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Spin dynamics of fermion particles in gravitational and electromagnetic fields

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Abstract. We present the results of a comprehensive study of the classical and quantum dynamics of spin $\frac{1}{2}$ Dirac fermion particles with dipole moments under the simultaneous action of arbitrary external fields, including the gravitational, inertial and electromagnetic ones. We demonstrate the complete consistency of the quantum and classical dynamics. Physical applications are discussed.

1. Introduction

Understanding the motion of fermion particles with dipole moments (electrons, protons, neutrons, neutrinos) in arbitrary electromagnetic, gravitational and inertial fields is important for many physical and astrophysical problems [1]. We continue the study of the classical and quantum dynamics of spinning particles with dipole moments in the framework of the general-relativistically covariant Dirac theory [2, 3, 4, 5, 6]. In this approach, the spacetime can have either the Riemannian or the post-Riemannian geometry. The exact Foldy-Wouthuysen transformation for the most general case of a fermion moving in combined configurations of arbitrary strong gravitational (inertial) and electromagnetic fields is derived. These results are used to obtain the quantum and quasiclassical equations of motion of fermion particles. Applications to the precision experiments are considered. In particular, effects of terrestrial gravity and rotation in the high-energy physics are discussed, the bounds on the new physical coupling parameters are obtained, and gravitational wave action on spin is analysed.

Our basic notations and conventions are the same as in [6].

2. Fermion in external classical fields

The relativistic theory of a fermion particle with spin $\frac{1}{2}$ is based on the Dirac Lagrangian

$$L = \frac{i\hbar}{2}\left(\overline{\psi}\gamma^\alpha D_\alpha\psi - D_\alpha\overline{\psi}\gamma^\alpha\psi\right) - mc\,\overline{\psi}\psi + \frac{\mu'}{2c}F_{\alpha\beta}\overline{\psi}\sigma^{\alpha\beta}\psi + \frac{\delta'}{2}G_{\alpha\beta}\overline{\psi}\sigma^{\alpha\beta}\psi. \tag{1}$$

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The first term accounts for the minimal coupling of a fermion with the mass m and the electric charge q to the external gauge fields – the electromagnetic potential A_i and the gravitational coframe and the Lorentz connection $(e_i^\alpha, \Gamma_i{}^{\alpha\beta})$ – via the covariant spinor derivative

$$D_\alpha\psi = e^i_\alpha \left(\partial_i\psi - \frac{iq}{\hbar}\,A_i\psi + \frac{i}{4}\Gamma_i{}^{\beta\gamma}\sigma_{\beta\gamma}\psi\right), \tag{2}$$

whereas the two Pauli terms describe the non-minimal coupling of the electromagnetic field strength $F_{\alpha\beta}, G_{\alpha\beta} = \frac{1}{2}\eta_{\alpha\beta\mu\nu}F^{\mu\nu}$ to an anomalous magnetic moment (AMM) and an electric dipole moment (EDM). The corresponding coupling parameters have the dimension $[\mu'] = [q\hbar/2m]$ of the magnetic dipole (nuclear magneton), $[\delta'] = [q\,l]$ of the electric dipole (charge times length). The anomalous magnetic moment can be written in terms of the magneton $\mu_0 = \frac{q\hbar}{2m}$, and in a similar way one can introduce a convenient unit of an electric dipole moment. A reasonable definition is the electric charge times the electron Compton length: $\delta_0 = q\,\frac{\hbar}{mc}$. Then one can write for both types of the dipole moments:

$$\mu' = a\,\frac{q\hbar}{2m}, \qquad \delta' = b\,\frac{q\hbar}{2mc}, \tag{3}$$

The dimensionless constant parameters $a = (g-2)/2$ (where g is the gyromagnetic factor) and b characterize the magnitude of the anomalous magnetic and electric dipole moments, respectively.

3. Electrodynamics in curved spacetime

A nontrivial feature of the system under consideration is the absence of superposition. The motion of a spinning particle only in the electromagnetic field or only in the gravitational field was extensively studied in the past. However, when both external fields are acting on a fermion together, their influence on spin and particle's trajectory is not just a sum of two separate effects. Indeed, whereas electromagnetic field affects only electric charges and currents, gravity is universal and it couples to all types of matter, including the electromagnetic field. Accordingly, a spinning particle feels gravity both directly –via the coframe and connection– and indirectly –via the electromagnetic field which carries an imprint of the curved spacetime geometry.

Denoting $x^i = (t, x^a)$ the local coordinates, we write the spacetime interval as

$$ds^2 = V^2c^2dt^2 - \delta_{\widehat{a}\widehat{b}}W^{\widehat{a}}{}_cW^{\widehat{b}}{}_d\,(dx^c - K^c cdt)\,(dx^d - K^d cdt). \tag{4}$$

The functions $V = V(x^i)$ and $K^a = K^a(x^i)$, and the 3×3 matrix $W^{\widehat{a}}{}_b = W^{\widehat{a}}{}_b(x^i)$ depend arbitrarily on the local spacetime coordinates. The total number of independent components is $1+3+9 = 13$, but in view of the rotation freedom $W^{\widehat{a}}{}_b \longrightarrow L^{\widehat{a}}{}_{\widehat{c}}W^{\widehat{c}}{}_b$ with $L^{\widehat{a}}{}_{\widehat{c}}(x^i) \in SO(3)$, this number is reduced to $13 - 3 = 10$. In other words, the line element (4) describes an arbitrary 4-dimensional geometry. Explicitly, the components of the metric g_{ij} read

$$g_{00} = c^2(V^2 - \underline{g}_{ab}K^aK^b), \qquad g_{0a} = c\underline{g}_{ab}K^b, \qquad g_{ab} = -\,\underline{g}_{ab}, \tag{5}$$

$$g^{00} = \frac{1}{c^2V^2}, \qquad g^{0a} = \frac{K^a}{cV^2}, \qquad g^{ab} = -\,\underline{g}^{ab} + \frac{1}{V^2}\,K^aK^b. \tag{6}$$

The spatial 3-dimensional metric is given by $\underline{g}_{ab} = \delta_{\widehat{c}\widehat{d}}W^{\widehat{c}}{}_aW^{\widehat{d}}{}_b$, and $\underline{g}^{ab} = \delta^{\widehat{c}\widehat{d}}W^a{}_{\widehat{c}}W^b{}_{\widehat{d}}$. The 3×3 matrix $W^b{}_{\widehat{a}}$ is inverse to $W^{\widehat{a}}{}_b$.

In Maxwell's theory, the electromagnetic field is described in terms of the field strength 2-form $F = \frac{1}{2}F_{ij}dx^i\wedge dx^j$ and the electromagnetic excitation 2-form $H = \frac{1}{2}H_{ij}dx^i\wedge dx^j$. These fundamental variables satisfy the generally covariant Maxwell equations [7]:

$$dF = 0, \qquad dH = J, \qquad H = \lambda_0\,\star F. \tag{7}$$

The constitutive relation is linear and local, with $\lambda_0 = \sqrt{\frac{\varepsilon_0}{\mu_0}}$ (where ε_0 and μ_0 are the electric and magnetic constants of the vacuum), and the star $\star$ denotes the Hodge duality operator of the spacetime metric. The current 3-form $J = \frac{1}{6}J_{ijk}dx^i \wedge dx^j \wedge dx^k$ describes the distribution of the electric charges and currents which are the sources of the electromagnetic field.

The equations (7) can be written in the equivalent vector form in terms of the components of the electric and magnetic fields, $\boldsymbol{E}_a = \{F_{10}, F_{20}, F_{30}\}$ and $\boldsymbol{B}^a = \{F_{23}, F_{31}, F_{12}\}$, and the components of the magnetic and electric excitations, $\boldsymbol{H}_a = \{H_{01}, H_{02}, H_{03}\}$ and $\boldsymbol{D}^a = \{H_{23}, H_{31}, H_{12}\}$. Identifying the components of the source 3-form J with the electric current density $\boldsymbol{J}^a = \{-J_{023}, -J_{031}, -J_{012}\}$ and the charge density $\rho = J_{123}$, we recast (7) into [8]

$$\boldsymbol{\nabla}\times\boldsymbol{E} + \dot{\boldsymbol{B}} = 0, \quad \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0, \qquad \boldsymbol{\nabla}\times\boldsymbol{H} - \dot{\boldsymbol{D}} = \boldsymbol{J}, \quad \boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho. \tag{8}$$

The influence of the inertia and gravity is encoded in the Maxwell-Lorentz constitutive relation between the electric and magnetic fields $\boldsymbol{E}, \boldsymbol{B}$ and the electric and magnetic excitations $\boldsymbol{D}, \boldsymbol{H}$:

$$D^a = \frac{\varepsilon_0 w}{V}\,\underline{g}^{ab}E_b - \lambda_0 \frac{w}{V}\,\underline{g}^{ad}\epsilon_{bcd}K^c\,B^b, \tag{9}$$

$$H_a = \frac{1}{\mu_0 w V}\left\{(V^2 - K^2)\underline{g}_{ab} + K_aK_b\right\}B^b - \lambda_0\frac{w}{V}\,\epsilon_{adc}K^c\,\underline{g}^{db}E_b. \tag{10}$$

Here $K_a = \underline{g}_{ab}K^b$, $K^2 = K_aK^a = \underline{g}_{ab}K^aK^b$, and $w = \det W^{\widehat{c}}{}_d$.

For the description of Dirac fermions, one needs a coframe e_i^α, such that $g_{\alpha\beta}e_i^\alpha e_j^\beta = g_{ij}$, with $g_{\alpha\beta} = \text{diag}(c^2, -1, -1, -1)$, and one should carefully distinguish the *holonomic* components $\boldsymbol{E}, \boldsymbol{B}$ of F_{ij}, from the *anholonomic* components $\mathfrak{E}, \mathfrak{B}$ of the Maxwell tensor $F_{\alpha\beta} = e^i_\alpha e^j_\beta F_{ij}$, defined as $\mathfrak{E}_a = \{F_{\widehat{1}\widehat{0}}, F_{\widehat{2}\widehat{0}}, F_{\widehat{3}\widehat{0}}\}$ and $\mathfrak{B}^a = \{F_{\widehat{2}\widehat{3}}, F_{\widehat{3}\widehat{1}}, F_{\widehat{1}\widehat{2}}\}$. In the Schwinger gauge $e_a^{\widehat{0}} = 0$ (also $e_{\widehat{a}}^{0} = 0$), $a = 1, 2, 3$, the coframe reads explicitly:

$$e_i^{\widehat{0}} = V\,\delta^{\,0}_i, \qquad e_i^{\widehat{a}} = W^{\widehat{a}}{}_b\left(\delta^b_i - cK^b\,\delta^{\,0}_i\right), \qquad a = 1, 2, 3. \tag{11}$$

Accordingly, the holonomic and and anholonomic electromagnetic fields are related via

$$\mathfrak{E}_a = \frac{1}{V}\,W^b{}_{\widehat{a}}(\boldsymbol{E} + c\boldsymbol{K}\times\boldsymbol{B})_b, \qquad \mathfrak{B}^a = \frac{1}{w}\,W^{\widehat{a}}{}_b\,\boldsymbol{B}^b, \tag{12}$$

Hereafter the vector product is defined by $\{\boldsymbol{A}\times\boldsymbol{B}\}_a = \epsilon_{abc}A^bB^c$ for any 3-vectors A^b and B^c.

4. Schrödinger and Foldy-Wouthuysen pictures for Dirac particle

To evaluate the covariant spinor derivative (2), we need the local Lorentz connection. For the general spacetime metric (4) with the tetrad (11), the connection components read explicitly

$$\Gamma_{i\,\widehat{a}\widehat{0}} = \frac{c^2}{V}\,W^b{}_{\widehat{a}}\,\partial_bV\,e_i{}^{\widehat{0}} - \frac{c}{V}\,\mathcal{Q}_{(\widehat{a}\widehat{b})}\,e_i{}^{\widehat{b}}, \qquad \Gamma_{i\,\widehat{a}\widehat{b}} = \frac{c}{V}\,\mathcal{Q}_{[\widehat{a}\widehat{b}]}\,e_i{}^{\widehat{0}} + \left(\mathcal{C}_{\widehat{a}\widehat{b}\widehat{c}} + \mathcal{C}_{\widehat{a}\widehat{c}\widehat{b}} + \mathcal{C}_{\widehat{c}\widehat{b}\widehat{a}}\right)e_i{}^{\widehat{c}}, \tag{13}$$

where we introduced

$$\mathcal{Q}_{\widehat{a}\widehat{b}} = g_{\widehat{a}\widehat{c}}W^d{}_{\widehat{b}}\left(\frac{1}{c}\dot{W}^{\widehat{c}}{}_d + K^e\partial_eW^{\widehat{c}}{}_d + W^{\widehat{c}}{}_e\partial_dK^e\right), \qquad \mathcal{C}_{\widehat{a}\widehat{b}}{}^{\widehat{c}} = W^d{}_{\widehat{a}}W^e{}_{\widehat{b}}\,\partial_{[d}W^{\widehat{c}}{}_{e]}, \tag{14}$$

and $\mathcal{C}_{\widehat{a}\widehat{b}\widehat{c}} = g_{\widehat{c}\widehat{d}}\mathcal{C}_{\widehat{a}\widehat{b}}{}^{\widehat{d}}$. The dot denotes the time derivative ($\dot{} = \partial_t$).

We derive the Dirac equation from the action principle with the Lagrangian (1), and after a rescaling of the wave function $\psi \longrightarrow (\sqrt{-g}e_{\widehat{0}}^{0})^{\frac{1}{2}}\psi$, we recast the resulting wave equation into a Schrödinger form $i\hbar\frac{\partial\psi}{\partial t} = \mathcal{H}\psi$ with the Hermitian (and self-adjoint) Hamiltonian

$$\begin{aligned}\mathcal{H} &= \beta mc^2 V + q\Phi + \frac{c}{2}\left(\pi_b\,\mathcal{F}^b{}_a\alpha^a + \alpha^a\mathcal{F}^b{}_a\pi_b\right) + \frac{c}{2}\left(\boldsymbol{K}\cdot\boldsymbol{\pi} + \boldsymbol{\pi}\cdot\boldsymbol{K}\right)\\ &\quad + \frac{\hbar c}{4}\left(\boldsymbol{\Xi}\cdot\boldsymbol{\Sigma} - \Upsilon\gamma_5\right) - \beta V\left(\boldsymbol{\Sigma}\cdot\boldsymbol{\mathcal{M}} + i\boldsymbol{\alpha}\cdot\boldsymbol{\mathcal{P}}\right).\end{aligned} \tag{15}$$

Here: $\alpha^a = \beta\gamma^a$ $(a, b, c, \dots = 1, 2, 3)$ and the spin matrices $\Sigma^1 = i\gamma^{\hat{2}}\gamma^{\hat{3}}, \Sigma^2 = i\gamma^{\hat{3}}\gamma^{\hat{1}}, \Sigma^3 = i\gamma^{\hat{1}}\gamma^{\hat{2}}$ and $\gamma_5 = i\alpha^{\hat{1}}\alpha^{\hat{2}}\alpha^{\hat{3}}$. Boldface notation is used for 3-vectors $\boldsymbol{K} = \{K^a\}$, $\boldsymbol{\alpha} = \{\alpha^a\}$, $\boldsymbol{\Sigma} = \{\Sigma^a\}$, $\boldsymbol{\pi} = \{\pi_a\}$. The latter is the kinetic momentum operator, $\boldsymbol{\pi} = -i\hbar\boldsymbol{\nabla} - q\boldsymbol{A}$ (with $A_0 = -\Phi$, and $\boldsymbol{A} = \{A_1, A_2, A_3\}$). The minimal coupling gives rise to the terms in (15) with the objects

$$\mathcal{F}^b{}_a = VW^b{}_{\widehat{a}}, \qquad \Upsilon = V\epsilon^{\widehat{a}\widehat{b}\widehat{c}}\Gamma_{\widehat{a}\widehat{b}\widehat{c}} = -V\epsilon^{\widehat{a}\widehat{b}\widehat{c}}C_{\widehat{a}\widehat{b}\widehat{c}}, \qquad \Xi^a = \frac{V}{c}\,\epsilon^{\widehat{a}\widehat{b}\widehat{c}}\Gamma_{\widehat{0}\widehat{b}\widehat{c}} = \epsilon_{\widehat{a}\widehat{b}\widehat{c}}\,\mathcal{Q}^{\widehat{b}\widehat{c}}, \tag{16}$$

whereas the nonminimal coupling is encoded in

$$\boldsymbol{\mathcal{M}}^a = \mu'\mathfrak{B}^a + \delta'\mathfrak{E}^a, \qquad \boldsymbol{\mathcal{P}}_a = c\delta'\mathfrak{B}_a - \mu'\mathfrak{E}_a/c. \tag{17}$$

The physical contents of the theory is revealed when one transitions from the Schrödinger picture (15) to the Foldy-Wouthuysen (FW) representation. We apply the general method developed in Refs. [9, 10, 11] to construct the FW transformation for the Dirac Hamiltonian (15) and to derive the FW Hamiltonian which is exact in all terms of the zero and the first orders in the Planck constant $\hbar$ without making assumptions about the weakness of the external fields.

Omitting the technical details (see [2, 3, 4, 5]) we find for the FW Hamiltonian [12]:

$$\mathcal{H}_{FW} = \beta\epsilon' + q\Phi + \frac{c}{2}\left(\boldsymbol{K}\cdot\boldsymbol{\pi} + \boldsymbol{\pi}\cdot\boldsymbol{K}\right) + \frac{\hbar}{2}\boldsymbol{\Pi}\cdot\boldsymbol{\Omega}_{(1)} + \frac{\hbar}{2}\boldsymbol{\Sigma}\cdot\boldsymbol{\Omega}_{(2)}, \tag{18}$$

where we introduced the 3-vector operators

$$\begin{aligned}\Omega^a_{(1)} &= \frac{mc^4}{2}\left\{\frac{1}{\mathcal{T}}, \{\pi_e, \epsilon^{abc}\mathcal{F}^e{}_b\mathcal{F}^d{}_c\partial_d\,V\}\right\}\\ &\quad + \frac{c^2}{8}\left\{\frac{1}{\epsilon'}, \{\pi_e, (2\epsilon^{abc}\mathcal{F}^d{}_b\partial_d\mathcal{F}^e{}_c + \delta^{ab}\mathcal{F}^e{}_b\,\Upsilon)\}\right\}\\ &\quad + \frac{qc^2}{4}\epsilon^{abc}\left\{\frac{1}{\mathcal{T}}, \left(\{\mathcal{F}^d{}_b, \pi_d\}V^2\mathfrak{E}_c - V^2\mathfrak{E}_b\{\mathcal{F}^d{}_c, \pi_d\}\right)\right\}\\ &\quad - \frac{c}{4\hbar}\epsilon^{abc}\left\{\frac{1}{\epsilon'}, \left(\{\mathcal{F}^d{}_b, \pi_d\}V\mathcal{P}_c - V\mathcal{P}_b\{\mathcal{F}^d{}_c, \pi_d\}\right)\right\},\end{aligned} \tag{19}$$

$$\begin{aligned}\Omega^a_{(2)} &= \frac{c^2}{8}\left\{\frac{1}{\mathcal{T}}, \left\{\{\pi_e, \mathcal{F}^e{}_b\}, \left\{\pi_f, \left[\epsilon^{abc}\left(\frac{1}{c}\dot{\mathcal{F}}^f{}_c - \mathcal{F}^d{}_c\partial_d K^f + K^d\partial_d\mathcal{F}^f{}_c\right)\right.\right.\right.\right.\\ &\quad \left.\left.\left.\left. - \frac{1}{2}\mathcal{F}^f{}_d\left(\delta^{db}\Xi^a - \delta^{da}\Xi^b\right)\right]\right\}\right\}\right\} + \frac{c}{2}\,\Xi^a - \frac{qc^2}{2}\left\{\frac{1}{\epsilon'}, V^2\mathfrak{B}^a\right\}\\ &\quad - \frac{2V}{\hbar}\mathcal{M}^a + \frac{c^2}{2\hbar}\left\{\frac{1}{\mathcal{T}}, \left\{\{\delta^{ab}\mathcal{F}^d{}_b\mathcal{F}^e{}_c V\mathcal{M}^c, \pi_d\}, \pi_e\right\}\right\},\end{aligned} \tag{20}$$

with the curly brackets $\{\ ,\ \}$ denoting anticommutators, and

$$\epsilon' = \sqrt{m^2c^4V^2 + \frac{c^2}{4}\delta^{ac}\{\pi_b, \mathcal{F}^b{}_a\}\{\pi_d, \mathcal{F}^d{}_c\}}, \qquad \mathcal{T} = 2\epsilon'^2 + \{\epsilon', mc^2V\}. \tag{21}$$

5. Quantum and classical spin dynamics

To find the dynamics of the spin, we evaluate the commutator of the polarization operator $\mathbf{\Pi} = \beta\mathbf{\Sigma}$ with the FW Hamiltonian (18):

$$\frac{d\mathbf{\Pi}}{dt} = \frac{i}{\hbar}[\mathcal{H}_{FW}, \mathbf{\Pi}] = \mathbf{\Omega}_{(1)} \times \mathbf{\Sigma} + \mathbf{\Omega}_{(2)} \times \mathbf{\Pi}. \tag{22}$$

The result describes the precession of the spin in the external gravitational and electromagnetic fields, and hence $\mathbf{\Omega}_{(1)}$ and $\mathbf{\Omega}_{(2)}$ have a clear meaning of the angular velocity operators.

It is straightforward to derive the corresponding semiclassical expressions by evaluating all anticommutators and neglecting the powers of $\hbar$ higher than 1 (the classical limit of the relativistic quantum mechanics is discussed in [13]). The equation (22) then gives rise to the semiclassical equations describing the precession of the average spin $\boldsymbol{s}$ vector:

$$\frac{d\boldsymbol{s}}{dt} = \mathbf{\Omega} \times \boldsymbol{s} = (\mathbf{\Omega}_{(1)} + \mathbf{\Omega}_{(2)}) \times \boldsymbol{s}, \tag{23}$$

$$\begin{aligned} \Omega^a_{(1)} &= \frac{c^2}{\epsilon'}\mathcal{F}^d{}_c\pi_d\Bigg[\frac{1}{2}\Upsilon\delta^{ac} - \epsilon^{abe}V\mathcal{C}_{be}{}^c + \frac{\epsilon'}{\epsilon' + mc^2V}\epsilon^{abc}W^e{}_{\widehat{b}}\partial_e V \\ &\quad + \frac{qV^2}{\epsilon' + mc^2V}\,\epsilon^{acb}\mathfrak{E}_b - \frac{2V}{c\hbar}\epsilon^{acb}\mathcal{P}_b\Bigg], \end{aligned} \tag{24}$$

$$\begin{aligned} \Omega^a_{(2)} &= \frac{c}{2}\,\Xi^a - \frac{c^3}{\epsilon'(\epsilon' + mc^2V)}\,\epsilon^{abc}\mathcal{Q}_{(bd)}\delta^{dn}\mathcal{F}^k{}_n\pi_k\mathcal{F}^l{}_c\pi_l \\ &\quad - \frac{qc^2V^2}{\epsilon'}\mathfrak{B}^a + \frac{2V}{\hbar}\left[-\mathcal{M}^a + \frac{c^2}{\epsilon'(\epsilon' + mc^2V)}\delta^{an}\mathcal{F}^c{}_n\pi_c\mathcal{F}^d{}_b\pi_d\mathcal{M}^b\right]. \end{aligned} \tag{25}$$

Here, in the semiclassical limit, we have $\epsilon' = \sqrt{m^2c^4V^2 + c^2\delta^{cd}\mathcal{F}^a{}_c\,\mathcal{F}^b{}_d\,\pi_a\,\pi_b}\,.$

Furthermore, from (18) we obtain the velocity operator in the semiclassical approximation:

$$\frac{dx^a}{dt} = \frac{i}{\hbar}[\mathcal{H}_{FW}, x^a] = \beta\,\frac{\partial\epsilon'}{\partial\pi_a} + cK^a = \beta\,\frac{c^2}{\epsilon'}\,\mathcal{F}^a{}_b\delta^{bc}\mathcal{F}^d{}_c\pi_d + cK^a, \tag{26}$$

which allows to identify the *anholonomic velocity* operator in the Schwinger frame (11) with

$$\beta\,\frac{c^2}{\epsilon'}\,\mathcal{F}^b{}_a\pi_b = \widehat{v}_a. \tag{27}$$

As a result, $\delta^{cd}\mathcal{F}^a{}_c\,\mathcal{F}^b{}_d\pi_a\pi_b = (\epsilon')^2\widehat{v}^2/c^2$, hence $(\epsilon')^2 = m^2c^4V^2 + (\epsilon')^2\widehat{v}^2/c^2$ and $\epsilon' = \gamma\, mc^2\,V$, thus, recalling that $\gamma = (1 - \widehat{v}^2/c^2)^{-1/2}$ is the Lorentz factor ($\widehat{v}^2 = \delta_{ab}\widehat{v}^a\widehat{v}^b$), we find:

$$\frac{\epsilon'}{\epsilon' + mc^2V} = \frac{\gamma}{1+\gamma}, \qquad \frac{c^3}{\epsilon'(\epsilon' + mc^2V)}\,\mathcal{F}^b{}_a\pi_b\mathcal{F}^d{}_c\pi_d = \frac{\gamma}{1+\gamma}\,\frac{\widehat{v}_a\widehat{v}_c}{c}. \tag{28}$$

Making use of (27) and (28) in (24) and (25), we then establish a fundamental agreement of the quantum and classical spin dynamics. Indeed, after the replacement of quantum operators by their classical counterparts, we recover (23) as the equation motion of the classical spin with the precession angular velocity $\mathbf{\Omega} = \mathbf{\Omega}_{(1)} + \mathbf{\Omega}_{(2)}$, obtained in the rest frame for the relativistic particle with mass m, electric charge q and dipole moments μ', δ' in the gravitational and electromagnetic fields, described by the dynamical system [6]

$$\frac{DU^\alpha}{d\tau} = \frac{dU^\alpha}{d\tau} + U^i\Gamma_{i\beta}{}^\alpha U^\beta = -\,\frac{q}{m}\,g^{\alpha\beta}F_{\beta\gamma}U^\gamma, \tag{29}$$

$$\begin{aligned} \frac{DS^\alpha}{d\tau} &= \frac{dS^\alpha}{d\tau} + U^i\Gamma_{i\beta}{}^\alpha S^\beta = -\,\frac{q}{m}\,g^{\alpha\beta}F_{\beta\gamma}S^\gamma \\ &\quad - \frac{2}{\hbar}\left[M^\alpha{}_\beta + \frac{1}{c^a}\,(M_{\beta\gamma}U^\alpha U^\gamma - M^{\alpha\gamma}U_\beta U_\gamma)\right]S^\beta. \end{aligned} \tag{30}$$

Here U^α is the 4-velocity and S^α is the spin 4-vector of the particle, and the polarization tensor is introduced by

$$M_{\alpha\beta} = \mu' F_{\alpha\beta} + c\delta'\, G_{\alpha\beta}. \tag{31}$$

Its components, $M_{\hat{0}\hat{a}} = c\mathcal{P}_a$ and $M_{\hat{a}\hat{b}} = \epsilon_{abc}\mathcal{M}^c$, are obviously identified with the 3-vectors $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{P}}$ which were defined in (17).

Quite remarkably, we can recast the precession angular velocity into a compact form

$$\boldsymbol{\Omega} = \frac{q}{m}\left[-\boldsymbol{\mathfrak{B}}_{\rm eff} + \frac{\gamma}{\gamma+1}\frac{\widehat{\boldsymbol{v}}\times\boldsymbol{\mathfrak{E}}_{\rm eff}}{c^2}\right] \tag{32}$$

by introducing the effective magnetic and electric fields

$$\boldsymbol{\mathfrak{B}}_{\rm eff} = \boldsymbol{\mathfrak{B}} + \frac{2m}{q\hbar}\left[\boldsymbol{\mathcal{M}} - \frac{\gamma^2}{c}\widehat{\boldsymbol{v}}\times\left(\boldsymbol{\mathcal{P}} - \frac{\widehat{\boldsymbol{v}}\times\boldsymbol{\mathcal{M}}}{c}\right)\right] + \frac{m}{q}\boldsymbol{\mathcal{B}}, \tag{33}$$

$$\boldsymbol{\mathfrak{E}}_{\rm eff} = \boldsymbol{\mathfrak{E}} - \frac{2m\gamma^2}{q\hbar}\widehat{\boldsymbol{v}}\times\left(\boldsymbol{\mathcal{M}} + \frac{\widehat{\boldsymbol{v}}\times\boldsymbol{\mathcal{P}}}{c}\right) + \frac{m}{q}\boldsymbol{\mathcal{E}}. \tag{34}$$

These expressions encompass the true magnetic and electric fields (the first terms on the right-hand sides), the nonminimal contributions due to AMM and EDM (the second terms), and the gravitomagnetic and gravitoelectric fields constructed from the metric of the spacetime:

$$\mathcal{B}^a = \frac{\gamma}{V}\left(-\frac{c}{2}\,\Xi^a - \frac{1}{2}\Upsilon\,\widehat{v}^a + \epsilon^{abc}V\mathcal{C}_{bc}{}^d\widehat{v}_d\right), \qquad \mathcal{E}^a = \frac{\gamma}{V}\delta^{ac}\left(c\mathcal{Q}_{(\widehat{c}\widehat{b})}\widehat{v}^b - c^2\, W^b{}_{\widehat{c}}\,\partial_b V\right). \tag{35}$$

6. Physical effects

The resulting general framework can be applied to the analysis of the possible physical effects of the spin dynamics. Let us briefly overview the three important applications.

6.1. Manifestations of terrestrial gravity and inertia in high-energy physics

The spacetime geometry [6], which correctly describes the terrestrial inertia and (weak) gravity of a rotating source, corresponds to the choice $W^{\hat{a}}{}_c = W\delta^{\hat{a}}_c$, and

$$V = 1 - \frac{GM}{c^2 r}, \qquad W = 1 + \frac{GM}{c^2 r}, \qquad \boldsymbol{K} = -\frac{\boldsymbol{\omega}\times\boldsymbol{r}}{c}. \tag{36}$$

Here $M = M_\oplus$, and $\boldsymbol{\omega} = (0, 0, \omega_\oplus)$, with $\omega_\oplus = \frac{2\pi}{T_\oplus} = 7.29\times10^{-5}{\rm s}^{-1}$. The inertial (due to rotation) and the gravitational fields of the Earth affect the motion of an elementary particle and its spin dynamics. This influence is not negligible and should be taken into account in high-energy physics experiments.

Earth's rotation manifests itself as the Coriolis and the centrifugal forces and produces the additional rotation of the spin. The corresponding corrections to particle's trajectory and the spin motion are rather small. Bigger oscillatory corrections average to zero, but the inhomogeneity of the inertial field gives rise to small non-oscillatory corrections which should be taken into consideration only for the storage ring EDM experiments.

Earth's gravity produces an additional force that acts on particle's momentum as well as an additional torque affecting the spin, although no direct electromagnetic effects caused by Earth's gravity were observed so far. The additional forces include the Newton-like force and the reaction force provided by a focusing system. The additional torques are caused by the focusing field and by the geodetic effect. In the storage ring EDM experiments, these forces and torques lead to the additional spin rotation about the radial axis. Theoretical analysis shows that terrestrial gravity can produce the same effect as deuteron's EDM of $\delta' = 1.5\times10^{-29}$ $e\cdot$cm in the planned dEDM experiment with magnetic focusing [14].

6.2. Limits on the non-Riemannian spacetime structure

Spin is a natural tool to probe the spacetime structure beyond Einstein's general relativity. Importantly, the torsion can be measured only by the matter with intrinsic spin [15, 16, 17]; orbital angular momentum (e.g., mechanically rotating gyroscopes) does not feel the torsion.

The spin of the Dirac particle is totally antisymmetric, and hence it can couple only to the axial torsion vector $\check{T}^\alpha = -\frac{1}{2}\eta^{\alpha\mu\nu\lambda}T_{\mu\nu\lambda} = \{\check{T}^{\hat{0}}, \check{T}^{\hat{a}}\}$. As a result, the hypothetical spin-torsion coupling modifies the Dirac Hamiltonian (15) via the additional terms [5]

$$-\frac{\hbar cV}{4}\left(\boldsymbol{\Sigma}\cdot\check{\boldsymbol{T}} + c\gamma_5\check{T}^{\hat{0}}\right). \tag{37}$$

We can straightforwardly take the spin-torsion coupling into account in (15) by the shift

$$\Upsilon \longrightarrow \Upsilon + Vc\check{T}^{\hat{0}}, \qquad \Xi^{\hat{a}} \longrightarrow \Xi^{\hat{a}} - V\check{T}^{\hat{a}}, \tag{38}$$

Making use of the theoretical framework established here, we can find observational bounds on spin-torsion coupling from the dynamics of freely precessing nuclear spins in a uniform magnetic field $\boldsymbol{B}$. In particular, using the data reported [18] for the experiment with ^{199}Hg and ^{201}Hg atoms devoted to the search of a hypothetical scalar-pseudoscalar interactions, we derive the restriction on the absolute value of the spacetime torsion:

$$\frac{\hbar c}{4}|\check{\boldsymbol{T}}|\cdot|\cos\Theta| < 2.2\times 10^{-21}\,\mathrm{eV}, \qquad |\check{\boldsymbol{T}}|\cdot|\cos\Theta| < 4.3\times 10^{-14}\,\mathrm{m}^{-1}. \tag{39}$$

Here Θ is the angle between $\boldsymbol{B}$ and the torsion $\check{\boldsymbol{T}}$. These bounds can be further improved by approximately one order on the basis of the more recent experimental data [19].

6.3. Spin in a gravitational wave

Fermion in the field of a gravitational wave was discussed earlier in [20, 21, 22].

In the local coordinates (t, x, y, z), the spacetime geometry of a weak gravitational wave is given by the metric (4) with

$$V = 1, \qquad \boldsymbol{K} = 0, \qquad W^{\hat{a}}{}_b = \begin{pmatrix} 1 + w_\oplus & w_\otimes & 0 \\ w_\otimes & 1 - w_\oplus & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{40}$$

The functions $w_\otimes = w_\otimes(\varphi)$ and $w_\oplus = w_\oplus(\varphi)$ of the phase $\varphi = \omega(t - z/c)$, describe the two independent polarizations of a plane gravitational wave with the frequency ω propagating along the z direction [23].

One can probe the gravitational wave by observing the spin dynamics in the magnetic resonance setup [12]. Assuming that the wave has only one polarization, $w_\oplus = 0$, whereas $w_\otimes = g_0\cos\varphi$ (with amplitude g_0), we can arrange the constant homogeneous magnetic field in the plane of the wave front: $\boldsymbol{B} = (B_0, 0, 0)$, with $B_0 =$const. Then (12) yields $\mathfrak{B} = (B_0, B_0 w_\otimes, 0)$, and hence we find the field configuration that reproduces the magnetic resonance conditions, when the spin is affected by the constant homogeneous magnetic field along x and an additional alternating field in the perpendicular plane (y, z). Then [24], we find

$$P_{-\frac{1}{2}} = \frac{\sin^2\{\omega_0 g_0(t - t_0)\Lambda/4\}}{\Lambda^2} \tag{41}$$

the probability of a flip, at time t, for the spin as compared to its initial opposite orientation at t_0. Here $\omega_0 = \frac{2(\mu_0+\mu')B_0}{\hbar}$ is the Larmor frequency, $\Lambda^2 = 1 + \frac{4(1-\xi)^2}{g_0^2}$ and $\xi = \frac{\omega}{\omega_0}\left(1 - \frac{g_0^2}{16\xi^2}\right)$.

We thus confirm the qualitative conclusions [20, 22] on the prospects of the gravitational wave searches using the magnetic resonance type experiments. The effect (41) is quadratic in the small quantity g_0, however, polarization effects which are linear in g_0 can possibly show up in the analysis of the spin components orthogonal to initial spin polarization [25, 26].

7. Discussion

Continuing the study of the dynamics of the quantum and classical Dirac fermions with spin $\frac{1}{2}$ and dipole moments in the gravitational and electromagnetic fields, we extended here our earlier findings in [27, 28, 29, 2, 3, 4] to the case of an arbitrary spacetime metric plus an arbitrary electromagnetic field. This is a nontrivial problem, because there is no direct superposition of the electromagnetic effects on spin with the gravity-spin effects, and the action of gravity on spin is twofold: via the coframe and connection (16) and via the electromagnetic field which gets modified (12) in the curved spacetime [12].

We use the general framework, in which we derived the exact FW Hamiltonian (18) and the quantum equations of motion, to analyse the effects of terrestrial gravity and rotation in the high-energy physics, to find the new strong bounds on the spacetime torsion, and to develop a magnetic resonance setup for spin in the gravitational wave.

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