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Characterizations of strongly entanglement breaking channels for infinite-dimensional quantum systems

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Abstract

Entanglement breaking (EB) channels, as completely positive and trace-preserving linear operators, sever the entanglement between the input system and other systems. In the realm of infinite-dimensional systems, a related concept known as strongly EB (SEB) channels emerges. This paper delves into characterizations of SEB channels, delineating sufficient conditions for a channel to be classified as SEB, especially with respect to the commutativity of its range. Moreover, we demonstrate that every closed self-adjoint subspace of trace-zero operators, with the trace norm, is the null space of a SEB channel.

Keywords: quantum channels, strongly entanglement breaking channels, infinite-dimensional quantum systems, commutative range, null space

1. Introduction

Entanglement breaking (EB) channels are channels which disrupt entanglement with other quantum systems. Specifically, when an EB channel is applied to a component of a composite system, any entanglement in the input state transforms into separable states. This concept was first introduced in [9] as an extension of the classical–quantum and quantum–classical channels

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considered in [7]. The characterization of these channels has been the subject of intensive research for finite-dimensional systems, as seen in works such as [2, 9, 12, 13]. Some of these findings have been expanded to infinite-dimensional systems in [3, 5, 6, 15].

We recall some notions in quantum information theory, see, e.g. [6, 10]. Consider infinite-dimensional Hilbert spaces H and K. Let $B(H)$ be the set of bounded linear operators on H, and let $B(H)^+$ denote the set of positive semi-definite operators in $B(H)$. Let $\mathcal{T}(H)$ be the space of trace-class operators on H. A state in $\mathcal{T}(H)$ is a self-adjoint positive and trace-one operator on H. Denote by $S(H)$ the set of states in $\mathcal{T}(H)$. A state $\rho \in S(H\otimes K)$ is called *separable* if it is a limit in the trace-norm of states of the form

$$\rho = \sum_{i=1}^{n} p_i g_i \otimes h_i, \quad \text{for some } g_i \in S(H), h_i \in S(K), p_i > 0, \text{ with } \sum_{i=1}^{n} p_i = 1. \tag{1.1}$$

In particular, if $\rho \in S(H\otimes K)$ can be expressed as an infinite sum of product states:

$$\rho = \sum_{i=1}^{\infty} p_i g_i \otimes h_i, \quad \text{for some } g_i \in S(H), h_i \in S(K), p_i > 0, \text{ with } \sum_{i=1}^{\infty} p_i = 1, \tag{1.2}$$

then ρ is said to be *countably separable*; see [6, 11]. For finite-dimensional systems, a separable state always has the form as outlined in (1.1). However, in infinite-dimensional systems, a separable state may not necessarily be countably separable, i.e. it may not have the form (1.2), see, e.g. [6].

A quantum channel $\Phi : \mathcal{T}(H) \to \mathcal{T}(K)$ is a completely positive and trace-preserving linear map. Specifically, a channel Φ is called *entanglement breaking* (EB in short) if, for any Hilbert space R and any state $\rho \in \mathcal{T}(R\otimes H)$, the output state $(I_R \otimes \Phi)(\rho)$ is always separable in $S(R\otimes K)$. Similarly, Φ is called *strongly entanglement breaking* (SEB in short) if $(I_R\otimes\Phi)(\rho)$ is always countably separable.

The following provides us with the structure of SEB channels, which we will utilize throughout this paper. It shows that a SEB channel always has a measure-and-prepare form (called the Holevo form) (1.3) and can be represented by an operator-sum of rank-one operations (1.4).

Theorem 1.1 ([5, theorem 3.1] [9, theorem 4]). *Let $\Phi : \mathcal{T}(H) \to \mathcal{T}(K)$ be a channel. The following are equivalent.*

(a) Φ *is a SEB channel.*
(b) Φ *has the form*

$$\Phi(X) = \sum_{k=1}^{\infty} R_k \operatorname{Tr}(F_k X) \tag{1.3}$$

for some states $R_k \in S(K)$ and positive operators $F_k \in B(H)^+$ such that $\sum_{k=1}^{\infty} F_k = I_H$.
(c) Φ *has an operator-sum representation by rank-one operations, i.e.*

$$\Phi(X) = \sum_{k=1}^{\infty} u_k v_k^* X v_k u_k^* = \sum_{k=1}^{\infty} (v_k^* X v_k) u_k u_k^*, \tag{1.4}$$

for some unit vectors $\{u_k\} \subseteq K$ and vectors $\{v_k\} \subseteq H$ such that $\sum_{k=1}^{\infty} v_k v_k^* = I_H$.

It is shown in [15, lemma 2.9] that a channel Φ is SEB if the output state $(I_H\otimes\Phi)(\rho_0)$ corresponding to the maximally entangled state ρ_0 in $S(H\otimes H)$ is countably separable. However, determining whether a state is separable (or countably separable) is indeed very challenging.

Therefore, it is interesting to find some conditions ensuring a channel is SEB other than the separability of the output states.

In finite-dimensional systems, if the range of a completely positive map is commutative, then it is EB; see [17, lemma III.1] and [19, corollary 3]. However, the validity of these assertions may not extend seamlessly to infinite-dimensional systems, particularly concerning SEB channels. Our contribution in theorem 2.3 establishes that the commutativity of the range serves as a sufficient condition to guarantee the SEB nature of a channel. Conversely, as demonstrated in proposition 2.5, we affirm that the dual of any SEB channel can be dilated to a completely positive map with commutative range.

In a recent study focusing on the structure of null spaces, the authors in [12] constructed some private subalgebras of certain classes of EB channels for finite-dimensional systems. In particular, they showed that every self-adjoint subspace of trace-zero matrices is the null space of an EB channel. The finding presented in theorem 3.2 expands upon this discovery, now encompassing applications to infinite-dimensional systems.

The article is structured as follows. In section 2, we delve into various conditions that characterize a channel as SEB. In section 3, we considered SEB channels whose null space is a given closed self-adjoint subspace of trace-zero operators for infinite-dimensional systems. Furthermore, we explore the attributes of the fixed point set of these channels concerning their representation via rank-one operations in (1.4). In section 4, we demonstrate some of the results in the previous sections to a class of single-mode Gaussian channels.

2. Some characterizations of SEB channels

As mentioned by Størmer [19], there is a natural duality between bounded linear maps on $B(H)$ and linear functionals on the tensor product $B(H)\hat{\otimes}\mathcal{T}(H)$. In the subsequent discussion, we leverage this duality within the context of SEB channels. It is demonstrated that a channel is SEB if and only if the corresponding linear functional satisfies a specific criterion associated with separability. We first revisit some notions outlined in [19].

Let $\Psi : B(H) \to B(H)$ be a bounded linear operator. Then a map $\widetilde{\Psi} : B(H)\hat{\otimes}\mathcal{T}(H) \to \mathbb{C}$ given by

$$\widetilde{\Psi}\left(Y\otimes X\right) = \mathrm{Tr}\left(\Psi\left(Y\right)X^T\right) \quad \text{for } X\in\mathcal{T}\left(H\right), Y\in B\left(H\right), \tag{2.1}$$

defines a linear functional on the projective tensor product space of $B(H)$ and $\mathcal{T}(H)$, where the transpose X^T defined by $\langle e_i|X^T|e_j\rangle = \langle e_j|X|e_i\rangle$ with respect to a fixed orthonormal basis $\{e_i\}$ of H. A linear functional φ on $B(H)\hat{\otimes}\mathcal{T}(H)$ is called *separable* (see [19]) if it belongs to the norm closure of elements of the form $\sum_{k=1}^n w_k\otimes\rho_k$ for some positive norm-one linear functional w_k on $B(H)$ and positive linear functional ρ_k on $\mathcal{T}(H)$. The separability of the corresponding linear functional $\widetilde{\Psi}$ is equivalent to the following characterization of the operator Ψ.

Proposition 2.1 ([19, theorem 2]). *Let Ψ be a completely positive operator and let $\widetilde{\Psi}$ be the linear functional defined in (2.1). Then $\widetilde{\Psi}$ is a separable linear functional if and only if Ψ is a limit (in bounded-weak topology) of operators of the form $x\mapsto\sum_{i=1}^n w_i(x)b_i$ for some positive norm-one linear functional $w_i\in B(H)^*$ and positive operators $b_i\in B(H)^+$.*

By leveraging the aforementioned duality, we proceed to establish the separability of the associated linear functional in relation to a SEB channel.

Proposition 2.2. *A quantum channel $\Phi : \mathcal{T}(H)\to\mathcal{T}(H)$ is SEB if and only if the linear functional $\widetilde{\Phi^*} : B(H)\hat{\otimes}\mathcal{T}(H)\to\mathbb{C}$ corresponding to the dual map Φ^* can be written as the form*

$\widetilde{\Phi^*} = \sum_{k=1}^{\infty} w_k \otimes \rho_k$ *for some weak** *continuous positive norm-one linear functional* w_k *on* $B(H)$ *and positive linear functional* ρ_k *on* $\mathcal{T}(H)$ *with* $\sum_{k=1}^{\infty} \rho_k(X) = \mathrm{Tr}(X)$ *for all* $X \in \mathcal{T}(H)$.

For convenience, we call a linear functional having the form as stated in the proposition 2.2 be *countably separable*. The proof aligns with the argument presented in [19, theorem 2] concerning SEB channels.

Proof. Suppose $\Phi : \mathcal{T}(H) \to \mathcal{T}(H)$ is a SEB channel. Recall that a dual map of Φ is the linear map $\Phi^* : B(H) \to B(H)$ satisfying $\mathrm{Tr}(\Phi(X)Y) = \mathrm{Tr}(X\Phi^*(Y))$ for all $X \in \mathcal{T}(H)$ and $Y \in B(H)$. If Φ has the form (1.3) then its dual map satisfies

$$\Phi^*(Y) = \sum_{k=1}^{\infty} \mathrm{Tr}(R_k Y) F_k, \tag{2.2}$$

for some states $R_k \in \mathcal{T}(H)$ and positive operators $F_k \in B(H)^+$. For $Y \in B(H)$ and $X \in \mathcal{T}(H)$,

$$\widetilde{\Phi^*}(Y \otimes X) = \mathrm{Tr}\left(\Phi^*(Y) X^T\right) = \sum_{k=1}^{\infty} \mathrm{Tr}(R_k Y)\,\mathrm{Tr}\left(F_k X^T\right) = \sum_{k=1}^{\infty} w_k(Y)\,\rho_k(X),$$

where

$$w_k(Y) = \mathrm{Tr}(R_k Y) \quad \text{and} \quad \rho_k(X) = \mathrm{Tr}\left(F_k X^T\right) \tag{2.3}$$

define a state in $B(H)^*$ and a positive linear functional on $\mathcal{T}(H)$, respectively. Then we have

$$\sum_{k=1}^{\infty} \rho_k(X) = \mathrm{Tr}\left(\left(\sum_{k=1}^{\infty} F_k\right) X^T\right) = \mathrm{Tr}\left(X^T\right) = \mathrm{Tr}(X).$$

Thus, $\widetilde{\Phi^*}$ is countably separable in $(B(H) \hat{\otimes} \mathcal{T}(H))^*$.

Conversely, suppose $\widetilde{\Phi^*}$ is countably separable and has the form $\widetilde{\Phi^*} = \sum_{k=1}^{\infty} w_k \otimes \rho_k$ as in the proposition. For each k, there exist states $R_k \in \mathcal{T}(H)$ and positive operators $F_k \in B(H)^+$ satisfying (2.3). The condition $\sum_{k=1}^{\infty} \rho_k(X) = \mathrm{Tr}(X)$ implies that $\sum_{k=1}^{\infty} F_k = I_H$. For each $X \in \mathcal{T}(H)$ and $Y \in B(H)$,

$$\mathrm{Tr}\left(\Phi^*(Y) X^T\right) = \widetilde{\Phi^*}(Y \otimes X) = \sum_{k=1}^{\infty} w_k(Y)\,\rho_k(X) = \mathrm{Tr}\left(\sum_{k=1}^{\infty} \mathrm{Tr}(R_k Y) F_k X^T\right).$$

Then Φ^* has the form as in (2.2), and hence Φ is a SEB channel. □

Herein, we present a sufficient condition adequate for identifying a channel as SEB, devoid of any reliance on the separability of states within the bipartite system. This condition asserts that if the range of a channel is commutative, then the channel qualifies as SEB.

Theorem 2.3. *Let* $\{e_i\}$ *be any orthonormal basis of a separable Hilbert space* H*. Let* $\Phi : \mathcal{T}(H) \to \mathcal{T}(H)$ *be a completely positive and trace-preserving channel. If the range of* Φ *is commutative, then* Φ *is a SEB channel. Under this condition, for any set of scalars* $\lambda_i > 0$ *with* $\sum_{i=1}^{\infty} \lambda_i = 1$*, the weighted Choi state* σ *of* Φ*, defined by*

$$\sigma := \sum_{i,j=1}^{\infty} \sqrt{\lambda_i \lambda_j}\, e_i e_j^* \otimes \Phi\left(e_i e_j^*\right),$$

is countably separable and can be written in the form

$$\sigma = \sum_{k=1}^{\infty} p_k \rho_k \otimes v_k v_k^*, \tag{2.4}$$

for some orthonormal set of vectors $\{v_k\}$ *in* H, *some states* ρ_k *in* $\mathcal{T}(H)$, *and some scalars* $p_k > 0$ *with* $\sum_{k=1}^{\infty} p_k = 1$. *Moreover, the channel* Φ *can be expressed as* $\Phi(X) = \sum_{k=1}^{\infty} R_k \operatorname{Tr}(XF_k)$, *where* $R_k = (Uv_k)(Uv_k)^*$ *for some unitary operator* U *in* $B(H)$, *and operators* $F_k \in B(H)^+$ *with a matrix representation as*

$$F_k = \left[\frac{p_k}{\sqrt{\lambda_i \lambda_j}} \langle e_i, \rho_k e_j \rangle \right]_{1 \leqslant i,j < \infty}.$$

Proof. We first note that if the range of a channel $\Phi : \mathcal{T}(H) \to \mathcal{T}(H)$ is commutative, then this range will consist of normal operators. Indeed, since Φ is positive, it maps self-adjoint operators to self-adjoint operators. Then for any operator X in $\mathcal{T}(H)$ of the form $X = X_1 + iX_2$ for some self-adjoint operators X_1 and X_2, we have $\Phi(X)^* = (\Phi(X_1) + i\Phi(X_2))^* = \Phi(X_1) - i\Phi(X_2) = \Phi(X^*)$. Thus, Φ preserves the involution. Since $\mathcal{T}(H)$ is a self-adjoint space, for each $X \in \mathcal{T}(H)$ we have $\Phi(X)\Phi(X)^* = \Phi(X)\Phi(X^*) = \Phi(X^*)\Phi(X) = \Phi(X)^*\Phi(X)$. Then $\Phi(X)$ is a normal operator for all $X \in \mathcal{T}(H)$.

We now employ some techniques introduced in [4, 15]. From the above paragraph, $\Phi(e_i e_j^*), i,j = 1, \ldots, \infty$, are mutually commuting, normal and trace-class (hence compact) operators. So they are simultaneously diagonalizable by some unitary operator U in $B(H)$. Then

$$(I_H \otimes U^*)\, \sigma\, (I_H \otimes U) = \sum_{i,j=1}^{\infty} e_i e_j^* \otimes D_{ij}, \tag{2.5}$$

for some diagonal operators D_{ij} in $B(H)$. Replacing $\Phi(\cdot)$ by $U\Phi(\cdot)U^*$, we can assume σ has the form as in the right side of (2.5). Hence σ can be written as

$$\sigma = \sum_k B_k \otimes P_k,$$

for some rank-one orthogonal projections P_k and positive operators B_k. On the other hand, since Φ is completely positive, the weighted Choi operator σ defined as in (2.4) is positive by [15, theorem 1.4]. It is easy to check that σ has trace-one, hence it is a state in $B(H \otimes H)$. Thus, σ has the form as (2.4) for some orthonormal set of vectors $\{v_k\}$ in H, some states $\{\rho_k\}$ in $\mathcal{T}(H)$ and some scalars $p_k \geqslant 0$ with $\sum_{k=1}^{\infty} p_k = 1$.

For each state $X = \sum_{i,j} e_{ij} \otimes X_{ij} \in S(H \otimes H)$, define the partial trace of X on the second component by $\operatorname{Tr}_2(X) = \sum_{i,j} e_{ij} \operatorname{Tr}(X_{ij})$. Since Φ is trace-preserving, taking the partial trace over the second component of (2.4), we arrive at

$$\sum_{i=1}^{\infty} p_i \rho_i = \sum_{i,j} \lambda_i e_i e_j^* \operatorname{Tr}\left(\Phi\left(e_i e_j^*\right)\right) = \sum_{i=1}^{\infty} \lambda_i e_i e_i^*. \tag{2.6}$$

From (2.4), for $x, y \in H$,

$$\sqrt{\lambda_i \lambda_j} \left\langle \Phi\left(e_i e_j^*\right) x, y \right\rangle = \left\langle \sigma\left(e_i \otimes x\right), \left(e_j \otimes y\right) \right\rangle = \sum_{k=1}^{\infty} \langle e_i, p_k \rho_k e_j \rangle \left\langle v_k v_k^* x, y \right\rangle.$$

Then

$$\Phi\left(e_ie_j^*\right)=\sum_{k=1}^{\infty}\frac{p_k}{\sqrt{\lambda_i\lambda_j}}\langle e_i,\rho_ke_j\rangle v_kv_k^*. \tag{2.7}$$

Following the proof of [15, lemma 2.9], we let F_k be the Schur product as

$$F_k=p_k\begin{pmatrix}\frac{1}{\lambda_1} & \sqrt{\frac{1}{\lambda_1\lambda_2}} & \vdots\\ \sqrt{\frac{1}{\lambda_2\lambda_1}} & \frac{1}{\lambda_2} & \vdots\\ \cdots & \cdots & \ddots\end{pmatrix}\circ\begin{pmatrix}(\rho_k)_{11} & (\rho_k)_{12} & \vdots\\ (\rho_k)_{21} & (\rho_k)_{22} & \vdots\\ \cdots & \cdots & \ddots\end{pmatrix}=\sum_{i,j=1}^{\infty}\frac{p_k}{\sqrt{\lambda_i\lambda_j}}(\rho_k)_{ij}e_ie_j^*,$$

where $(\rho_k)_{ij}=\langle e_i,\rho_ke_j\rangle$. We have $F_k\geqslant 0$ and $\sum_{k=1}^{\infty}F_k=I_H$ by (2.6). The map $\Psi(X)=\sum_{k=1}^{\infty}\mathrm{Tr}(XF_k)v_kv_k^*$ is well-defined and completely positive on $\mathcal{T}(H)$, see [15, lemma 2.5]. Equation (2.7) ensures that $\Psi(e_ie_j^*)=\Phi(e_ie_j^*)$ for all i,j. Hence, $\Phi=\Psi$ on $\mathcal{T}(H)$ from the continuity in trace norm of these maps, see the proof of [15, lemma 2.9] for details. Thus, Φ has the desired form. □

Remark 2.4. Theorem 2.3 shares a connection with a result by Størmer [19]. Specifically, corollary 3 and theorem 2 (along with proposition 2.1) in [19] imply that a completely positive operator with commutative range serves as a limit of *entanglement breaking maps*. These maps bear similarities to the dual maps in (1.3), albeit with a finite count of summands. In theorem 2.3, we demonstrate that a completely positive and trace-preserving channel with commutative range embodies a form characterized by an infinite summation akin to (1.3).

The subsequent demonstration proves an anticipation in [2]. It establishes that the dual map of any SEB channel can be dilated to a completely positive linear map with commutative range. Notably, the predual of an operator $\Psi:B(K)\to B(E)$ is represented by the operator $\Psi_*:\mathcal{T}(E)\to\mathcal{T}(K)$, which satisfies the following relationship:

$$\mathrm{Tr}(\Psi(Y)X)=\mathrm{Tr}(Y\Psi_*(X)),\quad \forall X\in\mathcal{T}(E),Y\in B(K).$$

Proposition 2.5. *If a channel $\Phi:\mathcal{T}(H)\to\mathcal{T}(K)$ is SEB, then there exist a Hilbert space E, an isometry $U:H\to E$, and a positive linear map $\Psi:B(K)\to B(E)$ with* range(Ψ) *being commutative such that $\Phi(X)=\Psi_*(UXU^*)$ for all $X\in\mathcal{T}(H)$.*

Proof. We follow the arguments in [2] which was for finite-dimensional systems. Let $\Phi^*:B(K)\to B(H)$ be a dual of Φ with a form

$$\Phi^*(Y)=\sum_{k=1}^{\infty}\mathrm{Tr}(R_kY)F_k,$$

for some states $R_k\in\mathcal{T}(K)$ and positive operators $F_k\in B(H)^+$ satisfying $\sum_{k=1}^{\infty}F_k=I_H$. Let l^∞ be the space of bounded sequences. Then l^∞ is an unital commutative C^*-algebra with pointwise sums and products, which can be identified with $C(\beta\mathbb{N})$, the space of continuous functions on the Stone-Čech compactification of $\mathbb{N}$.

We write Φ^* into the composition $\Phi^*=\eta\circ\gamma$ of unital maps $\gamma:B(K)\to l^\infty$ by $\gamma(Y)=\{\mathrm{Tr}(YR_k)\}_k$, and $\eta:l^\infty\to B(H)$ by $\eta(\{a_i\}_i)=\sum_{i=1}^{\infty}a_iF_i$. The positive linear map η is well-defined by conditions on $\{F_i\}_i$. Moreover, it is completely positive since l^∞ is a commutative C^*-algebra, see [16, theorem 3.11]. By the Stinespring dilation theorem, there exist a Hilbert space E, an isometry $U : H \to E$ and a $*$-homomorphism $\pi : l^\infty \to B(E)$ such that $\eta(X) = U^*\pi(X)U$. Then for each $Y \in B(K)$, we have

$$\Phi^*(Y) = \eta \circ \gamma(Y) = U^*(\pi \circ \gamma)(Y)U = U^*\Psi(Y)U,$$

where $\Psi = \pi \circ \gamma : B(K) \to B(E)$ is a weak*–weak* continuous positive mapping with commutative range since γ is so. For each $Y \in B(K)$ and $X \in \mathcal{T}(H)$,

$$\mathrm{Tr}(Y\Phi(X)) = \mathrm{Tr}(\Phi^*(Y)X) = \mathrm{Tr}(U^*\Psi(Y)UX) = \mathrm{Tr}(\Psi(Y)UXU^*) = \mathrm{Tr}(Y\Psi_*(UXU^*)).$$

Hence, $\Phi(X) = \Psi_*(UXU^*)$ as asserted. □

3. Null space, fixed point and multiplicative domain

In [12], the authors construct an EB channel that vanishes on a given subspace of trace-zero matrices as follows.

Proposition 3.1 ([12, proposition 3.1]). *Let H be a finite-dimensional Hilbert space. Let $\mathcal{N}$ be a self-adjoint subspace of trace-zero operators in B(H). Then there is an EB channel Φ on B(H) such that $\mathcal{N} = \{X \in B(H) : \Phi(X) = 0\}$.*

We demonstrate the validity of this assertion in the context of infinite-dimensional systems. Here, $\mathcal{T}(H)_0$ represents the subspace of trace-zero operators within $\mathcal{T}(H)$, and the null space of a channel Φ on $\mathcal{T}(H)$ is denoted by $\mathrm{null}(\Phi) := \{X \in \mathcal{T}(H) : \Phi(X) = 0\}$. Given that every channel operating on $\mathcal{T}(H)$ is continuous in the trace-norm topology, see [15, lemma 2.3], its null space must be closed within this topology.

Theorem 3.2. *Let H be a separable infinite-dimensional Hilbert space. Let $\mathcal{N} \subseteq \mathcal{T}(H)_0$ be a self-adjoint and closed (in trace-norm) subspace of trace-zero operators on H. Then there is a SEB channel Φ on $\mathcal{T}(H)$ such that* $\mathrm{null}(\Phi) = \mathcal{N}$.

Proof. Let $\mathcal{N}^\perp$ be a subspace of $B(H)$ defined by

$$\mathcal{N}^\perp = \{Y \in B(H) : \mathrm{Tr}(XY) = 0 \quad \text{for all } X \in \mathcal{N}\}.$$

The identity operator I_H is in $\mathcal{N}^\perp$ since $\mathcal{N} \subseteq \mathcal{T}(H)_0$. Let $\{e_k\}_k$ be an orthonormal basis of H. For each $Y \in \mathcal{N}^\perp$,

$$0 = \mathrm{Tr}(YX) = \sum_{i=1}^{\infty} \langle e_k, YXe_k\rangle = \sum_{k=1}^{\infty} \langle X^*Y^*e_k, e_k\rangle = \mathrm{Tr}(X^*Y^*) \quad \text{for all } X \in \mathcal{N}.$$

Since $\mathcal{N}$ is self-adjoint, i.e. $\mathcal{N} = \{X^* : X \in \mathcal{N}\}$, we have $Y^* \in \mathcal{N}^\perp$. Thus, $\mathcal{N}^\perp$ is a self-adjoint subspace of $B(H)$. We follow arguments in [20, lemma 2] to define a countable subset whose linear span is dense (in weak* topology) in $\mathcal{N}^\perp$.

Recall the duality between $(B(H), \|\cdot\|)$ and $(\mathcal{T}(H), \|\cdot\|_1)$ is defined by $\langle X, Y\rangle = \mathrm{Tr}(XY)$ for all $X \in \mathcal{T}(H)$ and $Y \in B(H)$. The weak* topology on $B(H)$ is the topology defined by the family of semi-norms $\{q_X : X \in \mathcal{T}(H)\}$, where $q_X(Y) = |\mathrm{Tr}(XY)|$ for all $Y \in B(H)$. In addition, since the Hilbert space H is separable, the trace-class $\mathcal{T}(H)$ is a separable Banach space with respect to the trace norm. This implies that the set

$$B := \left\{Y \in \mathcal{N}^\perp : \|Y - I_H\| \leqslant \frac{1}{2}\right\}$$

is weak* compact and metrizable. Hence, there exists a countable subset of self-adjoint operators $\{I_H, \tilde{F}_2, \tilde{F}_3, \ldots\}$ which is dense in B in the weak* topology. Since $\|I_H - \tilde{F}_k\| \leqslant \frac{1}{2}$, each self-adjoint operator $\tilde{F}_k$ is positive with the operator-norm $\|\tilde{F}_k\| \leqslant 2$. Let $F_k = \frac{1}{2^k}\tilde{F}_k$ for $k \geqslant 2$ and $F_1 = I_H - \sum_{k=2}^{\infty} F_k$. Then $\|I_H - F_1\| = \|\sum_{k=2}^{\infty} F_k\| \leqslant \sum_{k=2}^{\infty} \frac{1}{2^{k-1}} = 1$. Hence $\{F_k\}_{k=1}^{\infty}$ is a set of self-adjoint positive operators in $B(H)$ with $\sum_{k=1}^{\infty} F_k = I_H$ and its linear span is weak* dense in $\mathcal{N}^{\perp}$.

Let $R_k = e_k e_k^*$ be states in $B(H)$. Consider a SEB channel $\Phi : \mathcal{T}(H) \to \mathcal{T}(H)$ defined by

$$\Phi(X) = \sum_{k=1}^{\infty} R_k \operatorname{Tr}(F_k X).$$

Then $X \in \text{null}(\Phi)$ if and only if $\operatorname{Tr}(XF_k) = 0$ for all k. This happens exactly for $X \in (\mathcal{N}^{\perp})^{\perp} = \mathcal{N}$ since $\mathcal{N}$ is a closed subspace in trace-norm. Hence $\text{null}(\Phi) = \mathcal{N}$ as asserted. □

Throughout the remainder of this section, we delve into examining the fixed point set and the multiplicative domain of SEB channels by leveraging the operator-sum representation through rank-one operations as in (1.4). It is worth recalling that the fixed point set of a map $\Psi : Z \to Z$ is defined as $\text{Fix}(\Psi) := \{T \in Z : \Psi(T) = T\}$. For the details on the characterization of fixed point sets of quantum operations, refer to [1, 14].

Let $\Phi : \mathcal{T}(H) \to \mathcal{T}(H)$ be a SEB channel defined as in (1.4) by

$$\Phi(X) = \sum_{k=1}^{\infty} u_k v_k^* X v_k u_k^* = \sum_{k=1}^{\infty} E_k X E_k^*, \tag{3.1}$$

where $E_k = u_k v_k^*$ for $u_k, v_k \in H$ satisfying $\|u_k\| = 1$ for all k and $\sum_{k=1}^{\infty} v_k v_k^* = I_H$. Then $\Phi^* : B(H) \to B(H)$ is an unital completely positive operator with rank-one Kraus operations $\{E_k^* = v_k u_k^*\}$.

Suppose a projection $P \in \text{Fix}(\Phi^*)$. Then $\sum_{k=1}^{\infty} E_k^* P E_k = P$. By multiplying from both sides of this equation with $I_H - P$, we derive $PE_k(I_H - P) = 0$ for all k. Similarly, by multiplying both sides of equation $\Phi^*(I_H - P) = I_H - P$ by P, we get $(I_H - P)E_k P = 0$. Hence, $PE_k = E_k P$ for all k. Substitute $E_k = u_k v_k^*$ we obtain

$$Pv_k = Pv_k(u_k^* u_k) = P(v_k u_k^*)u_k = (v_k u_k^*)(Pu_k) = v_k(u_k^* P u_k).$$

Thus, $Pv_k = \lambda_k v_k$ for some scalar λ_k. Similarly, $Pu_k = \beta_k u_k$ for some scalar β_k. Hence, v_k and u_k are eigenvectors of P for all k. Let P, Q be projections in $\text{Fix}(\Phi^*)$. Then for each k, there are scalars λ_k, β_k such that $Pv_k = \lambda_k v_k$ and $Qv_k = \mu_k v_k$. Thus,

$$\begin{aligned} PQ &= PQ\left(\sum_{k=1}^{\infty} v_k v_k^*\right) = P\sum_{k=1}^{\infty} \mu_k v_k v_k^* = \sum_{k=1}^{\infty} \mu_k \lambda_k v_k v_k^* \\ &= \sum_{k=1}^{\infty} \lambda_k Q v_k v_k^* = \sum_{k=1}^{\infty} QP v_k v_k^* = QP. \end{aligned} \tag{3.2}$$

Then P and Q are commutative. Let $\mathcal{A} = \{A \in B(H) : [A, E_k] = [A, E_k^*] = 0\}$. Since $\mathcal{A}$ is spanned by its projections, $\mathcal{A}$ is a commutative von Neumann subalgebra of $\text{Fix}(\Phi^*)$. For infinite-dimensional systems, the inclusion $\mathcal{A} \subseteq \text{Fix}(\Phi^*)$ can be strict, see [1]. From the above observations, we have

Proposition 3.3. *Let Φ be a SEB channel on $\mathcal{T}(H)$ defined by (3.1) with rank-one operations $\{E_k = u_k v_k^*\}$. A projection P is a fixed point of Φ^* if and only if $[P, E_k] = 0$ for all k. In this case, all vectors u_k and v_k are eigenvectors of P. The set of projections in* $\text{Fix}(\Phi^*)$ *is commutative.*

The study of fixed point sets is useful for the investigation of the multiplicative domain of a map. Recall that a *multiplicative domain* of a map Φ on $\mathcal{T}(H)$ is the set (see e.g. [17])

$$\mathcal{M}_\Phi = \{A \in \mathcal{T}(H) : \Phi(AX) = \Phi(A)\Phi(X), \Phi(XA) = \Phi(X)\Phi(A) \text{ for all } X \in \mathcal{T}(H)\}.$$

Proposition 3.4. *Let Φ be a SEB channel on $\mathcal{T}(H)$ defined by (3.1) with rank-one operations $\{E_k = u_k v_k^*\}$. If a projection P is in the multiplicative domain $\mathcal{M}_\Phi$, then v_k are eigenvectors of P for all k. The set of projections in $\mathcal{M}_\Phi$ is commutative.*

Proof. It is known that if $P \in \mathcal{M}_\Phi$, then $P \in \text{Fix}(\Phi^* \circ \Phi)$. Indeed, for each $X \in \mathcal{T}(H)$, we have

$$\text{Tr}(PX) = \text{Tr}(\Phi(PX)) = \text{Tr}(\Phi(P)\Phi(X)) = \text{Tr}((\Phi^* \circ \Phi)(P)X).$$

Hence, $(\Phi^* \circ \Phi)(P) = P$. This implies $E_i^* E_j P = P E_i^* E_j P$ for all i, j. Substitute $E_i = u_i v_i^*$, we arrive at $P v_i = \lambda_i v_i$ for some scalar λ_i. The commutativity argument is similar as in (3.2). □

4. On single-mode Gaussian channels

In the following, we demonstrate some of the results discussed in the above sections to a class of single-mode bosonic Gaussian channels [18] which are widely considered in infinite-dimensional quantum systems.

We begin with some notations adopted from [8, 18]. Let $H = L^2(\mathbb{R})$ be the space of complex-valued square-integrable functions. For each $x, y \in \mathbb{R}$, we define unitary operators V_x and U_y on H by $V_x\psi(\xi) = \exp(\mathrm{i}\xi x)\psi(\xi)$ and $U_y\psi(\xi) = \psi(\xi + y)$ for $\psi \in L^2(\mathbb{R})$ and $\xi \in \mathbb{R}$. The unitary *Weyl representation* $W : \mathbb{R}^2 \to B(H)$ is defined by

$$W(z) = \exp\left(\frac{\mathrm{i}}{2}yx\right)V_x U_y, \; z = (x, y)^{\mathrm{T}} \in \mathbb{R}^2.$$

For each state $\rho \in \mathcal{T}(H)$, define a characteristic function on $\mathbb{R}^2$ by $\varphi_\rho(z) = \text{Tr}(\rho W(z))$. Let M_2 be the set of 2×2 real matrices. A state $\rho \in \mathcal{T}(H)$ is called a *Gaussian state* if its characteristic function φ_ρ has the form

$$\varphi_\rho(z) = \exp\left(\mathrm{i}m^{\mathrm{T}}z - \frac{1}{2}z^{\mathrm{T}}Cz\right), \tag{4.1}$$

for some vector $m \in \mathbb{R}^2$ and symmetric matrix $C \in M_2$. A quantum channel $\Phi : \mathcal{T}(H) \to \mathcal{T}(H)$ is called a *Gaussian channel* if there are parameters $K, \alpha \in M_2$ such that the characteristic function of $\Phi(\rho)$ satisfies

$$\varphi_{\Phi(\rho)}(z) = \varphi_\rho(Kz)\exp\left(-\frac{1}{2}z^{\mathrm{T}}\alpha z\right), \quad \forall z \in \mathbb{R}^2. \tag{4.2}$$

Equivalently,

$$\Phi^*(W(z)) = W(Kz)\exp\left(-\frac{1}{2}z^{\mathrm{T}}\alpha z\right). \tag{4.3}$$

Holevo [8, theorem 12.17] shows that (4.1) defines a state if and only if $C \geqslant \pm\frac{i}{2}\Delta$ where $\Delta = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. In this case, the characteristic function determines the operator via the following inverse formula

$$\rho = \frac{1}{2\pi}\int \varphi_\rho(z)W(-z)\,\mathrm{d}^2z, \tag{4.4}$$

where the integral converges weakly.

From [8, theorem 12.35], a Gaussian channel Φ defined by parameters (K,α) is EB if and only if α admits the decomposition $\alpha = \alpha_1 + \alpha_2$ where

$$\alpha_1 \geqslant \pm\frac{i}{2}K^{\mathrm{T}}\Delta K \quad \text{and} \quad \alpha_2 \geqslant \pm\frac{i}{2}\Delta. \tag{4.5}$$

For a given matrix $K \in M_2$, the channel corresponding to parameters (K,α_0), where α_0 is the minimal of α satisfying the condition (4.5) is called *entanglement breaking limit*. Classification of Gaussian channels for single-mode systems can be found in [8, theorem 12.41]. It was showed in [18, theorem 1] that all single-mode Gaussian channels at their EB limit admit only continuous-indexed (non-countable) set of rank-one Kraus operators and that is unique, except the *full-loss channel* $A_1(1)$ corresponding to the parameters $K = 0$ and $\alpha = I_2$. Therefore, these channels are EB but not SEB except the $A_1(1)$ channel.

Applying the equivalent condition in proposition 2.2, we can give another way to show that $A_1(1)$ channel is SEB by showing that the linear functional $\widetilde{\Phi^*}$ on $B(H)\hat{\otimes}\mathfrak{T}(H)$ defined in (2.1) corresponding to the channel Φ is countably separable. Indeed, for each $Y = W(z) \in B(H)$ and $X \in \mathfrak{T}(H)$, from equation (4.3) we have

$$\widetilde{\Phi^*}(Y\otimes X) = \mathrm{Tr}\left(\Phi^*(W(z))X^{\mathrm{T}}\right) = \mathrm{Tr}\left(W(Kz)X^{\mathrm{T}}\right)\exp\left(-\frac{1}{2}z^{\mathrm{T}}\alpha z\right) = \mathrm{Tr}(X)\exp\left(-\frac{1}{2}z^{\mathrm{T}}z\right).$$

Define a linear functional σ on $\mathrm{span}\{W(z) : z \in \mathbb{R}^2\}$ by extending linearly the map $\sigma(W(z)) = \exp(-\frac{1}{2}z^{\mathrm{T}}z)$. Then σ is a positive linear functional with $\sigma(I) = \sigma(W(0)) = 1$. By Hahn–Banach extension theorem, σ can be extended to a norm-one positive linear functional on $B(H)$. Let $w(X) := \mathrm{Tr}(X)$. Hence, $\widetilde{\Phi^*} = w\otimes\sigma$ is a countably separable linear functional on $B(H)\hat{\otimes}\mathfrak{T}(H)$.

From (4.2) and (4.4), we have

$$\Phi(\rho) = \frac{1}{2\pi}\int \varphi_\rho(Kz)\exp\left(-\frac{1}{2}z^{\mathrm{T}}\alpha z\right)W(-z)\,\mathrm{d}^2z. \tag{4.6}$$

In the case of a full-loss Gaussian channel with parameters $K = 0$ and $\alpha = I_2$,

$$\Phi(\rho) = \frac{1}{2\pi}\int \exp\left(-\frac{1}{2}z^{\mathrm{T}}z\right)W(-z)\,\mathrm{d}^2z := \rho_0,$$

where the state ρ_0 is independent of the input state ρ. Then Φ has the Holevo form $\Phi(X) = \mathrm{Tr}(X)\rho_0$ for X in the span of Gaussian states, hence it is a SEB channel and has the commutative range.

In the following, we examine the commutativity of the range of Gaussian channels corresponding to the parameter $K \neq 0$. For each state $\rho \in \mathfrak{T}(H)$, let f_ρ be the continuous function inside the integral (4.6) by

$$f_\rho(z) = \mathrm{Tr}(\rho W(Kz))\exp\left(-\frac{1}{2}z^{\mathrm{T}}\alpha z\right). \tag{4.7}$$

Following [8, section 5.3], for given states $\rho_1, \rho_2 \in \mathfrak{T}(H)$, we have

$$\Phi(\rho_1)\Phi(\rho_2) = \frac{1}{2\pi}\int \left(f_{\rho_1}\times f_{\rho_2}\right)(z)\,W(-z)\,\mathrm{d}^2z,$$

where

$$\left(f_{\rho_1}\times f_{\rho_2}\right)(z) = \frac{1}{2\pi}\int f_{\rho_1}(x)f_{\rho_2}(z-x)\exp\left(\frac{\mathrm{i}x\Delta z}{2}\right)\mathrm{d}^2x.$$

Since the transform $f \mapsto \frac{1}{2\pi}\int f(z)W(-z)\mathrm{d}^2z$ is injective, the equation $\Phi(\rho_1)\Phi(\rho_2) = \Phi(\rho_2)\Phi(\rho_1)$ implies that $(f_{\rho_1} \times f_{\rho_2})(z) = (f_{\rho_2} \times f_{\rho_1})(z), \forall z \in \mathbb{R}^2$. Equivalently,

$$\int [f_{\rho_1}(x)f_{\rho_2}(z-x) - f_{\rho_2}(x)f_{\rho_1}(z-x)]\exp\left(\frac{\mathrm{i}x\Delta z}{2}\right)\mathrm{d}^2x = 0. \tag{4.8}$$

Therefore the channel Φ has a commutative range if and only if equation (4.8) holds for all states $\rho_1,\rho_2 \in \mathcal{T}(H)$ and all $z \in \mathbb{R}^2$. An extreme case for (4.8) is that $f_{\rho_1}(x)f_{\rho_2}(z-x) = f_{\rho_2}(x)f_{\rho_1}(z-x)$ for all $x,z \in \mathbb{R}^2$. Substituting into (4.7) we get

$$\mathrm{Tr}(\rho_1 W(Kx))\,\mathrm{Tr}(\rho_2 W(K(z-x))) = \mathrm{Tr}(\rho_2 W(Kx))\,\mathrm{Tr}(\rho_1 W(K(z-x))). \tag{4.9}$$

Equation (4.9) being satisfied for all states $\rho_1,\rho_2 \in \mathcal{T}(H)$ implies that the following function is constant in the state $\rho \in \mathcal{T}(H)$ for all $x,z \in \mathbb{R}^2$,

$$\rho \mapsto \frac{\mathrm{Tr}(\rho W(Kx))}{\mathrm{Tr}(\rho W(K(z-x)))}.$$

This happens if and only if the parameter $K = 0$.

Now consider equation (4.8) with parameters $K = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\alpha = I_2$. With $z = (z_1,z_2)^{\mathrm{T}}$ and $x = (x_1,x_2)^{\mathrm{T}}$ in $\mathbb{R}^2$, we have $W(Kx) = V_{x_1}$, $W(K(z-x)) = V_{z_1-x_1}$, where V_{x_i} is the unitary operator in $B(H)$ defined above. The equation (4.8) becomes

$$\int [\mathrm{Tr}(\rho_1 V_{x_1})\,\mathrm{Tr}(\rho_2 V_{z_1-x_1}) - \mathrm{Tr}(\rho_2 V_{x_1})\,\mathrm{Tr}(\rho_1 V_{z_1-x_1})]\exp(-x_1^2 + z_1x_1)\exp\left(\frac{\mathrm{i}x_1z_2}{2}\right)\mathrm{d}x_1 = 0. \tag{4.10}$$

Let $g(x_1) = [\mathrm{Tr}(\rho_1 V_{x_1})\mathrm{Tr}(\rho_2 V_{z_1-x_1}) - \mathrm{Tr}(\rho_2 V_{x_1})\mathrm{Tr}(\rho_1 V_{z_1-x_1})]\exp(-x_1^2 + z_1x_1)$. Since the Fourier transform $g \to \mathcal{F}(g)$ defined by $\mathcal{F}(g)(z_2) = \int g(x_1)\exp\left(\frac{\mathrm{i}x_1z_2}{2}\right)\mathrm{d}x_1$ is injective, equation (4.10) implies that $g(x_1) = 0$ for all $x_1 \in \mathbb{R}$. Hence

$$\mathrm{Tr}(\rho_1 V_{x_1})\,\mathrm{Tr}(\rho_2 V_{z_1-x_1}) = \mathrm{Tr}(\rho_2 V_{x_1})\,\mathrm{Tr}(\rho_1 V_{z_1-x_1}), \quad \forall \rho_1,\rho_2 \in \mathcal{T}(H), \forall x_1,z_1 \in \mathbb{R}.$$

We get $\mathrm{Tr}(\rho V_{x_1}) = \mathrm{Tr}(\rho V_{z_1-x_1})$ for all $\rho \in \mathcal{T}(H)$ and all $x_1,z_1 \in \mathbb{R}$, that is impossible. Hence, the corresponding channel does not have a commutative range. With a similar argument, if the parameter $K \in M_2$ has rank one and $\alpha = I_2$, then the corresponding Gaussian channel does not have a commutative range.

In fact, by combining the sufficient condition stated in theorem 2.3 with the result in [18, theorem 1] referenced earlier, it can be deduced that the single-mode Gaussian channels corresponding to the parameter $K \neq 0$ do not have a commutative range.

We end this section with a discussion about the $A_2(1)$ Gaussian channel at the EB limit with parameters $K = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\alpha = I_2$, see [8, theorem 12.41]. In this case, the dual map Φ^* has a commutative range since for $z,z' \in \mathbb{R}^2$,

$$\Phi^*(W(z))\,\Phi^*(W(z')) = W(Kz)\,W(Kz')\exp\left(-\frac{1}{2}z^{\mathrm{T}}\alpha z - \frac{1}{2}z'^{\mathrm{T}}\alpha z'\right),$$

where $W(Kz)W(Kz') = \exp(-\frac{\mathrm{i}}{2}z^{\mathrm{T}}(K^{\mathrm{T}}\Delta K)z')W(Kz+Kz')$, see [8, section 12.2]. Here $W(Kz)W(Kz') = W(Kz+Kz') = W(Kz')W(Kz)$ since $K^{\mathrm{T}}\Delta K = 0$. Proposition 2.1 shows that Φ^* is the limit of operators in the Holevo form with finite terms. However, as mentioned before, Φ^* does not admit the Holevo form with countable summands since the $A_2(1)$ channel Φ is not SEB. This provides us an instant for the difference of proposition 2.1 in infinite dimensions with the finite-dimension cases where a unital completely positive map with commutative range always has the Holevo form, see e.g. [17, lemma III.1].

5. Conclusions

The paper establishes characterizations of SEB channels tailored to infinite-dimensional systems. It unveils properties akin to their finite-dimensional counterparts, as documented in works such as [2, 12, 17]. Notably, the paper demonstrates that a channel is SEB if its range is commutative. Furthermore, it reveals that any SEB channel can be dilated to the predual of a positive map with commutative range. Additionally, a SEB channel is constructed with a null space that aligns with a specified self-adjoint and closed subspace of trace-zero operators. The discussion also delves into the commutativity of projections within the fixed point set and the multiplicative domain of these channels. Finally, we illustrate our results to a class of single-mode bosonic Gaussian channels. The approach intertwines techniques from the finite-dimensional realm with expanded outcomes in operator theory tailored to infinite-dimensional systems, as expounded in [4, 15, 20].

Data availability statement

The data cannot be made publicly available upon publication because no suitable repository exists for hosting data in this field of study. The data that support the findings of this study are available upon reasonable request from the authors.

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