

# Reality from maximizing overlap in the periodic complex action theory

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We study the periodic complex action theory (CAT) by imposing a periodic condition in the future-included CAT where the time integration is performed from the past to the future, and extend a normalized matrix element of an operator  $\hat{O}$ , which is called the weak value in the real action theory, to another expression  $\langle \hat{O} \rangle_{\text{periodic time}}$ . We present two theorems stating that  $\langle \hat{O} \rangle_{\text{periodic time}}$  becomes real for  $\hat{O}$  being Hermitian with regard to a modified inner product that makes a given non-normal Hamiltonian  $\hat{H}$  normal. The first theorem holds for a given period  $t_p$  in a case where the number of eigenstates having the maximal imaginary part  $B$  of the eigenvalues of  $\hat{H}$  is just one, while the second one stands for  $t_p$  selected such that the absolute value of the transition amplitude is maximized in a case where  $B \leq 0$  and  $|B|$  is much smaller than the distances between any two real parts of the eigenvalues of  $\hat{H}$ . The latter proven via a number-theoretical argument suggests that, if our universe is periodic, then even the period could be an adjustment parameter to be determined in the Feynman path integral. This is a variant type of the maximization principle that we previously proposed.

Subject Index A60, B30

## 1. Introduction

In the usual quantum theory reality of action is implicitly imposed at first. Indeed, in the Feynman path integral, action is regarded as a phase of the integrand. However, there is a possibility that action also produces a scale factor in the integrand by taking a complex value. Such a complex action theory (CAT) [1]—an attempt to describe a quantum theory whose action is complex at a fundamental level but effectively looks real—has been investigated intensively with the expectation that the imaginary part of the action would give some falsifiable predictions [1–4]. Various interesting suggestions have been made for the Higgs mass [5], quantum-mechanical philosophy [6–8], some fine-tuning problems [9,10], black holes [11], de Broglie–Bohm particles and a cut-off in loop diagrams [12], a mechanism to obtain Hermitian Hamiltonians [13], the complex coordinate formalism [14], and the momentum relation [15,16]. The CAT is classified into two types. One is a special type of theory that we call “future-included”. In the future-included theory, not only the past state  $|A(T_A)\rangle$  at the initial time  $T_A$  but also the future state  $|B(T_B)\rangle$  at the final time  $T_B$  is given at first, and the time integration is performed over the whole period from the past to the future. This is in contrast to the other usual type of theory that we call “future-not-included”, where only the past state  $|A(T_A)\rangle$  is given at first, and the time integration is performed over the period between the initial time  $T_A$  and some specific time  $t$  ( $T_A$

$\leq t \leq T_B$ ). In Ref. [16] we clarified various interesting properties of the future-not-included CAT. However, in Ref. [17], we argued that, if a theory is described with a complex action, then such a theory is suggested to be the future-included theory, rather than the future-not-included theory. We encounter a philosophical contradiction in the future-not-included CAT as long as we respect objectivity.

In the future-included theory, what is expected to work as an expectation value for an operator  $\hat{O}$  is the normalized matrix element [1]<sup>1</sup>  $\langle \hat{O} \rangle^{BA} \equiv \frac{\langle B(t)|\hat{O}|A(t) \rangle}{\langle B(t)|A(t) \rangle}$ . Indeed, if we regard  $\langle \hat{O} \rangle^{BA}$  as an expectation value in the future-included theory, we obtain the Heisenberg equation, Ehrenfest's theorem, and a conserved probability current density [20,21]. Thus  $\langle \hat{O} \rangle^{BA}$  has very nice properties. Here we note that  $\langle \hat{O} \rangle^{BA}$  is generically complex even for Hermitian  $\hat{O}$  by its definition. On the other hand, if  $\langle \hat{O} \rangle^{BA}$  is desired to be an expectation value for  $\hat{O}$ , it has to be real, since we know that any observables should be real. Then how can we resolve this crucial problem?

In the CAT the imaginary parts of the eigenvalues  $\lambda_i$  of a given non-normal Hamiltonian<sup>2</sup>  $\hat{H}$  are supposed to be bounded from above for the Feynman path integral  $\int e^{\frac{i}{\hbar} S} \mathcal{D}\text{path}$  to converge. We can imagine that some  $\text{Im}\lambda_i$  take the maximal value  $B$ . We denote the corresponding subset of  $\{i\}$  as  $A$ . In Refs. [27,28], under this supposition, we answered the above question by proposing a theorem that states that, provided that an operator  $\hat{O}$  is  $Q$ -Hermitian, i.e., Hermitian with regard to a modified inner product  $I_Q$  that makes the given Hamiltonian normal by using an appropriately chosen Hermitian operator  $Q$ , the normalized matrix element defined with  $I_Q$  becomes real and time-develops under a  $Q$ -Hermitian Hamiltonian for the past and future states selected such that the absolute value of the transition amplitude defined with  $I_Q$  from the past state to the future state is maximized. We call this way of thinking the maximization principle. In Ref. [27] we gave a proof of the theorem in the case of non-normal Hamiltonians  $\hat{H}$  by considering that essentially only terms associated with the largest imaginary parts of the eigenvalues of  $\hat{H}$ , which belong to the subset  $A$ , contribute most significantly to the absolute value of the transition amplitude defined with  $I_Q$ , and that the normalized matrix element defined with  $I_Q$  for such maximizing states becomes an expression similar to an expectation value defined with  $I_Q$  in the future-not-included theory. This proof is based on the existence of imaginary parts of the eigenvalues of  $\hat{H}$ . In the case of the RAT we gave another proof in Ref. [28]. In Ref. [27] we found that via the maximization principle in the expansion of the resulting maximizing states  $|A(T_A)\rangle_{\text{max}}$  and  $|B(T_B)\rangle_{\text{max}}$  in terms of the eigenstates of  $\hat{H}$ ,  $|A(T_A)\rangle_{\text{max}} = \sum_{i \in A} a_i(T_A) |\lambda_i\rangle$ ,  $|B(T_B)\rangle_{\text{max}} = \sum_{i \in A} b_i(T_B) |\lambda_i\rangle$ , the absolute values of each component were found to be the same:  $|a_i(T_A)| = |b_i(T_B)|$  for  $\forall i \in A$ , while the phases were not so. This fact has partly motivated us to study a periodic universe. In addition, it would be interesting by itself to look for a possibility that our universe runs periodically, and also to see whether there still exists any kind of reality theorems on the expectation value for  $\hat{O}$  in such a periodic CAT.

In this letter, after briefly reviewing the future-included CAT and maximization principle, we study the periodic CAT. For simplicity let us now suppose that we obtained a periodic universe via the maximization principle for the past and future states  $|A(T_A)\rangle$  and  $|B(T_B)\rangle$ , or just consider it by imposing a periodic condition on the past and future states in the future-included

<sup>1</sup>The normalized matrix element  $\langle \hat{O} \rangle^{BA}$  is called the weak value [18] in the context of the real action theory (RAT), and it has been intensively studied. For details, see Ref. [19] and references therein.

<sup>2</sup>The Hamiltonian  $\hat{H}$  is generically non-normal, so it is not restricted to the class of PT-symmetric non-Hermitian Hamiltonians that were studied in Refs. [22–26].

CAT at first. Then the remaining quantity to be adjusted could be the period. If so, it would be very interesting that even the period is regarded as a parameter that is adjusted via the maximization principle. To proceed with this speculation, taking  $T_B = T_A + t_p$ , we simply impose the following periodic condition<sup>3</sup> on the future and past states in the future-included CAT:

$$|B(T_B)\rangle = |B(T_A + t_p)\rangle = |A(T_A)\rangle. \quad (1)$$

In the periodic CAT, extending a normalized matrix element of an operator  $\hat{O}$  to an expression such that various normalized matrix elements of  $\hat{O}$  are summed up with the weight of transition amplitudes, we introduce another normalized quantity  $\langle \hat{O} \rangle_{\text{periodic time}} \equiv \frac{\text{Tr}(e^{-\frac{i}{\hbar} \hat{H} t_p} \hat{O})}{\text{Tr}(e^{-\frac{i}{\hbar} \hat{H} t_p})}$  that is generically complex but expected to have a role of an expectation value for  $\hat{O}$ . We present two theorems stating that  $\langle \hat{O} \rangle_{\text{periodic time}}$  becomes real for  $Q$ -Hermitian  $\hat{O}$ . The first theorem holds for a given period  $t_p$ , even without any adjustment of it, in a case where the order of the subset  $A$  is just one, i.e., the number of eigenstates having the maximal imaginary part  $B$  of the eigenvalues of  $\hat{H}$  is just one, while the second one stands for  $t_p$  selected such that the absolute value of the transition amplitude  $|\text{Tr}(e^{-\frac{i}{\hbar} \hat{H} t_p})|$  is maximized in a case where  $B \leq 0$  and  $|B|$  is much smaller than the distances between any two real parts of the eigenvalues of  $\hat{H}$ . The second theorem, which is proven partly via a number-theoretical argument, suggests that, if our universe is periodic, then even the period could be an adjustment parameter to be determined in the Feynman path integral via such a variant type of the maximization principle that we proposed in Refs. [27,28].

## 2. Future-included complex action theory and maximization principle

The eigenstates of a given non-normal Hamiltonian  $\hat{H}$ ,  $|\lambda_i\rangle (i = 1, 2, \dots)$  obeying  $\hat{H}|\lambda_i\rangle = \lambda_i|\lambda_i\rangle$ , are not orthogonal to each other in the usual inner product  $I$ . In order to obtain an orthogonal basis, let us introduce a modified inner product  $I_Q$  [13,14] that makes  $\hat{H}$  normal with respect to it.<sup>4</sup> This enables  $|\lambda_i\rangle (i = 1, 2, \dots)$  to be orthogonal to each other with regard to  $I_Q$ , which is defined for arbitrary kets  $|u\rangle$  and  $|v\rangle$  as  $I_Q(|u\rangle, |v\rangle) \equiv \langle u|_Q v \rangle \equiv \langle u|Q|v\rangle$ . Here  $Q$  is a Hermitian operator that obeys  $\langle \lambda_i |_Q \lambda_j \rangle = \delta_{ij}$ . The Hamiltonian  $\hat{H}$  is diagonalized as  $\hat{H} = PDP^{-1}$ , where  $P = (|\lambda_1\rangle, |\lambda_2\rangle, \dots)$  and  $D = \text{diag}(\lambda_1, \lambda_2, \dots)$ . Using the diagonalizing operator  $P$ , we choose  $Q = (P^\dagger)^{-1}P^{-1}$ . Utilizing this  $Q$ , we introduce the  $Q$ -Hermitian conjugate  $\dagger^Q$  of an operator  $A$  by  $\langle \psi_2 |_Q A | \psi_1 \rangle^* \equiv \langle \psi_1 |_Q A^{\dagger^Q} | \psi_2 \rangle$ , so  $A^{\dagger^Q} \equiv Q^{-1}A^\dagger Q$ . Also, we define  $\dagger^Q$  for kets and bras as  $|\lambda\rangle^{\dagger^Q} \equiv \langle \lambda |_Q$  and  $(\langle \lambda |_Q)^{\dagger^Q} \equiv |\lambda\rangle$ . If  $A$  obeys  $A^{\dagger^Q} = A$ , we call  $A$   $Q$ -Hermitian. We note that, since  $P^{-1} = \begin{pmatrix} \langle \lambda_1 |_Q \\ \langle \lambda_2 |_Q \\ \vdots \end{pmatrix}$  obeys  $P^{-1}\hat{H}P = D$  and  $P^{-1}\hat{H}^{\dagger^Q}P = D^\dagger$ ,  $\hat{H}$  is  $Q$ -normal,  $[\hat{H}, \hat{H}^{\dagger^Q}] = P[D, D^\dagger]P^{-1} = 0$ .

$\hat{H}$  can be decomposed as  $\hat{H} = \hat{H}_{Qh} + \hat{H}_{Qa}$ , where  $\hat{H}_{Qh} = \frac{\hat{H} + \hat{H}^{\dagger^Q}}{2}$  and  $\hat{H}_{Qa} = \frac{\hat{H} - \hat{H}^{\dagger^Q}}{2}$  are  $Q$ -Hermitian and anti- $Q$ -Hermitian parts of  $\hat{H}$ , respectively.

In Ref. [27], we adopted the modified inner product  $I_Q$  for all quantities in the future-included CAT [1,20,21]. The future-included CAT is described by using the future state  $|B(T_B)\rangle$  at the

<sup>3</sup>Another periodic condition might be  $|A(T_A)\rangle = |A(T_A + t_p)\rangle = e^{-\frac{i}{\hbar} \hat{H} t_p} |A(T_A)\rangle$ . This means that  $e^{-\frac{i}{\hbar} \hat{H} t_p}$  has to have  $|A(T_A)\rangle$  as the eigenstate for its eigenvalue 1. Since  $|A(T_A)\rangle$  is supposed to be a generic state, we do not adopt such a periodic condition in this letter.

<sup>4</sup>Similar inner products are also studied in Refs. [22,23,29].

final time  $T_B$  and the past state  $|A(T_A)\rangle$  at the initial time  $T_A$ , where  $|A(T_A)\rangle$  and  $|B(T_B)\rangle$  are supposed to time-develop as follows:

$$i\hbar \frac{d}{dt} |A(t)\rangle = \hat{H} |A(t)\rangle, \quad (2)$$

$$i\hbar \frac{d}{dt} |B(t)\rangle = \hat{H}^{\dagger Q} |B(t)\rangle \quad \Leftrightarrow \quad -i\hbar \frac{d}{dt} \langle B(t)|_Q = \langle B(t)|_Q \hat{H}. \quad (3)$$

The normalized matrix element defined with the modified inner product  $I_Q$  is expressed as

$$\langle \hat{O} \rangle_Q^{BA} \equiv \frac{\langle B(t)|_Q \hat{O} |A(t)\rangle}{\langle B(t)|_Q A(t)\rangle}. \quad (4)$$

If we change the notation of  $\langle B(t)|$  such that it absorbs  $Q$ , it can be expressed simply as  $\langle \hat{O} \rangle^{BA}$  [1]. In the case of  $Q = 1$ , this corresponds to the weak value [18,19] that is well known in the RAT. If we regard  $\langle \hat{O} \rangle_Q^{BA}$  as an expectation value in the future-included CAT, then we obtain the Heisenberg equation, Ehrenfest's theorem, and a conserved probability current density [20,21]. Therefore, this quantity is a good candidate for an expectation value in the future-included CAT.

In Ref. [27] we proposed the following theorem.

**Theorem 1.** *As a prerequisite, assume that a given Hamiltonian  $\hat{H}$  is non-normal but diagonalizable and that the imaginary parts of the eigenvalues of  $\hat{H}$  are bounded from above, and define a modified inner product  $I_Q$  by means of a Hermitian operator  $Q$  arranged so that  $\hat{H}$  becomes normal with respect to  $I_Q$ . Let the two states  $|A(t)\rangle$  and  $|B(t)\rangle$  time-develop according to the Schrödinger equations with  $\hat{H}$  and  $\hat{H}^{\dagger Q}$  respectively:  $|A(t)\rangle = e^{-\frac{i}{\hbar} \hat{H}(t-T_A)} |A(T_A)\rangle$ ,  $|B(t)\rangle = e^{-\frac{i}{\hbar} \hat{H}^{\dagger Q}(t-T_B)} |B(T_B)\rangle$ , and be normalized with  $I_Q$  at the initial time  $T_A$  and the final time  $T_B$  respectively:  $\langle A(T_A)|_Q A(T_A)\rangle = 1$ ,  $\langle B(T_B)|_Q B(T_B)\rangle = 1$ . Next determine  $|A(T_A)\rangle$  and  $|B(T_B)\rangle$  so as to maximize the absolute value of the transition amplitude  $|\langle B(t)|_Q A(t)\rangle| = |\langle B(T_B)|_Q \exp(-i\hat{H}(T_B - T_A)) |A(T_A)\rangle|$ . Then, provided that an operator  $\hat{O}$  is  $Q$ -Hermitian, i.e., Hermitian with respect to the inner product  $I_Q$ ,  $\hat{O}^{\dagger Q} = \hat{O}$ , the normalized matrix element of the operator  $\hat{O}$  defined by  $\langle \hat{O} \rangle_Q^{BA} \equiv \frac{\langle B(t)|_Q \hat{O} |A(t)\rangle}{\langle B(t)|_Q A(t)\rangle}$  becomes real and time-develops under a  $Q$ -Hermitian Hamiltonian.*

We call this way of thinking the maximization principle. To prove this theorem in the case of the CAT<sup>5</sup>, we expand  $|A(t)\rangle$  and  $|B(t)\rangle$  in terms of the eigenstates  $|\lambda_i\rangle$  as follows:  $|A(t)\rangle = \sum_i a_i(t) |\lambda_i\rangle$ ,  $|B(t)\rangle = \sum_i b_i(t) |\lambda_i\rangle$ , where  $a_i(t) = a_i(T_A) e^{-\frac{i}{\hbar} \lambda_i(t-T_A)}$ ,  $b_i(t) = b_i(T_B) e^{-\frac{i}{\hbar} \lambda_i^*(t-T_B)}$ . Let us express  $a_i(T_A)$  and  $b_i(T_B)$  as  $a_i(T_A) = |a_i(T_A)| e^{i\theta_{a_i}}$  and  $b_i(T_B) = |b_i(T_B)| e^{i\theta_{b_i}}$ , and introduce  $T \equiv T_B - T_A$  and  $\Theta_i \equiv \theta_{a_i} - \theta_{b_i} - \frac{1}{\hbar} T \text{Re} \lambda_i$ . Since the imaginary parts of the eigenvalues of  $\hat{H}$  are supposed to be bounded from above for the Feynman path integral  $\int e^{\frac{i}{\hbar} S} \mathcal{D}\text{path}$  to converge, we can imagine that some of  $\text{Im} \lambda_i$  take the maximal value  $B$ , and denote the corresponding subset of  $\{i\}$  as  $A$ . Then,  $|\langle B(t)|_Q A(t)\rangle|$  can take the maximal value  $e^{\frac{1}{\hbar} BT}$  only under the following conditions:  $\Theta_i \equiv \Theta_c$  for  $\forall i \in A$ ,  $\sum_{i \in A} |a_i(T_A)|^2 = \sum_{i \in A} |b_i(T_B)|^2 = 1$ ,  $|a_i(T_A)| = |b_i(T_B)|$  for  $\forall i \in A$ ,  $|a_i(T_A)| = |b_i(T_B)| = 0$  for  $\forall i \notin A$ , and the states to maximize  $|\langle B(t)|_Q A(t)\rangle|$  are expressed as  $|A(t)\rangle_{\text{max}} = \sum_{i \in A} a_i(t) |\lambda_i\rangle$  and  $|B(t)\rangle_{\text{max}} = \sum_{i \in A} b_i(t) |\lambda_i\rangle$ . Introducing  $|\tilde{A}(t)\rangle \equiv e^{-\frac{i}{\hbar} (t-T_A) \hat{H}_{Qh}} |A(T_A)\rangle_{\text{max}}$ , which is normalized as  $\langle \tilde{A}(t)|_Q \tilde{A}(t)\rangle = 1$  and obeys

<sup>5</sup>The above proof depends on the existence of imaginary parts of the eigenvalues of  $\hat{H}$ , so it does not apply to the case where a given Hamiltonian is Hermitian, where there are no imaginary parts of the eigenvalues. For the proof in such a special case, see Ref. [28]. The maximization principle is reviewed in Refs. [30,31].

the Schrödinger equation  $i\hbar \frac{d}{dt} |\tilde{A}(t)\rangle = \hat{H}_{Qh} |\tilde{A}(t)\rangle$ , the normalized matrix element for  $|A(t)\rangle_{\max}$  and  $|B(t)\rangle_{\max}$  is evaluated as  $\langle \hat{O} \rangle_Q^{B_{\max} A_{\max}} = \langle \tilde{A}(t)|_Q \hat{O} |\tilde{A}(t)\rangle$ . Hence  $\langle \hat{O} \rangle_Q^{B_{\max} A_{\max}}$  is real for  $Q$ -Hermitian  $\hat{O}$ , and time-develops under the  $Q$ -Hermitian Hamiltonian  $\hat{H}_{Qh}$ :  $\frac{d}{dt} \langle \hat{O} \rangle_Q^{B_{\max} A_{\max}} = \frac{i}{\hbar} \langle [\hat{H}_{Qh}, \hat{O}] \rangle_Q^{B_{\max} A_{\max}}$ . Thus we have seen that the maximization principle provides both the reality of  $\langle \hat{O} \rangle_Q^{BA}$  for  $Q$ -Hermitian  $\hat{O}$  and the  $Q$ -Hermitian Hamiltonian.

### 3. Periodic complex action theory and maximization principle

In the future-included CAT, let us take  $T_B = T_A + t_p$ , and impose the periodicity condition (1). Then, since  $\langle B(t)|_Q$  is expressed as  $\langle B(t)|_Q = \langle A(T_A)|_Q e^{\frac{i}{\hbar} \hat{H}(t-T_B)}$ ,  $\langle \hat{O} \rangle_Q^{BA}$  defined in Eq. (4) is written as

$$\langle \hat{O} \rangle_Q^{BA} = \frac{\langle A(T_A)|_Q e^{\frac{i}{\hbar} \hat{H}(t-T_B)} \hat{O} e^{-\frac{i}{\hbar} \hat{H}(t-T_A)} |A(T_A)\rangle}{\langle A(T_A)|_Q e^{-\frac{i}{\hbar} \hat{H} t_p} |A(T_A)\rangle} \equiv \langle \hat{O} \rangle_Q^A. \quad (5)$$

Now we rewrite  $\langle \hat{O} \rangle_Q^A$  as follows:

$$\langle \hat{O} \rangle_Q^A \simeq \frac{\sum_n \langle \hat{O} \rangle_Q^{A_n} \langle A_n|_Q e^{-\frac{i}{\hbar} \hat{H} t_p} |A_n\rangle}{\sum_n \langle A_n|_Q e^{-\frac{i}{\hbar} \hat{H} t_p} |A_n\rangle} = \frac{\text{Tr} \left( e^{-\frac{i}{\hbar} \hat{H} t_p} \hat{O} \right)}{\text{Tr} \left( e^{-\frac{i}{\hbar} \hat{H} t_p} \right)}, \quad (6)$$

where we have taken a basis  $|A_n\rangle$  such that the state  $|A(T_A)\rangle$  maximizing  $|\langle A(T_A)|_Q e^{-\frac{i}{\hbar} \hat{H} t_p} |A(T_A)\rangle|$  is included. Weighting various normalized matrix elements  $\langle \hat{O} \rangle_Q^{A_n}$  by  $\langle A_n|_Q e^{-\frac{i}{\hbar} \hat{H} t_p} |A_n\rangle$  replaces maximizing  $|\langle A(T_A)|_Q e^{-\frac{i}{\hbar} \hat{H} t_p} |A(T_A)\rangle|$  crudely in a quantitative way. In addition we have used the cyclic property of  $\text{Tr}$ .

Based on the above evaluation, in the periodic CAT specified by the periodic condition (1), we propose our “expectation value” for  $\hat{O}$  by the following quantity:

$$\langle \hat{O} \rangle_{\text{periodic time}} \equiv \frac{\text{Tr} \left( e^{-\frac{i}{\hbar} \hat{H} t_p} \hat{O} \right)}{\text{Tr} \left( e^{-\frac{i}{\hbar} \hat{H} t_p} \right)}. \quad (7)$$

This quantity is generically complex by its definition, so it is unclear whether we can use it as an expectation value for  $\hat{O}$ . In addition, this quantity is independent of the time  $t$ , so the situation is like that in general relativity with an exact symmetry under translations in the time variable, where there is conservation of the total energy, which is even just zero, and averaging would lead to no time dependence. If we want to reintroduce the time  $t$  dependence, as we are accustomed to, we would have to introduce a clock variable  $T_{\text{clock}}(t)$  to be inserted in the normalized quantity of Eq. (7). In this letter, however, we will not be involved in it, but we concentrate on whether  $\langle \hat{O} \rangle_{\text{periodic time}}$  could be real, since the reality of  $\langle \hat{O} \rangle_{\text{periodic time}}$  is crucially important for our theory to be viable. Seeking a condition for  $\langle \hat{O} \rangle_{\text{periodic time}}$  to be real provided that  $\hat{O}$  is  $Q$ -Hermitian, we propose the following two theorems in special cases. In the first theorem, we consider a case where the order of the subset<sup>6</sup>  $A$  is just one for the given period  $t_p$ . In the second theorem, assuming that the maximal value  $B$  of the imaginary parts of the eigenvalues of  $\hat{H}$  is equal to or smaller than 0, and that  $|B|$  is much smaller than the distances between any two real parts of the eigenvalues of  $\hat{H}$ , we regard  $t_p$  as an adjustment parameter, which is to be selected such that the absolute value of the transition amplitude  $|\text{Tr} \left( e^{-\frac{i}{\hbar} \hat{H} t_p} \right)|$  is maximized. The second theorem is a variant type of Theorem 1 in the point that, on behalf of  $|B(T_B)\rangle$  and  $|A(T_A)\rangle$  that are constrained by the condition (1), the period  $t_p$  is used as an adjustment parameter.

<sup>6</sup>The subset  $A$  is given in the proof of Theorem 1.  $B$  is the maximal value of  $\text{Im} \lambda_n$ .

**Theorem 2.** *As a prerequisite, assume that a given Hamiltonian  $\hat{H}$  is non-normal but diagonalizable and that only one significant eigenstate of  $\hat{H}$  contributes essentially for a given fixed period  $t_p$ , and define a modified inner product  $I_Q$  by means of a Hermitian operator  $Q$  arranged so that  $\hat{H}$  becomes normal with respect to  $I_Q$ . Then, provided that an operator  $\hat{O}$  is  $Q$ -Hermitian, i.e., Hermitian with respect to the inner product  $I_Q$ ,  $\hat{O}^\dagger = \hat{O}$ ,  $\langle \hat{O} \rangle_{\text{periodic time}} \equiv \frac{\text{Tr}(e^{-\frac{i}{\hbar} \hat{H} t_p} \hat{O})}{\text{Tr}(e^{-\frac{i}{\hbar} \hat{H} t_p})}$  becomes real.*

**Theorem 3.** *As a prerequisite, assume that a given Hamiltonian  $\hat{H}$  is non-normal but diagonalizable, that the maximal value  $B$  of the imaginary parts of the eigenvalues of  $\hat{H}$  is equal to or smaller than zero, and that  $|B|$  is much smaller than the distances between any two real parts of the eigenvalues of  $\hat{H}$ , and define a modified inner product  $I_Q$  by means of a Hermitian operator  $Q$  arranged so that  $\hat{H}$  becomes normal with respect to  $I_Q$ . Then, provided that an operator  $\hat{O}$  is  $Q$ -Hermitian, i.e., Hermitian with respect to the inner product  $I_Q$ ,  $\hat{O}^\dagger = \hat{O}$ ,  $\langle \hat{O} \rangle_{\text{periodic time}} \equiv \frac{\text{Tr}(e^{-\frac{i}{\hbar} \hat{H} t_p} \hat{O})}{\text{Tr}(e^{-\frac{i}{\hbar} \hat{H} t_p})}$  becomes real for selected periods  $t_p$  such that  $|\text{Tr}(e^{-\frac{i}{\hbar} \hat{H} t_p})|$  is maximized.*

In preparation for proving these theorems, we first evaluate the numerator and denominator of the right-hand side of Eq. (7). The numerator is expressed as  $\text{Tr}(e^{-\frac{i}{\hbar} \hat{H} t_p} \hat{O}) = \sum_n \langle \lambda_n | Q e^{-\frac{i}{\hbar} \hat{H} t_p} \hat{O} | \lambda_n \rangle \simeq e^{\frac{B}{\hbar} t_p} \sum_{n \in A} \langle \lambda_n | Q \hat{O} | \lambda_n \rangle e^{-i\theta_n}$ , where we have used as a basis the set of eigenstates of the Hamiltonian  $\hat{H}$ ,  $|\lambda_n\rangle$ , which obeys the orthogonality and completeness relations:  $\langle \lambda_n | Q \lambda_m \rangle = \delta_{nm}$ ,  $\sum_m |\lambda_m\rangle \langle \lambda_m | Q = 1$ . In addition, we have introduced  $\theta_n \equiv \frac{1}{\hbar} \text{Re} \lambda_n t_p$ , and supposed that  $t_p$  is sufficiently large from a phenomenological point of view so that the terms coming from the subset  $A$  dominate most significantly. Similarly the denominator is evaluated as  $\text{Tr}(e^{-\frac{i}{\hbar} \hat{H} t_p}) \simeq e^{\frac{B}{\hbar} t_p} \sum_{n \in A} e^{-i\theta_n}$ . Thus  $\langle \hat{O} \rangle_{\text{periodic time}}$  is reduced to the following expression:

$$\langle \hat{O} \rangle_{\text{periodic time}} \simeq \frac{\sum_{n \in A} \langle \lambda_n | Q \hat{O} | \lambda_n \rangle e^{-i\theta_n}}{\sum_{n \in A} e^{-i\theta_n}}. \quad (8)$$

First let us prove Theorem 2 for a given fixed period  $t_p$  by assuming that the order of the subset  $A$  is one. We express the dominating eigenstate and eigenvalue associated with it as  $|\lambda_d\rangle$  and  $\lambda_d$  respectively. Then, since both the numerator and denominator of the right-hand side of Eq. (8) are composed of only one term associated with  $\lambda_d$ ,  $\langle \hat{O} \rangle_{\text{periodic time}}$  is expressed as

$$\langle \hat{O} \rangle_{\text{periodic time}} \simeq \langle \lambda_d | Q \hat{O} | \lambda_d \rangle. \quad (9)$$

This is real for  $Q$ -Hermitian  $\hat{O}$ , so we have proven Theorem 2.

Next let us prove Theorem 3 by utilizing the expression of Eq. (8) again. The function  $f(t_p) \equiv |\text{Tr}(e^{-\frac{i}{\hbar} \hat{H} t_p})|^2$  and its derivative with regard to  $t_p$  are evaluated as  $f(t_p) \simeq e^{\frac{2}{\hbar} B t_p} \sum_{n,m \in A} \cos \left\{ \frac{1}{\hbar} (\text{Re} \lambda_m - \text{Re} \lambda_n) t_p \right\}$  and  $\frac{df(t_p)}{dt_p} \simeq \frac{1}{\hbar} \sum_{n,m \in A} [2B \cos \left\{ \frac{1}{\hbar} (\text{Re} \lambda_m - \text{Re} \lambda_n) t_p \right\} - \sin \left\{ \frac{1}{\hbar} (\text{Re} \lambda_m - \text{Re} \lambda_n) t_p \right\} (\text{Re} \lambda_m - \text{Re} \lambda_n)] e^{\frac{2B}{\hbar} t_p}$ . Since we are assuming<sup>7</sup> that  $B \leq 0$  and  $|B|$  is much smaller than the distances between any two real parts of the eigenvalues of  $\hat{H}$ , the second term in the square brackets contributes significantly in the expression of  $\frac{df(t_p)}{dt_p}$ . Thus we find that, for  $\theta_i$  such that

$$\theta_i = \theta_c \pmod{2\pi} \text{ for } \forall i \in A \Leftrightarrow \text{Re} \lambda_i t_p = \hbar \theta_c \equiv C \pmod{2\pi \hbar} \text{ for } \forall i \in A, \quad (10)$$

<sup>7</sup>We note that  $B \leq 0$  has to be supposed so that  $|\text{Tr}(e^{-\frac{i}{\hbar} \hat{H} t_p})| = e^{\frac{B}{\hbar} t_p} |\sum_{n \in A} e^{-i\theta_n}|$  does not diverge when we seek  $t_p$  such that  $|\text{Tr}(e^{-\frac{i}{\hbar} \hat{H} t_p})|$  is maximized.

$\frac{d^2 f(t_p)}{dt_p^2} < 0$  and  $f(t_p)$  is maximized. In Eq. (10) we have introduced  $C \equiv \hbar\theta_c$ . If  $t_p$  satisfying Eq. (10) exist, the phase factor  $e^{-i\theta_n}$  becomes the same for  $\forall n \in A$  in Eq. (8), so  $\langle \hat{O} \rangle_{\text{periodic time}}$  is reduced to a simpler expression:

$$\langle \hat{O} \rangle_{\text{periodic time}} \simeq \frac{\sum_{n \in A} \langle \lambda_n | Q \hat{O} | \lambda_n \rangle}{\sum_{n \in A} 1}. \quad (11)$$

This is real for  $Q$ -Hermitian  $\hat{O}$ . Thus Theorem 3 will be proven.

#### 4. Proof of the existence of $t_p$ satisfying Eq. (10)

The existence of  $t_p$  satisfying Eq. (10) looks believable. Now we investigate it explicitly according to the order of the Hilbert space that is labeled by the subset  $A$ . First let us consider the case where the order of the Hilbert space is two. We express  $\text{Re}\lambda_i$  ( $i \in A$ ) as  $\{\text{Re}\lambda_i\} = \{\alpha_1, \alpha_2\}$ , where  $\alpha_1 < \alpha_2$ . In this case the condition (10) is expressed as  $\alpha_1 t_p = C + hm_1$  and  $\alpha_2 t_p = C + hm_2$ , where  $m_1$  and  $m_2$  ( $m_1 < m_2$ ) are integers that are to be chosen properly, and  $C$  is a constant ( $0 \leq C < h$ ). In order for  $t_p$  to obey these relations, there have to exist integers  $m_1$  and  $m_2$  that obey  $\alpha_1 t_p - hm_1 = \alpha_2 t_p - hm_2 = C \Leftrightarrow t_p = \frac{h(m_2 - m_1)}{\alpha_2 - \alpha_1}$  and  $\frac{\alpha_2}{\alpha_1} = \frac{C + hm_2}{C + hm_1}$  leading to  $C = \frac{h}{\alpha_2 - \alpha_1}(\alpha_1 m_2 - \alpha_2 m_1)$ . The condition  $0 \leq C < h$  is expressed as  $0 \leq \frac{1}{\alpha_2 - \alpha_1}(\alpha_1 m_2 - \alpha_2 m_1) < 1$ , i.e.,  $\alpha_2 m_1 \leq \alpha_1 m_2 < \alpha_2 m_1 + (\alpha_2 - \alpha_1)$ , which allows many pairs of  $(m_1, m_2)$ . Thus, in the  $B = 0$  case, we obtain many  $t_p$ , for which  $f(t_p)$  is maximized. In the very small  $|B| \neq 0$  case, because of the factor  $e^{\frac{2B}{\hbar}t_p}$  included in  $f(t_p)$ , most of such  $t_p$  become values for local maxima, and only the smallest  $t_p$ , i.e.,  $t_p$  for  $(m_1, m_2)$  giving the smallest  $m_2 - m_1$ , is selected.

In the case where the order of the Hilbert space is three, we express  $\text{Re}\lambda_i$  ( $i \in A$ ) as  $\{\text{Re}\lambda_i\} = \{\alpha_1, \alpha_2, \alpha_3\}$ , where we suppose that  $\alpha_1 < \alpha_2 < \alpha_3$ . In this case the condition (10) is expressed as  $\alpha_1 t_p = C + hm_1$ ,  $\alpha_2 t_p = C + hm_2$ , and  $\alpha_3 t_p = C + hm_3$ , where  $m_1$ ,  $m_2$ , and  $m_3$  ( $m_1 < m_2 < m_3$ ) are integers that are to be chosen properly, and  $C$  is a constant ( $0 \leq C < h$ ). In order for  $t_p$  to obey these relations, there have to exist integers  $m_1$ ,  $m_2$ , and  $m_3$  that obey  $\alpha_1 t_p - hm_1 = \alpha_2 t_p - hm_2 = \alpha_3 t_p - hm_3 = C \Leftrightarrow t_p = \frac{h(m_2 - m_1)}{\alpha_2 - \alpha_1} = \frac{h(m_3 - m_2)}{\alpha_3 - \alpha_2}$  and  $\frac{\alpha_2}{\alpha_1} = \frac{C + hm_2}{C + hm_1}$ , which leads to  $C = \frac{h}{\alpha_2 - \alpha_1}(\alpha_1 m_2 - \alpha_2 m_1)$ . Let us suppose that the ratio of  $\alpha_2 - \alpha_1$  to  $\alpha_3 - \alpha_2$  and that of  $\alpha_3 - \alpha_2$  to  $\alpha_3 - \alpha_1$  are rational numbers<sup>8</sup>, and express them as  $\frac{\alpha_2 - \alpha_1}{\alpha_3 - \alpha_2} = \frac{m_2 - m_1}{m_3 - m_2} = \frac{n_1}{d_1}$  and  $\frac{\alpha_3 - \alpha_2}{\alpha_3 - \alpha_1} = \frac{m_3 - m_2}{m_3 - m_1} = \frac{n_2}{d_2}$ , where  $n_i$  and  $d_i$  ( $i = 1, 2$ ) are positive and co-prime integers.<sup>9</sup> Since we have the relations  $(m_3 - m_2)n_1 = (m_2 - m_1)d_1$  and  $(m_3 - m_1)n_2 = (m_3 - m_2)d_2$ , we find  $(m_3 - m_2, m_2 - m_1) = k(d_1, n_1)$  and  $(m_3 - m_1, m_3 - m_2) = l(d_2, n_2)$ , where  $k$  and  $l$  are positive integers to be chosen properly. Then, we are led to the relation  $m_3 - m_2 = kd_1 = ln_2$ , so we find that  $k$  and  $l$  are expressed as  $(k, l) = a(n_2/\text{gcd}(n_2, d_1), d_1/\text{gcd}(n_2, d_1))$ , where  $a$  is a positive integer to be chosen properly. Thus we obtain  $m_3 - m_2 = an_2 d_1/\text{gcd}(n_2, d_1)$ ,  $m_2 - m_1 = an_1 n_2/\text{gcd}(n_2, d_1)$ , and  $m_3 - m_1 = ad_1 d_2/\text{gcd}(n_2, d_1)$ . Since the first and second relations provide  $m_3 - m_1 = an_2(d_1 + n_1)/\text{gcd}(n_2, d_1)$ , comparing this with the third relation, we obtain the relation  $d_1 d_2 = n_2(d_1 + n_1)$ , which leads to  $n_2 = d_1/\text{gcd}(d_1, n_1 + d_1)$  and  $d_2 = (n_1 + d_1)/\text{gcd}(d_1, n_1 + d_1)$ . In addition, we obtain  $m_2 = a \frac{n_1 n_2}{\text{gcd}(n_2, d_1)} + m_1$ ,  $m_3 = a \frac{n_2(n_1 + d_1)}{\text{gcd}(n_2, d_1)} + m_1$ . Thus we find that  $C = a \frac{h \alpha_1 n_1 n_2}{(\alpha_2 - \alpha_1) \text{gcd}(d_1, n_2)} - hm_1$ . The condition  $0 < C < h$  is expressed as  $0 < a \frac{\alpha_1 n_1 n_2}{(\alpha_2 - \alpha_1) \text{gcd}(d_1, n_2)} - m_1 < 1$ , which allows many pairs of  $(a, m_1)$ . On the other hand,  $t_p = \frac{ahn_1 n_2}{(\alpha_2 - \alpha_1) \text{gcd}(d_1, n_2)}$  is proportional to  $a$ . In the  $B = 0$  case, we

<sup>8</sup>In the case where both of them are not rational numbers, i.e., incommensurable, we approximate the irrational numbers to rational ones in their neighborhoods.

<sup>9</sup>We note that  $\text{gcd}(n_i, d_i) = 1$  for  $i = 1, 2$ , where  $\text{gcd}(a, b)$  is the greatest common divisor of integers  $a$  and  $b$ .

obtain many  $t_p$ . In the very small  $|B| \neq 0$  case, because of the factor  $e^{\frac{2B}{h}t_p}$ , the smallest  $a$  should be chosen, and  $m_1, m_2$ , and  $m_3$  are also determined. Thus the smallest  $t_p$  is selected.

Finally let us consider the general case where the order of the Hilbert space is  $n (n = 4, 5, \dots)$ . We express  $\text{Re}\lambda_i (i \in A)$  as  $\{\text{Re}\lambda_i\} = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ , where we suppose that  $\alpha_1 < \alpha_2 < \dots < \alpha_n$ .<sup>10</sup> In this case the condition (10) is expressed as  $\alpha_1 t_p = C + hm_1, \alpha_2 t_p = C + hm_2, \dots, \alpha_n t_p = C + hm_n$ , where  $m_1, m_2, \dots, m_n (m_1 < m_2 < \dots < m_n)$  are integers that are to be chosen properly, and  $C$  is a constant ( $0 \leq C < h$ ). In order for  $t_p$  to obey the above relations, there have to exist integers  $m_1, m_2, \dots, m_n$  that obey  $\alpha_1 t_p - hm_1 = \alpha_2 t_p - hm_2 = \dots = \alpha_n t_p - hm_n = C \Leftrightarrow t_p = \frac{h(m_{i+1} - m_i)}{\alpha_{i+1} - \alpha_i} (i = 1, 2, \dots, n-1)$  and  $\frac{\alpha_2}{\alpha_1} = \frac{C + hm_2}{C + hm_1}$ , which leads to  $C = \frac{h}{\alpha_2 - \alpha_1}(\alpha_1 m_2 - \alpha_2 m_1)$ . Let us suppose that the ratios of  $\alpha_i - \alpha_{i-1}$  to  $\alpha_{i+1} - \alpha_i (i = 2, 3, \dots, n-1)$  and of  $\alpha_n - \alpha_{n-1}$  to  $\alpha_n - \alpha_1$  are rational numbers<sup>11</sup>, and express them as  $\frac{\alpha_i - \alpha_{i-1}}{\alpha_{i+1} - \alpha_i} = \frac{m_i - m_{i-1}}{m_{i+1} - m_i} = \frac{n_{i-1}}{d_{i-1}} (i = 2, 3, \dots, n-1)$  and  $\frac{\alpha_n - \alpha_{n-1}}{\alpha_n - \alpha_1} = \frac{m_n - m_{n-1}}{m_n - m_1} = \frac{n_{n-1}}{d_{n-1}}$ , where  $n_j$  and  $d_j (j = 1, 2, \dots, n-1)$  are positive and coprime integers. Since we have the relations  $(m_{i+2} - m_{i+1})n_i = (m_{i+1} - m_i)d_i (i = 1, 2, \dots, n-2)$  and  $(m_n - m_1)n_{n-1} = (m_n - m_{n-1})d_{n-1}$ , we find  $(m_{i+2} - m_{i+1}, m_{i+1} - m_i) = k_i(d_i, n_i) (i = 1, \dots, n-2)$ , where  $k_j (j = 1, 2, \dots, n-1)$ , are positive integers to be chosen properly. We are led to the relations  $m_{i+1} - m_i = k_{i-1}d_{i-1} = k_i n_i (i = 2, \dots, n-1)$ , so we find that the pairs  $(k_i, k_{i+1}) (i = 1, 2, \dots, n-2)$ , are expressed as  $(k_i, k_{i+1}) = \{a_i / \text{gcd}(n_{i+1}, d_i)\} (n_{i+1}, d_i) (i = 1, \dots, n-2)$ , where  $a_j (j = 1, 2, \dots, n-2)$ , are positive integers to be chosen properly. Then we obtain  $k_1 = a_1 \frac{n_2}{\text{gcd}(n_2, d_1)}, k_i = a_{i-1} \frac{d_{i-1}}{\text{gcd}(n_i, d_{i-1})} = a_i \frac{n_{i+1}}{\text{gcd}(n_{i+1}, d_i)} (i = 2, 3, \dots, n-2)$ , and  $k_{n-1} = a_{n-2} \frac{d_{n-2}}{\text{gcd}(n_{n-1}, d_{n-2})}$ . These representations suggest that we have to choose  $a_1 = \ln_3 \frac{\text{gcd}(n_2, d_1)}{\text{gcd}(n_3, d_1)}, a_2 = l d_1 \frac{\text{gcd}(n_3, d_2)}{\text{gcd}(n_3, d_1)}$ , and  $a_i = l \frac{\prod_{k=1}^{i-1} d_k \text{gcd}(n_{i+1}, d_i)}{\prod_{l=4}^{i+1} n_l \text{gcd}(n_3, d_1)} (i = 3, 4, \dots, n-2)$ , which lead to  $k_1 = l \frac{n_2 n_3}{\text{gcd}(d_1, n_3)}, k_2 = l \frac{d_1 n_3}{\text{gcd}(d_1, n_3)}, k_3 = l \frac{d_1 d_2}{\text{gcd}(d_1, n_3)}$ , and  $k_i = l \frac{\prod_{k=1}^{i-1} d_k}{\prod_{l=4}^i n_l \text{gcd}(n_3, d_1)} (i = 4, 5, \dots, n-1)$ , where  $l$  is a positive integer to be chosen properly. Then, since  $m_2 = m_1 + l \frac{n_1 n_2 n_3}{\text{gcd}(n_3, d_1)}$ , we find  $C = h \left[ \frac{\ln_1 n_2 n_3 \alpha_1}{(\alpha_2 - \alpha_1) \text{gcd}(d_1, n_3)} - m_1 \right]$ , and the condition  $0 < C < h$  is expressed as  $0 < \frac{\ln_1 n_2 n_3 \alpha_1}{(\alpha_2 - \alpha_1) \text{gcd}(d_1, n_3)} - m_1 < 1$ , which allows many pairs of  $(l, m_1)$ . On the other hand, we find  $t_p = h l \frac{n_1 n_2 n_3}{(\alpha_2 - \alpha_1) \text{gcd}(n_3, d_1)}$ , which is proportional to  $l$ . In the  $B = 0$  case, we obtain many  $t_p$ . In the very small  $|B| \neq 0$  case, because of the factor  $e^{\frac{2B}{h}t_p}$ , the smallest  $l$  obeying the above inequality should be chosen, and  $m_1$  and  $m_i = \sum_{j=1}^{i-1} k_j n_j + m_1 (i = 2, \dots, n)$  are also determined. Thus the smallest  $t_p$  is selected.<sup>12</sup> Furthermore, in the case where the order of the Hilbert space is infinite, we can imagine obtaining selected  $t_p$  similarly by considering the infinite limit of  $n$  in the case where the order is  $n$ .

Now that we have proven the existence of  $t_p$  such that the condition (10) is satisfied and so  $|\text{Tr} \left( e^{-\frac{i}{h} \hat{H} t_p} \right)|$  is maximized,  $\langle \hat{\mathcal{O}} \rangle_{\text{periodic time}}$  defined by Eq. (7) has been found for such  $t_p$  to be reduced to the simpler expression given on the right-hand side of Eq. (11), which is real for  $Q$ -Hermitian  $\hat{\mathcal{O}}$ . Thus we have proven Theorem 3. Without considering the maximization principle, we do not have reality<sup>13</sup> for  $\langle \hat{\mathcal{O}} \rangle_{\text{periodic time}}$ . In Theorem 3 there can be many states that are degenerate with regard to the imaginary parts of the eigenvalues of  $\hat{H}$ , so Theorem 3

<sup>10</sup>In the case where there exists a subset  $\{i\}$  such that  $\alpha_i = \alpha_{i+1}$ , we just choose the integers  $m_i$  and  $m_{i+1}$  such that  $m_i = m_{i+1}$  and  $\alpha_i t_p - hm_i = C$  in the later argument.

<sup>11</sup>In the case where both of them are not rational numbers, we approximate the irrational numbers to rational ones in their neighborhoods, as we did in the previous case.

<sup>12</sup>The larger  $n$  is, the larger the selected  $t_p$  becomes.

<sup>13</sup>In the special case where the order of the Hilbert space labeled by the subset  $A$  is just one, Theorem 2 is applied and Eq. (11) corresponds to Eq. (9).

is highly nontrivial even in the  $B = 0$  case compared to Theorem 2 by including the RAT. Finally we provide a corollary as the simplest case in Theorem 3, where a given Hamiltonian is Hermitian.

**Corollary of Theorem 3** Assume that a given Hamiltonian  $\hat{H}$  is Hermitian. Then, provided that an operator  $\hat{O}$  is Hermitian,  $\hat{O}^\dagger = \hat{O}$ ,  $\langle \hat{O} \rangle_{\text{periodic time}} \equiv \frac{\text{Tr}(e^{-\frac{i}{\hbar} \hat{H} t_p} \hat{O})}{\text{Tr}(e^{-\frac{i}{\hbar} \hat{H} t_p})}$  becomes real for selected periods  $t_p$  such that  $|\text{Tr}(e^{-\frac{i}{\hbar} \hat{H} t_p})|$  is maximized.

## 5. Discussion

In this letter, after briefly reviewing our previous works, we studied the periodic complex action theory (CAT) that is obtained by imposing a periodic condition on the past and future states in the future-included CAT whose path runs over not only past but also future. In the periodic CAT, extending a normalized matrix element of an operator  $\hat{O}$ , which is called the weak value in the RAT, to an expression such that various normalized matrix elements of  $\hat{O}$  are summed up with the weight of transition amplitudes, we introduced in Eq. (7) another normalized quantity  $\langle \hat{O} \rangle_{\text{periodic time}}$  that is generically complex but expected to have a role of an expectation value for  $\hat{O}$ . Seeking a condition for  $\langle \hat{O} \rangle_{\text{periodic time}}$  to be real, we presented two theorems that hold in special cases. For a given period  $t_p$  that is supposed to be sufficiently large from a phenomenological point of view, eigenstates of the Hamiltonian  $\hat{H}$  that belong to the subset  $A$  contribute significantly in the traces in the expression of  $\langle \hat{O} \rangle_{\text{periodic time}}$  in Eq. (7), and thus  $\langle \hat{O} \rangle_{\text{periodic time}}$  is reduced to a simpler expression of Eq. (8).

In the first theorem (Theorem 2), considering a special case where the order of the subset  $A$  is just one, i.e., the number of eigenstates that have the maximal imaginary part  $B$  of the eigenvalues of  $\hat{H}$  is just one, we claimed that for a given period  $t_p$  the normalized quantity  $\langle \hat{O} \rangle_{\text{periodic time}}$  becomes real, provided that  $\hat{O}$  is  $Q$ -Hermitian, i.e., Hermitian with regard to the modified inner product  $I_Q$  that makes a given non-normal Hamiltonian  $\hat{H}$  normal. In this case, both the numerator and denominator in the expression of Eq. (8) are dominated by the contribution from just a single eigenstate of  $\hat{H}$ , so phase factors in both cancel each other. Thus, in this special case, we obtained the expression of  $\langle \hat{O} \rangle_{\text{periodic time}}$  in Eq. (9), and proved the theorem.

In the second theorem (Theorem 3), we considered another special case where  $B \leq 0$  and  $|B|$  is much smaller than the distances between any two real parts of the eigenvalues of  $\hat{H}$ , and claimed that, provided that  $\hat{O}$  is  $Q$ -Hermitian,  $\langle \hat{O} \rangle_{\text{periodic time}}$  becomes real for the period  $t_p$  selected such that the absolute value of the transition amplitude  $|\text{Tr}(e^{-\frac{i}{\hbar} \hat{H} t_p})|$  is maximized. We proved via a number-theoretical argument that this theorem holds except for the special case where the order of the Hilbert space labeled by the subset  $A$  is just one.<sup>14</sup> In the other generic cases where the order of the Hilbert space is equal to or larger than two, we showed that such  $t_p$  exist, for which  $\langle \hat{O} \rangle_{\text{periodic time}}$  becomes real. We argued that even the period  $t_p$  can become an adjustment parameter to be determined via such a variant type of the maximization principle that we proposed in Refs. [27,28]. This theorem suggests that, if our universe is periodic, then even the period could be fixed by our principle in the Feynman path integral.

<sup>14</sup>In this special case,  $\langle \hat{O} \rangle_{\text{periodic time}}$  is found to be real for a given  $t_p$ , as is shown by Theorem 2.

In the study in this letter, we have supposed that the whole universe is a closed time-like curve (CTC) and considered very special cases for simplicity. However, in general relativity, more complicated universes can be considered. It would be interesting to investigate such more intricate ones. Now we might have a question: can we propose any model that would lead to a periodic universe in practice via any kind of maximization principle? It would be intriguing if we could propose a model that results in an exactly periodic universe and also provides for  $t_p$  an order of magnitude identifiable with the age of our universe via any kind of maximization principle. In order to construct such a realistic model, it would be necessary to investigate the dynamics of the CAT in detail in some simple models. In Ref. [32] we formulated a harmonic oscillator model by introducing the two-basis formalism in the future-included CAT. It would be important to study the model further in detail. Furthermore, since our  $\langle \hat{\mathcal{O}} \rangle_{\text{periodic time}}$  is independent of a reference time  $t$ , it would also be interesting to provide it with the time  $t$  dependence by introducing a clock variable  $T_{\text{clock}}(t)$  to be inserted in the quantity. For this purpose we need to extend our series of reality theorems so that they hold not only for a single operator  $\hat{\mathcal{O}}$  but also for a product of operators. We would like to report such investigations in the future (K. Nagao and H. B. Nielsen, work in progress).

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