



Accelerated motion in general relativity: fate of the singularity

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Abstract Under general relativity, the paths of accelerated test particles are taken into account. It is examined whether such accelerations have any influence on the ‘singularity’ of the spacetime. The Raychaudhuri equation for the congruence of the time-like curves describing the paths of the accelerated particles is considered to calculate a few physical attributes. It is shown that if the acceleration of the test particles exceeds a particular value, then the congruences of the accelerated time-like curves do not encounter any singularity although the usual energy conditions are violated or modified. It is shown further that in the curved spacetime of general relativistic framework one may generate a system of transformations that is a generalization of the Rindler coordinates related to accelerated frame in the flat Minkowski spacetime. To show the influence of the acceleration of test particle on singularity of a particular spacetime the Schwarzschild spacetime is considered. Taking tidal deviation related acceleration term, it is shown that the acceleration may attain a specific value for which the modified Kretschmann scalar vanishes in a spherical neighbourhood of the singularity and thus the Schwarzschild singularity disappears. In the context of singularity as ‘geodesic incompleteness’ of the spacetime manifold it is also proved that prescribing an appropriate acceleration term on the maximal geodesic defined in a finite interval one may extend it up to infinite proper time and hence the spacetime becomes singularity free. Such results hold at the price of violating the usual energy conditions.

1 Introduction

The general relativity explaining the gravity in the true sense is so successfully that it has a wide range of applications in the precise measurement of planetary motions, galactic

and extra-galactic phenomena and also the evolution of the universe in the stationary as well as dynamical spacetime continuum. In the framework of that elegant theory there exists a situation while the causality breaks down and that is nothing but the singularity. It is supposed to be a condition in which the gravity is so intense that spacetime itself breaks down catastrophically; although that is not the only way to define a singularity.

In 1955 Penrose [1] proposed the idea of singularity in terms of incompleteness of spacetime manifold subject to certain conditions including the existence of trapped surface. In 1970 Hawking and Penrose [2] developed the ‘singularity theorem’ interpreting the gravitational singularity in the Big Bang situation. The singularities may be evident in all the black hole spacetime and in all the cosmological solutions that do not have a scalar field energy or a cosmological constant. In this article we concentrate on the spacelike singularity at which the spacetime curvature blows up. The blow up ratio of the curvature during gravitational collapse have been studied in details [3,4]. As the curvature is associated with gravity the curvature singularity corresponds to infinite gravity and there exist a number of possibilities of how such infinitely strong gravity manifests itself.

Now a pertaining question is whether that singularity lasts forever or there may be some way by which it is completely destroyed. We are strongly motivated by the event of black hole evaporation interpreting that if the entire black hole gets evaporated then its singularity must be diminished by some manner before the evaporation process is completed. We identify that the accelerated particles having some suitable value of acceleration may destroy the spacetime singularity.

The acceleration may be incorporated in the special relativity characterized by Minkowski spacetime. In 1973 Misner et al. [5] analyzed the accelerated motion in the framework of special relativity. Although, in general, the special relativity deals with inertial frame of reference, but that is

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also very much consistent in the non-inertial frame incorporating acceleration. In special relativity, a uniformly accelerating particle exhibits hyperbolic motion and therefore, a uniformly accelerating frame relative to which the particle is at rest may be chosen as its proper reference frame. Such hyperbolic coordinates are termed as Rindler coordinates [6] that are curvilinear in nature and correspond to the Rindler metric. The non-inertial frame characterized by the Rindler coordinates consists of the world lines of the accelerated particles. Unlike the standard spacetime coordinates the Rindler coordinates cover only a portion of the spacetime diagram, which is known as Rindler horizon.

The present work deals with the accelerated test particles, not in the framework of special relativity, but when general relativity comes into play. Finding the solution of the uniformly accelerated test particles in general relativity work in this arena have been carried out [7–9]. It has already been mentioned that the most remarkable feature of the general relativity is the existence of singularity which is absent in the Minkowski spacetime. It is intensively studied how the acceleration of the test particles affect the singularity. Another important point to be verified that the acceleration calculated in the framework of general relativity must be consistent with the expressions of the same found in the local flat spacetime with the aid of Gaussian normal coordinates.

The outline of the present study is therefore as follows: In Sect. 2 Raychaudhuri equation for the congruence of accelerated curves has been provided along with in two subsections we discuss the Raychaudhuri equation and the singularity as well as the proper acceleration and the accelerated frame. We provide the Kretschmann scalar and singularity in Sect. 3 with the discussions on acceleration in Schwarzschild spacetime in two different subsections. Acceleration and geodesic completeness are discussed in Sect. 4. Some concluding remarks are provided in Sect. 5.

2 Raychaudhuri equation for the congruence of accelerated curves

It is well known that the Raychaudhuri equation [10–16] plays a pivotal role in the singularity theorems [1, 17]. The most fascinating part of that elegant equation is that it has been established as a purely geometric relation, making no reference to the Einstein's equation [18] of general relativity. The Raychaudhuri equation is a relation between the Ricci tensor $R_{\mu\nu}$ and $\frac{d\Theta}{d\tau}$, where Θ represents how the ball of test particles moving along the curves is growing ($\Theta > 0$) or shrinking ($\Theta < 0$) at any point of time. Almost in all works the Raychaudhuri equation is considered in the perspective of geodesic congruence; but in this work it is taken in the framework of curves followed by accelerated test particles. Let u^μ , a tangent vector field to a timelike curve, represent

four-velocity of some massive fluid particles, and their integral curves of u^μ 's, in general, are not geodesics at all. The term $\mathcal{B}_{\mu\nu} = u_{\mu;\nu}$, a (0,2) tensor, can be decomposed as follows:

$$\mathcal{B}_{\mu\nu} = \frac{1}{3}\Theta\mathcal{P}_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}, \quad (1)$$

where $\mathcal{P}_{\mu\nu}$ represents a projection tensor that projects any vector in $T_p\mathcal{M}$ for the given vector field u^μ (at each point p) into a subspace of $T_p\mathcal{M}$ corresponding to the vector normal to u^μ . The traceless symmetric part $\sigma_{\mu\nu}$ represents the shear or distortion in the shape of the collection of test particles, where $\omega_{\mu\nu}$ is the traceless anti-symmetric part, called rotation. With a little bit of calculations the covariant derivative of $\mathcal{B}_{\mu\nu}$ relative to the proper time τ takes the following form

$$\frac{D\mathcal{B}_{\mu\nu}}{d\tau} = (u^\sigma u_{\mu;\sigma})_{;\nu} - \mathcal{B}_\nu^\sigma \mathcal{B}_{\mu\sigma} - R_{\lambda\mu\nu\sigma} u^\sigma u^\lambda. \quad (2)$$

Taking the trace of this equation the following expression is obtained

$$\frac{d\Theta}{d\tau} = -\frac{\Theta^2}{3} - \sigma_{\mu\nu}\sigma^{\mu\nu} - R_{\mu\nu}u^\mu u^\nu + (u^\nu u^\mu_{;\nu})_{;\mu}. \quad (3)$$

In the above equation the term $\omega_{\mu\nu}\omega^{\mu\nu}$ disappears by the Frobenius theorem according to which a congruence of timelike curves is hypersurface orthogonal if and only if $u_{[\mu;\nu}u_{\rho]} = 0$ (generalization of $\omega_{\mu\nu} = 0$). Equation (3) represents the Raychaudhuri equation for the congruence of timelike accelerated curves. The last term on right hand side represents the covariant derivative of the acceleration which may play a significant role that will be discussed subsequently.

For clarity, an important aspect arising due to the presence of acceleration term in the above equation should be discussed in brief. It is well known that the gravity is an attractive force, only due to the focussing phenomenon governed by $R_{\mu\nu}u^\mu u^\nu \geq 0$ (for any timelike vector u^μ), which is termed as *timelike convergence condition*. Such condition leads to *strong energy condition* $(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)u^\mu u^\nu \geq 0$, in which energy momentum tensor $T_{\mu\nu}$ is evolved via Einstein equation. For the consistency of the singularity theorem the *timelike convergence condition* should be modified into

$$R_{\mu\nu}u^\mu u^\nu \geq a^\mu_{;\mu}. \quad (4)$$

Such relation translates the classical energy condition in terms of pressure p and density ρ into

$$3p + \rho \geq \frac{1}{\sqrt{-g}}(\sqrt{-g}a^\mu)_{;\mu}. \quad (5)$$

It is straight forward to show that the acceleration term $a^\mu = u^\nu u^\mu{}_{;\nu}$ is orthogonal to the velocity vector u^μ . If u^μ is considered as purely timelike vector then a^μ must be spacelike, and it can be taken as $a^\mu = (0, \mathbf{a})$. Therefore Eq. (5) takes the form

$$a^\mu{}_{;\mu} = a^\mu{}_{,\mu} + \Gamma^\mu_{\mu\alpha} a^\alpha = a^\mu{}_{,\mu} + \frac{1}{\sqrt{-g}} a^\mu \partial_\mu \sqrt{-g} = \frac{1}{\sqrt{-g}} (\sqrt{-g} a^\mu)_{,\mu}$$

The metric $g_{\mu\nu}$ can be expressed as

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta} x^\alpha x^\beta + O(x^3)$$

In the locally flat spacetime $R_{\mu\alpha\nu\beta}(P) = 0$ at every point P therein and therefore, $g_{\mu\nu} = \eta_{\mu\nu}$. It makes the Eq. (5) as

$$3p + \rho \geq \frac{1}{\sqrt{-g}} \nabla \cdot (\sqrt{-g} \mathbf{a}). \quad (6)$$

In the locally flat spacetime such condition becomes

$$3p + \rho \geq \nabla \cdot \mathbf{a}. \quad (7)$$

The above Eq. (3) not only modifies the classical energy condition, but in some sense, it may also violate such condition. The Eq. (7) shows that the term $3p + \rho$ is greater or equal to the divergence of the acceleration three vector and the classical energy condition is recovered only at the source and sink of such acceleration vector field.

Now, Eq. (3) may be reformulated in a different manner as given hereunder:

$$\frac{D^2 u^\alpha}{d\tau^2} = q^2 u^\alpha, \quad (8)$$

where

$$q^2 = \frac{d\Theta}{d\tau} + \frac{\Theta^2}{3} + \sigma_{\mu\nu} \sigma^{\mu\nu} + R_{\mu\nu} u^\mu u^\nu. \quad (9)$$

In the following sub-section the fate of the singularity arising in the Raychaudhuri equation is explored. In the subsequent sub-section the expression of the proper acceleration as well as accelerating frame of reference are obtained.

2.1 Raychaudhuri equation and the singularity

Followed by the focusing theorem Raychaudhuri equation leads to a singularity in the congruence of curves, not necessarily being the geodesics. It physically signifies that the gravity, in some sense, is converging in nature. Here it is explored how the curves followed by accelerated particles affect the possible singularity that is inevitable for the initially converging geodesics. For the sake of simplicity in our

analysis we consider q as constant. Now one may set

$$k = q^2 - (\sigma_{\mu\nu} \sigma^{\mu\nu} + R_{\mu\nu} u^\mu u^\nu), \quad (10)$$

where, k may be taken as constant. The physical motivation for such consideration is that the analysis is carried out for the small scale of time for which the pressure and density are assumed to be constant, and therefore, $R_{\mu\nu} u^\mu u^\nu (= \rho + 3p)$ should remain constant in the current scenario. There is no harm to take the shear term as constant as well. Moreover, the term q representing the proper acceleration of the particle does not depend on time, although its components do.

At this juncture the following three cases may arise.

Case-I: $k < 0$, set $k = -\chi^2$

That includes the situation when there is no acceleration, i.e., $q = 0$ and therefore, the congruence consists of the geodesics. In general that is the situation in which there is a little acceleration experienced by the test particles. Under the situation Eq. (9) becomes

$$\frac{d\Theta}{d\tau} = -\frac{\Theta^2}{3} - \chi^2. \quad (11)$$

The analytical solution of the above differential equation gives

$$\Theta^{-1} = \frac{1}{\chi\sqrt{3}} \tan \left\{ \frac{\chi}{\sqrt{3}} \tau - \cot^{-1} \left(\frac{|\Theta_0|}{\chi\sqrt{3}} \right) \right\}, \quad (12)$$

where $\Theta_0 (< 0)$ is the initial contraction of the congruence of curves (or geodesics for $q = 0$). Equation (12) shows at the proper time $\tau_\infty = \frac{\sqrt{3}}{\chi} \cot^{-1} \left(\frac{|\Theta_0|}{\chi\sqrt{3}} \right)$ the Θ diverges to $-\infty$. That is nothing but the result of focusing theorem by which the singularity appears.

Case-II: $k = 0$

In this case the differential equation takes the following form

$$\frac{d\Theta}{d\tau} = -\frac{\Theta^2}{3}. \quad (13)$$

It can be solved with same initial condition $\Theta_0 (< 0)$ as

$$\Theta^{-1} = \frac{\tau}{3} - |\Theta_0|^{-1}. \quad (14)$$

Here also the Θ diverges to negative infinity at the proper time $\tau_\infty = \frac{3}{|\Theta_0|}$. Thus the focusing is followed to result a singularity.

Case-III: $k > 0$, set $k = \chi^2$.

The corresponding differential equation becomes

$$\frac{d\Theta}{d\tau} = \chi^2 - \frac{\Theta^2}{3}. \quad (15)$$

With same initial condition the solution takes the form

$$\Theta = \chi \sqrt{3} \tanh \left\{ \frac{\chi}{\sqrt{3}} \tau - \tanh^{-1} \left(\frac{|\Theta_0|}{\chi \sqrt{3}} \right) \right\}. \quad (16)$$

The initial converging condition of Θ exhibits the following nature in the subsequent stage.

$$\begin{aligned} \Theta &\leq 0, & 0 \leq \tau &\leq \sqrt{3} \tanh^{-1} \left(\frac{|\Theta_0|}{\chi \sqrt{3}} \right) \\ &> 0, & \tau &> \sqrt{3} \tanh^{-1} \left(\frac{|\Theta_0|}{\chi \sqrt{3}} \right). \end{aligned}$$

The Θ does not diverge not only for a finite value but for also the large value of τ . On contrary it converges to $\chi \sqrt{3}$ as $\tau \rightarrow \infty$ and therefore there is a stability of Θ at infinity. The most significant observation is there does not exist any singularity. That is very encouraging to observe the world lines of the accelerated particles do not encounter any singularity when q^2 exceeds $\sigma_{\mu\nu}\sigma^{\mu\nu} + R_{\mu\nu}u^\mu u^\nu$.

Thus when the acceleration is large enough compared to the contribution of energy momentum the singularity may be avoided. But it is worth mentioning that this picture may completely become nonphysical. For example, in the shear free situation the term $R_{\mu\nu}u^\mu u^\nu = 3p + \rho$, must not exceed the square of the proper acceleration term and then only the avoidance of singularity is possible. For the large density or pressure the contribution may exceed q^2 and the usual focusing phenomenon, as described in the Case-II occurs; encountering the singularity is inevitable.

2.2 Proper acceleration and the accelerated frame

In the special relativity the proper acceleration of a body gives rise to a non-inertial frame characterized by the Rindler coordinates. Here we consider accelerated curves in the framework of general relativity through Raychaudhuri equation. It is worth noting that the acceleration vector a^α is always normal to the velocity vector u^α and hence to the tangent space $T_p\mathcal{M}$, where p is any arbitrary point in the spacetime manifold \mathcal{M} . If we solve Eq. (8) in the Gaussian normal coordinates we obtain the expressions of velocity, acceleration and the coordinates respectively, as follows:

$$u^\alpha = u_*^\alpha \cosh q\tau + \frac{a_*^\alpha}{q} \sinh q\tau, \quad (17)$$

$$a^\alpha = qu_*^\alpha \sinh q\tau + a_*^\alpha \cosh q\tau, \quad (18)$$

$$x^\alpha = x_*^\alpha + \frac{u_*^\alpha}{q} \sinh q\tau + \frac{a_*^\alpha}{q^2} (\cosh q\tau - 1), \quad (19)$$

where u_*^α , a_*^α and x_*^α represent the initial values of the velocity, acceleration and coordinates respectively.

Equation (18) indicates $(a \cdot a)^{\frac{1}{2}} = q$ and therefore, q is simply the proper acceleration, whose high value makes the singularity disappears as shown in the Case-III of the previous sub-section. One may recover the Rindler coordinates in special relativity from Eq. (19).

Let us now choose the following set of initial values:

$$x_*^\alpha = (0, 0, 0, 0), \quad u_*^\alpha = (1, 0, 0, 0), \quad a_*^\alpha = (0, q, 0, 0). \quad (20)$$

The expressions of the coordinates, velocity and acceleration in terms of proper time and proper acceleration become

$$x^0 = \frac{1}{q} \sinh q\tau, \quad x^1 = \frac{1}{q} (\cosh q\tau - 1), \quad x^3 = x^4 = 0, \quad (21)$$

$$u^0 = \cosh q\tau, \quad u^1 = \sinh q\tau, \quad u^2 = u^3 = 0, \quad (22)$$

$$a^0 = q \sinh q\tau, \quad a^1 = q \cosh q\tau, \quad a^2 = a^3 = 0, \quad (23)$$

which are nothing but the said expressions in the Minkowski spacetime. Replacing $\frac{1}{q}$ by a coordinate variable ξ one may have a transformation from Minkowski coordinates (x^0, x^1) to Rindler coordinates (ξ, τ) in (1+1) dimensions. Using the same trick in the framework of general relativity a combine expression of the above transformations in the spacetime of (3+1) dimensions may be expressed as follows:

$$x^\alpha = \tilde{x}_1^\alpha \sinh(Q_\alpha \tilde{x}_0^\alpha) + \tilde{x}_2^\alpha [\cosh(Q_\alpha \tilde{x}_0^\alpha) - 1] + \tilde{x}_3^\alpha, \quad (24)$$

where

$$\begin{aligned} \tilde{x}_0^\alpha &= (\tau, 0, 0, 0), \quad \tilde{x}_1^\alpha = (\xi, 0, 0, 0), \quad \tilde{x}_2^\alpha = (0, \xi, 0, 0), \\ \tilde{x}_3^\alpha &= (0, 0, 0, 0), \quad Q_\alpha = (q, 0, 0, 0) \end{aligned} \quad (25)$$

With an arbitrary choice of $(\tilde{x}_0^\alpha, \tilde{x}_1^\alpha, \tilde{x}_2^\alpha, \tilde{x}_3^\alpha)$ a generalized Rindler coordinates under the scenario of different spacetime manifolds in general relativity may be obtained. Thus in general relativity through such transformation a new frame may be chosen where the test particles exhibit the accelerated motion and evidently such frame is different from that where the particles follow the geodesic motion.

3 Kretschmann scalar and singularity

The most awkward situation in the general relativity theory is the singularity, in particular the spacelike singularity, being evident in Schwarzschild spacetime, FRW spacetime etc. Such kind of singularity is characterized by the Kretschmann scalar $\mathcal{K} = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, an invariant quantity introduced by Erich Kretschmann [19] in the year 1951. It

is interesting to note that on the event horizon of any type of black hole such Kretschmann scalar is regular and therefore, the apparent singularity arising on the event horizon can be removed very easily by the proper choice of coordinate transformations. The real spacelike singularity arises either at the center of the Schwarzschild type of black hole or at the big bang, where the Kretschmann scalar is singular. Now is it possible to turn the Kretschmann scalar to be regular at the singularity by reconstructing the Riemann curvature term $R_{\alpha\beta\gamma\delta}$? If we consider the curves of accelerated particles instead of geodesics then the deviation equation becomes

$$\frac{D^2\zeta^\rho}{d\tau^2} = -R_{\beta\gamma\delta}^\rho u^\beta u^\delta \zeta^\gamma + (u^\kappa u_{\kappa}^\rho)_{;\gamma} \zeta^\gamma, \quad (26)$$

where ζ^ρ represents the deviation or separation vector between the neighbouring curves.

As the extent of Riemann curvature is measured by the tidal force $\frac{D^2\zeta^\rho}{d\tau^2}$ the right hand side of the equation (26) may be expressed in terms of a modified Riemann tensor as given hereunder

$$\frac{D^2\zeta^\rho}{d\tau^2} = -\bar{R}_{\beta\gamma\delta}^\rho u^\beta u^\delta \zeta^\gamma, \quad (27)$$

where

$$\bar{R}_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - a_{\alpha;\gamma} u_\beta u_\delta. \quad (28)$$

The acceleration term $a_\alpha = (u^\kappa u_{\kappa}^\alpha)_{;\gamma}$ modifies the curvature, but the question is whether it suffices to make the Kretschmann scalar to be regular near singularity. One may calculate such new Kretschmann scalar in terms of modified Riemann tensor, i.e., $\bar{\mathcal{K}} = \bar{R}_{\alpha\beta\gamma\delta} \bar{R}^{\alpha\beta\gamma\delta}$ as follows:

$$\bar{\mathcal{K}} = \mathcal{K} + g^{\gamma\rho} a_{\alpha;\rho} a_{\alpha;\gamma} - 2g^{\gamma\rho} R_{\alpha\beta\gamma\delta} a^{\alpha;\rho} u^\beta u^\delta. \quad (29)$$

The claim is that the particles following the curves will be accelerated with such a value of a^α so that the modified Kretschmann scalar $\bar{\mathcal{K}}$ vanishes near the singularity. Thus the singularity can be removed by the suitable acceleration associated with the congruence of curves. In the next sub-section that claim is verified with the illustration of Schwarzschild spacetime.

3.1 Acceleration in Schwarzschild spacetime

The Schwarzschild spacetime is a perfect example of a spherically symmetric vacuum solution of Einstein's equation. The corresponding metric is given by

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2. \quad (30)$$

The metric is evaluated in the plane where $\theta = \frac{\pi}{2}$ and $\phi = 0$. There exists a singularity at $r = 0$ since the corresponding Kretschmann scalar takes the form $\mathcal{K} = \frac{48m^2}{r^6}$. In the accelerated frame of reference the timelike condition of the velocity vector u^α , which is $u_\alpha u^\alpha = -1$, brings the expression of Eq. (30) in the following form.

$$\frac{(u^t)^2}{\left(1 - \frac{2m}{r}\right)^{-1}} - \frac{(u^r)^2}{\left(1 - \frac{2m}{r}\right)} = 1. \quad (31)$$

That represents a hyperbolic equation whose parametric form with parameter λ gives the expressions of u^t and u^r .

$$u^t = \frac{\cosh \lambda}{\left(1 - \frac{2m}{r}\right)^{\frac{1}{2}}}, \quad u^r = \left(1 - \frac{2m}{r}\right)^{\frac{1}{2}} \sinh \lambda, \quad u^\theta = u^\phi = 0. \quad (32)$$

It is worth noting that $\lambda = 0$, gives the velocity in Schwarzschild spacetime for a stationary object. In the framework of Schwarzschild metric the acceleration four vector is evaluated as

$$a^\alpha = u^r \frac{\partial u^\alpha}{\partial r} + \Gamma_{tt}^\alpha (u^t)^2 + 2\Gamma_{tr}^\alpha u^t u^r + \Gamma_{rr}^\alpha (u^r)^2 \quad (33)$$

From the Schwarzschild metric inserting the expressions of the affine connections the following contravariant space-time components of the acceleration are evaluated

$$a^t = \frac{m \sinh \lambda \cosh \lambda}{r^2 \left(1 - \frac{2m}{r}\right)}, \quad a^r = \frac{m \cosh^2 \lambda}{r^2}, \quad a^\theta = a^\phi = 0. \quad (34)$$

Similarly, the covariant expressions of the acceleration can be calculated to find the following terms

$$a_t = -\frac{m \sinh \lambda \cosh \lambda}{r^2}, \quad a_r = \frac{m \cosh^2 \lambda}{r^2 \left(1 - \frac{2m}{r}\right)}, \quad a_\theta = a_\phi = 0. \quad (35)$$

In addition to those components we can also calculate the proper acceleration as follows:

$$q = (a \cdot a)^{\frac{1}{2}} = \frac{m \cosh \lambda}{r^2 \left(1 - \frac{2m}{r}\right)^{\frac{1}{2}}}. \quad (36)$$

Such proper acceleration depending upon λ does not depend on r , but up to a certain extent. Eventually in the neighbourhood of $r = 0$ it becomes a slowly varying function of r and thus plays a crucial role to turn the modified Kretschmann scalar vanish. We must explore it.

To calculate $\bar{\mathcal{K}}$ from Eq. (29) we have to obtain the expressions $g^{\gamma\rho}a^\alpha{}_{;\rho}a_\alpha{}_{;\gamma}$ and $g^{\gamma\rho}R_{\alpha\beta\gamma\delta}a^\alpha{}_{;\rho}u^\beta u^\delta$ in terms of r . The various components of $a^\alpha{}_{;\beta}$ and $a_\alpha{}_{;\beta}$ are calculated elaborately in Appendix.

Utilizing that results one may simplify the two expressions, mentioned above, as given hereunder

$$g^{\gamma\rho}a^\alpha{}_{;\rho}a_\alpha{}_{;\gamma} = \frac{2m^4}{r^8} \cosh^4 \lambda \left[\frac{1}{\left(1 - \frac{2m}{r}\right)^2} + \frac{2r^2}{m^2} \left\{ 1 + \frac{m}{r\left(1 - \frac{2m}{r}\right)} \right\}^2 + \frac{2r^6}{m^2} \right] + \frac{2m^4}{r^8} \cosh^2 \lambda \left[1 - \frac{1}{\left(1 - \frac{2m}{r}\right)^2} \right], \quad (37)$$

and

$$g^{\gamma\rho}R_{\alpha\beta\gamma\delta}a^\alpha{}_{;\rho}u^\beta u^\delta = \frac{2m^4}{r^8} \cosh^4 \lambda \left[\frac{2r}{m} \left(1 - \frac{2m}{r}\right) + \frac{8r^2}{m^2 \left(1 - \frac{2m}{r}\right)^2} + \frac{6r}{m \left(1 - \frac{2m}{r}\right)^3} \right] + \frac{2m^4}{r^8} \cosh^2 \lambda \left[\frac{2r^4}{m^2} - \frac{2r}{m} \left(1 - \frac{2m}{r}\right) - \frac{2r}{m \left(1 - \frac{2m}{r}\right)^3} \right]. \quad (38)$$

It is to be noted that the Kretschmann scalar \mathcal{K} in the Schwarzschild spacetime is $\frac{48m^2}{r^6}$. Combining the results derived in Eqs. (37) and (38) the expression of the modified Kretschmann scalar $\bar{\mathcal{K}}$ from Eq. (29) is obtained as

$$\bar{\mathcal{K}} = \frac{2m^4}{r^8} [A(r) \cosh^4 \lambda - B(r) \cosh^2 \lambda + C(r)], \quad (39)$$

where

$$A(r) = 8 - \frac{4r}{m} + \frac{2r^2}{m^2} \left(\frac{1 - \frac{m}{r}}{1 - \frac{2m}{r}} \right)^2 - \frac{4r^6}{m^2} + \frac{1}{\left(1 - \frac{2m}{r}\right)^2}, \quad (40)$$

$$B(r) = 7 - \frac{4r}{m} \left\{ 1 + \frac{1}{\left(1 - \frac{2m}{r}\right)^3} \right\} + \frac{4r^4}{m^2}, - \frac{1}{\left(1 - \frac{2m}{r}\right)^2} \quad (41)$$

$$C(r) = \frac{24r^2}{m^2}. \quad (42)$$

It is time to specify the parameter λ arising from the hyperbolic equation (31). The λ plays a significant role in the perspective of the singularity. It may be considered as

constant outside a spherical neighbourhood of $r = 0$ with radius $r_\varepsilon \ll 2m$, but inside it is a slowly function of r . The λ has not been specified earlier, but now one may choose it in such a manner that the expression within the squared bracket of Eq. (39) vanishes within the said spherical neighbourhood of $r = 0$, that is

$$A(r) \cosh^4 \lambda - B(r) \cosh^2 \lambda + C(r) = 0, \quad r < r_\varepsilon. \quad (43)$$

Solving this equation and taking only the positive value the λ is obtained as

$$\lambda(r) = \cosh^{-1} \left\{ \sqrt{\frac{B}{2A}} \left(1 + \sqrt{1 - \frac{4CA}{B^2}} \right)^{\frac{1}{2}} \right\} \quad r < r_\varepsilon. \quad (44)$$

Within the said sphere the r is much smaller than $2m$ and hence the λ in Eq. (44) may be approximated to the following expression

$$\lambda \approx \cosh^{-1} \left(\frac{\sqrt{7}}{2\sqrt{2}} \right). \quad (45)$$

The significant aspect of such analysis is that the modified Kretschmann scalar $\bar{\mathcal{K}}$ vanishes and therefore, becomes regular. Although \mathcal{K} is diverging at $r = 0$, but the presence of an acceleration term with a particular value makes $\bar{\mathcal{K}}$ regular. Thus the spacetime singularity at the central portion of the Schwarzschild black hole disappears since the spacetime becomes regular. It should be noted that the Kretschmann scalar measuring total tidal force remains small if the spacetime is regular.

4 Acceleration and geodesic completeness

The most efficient and consistent definition of the singularity is geodesic incompleteness of the spacetime manifold. In other words, the world lines of the freely falling test particles cannot be extended beyond certain proper time. The singularity, in some sense, is an obstruction of extending a timelike geodesic. Before proceeding further we would define certain terms essential in the current context. More details are available in the book by Hawking and Ellis [20] and the lecture notes by Dafermos [21].

Let us consider $(\mathcal{M}, g_{\alpha\beta})$ to be a Lorentzian manifold having the metric $g_{\alpha\beta}$.

(i) Maximal curve:

A curve $\gamma : I \rightarrow \mathcal{M}$ is said to be maximal if there does not exist any curve $\tilde{\gamma} : \tilde{I} \rightarrow \mathcal{M}$ with $\tilde{I} \subset I$ such that $\gamma(\tau) = \tilde{\gamma}(\tau)$ for every $\tau \in \tilde{I}$.

(ii) Geodesic:

A curve $\gamma : I \rightarrow \mathcal{M}$ is said to be a geodesic if for every point $p \in \gamma$ the condition $u^\beta u^\alpha{}_{;\beta} = 0$ holds, where $u^\alpha \in T_p \mathcal{M}$.

(iii) Parallel transport of a vector:

If u^α represents a tangent vector along a curve $\gamma : I \rightarrow \mathcal{M}$ then a vector v^α satisfying the condition $u^\beta v^\alpha{}_{;\beta} = 0$ is said to be parallel transported along the curve γ .

Geometrically geodesic signifies the curve γ whose tangent vector u^α is parallel transported along itself.

(iv) Existence and uniqueness of geodesic:

For any point $p_0 \in \mathcal{M}$ and any vector $u_0^\alpha \in T_{p_0} \mathcal{M}$ there exists a unique maximal proper time parameterized geodesic $\gamma : I \rightarrow \mathcal{M}$ such that $\gamma(0) = p_0$ and $\frac{d\gamma}{d\tau} = u_0^\alpha$, where $I \subset \mathbb{R}$ is an open interval.

The geodesic equation $u^\beta u^\alpha{}_{;\beta} = 0$ is a second order ordinary differential equation. Therefore, the proof of the above, that is existence and uniqueness of geodesic, is simply followed from the Picard–Lindelöf theorem for the solutions of ODEs with prescribed initial conditions.

(v) Future end point:

A point p_* in \mathcal{M} is said to be future endpoint of a future-directed timelike (or null) curve $\gamma : I \rightarrow \mathcal{M}$ if, for each and every neighborhood O of the point p_* , there exists a $\tau_* \in I$ for which $\gamma(\tau) \in O$ for all $\tau \in I$ such that $\tau \geq \tau_*$.

A causal curve is future inextendible if it has no future endpoint.

(vi) Geodesic completeness:

A spacetime $(\mathcal{M}, g_{\alpha\beta})$ is geodesically complete if every maximal geodesic $\gamma : I \rightarrow \mathcal{M}$ is such that $I = (-\infty, \infty)$, that is the geodesic can be extended over infinite proper time parameter τ .

A spacetime is geodesically incomplete if there exists at least one maximal geodesic with future end point. When a timelike geodesic has a future end point then, followed by the consequence of geodesic incompleteness, the manifold must contain a spacetime singularity. If it is possible to overcome such end point by some means and extend the curve up to infinite proper time parameter the singularity can be removed from the spacetime manifold. In this context we may introduce the following theorem which may be named as *Removal of singularity*.

Theorem on removal of singularity:

A suitable choice of acceleration term prescribed on a maximal geodesic curve having a future end point at some value of proper time parameter makes the curve to be extended up to infinite proper time parameter with a suitable re-parameterization.

Proof: Let $\gamma : I \rightarrow \mathcal{M}$ be a maximal geodesic in the spacetime manifold $(\mathcal{M}, g_{\alpha\beta})$ with proper time parameter

$\tau \in I \subset (-\infty, \infty)$. Since γ is a geodesic the following condition must hold.

$$\frac{\partial u^\alpha}{\partial \tau} + \Gamma_{\mu\nu}^\alpha u^\mu u^\nu = 0. \quad (46)$$

Now the acceleration term is prescribed on each and every geodesic which is incomplete, that is ended up just before attaining a parametric value, say τ_* . Suppose τ_0 is the value of proper time parameter at which the acceleration comes into play. Obviously the curve remains no longer a geodesic as the following condition defined on the given curve replaces the condition represented by Eq. (46).

$$\frac{\partial u^\alpha}{\partial \tau} + \Gamma_{\mu\nu}^\alpha u^\mu u^\nu = a^\alpha \neq 0 \quad \tau_0 \leq \tau < \tau_*, \quad (47)$$

where $\Gamma_{\mu\nu}^\alpha$ is the affine connections associated with $g_{\mu\nu}$.

Since the geodesics, which fail to be complete, are assigned with the acceleration a^α they lose their geodesic property. At this juncture one may use a special type of re-parameterization $\tilde{\tau} = \tilde{\tau}(\tau)$ in such a manner that $\tilde{\tau} \rightarrow \infty$, whereas $\tau < \tau_*$. With such re-parameterization Eq. (47) turns into the following equation.

$$\frac{\partial u^\alpha}{\partial \tilde{\tau}} + \Gamma_{\mu\nu}^\alpha u^\mu u^\nu = 0, \quad (48)$$

and thus the accelerated curve becomes a geodesic. Such representation is possible if and only if the acceleration term a^α takes the following form by the said re-parameterization

$$a^\alpha = \frac{\partial u^\alpha}{\partial \tilde{\tau}} \left(\frac{d\tilde{\tau}}{d\tau} - 1 \right). \quad (49)$$

It is emphasized that the transformation $\tilde{\tau} = \tilde{\tau}(\tau)$ may not be unique, but it would be such that the above relation must be satisfied. We may give an example of one such transformation as

$$\tilde{\tau} = \tanh^{-1} \left(\frac{\tau}{\tau_*} \right), \quad \tau_0 \leq \tau < \tau_*. \quad (50)$$

Clearly the above transformation of the proper time shows that $\tilde{\tau} \rightarrow \infty$ as $\tau \rightarrow \tau_*$ and thus the proof of the above theorem follows.

5 Discussion and conclusion

Illustrating with different cases it is shown intensively in the Sect. 2.1 that when the proper acceleration q exceeds a certain value the paths of the test particles never find a singularity, but in case of the geodesic motion or the motion with little

acceleration of test particles the existence of singularity is inevitable. That infers two possibilities: either the paths of the accelerated particles get shifted from the track of the singularity or the singularity itself disappears. The shifting of path might be a possible explanation for a single accelerated particle following a single curve, but it is to be kept in our mind that the Raychaudhuri equation deals with the congruence of curves. Over a large region such congruence of curves of accelerated particles moving towards a black hole do not lead to the focusing and therefore, no singularity is evident in the said region. Thus the congruence of paths of unaccelerated particles (geodesics) moving towards the black hole must encounter a singularity through focusing, whereas the same of accelerated particles cannot find any singularity at all. Therefore, one may infer that the singularity inside the black hole, most likely, disappears. The cost of disappearing the singularity is to modify the classical energy condition, which is discussed in Sect. 2.

Such disappearance is verified more elaborately and exhaustively in the case of the Schwarzschild spacetime by calculating the Kretschmann scalar with adding an acceleration term. Considering the tidal equation for the accelerated curves a new Reimann tensor is constructed in the Sect. 3. In the Sect. 3.1 the acceleration term for the Schwarzschild spacetime is evaluated in terms of λ being a slowly varying function of r so that it may be thought as constant outside the spherical neighbourhood of the singularity at $r = 0$. The value of λ is such that the modified Kretschmann scalar associated with the modified Reimann curvature tensor arising due to the acceleration term must vanish inside a spherical neighbourhood of $r = 0$ with infinitesimal radius and so remains regular therein. Depending on the suitable choice of λ the singularity disappears and the Schwarzschild spacetime becomes regular at $r = 0$.

In the Sect. 4, the disappearance of singularity is discussed in the notion of geodesic incompleteness. The incompleteness of maximal timelike geodesic signifies that the history of the freely moving particles does not exist after a finite open interval of proper time. That is more robust criterion rather than that of infinite curvature to define a spacelike singularity. Therefore, essentially the singularity and its disappearance is also required to be explored in the light of geodesic incompleteness. In the said section it is shown in terms of a newly introduced theorem named as *removal of singularity*. It is shown that a maximal geodesic being unable to be complete may have an acceleration to be turned into non-geodesic curve (maximal), but consequently with a suitable value of acceleration a reparameterization of proper time may transform that accelerated curve into a geodesic that can be extended over infinite proper time. As a result no maximal geodesic will have the future end point, and the spacetime manifold becomes complete and singularity-free.

An observation of Eq. (49) is that the rate of change of new parameter $\tilde{\tau}$ relative to the old one may tend to infinity as $\tau \rightarrow \tau_*$. That is not a problem as the magnitude of $\frac{\partial u}{\partial \tilde{\tau}}$ becomes so small near τ_* that the product $\frac{\partial u}{\partial \tilde{\tau}} \frac{d\tilde{\tau}}{d\tau}$ makes the acceleration term a^α a finite quantity therein.

Thus the accelerated motion of the ball of test particles neighbouring the black hole destroys the singularity. The said particles may be the anti-particles formed by the vacuum fluctuations at the vicinity of the black hole event horizon. Here we have not carried out any quantum treatment as the whole work is based on the classical general relativity. At this juncture only what we can expect is that the particles may get accelerated by the process of pair productions. Thus one may conclude that the black hole singularity, is diminished when the test particles directed towards the singularity gains high value of acceleration. It may also be consistent with the phenomenon of black hole evaporation as its singularity disappears.

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Appendix

The various terms of $a^\alpha{}_{;\beta}$ as well as $a_\alpha{}_{;\beta}$ in the Schwarzschild spacetime are calculated precisely. At the beginning let us state different terms of affine connections $\Gamma^\rho_{\alpha\beta}$ of Schwarzschild solutions as follows:

$$\Gamma^t_{tr} = \frac{m}{r^2 \left(1 - \frac{2m}{r}\right)} \quad (\text{A1})$$

$$\Gamma^r_{tt} = \left(1 - \frac{2m}{r}\right) \frac{m}{r^2} \quad (\text{A2})$$

$$\Gamma^r_{rr} = -\frac{m}{r^2 \left(1 - \frac{2m}{r}\right)} \quad (\text{A3})$$

$$\Gamma_{\theta\theta}^r = \Gamma_{\phi\phi}^r = -r \left(1 - \frac{2m}{r} \right) \quad (\text{A4})$$

$$\Gamma_{r\theta}^\theta = \Gamma_{r\phi}^\phi = \frac{1}{r} \quad (\text{A5})$$

In the above result it is assumed $\theta = \frac{\pi}{2}$ and under such assumption different non-zero terms of $R_{\alpha\beta\gamma\delta}$ take the following forms.

$$R_{trtr} = -\frac{2m}{r^3} \quad (\text{A6})$$

$$R_{t\theta t\theta} = R_{t\phi t\phi} = \frac{m}{r} \left(1 - \frac{2m}{r} \right) \quad (\text{A7})$$

$$R_{r\theta r\theta} = R_{r\phi r\phi} = -\frac{m}{r \left(1 - \frac{2m}{r} \right)} \quad (\text{A8})$$

$$R_{t\theta t\theta} = \Gamma_{\phi\phi}^r = 2mr \quad (\text{A9})$$

The terms of $a^\alpha{}_{;\beta}$ and $a_\alpha{}_{;\beta}$ can be evaluated by using the following formula.

$$a^\alpha{}_{;\beta} = a^\alpha{}_{,\beta} + \Gamma_{\beta t}^\alpha a^t + \Gamma_{\beta r}^\alpha a^r \quad (\text{A10})$$

$$a_\alpha{}_{;\beta} = a_{\alpha,\beta} - \Gamma_{\alpha\beta}^t a_t - \Gamma_{\alpha\beta}^r a_r \quad (\text{A11})$$

Thus the various non-zero terms of the acceleration become

$$a^t{}_{;t} = \frac{m^2}{r^4 \left(1 - \frac{2m}{r} \right)} \cosh^2 \lambda \quad (\text{A12})$$

$$a_t{}_{;t} = -\frac{m^2}{r^4} \cosh^2 \lambda \quad (\text{A13})$$

$$a^t{}_{;r} = \frac{m^2}{r^4 \left(1 - \frac{2m}{r} \right)^2} \sinh \lambda \cosh \lambda \quad (\text{A14})$$

$$a_t{}_{;r} = \frac{m^2}{r^4 \left(1 - \frac{2m}{r} \right)} \sinh \lambda \cosh \lambda \quad (\text{A15})$$

$$a^r{}_{;t} = \frac{m^2}{r^4} \sinh \lambda \cosh \lambda \quad (\text{A16})$$

$$a_r{}_{;t} = \frac{m^2}{r^4 \left(1 - \frac{2m}{r} \right)} \sinh \lambda \cosh \lambda \quad (\text{A17})$$

$$a^r{}_{;r} = -\left[\frac{2m}{r^3} + \frac{m^2}{r^4 \left(1 - \frac{2m}{r} \right)} \right] \cosh^2 \lambda \quad (\text{A18})$$

$$a_r{}_{;r} = -\frac{1}{\left(1 - \frac{2m}{r} \right)} \left[\frac{2m}{r^3} + \frac{m^2}{r^4 \left(1 - \frac{2m}{r} \right)} \right] \cosh^2 \lambda \quad (\text{A19})$$

$$a^\theta{}_{;\theta} = a^\phi{}_{;\phi} = \frac{m^2}{r^3} \cosh^2 \lambda \quad (\text{A20})$$

$$a_\theta{}_{;\theta} = a_\phi{}_{;\phi} = \frac{m^2}{r^3} \cosh^2 \lambda \quad (\text{A21})$$

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