathcal{G}$ and $\mathcal{C}$ to $\mathcal{C}'$, the space of irreducible connections [10].

$\mathcal{C}(\xi)$ is an affine space modelled on $A^1(\xi) = \Gamma(T^*B \otimes E)$ where E is the bundle of Lie algebras of ξ (see below); once an arbitrary but fixed connection ω_0 is chosen in $\mathcal{C}(\xi)$ (base point) $\mathcal{C}(\xi) = \mathcal{C}^0(\xi)$ becomes an infinite dimensional real vector space (with origin in ω_0) and then an infinite dimensional differentiable manifold which is contractible to a point and therefore has vanishing cohomology groups except in dimension zero. We nevertheless expect that the *double complex* to be defined later will contain non-trivial information concerning any quantum gauge theory defined on ξ.

2.2. Associated bundles

Associated with ξ, there exist two *canonically* associated bundles: $\xi_G : G \to P \times_G G = F \overset{\pi_G}{\to} B$: the bundle of Lie groups of ξ and $\xi_{\mathbf{g}} : \mathbf{g} \to P \times_G \mathbf{g} = E \overset{\pi_{\mathbf{g}}}{\to} B$: the bundle of Lie algebras of ξ, where one has the left *adjoint action* of G on G and on $\mathbf{g}$ respectively. It can be shown that $\mathcal{G}(\xi) \cong \Gamma(F)$ and then $Lie\mathcal{G}(\xi) \cong \Gamma(E)$. We are interested in the Chevalley- Eilenberg [3] cohomology of $Lie\mathcal{G}(\xi)$ with coefficients in the differential forms on $\mathcal{C}(\xi)$, however it should be remarked here that the original definition of Lie algebra cohomology in reference [3] was for finite dimensional Lie algebras, while $Lie\mathcal{G}(\xi)$ is infinite dimensional.

2.3. Vector spaces of sections, covariant derivative and exterior covariant derivative

For $p = 0, 1, \ldots, dimB$ one defines the vector spaces $A^p = \Omega^p(E) := \Gamma(\Lambda^p T^* B \otimes E)$ of p-differential forms on B with values in E *i.e.* if $X_1, \ldots, X_p \in Vect(B)$ and $s \in A^p$, then $s(X_1, \ldots, X_p) \in \Gamma(E)$ (so $\Omega^p(E) \cong \Omega^p(B) \otimes_{C^\infty(B,R)} \Gamma(E)$); in particular $A^0 = \Gamma(E)$. There is a bijection between $\Gamma(E)$ and the set of equivariant maps $\gamma : P \to \mathbf{g}$, *i.e.*, smooth maps such that $\gamma(pg) = g^{-1}\gamma(p)$, defined as follows. If $s \in \Gamma(E)$ then $\gamma_s : P \to \mathbf{g}$ is given by $\gamma_s(p) = v$, where $s(b) = [p, v]$; if $\gamma : P \to \mathbf{g}$, then $s_\gamma \in \Gamma(E)$ is given by $s_\gamma(b) = [p, \gamma(p)]$, where $p \in \pi^{-1}(b)$. If $s \in \Gamma(E)$, $X \in Vect(B)$, and $\tilde{X} \in Vect(P)$ is the lifting of X by a connection ω, then the *covariant derivative, of s with respect to ω in the direction of* X, is defined by $\nabla^\omega_X s := s_{\tilde{X}(\gamma_s)}$, where $\tilde{X}(\gamma_s) := d\gamma_s(\tilde{X})$. In general if $f : P \to \mathbf{g}$ is equivariant, $Y \in Vect(P)$ and $\omega \in \mathcal{C}(\xi)$ then the covariant derivative of f with respect to ω in the direction Y is defined as $Df(Y) := df(horY)$, hence $\tilde{X}(\gamma_s) = D\gamma_s(\tilde{X})$. We define $d^\omega : A^0 \to A^1$ by $d^\omega s(X) := \nabla^\omega_X s$ and $\nabla^\omega : Vect(B) \times \Gamma(E) \to \Gamma(E)$ by $\nabla^\omega(X, s) := \nabla^\omega_X s$. The operator ∇^ω is a linear connection on E, *i.e.*, ∇^ω is $C^\infty(B, \mathbf{R})$- linear with respect to X but satisfies the Leibnitz rule $\nabla^\omega(X, fs) = f\nabla^\omega(X, s) + X(f)s$ with respect to s. The *covariant differential (linear) operator* $d^\omega : A^0 \to A^1$, extends to the *exterior covariant derivative (linear) operator* $\{\mathcal{D}^\omega_p\}_{p=0}^{dimB}$ in the same way as the De Rham exterior derivative extends the ordinary differential, namely $\mathcal{D}^\omega_0 = d^\omega$ and for $1 \leq p \leq dimB$, $\mathcal{D}^\omega_p(\alpha)(X_1, \ldots, X_{p+1}) = \sum_{i=1}^{p+1}(-1)^{i+1}\nabla^\omega_{X_i}(\alpha(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1}))+$
$\sum_{1\leq i<j\leq p+1}(-1)^{i+j}\alpha([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1})$. In general

$A^0 \xrightarrow{d^\omega} A^1 \xrightarrow{\mathcal{D}_1^\omega} A^2 \xrightarrow{\mathcal{D}_2^\omega} \ldots \xrightarrow{\mathcal{D}_{n-1}^\omega} A^n \to 0$ fails to be a complex *i.e.* $\mathcal{D}_{p+1}^\omega \circ \mathcal{D}_p^\omega \neq 0$, unless ω is *flat i.e.* $\mathcal{R}^\omega = 0$ where $\mathcal{R}^\omega : Vect(B) \times Vect(B) \to End(\Gamma(E))$, $\mathcal{R}^\omega(X,Y)(s) := (\nabla_X^\omega \circ \nabla_Y^\omega - \nabla_Y^\omega \circ \nabla_X^\omega - \nabla_{[X,Y]}^\omega)(s)$ is the *curvature* of ∇^ω; so $\mathcal{R}^\omega \in \Gamma(\Lambda^2 T^*B \otimes Hom(E,E)) = \Omega^2(Hom(E,E))$ is an obstruction to have such a complex.

Assuming a compact, connected and 1-connected Lie group G and a compact and orientable base space B, we can define a positive definite non degenerate inner product on each of the A^p's, namely $< \alpha, \beta >_p := - \int_B tr(\alpha \wedge * \beta)$ [11], thus the A^p's become pre-Hilbert spaces; however they are not in general complete and to have Hilbert spaces one needs to specify a completion. Notice that the inner products induce norms $\| \alpha \|_p^2 := < \alpha, \alpha >_p$, so that the A^p's are normed vector spaces in a natural way.

2.4. Sobolev completions

Given $\omega \in \mathcal{C}(\xi)$ and ∇^ω the associated linear connection on $\xi_{\mathfrak{g}}$, the *Sobolev k-norm* on $\Omega^p(E)$ is defined by [8] $\| \phi \|_{p,k}^2 := \| \phi \|_p^2 + \| \mathcal{D}_p^\omega \phi \|_{p+1}^2 + \ldots + \| \mathcal{D}_{p+k-1}^\omega \circ \ldots \circ \mathcal{D}_p^\omega \phi \|_{p+k}^2$. Different connections are equivalent in the sense that they lead to equivalent norms *i.e.* to topologically isomorphic vector spaces. The completion of $\Omega^p(E)$ with respect to this norm *i.e.* the set of formal limits of all Cauchy sequences in $\Omega^p(E)$ is called the *Sobolev k-norm completion* of $\Omega^p(E)$ and is denoted by $\Omega_k^p(E)$. So in particular $(Lie\mathcal{G}(\xi))_k = \Omega_k^0(E)$ and $\mathcal{C}_k^0(\xi) \cong \Omega_k^1(E)$. The Sobolev completion of $\mathcal{G}(\xi)$ is made through the following considerations. Let E_1, E_2 be smooth K-vector bundles over X ($K = \mathbf{R}$ or $\mathbf{C}$), then we have an isomorphism μ: $E_1^* \otimes E_2 \to Hom_K(E_1, E_2)$, given by $\mu(\alpha \otimes \omega)(v) = \alpha(v)\omega$. In particular for $E_1 = E_2 = V$, one has an isomorphism of vector bundles $V^* \otimes_K V \xrightarrow{\cong} Hom_K(V,V) \equiv End_K(V)$. A smooth map $h : V \to V$ is called *bundle map* if: *i)* $\pi \circ h = \pi$, where $\pi : V \to X$ is the projection; *ii)* $h|V_x : V_x \to V_x$ is linear, where $V_x = \pi^{-1}(x)$. We denote by $Map_X(V)$ the set of bundle maps. There is a bijection $\Psi : Map_X(V) \to \Gamma(End_K(V))$, given by $\Psi(h)(x) = h|V_x$, and with inverse $\Psi^{-1}(s)(v) = s(\pi(v))(v)$. There is a monoid structure on $Map_X(V)$ given by composition of bundle maps and a monoid structure on $\Gamma(End_K(V))$ given by $(s_1 \cdot s_2)(x) = s_1(x) \circ s_2(x)$. Clearly Ψ is a monoid isomorphism.

Let $K^n \to V = P \times_G K^n \to X$ be a vector bundle over X associated with the principal fiber bundle ξ: $G \to P \xrightarrow{\pi} X$, and a linear representation of G, $G \times K^n \to K^n$, then it can be easily verified that $\rho : \mathcal{G}(\xi) \to Map_X(V)$, given by $\rho(f)([p, \vec{x}]) := [f(p), \vec{x}]$ is a monoid monomorphism; therefore one has the following composition

$$\mathcal{G}(\xi) \xrightarrow{\rho} Map_X(V) \xrightarrow{\Psi} \Gamma(End_K(V)).$$

Clearly if $s = \Psi \circ \rho(f)$ *i.e.* if $s \in im(\Psi \circ \rho)$ then s has an inverse $s^{-1} = \Psi \circ \rho(f^{-1})$, so $\mathcal{G}(\xi)$ is isomorphic (as a group) to its image $\Psi \circ \rho(\mathcal{G}(\xi)) \hookrightarrow \Gamma(End_K(V))$. Let $<\,,\,>_x$ be an inner product in V_x, then the associated inner product in V_x^*, $<\alpha,\beta>_{*x} = <\lambda^{-1}(\alpha),\lambda^{-1}(\beta)>_x$, where $\lambda : V_x \to V_x^*$ is the vector space isomorphism $\lambda(v)(v') = <v,v'>_x$, induces the inner product $<\,,\,>_{\otimes x}$ in $V_x^* \otimes_K V_x$ as the linear extension of $<\alpha \otimes_K v, \alpha' \otimes_K v'>_{\otimes x} = <\alpha,\alpha'>_{*x} <v,v'>_x$. Thus one has the inner product $<\,,\,>_{End_x}$ in $End_K(V_x)$ given by $<\phi_{1x},\phi_{2x}>_{End_x} = <\mu_x^{-1}(\phi_{1x}),\mu_x^{-1}(\phi_{2x})>_{\otimes x}$.
Let $K = \mathbf{R}$, $V = E$ and $X = B$; for each $p = 0,1,...,dimB$ the inner product in $\Omega^p(E)$ defines an inner product in $\Omega^p(End_{\mathbf{R}}(E)) = \Gamma(\Lambda^p T^*B \otimes End_{\mathbf{R}}(E))$ for which one defines a Sobolev k-norm analogous to that defined for $\Omega^p(E)$; in particular *the Sobolev k-norm completion of* $\mathcal{G}(\xi)$, denoted by $\mathcal{G}(\xi)_k$, is the closure of $\mathcal{G}(\xi)$ it with respect to the Sobolev k-norm completion of $\Gamma(End_{\mathbf{R}}(P \times_G \mathbf{g}))$[8].

3. Cohomology of Lie algebras

3.1. Chevalley-Eilenberg cohomology

Let $\mathbf{g}$ and V be a finite dimensional Lie algebra and a finite dimensional vector space respectively, both over the field K ($K = \mathbf{R}$ or $\mathbf{C}$), and let $\phi : \mathbf{g} \to End_K(V)$ be a representation of $\mathbf{g}$ on V. Define $C^0 := V$ and $C^p := \{\alpha : \mathbf{g} \times \ldots \times \mathbf{g}(p\ times) \to V,\ \alpha\ multilinear\ alternating\}$ for $p = 1,2,\ldots$. Define K-linear operators $\delta^p : C^p \to C^{p+1}$, by $\delta^p(\alpha)(a_1,\ldots,a_{p+1}) := \sum_{i=1}^{p+1}(-1)^{i+1}\phi(a_i)(\alpha(a_1,\ldots,\hat{a}_i,\ldots,a_{p+1})) +$
$\sum_{1\le i<j\le p+1}(-1)^{i+j}\alpha([a_i,a_j],\ldots,\hat{a}_i,\ldots,\hat{a}_j,\ldots,a_{p+1})$. One can show [3] that $\delta^{p+1} \circ \delta^p = 0$, therefore one has a *cochain complex* $0 \to C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} C^2 \xrightarrow{\delta^2} \ldots$ with p-cocycles $Z^p = ker\delta^p$ and p-coboundaries $B^p = im\delta^{p-1} \subset Z^p$. One defines the *Chevalley-Eilenberg cohomology of* $\mathbf{g}$ *with respect to the representation* ϕ *of* $\mathbf{g}$ *on* V ("with coefficients in V") as the graded group $H^*_{CE}(\mathbf{g},\phi,V;K) := \{H^p_{CE}(\mathbf{g},\phi,V;K)\}_{p=0}^{\infty}$, where $H^p_{CE}(\mathbf{g},\phi,V;K) := Z^p/B^p$.

3.2. Double complexes and total cohomology

A *double cochain complex* is a triple (C,∂,d), where $C = \{C^{p,q}\}, p,q = 0,1,2,\ldots$is a set of abelian groups, and $d = \{d^{p,q} : C^{p,q} \to C^{p,q+1}\}$ and $\partial = \{\partial^{p,q} : C^{p,q} \to C^{p+1,q}\}$ are *differentials* *i.e.* group homomorphisms satisfying $d^2 = \partial^2 = 0$, such that the following diagrams are commutative:

$$\begin{array}{ccc} C^{p,q} & \xrightarrow{d^{p,q}} & C^{p,q+1} \\ \partial^{p,q}\downarrow & & \downarrow\partial^{p,q+1} \\ C^{p+1,q} & \xrightarrow{d^{p+1,q}} & C^{p+1,q+1} \end{array}$$

A double cochain complex (C, ∂, d) naturally induces a (simple) cochain complex (K^*, D) as follows: for $n = 0, 1, 2, \ldots$ one defines the abelian groups $K^n := \oplus_{p+q=n} C^{p,q}$ and the operators $D^n := \oplus_{p+q=n}(\partial^{p,q} \oplus (-1)^p$ $d^{p,q})$. (K^*, D) is usually called the *total (cochain) complex* associated with the double (cochain) complex (C, ∂, d) [7]. The cohomology of (K^*, D), namely $H^*(K^*, D) := \{H^n(K^*, D)\}_{p=0}^{\infty}$ with $H^n(K^*, D) := kerD^n/imD^{n-1}$ is called the *total cohomology of the double (cochain) complex* (C, ∂, d).

3.3. Group actions and double complexes

Proposition: Let G be a Lie group, M a C^∞ manifold and $M \times G \xrightarrow{\psi} M$, $\psi(x, g) = xg$ a smooth action of G on M. Then there exists a double cochain complex involving $\mathbf{g} = LieG$ and $\Omega(M)$, the differential forms on M.

Proof:

i) $A \in \mathbf{g}$ induces the fundamental field of A in M, $A^* \in Vect(M)$ defined by $A^*_x(f) := d/dt(f(xexptA))\,|_{t=0}$.

ii) The Lie derivative of a tensor τ on M with respect to A^* is given by $\mathcal{L}_{A^*}\tau := d/dt(\Phi_t^*\tau)\,|_{t=0}$ where Φ_t is the flow of A^* and $\Phi_t^*\tau$ is the pull-back of τ by Φ_t, in particular this is valid for forms of arbitrary order and $\mathcal{L}_{A^*}\alpha \in \Omega^p M$ if $\alpha \in \Omega^p M$.

iii) Fix p in the set $\{0, 1, \ldots, n = dimM\}$ and define the infinite set of vector spaces $C_p^0 := \Omega^p M$ and $C_p^\nu := \{\mathbf{g} \times \ldots \times \mathbf{g}(\nu\ times) \xrightarrow{\alpha} \Omega^p M, \alpha\ alternating\}$, for $\nu = 1, 2, \ldots$.

iv) Since $\mathcal{L}_{[A,B]^*} = [\mathcal{L}_{A^*}, \mathcal{L}_{B^*}], \phi_p : \mathbf{g} \to End_{\mathbf{R}}(\Omega^p M)$, given by $\phi_p(A) := \mathcal{L}_{A^*}$ is a representation of $\mathbf{g}$ on $\Omega^p M$, and then $C_p^0 \xrightarrow{\delta_p^0} C_p^1 \xrightarrow{\delta_p^1} C_p^2 \xrightarrow{\delta_p^2} \ldots$ is a cochain complex with coboundary $\{\delta_p^\nu\}_{\nu=0}^{\infty}$, defined by $\delta_p^\nu(\alpha)(A_0, \ldots, A_\nu) =$
$\sum_{i=0}^{\nu}(-1)^i \mathcal{L}_{A_i^*}(\alpha(A_0, \ldots, \hat{A}_i, \ldots, A_\nu$
$)) + \sum_{0 \le i < j \le \nu}(-1)^{i+j}\alpha([A_i, A_j], A_0, \ldots, \hat{A}_i, \ldots, \hat{A}_j, \ldots, A_\nu)$. Therefore one has the Chevalley- Eilenberg cohomology of $\mathbf{g}$ with coefficients in $\Omega^p M$, $H^*_{CE}(\mathbf{g}, \phi_p, \Omega^p M; \mathbf{R}) = \{H^\nu_{CE}(\mathbf{g}, \phi_p, \Omega^p M; \mathbf{R})\}_{\nu=0}^{\infty}$.

v) Let $d_p^\nu : C_p^\nu \to C_{p+1}^\nu$ be defined by $d_p^\nu(\alpha)(A_1, \ldots, A_\nu) = d(\alpha$ $(A_1, \ldots, A_\nu))$, where d is the exterior derivative operator on M. Since $d\mathcal{L} = \mathcal{L}d$, we have a double cochain complex (C, δ, d) given by the following lattice of commuting diagrams:

$$
\begin{array}{ccccccccc}
C_0^0 & \xrightarrow{d_0^0} & C_1^0 & \xrightarrow{d_1^0} & C_2^0 & \xrightarrow{d_2^0} \ldots \xrightarrow{d_{n-2}^0} & C_{n-1}^0 & \xrightarrow{d_{n-1}^0} & C_n^0 \\
\delta_0^0 \downarrow & & \delta_1^0 \downarrow & & \delta_2^0 \downarrow & & \delta_{n-1}^0 \downarrow & & \delta_n^0 \downarrow \\
C_0^1 & \xrightarrow{d_0^1} & C_1^1 & \xrightarrow{d_1^1} & C_2^1 & \xrightarrow{d_2^1} \ldots \xrightarrow{d_{n-2}^1} & C_{n-1}^1 & \xrightarrow{d_{n-1}^1} & C_n^1 \\
\delta_0^1 \downarrow & & \delta_1^1 \downarrow & & \delta_2^1 \downarrow & & \delta_{n-1}^1 \downarrow & & \delta_n^1 \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

Therefore, the action $M \times G \xrightarrow{\psi} M$ has a naturally associated cohomology $H^*(K^*, D)$, namely that of the total (cochain) complex (K^*, D) associated with the double cochain complex $(\mathcal{C}, \delta, d)$ as specified in the previous subsection. *q.e.d.*

4. BRST cohomology

4.1. Total BRST cohomology of a principal fiber bundle

Using the results of the previous section and taking into account suitable Sobolev completions in the infinite dimensional case, one considers the case where $G = \mathcal{G}(\xi)$ and $M = \mathcal{C}(\xi)$; so for each $p = 0, 1, 2, \ldots$ one has the representation of the Lie algebra of the gauge group on the differential p-forms on $\mathcal{C}(\xi)$, $\phi_p : A^0 \to End_{\mathbf{R}}(\Omega^p \mathrm{C}(\xi))$, $\phi_p(s) := \mathcal{L}_{s^\bullet}$. It is usual to give to $\mathcal{C}(\xi)$ and A^0 respectively Sobolev k- and $(k+1)$- norm completions with $k \geq (dim B/2) + 1$ which guarantee that the action $\mathcal{C} \times \mathcal{G} \to \mathcal{C}$ extends to a smooth action $\mathcal{C}_k \times \mathcal{G}_k \to \mathcal{C}_k$ [8]. Notice however that this condition does not fix the value of k and so the results of the BRST cohomology might depend on it. For $\nu = 1, 2, \ldots$ one defines the spaces $C_p^\nu(\xi) := \{$*alternating continuous functions* $A^0 \times \ldots \times A^0(\nu$ *times*$) \xrightarrow{\alpha} \Omega^p \mathcal{C}(\xi)\}$, and $C_p^0(\xi) := \Omega^p \mathcal{C}(\xi)$; the continuity of α is defined as follows: if $\alpha \in C_p^\nu(\xi)$ then for all $\omega \in \mathcal{C}_k$ and all $\xi_1, \ldots, \xi_p \in Vect(\mathcal{C}_k)$ the maps $\alpha_{\omega, \xi_1, \ldots, \xi_p} : A^0 \times \ldots \times A^0(\nu$ *times*$) \to \mathbf{R}$, where $\alpha_{\omega, \xi_1, \ldots, \xi_p}(s_1, \ldots, s_\nu) := \alpha(s_1, \ldots, s_\nu)(\xi_1, \ldots, \xi_p)(\omega)$, are continuous.

The first column in the double cochain complex

$$
\begin{array}{ccccc}
C_0^0(\xi) & \xrightarrow{d_0^0} & C_1^0(\xi) & \xrightarrow{d_1^0} & \ldots \\
\delta_0^0 \downarrow & & \delta_1^0 \downarrow & & \\
C_0^1(\xi) & \xrightarrow{d_0^1} & C_1^1(\xi) & \xrightarrow{d_1^1} & \ldots \\
\delta_0^1 \downarrow & & \delta_1^1 \downarrow & & \\
\vdots & & \vdots & &
\end{array}
$$

i.e. the one corresponding to 0-forms on $\mathcal{C}(\xi)$ leads to the usual BRST cohomology of the principal fiber bundle ξ [1] $H^*_{BRST}(\xi) = \{H^\nu_{BRST}(\xi)\}_{\nu=0}^\infty$ where $H^\nu_{BRST}(\xi) = ker\delta_0^\nu / im\delta_0^{\nu-1}$ $(\delta_0^{-1} = 0)$. ν is identified with the ghost number and the coboundary $\{\delta_0^\nu\}_{\nu=0}^\infty$ with the BRST nilpotent operator of any quantum field theory defined on ξ. The remaining columns for p=1,2,... have been here formally defined and lead to a total complex (K^*, D) with cohomology $H^*(K^*, D)$ which we call the *total BRST cohomology of the principal fiber bundle* ξ, and denote by $\mathcal{H}^*_{BRST}(\xi)$. We do not yet know the possible physical meaning of these additional cohomology groups; however the fact that for the case $p = 0$ the local version of this cohomology has an interpretation in terms of anomalies and since these groups are in principle computable with the technique of spectral sequences, their definition is of interest for further investigation.

5. Spectral sequences

5.1. Spectral sequence of a filtered complex

We give the basic definitions and results, for more details the reader may consult reference [9].

a) Let (K^*, d) be a *cochain complex*, $(K^*, d) = \{K^n, d^n : K^n \to K^{n+1}\}_{n \geq 0}$. Let $B^n = imd^{n-1} =$ n-coboundaries $\subset Z^n = kerd^n =$ n-cocycles, then the *Cohomology* $H^*(K^*)$ of (K^*, d) is given by $H^n(K^*) := Z^n/B^n$.

b) A *Filtration* F^*K^* of (K^*, d) is a sequence of subcomplexes $K^* = F^0K^* \supset F^1K^* \supset F^2K^* \supset \ldots \supset F^pK^* \supset \ldots$ so that $d : F^pK^* \to F^pK^*$. At each level in K^* one has $K^n = F^0K^n \supset F^1K^n \supset F^2K^n \supset \ldots \supset F^pK^n \supset \ldots$ and $d^n : F^pK^n \to F^pK^{n+1}$. (K^*, d) is said to be *finitely filtered* if for each n there exists an m such that $F^mK^n = 0$. We have short exact sequences of complexes $0 \to F^{p+1}K^* \to F^pK^* \to F^pK^*/F^{p+1}K^* \to 0$ which means that for each $p = 0, 1, 2, \ldots$ one has the array

$$\begin{array}{ccccccccc}
0 & \to & F^{p+1}K^0 & \to & F^pK^0 & \to & F^pK^0/F^{p+1}K^0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \Delta_p^0 & & \\
0 & \to & F^{p+1}K^1 & \to & F^pK^1 & \to & F^pK^1/F^{p+1}K^1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \Delta_p^1 & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

where the coboundary $\Delta_p^n : F^pK^n/F^{p+1}K^n \to F^pK^{n+1}/F^{p+1}K^{n+1}$ for $n = 0, 1, 2, 3, \ldots$ is induced by d^n. Each sequence gives a long exact cohomology sequence

$$\begin{array}{ccccc}
 & & H^*(F^pK^*) & & \\
 & i \nearrow & & \searrow \pi & \\
H^*(F^{p+1}K^*) & & \overset{\delta}{\leftarrow} & & H^*(F^pK^*/F^{p+1}K^*)
\end{array}$$

where δ is a morphism of degree one.

Given a subcomplex $L^* \subset K^*$ we define a filtration F^*K^* by $F^0K^* = K^*$, $F^1K^* = L^*$ and $F^2K^* = 0$. The idea of the spectral sequences is that they play with respect to filtrations a róle analogous to that played by the long exact cohomology sequence with respect to subcomplexes.

c) The filtration of K^*, F^*K^*, gives a filtration of $H^*(K^*)$ as follows. Since F^pK^* is a subcomplex of K^*, the inclusion induces, for each $n \geq 0$, a homomorphism $i_p^n : H^n(F^pK^*) \to H^n(K^*)$, then the filtration is defined by $F^pH^n(K^*) = im\ i_p^n$.

d) A Spectral sequence consists of the following:

i) An infinite sequence of *bigraded abelian groups* $E_r = \{E_r^{p,q}\}_{p,q\geq 0}$, with $r \geq 0$,

ii) *differentials* of degree $(r, 1-r)$ *i.e.* group endomorphisms $d_r : E_r \to E_r$, $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ with $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$,

iii) *Cohomology relations* $E_{r+1} \cong H^*(E_r)$ *i.e.* $E_{r+1}^{p,q} \cong H^{p,q}(E_r)$ where $H^{p,q}(E_r) = ker d_r^{p,q}/im d_r^{p-r,q-1+r}$.

e) One says that a spectral sequence $(E_r, d_r)_{r=0}^{\infty}$ *converges finitely* to a filtered graded group H^*, provided: *i*) for each p and q, there exists an integer r_0 such that $E_r^{p,q} \cong E_{r_0}^{p,q}$ for $r \geq r_0$; *ii*) $E_\infty^{p,q} \cong F^pH^{p+q}/F^{p+1}H^{p+q}$, where $E_\infty^{p,q} \equiv E_{r_0}^{p,q}$.

f) *Proposition*: If K^* is a complex finitely filtered by F^*K^*, then there exists a spectral sequence $(E_r, d_r)_{r=0}^{\infty}$ converging finitely to $H^*(K^*)$ with:

i) $E_0^{p,q} = F^pK^{p+q}/F^{p+1}K^{p+q}$,

ii) $E_1^{p,q} = H^{p+q}(F^pK^*/F^{p+1}K^*)$.

5.2. Application to double (cochain) complexes

For the total complex (K^*, D) associated with a double cochain complex (C, ∂, d) as defined in subsection 3.2 it can be verified the following:

i) K^* has *two* canonical filtrations $F_1^*K^*$ and $F_2^*K^*$ with $F_1^0K^* = K^*$ $(F_2^0K^* = K^*), ..., F_1^lK^*$ $(F_2^mK^*)$: the standard complex associated with the doble complex obtained from C after making zero the first l (m) rows (columns) of C, $(l, m = 1, 2, ...)$

ii) The corresponding spectral sequences both converge finitely to $H^*(K^*)$.

In particular this result holds for the BRST double complex of subsection 4.1 and hence $\mathcal{H}^*_{BRST}(\xi)$ can be computed.

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A METHOD TO COMPARE OPERATORS APPLICATIONS TO SCHRÖDINGER AND DIRAC OPERATORS

MANLIO BORDONI
Dipartimento di Matematica "G. Castelnuovo"
Universita degli Studi di Roma "La Sapienza"
Piazzale Aldo Moro 2, 00185 Roma, Italia

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1. Introduction

The main reference for this talk is our paper [1] which is going to appear in Math. Ann. (1994).

Let H and H' be Hilbert spaces and let $\varpi : H' \longrightarrow H$ be any mapping. We say that two self-adjoint semibounded operators $T' : H' \longrightarrow H'$ and $T : H \longrightarrow H$ satisfy *Kato's property* with respect to ϖ if ϖ maps the domain of T' into the domain of T and if

$$Q_T(\varpi f) \leq Q_{T'}(f) \quad \forall f \in \text{domain of } T'$$

where Q_T and $Q_{T'}$ are the quadratic forms associated with T and T' respectively.

PROBLEM 1. *Comparison between the spectra of T and T'.*

We translate this problem into another one. Universal tools in Hilbertian analysis are:

- CAUCHY-SCHWARZ INEQUALITY
- PITHAGORAS ORTHOGONAL DECOMPOSITION THEOREM .

PROBLEM 2. *Is it possible to improve them? And, in what sense?*

More precisely, we ask if exists *universal constant* $0 < C(N,k) < 1$ such that for any couple of vector subspaces $\mathcal{E} \subset H'$ and $\mathcal{K} \subset H$ whose dimensions are N and k resp., the following inequalities hold:

$$\int_{\mathcal{E}} \langle \varpi(u), u \rangle dG(u) \;\leq\; C(N,1)\|v\| \int_{\mathcal{E}} \|\varpi(u)\| dG(u) \tag{1}$$

$$\int_{\mathcal{E}} \|P_{\mathcal{K}^\perp} \varpi(u)\| dG(u) \leq (1 - C(N,k)) \int_{\mathcal{E}} \|\varpi(u)\| dG(u) \tag{2}$$

where dG is the Gaussian measure on $\mathcal{E}$ and $P_{\mathcal{K}^\perp}$ denotes the orthogonal projection onto the orthogonal complement $\mathcal{K}^\perp$ of $\mathcal{K}$ in H.

An answer YES to Problem 2 and MIN-MAX principle give comparison theorems between spectra of operators (see for instance [4]).

The answer to Problem 2 is yes if ϖ is linear and $\dim \mathcal{E} > \dim \mathcal{K}$ and the proof is trivial in this case.

The answer is also yes if ϖ is *non linear* in the following case. Let $(M,\mu), (M',\mu')$ be measure spaces and let $\pi : M' \longrightarrow M$ be any mapping verifying *Fubini's property*:

$$\int_{M'} f = \int_M \left(\int_{\pi^{-1}(x)} f\,|_{\pi^{-1}(x)} \right).$$

We define $\varpi : L^2(M',\mu') \longrightarrow L^2(M,\mu)$ by

$$(\varpi f)(x) = \left(\int_{\pi^{-1}(x)} (f\,|_{\pi^{-1}(x)})^2 \right)^{1/2}. \tag{3}$$

Then we have the following:

HILBERTIAN THEOREM. *Let T, T' be self-adjoint semibounded operators acting on $L^2(M)$ and $L^2(M')$ resp. If T, T' verify Kato's property with respect to ϖ defined by (3), then there exists C such that (1),(2) hold (we shall give later an explicit expression for C).*

A first partial version of this theorem was given by Meyer [9] in 1982, a second one is due to Gallot and Meyer [6], and the general version here presented is in [1].

2. Consequences on the Spectra.

We shall use the same symbols and assumptions as in section 1. Let $\mathcal{E}$ be a vector subspace of $L^2(M')$ and let $\mathcal{E}_x$ be its image in $L^2(\pi^{-1}(x))$ by the mapping $f \longmapsto f\,|_{\pi^{-1}(x)}$, which is defined for almost every $x \in M$: we set $\mathcal{E}_x = \{0\}$ if f is not defined on $\pi^{-1}(x)$. We define the *rank* of $\mathcal{E}$ to be the essential supremum of the dimensions of $\mathcal{E}_x$, i.e.

$$\operatorname{rank} \mathcal{E} = \inf_{A \in \mathcal{A}} \left(\sup_{x \in M \setminus A} \dim \mathcal{E}_x \right)$$

where $\mathcal{A}$ is the class of all subsets in M of zero measure. Notice that the rank of $\mathcal{E}$ may be much smaller than the dimension of $\mathcal{E}$ (cfr. [1]).

THEOREM 1. [1]. *Under the same assumptions as in Hilbertian theorem, we have*

$$\lambda_N(T') \geq (1 - C(p))\lambda_1(T) + C(p)\lambda_{k+1}(T) \qquad (i)$$

$$\frac{1}{N}\sum_{i=1}^{N}\lambda_i(T') \geq (1 - C(p) - B(p))\lambda_1(T) + B(p)\left(\frac{1}{k}\sum_{j=1}^{k}\lambda_j(t)\right) + C(p)\lambda_{k+1}(T) \qquad (ii)$$

where N is any positive integer, p is the rank of the subspace $\mathcal{E} \subset L^2(M')$ spanned by the first N eigenfunctions of T', k is the integer part of $\frac{N}{p+1}$, and where

$$C(p) = \frac{1}{8(p+1)^2}, \quad B(p) = \frac{1}{2(p+1)}.$$

Let us consider now a Riemannian vector bundle $(E, \langle\cdot,\cdot\rangle) \longrightarrow (M, \mu)$. We denote by $L^2(M, E)$ the Hilbert space of measurable sections $s : M \longrightarrow E$ associated to the $L^2(M, E)$-norm

$$\|s\|_{L^2(M,E)} = \left(\int_M (\langle s(x), s(x)\rangle \, d\mu(x)\right)^{\frac{1}{2}}.$$

THEOREM 2, [6]. *Let T, T' be self-adjoint semibounded operators acting on $L^2(M)$ and $L^2(M, E)$ resp. If T, T' verify Kato's property with respect to the mapping $L^2(M, E) \longrightarrow L^2(M)$ which associates to a sections s the function $|s|(x) = \langle s(x), s(x)\rangle^{\frac{1}{2}}$, then, (i), (ii) of theorem 1 hold with the same constants and where p is now the usual rank of E, i.e. the dimension of the fibers.*

REMARK. If T, T' have non discrete spectra, the results of theorems 1 and 2 apply to the discrete parts of them (i.e. the parts lying under the essential spectra).

3. Applications: General Underlying Philosophy.

We assume now that (M, g) is a Riemannian manifold. Let $E \longrightarrow M$ be a vector bundle on M, endowed with a scalar product $\langle\cdot,\cdot\rangle$ and with a compatible connection D. Let $\mathcal{R}$ be a field of symmetric endomorphisms of the fibers, we denote $\mathcal{R}_{min}(x)$ the minimum eigenvalue of the endomorphism $\mathcal{R}_x : E_x \longrightarrow E_x, x \in M$.

We apply theorems 1,2 to compare the following operators:

Dirac-type operator $T' = D^*D + \mathcal{R}$. (D^*D is the so called rough Laplacian), *Schrödinger operator* $T = \Delta_M + \mathcal{R}_{min}$. ($\Delta_M$ is the usual Laplace-Beltrami operator and $\mathcal{R}_{min}$ is the potential function).

Examples of operators of type $D^*D + \mathcal{R}$ are all "natural Laplacians" on fiber bundles, specially those obtained by Bochner-Weitzenböck formulas according to Bourguignon [2].

To improve the estimates (*i*), (*ii*) instead to use the given connection D (the Levi-Civita one, for instance), we introduce a modified connection D' and we calculate the corresponding Bochner-Weitzenböck formulas. We consider a trivial line bundle $E_1 \longrightarrow M$ over M and a never vanishing section $e : M \longrightarrow E_1$. We define on $E' = E \oplus E_1 \longrightarrow M$ the metric $\langle \cdot, \cdot \rangle'$ by

$$\langle X, Y \rangle' = \langle X, Y \rangle \quad \forall X, Y \in E,$$
$$\langle X, e \rangle' = 0 \quad , \quad \langle e, e \rangle' = 1.$$

The connection D' on E' defined by

$$D'_X Y = D_X Y - k \langle X, Y \rangle e \quad , \quad D'_X e = kX$$

(where k is a scalar to choose *a posteriori* in order to optimize the results) is compatible with the metric $\langle \cdot, \cdot \rangle'$.

4. Hodge-De Rham Operator.

We consider $E = \Lambda^p M$, the bundle of differential p-forms on the n-dimensional manifold (M, g) endowed with the usual scalar product and the Levi-Civita connection. The operator

$$\mathcal{D} = d + \delta$$

where d is the differential and δ is the codifferential, is such that its square is the Hodge-de Rham operator:

$$\mathcal{D}^2 = \Delta^p.$$

In this case the classical Bochner-Weitzenböck formula is

$$\mathcal{D}^2 = D^* D + \mathcal{R}$$

where $\mathcal{R}$ is explicitely expressed in terms of the *curvature* of M (see for instance [5]). We use the method illustrated in section 3 to obtain:

THEOREM 3, [1]. *Let* $\{\lambda_i(\mathcal{D})^2\}$ *be the eigenvalues of the Hodge-de Rham operator acting on closed differential p-forms. Then for any positive integer* N *we have:*

$$\lambda_N(\mathcal{D})^2 \geq p(n-p+1)\rho + \frac{1}{2N} \sum_{j=1}^{k} \lambda_j(\Delta_M) + C_1(p)\lambda_{k+1}(\Delta_M), \qquad (i)$$

$$\frac{1}{N}\sum_{i=1}^{N}\lambda_i(\mathcal{D})^2 - \frac{2}{n+1}\left(\frac{1}{N}\sum_{i=1}^{N}\lambda_i(\mathcal{D})\right)^2 \geq$$

$$\geq \left(1-\frac{2}{n+1}\right)p(n-p+1)\rho + \frac{1}{2N}\sum_{j=1}^{k}\lambda_j(\Delta_M) + C_1(p)\lambda_{k+1}(\Delta_M) \qquad (ii)$$

where ρ is a lower bound of the curvature operator, and where

$$k = \left[\frac{N}{\binom{n+1}{p}+1}\right], \qquad C_1(p) = \frac{1}{8\left(\binom{n+1}{p}+1\right)^2}.$$

For closed 1-forms, the inequalities may be improved by replacing $p(n-p+1)\rho$ by $\frac{n}{n-1}Ric_{min}$, where Ric_{min} is the minimum eigenvalue of the Ricci curvature tensor.

To illustrate the type of results obtained, we point out some consequences. SHARPNESS PROPERTY. There are at most $\binom{n+1}{p}$ eigenvalues of $\mathcal{D}^2 = \Delta^p$ lying in the interval $[p(n-p+1)\rho, p(n-p+1)\rho + C_1(p)\lambda_2(\Delta_M)[$. This is sharp: for the canonical sphere, Δ^p has exactly $\binom{n+1}{p}$ eigenvalues equal to $p(n-p+1)\rho$ ($\rho =$ constant).

PARTICULAR CASE $N = 1$. In this case our theorem implies the previous known estimates on the first eigenvalue of Δ^p (Gallot and Meyer, [5] and [6]).

TOPOLOGICAL COROLLARY. Let M be compact and let $b_p(M)$ denote its p^{th} Betti number. Then *for any metric* on M we have:

$$\sum_{j=1}^{k}\lambda_j(\Delta_M) + 2C_1(p)b_p(M)\lambda_{k+1}(\Delta_M) \leq -2\frac{n-1}{n+1}p(n-p+1)\rho b_p(M); \quad (a)$$

$$\sum_{j=1}^{k}\lambda_j(\Delta_M) + 2C_1(p)b_1(M)\lambda_{k+1}(\Delta_M) \leq -2\frac{n}{n+1}Ric_{min}b_1(M) \qquad (b)$$

where $k = \left[\frac{b_p}{\binom{n+1}{p}+1}\right]$.

5. Dirac Operator.

We assume now that M is a spin manifold and we consider the bundle of spinors $S \longrightarrow M$, endowed with the connection D induced on S by the Levi-Civita connection. The classical *Dirac operator* acting on spinors is defined by

$$\mathcal{D} = \sum_{i=1}^{n} e_i \cdot D_{e_i}$$

where $\{e_i\}$ is an orthonormal basis of tangent vectors and " $\cdot$ " denotes the Clifford multiplication. The analogous of the Bochner-Weitzenböck formula is in this case the *Lichnerowicz formula*

$$\mathcal{D}^2 = D^* D + \frac{1}{4} scal$$

where *scal* is the scalar curvature of M (see [8]). We use the modified connection D' of section 3 (introduced in this case by Hijazi [7] and the corresponding Lichnerowicz's formula to obtain:

THEOREM 4, [1]. *For any set $\{\lambda_i(\mathcal{D})\}_{i\in I}$ of eigenvalues of the Dirac operator $\mathcal{D}$ acting on spinors, we have*

$$\begin{aligned} \tfrac{1}{\#I}\sum_{i\in I}\lambda_i(\mathcal{D})^2 - \tfrac{1}{n}\left(\tfrac{1}{\#I}\sum_{i\in I}\lambda_i(\mathcal{D})\right)^2 &\geq (1-a-b)\lambda_1(\Delta_M + \tfrac{1}{4}scal) \\ &+ b\left(\sum_{j=1}^{k}\lambda_j(\Delta_M + \tfrac{1}{4}scal)\right) + a\lambda_{k+1}(\Delta_M + \tfrac{1}{4}scal) \end{aligned} \qquad (a)$$

$$\begin{aligned} \sup_{i\in I}\lambda_i(\mathcal{D})^2 &\geq \tfrac{n}{n-1}\Big((1-a)\lambda_1(\Delta_M + \tfrac{1}{4}scal) + a\lambda_{k+1}(\Delta_M + \tfrac{1}{4}scal)\Big) \\ &\geq \tfrac{n}{n-1}\left(\tfrac{1}{4}\min_{x\in M} scal(x) + a\lambda_{k+1}(\Delta_M)\right) \end{aligned} \qquad (b)$$

where $k = \left[\#I(2^{[n/2]}+1)^{-1}\right]$, $a = \frac{1}{8}(2^{[n/2]}+1)^{-2}$ and $b = \frac{k}{2\#I}$.

Also in this case, we underline some consequences.
SHARPNESS PROPERTY. There are at most $2^{[n/2]}$ eigenvalues of $\mathcal{D}^2$ in $[\frac{n}{4(n-1)}\min_{x\in M} scal(x), \frac{n}{4(n-1)}\min_{x\in M} scal(x) + \frac{n}{n-1}a\lambda_1(\Delta_M)[$. This is sharp: for the flat torus there are exactly $2^{[n/2]}$ eigenvalues of $\mathcal{D}^2$ equal to $\frac{n}{n-1}\min_{x\in M} scal(x) = 0$.
PARTICULAR CASE on the first eigenvalue: we have

$$\lambda_1(\mathcal{D})^2 \geq \frac{n}{4(n-1)}\min_{x\in M} scal(x)$$

which is the famous Friedrich theorem, [3] 1980.

Let us recall that the $\hat{A}$-genus $\hat{A}(M)$ of a manifold M is a topological invariant which is the index of the Dirac operator.

TOPOLOGICAL COROLLARY. If the $\hat{A}$-genus of M is not trivial, the spectrum of Δ_M satisfies

$$\frac{1}{2}\sum_{j=1}^{k}\lambda_j(\Delta_M)+\hat{A}(M)a\lambda_{k+1}(\Delta_M)\leq -\frac{\hat{A}(M)}{4}\min_{x\in M} scal(x)$$

where $k=\left[\hat{A}(M)(2^{[n/2]}+1)^{-1}\right]$.

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TRANS–SYMMETRIC SPACES

LIUDMILA SABININA*
Instituto de Matemáticas. UNAM. Area de la Investigación Científica, Circuito Exterior, C.U., C.P. 04510 México D.F. México
and
Department of Mathematics, The University of Oklahoma, 601 Elm Avenue, Room 423, Norman, OK 73019-0315.
lsabinin@nsfuvax.math.uoknor.edu

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1. Introduction

In [7] we have introduced the notion of transsymmetric spaces. This notion permits us to describe different classes of spaces with symmetries.

Definition 1.1. Let M be a C^∞-manifold, $\{\sigma_x\}_{x\in M}$ be a family of local diffeomorphisms $\sigma_x : M \to M$ and $e : M \to M$ be a map defined on M by $e(x) = \sigma_x^{-1}x$. The pair $(M, \{\sigma_x\}_{x\in M})$ is called a **transsymmetric space** if the following conditions hold:

(i) $\sigma_x \circ \sigma_y = \sigma_{\sigma_x y} \circ \sigma_{e(x)}$,

(ii) $\rho_{e(x)} : M \to M$ is a local diffeomorphism and $\rho_{e(x)}b \overset{def}{=} \sigma_b e(x)$.

Let us give some examples:

1.2 *Let G be a Lie group and let $\{L_x\}_{x\in G}$ be a family of left translations $(L_x y = x \cdot y)$. Then $\{G, \{L_x\}_{x\in G}\}$ is a transsymmetric space.*

In this case the map $e: M \to M$ is the constant map such that $e(a) = e_G, \forall a \in G$ where e_G is the neutral element of G.

1.3 *Every symmetric space and regular s-space is a transsymetric space.*

Let M be a C^∞ manifold and $\mu : M \times M \to M$, $\mu(x,y) = x \cdot y$ be a differentiable multiplication. The space M with the multiplication μ is said to be a **symmetric space** if the following conditions hold:

(1) $x \cdot x = x$

(2) $x \cdot (x \cdot y) = y$

(3) $x \cdot (y \cdot z) = (x \cdot y)(x \cdot z)$

(4) Every point x has a neighborhood U such that $x \cdot y = y$ implies that $y = x$ for all $y \in U$.

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The notion of symmetric spaces is due to E. Cartan and reformulated by Loos as pair (M, μ) with conditions (1)–(4) in [3]. Loos has shown that in M there exists an affine connection ∇ such that $T = 0, \nabla R = 0$, where T and R are tensor fields of torsion and curvature, respectively. If M is connected, then (M, ∇) is an affine symmetric space and $\{s_x\}_{x\in M}$ is a family of geodesic symmetries, where $s_x y = x \cdot y = \mu(x, y)$.

Later, Kowalski [2] defined generalized symmetric spaces or regular s-spaces.

Let M be a C^∞-manifold with a family of maps $(\sigma_x)_{x\in M}$. The space M is said to be a **regular** s**-space** (from here regular s-space we will call s-space) if the following conditions hold:

(1) $s_x x = x$,

(2) s_x is a diffeomorphism,

(3) $s_x \circ s_y = s_{s_x y} \circ s_x$,

(4) $(s_x)_\star$ has only one fixed vector, the zero vector.

Condition (2) from the definition of symmetric space is equivalent to $s_x^2 = id$, and it implies (2)' . Kowalski modified this definition, by changing (2) into (2)', and (4) to (4)', respectively. If conditions (1) and (2)' hold. Then conditions (4), (4)' and $(s_x)_\star - Id_{\star,x}$ - is non singular are equivalent.

Fedenko [1], and Kowalski [2] demonstrated that there exists a unique affine connection $\hat{\nabla}$ on a manifold M with (1)'-(4)' such that s_x is an affine transformation , i.e.,$(s_x)_\star \hat{\nabla}_X Y = \hat{\nabla}_{(s_x)_\star X}(s_x)_\star Y$ and $\hat{\nabla} T = 0, \hat{\nabla} R = 0, \hat{\nabla} S = 0, S_x = (s_x)_{\star,x}$, where the group Tran $M =< \{s_x \circ s_y^{-1}\} > \subset Aut < M, \cdot >, x \cdot y = s_x y$, acts transitively on M, and M is the homogeneous space G/G_a.

Symmetric spaces and regular s-spaces are transsymmetric spaces. In this case, the desired map e is the identity map on M. Condition (3) is called the left-distributive identity. The condition (ii) holds in virtue of (4)'. This fact we will prove below.

1.4 Discrete example.

Let us consider triangulations of compact, connected, orientable surfaces with the following conditions:

(a) All pairs of points are connected by a segment,

(b) Every segment belongs only to 2 triangles.

Then a binary operation can be defined on the set of vertices of M by the following rule: Suppose the segment $ab \subset \Delta abc$, the orientation of which coincides with the orientation of the surface and the segment ba belongs to a triangle Δbad, the orientation of which coincides with the orientation of the surface. Then we define $a \cdot b = c$, $b \cdot a = d$ and $a \cdot a = a \quad \forall a \in M$.

This triangulation was used by Ringel in his intent to solve the four color problem [6].

1.5 Proposition *If $p = 7 \, mod \, 12$ and p-is a prime, then there exists a triangulation M with p-vertices and this triangulation is a s-space.*

Proof: Let n be the number of vertices of the triangulation M. Then the number of segments is C_n^2 and the number of triangles is $\frac{2}{3}C_n^2$. By Euler's formula it is only possible for n to have the following values, $n = 0, 3, 4, 7 \, mod \, 12$. By construction we have that $ab \neq ba$ for all $a, b \in M$ and $ab = c$ implies $bc = a$ and $ca = b$. In [4] Mendelsohn has shown that $\langle M, \cdot \rangle$ is a quasigroup and for $p = 1 \, mod \, 3$ and p a prime such a quasigroups exist. The binary operation can be expressed by $a \cdot b = \lambda a + (1 - \lambda)b$ in the field $GF(p)$. Then $\langle M, \cdot \rangle$ is left-distributive. Moreover if p is a prime , then M does not have non-trivial subquasigroups. So if we compare our values of number of vertices for $p = 7 \, mod \, 12$, p a prime we have a triangulation $\langle M, \cdot \rangle$ with the following conditions:

(1) $\langle M, \cdot \rangle$ is a quasigroup , in particular $ab = cb$ implies $a = c$ and $ab = ad$ implies $b = d$,

(2) $ab = c$ implies $bc = a$,

(3) $ab \neq ba$

(4) $a(bc) = (ab)(ac)$, in particular $a^2 = a$

It is easy to see that the finite quasigroup $\langle M, \cdot \rangle$ with the condition (4) is an s-space.

This means that the graph is an invariant under the action of the symmetries s_x for all $x \in M$. Now consider the case $M = 7$ (triangulation of the torus). By Kowalski the group of automorphisms acting transitively on M can be calculated by $\langle \{s_0^{-1} \circ s_x\}_{x \in M} \rangle$. In this particular case we have that such a group is Z_7.

Cayley table gives the operation $x \cdot y = s_x y$.

$\cdot$	a	b	c	d	e	f	g
a	a	g	e	c	b	d	f
b	e	b	g	f	d	c	g
c	d	f	c	g	a	e	b
d	f	e	a	d	g	b	c
e	c	a	f	b	e	g	d
f	g	d	b	a	c	f	e
g	b	c	d	e	f	a	g

1.6 *The next example shows that the class of transsymmetric spaces is greater than the classes of examples 1.2 and 1.3.*

Let $(G, \cdot)$ be a commutative Lie group. We define the symmetry $\sigma_x y = x^{-1} \cdot y$. Then $\{G, \{\sigma_x\}_{x \in G}\}$ with the map $e(x) = x^2 = x \cdot x$ is a transsymmetric space.

Indeed (i) holds:

$\sigma_x \circ \sigma_y z = x^{-1} \cdot y^{-1} \cdot z$

$\sigma_{\sigma_x y} \circ \sigma_{e(x)} z = (x^{-1} \cdot y)^{-1} \cdot (x^{-2} \cdot y) = y^{-1} \cdot x \cdot x^{-2} \cdot y = x^{-1} \cdot y^{-1} \cdot z.$
It is obvious that (ii) holds because $\rho_{e(x)} z = z^{-1} \cdot x^2$ in the group G.

2. Some Properties of Transsymmetric Spaces

Lemma 2.1. *Let $(M, (\sigma_x)_{x \in M})$ be a transsymmetric space and $e(x) = \sigma_x^{-1} x$. Then $e \circ \sigma_x = \sigma_{e(x)} \circ e$ and the condition (ii) is equivalent to (ii)' $(\sigma_x \circ e)_{\star,x} - Id_{\star,x}$- is non-singular.*

Proof: By virtue of (i), we have that $\sigma_x \circ \sigma_y z = \sigma_{\sigma_x y} \circ \sigma_{e(x)} z$. Let us substitute $z = \sigma_y^{-1} y = e(y)$. We get $\sigma_x y = \sigma_{\sigma_x y} \circ \sigma_{e(x)} e(y)$ then $\sigma_{\sigma_x y}^{-1} \sigma_x y = \sigma_{e(x)} e(y)$ or $e \circ \sigma_x = \sigma_{e(x)} \circ e$. Now let us differentiate both sides of the identity $x \cdot e(x) = x$ at the point x, where $\mu(x, y) = x \cdot y$ is a function in two variables. Then we get $(\sigma_x \circ e)_{\star,x} + (\rho_{e(x)})_{\star,x} = Id_{\star,x}$. $(\rho_{e(x)_{\star,x}})$ is non-singular by virtue of (ii). Then (ii) is equivalent to (ii)'

Remark 2.2. *Condition (4) from the definition of symmetric spaces (example 1.3) means that locally there exists r_x^{-1}, where $r_x y = y \cdot x$ and so (4) is equivalent to (4)'. Let us denote $\sigma_x \circ e = \psi_x$. Then we get the condition (ii) in the next form*

$$(\psi_x)_{\star,x} - Id_x \quad is\, non - singular$$

On the transsymmetric space $(M, (\sigma_x)_{x \in M})$ the structure of a manifold with the reductive affine conection is determined. More exactly

Theorem 2.3. *Let $(M, (\sigma_x)_{x \in M})$ be a transsymmetric space. Then on M there exists an unique affine connection ∇ such that $(\sigma_x)_{x \in M}$ are local automorphisms of ∇ , $(\sigma_x)_\star \nabla_X Y = \nabla_{(\sigma_x)_\star X} (\sigma_x) \star Y$, $\nabla T = 0, \nabla R = 0$. If we denote $(\sigma_x)_{\star, e(x)}: T_{e(x)} M \to T_x M$ by S_x, then $S_a \nabla_{e_\star X_{e(a)}} Y = \nabla_{X_a} (SY), (SY)_b = S_b Y_{e(b)}$ or shorter $S \nabla_{e_\star X} Y = \nabla_X (SY)$.*

This affine connection can be expressed by:

$$(\nabla_X Y)_b = \frac{d}{dt} \{[(P^b_{x(t)})_{\star,b}]^{-1} Y_{x(t)}\}|_{t=0}$$

Where $P^b_x = \sigma_{(\rho^{-1}_{e(b)} x)} \circ \sigma_x^{-1}$, $x(0) = b$ and $x(0) = X_b$.

For the proof see [7].

Theorem 2.4. [7] *Let (M, ∇) be a manifold with an affine connection ∇ and let $(\sigma_x)_{x \in M}$ be a family of local automorphisms of the affine connection ∇ such that $e \circ \sigma_x = \sigma_{e(x)} \circ e$ holds, where $e(x) = \sigma_x^{-1} x$, $S \nabla_{e_\star X} Y = \nabla_X (SY)$. Then the condition (i) holds.*

Since a transsymmetric space determines an affine connection, it can be characterized in terms of geodesics loops. Let us consider the local geodesic loop $\langle N_a, \cdot_a a \rangle$ of the transsymmetric space $(M, (\sigma_x)_{x \in M})$ which is defined in a normal convex neighborhood $N_a \subset M$ of the point a by the rule:

$x \cdot_a y = Exp_x \circ \tau_{a,x} \circ (Exp_a)^{-1} y$, where $Exp_a : T_a M \to M$, $\tau_{a,x}$ is a parallel translation from the point a to x.

Lemma 2.5. *Let $\langle N_a, \cdot_a, a \rangle$ be a geodesic loop of a transsymmetric space $(M, (\sigma_x)_{x \in M})$. Then*

$$\sigma_b = L_b^a \circ (L_{\psi_a b}^a)^{-1} \circ \sigma_a \qquad (\star)$$

where $\psi_a = \sigma_a \circ e, L_b^a c = b \cdot_a c$.

Proof: We can get the identity $(\star)$ from $\nabla_X(SY) = S(\nabla_{e_* X} Y)$, considering $Y_{e(x(t))}$ parallel along the smooth curve $e(x(t))$ and $x(0) = a$, $x(1) = b$, $\dot{x}(t) = X$. We have

$$(\sigma_b)_{*,e(b)} \circ \tau_{e(a),e(b)} = \tau_{a,b} \circ (\sigma_a)_{*,e(a)}$$

or, equivalently

$$\sigma_b \circ L_{e(b)}^{e(a)} = L_b^a \circ \sigma_a$$

Futher $\sigma_a (b \cdot_c d) = \sigma_a \cdot_{\sigma_a c} \sigma_a d$ and so $\sigma_b \circ L_{e(b)}^{e(a)} = (\sigma_b \circ \sigma_a^{-1}) \circ (L_{\sigma_a e(b)}^a \circ \sigma_a$. Finally we get $(\star)$. For details see [6,7].□

Theorem 2.6. *Let $\langle N_a, \cdot_a \rangle$ be a geodesic loop of a transsymmetric space $(M, (\sigma_x)_{x \in M})$. Then the following conditions hold:*

(1) $x^k \cdot_a (x^l \cdot_a y) = x^{k+l} \cdot_a y$,

(2) $l_{x,y}(z \cdot_a w) = l_{x,y} z \cdot_a l_{x,y}(w)$, where $l_{x,y} = (L_{x \cdot_a y}^a)^{-1} \circ L_x^a \circ L_y^a$,

(3) $\psi_a (x \cdot_a y) = \psi_a x \cdot_a \psi_a y$

(4) $(\psi_a)_{*,a} - Id_{*,a}$ is non-singular,

(5) $L_x^a \circ (L_{\psi_a x}^a)^{-1} \circ L_z^a \circ (L_{\psi_a z}^a)^{-1} = L_{L_x^a (L_{\psi_a x}^a)^{-1} z}^a \circ (L_{L_{\psi_a x}^a (L_{\psi_a^2 x}^a)^{-1} \psi_a z}^a)^{-1} \circ L_{\psi_a x}^a \circ (L_{\psi_a^2 x}^a)^{-1}$,

(6) $l_{\psi_a y, \psi_a z} = l_{y,z}$,

(7) $\psi_a \circ \sigma_a = \sigma_a \circ L_{e(a)}^a \circ (L_{\psi_a e(a)}^a)^{-1} \circ \psi_a$,

(8) $\sigma_a \circ L_x^a = L_{\sigma_a a}^a \circ L_{(L_{\sigma_a a}^a)^{-1} \sigma_a x}^a \circ (L_{\sigma_a a}^a)^{-1}$.

And, conversely, let $\langle N, \cdot, a \rangle$, be a local smooth loop with a local diffeomorphism $\sigma_a : N \to N$, a local map $\psi_a : N \to N$ and such that the conditions (1)-(8) hold. Then $\langle N, \cdot, a \rangle$ is isomorphic to a geodesic loop of a transsymmetric space.

Proof: The family $(\sigma_x)_{x \in M}$ of symmetries is obtained by formula $(\star)$ from Lemma 2.5. The identities (1) and (2) hold for geodesic loops of manifolds M with reductive affine connection ∇ [5]. (3) means that ψ_a is an endomorphism of a geodesic loop $\langle N_a, \cdot_a, a \rangle$ and it follows from the more general fact: $\sigma_b (x \cdot_a y) = \sigma_b x \cdot_{\sigma_b a} \sigma_b y$ and $e(x \cdot_a y) = e(x) \cdot_{e(a)} e(y)$. (4) is equivalent to the condition (ii) of the definition of transsymmetric spaces (remark 2.2). The condition (5) we will call the ψ-identity and it is obtained from (i) using $(\star)$

from Lemma 2.5. The identities (6)-(8) have a technical character, see [8] for details. □

Let us remark that the existence of these identities is a sufficient conditions for a local loop with σ_a and ψ_a to be isomorphic to a geodesic loop of a transsymetric space.

Remark 2.7 *The same conditions (1)-(5) hold for a geodesic loop of a regular s-space (example 1.3), but in case of an s-space instead of the endomorphism ψ_a we have the automorphisms $s_x \in \{s_x\}_{x\in M}$ [5] . So for proving (3)-(5) of theorem 2.6 it enough to note that the family $\{\psi_x\}$ defines a structure of a left-distributive groupoid on M.*

Proposition 2.8 *Let (M, σ_x) be a transsymmetric space and $\psi_x = \sigma_x \circ e$, where $\sigma_x y = x \cdot y$. Then $\psi_x \circ \psi_y = \psi_{\psi_x y} \circ \psi_x$.*

Proof: $\psi_x \circ \psi_y z = \psi_x(y \cdot e(z)) = x \cdot e(y \cdot e(z)) = x \cdot (e(y) \cdot e^2(z)) = (x \cdot e(y))(e(x) \cdot e^2(z)) = \psi_x y \cdot e(\psi_x z) = \psi_{\psi_x y}\psi_x z$. □

Corollary 2.9 *If $e(x) = \sigma_x^{-1} x$ is invertible, then $(M, (\psi_x)_{x\in M})$ is an s-space.*

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A GEOMETRIC STRUCTURE FOR THE LORENTZ-DIRAC EQUATION

GEOFFREY MARTIN
Department of Mathematics
University of Toledo, Toledo OH 43606
email: gmartin@uoftol.uoft02.edu

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This report describes a class of dynamical structure in which the classical Lorentz force law and the Lorentz-Dirac equation have the same formal description. The construction to be given shows how to obtain the Lorentz-Dirac equation in the spirit of Souriau [5], and perhaps even more importantly, associates a geometric structure with the radiation reaction. These results arise from a new approach to parameter independent ordinary differential equations on a manifold M. Rather than study such systems from the point of view of exterior differential systems, I shall use tangent bundle techniques. Given a smooth curve $\gamma: \mathbb{R} \to M$ there are two parameter independent lifts of γ to TM. The first is achieved by projecting $\dot{\gamma}$ onto the projective tangent bundle PM. The second is obtained by associating the lift of γ with the 2-dimensional manifold $S = \{v \in TM | v = s\dot{\gamma}(t), s, t \in \mathbb{R}\}$. For each type of lift there is a similar construction for higher derivatives of γ. In the first case, k^{th} derivatives of γ can be lifted to the k^{th} projective tangent bundle of M, $P^{(k)}M$; while in the second case the curve is lifted to the multivector bundle $\Lambda^k(\Lambda^{k-1}(\ldots\Lambda^2(TM)\ldots))$. An explicit relation between these two parameter independent lifts will be given, and it is this relation that leads to a class of dynamical structures that includes both the Lorentz force law and the Lorentz-Dirac equation. The principle that unites both systems is that they are both associated with the perturbation of a canonical differential form in the sence of Souriau [5]. In the case of the Lorentz force law this form is the canonical 2-form on T^*M. The Lorentz-Dirac equation arises from a perturbation of a canonical 3-form generated by the difference between the acceleration bivector and the Faraday tensor.

1. Projective Derivatives

This section and the next introduce two distinct constructions of a parameter independent derivative of an immersion $\gamma : \mathbb{R} \to M$ into a smooth manifold

M. The first construction starts with the standard representation of the k^{th} derivative of γ as a curve in the k^{th} tangent space of M, $T^{(k)}M$. The space $T^{(k)}M$ is a submanifold of $T^kM = T(T^{k-1}M)$ determined by the natural geometric structures on T^kM. Recall that $T^{k+1}M$ as a bundle over T^kM possesses $k+1$ distinct vector bundle structures defined by the vector bundle maps $\pi_j^k: T^{k+1}M \to T^kM$ where $j \in (0,\ldots,k)$. Here the maps π_j^k are defined recursively for $j \in (1,\ldots,k)$ by $\pi_j^k = \pi_{j-1\,*}^{k-1}$ and $\pi_0^k: T(T^kM) \to T^kM$ is the tangent bundle map. Further, these bundle maps together determine k distinct smooth involutions of $T^{k+1}M$, S_j^k for $j \in (0,\ldots,k-1)$. The involution S_0^k is a natural map that satisfies $\pi_1^k \circ S_0^k = \pi_0^k$. For $j > 0$, S_j^k is defined recursively by $S_j^k = S_{j-1\,*}^{k-1}$. The involution S_j^k, when expressed in local coordinates induced from M, effects an exchange of components of the coordinate map.

The $k+1$ order tangent bundle $T^{(k+1)}M$ is defined to be the fixed point set of the k involutions $S_0^k,\ldots,S_{k-1}^k$; that is $T^{(k+1)}M = \{v \in T^{k+1}M | S_j^k v = v, j \in (0,\ldots,k-1)\}$. Now an immersion $\gamma: \mathbb{R} \to M$ determines an immersion $\gamma^{(k)}: \mathbb{R} \to T^{(k)}M$. Here $\gamma^{(1)} = \dot{\gamma}$ and $\gamma^{(k)} = \dot{\widehat{\gamma^{(k-1)}}}$. In fact, for every $p \in T^{(k)}M$ there is a parameterized curve $\gamma : \mathbb{R} \to M$ such that for some $t_0 \in \mathbb{R}$, $\gamma^{(k)}(t_0) = p$.

On $T^{(k)}M$ there is a natural action $Gl_k(\mathbb{R})$, the k^{th} prolongation of the general linear of $\mathbb{R}$. This group acts freely and properly discontinuously on the complement of $\tau^{-1}(0_M)$ where $\tau: T^{(k)}M \to TM$ is the natural surjection, and 0_M is the the zero section of TM [4]. The quotient manifold $P^{(k)}M = (T^{(k)}M - \tau^{-1}(0_M))/Gl_k(\mathbb{R})$ is the k^{th} projective tangent space of M. If $\gamma: \mathbb{R} \to M$ is an immersion and $P: T^{(k)}M - \tau^{-1}(0_M) \to P^{(k)}M$ is the projection, then $P\gamma^{(k)}$ is easily seen to be a parameter independent object along γ. To construct a realization of $P\gamma^{(k)}$ introduce a hypersurface $S \subset TM$ that possesses the following properties

1. If S is to provide a model for the projectivized tangent space at each point then S must be transverse to the vertical bundle VTM; that is, $TS + VTM|_S = T^2M|_S$.
2. For every $v \in S$, any ray in $TM_{\pi(v)}$ that intersects S must intersect S in a unique point.
3. For every $v \in S$, if $i: TM_{\pi(v)} \to VTM_v$ is the natural identification, then $i(v) \notin TS_v$.

Notice that properties 2. and 3. imply that for each $v \in S$ there is a unique 1-form λ_v such that $\ker(\lambda_v) = TS_v$ and $\lambda_v(v) = 1$. The fact that such hypersurfaces define a unique 1-form for each $v \in S$ has the consequence that there is a map $\ell: S \to T^*M$ defined by $\ell(v) = i^*\lambda_v|_{VTM}$. The map ℓ is simply the Legendre map associated with S. Suppose that $u \in TM$ with the property that there is $v \in S$ so that $u = tv$ for some $t > 0$. Since v is uniquely determined by u, let $v = Pu$. It is easy to see that $t = 1/\ell(Pu)(u)$ and so $u/\ell(Pu)(u) \in S$. This fact motivates the following definition.

DEFINITION 1.1. *The* k^{th}-projective derivative *of a parameterized curve* $\gamma : \mathbb{R} \to M$ *associated with a hypersurface* S *that satisfies properties 1, 2, and 3 is defined recursively by the expressions:*

$$\delta^{(k)}\gamma = \frac{1}{\ell(P\dot{\gamma})(\dot{\gamma})}\dot{\overbrace{\delta^{(k-1)}\gamma}} \quad \text{and} \quad \delta^{(1)}\gamma = \frac{1}{\ell(P\dot{\gamma})(\dot{\gamma})}\dot{\gamma} \tag{1.1}$$

PROPOSITION 1.1. *For any curve* $\gamma: \mathbb{R} \to M$, $\delta^{(k)}\gamma: \mathbb{R} \to T^{(k)}M$ *and if* $\sigma: \mathbb{R} \to \mathbb{R}$ *is a diffeomorphism then* $\delta^k(\gamma \circ \sigma)(t) = \delta^{(k)}\gamma(\sigma(t))$.

The proof of this proposition can be found in [4]. Since $\delta^{(k)}\gamma$ is determined by γ and its first k–derivatives, propositition 1.1 implies that definition 1.1 gives a map $\Delta^k: T^{(k)}M \to T^{(k)}M$ with the property that $\Delta^k(\gamma^{(k)}) = \delta^{(k)}\gamma$. Proposition 1.1 shows that Δ^k is a modular map for the action of $Gl_k(\mathbb{R})$ on $T^{(k)}M$, and so gives a realization of the kth parameter indepedent derivative. This representation depends on the choice of the hypersurface S. However, if the values of Δ^k are only required along a fixed curve γ, then any surface may be replaced with a hyperplane. This is done by choosing a 1-form λ on M defined in a neighborhood of γ with the property that if $\ell: S \to T^*M$ is the Legendre map defined by the given hypersurface, then $\lambda(\gamma(t)) = \ell(P(\dot{\gamma}))$. Because of this fact the constructions in the next section will only involve hyperplanes.

2. Multisymplectic Realizations

This section introduces the parameter independent derivatives of an immersion $\gamma: \mathbb{R} \to M$ obtained from the submanifold $S_1 = \{v \in TM | v = s\dot{\gamma}(t), s, t \in \mathbb{R}\}$. Since S_1 is 2-dimensional the standard coordinates on $T\mathbb{R}$ determine a 2-vector field along S_1 that can be viewed as a section of $\Lambda^2(TM)$ along S_1. Reparameterization of γ scales this 2-vector field and so determine a 3-dimensional subspace S_2 of $\Lambda^2(TM)$. The submanifold S_2 in turn possesses a natural 3-vector field which can be viewed as a section of $\Lambda^3(\Lambda^2(TM))$. In this manner γ determines an $n+1$-dimensional submanifold S_n of $M^n = \Lambda^n(\Lambda^{n+1}(\ldots\Lambda^2(TM)\ldots))$. We may introduce a similar extension of M by form bundles; $M_n = \Lambda_n(\Lambda_{n-1}(\ldots\Lambda_2(T^*M)\ldots))$. To construct parameter independent lifts of γ to S_n and to introduce dynamical structures, an identification of M_n and M^n is required.

To define an identification map $i_n: M_n \to M^n$, assume that M possesses a metric g and let $\ell: T^*M \to TM$ be the identification induced by g. First, let $i_1 = \ell$ and for $n > 1$ construct i_n as follows. Suppose that for $k \leq n$, i_k has been defined and suppose a metric has been constructed on M_{n-1}.

Using the horizontal distribution of the Levi Civita connection in $\Lambda_n(M_{n-1})$, the metric on M_{n-1} can be lifted by various procedures [7] to a metric on M_n. The lifted metric defines an identification map $\ell': T^*M_n \to TM_n$. The identification of M_{n+1} and M^{n+1} is then defined in terms of i_n and ℓ' by the composition $i_{n+1} = \Lambda^{n+1} i_{n*} \circ \Lambda_{n+1}\ell'$.

In this section this identification is used to construct a natural set of coordinates on M^n when $M = \mathbb{R}$. Denote this space by $(\mathbb{R})^n$ and the dual object by $(\mathbb{R})_n$. First observe that the fiber of $\Lambda_n((\mathbb{R})_{n-1}) \xrightarrow{\pi} (\mathbb{R})_{n-1}$ is one dimensional. Now, because $(\mathbb{R})_{n-1}$ is itself an $(n-1)$–form bundle, there is a canonical nonvanishing n–form ω_n on $(\mathbb{R})_{n-1}$ that spans the fiber of $(\mathbb{R})_n$. As a result, if $\mu \in (\mathbb{R})_n$, there is $\lambda \in (\mathbb{R})_{n-1}$ with $\pi(\mu) = \lambda$ and $\mu = (\lambda, t_n\omega_n)$. Inductively one obtains that $\mu = (t_0, t_1\omega_1, \ldots, t_n\omega_n)$. Using the natural euclidean structure on $\mathbb{R}$, an identification map $i_n: (\mathbb{R})_n \to (\mathbb{R})^n$ is constructed as in the last paragraph. Hence, the n-form ω_n determines an n-vector field $\omega^n = i_n\omega_n$ and so an element $v \in (\mathbb{R})^n$ can in a similar manner be written as $v = (t_0, t_1\omega^1, \ldots, t_n\omega^n)$. This decomposition of $(\mathbb{R})^n$ defines a chart $\tau_n: (\mathbb{R})^n \to \mathbb{R}^{n+1}$ given by $\tau_n(v) = (t_0, t_1, \ldots, t_n)$. It can be easily seen that, in terms of the standard coordinate vector fields $(\frac{\partial}{\partial t_0}, \ldots, \frac{\partial}{\partial t_{n-1}})$ on $\mathbb{R}^n$, $\omega^n = \tau_{n-1}^{-1}{}_*(\frac{\partial}{\partial t_{n-1}} \wedge \ldots \wedge \frac{\partial}{\partial t_0})$. Using this fact, a point $v \in (\mathbb{R})^n$ may be represented by the expression

$$v = t_n{}^n \tau_{n-1}^{-1}{}_*(\frac{\partial}{\partial t_{n-1}} \wedge \ldots \wedge \frac{\partial}{\partial t_0})_{\tau_{n-1}^{-1}(t_0,\ldots,t_{n-1})}. \tag{2.1}$$

Suppose now that $\gamma: \mathbb{R} \to M$ is an immersion. Clearly γ induces a map $\gamma_*: (\mathbb{R})^n \to M^n$ which is easily seen to be an immersion. However, this map is computationally inconvenient. To obtain a more useful object, compose γ_* with a map $\rho: \mathbb{R}^n \to \mathbb{R}^n$; the components of which are defined recursively as follows. Let $\rho = (\rho_0, \ldots, \rho_{n-1})$, and define $\rho_0(t_0, \ldots, t_{n-1}) = t_0$, $\rho_1(t_0, \ldots, t_{n-1}) = t_1$, and for $1 \leq k \leq n-1$,

$$\rho_k(t_0, \ldots, t_{n-1}) = t_k{}^k \frac{\partial \rho_{k-1}}{\partial t_{k-1}} \cdots \frac{\partial \rho_1}{\partial t_1}(t_0, \ldots, t_{n-1}). \tag{2.2}$$

Observe that each component ρ_k for $k \geq 1$ is a monomial in the variables $t_0, \ldots t_k$ of degree 2^{k-1}. The desired parameterization of $\gamma_*((\mathbb{R})^n)$ is obtained from a modification of (2.1) as follows.

$$G_\gamma^n(t_0, \ldots, t_n) = t_n{}^n \gamma_*(\tau_{n-1}{}^{-1} \circ \rho)_*(\frac{\partial}{\partial t_{n-1}} \wedge \ldots \wedge \frac{\partial}{\partial t_0})_{\tau_{n-1}{}^{-1} \circ \rho(t_0,\ldots,t_{n-1})} \tag{2.3}$$

The reason for introducing this singular parameterization is that G_γ^n has a simple scaling property; as the following proposition shows. The proof can be found in [4].

PROPOSITION 2.1. *Let $\sigma: \mathbb{R} \to \mathbb{R}$ be a diffeomorphism and let $\sigma_*: (\mathbb{R})^n \to (\mathbb{R})^n$ be the induced map. Define $s: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by $s(t_0, \ldots, t_n) = (\sigma(t_0), \dot\sigma(t_0)t_1, \ldots, \dot\sigma(t_0)t_n)$, then $G_{\gamma\circ\sigma}^n = G_\gamma^n \circ s$.*

It is easy to see that given a 1-form λ on M this proposition implies that

$$G^n_\gamma(t) = G^n_\gamma(t, \frac{1}{\lambda(\dot\gamma)}(t), \ldots, \frac{1}{\lambda(\dot\gamma)}(t))$$

is independent of parameterization and so defines a parameter independent derivative.

3. Equivalence

The objective of this section is to explain the relation between the n^{th} projective derivative of γ associated with a 1-form λ and the map $G^n_\gamma(t)$ introduced in the last section. The derivation of the relation between $\delta^{(n)}\gamma$ and G^n_γ requires several additional geometric structures. The definition and manipulation of these objects is somewhat complicated, and so the reader may wish to concurrently examine example 2.1 at the end of this section. First observe that points of M^n of the form $G^n_\gamma(t, \frac{1}{\lambda(\dot\gamma)}(t), \ldots, \frac{1}{\lambda(\dot\gamma)}(t))$ are the image by G^n_γ of the subspace $W \subset I\!R^{n+1}$ spanned by $\{(1,0,\ldots,0),(0,1,\ldots,1)\}$. Now the annihilator $\mathrm{ann}(W) \subset I\!R^{n+1^*}$ is spanned by the 1-forms $\{dt_n - dt_{n-1}, \ldots, dt_2 - dt_1\}$. Proposition 3.1 shows that the pullback of these 1-forms have the scaling property

$$(G^n_{\gamma\circ\sigma})^{-1^*}(dt_{i+1} - dt_i)_{G^n_{\gamma\circ\sigma}(t_0,\ldots,t_n)} = (G^n_\gamma)^{-1^*} s^{-1^*}(dt_{i+1} - dt_i)_{(t_0,\ldots,t_n)}$$

$$= \frac{1}{\dot\sigma(t_0)}(G^n_\gamma)^{-1^*}(dt_{i+1} - dt_i)_{G^n_\gamma(s(t_0,\ldots,t_n))}.$$

Consequently, the 1-forms $f_i = \lambda(\dot\gamma)(G^n_\gamma)^{-1^*}(dt_{i+1} - dt_i)$ defined along $G^n_\gamma(W)$ with values in $T^*G^n_\gamma(I\!R^{n+1})$ are parameter invariant 1-forms that annihilate $TG^n_\gamma(W)$.

The next required structure is a natural diffeomorphism between TT^*M and T^*TM. This diffeomorphism, denoted by $C: TT^*M \to T^*TM$, is constructed in a manner that is similar to the construction of the natural involution of T^2M described in the last section [6], [1]. In this case the map C determines a pair of compositions. First, C satisfies $\pi' \circ C = \pi_*$ where $\pi': T^*TM \to TM$ is the bundle map and the map $\pi_*: TT^*M \to TM$ is induced by $\pi: T^*M \to M$. Second, $j \circ C = \pi''$ where $\pi'': TT^*M \to T^*M$ is the bundle map, and the submersion $j: T^*TM \to T^*M$ is defined in terms of the natural identification $i: TM_{\pi(p)} \to VTM_p$ as follows. For $\lambda \in T^*(TM)_p$ let $j(\lambda) = i^*(\lambda|_{VTM_p})$. The identification C is useful because a 1-form λ on M can be lifted to a 1-form λ^1 on TM by the relation $\lambda^1(v) = C\lambda_*(v)$ for $v \in TM$. This lift is generally referred to as the complete lift of λ [7]. In this construction the 1-form λ^1 will also be lifted to a 1-form λ^n on M^n by pulling back with the projection $\pi: M^n \to TM$; that is $\lambda^n = \pi^*\lambda^1$.

Now λ^n can be restricted to $TG^n_\gamma(\mathbb{R}^{n+1})$ and along with the 1-forms $f_1, \ldots, f_n$ determines a system of n independent 1-forms in $T^*G^n_\gamma(\mathbb{R}^{n+1})$ along $G^n_\gamma(W)$. This frame can be used to derive the identity stated in the following proposition.

PROPOSITION 3.1. $\frac{1}{\lambda(\dot{\gamma})}\frac{d}{dt}G^n_\gamma(t) = \iota(\lambda^n)\iota(f_1)\ldots\iota(f_{n-1})G^{n+1}_\gamma(t)$.

The proof of this proposition can be found in [4]. However, essential to the demonstration of this relation is the following identity.

$$\iota(f_1)\ldots\iota(f_{n-1})G^{n+1}_\gamma(t) = s^2\Big(\sum_{i=1}^{n} G^n_{\gamma_*}\tfrac{\partial}{\partial t_i}\Big) \wedge G^n_{\gamma_*}\tfrac{\partial}{\partial t_0} \tag{3.1}$$

To obtain a relation between $\delta^{(n)}\gamma$ and G^n_γ from proposition 3.1. requires the following lemma concerning bilinear pairings in vector bundles.

LEMMA 3.1. *Let E_1, E_2, and E_3 be vector bundles over M with bundle maps π_1, π_2, and π_3. If $\mu: E_1 \oplus E_2 \to E_3$ is bilinear, then the differential of μ, $\mu_*: TE_1 \oplus TE_2 \to TE_3$, is bilinear relative to the bundle structures determined by π_{1*}, π_{2*}, and π_{3*}. Also, if $t_s: TE_i \to TE_i$ is multiplication in the fiber of $TE_i \to E_i$ for $i = (1,2,3)$, then for $v \in TE_1$ and $w \in TE_2$ with $\pi_{1*}(v) = \pi_{2*}(w)$, $\mu_*(t_s v, t_s w) = t_s\mu_*(v,w)$.*

To proceed, fix a parameterized curve $\gamma: \mathbb{R} \to M$, and extend the 1-forms f_i to 1-forms $\tilde{f}_i$. Using this extension it is possible to restate proposition 3.1 in terms of a bilinear pairing $\mu_{n+1}: M^{n+1} \oplus T^*M^n \to TM^n$ where M^{n+1}, T^*M^n, and TM^n are vector bundles over M^n. The pairing μ_{n+1} is defined for each $p \in M^n$ to have the value

$$\mu^{n+1}(w,h) = \iota(h)\iota(\tilde{f}_1)\ldots\iota(\tilde{f}_{n-1})w$$

on $w \in {M^{n+1}}_p$ and $h \in {T^*M^n}_p$. Applying lemma 3.1 to μ^{n+1}, r times and each time composing the second argument with the identification $C: TT^*T^{s-1}M \to T^*T^sM$ leads to a bilinear pairing

$$\mu^{(r)}_{n+1}: T^rM^{n+1} \oplus T^*T^rM^n \to T^{r+1}M^n.$$

The next step is to lift the 1-form λ^n to T^rM^n. This is done recursively by defining $\delta^{(r)}{\lambda^n}_w \in T^*T^rM$ by the expression $\delta^{(r)}{\lambda^n}_w = C(\delta^{(r-1)}{\lambda^n}_*(w))$, where of course $\delta^{(0)}\lambda^n = \lambda^n$. For a fixed n and r define $\iota(\delta^{(r)}\lambda^n): T^rM^{n+1} \to T^{r+1}M^n$ by

$$\iota(\delta^{(r)}\lambda^n)(w) = \mu^{(r)}_n(w, \delta^{(r)}\lambda^n). \tag{3.2}$$

THEOREM 3.1. *If $\gamma: \mathbb{R} \to M$ is a parameterized curve and if λ is a 1-form on M, then the $n+1^{th}$ projective derivative of γ relative to λ is given by*

$$\delta^{(n+1)}\gamma(t) = \iota(\delta^{(n-1)}\lambda^1)\iota(\delta^{(n-2)}\lambda^2)\ldots\iota(\lambda^n)\iota(f_1)\ldots\iota(f_{n-1})G_\gamma^{n+1}(t) \quad (3.3)$$

The proof of Theorem 3.1 is an induction argument involving an upper triangular diagram. The rows of the diagram represent the composition of map found on the right hand side of (3.3) and the columns correspond to projective derivatives. The first step in the induction uses Proposition 3.1 . For more details see [4].

Example 2.1 The point of this example it to express in local coordinates the various 1–forms and multi-vector fields involved in the calculation of the projective derivative from (3.3). Consider the case where $n = 2$. For the purpose of this calculation let $M = \mathbb{R}^k$. The construction described above then occurs in $\Lambda^2(T\mathbb{R}^k)$ which can be decomposed as $\mathbb{R}^k \times \mathbb{R}^k \times \Lambda^2(\mathbb{R}^k \times \mathbb{R}^k)$. Let $\gamma\colon \mathbb{R} \to \mathbb{R}^k$ be an immersion and let λ be a 1-form defining a local projective model for $T\mathbb{R}^k$. In the following denote the Jacobian of λ by $D\lambda$. Theorem 3.1 states that $\delta^{(3)}\gamma = \iota(\delta\lambda)\iota(\lambda^2)\iota(f^1)G_\gamma^3$. To verify this identity in local coordinates first use (3.1) to express $\iota(f^1)G_\gamma^3 \in \Lambda^2(\Lambda^2(T\mathbb{R}^k))$. This step requires $G_\gamma^2(t_0,t_1,t_2) \in \Lambda^2(T\mathbb{R}^k)$ which can be seen to be

$$G_\gamma^2(t_0,t_1,t_2) = \Big(\gamma(t_0), t_1\dot\gamma(t_0), {t_2}^2((0,\dot\gamma(t_0)) \wedge (\dot\gamma(t_0), t_1\ddot\gamma(t_0))\Big).$$

To simplify, the variable t_0 will be taken to be implicit in the following expressions. From this identity it is easily seen that the map G_γ^2 determines the following tangent vectors in $T\Lambda^2(\mathbb{R}^k)_{G_\gamma^2(t_0,t_1,t_2)}$.

$$\left(G^2_{\gamma *}\frac{\partial}{\partial t_0}\right)_{G_\gamma^2(t_0,t_1,t_2)} = \Big(\dot\gamma, t_1\ddot\gamma, {t_2}^2\Big((0,\ddot\gamma)\wedge(\dot\gamma,0) + (0,\dot\gamma)\wedge(\ddot\gamma, t_1\gamma^{(3)})\Big)\Big) \quad (3.4)$$

$$\left(G^2_{\gamma *}\frac{\partial}{\partial t_1}\right)_{G_\gamma^2(t_0,t_1,t_2)} = \Big(0, \dot\gamma, {t_2}^2(0,\dot\gamma)\wedge(0,\ddot\gamma)\Big) \quad (3.5)$$

$$\left(G^2_{\gamma *}\frac{\partial}{\partial t_2}\right)_{G_\gamma^2(t_0,t_1,t_2)} = \Big(0,0,2t_2(0,\dot\gamma)\wedge(\dot\gamma, t_1\ddot\gamma)\Big) \quad (3.6)$$

Evaluating these expressions at $(t, s(t), s(t))$ where as before $s(t) = 1/\lambda(\gamma(t))$ and substituting into (3.1) gives

$$\iota(f_1)G_\gamma^3(t)_{G_\gamma^2} = s^2\Big(0,\dot\gamma, 2s(0,\dot\gamma)\wedge(\dot\gamma, s\ddot\gamma) + s^2(0,\dot\gamma)\wedge(0,\ddot\gamma)\Big)$$
$$\wedge\Big(\dot\gamma, s\ddot\gamma, s^2\Big((0,\ddot\gamma)\wedge(\dot\gamma,0) + (0,\dot\gamma)\wedge(\ddot\gamma, s\gamma^{(3)})\Big)\Big).$$

Next, $\iota(f_1)G_\gamma^3(t)$ is contracted with λ^2 and $\delta^1\lambda$ evaluated at $G_\gamma^2(t) \in \Lambda^2(\mathbb{R}^k)$ and $\delta^{(2)}\gamma \in T^2M$ respectivily. Since $\delta^{(2)}\gamma = (\gamma, s\dot\gamma, s\dot\gamma, \dot{\widehat{s s\dot\gamma}})$,

$$\lambda^2{}_{G_\gamma^2} = (sD\lambda(\dot\gamma), \lambda, 0)$$

$$\delta\lambda^1{}_{\delta^2\gamma} = (D^2\lambda(s\dot\gamma, s\dot\gamma) + D\lambda(\dot{\widehat{s s\dot\gamma}}), sD\lambda(\dot\gamma), sD\lambda(\dot\gamma), \lambda)$$

A calculation now shows that $\iota(\lambda^2)\iota(f_1)G^3_\gamma(t) \in T\Lambda^2(I\!R^k)$ is given by

$$\iota(\lambda^2)\iota(f_1)G^3_{\gamma G^2_\gamma} = \Big(s\dot{\gamma}, \dot{\widehat{ss\gamma}}, s^3\left((0,\dot{\gamma})\wedge(\ddot{\gamma}, s\gamma^{(3)}) + (0,\ddot{\gamma})\wedge(\dot{\gamma},0)\right) \tag{3.7}$$

$$-s^3\left(D\lambda(\dot{\gamma})(\dot{\gamma}) + \lambda(\ddot{\gamma})\right)\left(2s(0,\dot{\gamma})\wedge(\dot{\gamma},0) + 3s^2(0,\dot{\gamma})\wedge(0,\ddot{\gamma})\right)\Big)$$

The final interior product defined by (3.2) contracts the base point G^2_γ with λ^1 to give the base point of the image $\delta^{(2)}\gamma$. To obtain the tangent vector at the base point contract the first two coordinate blocks of $\delta\lambda^1$ with G^2_γ and add this to the contraction of the last two blocks of $\delta\lambda^1$ with the 2-vector component of (3.7) to give the last two blocks of $\delta^{(3)}\gamma$.

4. Applications to Dynamical Structures

Theorem 4.1 can be used to define a geometric structure that describes projectively invariant differential equations on a manifold M. This formalism will be illustrated by deducing the Lorentz force law and the Lorentz-Dirac equation from this structure. The geometric objects needed to define the geometric structure associated with an invariant $k+1^{th}$ order system are a metric on M, a certain $k+1$-vector field on M^k, and a special extension of the 1-forms $f_1, \ldots, f_{k+1}$. The required $k+1$ vector fields where introduced in [3].

DEFINITION 4.1. *A differential $k+1$-form ($k+1$-vector field) Ω on N is said to be* almost multi-symplectic *if there exists a subbundle W of TN (T^*N) such that for all $p \in N$*

1. *Ω is nondegenerate; that is if for $v \in TN_p$ ($v \in T^*N_p$), $\iota(v)\Omega_p = 0$ then $v = 0$.*
2. *For $u, v \in W_p$, $\iota(u \wedge v)\Omega_p = 0$.*
3. $\dim(W) = \dim(\Lambda_k(TN/W))$ *and* $\dim(W) > \dim(TN/W)$, ($\dim(W) = \dim(\Lambda^k(T^*N/W))$ *and* $\dim(W) > \dim(T^*N/W)$).

A word of caution concerning the terminology is required. Other authors use this term to describe a more general class of differential forms namely those that are closed and nondegenerate. Almost multi-symplectic $k+1$ vector fields on $\Lambda^k(N)$ can be obtained from a metric g on N and the canonical $k+1$-form μ_0 on $\Lambda_k(N)$ by the procedure described at the beginning of section 3. Let $H \subset T\Lambda_k(N)$ be the horizontal distribution of the Levi Civita connection in $\Lambda_k(N)$. Using H, the metric on N can be lifted to a metric on $\Lambda_k(N)$ with Legendre map $\tilde{\ell}: T\Lambda_k(N) \to T^*\Lambda_k(N)$. It is easy to see that $\Theta_0 = (\wedge^{k+1}\tilde{\ell})\mu_0$ is an almost multi-symplectic $k+1$ vector field. It is shown in [3] that all multi-symplectic vector fields that have the vertical bundle as their defining distribution can be expressed in terms of an map $A: H \to V\Lambda_k(N)$ and the projection $P: T\Lambda_k(N) \to H$ by the expression

$$\Theta(\lambda_0, \ldots, \lambda_k) = \sum_{i=0}^{k} k!\Theta_0((P + AP)^T\lambda_0, \ldots, \lambda_i, \ldots, (P + AP)^T\lambda_k), \quad (4.1)$$

where $\lambda_0, \ldots, \lambda_1$ are 1-forms on $\Lambda_k(N)$. In the case where $N = TM$ and $k = 1$, such 2-vector fields are of the form $\Theta = \Theta_0 + E$ where E is a 2-vector field supported on VTM.

The metric q on M is employed in various constructions. First, along each curve $\gamma: I\!R \to M$ on which q is nondegenerate introduce vector field X defined in a neighborhood of γ with the property that $\dot\gamma(t) = X(\gamma(t))$. The 1-form λ appearing in Theorem 4.1. is determined by X and q by the expression $\lambda = \mathrm{sign}(q(X,X))(1/\|X\|)\ell X$ where $\|X\| = \sqrt{|q(X,X)|}$ and where ℓ is the Legendre map of q. Second, the Levi Civita connection associated with q permits the identification of structures on $T^{(k)}M$ or M^k with structures on M. In this regard the following lemma is crucial. Recall that if $S: T^2M \to T^2M$ is the natural involution, then the complete lift of a vector field X on M to TM is given by $\overline{X} = SX_*$ and similarly the complete lift of a 1-form λ on M is given by $\overline{\lambda} = C\lambda_*$. Also let $\imath X$ be the vertical lift of X; that is for $v \in TM$ $\imath X_v = i(X_{\pi(v)})$. If Θ_0 is the canonical two vector field constructed above, then let $\Lambda_0 = \ell^{-1}{}_*\Theta_0$.

LEMMA 4.1. *The Levi Civita covariant derivative* ∇ *of a vector field* X *satisfies*

$$\imath(\nabla_v X) = \frac{1}{2}(\overline{X}_v - \iota(\overline{\ell X}_v)\Lambda_0). \quad (4.2)$$

Using the given multi-symplectic vector field Λ on M^k and the metric q, it is a conjecture that there is an extension of the 1-forms $f_1, \ldots, f_{k-1}$ defined on $G^k_\gamma(I\!R^{k+1})$ to 1-forms $\hat f_1, \ldots, \hat f_{k+1}$ on M^k such that the identity

$$\iota(\delta^{(n-1)}\lambda^1) \ldots \iota(\delta\lambda^{n-1})\iota(\lambda^n)\iota(\hat f_1) \ldots \iota(\hat f_{n-1})(G^{n+1}_\gamma(t) - \Lambda_{G^n_\gamma(t)}) = 0, \quad (4.3)$$

determines a system of $k+1^{th}$ order parameter independent ordinary differential equations on M. The existence of such extensions is of course trivial when $k = 1$, and the conjecture will be shown to be true for $k = 2$. To illustrate the type of systems that arise from (4.3) consider first the case where $k = 1$. In this case (4.3) reduces to

$$\iota(\lambda^1)(G^2_\gamma(t) - \Lambda_{G^1_\gamma(t)}) = 0, \quad (4.4)$$

where $\Lambda = \Lambda_0 + \imath E$ and E is a 2-vector field on M. The following proposition follows from a calculation based on lemma 4.1 and shows that (4.4) is equivalent to the Lorentz force law.

PROPOSITION 4.1. *A curve $\gamma\colon \mathbb{R} \to M$ is a solution of (4.4) if and only if it is a solution of the projective Lorentz force law*
$(1/\|\dot\gamma\|^2)\nabla_{\dot\gamma}\dot\gamma^{\perp} = -\frac{1}{2}\iota((1/\|\dot\gamma\|)\ell\dot\gamma)E$.

The case for $k = 2$ is considerably more complicated. In this case (4.3) reduces

$$\iota(\delta\lambda^1)\iota(\lambda^2)\iota(\hat f)(G^3_\gamma(t) - \Lambda_{G^2_\gamma(t)}) = 0, \tag{4.5}$$

where Λ is a multi-symplectic 3-vector field on $\Lambda^2(TM)$ constructed using (4.1). This identity will yield a parameter invariant third order system if the 1-form f_1 can be extended to a 1-form $\hat f$ so that

$$\pi_*\iota(\lambda^2)\iota(\hat f)\Lambda_{G^2_\gamma(t)} = \iota(\lambda^1)G^2_\gamma(t). \tag{4.6}$$

Also it is necessary that the metric q on M be lifted to a neutral metric on TM. That (4.6) can be satisfied is the content of the following theorem.

THEOREM 4.1. *Let $\gamma\colon \mathbb{R} \to M$ be a nondegenerate curve is a pseudo-riemannian manifold M such that $\nabla_{\dot\gamma}\dot\gamma(t)$ and $\dot\gamma(t)$ are independent for all t. The 1-form f_1 with values in $T^*G^2_\gamma(\mathbb{R}^3)_{G^2_\gamma(t)}$ can be extended to a 1-form $\hat f$ with values in $T^*\Lambda^2(TM)_{G^2_\gamma(t)}$ that satisfies (4.6).*

Proof. If $E \to N$ is a vector bundle with connection ∇, denote the horizontal lift of a vector field X on N to a vector field on E by $\widetilde X$. To construct the 1-form $\hat f$ requires a lifted metric g on TM. To obtain a consistent third order system it is necessary that q be lifted to a neutral metric g on TM given by $g = -\imath q \oplus \widetilde q$. The Levi Civita covariant derivative D for g is related to the Levi Civita covariant derivative ∇ for q by the following expressions [2]. If X, Y, U, V are vector fields on M and if R is the curvature of ∇, then for $v \in TM$

$$\begin{aligned} D_{\imath V}\widetilde X_v &= \tfrac{1}{2}\widetilde{R(v,V)}X & D_{\imath U}\imath V &= 0 \\ D_{\widetilde Y}\widetilde X_v &= \tfrac{1}{2}\imath R(Y,X)v + \widetilde{\nabla_Y X} & D_{\widetilde X}\imath V &= \imath\nabla_X V + \tfrac{1}{2}\widetilde{R(v,V)}X. \end{aligned} \tag{4.7}$$

In the following, $\gamma(t)$ will be assumed to be time-like; that is $q(\dot\gamma(t),\dot\gamma(t)) < 0$. The curve γ induces the maps $G^1_\gamma(t_0,t_1) = t_1\dot\gamma(t_0)$ and

$$G^2_\gamma(t_0,t_1,t_2) = {t_2}^2 G^1_{\gamma*}\tfrac{\partial}{\partial t_1} \wedge G^1_{\gamma*}\tfrac{\partial}{\partial t_0}.$$

To construct an extension of f_1, first write the coordinate vector fields $G^2_{\gamma*}\frac{\partial}{\partial t_0}$, $G^2_{\gamma*}\frac{\partial}{\partial t_1}$, and $G^2_{\gamma*}\frac{\partial}{\partial t_2}$, in terms of the splitting of $T\Lambda^2(TM)$ and T^2M defined by the connections D and ∇. Observe that

$$G^2_{\gamma*}\frac{\partial}{\partial t_i} = \imath {t_2}^2 D_{\frac{\partial}{\partial t_i}} G^1_{\gamma*}\frac{\partial}{\partial t_1} \wedge G^1_{\gamma*}\frac{\partial}{\partial t_0} + \widetilde{G^1_{\gamma*}\frac{\partial}{\partial t_i}}. \tag{4.8}$$

As above $\imath$ is the vertical lift to $V\Lambda^2(TM)$. Since $G^1_{\gamma *}\frac{\partial}{\partial t_0} = \tilde{\dot\gamma} + t_1\imath\nabla_{\dot\gamma}\dot\gamma$ and $G^1_{\gamma *}\frac{\partial}{\partial t_1} = \imath\dot\gamma$ the relations (4.7) and (4.8) give

$$\begin{aligned}
G^2_{\gamma *}\frac{\partial}{\partial t_0} &= (\tilde{\dot\gamma} + \widetilde{t_1\imath\nabla_{\dot\gamma}\dot\gamma}) \\
&\quad + {t_2}^2\imath\Big(\imath\nabla_{\dot\gamma}\dot\gamma \wedge \tilde{\dot\gamma} + \imath\dot\gamma \wedge (\widetilde{\nabla_{\dot\gamma}\dot\gamma} - {t_1}^2 R(\dot\gamma, \widetilde{\nabla_{\dot\gamma}\dot\gamma})\dot\gamma + t_1\imath\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma)\Big) \\
G^2_{\gamma *}\frac{\partial}{\partial t_1} &= \imath\tilde{\dot\gamma} + {t_2}^2\imath(\imath\dot\gamma \wedge \imath\nabla_{\dot\gamma}\dot\gamma) \\
G^2_{\gamma *}\frac{\partial}{\partial t_2} &= 2t_2\imath\Big(\imath\dot\gamma \wedge (\tilde{\dot\gamma} + t_1\imath\nabla_{\dot\gamma}\dot\gamma)\Big) \qquad (4.9)
\end{aligned}$$

Since $f_1 = \lambda(\dot\gamma)(G^{2\,-1*}_\gamma dt_2 - G^{2\,-1*}_\gamma dt_1)$, to extend f_1, first extend $G^{2\,-1*}_\gamma dt_1$ and $G^{2\,-1*}_\gamma dt_2$ to τ_1 and τ_2 along $G^2_\gamma(t)$. At this point, set $t_0 = t$ and $t_1 = t_2 = 1/\lambda(\dot\gamma) = 1/\|\dot\gamma\|$. Also, recall that the standard normalization of the exterior algebra implies that if ν is a 2-form and if $u \wedge v$ is a simple 2-vector then the dual pairing of ν with $u \wedge v$ is given by $\langle \nu, u \wedge v\rangle = 2\nu(u,v)$. If $\omega_0 = \ell^*\mu_0$ where ℓ is the Legendre map determined by q on M and if $\hat\ell\colon T^2M \to T^*TM$ is the Legendre map determined by g on TM, then a calculation shows that τ_1 and τ_2 can be defined to depend on two arbitrary functions $a(t)$ and $d(t)$ and have the form

$$\begin{aligned}
\tau_1 &= \frac{1}{\|\dot\gamma\|^3}\hat\ell\widetilde{\nabla_{\dot\gamma}\dot\gamma} - \frac{1}{\|\dot\gamma\|^2}\hat\ell\imath\dot\gamma \\
\tau_2 &= \frac{1}{\|\dot\gamma\|^3}\hat\ell\widetilde{\nabla_{\dot\gamma}\dot\gamma} + \imath(a\omega_0 + \frac{1+4a\|\dot\gamma\|}{4\|\dot\gamma\|^3}\hat\ell\tilde{\dot\gamma} \wedge \hat\ell\imath\dot\gamma + d\hat\ell\tilde{\dot\gamma} \wedge \hat\ell\widetilde{\nabla_{\dot\gamma}\dot\gamma})
\end{aligned}$$

Consequently, $\hat f = \|\dot\gamma\|(\tau_2 - \tau_1)$ is an extension of f_1. The functions a and d are fixed by imposing (4.6). This computation requires several observations. First, denote by $\tilde\ell$ the Legendre map associated with the lifted metric on $\Lambda^2(TM)$. Now, $\hat\ell$ induces a map $\hat\ell\colon \Lambda^2(TM) \to \Lambda_2(TM)$. Suppose that Λ is a 3-vector field on $\Lambda^2(TM)$ given by (4.1) and $\Lambda_0 = \hat\ell^{-1}{}_*\tilde\ell^{-1}\beta_0$ where β_0 is the canonical 3-form on $\Lambda_2(TM)$. If ν is a 1-form on $\Lambda^2(TM)$ then ν decomposes into vertical and horizontal components as $\nu = \nu_V + \nu_H$. Since $\pi_*\iota(\nu)\Lambda = \pi_*\iota(\nu_V)\Lambda = \pi_*\iota(\nu)2\Lambda_0$, (4.6) only needs to be verified for $2\Lambda_0$. Next observe that if Σ is a 2-vector field on TM then $\pi_*\iota(\imath(\hat\ell\Sigma))\Lambda_0 = \Sigma$. Consequently,

$$\begin{aligned}
\pi_*\iota\left(\imath(\hat\ell\Sigma)\right)\Lambda_0 &= \pi_*\hat\ell^{-1}{}_*\tilde\ell^{-1}\iota(\tilde\ell^{-1}\imath(\Sigma))\beta_0 \\
&= \pi_*\tilde\ell^{-1}\iota(\imath(\hat\ell\Sigma))\beta_0 = \pi_*\tilde\ell^{-1}\pi^*\hat\ell\Sigma = \Sigma.
\end{aligned}$$

These facts imply that

$$\pi_*\iota(\hat f)\Lambda = 2\Big(a\|\dot\gamma\|\hat\ell^{-1}\omega_0 + \frac{1+4\|\dot\gamma\|a}{4\|\dot\gamma\|^2}\tilde{\dot\gamma} \wedge \imath\dot\gamma + d\|\dot\gamma\|\tilde{\dot\gamma} \wedge \widetilde{\nabla_{\dot\gamma}\dot\gamma}\Big). \qquad (4.10)$$

To determine a and d, (4.10) must be contracted with λ^1 where $\lambda = -(1/\|X\|)\ell X$ as defined above. Lemma 2.1 can be used to give an invariant expression for λ^1 since for any vector field X (4.2) implies that

$$\overline{\ell X}_v = 2\iota(\nabla_v X)\omega_0 - \iota(\overline{X})\omega_0.$$

Also for any smooth function f, it is easy to see that $\overline{fX}_v = df(v)\iota X + \pi^* f\overline{X}_v$. Using the fact that $\overline{X}_{\dot\gamma/\|\dot\gamma\|} = \widetilde{\dot\gamma} + \iota\nabla_{\dot\gamma/\|\dot\gamma\|}\dot\gamma$, it is now easy to see that

$$\lambda^1{}_{\frac{\dot\gamma}{\|\dot\gamma\|}} = \iota(\widetilde{\frac{\dot\gamma}{\|\dot\gamma\|}})\omega_0 - \iota(\iota\nabla_{\frac{\dot\gamma}{\|\dot\gamma\|}}\frac{\dot\gamma}{\|\dot\gamma\|})\omega_0. \tag{4.11}$$

Since $\nabla_{\dot\gamma/\|\dot\gamma\|}\frac{\dot\gamma}{\|\dot\gamma\|} = \frac{1}{\|\dot\gamma\|^2}\nabla_{\dot\gamma}\dot\gamma^\perp$, where $\nabla_{\dot\gamma}\dot\gamma^\perp$ is the component of $\nabla_{\dot\gamma}\dot\gamma$ orthogonal to $\dot\gamma$, a calculation shows that if $a = -1/2\|\dot\gamma\|$ and $d = \frac{3}{4}$ $(1/q(\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma^\perp))$, then (4.6) is satisfied.

The fact that (4.6) and the neutral lift of q lead to consistent third order systems is contained in the proof of the fact that if bundle map A in (4.1) is chosen properly then (4.5) is equivalent to the Lorentz-Dirac equation. The Lorentz-Dirac equation is a third order system on $n+1$ dimensional Minkowski space $(\mathbb{R}^{n+1}, \langle\cdot,\cdot\rangle)$ that describes the world line $\gamma: \mathbb{R} \to \mathbb{R}^{n+1}$ of a particle accelerating in an external electromagnetic field represented by a 2-vector field E on $\mathbb{R}^{n+1}$ and has the form.

$$m_e\frac{\ddot\gamma^\perp}{\|\dot\gamma\|^2} = \frac{e}{c}\iota(\frac{\ell\dot\gamma}{\|\dot\gamma\|})(E + \frac{e}{4\pi c^4}e_-) \tag{4.12}$$

Here e_- is the self-force and the constants m_e, e, and c are the renormalized mass of the electron, the charge of the electron, and the speed of light. To simplify matters, choose units so that $e/c = e/4\pi c^4 = -1$ and $m_e = 1$. Recall that the self force e_- is given by the expression.

$$e_-(\dot\gamma(t)) = \frac{2}{3}\frac{1}{\|\dot\gamma\|^6}(3\langle\ddot\gamma,\dot\gamma\rangle\dot\gamma\wedge\ddot\gamma - \langle\dot\gamma,\dot\gamma\rangle\dot\gamma\wedge\gamma^{(3)})(t) \tag{4.13}$$

To represent the various spaces involved in the demonstration of equivalence construction, introduce the natural splitting of $T\mathbb{R}^{n+1}$ and the coordinate notation of example 2.1; that is, at a general base point p, $T\mathbb{R}^{n+1}_p = \mathbb{R}^{n+1}\oplus\mathbb{R}^{n+1}$, $T^2\mathbb{R}^{n+1}_p = T\mathbb{R}^{n+1}_{\pi(p)}\oplus T\mathbb{R}^{n+1}_{\pi(p)}$, $T\Lambda^2(T\mathbb{R}^{n+1})_p = \Lambda^2(T\mathbb{R}^{n+1}_{\pi(p)})\oplus T^2\mathbb{R}^{n+1}_{\pi(p)}$ and $T^3\mathbb{R}^{n+1}_p = T^2\mathbb{R}^{n+1}_{\pi(p)} \oplus T^2\mathbb{R}^{n+1}_{\pi(p)}$.

THEOREM 4.2. *Let* $\gamma: \mathbb{R} \to \mathbb{R}^{n+1}$ *be a time-like curve, and* $A(t): T^2\mathbb{R}^{n+1}_{\delta^2\gamma(t)} \to \Lambda^2(T\mathbb{R}^{n+1})$ *be defined by*

$$A(t) = \frac{3}{2}\left((0, \frac{\dot\gamma}{\|\dot\gamma\|}) \wedge (0, \frac{\ddot\gamma^\perp}{\|\dot\gamma\|^2}) - \iota E_{\gamma(t)}\right) \otimes \pi'^*\lambda, \tag{4.14}$$

where $\pi' : T^2\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is the projection. If Λ is the 3-vector field determined by A and the natural splitting of $T\Lambda^2(\mathbb{R}^{n+1})$ given above, then (4.5) is equivalent to (4.12).

Proof. The proof will be broken into two segments. First, the case where $A = 0$ will be considered and then the general case will be treated. In the first case $\Lambda = \Lambda_0$. The key ingredients in this calculation are the 1-forms
$\hat{f}_{G^2_\gamma(t)} = \left(0, \frac{\ell\dot\gamma}{\|\dot\gamma\|}, -\frac{1}{2}\omega_0 - \frac{1}{4}\frac{1}{\|\dot\gamma\|^2}(\ell\dot\gamma, 0) \wedge (0, \ell\dot\gamma) + \frac{3}{4}\frac{\|\dot\gamma\|^2}{\langle\ddot\gamma, \ddot\gamma^\perp\rangle}(\ell\dot\gamma, 0) \wedge (\ell\ddot\gamma, 0)\right)$
$\lambda^2_{G^2_\gamma(t)} = -\left(\frac{\ell\ddot\gamma^\perp}{\|\dot\gamma\|^2}, \frac{\ell\dot\gamma}{\|\dot\gamma\|}, 0\right).$

Denote by Q and P the vertical and horizontal projection associated with the splittings of both $T\Lambda^2(\mathbb{R}^{n+1})$ and $T^3\mathbb{R}^{n+1}$. Using theorem 5.1 it is clear that $P\iota(\lambda^2)\iota(\hat f)2\Lambda_0 = \iota(\lambda^1)G^2_\gamma(t)$. If $\hat f$ is decomposed as $\hat f = Q^T\hat f + P^T\hat f$, where $P^T\hat f = (0, (1/\|\dot\gamma\|)\ell\dot\gamma, 0)$, then it is easy to see that

$$Q\iota(\lambda^2)\iota(\hat f)2\Lambda_0 = \iota(\lambda^2)\iota(P^T\hat f)2\Lambda_0 = -\imath\left(\left(\frac{\ddot\gamma^\perp}{\|\dot\gamma\|^2}, 0\right) \wedge \left(0, \frac{\dot\gamma}{\|\dot\gamma\|}\right)\right).$$

To compute $\iota(\delta\lambda^1)\iota(\lambda^2)\iota(\hat f)2\Lambda_{0\,\iota(\lambda^1)G^2_\gamma(t)}$, evaluate $\delta\lambda^1$ at $\iota(\lambda^1)G^2_\gamma(t) = \delta^2\gamma$ and use (3.2). A calculation shows that

$$\delta\lambda^1_{\delta^2\gamma} = \Big(\frac{1}{\|\dot\gamma\|^5}(\|\dot\gamma\|^2\ell\gamma^{(3)} + \langle\gamma^{(3)}, \dot\gamma\rangle\ell\dot\gamma + 3(\langle\ddot\gamma, \dot\gamma\rangle\ell\ddot\gamma - \langle\ddot\gamma, \ddot\gamma\rangle\ell\dot\gamma) - 4\frac{\|\ddot\gamma \wedge \dot\gamma\|^2}{\|\dot\gamma\|^2}\ell\dot\gamma),$$

$$\frac{\ell\ddot\gamma^\perp}{\|\dot\gamma\|^2}, \frac{\ell\ddot\gamma^\perp}{\|\dot\gamma\|^2}, \frac{\ell\dot\gamma}{\|\dot\gamma\|}\Big), \tag{4.15}$$

where $\|\ddot\gamma \wedge \dot\gamma\|^2 = \langle\dot\gamma, \dot\gamma\rangle\langle\ddot\gamma, \ddot\gamma\rangle - \langle\dot\gamma, \ddot\gamma\rangle^2$: Now using the decomposition of $T^3\mathbb{R}^{n+1}$ given above $\delta\lambda^1$ can be decomposed as $\delta\lambda^1 = P^T\delta\lambda^1 + Q^T\delta\lambda^1$. From the proof of lemma 4.1 it can be seen that

$$Q\iota(\delta\lambda^1_{\delta^2\gamma})\iota(\lambda^2)\iota(\hat f)\Lambda_{G^2_\gamma(t)} = \iota(Q^T\delta\lambda^1_{\delta^2\gamma})Q\iota(\lambda^2)\iota(\hat f)\Lambda_{0G^2_\gamma(t)} + \iota(P^T\delta\lambda^1_{\delta^2\gamma})G^2_\gamma(t).$$

Using the above expressions it can be shown that $\iota(P^T\delta\lambda^1_{\delta^2\gamma})G^2_\gamma(t) = 0$ and so

$$Q\iota(\delta\lambda^1_{\delta^2\gamma})\iota(\lambda^2)\iota(\hat f)\Lambda_{0G^2_\gamma(t)} = \left(\frac{\ddot\gamma^\perp}{\|\dot\gamma\|^2}, -\frac{\|\ddot\gamma \wedge \dot\gamma\|^2}{\|\dot\gamma\|^7}\dot\gamma\right).$$

However, it is easy to see that $\delta^{(3)}\gamma : \mathbb{R} \to T^3\mathbb{R}^{n+1}$ decomposes as

$$Q\delta^{(3)}\gamma = \Big(\frac{\ddot\gamma^\perp}{\|\dot\gamma\|^2}, \frac{1}{\|\dot\gamma\|^5}(\|\dot\gamma\|^2\gamma^{(3)} + \langle\gamma^{(3)}, \dot\gamma\rangle\dot\gamma + 3(\langle\ddot\gamma, \dot\gamma\rangle\ddot\gamma - \langle\ddot\gamma, \ddot\gamma\rangle\dot\gamma)$$

$$-4\frac{\|\ddot{\gamma}\wedge\dot{\gamma}\|^2}{\|\dot{\gamma}\|^2}\dot{\gamma}))$$

$$P\delta^{(3)}\gamma = \Big(\frac{\dot{\gamma}}{\|\dot{\gamma}\|}, \frac{\ddot{\gamma}^{\perp}}{\|\dot{\gamma}\|^2}\Big).$$

Since by construction $\pi_*\iota(\delta\lambda^1_{\delta^2\gamma})\iota(\lambda^2)\iota(\hat{f})\Lambda_{0G^2_\gamma(t)} = \delta^{(2)}\gamma$, it follows that

$$P(\delta\lambda^1_{\delta^2\gamma})\iota(\lambda^2)\iota(\hat{f})\Lambda_{0G^2_\gamma(t)} = (\frac{\dot{\gamma}}{\|\dot{\gamma}\|}, \frac{\ddot{\gamma}^{\perp}}{\|\dot{\gamma}\|^2}).$$

Therefore (4.13) gives

$$\iota(\delta\lambda^1_{\delta^2\gamma})\iota(\lambda^2)\iota(\hat{f})(G^3_\gamma(t) - 2\Lambda_{0G^2_\gamma(t)}) = -\frac{3}{2}\imath\Big((0, \iota(\frac{\ell\dot{\gamma}}{\|\dot{\gamma}\|})e_-(t))\Big). \tag{4.16}$$

Now suppose that A has the form given by (4.14). Note that if ν is a 1-form on $\Lambda^2(T\mathbb{R}^{n+1})$ with $\nu(V\Lambda^2(T\mathbb{R}^{n+1})) = 0$, then clearly $(P+A)^T\nu = 0$. Also, since $\hat{f}(\imath(VT\mathbb{R}^{n+1}\wedge VT\mathbb{R}^{n+1}) = 0$, it follows that $(P+A)^T\hat{f} = P^T\hat{f}$. These identities and the definition (4.1) imply that for any 1-form ν

$$\iota(\lambda^2)\iota(\hat{f})\Lambda(\nu) = \Lambda_0(P^T\hat{f}, \lambda^2, \nu) + \Lambda_0(P^T\hat{f}, \lambda^2, (P+A)^T\nu)$$
$$+\Lambda_0(\hat{f}, \lambda^2, (P+A)^T\nu).$$

To evaluate this identity observe that if ν is a section of $\mathrm{ann}(V\Lambda^2(T\mathbb{R}^{n+1}))$ or a section of $\imath(\ell HT\mathbb{R}^{n+1}\wedge \ell HT\mathbb{R}^{n+1}) \oplus \imath(\ell HT\mathbb{R}^{n+1}\wedge \ell VT\mathbb{R}^{n+1})$, then $\iota(\lambda^2)\iota(\hat{f})\Lambda(\nu) = \iota(\lambda^2)\iota(\hat{f})\Lambda_0(\nu)$. On the other hand, if ν is a section of $\imath(\ell VT\mathbb{R}^{n+1}\wedge \ell VT\mathbb{R}^{n+1})$,

$$(P+A)^T\nu = \nu\left((0, \frac{\ddot{\gamma}^{\perp}}{\|\dot{\gamma}\|^2})\wedge(0, \frac{\dot{\gamma}}{\|\dot{\gamma}\|}) - \imath E\right)\pi'^*\lambda,$$

and so

$$\iota(\lambda^2)\iota(\hat{f})\Lambda = \iota(\lambda^2)\iota(\hat{f})2\Lambda_0 + \frac{3}{2}\left((0, \frac{\ddot{\gamma}^{\perp}}{\|\dot{\gamma}\|^2})\wedge(0, \frac{\dot{\gamma}}{\|\dot{\gamma}\|}) - \imath E\right)2\Lambda_0(\hat{f}, \lambda^2, \pi'^*\lambda). \tag{4.17}$$

The constant $2\Lambda_0(\hat{f}, \lambda^2, \pi'^*\lambda)$ is found to be $2\Lambda_0(\hat{f}, \lambda^2, \pi'^*\lambda) = \lambda((1/\|\dot{\gamma}\|^2)\ell\dot{\gamma}) = 1$. Consequently, (4.17), (4.16), and (4.15) now imply the equivalence of (4.5) and (4.12).

One interesting feature of this construction is that since the coefficient of the self force is positive, the sign for the generalized Lorentz force A, must be chosen as in (4.14). Consequently, if the generalized Lorentz force is to be expressed by a difference, then the geometric structure imposes a negative charge on particles represented by this construction. Another interesting feature of this construction is that in order to obtain a consistent system of third order equations from (4.5), one is forced to lift the metric on M to a neutral metric on TM. This choice of lifted metric is also required in a nonlinear extension of Maxwell's equations. Finally, observe that the construction of theorem 4.2 is easily generalized to non-flat metrics.

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POLYNOMIAL LIE SUPER ALGEBRAS IN COMPOSITE MODELS WITH INTERNAL SYMMETRIES

VALERY P. KARASSIOV
Lebedev Physical Institute, Leninsky prospect 53,
Moscow 117924 RUSSIA
karas@sci.fian.msk.su

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Abstract. Generalized dual pairs of complementary groups and Lie-like (super)algebras obtained via a generalized Jordan mapping, are used for describing both invariance and dynamic symmetry of quantum composite models with internal symmetries. Applications of polynomial Lie algebras $sl_{pd}(2)$ are given for solving physical tasks in nonlinear many-body physics models having symmetry groups G_{inv}. Within this approach new classes of orthogonal polynomials are revealed. They are related to abelian (hyperelliptic) functions arising as some quasiclassical solutions of nonlinear generalizations of the Bloch equations. We also define some specific q-analogs of elliptic functions.

1. Introduction

Recently some new Lie-algebraic structures (quantum groups, W-algebras, deformed oscillator algebras, etc.) have been displayed in different areas of modern physics (see, e.g., [1-10] and references therein). All these objects, called as nonlinear or deformed Lie algebras "g_d" [6], may be considered as extensions $g_d = h + v$ of usual Lie algebras $h = \{E_c\}$ by their ireducible tensor operators $v = \{V_c\}$ satisfying the commutation relations(CRs)

$$[E_a, V_b] = \sum_c \tau^c_{ab} V_c,$$

$$[V_a, V_b]_\pm = f_{ab}(E_c), V_a \in v, E_c \in h$$

where τ^c_{ab} are matrix elements of operators V_c and $f_{ab}(E_c)$ are some power series in generators of the subalgebra "h" only (the so-called coset construction). Until quite recently such deformed Lie algebras g_d were examined mainly in context of quantum field theory and statistical physics models [3-6]. But results of the recent papers [6-10] show their use in other areas of modern quantum physics. Specifically, in [8, 9] we showed that deformed Lie (super)algebras g_{pd} arise in a natural manner in composite many-body physics models with Hamiltonians H having invariance groups

$G_{inv}([H,G_{inv}] = 0)$ and presented (via a G_{inv}-invariant generalization of Jordan mapping) as linear forms in elements of finite sets $I(G_{inv})$ of basic invariants of groups G_{inv}. The sets $I(G_{inv})$ endowed with commutators $[A,B]_{\pm} = AB \pm BA$ generate, in general, nonlinear Lie (super)algebras g_{pd} with the above structure functions $f_{ab}(E_c)$ being polynomials. Algebras g_{pd} retain certain properties of familiar Lie algebras [6, 9] and form together with G_{inv} generalized Weyl-Howe's dual pairs $(D_1 = G_{inv}, D_2 = g_{pd})$ [9] which act complementarily on the Hilbert spaces $L(H)$ of quantum states of models under study, i.e. there are decompositions

$$L(H) = \sum_{[l_i]} \oplus L([l_i]) \qquad (1)$$

of $L(H)$ into direct sums of the subspaces $L([l_i])$ which are invariant with respect to actions of both D_1 and D_2 (the label $[l_i]$ specifies irreducible representations (irreps) of both D_1 and D_2 simultaneously). All that opens up some ways of applications of the g_{pd} formalism to solving physical tasks by analogy with usual Lie algebras (cf.[11-13]). However, there exist some peculiarities of applications of nonlinear Lie (super)algebras in comparison with those of familiar Lie algebras. Specifically, for algebras g_{pd} there do not exist satisfactory definitions of the Wigner D-function analogs via matrix elements of g_{pd} exponentials (which are not analytical) or of the group orbit type generalized coherent states [9]. Therefore, for solving both spectral and evolution tasks we must use only direct algebraic methods. On this way we get some new classes of special functions exploited in quantum physics and , in a sense, connected with quantization schemes on algebraic varities. For more full list of such models we refer to papers [8, 9]. Below we discuss these items by analyzing some simplest models with essentially nonlinear Hamiltonians, whose dynamic symmetry algebras g^{ds} are nonlinear Lie algebras $sl_{pd}(2)$ [9].

2. Polynomial Lie algebras $sl_{pd}(2)$ in some non-linear models of quantum physics

Let us consider quantum models with Hamiltonians of the form

$$H = \omega_1 a_1^+ a_1 + \omega_2 a_2^+ a_2 + b(a_1^+)^n (a_2)^m + b^*(a_1)^n (a_2^+)^m, 0 \le m \le n \qquad (2)$$

where non-quadratic parts of H describe specific scaterring processes: creation/absorption of multiboson clusters in external classical ($m = 0$, $n \neq 0$) and quantized ($m \neq 0$, $n \neq 0$) fields [9] (b are some constants or time-dependent functions and $[a_i, a_j^+] = \delta_{ij}, [a_i^{(+)}, a_j^{(+)}] = 0$). Evidently, the Hamiltonians (2) have the invariance groups $G_{inv}(H) = C_n \times C_m \times U(1)$ where $C_k = \{c_{lk} = exp(i2\pi l/k) : a_j^+ \to c_{lk} a_j^+\}$, $U(1) = exp(i\alpha R), R =$

$a_1^+a_1/n + a_2^+a_2/m$ $(m \neq n)$ (or $R = a_2^+a_2$ for $m = 0$). We note, that groups C_n and C_m describe some internal symmetries of two oscillator subsystems whereas $U(1)$ characterizes their coupling. Introducing (with the help of a generalized Jordan mapping) the G_{inv} invariant quantities $Y_0 = (a_1^+a_1 - a_2^+a_2)/(m+n)$ (or $a_1^+a_1/n$ for $m = 0$), $Y_+ = (a_1^+)^n(a_2)^m$, $Y_- = (Y_+)^+$, one may rewrite (2) in the form [9]

$$H = aY_0 + bY_+ + b^*Y_- + C, [Y_\alpha, C] = 0 \qquad (3)$$

where Y_α are generators of Lie-like algebras $g_{pd}(H)$ which satisfy CRs

$$[Y_0, Y_\pm] = \pm Y_\pm, [Y_-, Y_+] = \psi_{n,m}(Y_0) = \phi_{n,m}(Y_0+1) - \phi_{n,m}(Y_0) \qquad (4)$$

with $\phi_{n,m}(Y_0) = (n[Y_0 + \frac{mR}{m+n})^{(n)}(m[\frac{nR}{m+n} - Y_0 + 1])^{(m)}$, $A^{(B)} = A(A-1)...(A-B+1)$. Eqs (4) resemble the CR for the familiar Lie algebra $sl(2)$ that allowed us to identify $g_{pd}(H)$ as two (noncompact and compact) deformations $sl_{pd}^{(n,m)}(2)$ of the Lie algebra $sl(2)$ distinguished by some features of their irreps on $L(H)$ and referred to as $su_{pd}^{(n)}(1,1)$ and $su_{pd}^{(n,m)}(2)$ (for $m \neq 0$) [9]. It is easy to check that these algebras $sl_{pd}^{(n,m)}(2)$ have the Casimir operators $C_2^{(n,m)}(2) = -Y_+Y_- + \phi_{n,m}(Y_0)$ that is a specific deformation of the usual $sl(2)$ Casimir operator $C_2(2) = \pm E_+E_- + E_0^{(2)}$ [11]. This allows us to develop a theory of the $sl_{pd}^{(n,m)}(2)$ representations by analogy with that of usual Lie algebras [6]. Specifically, using the above boson realization for $su_{pd}^{(n)}(1,1)$ and $su_{pd}^{(n,m)}(2)$ and the decomposition (1) for $D_1 = C_m \times C_n \times U(1), D_2 = sl_{pd}^{(n,m)}(2)$ we can determine all irreps of $sl_{pd}^{(n,m)}(2)$ which act on $L(H) = L_F(k) = Span\{|\{n_\alpha\}>= \prod_\alpha (a_\alpha^+)^{n_\alpha}|0>\}$. Namely, the algebras $su_{pd}^{(n)}(1,1)$ have on the space $L_F(1)$ only "n" infinite-dimensional irreps $D([l_1])$ specified by the lowest weights $l_1 = k/n, k = 0,1,...,n-1$, and lowest vectors $|[l_1]>= (a_1^+)^k|0>, Y_0|[l_1]>= l_1|[l_1]>, Y_-|[l_1]>= 0$, whereas the algebras $su_{pd}^{(n,m)}(2)$ have on $L_F(2)$ an infinite number of finite-dimensional irreps $D([l_1l_2])$ specified by the lowest weights $l_1 = (k-s)/(n+m)$, eigenvalues $l_2 = k/n + s/m$ of the above operator $R, k = 0,1,...,n-1, s = 0,1,...$ and the lowest vectors $|[l_1l_2]>= (a_1^+)^k(a_2^+)^s|0>, Y_0|[l_i]>= l_1|[l_i]>, R|[l_i]>= l_2|[l_i]>, Y_-|[l_i]>= 0$. All other basic vectors of the $su_{pd}^{(n)}(1,1)$ and $su_{pd}^{(n,m)}(2)$ irreps are constructed by means of action of raising operators $(Y_+)^t$ on the lowest vectors [9]. We note that, in fact, Hamiltonians of a lot of quantum many-body physics models may be expressed in the form (3)-(4) (with replacing concrete structure polynomials $\phi_{n,m}(x)$ by other polynomials determined by model Hamiltonians) [8, 9]. Without dwelling on a description of appropriate generalized Jordan mappings in detail (see, e.g., [8, 9]) we write down only expressions for Hamiltonians of two widespread classes of quantum optics models

$$H' = b\sigma_+(j)(a_1)^n + b^*\sigma_-(j)(a_1^+)^n \tag{5a}$$

$$H'' = \sum_{i=1}^{n} \omega_i a_i^+ a_i + \omega_0 a_0^+ a_0 + b(a_1^+ ... a_n^+)(a_0)^m + b^*(a_1 ... a_n)(a_0^+)\ m \tag{5b}$$

are generalizations of Hamiltonian eqs (2) and describe multiphoton Dicke models and frequency conversion processes.

3. Algebras $sl_{pd}(2)$ in solving physical tasks and new classes of orthogonal polynomials

By analogy with the case $n \leq 2$ [11-13] one can expect that the theory of algebras $sl_{pd}^{(n,m)}(2)$ may be useful for treating different problems with Hamiltonians (3). But for lack of simple formulas for disentangling exponents $\exp(\sum d_i Y_i)$ [9] one cannot apply the orbit coherent states techniques (or simililar ones) [12] for diagonalizing H or for finding appropriate time-evolution operators $U_H(t;t_0)$ as it is the case for usual Lie algebras. Nevertheless, there exist some possibilities of applications of the g_{pd} formalism to solving these tasks. One way of applications is related to finding eigenstates of the stationary Schrodinger equation $H|E^a >= E^a|E^a >$ with H from eq.(3) on the invariant subspaces $L([l_i])$ in the form [8, 9]

$$|E >= u(Y_+;E)|[l_i] >= \sum_f Q_f(E)(Y_+)^f|[l_i] > \tag{6}$$

that corresponds to the diagonalization scheme [14] of any elements of the usual Lie algebra $sl(2)$. For determining the function $u(Y_+;E)$ in (6) one can use either the Bargmann-type representation $Y_+ = z, Y_0 = (zd/dz+l_1), Y_- = z^{-1}\phi_{n,m}(zd/dz + l_1) = \sum_{s=1}^{n} c^s z^{(s-1)} d/dz^s$ of the $sl_{pd}^{(n,m)}(2)$ algebra [9] or a calculation of the coefficients $Q_f(E)$ with the help CRs (4). When using the first way we find that functions $u(z;E)$ satisfy differential equations of the Fuchs type,

$$\{al_1 - \lambda(E) + azd/dz + bz + b^*z^{-1}\phi_{n,m}(zd/dz + l_1)\}u(z;E) = 0 \tag{7}$$

which resemble equations for higher hypergeometric functions ${}_pF_q(...;z)$ [15]. We note that in general solutions of eqs (7) expressed, for example, in the terms of contour or multiple integrals [15] determine some new orthogonal functions similar to D-functions for $sl(2)$. Because of the of the occurrence of higher derivatives $d^s u/dz^s$ in eqs (7) these functions are singular that corresponds to the aforementioned non-analytical nature of D-function analogs for $sl_{pd}^{(n,m)}(2)$. The second way is related to solving finite-difference equations

$$[(l_1+f)a-\lambda]Q_f + bQ_{f-1} + b^*\phi_{n,m}(l_1+f+1)Q_{f+1} = 0, f = 0,1,... \quad (8)$$

for the coefficients $Q_f = Q_f(E)$ where $\phi_{n,m}(x)$ is the $sl_{pd}^{(n,m)}(2)$ structure polynomial from (4) and the spectral parameter $\lambda = E - c$ comprises both the energy eigenvalue and that of the invariant operator C which is constant on the whole $L([l_i])$. Difference equations (8) belong to the hypergeometric type [15] and for structure polynomials $\phi_{n,m}(x)(m+n=2)$ related to the usual $sl(2)$ their solutions are expressed in terms of classical orthogonal polynomials in the variable λ taking its values on homogeneous lattices $\{\lambda_k = \lambda_0 + k\Delta\}$ [14, 15]. In the case of $sl_{pd}^{(n,m)}(2)$ we get in such a manner new classes of orthogonal polynomials $\pi_f^\phi(\lambda) = (b^*)^f Q_f(\lambda)\prod_{s=0}^{f-1}\phi_{n,m}(l_1+f-s)$ on inhomogeneous, in general, lattices $\{\lambda_k\}$ related to energy spectra of H [9]. Indeed, from eqs (8) one easily gets a standard form [15]

$$\pi_{f+1}^\phi(\lambda) = [\lambda-(l_1+f)a]\pi_f^\phi(\lambda) - |b|^2\phi_{n,m}(l_1+f)\pi_{f-1}^\phi(\lambda) \quad (8')$$

of three-term recurrence relations for polynomials $\pi_f(\lambda)$, and their orthogonality follows from the completness property of eigenstates (6). However, unlike the sets of classical orthogonal polynomials [15], we cannot extract from these results a simple closed Rodriguez formula for polynomials $\pi_f(\lambda)$, and to find its suitable generalizations is one of important tasks to be solved for arbitrary structure polynomials $\phi_{n,m}(x)$. Nevertheless, the $sl_{pd}^{(n,m)}(2)$ formalism enables to find lattices $\{\lambda_k\}$ on which polynomals $\pi_f(\lambda)$ are determined. For the $su_{pd}^{(n)}(1,1)$, when all spaces $L([l_i])$ are infinite- dimensional, the concrete forms of these lattices can be found by solving characteristic equations

$$F_{l_1}(\lambda) = 0 \quad (9)$$

for tridiagonal matrices $||F_{ij}||$ of coefficients in eqs (8) where $F_{l_1}(\lambda)$ belong to a set of functions $F_{\ell_1+k}(\lambda)$ which are determinants of matrices obtained from $||F_{ij}||$ by cancelling first k rows and columns. These functions satisfy recurrence relations

$$[a(k-1)-\lambda]F_{l_1+k}(\lambda) = F_{l_1+k-1}(\lambda) + |b|^2\phi_{n,m}(k+l_1)F_{l_1+k+1}(\lambda) \quad (10)$$

with asymptotic boundary condition

$$\lim_{k\to\infty} F_{l_1+k}(\lambda) = 0 \quad (11)$$

In the case of $su_{pd}^{(n,m)}(2)$ all irreps on $L([l_i])$ are two-side bounded and for determining these lattices and spectra $E^a = E^a(l_i)$ one can use instead of eqs (8)-(10) the condition

$$\phi_{n,m}(l_1+d)Q_d(E) = 0 \Longrightarrow [(l_1+d-1)a-\lambda]Q_{d-1}+bQ_{d-2} = 0, Q_{-1}(E) = 0, \quad (12)$$

where $d = d(l_1 l_2)$ is the $L([l_i])$ dimension. So, solving eqs (7)-(12) we find eigenfunctions and energy spectra of Hamiltonians (2) in terms of new special functions related to the algebras $sl_{pd}^{(n,m)}(2)$ (cf. well-known relationships between the Gaussian hypergeometric functions $_2F_1(...;z)$ and the usual algebra $sl(2)$ [11, 14]). We also note that from the eq.(7) one can find in a familiar manner [11] its nonstationary analog defining solutions $u(z;t)$ of an appropriate nonstationary Schrodinger equation and related time-evolution operators [9]. Another way of applications of the $sl_{pd}^{(n,m)}(2)$ formalism to solving physical tasks is related to analysis of equations

$$dY_0/dt = b_0Y_+ + b_0^*Y_-, dY_\pm/dt = a_\pm Y_\pm + b_\pm\psi_{n,m}(Y_0) \quad (13)$$

which follow from the Heisenberg equations $i\hbar dY_\alpha/dt = [Y_\alpha, H]$ with H given by eq. (3): $b_0 = b/i\hbar$, $a_\pm = \mp a/i\hbar$, $b_+ = -b^*/i\hbar = (b_-)^*$. These equations coincide with Bloch equations [16] in the case $m+n=2$, and, therefore, may be named as generalized Bloch equations. The familiar Bloch equations are linear in operators $Y_\alpha(t)$, and their solutions are given by linear combinations of initial operator $Y_\alpha = Y_\alpha(0) \in sl(2)$ [9]. However, in the general case $m+n>2$ eqs (13) are nonlinear in $Y_\alpha(t)$, and their solutions may be given by only power series

$$Y_\alpha(t) = \sum_{k\geq 0}[A_{k\alpha}(Y_0;t)(Y_-)^k + (Y_+)^k B_{k\alpha}(Y_0;t)] \quad (14)$$

where operator functions $A(...)$ and $B(...)$ are determined from some differential-diference equations. The solution of eqs (13) is also reduced to solving the only nonlinear equation

$$d^2Y_0(t)/dt^2 = AC_1 - \hbar^2A^2Y_0(t) + pB\psi_{n,m}(Y_0(t)) \quad (15)$$

where $A = a/\hbar^2$, $B = 2 \mid b \mid^2 /\hbar^2$, $C_1 = (H-C)$ is an integral of motion. In the mean-field approximation given by the condition $< \psi_{n,m}(Y_0) >_{mf} = \psi_{n,m}(< Y_0 >_{mf})$ we get from eq. (15) for the c-number function $y(t) =< Y_0(t) >_{mf}$ the equation

$$(dy/dt)^2 = 2[A < C_1 > y(t) - \frac{1}{2}\hbar^2A^2(y(t))^2 + B\int^{y(t)} dy\psi_{n,m}(y) + D] \quad (16)$$

constants a, b in terms of hyperelliptic integrals (cf. [17] where elliptic integrals were first used in trilinear models) defining special abelian functions

[18,19]. Therefore, exact operator solutions of eqs (13), (15), in the form (14), seem to determine operator analogs of abelian functions which, perhaps, are related to the problems of quantization on algebraic (abelian) varieties [18]. It is also of interest to examine interrelations of these results with possibilities of solving algebraic equations (12) in terms of theta-constants and hyperelliptic integrals [19] as well as connections between solutions of eq. (16) and orthogonal polynomials $\{\pi_f(\lambda)\}$. For this aim one can, e.g., compare quasiclassical solutons $y(t)$ with exact quantum expectations $< Y_0(t) >=< \psi(t) \mid Y_0 \mid \psi(t) >$ obtained by using expansions of $\mid \psi(t) >= \sum_k c_k exp(-iE_k t/\hbar) \mid E_k >$ in eigenstates $\mid E_k >$ from eq. (6).

4. Concluding remarks

In conclusion we point out that the similar analysis of multimode versions of the Hamiltonians (2) leads also to polynomial Lie algebras g_{pd} with the coset structure $g_{pd} = h + v$ and $h = u(m)$ [8, 9]. But for using appropriate generalizations of the scheme (6)-(16) we need an additional work for separating variables. We also note that using generalized Holstein-Primakoff mappings [13, 9] one can determine some relations between familiar Lie algebras and both polynmially and q-deformed Lie algebras g_d in order to display different exotic states and phenomena [7] in realistic multi- particle models as well as to determine their asymptotic behaviours [9]. Specifically, we may compare the Hamiltonians (3) with some "distorted" Hamiltonians

$$H_D = \bar{a}V_0 + \bar{b}V_+ + \bar{b}^* V_- + \bar{C}, [\bar{C}, V_\alpha] = 0 \qquad (17)$$

which are linear in generators V_α of certain 3-dimensional Lie algebras $g(H_D)$ or q-deformed Lie algebras $g_q(H_D)$. Here with algebras $g(H_D)$ and $g_q(H_D)$ are related to $sl_{pd}^{(n,m)}(2)$ via the Holstein-Primakoff type mapping [9, 20]

$$sl_{pd}^{(n,m)}(2) \to (g(H_D)/g_q(H_D)) = \{V_\alpha, \alpha = 0, \pm : [V_0, V_\pm]$$
$$= \pm V_\pm, [V_-, V_+] = \Psi(V_0),$$
$$V_0 = Y_0 + \mu, V_+ = Y_+ f(Y_0), V_- = f(Y_0)Y_-\} \qquad (18)$$

where $\Psi(V_0)$ are structure functions of algebras $g(H_D)$ or $g_q(H_D)$ and functions $f(Y_0)$ are determined from the equations

$$\gamma(Y_0) - \gamma(Y_0 - 1) = \Psi(Y_0 + \mu), \gamma = \mid f(Y_0) \mid^2 \phi_{n,m}(Y_0 + 1) \qquad (19)$$

with $\phi_{n,m}(Y_0)$ being the structure polynomial in eq. (4). For example, taking $g(h_D) = sl(2) = su(2)/su(1,1)$ (with $\Psi(V_0) = \mp 2V_0$ in eq. (19)) we get

$$\mid f(Y_0) \mid^2 = [\lambda \mp (Y_0 + \mu + 1)^{(2)}]/\phi_{n,m}(Y_0 + 1) \qquad (20)$$

where constant operators λ, μ in eq. (20) are found from conditions of a "canonical" behaviour of operators V_α of the $sl(2)$ irreps on the subspaces $L([l_i])$ [9]. Solutions of such "linearized" models with "distorted" Hamiltonians (17) and $g(H_D) = sl(2)$ implemented along the lines of eqs (6)-(13), e.g., with the help of Lie-algebraic and group-theoretical methods [11-13] may be viewed at an adequate choice of parameters "$\bar{a}$", "$\bar{b}$" in (17) as specific smooth (analytical) approximations modulating exact (generally, non-analytical [9]) solutions of models with Hamiltonians (3). As proximity measures of such approximations, one may use relative moments

$$\delta_p(H, H_D) = | Tr(H - H_D)^p / Tr(H)^p |, p = 1, 2, \ldots \tag{21}$$

where traces are calculated over invariant subspaces $L([l_i]) \subset L(H)$ or whole spaces $L(H)$ [9]. Choosing in eq. (18) $\Psi(V_0) = (q^{V_0} - q^{-V_0})/(q^{1/2} - q^{-1/2})$ we can implement a similar analysis along the lines eqs (6)-(16) for Hamiltonians (17) with $g_q(H) = sl_q(2)$. On this way we can both obtain new orthogonal polynomials within schemes (6)-(12) and define specific q-analogs of abelian (elliptic) functions within appropriate generalizations of schemes (13)-(16). From the physical viewpoint solutions of such q-analogs of models (3) may be considered as exotic approximations of solutions of original models(cf. [7]). So, starting from Hamiltonians (2),(5) we may obtain both their exact solutions (with the help of the $sl_{pd}(2)$ formalism) and two kinds of their approximations (via the generalized Holstein-Primakoff mapping (18)). Using measures (21) and some minimizing principles we can determine both parameters $\bar{a}$, $\bar{b}$ in (17) and value of q providing maximal proximity of "distorted" models to original ones. Such comparisons may be useful in examining some dynamic peculiarities of models (2), (5), in particular, in determining conditions of a realization of their regular (periodic and quasiperiodic) or nearly stochastic regimes (cf. [21, 22]). An alternative approach is given in [23]: using $sl(2)$ solutions for solving both spectral and evolution tasks associated with $sl_{pd}(2)$. Other results along the lines outlined above can be found in [8, 9]. It is of interest to compare these approaches with other, see, e.g., [24-28] and references therein.

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List of Participants

Marcelo **Aguilar**
Instituto de Matemáticas
Universidad Nacional Autónoma
de México
Ciudad Universitaria
04510 México, D.F.
socolov@roxanne.nuclecu.unam.mx

Keiichi **Akama**
Department of Physics
Saitama Medical College
Kawakado, Moroyama, Saitama
350-04, Japan
akama@jpnutins

Manlio **Bordoni**
Dipart. di Matematica "G. Castelnuovo"
U. degli Studi di Roma "La Sapienza"
Piazzale Aldo Moro 2
00185 Roma, Italia
bordoni@sci.uniroma1.it
Fax 39 6 49 91 32 87

Howard E. **Brandt**
Army Research Laboratory
2800 Powder Mill Road
Adelphi, Maryland 20783-1197, U.S.A.
hbrandt@lamp0.arl.army.mil
Fax: 301 394 4704

Mauro **Bregola**
Dipartimento di Fisica
Via Paradiso 12
44100 Ferrara, Italy
ferrario@ferrara.infn.it
Fax 0038-532-78-18-10

Paolo **Budinich**
Interdisciplinary Lab. for Natural &
Humanistic Sciences and
Int. Centre for Theoretical Physics
Strada Costiera 11, 34014 Trieste, Italy
budinich@tsmi19.sissa.it
Fax 39 40 224559

Krystyna **Bugajska**
Dept. of Mathematics
York University
4700 Keebe St. North York
Ontario, M3J 1P3, CANADA
bugajska@vm1.yorku.ca

Daniel **Cangemi**
Department of Physics, UCLA
405 Hilgard Ave.
Los Angeles, CA 90024, U.S.A.
cangemi@pierre.mit.edu
cangemi@Mitlns.bitnet
Fax 617 253 8674

Chryssomalis **Chryssomalakos**
Department of Physics and
Theoretical Physics Group
Lawrence Berkeley Laboratory
University of California
Berkeley, CA 94720, U.S.A.
cryss@physics.berkeley.edu

Paolo **Cotta-Ramusino**
Dipartimento di Fisica
Universita di Milano
Via Celoria 16, 201333 Milano, Italy
cotta@vaxmi.mi.infn.it
ramusino@itnvax.cineca.it
Fax: 39 2 2392425; 39 2 2392480

Louis **Crane**
Department of Mathematics
Kansas State University
Manhattan, KS 66506-2602
U.S.A.

Louise **Dolan**
Deptartment of Physics
University of North Carolina
Chapel Hill NC 27599-3255, U.S.A.
dolan@augustus.physics.unc.edu
Fax: 919 962 0480

Thibault **Damour**
IHES
France
damour@frihes61
fax: 33-1-69-07-60-24

Wolfgang **Drechsler**
Max-Planc-Institut für Physik
Werner Heisenberg Institut
P.O. Box 401212
80712 Munich, Germany
wdd-dmumpiwh@tecmtyum.mty.itesm.mx
Fax: 3322-67-04

F. **Delduc**
Laboratoire de Physique Théorique
ENSLAPP
Groupe de Lyon, ENS Lyon
46 Allée d'Italie
F-69364 LYON CEDEX 07, France
sorba@frlapp51

Gerald **Dunne**
Physics Deptartment
University of Connecticut
2152 Hillside Rd.
Storrs CT 06269, U.S.A.
dunne@hep.phys.uconn.edu
fax: (203)486-33-46

Alfred W. **Differ**
Department of Physics
American River College
Sacramento, CA 95841
U.S.A.
wpezzaglia@scuacc.scu.edu

Mićo **Đurđević**
Faculty of Physics,
P.O. Box 550, Studentski TRG 12
Beograd, Serbia, Yugoslavia
miko@unamvm1.dgsca.unam.mx
Fax: 38-11-632-133

Aristophanes **Dimakis**
Department of Mathematics
University of Crete
GR-71409 Iraklion
Greece
hoissen@hal9000.theo-phys.gwdg.de

Sergio **Ferrara**
Theory Division
Cern 1211
Geneva, 23, Switzerland.
ferraras@cernvm, vxcern::ferraras
Fax: 41 22 17823914

Geoffrey **Dixon**
Deptartment of Physics
Brandeis University
Waltham, MA 02254 U.S.A. and
Dept. of Mathematics, U. of Massachusetts,
Boston, MA 02125
dixon@binah.cc.brandeis.edu

Carlo **Ferrario**
Dipartimento di Fisica
Via Paradiso 12
44100 Ferrara, Italy
ferrario@ferrara.infn.it
fax: 0038-532-78-18-10

Frank **Flaherty**
Deptartment of Mathematics
Orègon State University
Corvallis, Oregon 97331-4605
U.S.A.
flaherty@math.orst.edu

Michael **Forger**
Dept. de Mat. Aplicada, Inst. de Mat. e Estatistica
Universidade de Sao Paulo
Cx. Postal 20570
BR 01498-970 Sao Paulo, SP Brazil
forger@ime.usp.br

Davide **Franco**
S.I.S.S.A.
Via Beirut 4
34100 Trieste, Italy
reina@itssissa.bitnet

L. **Frappat**
Laboratoire de Physique Théorique
ENSLAPP
Groupe d'Annecy, LAPP, BP 110
F - 74941 Annecy-le-Vieux Cedex
France
sorba@frlapp51

Paolo **Furlan**
Dipartimento di Fisica Teorica
dell' Universitá di Trieste
and I.N.F.N.
Sezione di Trieste, Italy
furlan@trieste.infn.it
fax: 39 40 224601

Thomas **Gisiger**
Laboratoire de Physique Nucléaire
Université de Montréal
Montréal, Quebec, Canada, H3C 3J7
paranj@lps.umontreal.ca
fax: (514)343-62-15

Magdalena **Guslew-Czudżak**
University of Wroclaw
Institute of Theoretical Physics
Plac Maxa Borna 9
50204 Wrocław, Poland
magc@proton.ift.uni.wroc.pl

Leopold **Halpern**
Florida State University
Department of Physics
Tallahassee, FLA 32306
U.S.A.
Fax: (904) 644-86-30

Gerald **Harnett**
Department of Mathematics
Florida Atlantic University
Boca Raton, FL 33431-0991, U.S.A.
harnett@acc.fau.edu
Fax: (407)367-2436

Jonahatan **Huntley**
Department of Mathematics
Banuch College, CUNY, Box 515
New York, NY 10010
U.S.A.
huntley@kronecker.baruch.cuny.edu

Alexei P. **Isaev**
Bogolubov Theoretical Laboratory
JINR, Dubna
Moscow, SU 101 000, Russia
isaevap@theor.jinrc.dubna.su
fax: (7095)975-23-81

Tatiana A. **Ivanova**
Laboratory of Theoretical Physics
Joint Institute for Nuclear Research (Dubna)
P.O. Box 79, Moscow 101 100, Russia
popov@theor.jinrc.dubna.su
fax: 7096 2165084 / 7095 9752381

Valery Pavlovich **Karassiov**
Lebedev Physical Institute
Leninsky Prospect 53
Moscow 117924, Russia
karas@sci.fian.msk.su

J. M. **López-Romero**
Departamento de Física
Centro de Investigación y
Estudios Avanzados del IPN
Apdo. Postal 14-740, 07000 México D.F.
socolov@roxanne.nuclecu.unam.mx

Louis H. **Kauffman**
Department of Mathematics
Statistics and Computer Science
University of Illinois at Chicago
851 South Morgan Street
Chicago, Illinois 60607-7045, U.S.A.
u10451@uicvm.uic.edu; Fax (312)996-1491

Jaime **Keller**
F.Q., División de Ciencias Básicas and
FES-Cuautitlán, U.N.A.M.
Apartado 70-528
04510 México, D.F., México
keller@redvax1.dgsca.unam.mx
fax (525) 616 2010 / (525) 872 0984

Achim **Kempf**
Deptartment of Applied Mathematics and
Theoretical Physics
University of Cambridge
Cambridge CB3 9EW, United Kingdom
a.kempf@damtp.cambridge.ac.uk
fax: 0223 33-79-18

Thomas **Kerler**
Department of Mathematics
Harvard University
Cambridge, MA 02138, U.S.A.
kerler@math.harvard.edu
fax: (671)695-51-32

Klaus **Lang**
Fachbereich Physik
Universität Kaiserslautern
67653 Kaiserslautern
Germany
lang@gypsy.physik.uni-kl.de

Shan **Majid**
Dept. of Applied Mathematics
& Theoretical Physics
University of Cambridge
Cambridge CB3 9EW, U. K.
magid@damtp.cambridge.ac.uk
fax: 0223-33-79-18

Geoffrey **Martin**
Department of Mathematics
University of Toledo
Toledo, OH 43606, U.S.A.
gmartin@uoftol.uoft02.edu
fax: (419)537-21-57

Jouko **Mickelsson**
Dept. of Theoretical Physics
Royal Institute of Technology
Stockholm S-100 44, Sweden
mickelsson@finuhcb.earn
jouko@theophys.kth.se

Folkert **Müller-Hoissen**
Institut für Theoretische Physik
Universität Göttingen
Bunsenstrasse 9
D-37073 Göttingen, Germany
hoissen@theo-phys.gwdg.de
fax: (551)39-96-31

Werner **Nahm**
Physikalisches Institut der
Universität Bonn
Nussalle 12
D-5300 Bonn 1, Germany
nahm@pib1.physik.uni-bonn.de
fax: 49-(228)-737869

Ichiro **Oda**
Department of Physics
Ochanomizu University
2-1-1, Otsuka, Bunkyo-ku
Tokyo, 112, Japan
akama@jpnutins

Zbigniew **Oziewicz**
FES-Cuautitlán, U.N.A.M.
Apartado Postal 25
Cuautitlán Izcalli 54700
Edo. de México, México
oziewicz@redvax1.dgsca.unam.mx
oziewicz@proton.ift.uni.wroc.pl
Fax Poland 0048 71 214 454

Manu **Paranjape**
Laboratoire de
Physique Nucléaire
Université de Montréal
Montréal, Quebec, Canada, H3C 3J7
paranj@lps.umontreal.ca
fax: (514)343-62-15

Vladimir G. **Pestov**
Department of Mathematics
Victoria University of Wellington
P.O. Box 600
Wellington, New Zealand
vladimir.pestov@vuw.ac.nz
fax: 64-4-4712-070

William **Pezzaglia** Jr.
Department of Physics
Santa Clara University
Santa Clara, CA 95053, U.S.A.
wpezzaglia@scuacc.scu.edu
fax: (510)376-4027

A. D. **Popov**
Laboratory of Theoretical Physics
Joint Institute for
Nuclear Research (Dubna)
P.O. Box 79, Moscow 101 100, Russia
popov@theor.jinrc.dubna.su
fax: (7-09621) 65-084/(7095)975-23-81

E. **Ragoucy**
Laboratoire de Physique Théorique
ENSLAPP
Groupe d'Annecy, LAPP, BP 110
F - 74941 Annecy-le-Vieux Cedex
France
sorba@frlapp51

Cupatitzio **Ramirez**
Facultad de Ciencias
Físico Matemáticas
Universidad Autónoma de Puebla
Apartado 1364, Puebla, Pue., México
q79@vm.urz.uni.heidelberg.de
hsalazar@fnalv.fnal.gov

Enrique **Ramirez de Arellano**
Depto. de Matematicas
CINVESTAV-IPN
Apartado 14740
México, D.F. 07000
eramirez@cinvesmx.bitnet
fax: (525)752-64-12

Cesare **Reina**
S.I.S.S.A.
Via Beirut 4
34100 Trieste
Italy
reina@tsmi19.sissa.it
fax: 0039 40 37 87441

Suemi **Rodriguez-Romo**
Centro de Investigaciones Teóricas
FES-Cuautitlan, U.N.A.M.
Apdo. 142, Cuautitlán Izcalli
Edo. de México 54700, México
suemi@unamvm1.dgsca.unam.mx
suemi@fis.cinvestav.mx

Werner **Rühl**
Fachbereich Physik
Universität Kaiserslautern
67653 Kaiserslautern
Germany
lang@gypsy.physik.uni-kl.de
fax: (0631)205-33-00

Liudmila **Sabinina**
Instituto de Matemáticas
Area de la Invest. Cientifica U.N.A.M.
Circuito Exterior, Ciudad Universitaria
04510 México, D.F., México.
lsabinin@nsfuvax.math.uoknor.edu

Mikhail **Saveliev**
Institute for High Energy Physics
142284, Protvino, Moscow Region
Russia
saveliev@mx.ihep.su
Fax: 7-095-230-23-37

Peter **Schupp**
Dept. of Physics, U. of California, and
Theoretical Physics Group
Lawrence Berkeley Lab., U. of California
Berkeley, CA. 94720, U.S.A.
schupp@physics.berkeley.edu
fax: (805)8.93.24.31

Michael V. **Shapiro**
Depto. de Matematicas
ESFM, IPN
Apdo. Postal 14-740
07000 México, D.F.
mshapiro@cinvesmx.bitnet
fax: (525)752-64-12

Garret **Sobczyk**
Universidad de las Américas
Depto. de Físico-Matemáticas
Apartado Postal #100
Santa Catarina Mártir
72820 Puebla, Pue., México
sobczyk@udlapvms.pue.udlap.mx

Miguel **Socolovsky**
Instituto de Ciencias Nucleares
U.N.A.M.
Circuito Exterior, C.U.
Apdo. 70-543, 04510 México, D.F.
socolovs@roxanne.nuclecu.unam.mx

Frank **Sommen**
Dept. of Mathematical Analysis
University of Gent
Galglaan 2, B-9000 Gent, Belgium
fs@cage.rug.ac.be
Fax: 32 91 64 49 87

Paul **Sorba**
Laboratoire de Physique Théorique
ENSLAPP
Groupe de Lyon, et Groupe d'Annecy
ENS Lyon, 46 Allée d'Italie
F-69364 LYON CEDEX 07, France
sorba@frlapp51; Fax (Lyon) 337272-80-80
fax (Annecy) 33 50 27 94 95

Vyacheslav **Spiridonov**
Centre de Recherches Mathématiques
C.P. 6128-A
Montréal, Québec, H3C 3J7
Canada
spiridonov@lps.umontreal.ca
fax (514) 343-6215

Yassen S. **Stanev**
The Erwin Schrödinger
International Institute
for Mathematical Physics (ESI)
Vienna, Austria
furlan@trieste.infn.it

Allina **Suzko**
Joint Inst. for Nucl. Research, Dubna
P.O Box 79, Moscow, Russia, and
Institute of Radiation Physical &
Chemical Problems,
Byelorussian Acad. of Sciences, Minsk
suzko@theor.jinrc.dubna.su

Earl J. **Taft**
Department of Mathematics
Rutgers University
New Brunswick NJ 08903
U.S.A.
fax: (908)932-55-30

Mark **Temple-Raston**
Department of Physics
Bishop's University
Lennoxville, Quebec
CANADA
temple@vax2.concordia.ca

Paul **Watts**
Dept. of Physics, U. of California
& Theoretical Physics Group
Lawrence Berkeley Lab.
University of California
Berkeley, CA. 94720, U.S.A.
schupp@physics.berkeley.edu

Ivan. T. **Todorov**
The Erwin Schrödinger
International Institute
for Mathematical Physics (ESI)
Vienna, Austria
furlan@trieste.infn.it

Gregory P. **Wene**
U. of Texas at San Antonio
12513 Elm Country Ln
San Antonio, Texas 78230
U.S.A.
wene@ringer.cs.utsa.edu
fax: (512)-691-4439

Luis **Urrutia**
Departamento de Física
Universidad Autónoma
Metropolitana-Ixtapalapa
México, D.F.
México
luis@xanum.uam.mx

Julius **Wess**
Sektion Physik, LMU Münche
Theresienstr. 37
80333 München, Germany
jup@dmumpiwh
fax: (89)322-6704

Nikolai L. **Vasilevski**
Depto. de Matemáticas
CINVESTAV, IPN
Apdo. Postal 14-740
07000 México, D.F.
nvasilev@cinvesmx.bitnet
fax: (525)7-52-64-12

L.C.R. **Wijewardhana**
Dept. of Physics, M.L. 11
University of Cincinnati
Cincinnati, Ohio, 45221
U.S.A.
rohana@ucbeh.san.vc.edu

Luc **Vinet**
Centre de Recherches
Mathématiques, C.P. 6128-A
Montréal, Québec, H3C 3J7
Canada
spiridonov@lps.umontreal.ca

Robert **Yamaleev**
Dubna, JINR, LCTA
Head Post Office, P.O. Box 79
101000 Moscow, Russia
yamal@lcta.jinrc.dubna.su
fax: 7.095.9.75.23.81

Alexander **Voronov**
Department of Mathematics
University of Pennsylvania
Philadelphia, PA 19104-6395
U.S.A.
voronov@math.princeton.edu
Fax: (215) 898-20-10

David **Yetter**
Department of Mathematics
Kansas State University
Manhattan, Kansas 66506-2602
U.S.A.
dyetter@mtha.ksu.edu

Alexei **Zhedanov**
Physics Department
Donetsk University
Donetsk, 340055
Ukraine
spiridonov@lps.umontreal.ca

Bruno **Zumino**
Dept. of Physics, U. of California
& Theoretical Physics Group
Lawrence Berkeley Lab.
University of California
Berkeley, CA. 94720, U.S.A.
schüpp@physics.berkeley.edu

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