Applications of Quantum Field Theory, From the Formal to the Phenomenological

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Abstract

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Much of the beautiful and complex phenomenology of the Standard Model can be traced back to the presence of chiral symmetries, both global and local, and nonperturbative phenomena. Without chiral symmetries (and the anomalous ways that they can be broken), there would be no flavor physics or CP violation. Without a strongly coupled sector, the richly intricate structure of the hadron spectrum would be lost. After introducing the Standard Model and some of its phenomenological aspects, as well as some of the tools that are used to explore its structure, I will focus on my contributions to several open problems in flavor physics, lattice field theory and astrophysics.

TABLE OF CONTENTS

LIST OF FIGURES

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Chapter 1

INTRODUCTION

The Standard Model of Particle Physics, developed throughout the 20^{th} century, has proven extremely successful at explaining experimental observations, as well as predicting previously unseen phenomena. With the discovery of the Higgs boson in 2012, the discovery of the particle content and interactions of the Standard Model is complete. And yet, we know that there must be physics beyond the Standard Model. For one, the Standard Model does not provide a theory of quantum gravity, nor does it offer a natural explanation for neutrino masses. Beyond that, there are questions about how the parameters of the Standard Model arise. For example, the Higgs is unnaturally light, as there is no symmetry in the Standard Model that protects it from additive renormalization. There are also multiple hierarchies in the flavor structure of the Standard Model: fermion masses range over from sub-eV (for neutrinos) to 170 GeV (for the top quark) and the angle that parametrizes the mixing between the first two families is much larger than those that parameterize mixing with the third family. These questions can only be addressed by Beyond the Standard Model (BSM) physics. However, without looking to BSM physics, there are important questions about the structure and properties of the Standard Model. For example, the full phase diagram of QCD cannot currently be explored from first principles; the physics of the neutron stars, the early universe and heavy-ion collisions all lie in the sections of the phase diagram that is inaccessible. Beyond this, we cannot even state with full certainty the basic building of the Standard Model, as we do not have a nonperturbative regulator that can tame the UV divergences of chiral gauge theories.

To explore the various formal and phenomenological aspects of both chiral and strongly couple theories, I will begin with a synopsis of the Standard Model in Sec. 1.1. I will then delve deeper into two aspects of the Standard Model – the behavior of QCD at low energies in Sec. 1.2 and the flavor structure of the Standard Model in Sec. 1.3. From here, I will shift gears and focus more on the formal aspects of quantum field theory, specifically on lattice field theory. Sec. 1.4 will have a brief review of regularization and renormalization. I will give a brief overview of Lattice Field Theory in Sec. 1.5 and then discuss the Monte Carlo simulation of lattice field theories as a way to calculate strongly coupled matrix elements in Sec. 1.6. Lastly, I will bring together the ideas of chiral symmetries and nonperturbative physics in discussing the difficulties of using the lattice as a nonperturbative regulator for chiral gauge theories in Sec. 1.7.

In the remaining chapters, I will detail my contributions in these topics. In Ch. 2, I will discuss a new paradigm for having BSM models with non-trivial flavor structure. This work [1], in an expanded form, will be submitted to a journal. In Ch. 3, I will present work on a mechanism to generate the large magnetic fields of neutron stars; this work [2] has appeared previously in the Physical Review D. Ch. 4 is dedicated to discussing the mitigation of the sign and noise problems in the NJL model in three dimension; this work [3] appeared previously in Physical Review D. Lastly, Ch. 5 is dedicated to the question of how to use a lattice regulator for chiral gauge theories. This work [4] has been published in the Physical Review Letters.

In this dissertation, I will assume that the reader is familiar with the basics of quantum field theory, including Feynman diagrams, regularization and renormalization, Lie Groups and other such things one would cover in a graduate course, though I will provide a brief review of key topics. When working in Minkowski space, I use the mostly positive metric, as it is easier to analytically continue into Euclidean space.

1.1 The Standard Model of Particle Physics

The quantum field theory that codifies all known elementary particles and their interactions via the strong, weak and electromagnetic forces is the Standard Model of Particle Physics. Before delving into the phenomenological details, let me introduce the matter and gauge content[1]. The gauge group of the Standard Model is $SU(3)_c \times SU(2)_L \times U(1)_Y$, where $SU(3)_c$ is the gauge group of quantum chromodynamics (QCD) and $SU(2)_L \times U(1)_Y$ is the electroweak gauge group. Ignoring other interactions, the pure gauge Lagrangian is

$$\mathcal{L}_{\text{Gauge}} = -\frac{1}{4}G^A_{\mu\nu}G^{A\mu\nu} - \frac{1}{4}W^A_{\mu\nu}W^{A\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{g\theta N_F}{32\pi^2}G^{C\mu\nu}\tilde{G}^C_{\mu\nu} \tag{1.1}$$

where N_F is the number of (quark) flavors, $G^C_{\mu\nu}$, $W^C_{\mu\nu}$ and $B_{\mu\nu}$ are the strength energy tensors of $SU(3)_c, SU(2)_L$ and $U(1)_Y$, respectively:

$$\begin{aligned} G^C_{\mu\nu} &= \partial_\mu G^C_\nu - \partial_\nu G^C_\mu + g_3 f_3^{ABC} G^A_\mu G^B_\nu \\ W^C_{\mu\nu} &= \partial_\mu W^C_\nu - \partial_\nu W^C_\mu + g_2 f_2^{ABC} W^A_\mu W^B_\nu \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \end{aligned} \tag{1.2}$$

where $A, B, C = 1, 2, \ldots N^2 - 1$ are $SU(N)$ (adjoint representation) index, $g_{2,3}$ are the gauge coupling for $SU(2)_L$ and $SU(3)_c$, respectively and $f^{ABC}_{2,3}$ are their structure constants. Our normalization convention is

$$\operatorname{Tr} T^a_R T^b_R = \frac{1}{2}\delta^{ab} \qquad [T^a_R, T^b_R] = i f^{abc} T^c_R \tag{1.3}$$

where R indicates the representation. Of particular interest to us is the adjoint representation, which is specified by $N^2 - 1$ Hermitian matrices of dimension $N \times N$:

$$\left(T^A_{\text{adj}}\right)_{ab} = -i f^{Aab} \tag{1.4}$$

where we will leave off of the 'adj' subscript from here on out. When discussing the phenomenology of the Standard Model, I work in Minkowski spacetime.

The Standard Model contains five types of Weyl fields. Two are left-handed fields, q_L and ℓ_L, and three right-handed fields, u_R, d_R and e_R. For simplicity, we use Dirac notation. Under the gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$, the quark fields, q_L, u_R, and d_R are in the representations $(3, 2, +1/6)$, $(3, 1, 2/3)$, and $(3, 1, -1/3)$; the lepton ℓ_L and e_R are in the

[1] This section uses the conventions of [5] and is inspired by its presentational choices

representation $(1, 2, -1/2)$, and $(1, 1, -1)$. Furthermore, there are three copies of each field, which we will see gives rise to both nontrivial flavor structure and CP violation. The charge assignments are also summarized in Table 1.1. Our convention for relating $SU(2)_L$ isospin, T_3, hypercharge, Y, and electric charge, Q, is $Q = T_3 + Y$.

The kinetic term for the quarks and leptons is

$$\begin{aligned}\mathcal{L}_\psi &= i\bar{q}_L^{ai}\gamma^\mu\left(D_\mu q_L\right)^{ai} + i\bar{u}_R^a\gamma^\mu\left(D_\mu u_R\right)^a + i\bar{d}_R^a\gamma^\mu\left(D_\mu d_R\right)^a \\ &+ i\bar{\ell}_L^i\gamma^\mu\left(D_\mu \ell_L\right)^i + i\bar{e}_R\gamma^\mu\left(D_\mu e_R\right)\end{aligned} \tag{1.5}$$

where $a, b, c = 1, 2, 3$ are $SU(3)_c$ indices, while $i, j = 1, 2$ are $SU(2)_L$ indices. Note that we are using the convention where

$$\ell_L \equiv \frac{1-\gamma_5}{2}\ell \qquad e_R \equiv \frac{1+\gamma_5}{2}e \tag{1.6}$$

and ℓ, e are both Dirac fermions. As all the fermions are either singlets or are in the fundamental representation, the form of the covariant derivatives can be easily found:

$$\begin{aligned}\left(D_\mu q_L\right)^{ai} &= \partial_\mu q_L^{ai} - ig_3 G_\mu^A\left(T_3^A\right)^{ab} q_L^{bi} - ig_2 W_\mu^A\left(T_2^A\right)^{ij} q_L^{qj} - ig_1 B_\mu\left(+\tfrac{1}{6}\right) q_L^{ai} \\ \left(D_\mu \ell_L\right)^i &= \partial_\mu \ell_L^i - ig_2 W_\mu^A\left(T_2^A\right)^{ij}\ell_L^j - ig_1 B_\mu\left(-\tfrac{1}{2}\right)\ell_L^i \\ \left(D_\mu u_R\right)^a &= \partial_\mu u_R^a - ig_3 G_\mu^A\left(T_3^A\right)_R^{ab} u_R^b - ig_1 B_\mu\left(\tfrac{2}{3}\right) u_R^a \\ \left(D_\mu d_R\right)^a &= \partial_\mu d_R^a - ig_3 G_\mu^A\left(T_3^A\right)^{ab} d_R^b - ig_1 B_\mu\left(-\tfrac{1}{3}\right) d_R^a \\ D_\mu e_R &= \partial_\mu e_R - ig_1 B_\mu\left(-1\right)\bar{e}_R\end{aligned} \tag{1.7}$$

where $T_{2,3}^A$ are the generators of $SU(2)_L$ and $SU(3)_c$ respectively. A key property of the Standard Model is that it has both vector and chiral gauge interactions. The strong sector is a vector theory, in that left-handed and right-handed quarks transform the same under $SU(3)_c$; the electroweak sector is a chiral theory, as left-handed and right-handed fields transform differently under $SU(2)_L \times U(1)_Y$. Parity violation in the electroweak sector was first seen in the decay of ^{60}Co [6].

While the chiral nature of the electroweak interactions explained the parity violation in the decay of ^{60}Co, it gave birth to a new conundrum. It has been known since the late 1800s

		$SU(3)_c$	$SU(2)_L$	$U(1)_Y$
Quarks	$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	3	2	1/6
	u_R	3	1	2/3
	d_R	3	1	-1/3
Leptons	$\ell_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	1	2	-1/2
	e_R	1	1	-1
Scalar	φ	1	2	1/2

Table 1.1: Charge assignments of the quarks, leptons and scalar field under the Standard Model gauge group

that the electron is massive, due to J.J. Thomson's cathode ray experiments [7]; however, fermion mass terms are forbidden by electroweak gauge invariance. The quark sector displays the same behavior, where the massiveness of the (light) quarks can be inferred from the non-zero mass of the pions and other pseudo-scalar mesons, as will be explained in Sec. 1.2. With conventional fermion mass terms forbidden in the Standard Model, quark and lepton masses must be generated via a different mechanism. This mechanism requires the addition of a charged scalar that acquires a vacuum expectation value (vev); this mechanism also results in the spontaneous breaking of electroweak symmetry.

In order to construct a gauge singlet out of two fermion fields and a scalar, said scalar must be only under the chiral gauge groups, as vector gauge theories have no restrictions on fermion masses. The scalar φ is in the representation $(1, 2, 1/2)$, as shown in Table 1.1. To see this field acquires a vev, we look at its kinetic and potential terms:

$$\mathcal{L}_\varphi = -D^\mu \varphi^\dagger D_\mu \varphi + \frac{1}{2}\mu^2 \varphi^\dagger \varphi - \frac{\lambda}{4}\left(\varphi^\dagger \varphi\right)^2 \tag{1.8}$$

where, since the scalar carries electroweak charge,

$$(D_\mu \varphi)^i = \partial \varphi_i - i g_2 W_\mu^A \left(T_2^A\right)^{ij} \varphi_j - i g_1 B_\mu \left(\tfrac{1}{2}\right) \varphi^i \tag{1.9}$$

The minimum of the scalar potential has a nonzero value if $\mu, \lambda > 0$: $|\langle\varphi\rangle| = \mu/\sqrt{\lambda}$. The phase of the vev of ϕ depends on the choice of gauge and so is physically irrelevant. The scalar field can be parameterized as

$$\varphi(x) = \frac{e^{i\zeta^i(x)\sigma^i/2v}}{\sqrt{2}} \begin{pmatrix} 0 \\ h(x) + v \end{pmatrix} \tag{1.10}$$

where $\zeta^i(x), h(x)$ are the four real degrees of freedom of an $SU(2)$ doublet, σ^i are the Pauli matrixes and v is the vev of φ. However, it is easiest to see the particle content of the theory after electroweak symmetry breaking by performing a gauge transform such that

$$\varphi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ h(x) + v \end{pmatrix} \tag{1.11}$$

and corresponding transformations of the gauge fields; this is called unitary gauge. Note that the field $h(x)$ is the Higgs boson, discovered at the LHC in 2012 [8,9]. When $\langle\varphi\rangle \neq 0$, the gauge symmetry $SU(2)_L \times U(1)$ is spontaneously broken, as can be seen by looking at the gauge boson mass matrix in the kinetic term of the scalar field:

$$D_\mu \varphi^\dagger D^\mu \varphi \supset \frac{v^2 g_2^2}{8} \left(\left(W^1\right)^2 + \left(W^2\right)^2 \right) + \frac{v^2}{8} \left(g_2 W^3 - g_1 B\right)^2 \tag{1.12}$$

Notice that only one gauge boson is massless; the other three acquire a mass. The heavy charged and neutral gauge bosons are the $W^\pm$ and Z gauge bosons and we identifying the remaining massless gauge boson as the photon, A_μ. The electric charge is $e = g_2 \sin\theta_W$, where $\theta_W = \tan^{-1}(g_1/g_2)$. The masses of the heavy gauge bosons are

$$M_W = \frac{g_2 v}{2} \qquad M_Z = \frac{M_W}{\cos\theta_W} \tag{1.13}$$

and the two different gauge boson bases are related by

$$Z_\mu = \cos\theta_W W_\mu^3 - \sin\theta_W B_\mu \qquad A_\mu = \sin\theta_W W_\mu^3 + \cos\theta_W B_\mu$$
$$W_\mu^\pm = \frac{W_\mu^1 \mp W_\mu^2}{\sqrt{2}}$$

The number of degrees of freedom is the same before and after electroweak symmetry breaking. When $\langle\varphi\rangle = 0$ the four gauge bosons have two degrees of freedom each, corresponding to the two helicities of a massless spin one field, while the scalar field has four real degrees of freedom. After φ acquires a vev, the scalar field only has one degree of freedom, $h(x)$, but the three massive gauge bosons now have an additional degree of freedom each. It is often said that the three goldstone bosons that one would expect to find after spontaneous symmetry breaking have been eaten by the gauge bosons, causing them to be heavy.

Lastly, the scalar field couples to the fermions via

$$\mathcal{L}_{Yukawa} = -y_d^{IJ}\bar{q}_L^{iI}\varphi_i d_R^J - y_u^{IJ}\epsilon^{ij}\bar{q}_L^{iI}\varphi_j^\dagger u_R^J - y_e^{IJ}\ell_L^{iI}\varphi_i e_R^J + \text{h.c} \tag{1.14}$$

where $I, J = 1, \ldots, n_f$ are family indices, $y_{e,u,d}$ are the Yukawa coupling matrixes and 'h.c' stands for 'hermitian conjugate'; in the Standard Model, $n_f = 3$. Notice that the neutrinos remain massless after electroweak symmetry breaking, due to the lack of a field for the right handed neutrno. The Yukawa couplings are the only objects in the Standard Model Lagrangian that are not flavor universal and so, as we shall see in Sec. 1.3, they encode all of the flavor structure of the Standard Model. The four Lagrangians discussed above combine to give the Lagrangian of the Standard Model:

$$\mathcal{L}_{SM} = \mathcal{L}_{\text{Gauge}} + \mathcal{L}_\varphi + \mathcal{L}_\psi + \mathcal{L}_{\text{Yukawa}} \tag{1.15}$$

With the discovery of the Higgs, all the interactions and particle content of the Standard Model have been quantified and theoretical predictions of the Standard Model prediction are in good agreement with experiment. However, this by no means indicates that particle physics is complete. The nature of Dark Matter, whose existence was initially inferred from galactic rotation curves [10], and the massiveness of neutrinos, necessary for the observed phenomena of neutrino oscillations [11, 12], cannot be explained by Standard Model physics alone. While we will not discuss these topics further, it is important to note that we know that our understanding of particle physics is incomplete, even without considering gravity.

Before moving on to the phenomenological properties of the Standard Model, let me detail some of its important global symmetry properties, starting with discrete symmetries. Specifically, there are three discrete symmetries in which we are interested: charge conjugation (C), parity (P) and time reversal (T). It has been proved that any general Lorentz-invariant quantum field theory that has a Hermitian Hamiltonian cannot violate CPT [13–15], though it can violate any of the symmetries individually. Both CP and T violation has been observed in the electroweak sector. CP violation has been seen in the kaon [16–18] and the B-meson systems, [19–24] while T violation has only been seen in the B-meson system [25]. CPT violation has not been detected. Experimentally, the strong interactions preserve C, P and T individually, even though the Lagrangian for $SU(3)$ Yang Mills with no massless fermions does contain a CP-violating term. This problem, the so-called 'Strong CP Problem' is one of the major unsolved problems of particle physics. We will discuss CP violation is both the electroweak and strong sectors in the following sections.

There are also several continuous symmetries that I would like to mention. The first is fermion number, or perhaps more carefully, baryon number B and lepton number L. Looking at the Standard Model Lagrangian, it appears that it is invariant under both B, which rotates quark fields by the same phase and antiquarks by the opposite phase, and L, which rotates leptons in a similar manner. However, this 'tree-level' result is violated by loop corrections and neither B nor L is a symmetry of the Standard Model, though $B - L$ is. This is one example of an anomalous symmetry, where a symmetry that is present on the classical level, is violated by quantum mechanical effects. We will return to the idea of anomalous symmetries throughout the following work. Lastly, in the limit that the quarks are massless, QCD has a global $SU(N_f)_R \times SU(N_f)_L$, where $SU(N)_{L(R)}$ acts on only the N_f massless left(right)-handed fermions. This symmetry, and the observation that quark masses break this symmetry down to its diagonal subgroup $SU(N_f)_V$, play an important role in understanding the hadronic sector. We will discuss this in the next section.

1.2 Quantum Chromodynamics at Low Energy

The question of how the seemingly simply QCD Lagrangian can give rise to the rich and complex structure of the hadron spectrum is one of the major unsolved problems of modern day particle physics.[2] The difficulty in finding a satisfactory answer is due to the fact that QCD becomes strongly coupled at low energies – at a few GeV, one can no longer trust perturbative calculations. However, all hope of understanding the hadron spectrum is not lost. As the symmetries of the ultraviolet theory, with quarks and gluons as the dynamical degrees of freedom, are reflected in the low energy spectrum, it is possible to derive effective theories that describe the interactions of low mass hadrons. Additionally, numerical simulation of the QCD path integral using Monte Carlo methods allows for the nonperturbative evaluation of matrix elements. In this section, I will focus on the effective theory approach; the topic of numerical simulation using lattice methods will wait until Sec. 1.5.

Let us begin by considering QCD with only three light quark flavors, as this will allow us to understand several key features of the light meson spectrum:

$$\mathcal{L} = -\frac{1}{4} G^C_{\mu\nu} G^C_{\mu\nu} + \bar{u}\left(i\not{D} - m_u\right) u + \bar{d}\left(i\not{D} - m_d\right) d + \bar{s}\left(i\not{D} - m_s\right) s \tag{1.16}$$

For $m_{u,d,s} \neq 0$, the global symmetry of this Lagrangian is $U(3)_V$, with both left handed and right handed fields transforming in the same manner.[3] Also note that the global $SU(3)_V$ contained in $U(3)_V$ is the flavor symmetry and unrelated to the $SU(3)_c$ gauge symmetry. Since the up, down and strange quark all have small masses, let us set $m_{u,d,s} = 0$ for now; we will return to the question of whether this is a reasonable approximation in a little bit. With massless quarks, the global symmetry group is enlarged to $U(1)_V \times U(1)_A \times SU(3)_L \times SU(3)_R$.

[2] It is also an important question in mathematics – so much so that one of the Clay Institutes Millennium Problems concerns proving the existence of the mass gap in the pure gauge theory [26]

[3] Subscript V stands for vector – left and right handed fields have the same transformation properties; A will stand for axial, where left and right have inverse transformation properties

The quarks transform as

$$\begin{aligned} SU(3)_L \times SU(3)_R : &\quad q_L \to \tilde{L} q_L, \quad q_R \to \tilde{R} q_R, \qquad \tilde{L}, \tilde{R} \in SU(3) \\ U(1)_V : &\quad q_L \to e^{i\beta} q_L, \quad q_R \to e^{i\beta} q_R, \qquad \beta = [0, 2\pi] \\ U(1)_A : &\quad q_L \to e^{i\alpha} q_L, \quad q_R \to e^{-i\alpha} q_R, \qquad \alpha = [0, 2\pi] \end{aligned} \tag{1.17}$$

However, this full symmetry is not respected by nature. Instead, the vacuum spontaneously break chiral symmetry, resulting in a non-zero quark condensate,

$$\langle \bar{q}_{RJ} q_{LI} \rangle = \Lambda_\chi^3 \delta_{IJ} \,, \tag{1.18}$$

where $I, J = 1, 2, 3$ are flavor indices. This condensate breaks the global symmetry down to $U(3)_L \times U(3)_R / U(3)_V$. By Goldstone's theorem, with the spontaneous breaking of the chiral symmetry, the low energy spectrum of the theory should contain nine massless pseudoscalars. Does the hadronic spectrum of QCD contain such states?

The lightest states in the hadronic spectrum are mesons, with masses

$$\begin{aligned} m_{\pi^0} = 135 \text{ MeV}, \quad m_{\pi^\pm} = 140 \text{ MeV}, \quad m_{K^\pm} = 494 \text{ MeV}, \\ m_{K^0, \bar{K}^0} = 498 \text{ MeV}, \quad m_\eta = 548 \text{ MeV} \,; \end{aligned} \tag{1.19}$$

The mass scale of these mesons is well separated from that of the baryons, as the lightest baryons, the proton and the neutron, have a mass of a GeV. These mesons are not exactly massless, as one would expect due to Goldstone's theorem, since the quark masses, which explicitly break chiral symmetry, are not-zero. The mass of the up, down, and strange quarks, using dimensional regularization and $\overline{\text{MS}}$ with a renormalization scale of $\mu = 2$ GeV [27], are

$$m_u = 2.3^{+0.7}_{-0.5} \text{ MeV} \qquad m_d = 4.8^{+0.5}_{-0.3} \text{ MeV} \qquad m_s = 95 \pm 5 \text{ MeV} \,. \tag{1.20}$$

If this explicit breaking is small compared to the spontaneous breaking, then the effects of the quark masses can be treated as a small perturbation. The scale of spontaneous chiral symmetry breaking can be estimated by the mass of the lightest baryons, as their mass is not protected by symmetries.[4] As all three quarks have a mass that is much lighter than a

[4] It is a generic property of spontaneous symmetry breaking that particles acquire a mass on the order of the symmetry breaking scale, unless the mass is protected by symmetries. Even in the limit of massless quarks, baryons stay heavy, while the mesons become massless

GeV, it is a reasonable approximation to treat the explicit breaking of the chiral symmetry by quark mass as a small perturbation to spontaneous symmetry breaking. Thus, these light eight mesons are pseudo-Goldstone bosons. But what about the ninth Goldstone bosons one would expect, given the symmetry breaking pattern described above?

There is one more pseudoscalar state, the η', with a mass of 956 MeV; however, due to its heaviness, η' cannot be considered a pseudo-Goldstone boson. If the η' and the eight light mesons all originate from the same symmetry breaking, then the masses of the η' and the pions are related: $m_{\eta'} \leq \sqrt{3} m_\pi$ [28]. This is clearly violated in nature. Nor is it possible that $U(1)_A$, which is the global symmetry whose spontaneous breaking results in a massless η', remains unbroken at low energies. If it was, the hadron spectrum would contain baryon parity doublets. This contradiction between the symmetries of the Lagrangian and the observed hadron spectrum is called the $U(1)_A$ problem; its resolution lies in the realization that $U(1)_A$ is not a symmetry of QCD. We will return to this question later in this section. For now, let us assume that the global symmetry above the GeV scale is just $U_V \times SU(3)_L \times SU(3)_R$.

With this understanding of the global symmetry structure of QCD, it is possible to derive an effective theory, called chiral perturbation theory, for the eight light mesons. In order to construct the Lagrangian for this effective theory, we first need to introduce a field that contains the mesons. First, notice that under $SU(3)_L \times SU(3)_R$, the chiral condensate transforms as

$$\langle \bar{q}_{RJ} q_{LI} \rangle \rightarrow L_{II'} R^\dagger_{J'J} \langle \bar{q}_{RJ} q_{LI} \rangle = \Lambda_\chi^3 (LR^\dagger)_{IJ} \tag{1.21}$$

Such a rotation does not change the vacuum energy, but shifts the theory from one vacua to another; said differently, these transformations rotate the field along the direction of constant (minimum) potential. Recalling that Goldstone bosons are the excitations of a field precisely in this direction and so the light mesons transform in a way that is analogous to the chiral condensate. The form of this field is further constrained by recalling the charges of the mesons under $SU(3)_V$. For example, the pions are an isospin triplet, where isospin is the

$SU(2)$ subgroup. Therefore, the mesons can be parameterized by

$$\Sigma \equiv e^{2i\boldsymbol{\pi}/f}\,, \qquad \boldsymbol{\pi} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \overline{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix} \tag{1.22}$$

where $\boldsymbol{\pi}$ contains the eight meson fields and f is the decay constant, which can be found by looking at the charged pion decay $\pi \to \nu\mu$. Under $SU(3)_L \times SU(3)_R$, Σ transforms as

$$\Sigma \to L\Sigma R^\dagger \tag{1.23}$$

The Lagrangian is constructed by requiring invariance under the symmetries of QCD, including $SU(3)_L \times SU(3)_R$. For example, the kinetic term for the mesons arises from

$$\mathcal{L}_4 = -\frac{f^2}{4}\mathrm{Tr}\,\partial_\mu\Sigma^\dagger\partial^\mu\Sigma \tag{1.24}$$

where the prefactor is chosen such that the mesons have the canonically normalized kinetic term and index '4' indices that this Lagrangian contains only dimension four operators. As this is an effective low energy theory, all higher order terms that are allowed by the symmetries must be included. Examples of such terms are

$$\mathcal{L}_6 \supset \ell_1 \mathrm{Tr}\,\left(\partial_\mu\Sigma^\dagger\partial^\mu\Sigma\right)^2 + \ell_2 \mathrm{Tr}\,\left(\partial_\mu\Sigma^\dagger\partial_\nu\Sigma\right)\left(\partial^\mu\Sigma^\dagger\partial^\nu\Sigma\right) \tag{1.25}$$

where ℓ_i are low energy coefficients whose size can be estimated using naive dimensional analysis (NDA) [29]; their values can be found by matching to experimental data [30–32] or lattice QCD simulations [33, 34]. Since the Lagrangian contains all higher order terms, a power counting scheme is necessary to know which terms are necessary to include and which ones can be safely dropped to achieve the desired precision. A commonly used scheme is Weinberg's momentum scheme [35], which is essentially a derivative expansion; quark masses and gauge interactions can also be included into this power counting scheme, as will be explained below. Since the typical momentum of an on-shell particle is given by its mass and the only other energy scale in the problem, the chiral condensate Λ_χ, is large compared to meson mass scale, a reasonable expansion parameter is p/Λ_χ. Thus, higher dimensional terms such as those contained in $\mathcal{L}_6$ will typically be smaller than lower dimensional ones, such as those in $\mathcal{L}_4$.

Non-zero quark masses can be incorporated into the effective Lagrangian by treating the quark mass matrix, M, as a spurion. In this method, a field that is not dynamical is imbued with the necessary transformations properties such that the symmetry that it breaks remains unbroken.[5] The quark mass term $\bar{q}_R M q_L$ breaks the chiral symmetry $SU(3)_L \times SU(3)_R$ unless M transforms as $M \to RML^\dagger$. Therefore, the effects of explicit breaking of chiral symmetry on the meson spectrum can be analyzed in the low energy theory by introducing the field M into the effective Lagrangian and requiring that it transforms in this way. For example, the four dimensional operator

$$\Lambda_\chi^2 f^2 \left(\frac{c}{2\Lambda_\chi} \mathrm{Tr}\, M\Sigma + \mathrm{h.c} \right) \tag{1.26}$$

must now be included in $\mathcal{L}_4$. It gives the leading order contribution to the meson masses; for example, $m_{\pi^\pm}^2 = (m_u + m_d)\,\tilde{\Lambda}$ where $c\Lambda_\chi \equiv \tilde{\Lambda}$. This result indicates that the light mesons are strongly bound, as the mass of a weakly bound systems depend linearly on the masses of its constituents.

As before, all higher order terms that are singlets under $SU(3)_L \times SU(3)_R$ must be included. For example, $\mathcal{L}_6$ include terms like

$$\ell_4 \mathrm{Tr}\, \left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma \right) \mathrm{Tr}\, \left(M\Sigma + \mathrm{h.c} \right) \tag{1.27}$$

where the label of the low-energy coefficient follows the standard choice given in Ref. [30,31]. The power counting scheme must be expanded to account for insertions of the quark mass matrix. As the typical momentum scale of the mesons is given by their mass and the meson masses squared are linear in the quark masses, it follows that $\mathcal{O}(\partial^2/\Lambda_\chi^2) \sim \mathcal{O}(\tilde{\Lambda}M/\Lambda_\chi^2)$. This relation can be used to determine the relative importance of various operators.

[5] In the Standard Model, the quark mass matrix is dynamical above the electroweak symmetry breaking scale, but non-dynamical below.

Electromagnetism can be added in a similar way. First, the partial derivative is promoted to a covariant derivative, in order to maintain gauge invariance. Additionally, the gauge generators themselves are treated like spurions. For example, the charge matrix for the three light quarks is

$$Q = \frac{1}{3}\begin{pmatrix} 2 & & \\ & -1 & \\ & & -1 \end{pmatrix} \tag{1.28}$$

which breaks $SU(3)_L \times SU(3)_R$. To find how this charge matrix should transform in the low energy theory, look at the electromagnetic interactions of the quarks:

$$\mathcal{L} \supset ieA_\mu \left(\bar{q}_L \gamma^\mu Q q_L + \bar{q}_R \gamma^\mu Q q_R\right) \tag{1.29}$$

In order to maintain gauge invariance, $Q_L \to L Q_L L^\dagger, Q_R \to R Q_R R^\dagger$. This gives rise to another dimension four operator,

$$\mathcal{L}_4 \supset \frac{\alpha}{4\pi} \Lambda_\chi^2 f^2 \,\mathrm{Tr}\, Q_L \Sigma Q_R^\dagger \Sigma^\dagger \tag{1.30}$$

where the prefactor is again approximated using NDA. This term gives rise to the mass splitting between the charged and neutral pions. With the inclusion of Q_{LR} and M, the power counting scheme must be expanded to correctly account for these terms, though we will not discuss this further.

Let us now return to the question of the $U(1)_A$ problem. Recall that there is a contradiction concerning the global $U(1)_A$ symmetry in the hadron spectrum. If $U(1)_A$ is unbroken, the baryons must come in parity doublets; if $U(1)_A$ is spontaneously broken, then there must be nine light pseudoscalar states. As neither of these is true experimentally, one must look more carefully at the properties of QCD with respect to this symmetry. Doing so leads to the discover that while the Lagrangian is invariant under $U(1)_A$ transformations, the path integral measure is not. This anomalous breaking was first shown in Adler, Bell and Jackiw [36, 37] and further elucidated by Fujikawa [38, 39]. To build a more intuitive picture of anomalies, which will prove useful in later sections, let us look a simple derivation of the $U(1)_A$ anomaly in two dimensional QED.

The nonconservation of the $U(1)_A$ current, j^5_μ, in two dimensional QED can be derived by considering the behavior of fermions in a background field. Specifically, let us analyze what happens to the ground state of single flavor of massless Dirac fermions as we adiabatically turn on an electric field and then turn it off again. The dispersion relationship for fermions in this system is $E = \pm p$. It is also important to note that in two dimension, the direction of propagation defines chirality: right-moving particles have positive chirality, whereas left-moving particles have negative chirality.

The initial ground state, before the electric field is turned on, is simply a filled Dirac sea, as shown on the left-handed panel of Fig. 1.1. This ground state has both $Q = 0$ and $Q_A = 0$, where Q is the fermion number charge, $Q \equiv n_R + n_L$, and Q_A is the axial charge, $Q_A = n_R - n_L$. Note that left-moving particles has $n_L = 1$, while a left-moving antiparticle has $N_L = -1$. As the electric field is turned on, right-moving particles are accelerated out of the vacuum and left-moving particles are pushed deeper into the Dirac sea. Therefore, once the electric field has been turned off, the ground state contains right-moving particles and left-moving anti-particles (where we associate holes in the Dirac see with anti-particles); this new ground state is shown on the right-handed panel of Fig. 1.1. Assuming that the Dirac sea is infinitely deep, both N_L and N_R have changed due to the electric field. The new ground state has $Q = 0$, as the same number of right-moving particles and left-moving antiparticles have popped out of the vacuum. This is consistent with the conservation of the $U(1)$ current, $\partial_\mu j^\mu = 0$. However, since both $N_L = -N_R = \tilde{n}$, the axial charge has changed. The $U(1)_A$ current is not conserved and thus $U(1)_A$ is not a symmetry of the theory!

The change in the axial charge can be calculated by considering the theory in a finite volume, $L \times L$, and imposing periodic boundary conditions, which results in quantized momenta, $p_n = 2\pi n/L$. Recalling that the momentum of the fermions is directly related to

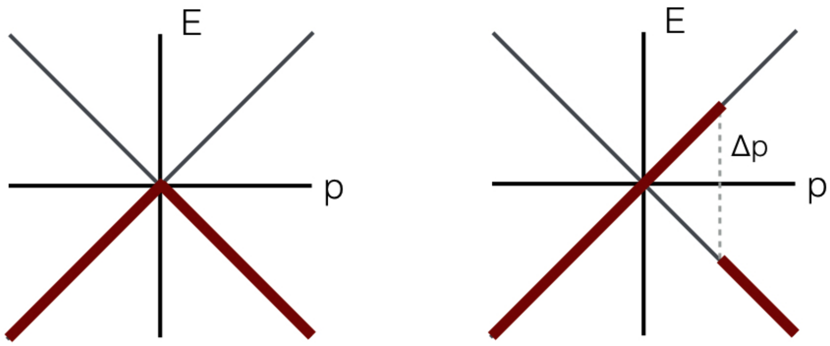

Figure 1.1: The ground state of two dimensional fermions before turning on an electric field is shown on the left; on the right is the ground state after an electric field is turned on and off adiabatically

their chirality, the change in the axial charge is

$$\begin{aligned}\Delta Q_A &= 2 \times \frac{\Delta p}{2\pi/L} \\ &= \frac{e}{\pi}\int dx\, dt\, E(t) \\ &= \frac{e}{2\pi}\int d^2x \epsilon_{\mu\nu} F^{\mu\nu} \end{aligned} \tag{1.31}$$

where the relation $E = F_{01} = -F_{01}$ has been used. From this, the divergence of the axial current is

$$\partial_\mu j_5^\mu = \frac{e}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu} \tag{1.32}$$

as $\Delta Q_A = \int d^2x\, \partial_\mu j_5^\mu$. While this derivation does not work for four dimensional theories, one can either calculate the one-loop correction to the axial current (the famous triangle diagrams of Adler-Bell-Jackiw) or calculate the change in the path integral measure due to an axial rotation (the Fujikawa method) to derive the divergence of the axial current for QED and QCD:

$$\begin{aligned}\partial_\mu j^\mu_{5,\mathrm{QED}} &= -\frac{e^2}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} \\ \partial_\mu j^\mu_{5,\mathrm{QCD}} &= -\frac{e^2}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\, G^{\mu\nu}G^{\rho\sigma}\end{aligned} \tag{1.33}$$

As the violation of $U(1)_A$ only occurs at the quantum level,[6] it is an example of an anomalous symmetry. Looking back at the calculation of the change in axial charge in the two dimensional QED example, we had to assume that the Dirac sea is infinitely deep. If it only had a finite depth, i.e. a fixed number of negative energy states, $N_{L,R}$ would not be changed by the action of the electric field and $\Delta Q_A = 0$. The existence of an infinitely deep Dirac sea, also called a Hilbert hotel, is purely a quantum phenomenon. This observation, that anomalies are only present in theories with an infinite number of degrees of freedom, will come into play when discussing the difficulty of lattice regulated chiral gauge theories in Sec. 1.7.

Returning back to the $U(1)_A$ problem and the mystery of the missing ninth pseudo-Goldstone boson or baryon parity doublets. The resolution is that $U(1)_A$ is not a true symmetry of QCD and the global symmetry before chiral symmetry breaking is simply $U(1)_V \times SU(3)_L \times SU(3)_R$. The discovery of the anomalous nature of $U(1)_A$, while resolving the conundrum in the hadron spectrum, gave birth to a new problem. The QCD Lagrangian shown above is missing a term,

$$\mathcal{L}_\theta = -N_f \theta \frac{g}{32\pi^2} G^{C\mu\nu} \tilde{G}^C_{\mu\nu} \tag{1.34}$$

where N_f is the number of massive fermions. Under an axial rotation on the quark masses with phase α, the value of θ is shifted by $\theta \to \theta - 2N_f\alpha$, which shows explicitly that $U(1)_A$ is not a symmetry of QCD. Additionally, this term arises due to the fact that $SU(3)_c$ in four dimensions has nontrivial topology, due to the fact that the profiles of different gauge field configurations at infinity cannot all be related by a continuous gauge transform. 't Hooft showed that these same configurations give rise to an effective operator that has the correct quantum numbers for an η' mass [40,41], allowing it to be much heavier than expected. The cause of the heaviness of the η' can also be elucidated by looking at large N_c QCD [42].

Now, for the new mystery. Notice that this θ-term violates CP invariance and is the

[6] The axial anomaly cannot be seen in tree-level calculations. In four dimensions, it is seen either in loop corrections or the path integral measure

only term in the QCD Lagrangian that does so. Therefore, the amount of CP violation in the strong sector is quantified $\bar{\theta} \equiv \theta + \arg\det M$, which is reparameterization invarient. The amount of CP violation can be quantified by looking at such observables as the neutron electric dipole moment (EDM), where the dependence of the neutron EDM on $\bar{\theta}$ can be found by expanding chiral perturbation theory to include baryons [43]. Current bounds [27] on the neutron EDM are $d_n < 2.9 \times 10^{-26}\,\text{e cm}$, which translates into $\bar{\theta} \lesssim 10^{-10}$, ignoring contributions from higher dimensional operators. Since in the Standard Model, $\bar{\theta} = [0, 2\pi]$, this is an incredible amount of fine tuning. The question of why the CP violation in the strong sector is so unnaturally small is called the Strong CP problem. There are several proposed resolutions, such as the existence of massless colored fermions or the axion [44–47], though no solution has been experimentally verified.

The global $U(1)_A$ anomaly does not only explain the lack of both a ninth PGB and baryon parity doublets. It might also play a key role in an open problem in astrophysics, concerning the magnetic fields of neutron stars. Based on experimental observation, a certain class of neutron stars, called magnetars, have extremely large surface magnetic fields, $B_s \simeq 10^{14} - 10^{15}$ Gauss; for reference, the Earth's magnetic field is about a Gauss. It is not yet known definitively how these large magnetic fields are generated. In Ch. 3, I will discuss a proposal that makes use of the global $U(1)_A$ anomaly. The mechanism depends on the fact that during core collapse, protons are converted into neutrons via electron capture; however, as this process is mediated by the weak interactions, only left-handed electrons are captured. This creates an imbalance between the left-handed and right-handed fermions, which can be converted into a magnetic field via the anomaly. I will focus on the question of whether a magnetic field of sufficient strength can be generated if the mass of electron is not assumed to be irrelevant.

1.3 Flavor Physics

In the Standard Model, the Yukawa couplings are the only parameters that break flavor universality and so all of the flavor structure must be encoded in them. For three families of fermions and ignoring the neutrino masses, the Yukawa couplings contain nine masses, three mixing angles and one CP violating phase. The values of these parameters can only be determined experimentally and so one of the major open problems in particle physics is how this particular flavor structure arises. This question is especially pertinent, as the flavor structure itself is highly nontrivial. The quark masses range from a few MeV to almost 200 GeV; the mixing between the three families is almost completely isolated to mixing between the two lightest families.

In order to address the generation of the Standard Model's flavor structure, one must look towards Beyond the Standard Model (BSM) physics. However, constructing models that attempt to do so, particularly models that also have new physics at the TeV scale (and are thus testable) is not easy. This is due to the stringent constraints imposes by precision electroweak experiments. In this section, I will first look at the flavor structure of the Standard Model in more detail, before discussing some relevant flavor phenomenology. I will mostly focus on the neutral kaon system, as not only does it provide a clear example of the Standard Model's highly nontrivial flavor structure, but also as it provides several constraints that prove difficult for BSM models to overcome. This section provides the background and the motivation for the longer discussion of Ch. 2 which details work on a BSM flavor model called Little Flavor.

After electroweak symmetry breaking, the Yukawa interactions have the form

$$\mathcal{L}_{Yukawa} = -\frac{y_d^{IJ}}{\sqrt{2}}\,(v+h)\,\bar{d}_L^I d_R^J - y_u^{IJ}\,(v+h)\,\bar{u}_L^I u_R^J + \text{h.c.} \tag{1.35}$$

where $I, J = 1, 2, \ldots n_f$ are flavor indices and we focus only on the quark sector. We ignore the lepton sector for now, as the Standard Model has no lepton flavor violation due to the lack of a right handed neutrino field. It is know that neutrinos are not massless, due to neutrino flavor oscillations, and so the Standard Model must be, at the minimum, augmented with new interactions that allow for neutrino masses. However, I will ignore this question for most of the remainder of this section.[7]

[7]If the Standard Model is only extended to include neutrinos masses and no other new physics, lepton

The later two terms in Eq. 1.35 are quark mass terms

$$M_u^{IJ} = \frac{y_u^{IJ} v}{\sqrt{2}} \qquad\qquad M_d^{IJ} = \frac{y_d^{IJ} v}{\sqrt{2}} \tag{1.36}$$

which do not have to be hermitian, and in fact, are not. They are diagonalized by introducing four unitary matrices: $\mathcal{U}_L, \mathcal{U}_R, \mathcal{D}_L, \mathcal{D}_R$

$$d'_L = \mathcal{D}_L d_L \qquad d'_R \mathcal{D}_R d_R \qquad u'_L \mathcal{U}_L u_L \qquad u'_R \mathcal{U}_R u_R \tag{1.37}$$

where d', u' are the quark fields in the mass basis. These matrices are chosen such that $\mathcal{D}_R y_d \mathcal{D}_L^\dagger$ and $\mathcal{U}_R y_u \mathcal{U}_L^\dagger$ are diagonal with real, positive eigenvalues. Since the neutral currents only couple quarks of the same flavor, this change of basis does not affect them. This is not true for the charged currents, as they couple quarks of different electric charge. In the mass basis, the charged current contribute to the Lagrangian as

$$\mathcal{L} \supset \frac{1}{\sqrt{2}} g_2 \left[W_\mu^+ \, \bar{u}_{LI} V_{IJ} \gamma^\mu d_{RJ} + W_\mu^- \, d_{LI}^\dagger \left(V^\dagger \right)_{IJ} \gamma^\mu u_{LJ} \right] \tag{1.38}$$

where $V \equiv \mathcal{U}_L \mathcal{D}_L^\dagger$ is the Cabibbo-Kobayasi-Maskawa (CKM) matrix [48, 49]. This matrix, along with the six quark masses, contains all the flavor structure of the Standard Model.

As the CKM matrix encodes all of the flavor changing processes, it is important to quantify the number of physically relevant parameters it contains. A general $n_f \times n_f$ unitary matrix has n_f^2 free parameters, of which $n_f(n_f - 1)/2$ are angles and $n_f(n_f + 1)/2$ are phases. However, since the charged currents only couple to left handed fields, $2n_f - 1$ of these phases can be eliminated by the rotating the left-handed and right-handed quark fields by the same phase.[8] For n_f generations, the CKM matrix contains $n_f(n_f - 1)/2$ angles and $(n_f - 1)(n_f - 2)/2$ phases; for three generations, this is three angles and one phase. A standard parameterization [27] of CKM matrix in terms of these parameters is

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \tag{1.39}$$

flavor violating processes remain unobservable, as they suppressed by the minuscule neutrino mass.

[8]If the right handed quarks also coupled to charged currents, these phases would reappear in the right handed charged currents

where $s_{ij} = \sin\theta_{ij}$ and $c_{ij} = \cos\theta_{ij}$; experimentally, $s_{13} \ll s_{23} \ll s_{12} \ll 1$. Since there is only one phase, δ, in the CKM matrix, this phase is responsible for all flavor changing CP violation.[9] As mentioned previously, there is no CKM-like matrix in the lepton sector as long as no right handed neutrino field is introduced, as the matrix always be reparameterized to be unity. When a mechanism for generating neutrino masses is added to the Standard Model, the lepton sector also has a CKM like matrix, called the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix [11, 12].

Due to the existence of the CKM matrix, quark flavor mixing occurs via interactions with the W boson. As stated above, the quark coupling to the Z is flavor universal: the Standard Model has no tree-level flavor changing neutral currents(FCNC). However, there are loop level contributions to FCNC via the exchange of W's. These contributions are extremely suppressed due to the Glashow-Iliopoulos-Maiani (GIM) mechanism [50] and so place tight bounds on BSM models of flavor physics. These FCNCs can be seen, for example, in neutral kaon oscillations.

Ignoring electroweak interactions, kaons are doublets of strong isospin and stable:

$$K^+ = u\bar{s} \qquad K^0 = d\bar{s} \qquad \overline{K}^0 = s\bar{d} \qquad K^- = s\bar{u} \tag{1.40}$$

Once electroweak interactions are turned on, both K^0 and $\overline{K}^0$ are able to decay into two pions (either $\pi^+\pi^0$ or $\pi^0\pi^0$) or three pions ($\pi^+\pi^-\pi^0$ and $\pi^0\pi^0\pi^0$). Under CP, the neutral kaons transform as

$$CP|K^0\rangle = \eta|\overline{K}^0\rangle \qquad CP|\overline{K}^0\rangle = \bar{\eta}|K^0\rangle \tag{1.41}$$

where the convention here is that $\eta = -1$ and so the CP eigenstates are

$$|K_1\rangle = \frac{|K^0\rangle - |\overline{K}^0\rangle}{\sqrt{2}} \qquad |K_2\rangle = \frac{|K^0\rangle + |\overline{K}^0\rangle}{\sqrt{2}} \tag{1.42}$$

However, since the Standard Model does not preserve CP, the mass eigenstates are a mixture of the CP-even and CP-odd eigenstates,

$$|K_S\rangle = \frac{1}{\sqrt{1+|\epsilon|^2}}\left(|K_1\rangle + \epsilon|K_2\rangle\right) \qquad |K_L\rangle = \frac{1}{\sqrt{1+|\epsilon|^2}}\left(|K_2\rangle + \epsilon|K_1\rangle\right) \tag{1.43}$$

[9] CP violation in the strong sector does not change flavor

where the L and S stand for 'long' and 'short'. As $\epsilon \ll 1$, K_S decays mostly to two pions and K_L intro three pions; the choice of 'long' and 'short' is due to the fact that the two pion decay has a much shorter mean lifetime.

The masses of K_L and K_s are not degenerate due to flavor changing processes. The mass difference can be calculated using an effective Hamiltonian; to simplify the analysis, the third family of quarks shall be ignored for now. As there is no CP-violation in the Standard Model with only two quark families, the theory obeys both CP and CPT invariance. The effective Hamiltonian for the neutral kaon system in the strong interaction basis is

$$\mathcal{H} = \begin{pmatrix} M_0 - i\frac{1}{2}\Gamma_0 & M_{12} - \frac{1}{2}i\Gamma_{12} \\ M_{12} - \frac{1}{2}i\Gamma_{12} & M_0 - i\frac{1}{2}\Gamma_0 \end{pmatrix} \tag{1.44}$$

where M encodes all unitary interactions, i.e. those whose external states are only $|K^0\rangle, |\overline{K}^0\rangle$. Γ contains interactions that allow the decay of $|K^0\rangle, |\overline{K}^0\rangle$ into states not included in the space of this Hamiltonian, such as the decay into the pions. The parameters M_0, M_{12}, Γ_0 and Γ_{12} are all real, due to CPT and CP invariance. By comparing the experimental values for the mass differences between the two eigenstates of this system,

$$M_S - \frac{i}{2}\Gamma_S = M_0 - M_{12} - i\frac{\Gamma_0 - \Gamma_{12}}{2} \qquad M_L - \frac{i}{2}\Gamma_L = M_0 + M_{12} - i\frac{\Gamma_0 + \Gamma_{12}}{2} \tag{1.45}$$

and Standard Model prediction, the contributions to this $\Delta S = 2$ FCNC from BSM models of flavor can be constrained. To see just how stringent this constraint, we need to calculate the leading order Standard Model contributions, which are the famous 'Box Diagrams', shown in Fig. 1.2

As the third family of quarks is being ignored for this analysis, the scattering amplitude has contributions from the exchange of two up quarks, two charm quarks or one up quark and one charm quark,

$$\begin{aligned} \mathcal{M}^{K\overline{K}} &= \mathcal{M}_{uu} + \mathcal{M}_{cu} + \mathcal{M}_{uc} + \mathcal{M}_{cc} \\ &= (V_{ud}V_{us}^*)(V_{ud}V_{us}^*)\mathcal{F}(m_u, m_u) + (V_{cd}V_{cs}^*)(V_{ud}V_{us}^*)\mathcal{F}(m_c, m_u) \\ &+ (V_{ud}V_{us}^*)(V_{cd}V_{cs}^*)\mathcal{F}(m_u, m_c) + (V_{cd}V_{cs}^*)(V_{cd}V_{cs}^*)\mathcal{F}(m_c, m_c) \end{aligned} \tag{1.46}$$

where in the second line the dependence on CKM matrix elements has been stripped off of $\mathcal{M}_{IJ}$. $F(m_I, m_J)$ is the Feynman integral that must be evaluated for the one loop box diagram. As the CKM matrix is unitary

$$V_{ud}V_{us}^* + V_{cd}V_{cs}^* = 0 \tag{1.47}$$

If the up and charm masses were degenerate, $m_{u,c} = m$, and the one loop Standard Model contribution to $\Delta S = 2$ would be identically zero!

$$\begin{aligned}\mathcal{M}^{K\overline{K}} &= (2(V_{ud}V_{us}^*)(V_{ud}V_{us}^*) - (V_{ud}V_{us}^*)(V_{ud}V_{us}^*))\, F(m,m) \\ &= 0\end{aligned} \tag{1.48}$$

This is the GIM mechanism, which was used in the prediction of the charm quark [51, 52].

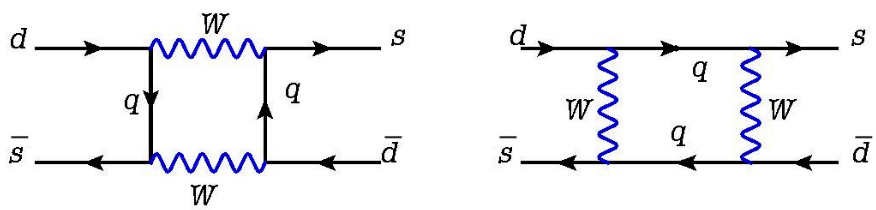

Figure 1.2: Leading order Standard Model contribution to neutral kaon mass difference

In reality, the charm and the up quark mass are not degenerate; however, both are much lighter than the mass of the W boson and the scale of electroweak breaking. Therefore, this explicit symmetry breaking is small and the matrix element is expected to be small due to symmetry protection. Assuming $m_u \sim m_d \ll m_c \ll M_W$, the mass difference is

$$M_L - M_S = \Delta M = \frac{G_F^2 m_c^2}{3\pi^2} f_K m_K \sin^2\theta_c \cos^2\theta_c \tag{1.49}$$

where $\theta_c \simeq \theta_{12}$ is the Cabibbo angle, G_F is Fermi's constant, m_c is the charm quark mass and m_k, f_k are the kaon mass and decay constant, respectively. If the third family is not ignored, ΔM also has a contribution from the third family that goes as

$$\frac{G_F^2 m_t^2}{3\pi^2} f_K m_K |V_{td}V_{ts}^*| \tag{1.50}$$

which, due to the smallness of $V_{td,ts}$, is smaller than Eq. 1.49. The convention for the decay constant is

$$\langle 0|\bar{d}\gamma^{\mu}(1-\gamma_5)s|\overline{K}^0\rangle \equiv i\sqrt{2}f_k p_K^{\mu} \tag{1.51}$$

where p_K^{μ} is the momentum of $\overline{K}^0$. Using experimental values for G_F, m_K, θ_c and f_k, along with the current mass extracted from the masses and decay rates of heavy hadrons [27], the neutral kaon mass difference is

$$\Delta M \sim 6.8 \times 10^{-12}\ \text{MeV} \qquad \Delta M_{\text{exp}} = (3.484 \pm 0.006) \times 10^{-12}\ \text{MeV} \tag{1.52}$$

The mass difference[10] is remarkably small, considering that $m_K \sim 500$ MeV. Since the explicit breaking of the quark mass degeneracy due to $m_c - m_u \neq 0$ is small compared to the scale of electroweak symmetry breaking, the GIM mechanism provides incomplete protection for ΔM.

With the calculation of the Standard Model contribution to the $\Delta S = 2$ FCNCs, the contribution from BSM can now be constrained. This can be done by constructing an effective Hamiltonian for the BSM model,

$$\mathcal{H}_{BSM} = \sum_{i=1}^{N} \mathcal{C}_i \mathcal{O}_i \tag{1.53}$$

where $\mathcal{O}_i$ are the operators of the effective Hamiltonian and $\mathcal{C}_i$ are the Wilson coefficients. The operators and coefficients are found in the following manner. Consider a BSM model where quarks interact via the exchange of some new heavy particles, for example, a W' or Z'. At energy scales below their masses, the new particles are integrated out; this results in a set of four-fermion operators.[11] As there is a finite number of unique four-fermion operators, to

[10] One may worry about the factor of two difference between the experimental value and the 'back of the envelope' theoretical prediction of ΔM. Once the third family is included, as well as long-distance contributions, experiment and prediction match; see [53–55] and references therein.

[11] This is analogous to Ferm's theory of the weak interactions [56, 57], where the four fermion operators are actually the result of W boson exchange (though the W boson was several decades away from being predicted or discovered). For an English translation of Fermi's original work, see [58].

define an effective Hamiltonian correctly, one must choose a basis of operators. For example, for the $\Delta S = 2$ process that contributes to neutral kaon oscillations, the operators of the effective Hamiltonian are

$$\begin{aligned}
Q_1 &= \left(\bar{d}^a_L \gamma_\mu s^a_L\right)\left(\bar{d}^b_L \gamma_\mu s^b_L\right), & \tilde{Q}_1 &= \left(\bar{d}^a_R \gamma_\mu s^a_R\right)\left(\bar{d}^b_R \gamma_\mu s^b_R\right), \\
Q_2 &= \left(\bar{d}^a_L s^a_R\right)\left(\bar{d}^b_L s^b_R\right), & \tilde{Q}_2 &= \left(\bar{d}^a_R s^a_L\right)\left(\bar{d}^b_R s^b_L\right), \\
Q_3 &= \left(\bar{d}^a_L s^b_R\right)\left(\bar{d}^b_L s^a_R\right), & \tilde{Q}_3 &= \left(\bar{d}^a_R s^b_L\right)\left(\bar{d}^b_R s^a_L\right), \\
Q_4 &= \left(\bar{d}^a_L s^a_R\right)\left(\bar{d}^b_R s^b_L\right), \\
Q_5 &= \left(\bar{d}^a_L s^b_R\right)\left(\bar{d}^b_R s^a_L\right),
\end{aligned} \tag{1.54}$$

where $a, b = 1, 2, 3$ are color indices and the parentheses are used to visually separate the two quark bilinears present in each operator.

The Wilson coefficients, $\mathcal{C}_i$ are model dependent – they are found by matching matrix elements in the full theory to matrix elements in the effective theory at some matching scale. This matching scale is usually taken to be below the mass scale of the new particles, but above the mass scale of any Standard Model field. As these coefficients depend on the parameters of the BSM model, such as coupling constants and particle masses, comparing their values to constraints derived from experiment allows one to constrain the range of parameters for the BSM model. A thorough example of this process is given in [59].

To demonstrate how severely the neutral kaon mass difference constraints BSM models, consider a BSM model that has no mechanism for FCNC suppression and whose only non-zero Wilson coefficient is $\mathcal{O}_1$. For this theory, the Wilson coefficient is $\mathcal{O}(1)$, as there are no symmetries or mechanisms, such as the GIM mechanism, to suppress it. Therefore, the effective Hamiltonian is

$$\mathcal{H}_{\text{eff}} \sim \frac{1}{\Lambda^2_{NP}} \left(\bar{d}_L \gamma^\mu s_L\right)\left(\bar{d}_L \gamma^\mu s_L\right) \tag{1.55}$$

where parentheses are again used for visual clarity and Λ_{NP} is the scale of new physics; for example, it could be the mass of the new particles. This model's contribution to the kaon mass difference is roughly

$$\Delta M_{\text{BSM}} \sim \frac{f^2_K}{\Lambda^2_{NP}} m_K, \tag{1.56}$$

Operator	Bounds on Λ in TeV ($c_{NP}=1$)	Bounds on c_{NP} (Λ =1 TeV)	Observables
$(\bar{s}_L\gamma_\mu d_L)^2$	$(9.8\times10^2)+i\,(1.6\times10^4)$	$(9.0\times10^{-7})+i\,(3.4\times10^{-9})$	$\Delta m_K,\epsilon_K$
$(\bar{s}_R d_L)\,(\bar{s}_L d_R)$	$(1.8\times10^4)+i\,(3.2\times10^5)$	$(6.9\times10^{-9})+i\,(2.6\times10^{-11})$	$\Delta m_K,\epsilon_K$
$(\bar{c}_L\gamma_\mu u_L)^2$	$(1.2\times10^3)+i\,(2.9\times10^3)$	$(5.6\times10^{-7})+i\,(1.0\times10^{-7})$	$\Delta m_D, \lvert q/p\rvert, \phi_D$
$(\bar{c}_R u_L)\,(\bar{c}_L u_R)$	$(6.2\times10^3)+i\,(1.5\times10^4)$	$(5.7\times10^{-8})+i\,(1.1\times10^{-8})$	$\Delta m_D, \lvert q/p\rvert, \phi_D$
$(\bar{b}_L\gamma_\mu d_L)^2$	$(6.6\times10^2)+i\,(9.3\times10^2)$	$(2.3\times10^{-6})+i\,(1.1\times10^{-6})$	$\Delta m_{B_d}, S_{\psi K_s}$
$(\bar{b}_R d_L)\,(\bar{b}_L d_R)$	$(2.5\times10^3)+i\,(3.6\times10^3)$	$(3.9\times10^{-7})+i\,(1.9\times10^{-7})$	$\Delta m_{B_d}, S_{\psi K_s}$
$(\bar{b}_L\gamma_\mu s_L)^2$	$(1.4\times10^2)+i\,(2.5\times10^2)$	$(5.0\times10^{-5})+i\,(1.7\times10^{-5})$	$\Delta m_{B_s}, S_{\psi\phi}$
$(\bar{b}_R s_L)\,(\bar{b}_L s_R)$	$(4.8\times10^2)+i\,(8.3\times10^2)$	$(8.8\times10^{-6})+i\,(2.9\times10^{-6})$	$\Delta m_{B_s}, S_{\psi\phi}$

Table 1.2: Bounds on several flavor-changing four-fermion operators, where the Wilson coefficient is assumed to be c^n_{NP}/Λ^2. The right most column details the main observables that are used to constrain the Wilson coefficients.

which sets a lower bound on the scale of new physics of $\Lambda_{NP}\gtrsim 10^3$ TeV, assuming that the BSM contribution cannot be any larger than the Standard Model contribution and we have used the bounds from Ref. [55]. Other constraints, coming from the neutral kaon, B meson and other system also give similarly high scales for new physics; see Table 1.2; the bounds are from Ref [55]. While there is nothing inherently wrong about such a high scale for new physics, it does mean that if there is new physics at the TeV scale, it will not contain any new flavor structure and so will be unable to resolve the question of how the flavor structure of the Standard Model is generated. If is clear that if a TeV scale model of new physics hopes to help elucidate this question, it must have mechanisms for suppressing FCNCs.

1.4 Regularization and Renormalization

One of the first stumbling blocks in the formal development of Quantum Field Theory was the realization that naive calculations of matrix elements led to unphysical divergences due to divergences in loop diagrams. For example, due to interactions with photons, the electron self-energy $\Sigma(p)$ receives a correction of the form

$$i\Sigma(p) = e^2 \int \frac{d^4\ell}{(2\pi)^4} \gamma^\nu S(p+\ell)\, \gamma^\mu \Delta_{\mu\nu}(\ell) \tag{1.57}$$

where $S(k), \Delta(k)$ are the electron and photon propogators, respectively, in the mostly plus Minkowski metric,

$$S(k) = \frac{-i\,(-\not{k} + m)}{k^2 + m^2 - i\epsilon} \qquad \Delta(k) = \frac{-ig_{\mu\nu}}{k^2 + m_\gamma^2 - i\epsilon} \tag{1.58}$$

and the theory has been gauged fixed using Feynman gauge. The fictitious photon mass, m_γ, used as an infrared (IR) cutoff and must be taken to zero after renormalization. Using the standard Feynman parameter method and shifting the loop momenta $q = \ell + xp$, the self-energy can be written as

$$i\Sigma(p) = e^2 \int_0^1 dx \int \frac{d^4q}{(2\pi)^4} \frac{-(1-x)\not{p} - 4m}{\left(q^2 + x(1-x)p^2 + xm^2 + (1-x)m_\gamma^2\right)^2} \tag{1.59}$$

where the term linear in $\not{q}$ evaluates to zero, due to the rotational invariance of the integral measure and denominator. At large loop momenta, the integral is divergent[12] and Σ contributes an infinitely large correction to the electron self-energy. Such an answer is not physical.

The electron self-energy can be made finite by controlling the ultraviolet (UV) behavior of the integral. One way to do this is by simply placing a hard cutoff on the loop momenta, taking the maximum momenta to have some large, but finite, value. Another way is by adding a new particle to the theory, one that has a large mass and the same gauge couplings as the photon, but opposite statistics [60–63]. Adding such a field, referred to as a Pauli-Villars field, has the effect of shifting the photon propagator in the electron self-energy diagram by

$$\frac{1}{k^2 + m_\gamma^2 - i\epsilon} \implies \frac{1}{k^2 + m_\gamma^2 - i\epsilon} - \frac{1}{k^2 + \Lambda^2 + m_\gamma^2 - i\epsilon}, \tag{1.60}$$

which cut offs the integral for $q \gtrsim \Lambda$. Yet another possibility is dimensional regularization ('dim-reg'), where the divergent integral is analytically continued to arbitrary dimension [64].

[12] Specifically, log divergent

As decreasing the dimension always increases the convergence of an integral, the integral's divergences can be quantified in inverse powers of $4-d$. Methods that modify the UV physics of a theory[13] in order to have finite results for physical observables are called regulators. A fourth regulator, the lattice, is discussed in the remaining sections of this chapter.

A good regulator must not only control the UV divergences of a theory, but it must also leave the symmetry structure of the theory unaltered. For example, a regulated vacuum polarization diagram, also knows as the photon self-energy, must obey the Ward-Takahashi identity [66, 67]

$$q^\mu \Pi_{\mu\nu} = 0 = q^\nu \Pi_{\mu\nu} \tag{1.61}$$

to all orders in the gauge coupling. This identity is a statement about the conservation of the current associated with a gauge transformation; if Eq. 1.61 is violated, so is gauge invariance. With a hard cutoff in place, the vacuum polarization for QED is

$$\Pi^{\mu\nu} \propto e^2 \Lambda g^{\mu\nu} \tag{1.62}$$

and so the theory has a broken gauge symmetry. Therefore, a hard cutoff, which also violates Lorentz invariance, cannot be used as a regulator for gauge theories with fermions.

Using dimension regularization, the electron self-energy, now finite for $\epsilon \neq 0$, is

$$\Sigma(p) = -\frac{e^2}{8\pi^2} \int_0^1 dx \, \left[(2-\epsilon)(1-x)\not{p} + (4-\epsilon)\, m\right] \left[\frac{1}{\epsilon} - \frac{1}{2}\log\left(B/\mu\right)\right] \tag{1.63}$$

where $B = x(1-x)p^2 + xm^2 + (1-x)m_\gamma^2$ and $\epsilon = 4-d$; this ϵ should not be confused with the one in the fermion and photon propagators which is responsible for providing the pole prescription. The scale μ is called the renormalization scale.

The dependence on a regulator, and thus the divergences in physical observables, can be removed via renormalization, which is a rescaling of fields, masses and coupling constants. Renormalization is also necessary to ensure that time ordered n-point correlation functions

[13] For a discussion of how dimensional regularization controls both the UV and IR divergences of a theory, see Ref. [65]

have the necessary analytic structure, as the Lehman-Symanzik-Zimmermann (LSZ) reduction formula relates the time-ordered correlation functions to S-matrix elements based on the analytic structure of both [68]. For example, the electron self-energy one loop diagram shifts the location of the pole of the two-point function away from m, the parameter that appears in the Lagrangian; this also shifts the residue, which is related to the probability amplitude of creating a single particle state out of the vacuum.

The renormalized Lagrangian for QED has the form

$$\mathcal{L} = -\frac{Z_3}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}\left(iZ_2\partial\!\!\!/ + eZ_1 A\!\!\!/ - Z_m m\right)\psi \tag{1.64}$$

where Z_1, Z_2, Z_3, Z_m are the electric charge, electron field strength, photon field strength and electron mass renormalization, respectively. This renormalized Lagrangian can be rewritten in terms of the original unrenormalized Lagrangian, which has UV divergences, and a finite number of counterterms,[14]

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}\left(iD\!\!\!\!/ - m\right)\psi \\ &- \frac{\delta_3}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}\left(i\delta_2\partial\!\!\!/ + e\delta_1 A\!\!\!/ - \delta_m m\right)\psi \end{aligned}\tag{1.65}$$

where $\delta_i = Z_i - 1$ are the counterterm coefficients. These counterterm coefficients can be determined by requiring that physical observables be finite once the regulator is removed; the manner in which this is done depends on the renormalization scheme.

One choice is the 'on-shell' renormalization scheme, where the counterterms are chosen such that the following four conditions are satisfied,

$$\Sigma(p = im) = 0\,, \quad \partial_p\,\Sigma(p)\big|_{p=im} = 0\,, \quad \Pi(q^2 = 0) = 0\,, \quad ie\Gamma^\mu(p' = p) = -ie\gamma^\mu\,, \tag{1.66}$$

where Γ^μ is the electron vertex correction. The first condition sets the electron mass to be m, the next two fix the electron and photon propagator residues to be one, and the last one sets the electric charge to be e. Another renormalization scheme, which is used for dimensional

[14]QED is a renormalizable theory and this has a finite number of counterterms

regularization, is the minimal subtraction (MS) scheme, where the counterterm coefficients are chosen to cancel off the divergences and nothing more; in modified minimal subtraction, $\overline{\text{MS}}$, the coefficients also subtract off certain finite terms. Returning to the example of the electron self-energy, in $\overline{\text{MS}}$, the electron self-energy Σ is finite and independent of ϵ at $\mathcal{O}(e^4)$ if δ_m is set to be

$$\delta_m = -\frac{e^2}{8\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right) + \mathcal{O}(e^4)\,. \tag{1.67}$$

A well-defined QFT, one that predicts finite values for physical observables, needs to be both regulated and renormalized. The methods of doing so that are described in this section are all perturbative and can be applied to the Standard Model, which is known to be perturbatively renormalizable.[15] The behavior of the Standard Model with regards to nonperturbative regularization and renormalization is not fully resolved and will be discussed in the following sections.

1.5 Basics of Lattice Field Theory

Lattice field theories are quantum field theories in Euclidean spacetime where spacetime itself is discretized, forming a lattice [69]. There are several key benefits to working with a discretized, instead of continuous, theory:

- QFTs defined on the lattice are mathematically rigorous, as the full path integral is regulated and does not suffer from such problematic features as an infinite dimensional Hilbert space.
- The path integral of certain lattice field theories is amenable to numerical simulation, allowing for the evaluation of matrix elements in some strongly coupled theories
- The lattice acts as a nonperturbative regulator, controlling divergences both in the Lagrangian and in the measure

[15] Though there is some question of whether dimensional regularization successfully regulates the electroweak sector to all orders, as no explicit proof exists.

These properties will be demonstrated in the following several sections. This section focuses on how to construct a lattice version of a continuous Euclidean path integral and is heavily influenced by Refs. [70] and [71]. Of particular important is the manner in which scalar, fermion and gauge fields are implemented on the lattice, as well as the way in which the path integral measure is regulated. The following section will address the question of numerical simulation.

Take the simplest example of QFT, a real single scalar field. The action for this theory is

$$S = \int_{\tau_1}^{\tau_2} dx_4 \int d^3x \, \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 \tag{1.68}$$

where $x = (\mathbf{x}, x_4)$ and x_4 is the imaginary time coordinate. The partition function is

$$Z = \int D\phi e^{-S} \tag{1.69}$$

where $D\phi$ stands for the integral over all functions that are periodic over the time extend. For continuous spacetime, there are an infinite number of such paths.

To discretize, constrain x_μ to a hypercubic lattice, $x_\mu = an_\mu$, where a is the lattice spacing and has units of length. Instead of a smooth function over spacetime, the field is now defined by its value at each discrete spacetime point. $\phi(x) \to a\hat{\phi}_n$. The variable n labels each site on the lattice and $\hat{\phi}_n$ is dimensionless. The quadratic and quartic terms are therefore

$$\begin{aligned} m^2\phi(x) &\rightarrow \frac{1}{a^4}\hat{m}\hat{\phi}_n \\ \lambda\phi^4 &\rightarrow \frac{1}{a^4}\hat{\lambda}\hat{\phi}^4 \end{aligned} \tag{1.70}$$

To discretize the kinetic term, recall that a derivative is defined by

$$\partial_x f(x) = \lim_{\delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{1.71}$$

from which it is clear that

$$\partial_\mu \phi(x) \rightarrow \frac{1}{a^2}\left(\hat{\phi}_{n+\hat{\mu}} - \hat{\phi}_n\right) \tag{1.72}$$

where $\hat{\mu}$ defines a single step in the x_μ direction. Lastly, the integral over the spacetime four-volume simply becomes a sum over all lattice points:

$$\int d^4x \;\; \rightarrow \;\; a^4 \sum_n \tag{1.73}$$

Putting this all together, the action is

$$S = \sum_n \left(\hat{\phi}_{n+\hat{\mu}} - \hat{\phi}_n\right)\left(\hat{\phi}_{n+\hat{\mu}} - \hat{\phi}_n\right) + \frac{\hat{m}^2}{2}\hat{\phi}^2 + \frac{\hat{\lambda}}{4}\hat{\phi}^4 \tag{1.74}$$

where the sum over $\hat{\mu}$ is implied, as in standard Einstein convention and all 'hatted' variables are dimensionless. This action is not unique, as terms that are proportional to higher powers of the lattice spacing may be added without changing the continuum behavior of the theory. The Lagrangian is regulated as the discretized theory has a cutoff: $\Lambda = 1/a$. Since the minimum 'step' that one can make on the lattice is given by the lattice spacing $x_\mu^{min} = a$, the maximum momentum allowed is $p_\mu^{max} = \pi/a$. Thus, all UV divergences are now controlled and the theory can be renormalized using standard methods.

The path integral measure is trivial to discretize, as the continuum measure is only defined as the limit of zero time slice separation:

$$D\phi(x) \rightarrow \prod_n \int_{\infty}^{-\infty} d\hat{\phi}_n \tag{1.75}$$

The measure is now defined in terms of well-defined ordinary integrals over the field ϕ_n and thus avoids the difficulty of convergence faced by a perturbative treatment of a continuum theory. With the path integral measure now well-defined, it is clear why the lattice serves not only as a regulator, but as a nonperturbative one as well. Motivated by the original derivation due to Feynman [72], an intuitive picture of the path integral measure is as a sum over all possible paths.

An action containing fermionic degrees of freedom can be discretized in the same manner as the scalar fields, i.e. the fermion fields are defined by their value at each discrete spacetime point. However, the treatment of fermionic fields differs in two important ways. First, fermions obey anticommutation relations and so cannot be described by ordinary c-numbers;

much like in the continuum, they must be represented by Grassman numbers. Grassman numbers are not well-suited for numerical evaluation, as they can only be represented in terms of c-numbers by matrices. In general, a set of n Grassman numbers can be represented by n matrices of size $2^n \times 2^n$. This difficulty is easily overcome for Dirac fermions by integrating out the fermions before evaluating the remainder of the path integral. This done using the identity that

$$\int D\psi D\bar{\psi} e^{\bar{\psi}D\psi} = \mathrm{Det}D \tag{1.76}$$

This step, however, is highly nontrivial for chiral fermions, as will be discussed in later sections. The second difference has to do with the fermion doubling and will pose a more serious complication.

The fermion doubling problem arises due to the change in spacetime symmetry, from $SO(4)$ to hypercubic, once spacetime is discretized. The effect of this alteration can be seen by looking at the fermion propagator, derived from the naive fermion action: [73]

$$\begin{aligned} S &= \frac{1}{2}\sum_{n,\hat{\mu}} \bar{\psi}_n \gamma^\mu \left(\psi_{n+\hat{\mu}} - \psi_{n-\hat{\mu}}\right) + \sum_n m_0 \bar{\psi}_n \psi_n \\ &= \int \frac{d^4k}{(2\pi)^4)} \psi_n \left(i\gamma_\mu \sin k_\mu + m_0\right)\psi(k) \end{aligned} \tag{1.77}$$

The site are again labeled by n and $\hat{\mu}$ is a single step in any of the four directions and where all fields and couplings are dimensionless.[16] The propagator for a naive lattice fermion is therefore

$$\frac{1}{i\gamma_\mu \sin k_\mu + m_0} . \tag{1.78}$$

If m_0 is taken to zero, this propagator has a pole not only at $k_\mu = 0$, but also at $k_\mu = \pi$; working in d dimensions, there are 2^d massless modes. This is in sharp contrast to the continuum, which only ever has one massless mode.

[16] The hatted convention has been dropped for clarity.

The unwanted $2^d - 1$ additional zero modes, called doublers, can be removed by introducing a Wilson term [73]

$$\mathcal{L}_W = -\frac{r}{2}\,\bar{\psi} D_\mu D^\mu \psi\,, \tag{1.79}$$

which gives the doublers a mass that is $\mathcal{O}(1/a)$; the cost of doing so is that chiral symmetry is broken. Without chiral symmetry, all the fermion masses are additively renormalized; their mass shifts by $\delta_m \sim \alpha r/\left(4\pi a\right)$ due to interactions with the gauge fields. The amount of additive renormalization can be reduced [74–76] by using a Symanzik improved action. The action is improved [77, 78] by adding higher order operators with carefully tuned coefficients to eliminate discretization errors in correlation function. For an example of discretization error, expand Eq. 1.78 for small a:

$$\frac{1}{i\gamma_\mu \sin a\hat{k}_\mu + a\hat{m}_0} = \frac{1}{a\left(\hat{k}_\mu\gamma^\mu + \hat{m}_0\right)} + \frac{a}{6}\frac{\hat{k}^2}{\hat{m}_0^2 + \hat{k}^2}\hat{k}_\mu\gamma^\mu + \mathcal{O}(a^3) \tag{1.80}$$

and note that this differs from the continuum result by $\mathcal{O}(a)$. However, no choice in action allows for the simultaneous removal of the doublers and restoration of chiral symmetry. Nielsen and Ninomiya [79], showed that doublers exist in any theory of fermions that is local, hermitian and has both chiral and translational invariance. The question of how to overcome this no-go theorem and thus have massless fermions without doublers will be addressed in Sec. 1.7.

Gauge symmetry can be introduced into a lattice theory in a manner similar to the continuum. For fermions that are gauged under a vector-like symmetry, the field at each site transforms as

$$\psi_n \to \Omega_n \psi_n \qquad \bar{\psi}_n \to \bar{\psi}_n \Omega_n^\dagger \tag{1.81}$$

where Ω is an element of the gauge group; for QCD, $\Omega \in SU(3)$. Much like in the continuum, the mass term is invariant under this transformation while the kinetic term is not, as

$$\begin{aligned} \bar{\psi}_n \psi_n &\to \bar{\psi}_n \psi \\ \bar{\psi}_{n+\hat{\mu}} \psi_n &\to \bar{\psi}_{n+\hat{\mu}} \Omega_{n+1}^\dagger \Omega_n \psi_n \end{aligned} \tag{1.82}$$

The second term can be made gauge invariant if a new field, $U_{n\mu}$, is introduced into the bilinear,

$$\bar{\psi}_{n+\hat{\mu}} U_{n\mu} \psi_n \rightarrow \bar{\psi}_{n+\hat{\mu}} \Omega^\dagger_{n+1} U'_{n,\mu} \Omega_n \psi_n \tag{1.83}$$

where $U_{n\mu}$ is also an element of the gauge group. Gauge invariance is maintained if

$$U_{n,\mu} \rightarrow \Omega^\dagger_{n+\hat{\mu}} U_{n\mu} \Omega_n \tag{1.84}$$

Since $U_{n,\mu}$ connects fermions on two different sites, it is commonly referred to as a link variable. It must also carry two indices; the convention here is that n labels the site and μ the direction in which the field acts. The oppositely oriented link ($\hat{\mu} \rightarrow -\hat{\mu}$) and the original link are related by

$$U_{n,-\mu} \equiv U^\dagger_{n-\hat{\mu},\mu} \tag{1.85}$$

The naively discretized action for a gauged fermion is therefore

$$S = \sum_n a \left(\bar{\psi}_n U_{n\mu} \psi_{n+\hat{\mu}} - \bar{\psi}_n U_{n,-\mu} \psi_{n-\hat{\mu}} \right) + m_0 \bar{\psi}_n \psi_n \tag{1.86}$$

The connection between the link variable and the continuum gauge group can be found by again considering the transformation of both under gauge transforms. As the gauge field is an element of the algebra, the two can be connected via an exponential map. More concretely, consider the continuum gauge transporter,

$$\mathcal{G}(x, y) = \text{Pexp} \left(i \int_{C_{xy}} A \cdot ds \right) \tag{1.87}$$

which is a path ordered exponential integral along the curve C_{xy} from x to y. Under gauge transformations, the gauge transporter transforms as

$$\mathcal{G}(x, y) \rightarrow \Omega(x) \mathcal{G}(x, y) \Omega(y)^\dagger \tag{1.88}$$

which is the continuum version of the link variable's transformation property. Therefore, the link variable $U_{n,\mu}$ can be associated with the gauge transporter, where C_{xy} is a single step from site n to site $n + \hat{\mu}$,

$$U_{n\mu} = \exp\left(i a A_{n\mu} \right) \tag{1.89}$$

where $A_{n\mu}$ is the gauge field A_μ at site n. The path ordering has been eliminated at the cost of $\mathcal{O}(a)$ errors on the evaluation of the integral in Eq. 1.87,

$$\int_{C_{xy}} A \cdot ds = aA_{n\mu} + \mathcal{O}(a^2)\,. \tag{1.90}$$

To construct the gauge action, recall that the gauge transformation properties of the link variable,

$$U_{n,\mu} \to \Omega^\dagger_{n+\hat{\mu}} U_{n\mu} \Omega_n \tag{1.91}$$

imply that a path of links that starts with link $U_{n\mu}$ and ending with link $U(m, \nu,$, $P[U]_{n,\mu;m,\nu}$ transforms as

$$P[U]_{n,\mu;m,\nu} \to \Omega_m P[U]_{n,\mu;m,\nu} \Omega^\dagger_n \tag{1.92}$$

Thus, a closed path is gauge invariant due to $\Omega^\dagger_n \Omega_n = \mathbb{1}$. Defining a plaquette, $U_{n,\mu\nu}$ as the shortest, nontrivial loop,

$$\begin{aligned} U_{n,\mu\nu} &= U_{n,\mu} U_{n+\hat{\mu},\nu} U_{n+\hat{\mu}+\hat{\nu},-\mu} U_{n+\hat{\nu},-\nu} \\ &= U_{n,\mu} U_{n+\hat{\mu},\nu} U^\dagger_{n+\hat{\nu},\mu} U^\dagger_{n,\nu} \end{aligned} \tag{1.93}$$

Wilson [69] proposed

$$S = \frac{2}{g} \sum_{n,\mu<\nu} \mathrm{ReTr}\,\left[\mathbb{1} - U_{n,\mu\nu}\right] \tag{1.94}$$

for the discretized version of the Yang-Mills gauge action. By taking the definition of the link variable in terms of the algebra-valued gauge field, Eq. 1.89, one can show that the Wilson action reduces to the continuum Yang-Mills action in the limit of $a \to 0$. This action is also not unique, as terms proportional to powers of the lattice spacing can be added into the action, à la Symanzik improvement.

1.6 Monte Carlo Evaluation of Lattice Field Theories

One of the benefits of using a lattice regulator is that it allows for the numerical simulation of the full path integral, which is not only key for understanding the properties of strongly coupled theories. Perturbative evaluation of the path integral is not only invalid at strong coupling, but it also misses important nonperturbative effects such as those due to topology. Numerical simulation is currently the only way to calculate strongly coupled matrix elements with quantifiable errors and so is also necessary for comparing Standard Model prediction to experimental observations. However, not all theories are amenable to numerical simulation. In this section, I will introduce the technique of Monte Carlo simulation of path integrals and detail some of the ways that these methods may fail.

The expectation value of an operator in pure Yang-Mills is

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int DU e^{-S(U)} \mathcal{O}(U) \tag{1.95}$$

where U specifies the value of all the links on the lattice and is commonly referred to as a gauge field configuration. Given a set of configurations $\{U\}$ of size N that are distributed according to the probability $P[U] = e^{-S(U)}$, the expectation value of an operator can be approximated by the average of said observable evaluated on the set of the gauge field configurations. Since the action $S(U)$ is real and bounded from below, $P[U]$ is always real and positive and therefore can be be used as a probability distribution.

Numerical simulation involves not only evaluating the sum

$$\langle \mathcal{O} \rangle \approx \frac{1}{N} \sum_{n=1}^{N} \mathcal{O}(U(n)) \tag{1.96}$$

where $U(n)$ is defined to be the n^{th} configuration in the set $\{U\}$, but also generating the set of configurations. Each element of the set is constructed by taking an arbitrary initial configuration and evolving it stochastically through a sequence of configurations until eventually it reaches a configuration that is distributed according to $P[U]$:

$$\widetilde{U}_0 \to \widetilde{U}_1 \to \widetilde{U}_2 \to \cdots \to U(n) \tag{1.97}$$

where $\widetilde{U}_i$ is an element of the gauge group, but not distributed according to $P[\widetilde{U}]$. This process of stochastic evolution is called a Markov chain. There are different methods of advancing the Markov chain from configuration $\widetilde{U}_{i-1}$ to $\widetilde{U}_i$, such as the Metropolis algorithm [80], Langevin methods [81, 82], heat bath algorithim [83], and Hybrid Monte Carlo [84]

With the addition of fermions, the expectation value of an operator in a theory with fermions is given by

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int DU \Delta(U) e^{-S(U)} \mathcal{O}(U) \tag{1.98}$$

where $\Delta(U)$ results from integrating out the fermions, as in Eq. 1.76. Following the same logic as in the pure gauge case, the probability distribution for generating gauge configurations is $P[U] = \Delta U e^{-S(U)}$. While the exponential factor is still guaranteed to be real and positive, the phase of ΔU is theory dependent. For a massive Dirac fermion, $\Delta(U) = \det(D)$, where

$$D = \not{D} + m - \frac{r}{2} D^2 \tag{1.99}$$

As D obeys γ_5-hermiticity, $\gamma_5 D \gamma_5 = D^\dagger$, its eigenvalues come in complex conjugate pairs and so $\Delta(U)$ is real and positive.

It is no longer true that $\Delta(U)$ is real and positive once a baryon chemical potential is turned on as $D(\mu)$ no longer obeys γ_5-hermiticity, where

$$D(\mu) = \not{D} + m - \frac{r}{2} D^2 + \mu\gamma_4 \,. \tag{1.100}$$

In particular, the determinant of $D(\mu)$ obeys the relation

$$\begin{aligned} [\mathrm{Det}\, D(\mu)]^* &= \mathrm{Det}\, D(\mu)^\dagger \\ &= \mathrm{Det}\, \gamma_5 \mathrm{Det}\, \left(D(0)^\dagger + \mu\gamma_4\right) \mathrm{Det}\, \gamma_5 \\ &= \mathrm{Det}\, D(-\mu) \end{aligned} \tag{1.101}$$

Therefore $P[U]$ is no longer real and positive and cannot be used as a probability distribution. This 'sign problem' does not allow for use of Monte Carlo methods for the numerical simulation of finite density nuclear matter, though at small baryon chemical potential, the problem may be circumvented [85]. There have been several proposals, such as reweighting [86, 87], working at imaginary chemical potential [88–90] or using complex Langevin methods [91–93], that have given access to certain portions of the QCD phase diagrams. However, none have completely alleviated the problems. One can also attempt to work in the canonical formulation, though in this formulation, one is faced with bad signal to noise rations. In Ch. 4, I discuss a model that has two equivalent formulations, only one of which has a sign problem.

1.7 Chiral Symmetry on the Lattice

Regularization and renormalization are key aspects of defining a well-behaved QFT. For a consistent QFT there must also be a regulator that works beyond perturbation theory, as a path integral cannot be defined purely in terms of perturbative behavior; the lattice is the only known regulator with this property. As discussed in Sect. 1.5, it is known how to implement vector gauge theories, such as QCD, on the lattice. Chiral gauge theories are a different matter. There have been many proposals for a lattice regulated chiral gauge theory,[17] though none have proved successful. Without a successful proposal, the Standard Model, and specifically, the electroweak sector, is not on firm formal footing. Additionally, without a regulator that can 'see' the entire spectrum of a chiral gauge theory theory, nor experiments that can probe its nonperturbative nature, the basic building blocks of chiral gauge theories cannot be known. It could be that there is new physics in the Standard Model, hidden by the weakly coupled nature of the electroweak sector.[18]

There are two key problems that must be overcome to have a lattice regulated chiral gauge theory. The first is the question of how to remove fermion doublers without violating chiral symmetry. The Nielsen-Ninomiya theorem [79] states that for a fermion action in $2n$ Euclidean spacetime dimensions,

$$S = \int \frac{d^{2n}p}{(2\pi)^{2n}} \bar{\psi}_{-\mathbf{p}} \widetilde{D}(\mathbf{p}) \psi_{\mathbf{p}} \tag{1.102}$$

[17]See Ch. 5 for details.

[18]Here, nonperturbative is taken to mean anything that cannot be seen in perturbation theory, even after summing to all orders. For example, a single instanton contributes to the Euclidean path integral with weight e^{-S_0}, where $S_0 \sim 1/\alpha$. This behavior cannot be captured by expanding the path integral in a Taylor series around $\alpha = 0$.

where $\mathbf{p}$ is spatial momentum, there is no operator $\widetilde{D}(\mathbf{p})$ that results in a local and hermitian theory with both chiral and translational invariance. An equivalent statement is that no operator $\widetilde{D}(\mathbf{p})$ exists that simultaneously satisfies the four following conditions,

1. $\widetilde{D}(\mathbf{p})$ is periodic analytic in p_μ, with period $2\pi/a$
2. $\widetilde{D}(\mathbf{p}) \propto \gamma_\mu p_\mu$ for $a|p_\mu| << 1$
3. $\widetilde{D}(\mathbf{p})$ is invertible everywhere except $p_\mu = 0$
4. $\{\gamma_5, \widetilde{D}(\mathbf{p})\} = 0$

where the first allows for a local Fourier transform, the next two for the successful removal of all the doublers and the last one for chiral symmetry [94].

The second problem lies in the fact that gauge anomalies play an important role in the continuum: a gauge theory is consistent if and only if all gauge anomalies cancel, as any non-zero gauge anomalies result in a non-renormalizable and non-unitary theory [95, 96]. However, anomalies do not exist on the lattice, due to its finite number of degrees of freedom; neither gauge nor global symmetries can be anomalous. Any successful proposal for a lattice regulated chiral gauge theory must have some mechanism that not only distinguishes anomaly-free gauge theories from anomalous ones, but also one that mirrors the inconsistencies seen in anomalous continuum chiral gauge theories.

These two problems are partially addressed with the advent of domain-wall fermions [97], a theory of $(2n+1)$-dimensional fermions with masses that depend on the extra dimension,

$$S = \int d^{2n}x ds\, \bar{\Psi} \left[\partial\!\!\!/_{2n+1} + m(s)\right] \Psi \qquad m(s) = \Lambda\epsilon(s) \tag{1.103}$$

where s is a extra dimension of length $2L$ and $\epsilon(s) = \mathrm{sgn}(s)$. The $2n+1$-dimensional fermion can be decomposed into a tower of $2n$-dimension fermions by

$$\Psi = \sum_{n=0}^{\infty} b_n(s) P_+ \psi_n(x) + f_n(s) P_- \psi_n(x) \qquad P_\pm = \frac{1 \pm \gamma_5}{2} \tag{1.104}$$

where $f(s)$ and $b(s)$ satisfy

$$\left[\partial_s + m(s)\right] b_n = \mu_n f_n \qquad \left[\partial_s - m(s)\right] f_n = \mu_n b_n \,. \tag{1.105}$$

This results in a $2n$ dimensional theory

$$S = \int d^{2n}x \; \bar{\psi}_0 \left(\partial\!\!\!/_{2n} + \mu_0\right) \psi + \sum_{k=1}^{\infty} \bar{\psi}_k \left(\partial\!\!\!/_{2n} + \mu_k\right) \psi_k \tag{1.106}$$

with fermion masses

$$\mu_0 \sim 2\Lambda e^{-2\Lambda L} \qquad \mu_k = \sqrt{\left(\frac{\pi k}{L}\right)^2 + \Lambda^2}\,. \tag{1.107}$$

The light modes are localized on the domain walls

$$b_0 \sim e^{-\Lambda|s|} \qquad f_0 \sim e^{\Lambda|s|} \tag{1.108}$$

and μ_0 is exponentially small for $\Lambda L \gg 1$. The existence of the mass gap allows for the heavy fermions to be integrated out and the theory at momentum scales $p \ll \Lambda$ is simply the theory of a light non-interacting Dirac fermion.

The theory is gauged by adding $2n$-dimensional gauge fields such that the action is

$$S \;\; = \;\; \int d^{2n}xds \, \bar{\Psi} \left[D\!\!\!\!/_{2n}^{(R)} + \gamma_{2n+1}\partial_s + m(s) \right] \Psi \tag{1.109}$$

where $D\!\!\!\!/_{2n}^{(R)}$ is the covariant derivative for a fermion in the gauge group representation R. Since the gauge field is independent of s and thus does not affect Eq. 1.105, the $2n$-dimensional action is simply Eq. 1.106 with $\partial_\mu \rightarrow D_\mu^{(R)}$. Most importantly, the addition of the gauge field does not delocalize the light modes and so the mass gap is protected.

With the theory now gauged, the low energy effective theory needs to reproduce the $U(1)_A$ anomaly correctly. The gauged low-energy theory is derived by once again integrating out the heavy fermions. Due to their heavy mass, one might expect that these fermions decouple completely and their effects are only seen in higher order operators. However, as shown by Callen-Harvey [98], integrating out the heavy fermions results in a residual term. This term gives rise to a current that flows from one wall to the other and which has precisely the form necessary to give rise to the axial anomaly in the low-energy theory,

The theory can be discretized and the doublers removed by the addition of a Wilson term, which alters the localization of the modes on the domain walls. The number of light modes depends on the ratio of $m/|r|$; if the ratio is too large, the modes delocalize and the theory contains no light degrees of freedom [97, 99]. In the limit of an infinitely large extra dimension, the light modes become exactly massless and the number of zero modes is topologically protected [100]. Therefore, with an infinite extra dimension, it appears as though fermion doublers can be successfully decoupled, the global axial anomaly is correctly reproduced and chiral symmetry is respected. How is this possible in the context of the Nielsen-Ninomiya theorem?

This question is most easily addressed in the overlap formalism [101, 102]. The formalism relies on the realization that if the extra dimension is considered to be time, then the path integral for the action given in Eq. 1.103 is the overlap of the eigenstates of the Hamiltonian $\mathcal{H}(-m)$ and the Hamiltonian $\mathcal{H}(m)$, where

$$\mathcal{H}(M) = \gamma_{2n+1} \left(\not{D}_{2n} + M \right) \tag{1.110}$$

If the extra dimension is taken it be infinite, all other eigenstates are projected out and so the path integral is the overlap of the ground state of $\mathcal{H}(-m)$ with the ground state of $\mathcal{H}(m)$.

A masslass Dirac fermion can be represented by $\mathcal{Z} = |\langle \Omega, m_2 | \Omega, -m_1 \rangle|^2$, with $0 < m_1 < 2r$ and m_2 arbitrary. Designating the eigenstates of $\mathcal{H}(M)$ as $|n, M\rangle$, the overlap of any two eigenstates is given by

$$\langle n, m_2 | n', -m_1 \rangle \equiv U_{nn'} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}_{nn'} \tag{1.111}$$

where U is a unitary matrix and the block structure is in the γ-matrix space. In the same basis, the Hamiltonian is

$$\mathcal{H}(M) = \begin{pmatrix} -\frac{r}{2} D^\mu D_\mu + M & D_\mu \sigma_\mu \\ (D_\mu \sigma_\mu)^\dagger & \frac{r}{2} D^\mu D_\mu - M \end{pmatrix} \qquad \sigma_\mu = \{i, \vec{\sigma}\} \tag{1.112}$$

If m_2 to taken to be infinite, U also diagonalizes $\mathcal{H}(-m_1)$,

$$\mathcal{H}(-m_1) U = U \lambda \tag{1.113}$$

where the upper block of λ contains only positive eigenvalues and the lower only negative, as the eigenvalues of $\mathcal{H}(-m_1)$ are all paired. The path integral is therefore

$$\begin{aligned}\mathcal{Z} &= |\langle \Omega, m_2 \to \infty | \Omega, -m_1 \rangle|^2 \\ &= |\det U_{22}|^2 \\ &= \det\left(\frac{1+\gamma_5 \epsilon(\mathcal{H}(-m_1))}{2}\right) \end{aligned} \tag{1.114}$$

where the determinant is due to the definition of the ground state as the Slater determinant of all negative energy one-particle wavefunctions. The function ϵ is the sign function of a matrix,

$$\epsilon(\mathcal{H}(M)) \equiv \frac{\mathcal{H}(M)}{\sqrt{\mathcal{H}(M)\mathcal{H}(M)^\dagger}}\,. \tag{1.115}$$

From Eq. 1.76, it is known that the path integral is proportional to the determinant of the fermion operator and so

$$\det\left(\frac{1+\gamma_5 \epsilon(\mathcal{H}(-m_1))}{2}\right) \propto \det D \tag{1.116}$$

The operator that satisfies this relation,

$$\begin{aligned} D &= \frac{1+\gamma_5 \epsilon(\mathcal{H}(-m_1))}{} \\ &= 1 + \frac{D_w - m}{\sqrt{(D_w-m)^\dagger (D_w - m)}}\,, \qquad D_w = D_\mu \gamma_\mu - \frac{r}{2} D_\mu D^\mu \end{aligned} \tag{1.117}$$

is the overlap operator [103,104]. It can also be derived explicitly from domain-wall fermions with an infinite extra dimension [105,106].

Returning to the Nielson-Ninomiya theorem, the overlap operator satisfies the first three necessary conditions [107]. The fourth condition is violated as

$$\{\gamma_5, D\} = a D \gamma_5 D \tag{1.118}$$

However, this is precisely the Ginsparg-Wilson relation [108], derived before the advent of domain-wall fermions using block averaging of a continuum theory to create a lattice theory.

When first written down, it was proposed that any fermion operator in a theory tuned to the chiral fixed point would obey said relation. For the Green's function $S(x, y)$ of the fermion operator, the Ginsparg-Wilson relation is

$$S(x,y)\gamma_5 + \gamma_5 S(x,y) = a\gamma_5\delta(x-y) \tag{1.119}$$

which indicates that chiral symmetry is preserved at all non-zero distances; its violation at zero-distances is exactly what is necessary to give rise to the correct axial anomaly.

The overlap operator is the only known operator that obeys the Ginsparg-Wilson relation and so it is reasonable to use it, and thus domain-wall fermions with an infinite extra dimension, as the starting point for a lattice regulated chiral gauge theory. In order to do so, all the fermions on one of the walls have to decouple, as these fermions are parity doublets of the fermions on the other wall. Additionally, there must be a mechanism that can distinguish between a theory where the fermions on one wall are in an anomaly-free representation and a theory where the fermions are in an anomalous representation. I will discuss, in Ch. 5, a recent proposal for how to achieve a lattice regulated chiral gauge theory using domain-wall fermions and a gauge field whose profile in the extra dimension obeys a gauge covariant flow equation.

Chapter 2

LITTLE FLAVOR AND $U(2)$ FAMILY SYMMETRY[1]

The origin of flavor in the Standard Model (SM) remains one of the great mysteries of particle physics. While patterns seem to exist in the quark masses and mixings, they have not led to any particularly convincing models of the generation of the SM flavor structure. Furthermore, the absence of observed electric dipole moments, highly suppressed rare decays, such as $\mu \to 3e$, and the small neutrino masses all provide circumstantial evidence that the next new physics may lie at extremely short distance scales, out of the reach of collider experiments, and quite possibly precision experiments. However, the large suppression of flavor changing neutral currents (FCNC) sends a more ambiguous message. It can certainly be explained by a dearth of new physics until high energy scales, but it could also be compatible with experimentally accessible physics provided there is some approximate symmetry that ensures that neutral currents are approximately flavor-diagonal in the mass eigenstate basis. As this is exactly what suppresses FCNC in the SM, such a possibility may not be far-fetched.

The operative symmetry in the SM and in some models of physics beyond the SM, such as Minimal Flavor Violation (MFV) models, is an approximate chiral $U(2)^5$ symmetry acting on the lightest two families of quarks and leptons. An interesting alternative, which occurs in Extended Technicolor and some supersymmetric models, is a much smaller vector $U(2)$ symmetry acting on the lightest two families of quarks and another $U(2)$ acting on the leptons. Models with a vector $U(2)$ are more interesting phenomenologically as they typically allow for observable rare lepton decays, suppressed by the square of the muon mass, not by the much smaller neutrino masses. Such a vector symmetry does not by itself

[1]This work, Ref [1], will be submitted to a journal.

completely account for the smallness of FCNCs and an additional suppression mechanism must be present. I will first summarize how these symmetries are implement and then, using effective field theory (EFT) techniques, show how a new paradigm for Beyond the Standard Model (BSM) physics based on Little Flavor [109] provides a viable realization of approximate $U(2)^2$ flavor symmetry.

2.1 Symmetries and FCNC

The $U(2)^5$ approximate flavor symmetry of the SM corresponds to independent rotations between first- and second-family fermions in the five different SM representations q, ℓ, u^c, d^c, e^c. Restricting the analysis to a two-family version of the SM, this symmetry is broken by the Yukawa matrices which act as spurions transforming as

$$Y_{u,d} \to U_q Y_{u,d} U_{u,d}^\dagger \;, \qquad Y_e \to U_\ell Y_e U_e^\dagger \;. \tag{2.1}$$

with U_i being $U(2)$ matrices. The $\Delta S = 2$ operator that gives rise to mass difference in the kaon system comes from a one-loop diagram, as discussed in Sec. 1.3. Its size may be estimated by writing down the lowest order (in powers of the Yukawa matrices) four quark operator that is both consistent with the symmetry $U(2)^5$ and gives rise to a $\Delta S = 2$ transition. Working in the mass basis of the down quarks, Y_d is diagonal and the operator must, by symmetry, be proportional to

$$\frac{\left(Y_u Y_u^\dagger\right)_{12}^2}{16\pi^2 G_F} \left(\bar{d}_L \gamma^\mu s_L\right) \left(\right)\bar{d}_L \gamma_\mu s_L\big) \tag{2.2}$$

As quarks become massless in the $Y_{u,d} \to 0$ limit, the operator need not be analytic in Y. In fact the box diagram has IR singularities in that limit, and thus the prefactor is of the form

$$\frac{(Y_u Y_u^\dagger)_{12}^2}{16\pi^2 \mathrm{Tr}\, Y_u Y_u^\dagger} \,. \tag{2.3}$$

The dominant breaking of $U(2)^3$ symmetry in the quark sector is due to the charm mass, and so the SM result is

$$M_{12} \propto G_F^2 \sin^2\theta_c m_c^2 \,, \tag{2.4}$$

which is consistent with the Eq. 1.49. In the realistic three family case the operative symmetry is still an approximate $U(2)^5$ and not $U(3)^5$, as the top quark is heavy, but the mixing between the third family and the two light families is small. Two additional doublet spurions account for the coupling of the third family to the first two, namely

$$\mathcal{T} = \lambda_t \begin{pmatrix} V_{td}^* \\ V_{ts}^* \end{pmatrix} \qquad \mathcal{B} = \begin{pmatrix} \lambda_u V_{ub} \\ \lambda_c V_{cb} \end{pmatrix} \tag{2.5}$$

where λ_i are quark Yukawa couplings. These spurions contribute to M_{12} a term proportional to$\left(\mathcal{T}^\dagger\mathcal{T} + \mathcal{B}^\dagger\mathcal{B}\right)_{12}$. As all the CP violation is contained in $\mathcal{T}, \mathcal{B}$, the leading order contribution to ϵ_K is completely determined by this operator; due to the smallness of the third family mixing angles, it is not the dominant contribution to ΔM_K.

In Minimal Flavor Violation, one assumes that there are no new sources of flavor violation in the BSM model and that the effective theory also obeys an $U(2)^5$ symmetry [110, 111]. The contributions to $\Delta S = 2$ processes from new physics at scale Λ is therefore proportional to $\mathcal{C}_1 = (Y_u Y_u^\dagger)_{12}^2/\Lambda^2$. With two families,

$$\mathcal{C}_1 = m_c^4 \theta_c^2/\Lambda\,, \tag{2.6}$$

which sets the bound $\Lambda \gtrsim \mathcal{O}(10\text{ GeV})$. With three families. the largest contribution is $\mathcal{C}_1$ involves four insertions of the top quark Yukawa coupling and so

$$\mathcal{C}_1 \sim \lambda_t^4 V_{td}^{*2} V_{ts}^2/\Lambda^2\,. \tag{2.7}$$

This gives a much higher bound on the scale of new physics, $\Lambda \geq \mathcal{O}(5\text{ TeV})$. The MFV paradigm also gives bounds from rare lepton decays, once neutrino masses are introduced. For Dirac neutrinos, rare lepton decays are suppressed by at least two powers of the neutrino Yukawa coupling, in complete analogy with the quark sector. For masses generated via a seesaw mechanism, rare lepton decays are only observable if the scale of lepton number violation is vastly different than the scale of lepton flavor violation [112].

In Extended Technicolor (ETC), the approximate family symmetry is reduced to a vector-like $U(2)$ in the quark sector. This symmetry is gauged in order to generate SM fermion masses via radiative corrections. The mass of the fermions is given by

$$m_q = g_{ETC}^2 \langle \overline{F} F \rangle / 2 M_{ETC}^2 \tag{2.8}$$

where $\langle \overline{F} F \rangle$ is the condensate of the technifermions and M_{ETC} is the mass of the gauge bosons that link SM fermions to technifermions. The gauged $U(2)$ also gives rise to tree level FCNC four-quark operators proportional to $\cos^2\theta\, g_{ETC}^2/\overline{M}_{ETC}^2$, where θ is some mixing angle and $\overline{M}_{ETC}$ is the mass of the gauge bosons that only mediate between SM fermions [113,114]. Whereas the exact relative size of M_{ETC} and $\overline{M}_{ETC}$ depends on the details of ETC symmetry breaking, they will be on the same order,[2] which implies that the coefficient of the $\Delta S = 2$ operator can be no smaller than

$$\cos^2\theta\, m_c / \langle \overline{F} F \rangle \,. \tag{2.9}$$

For small θ, this is three orders of magnitude too large, and the theory requires an extra suppression mechanism, such as “walking" [115].

In supersymmetric models, the vector $U(2)$ violating spurions that give rise the the SM quark masses also generate the squark mass differences, though not the flavor universal squark mass, $m_{\tilde{q}}$ [116–119]. Working in the mass basis of the down quarks, the $U(2)$ symmetry constrains the SUSY contribution to M_{12} to be proportional to

$$(V \delta_{\tilde{q}}^2 V^\dagger)(U \delta_{\tilde{q}}^2 U^\dagger)/m_{\tilde{q}}^6 \,, \tag{2.10}$$

where $\delta_{\tilde{q}}^2$ is the mass squared difference for first two families of squarks and V, U are superfield rotation matrices that diagonalize the mass squared and trilinear terms. V, U and $\delta_{\tilde{q}}^2$ are related to the spurions that generate the SM quark masses, and one finds the constraint of $m_{\tilde{q}} \gtrsim 2.5$ TeV [59, 120] is adequate to supply the needed suppression.

An alternative method, based on a low energy effective description of Little Flavor [109], utilizes an approximate $U(2)$ horizontal symmetry whose spurions also transform under a

[2] In a weakly coupled theory, group theoretic considerations allow for $M_{ETC} \gg \overline{M}_{ETC}$ but not the reverse, since the generators associated with gauge bosons of mass $\overline{M}_{ETC}$ can form a closed algebra, while those associated with M_{ETC} cannot. Strong interactions have been invoked in Walking Technicolor [115] to give rise to $M_{ETC} \ll \overline{M}_{ETC}$, reconciling ETC with FCNC constraints.

nonlinearly realized symmetry of the Higgs sector. This allows for FCNCs that are compatible with experimental constraints while predicting new TeV scale physics, as well as protecting the Higgs mass from large quadratic divergence, via the Composite Higgs [121–124] and Little Higgs mechanisms [125]. Note that the Little Higgs cancellations are incomplete in the effective theory considered here, where heavy fermions have been integrated out. The model predicts rare leptonic decays that are suppressed by charged lepton masses, rather than the much smaller neutrino masses, and therefore potentially observable.

2.2 Two families of quarks

Consider a model with the following content: two families of SM quarks q_L^i, u_R^i, d_R^i, $i = 1, 2$, and a 4×4 unitary nonlinear sigma field Σ which contains two composite Higgs doublets $H_{u,d}$ as well as other pseudo Goldstone bosons. The theory has an approximate global symmetry

$$\begin{aligned} G_{\text{global}} &= [SU(2)_L \times SU(2)_R \times U(1)_{B-L} \times U(2)_f] \\ &\quad \times [SU(4)_L \times SU(4)_R] \end{aligned} \tag{2.11}$$

where the subgroup in the first bracket acts on the quarks, $U(2)_f$ is a horizontal family symmetry, and the group in the second bracket acts on the Σ field; color and the anomalous $U(1)_A$ symmetry are ignored, as they play no role in the model. A $\mathcal{G}_a \times \mathcal{G}_b$ subgroup of the global symmetry is gauged, where $\mathcal{G}_{a,b}$ are independent $SU(2) \times U(1)$ groups embedded in the global symmetry as

$$\begin{aligned} SU(2)_a &\subset SU(2)_L \times SU(4)_L \ , \\ U(1)_a &\subset SU(2)_R \times U(1)_{B-L} \times SU(4)_L \ , \\ SU(2)_b \times U(1)_b &\subset SU(4)_R \end{aligned} \tag{2.12}$$

The quarks have the usual SM quantum numbers under $SU(2)_a \times U(1)_a$

$$q_L^i = 2_{\frac{1}{6}} \ , \quad u_R^i = 1_{\frac{2}{3}} \ , \quad d_R^i = 1_{-\frac{1}{3}} \ , \tag{2.13}$$

and are neutral under $SU(2)_b \times U(1)_b$. The gauge generators act on the Σ field as

$$\delta\Sigma = iX_a\Sigma - i\Sigma X_b \tag{2.14}$$

Denoting the four X generators as $\{gT^i,\, g'Y\}$ for both gauge groups, their representations on the Σ field are given by

$$T^i = \tfrac{1}{2}\begin{pmatrix} \sigma_i & \\ & 0 \end{pmatrix}, \quad Y = \tfrac{1}{2}\begin{pmatrix} 0 & \\ & \sigma_3 \end{pmatrix} \,. \tag{2.15}$$

The X generators act as spurions breaking the global symmetry, with X_a transforming as (adjoint $\times$ 1) $\oplus$ (1 $\times$ adjoint) under the $[SU(2)_L \times SU(2)_R] \times SU(4)_L$ symmetry, and X_b transforming as the adjoint under $SU(4)_R$. Before the introduction of additional spurions, the EFT is quite simple: the usual gauged chiral Lagrangian for Σ, quark kinetic terms, and fermion-meson interactions, which are $\mathcal{O}(\alpha)$, that are required as counterterms. The chiral Lagrangian is characterized by decay constant f and cutoff $\Lambda \sim 4\pi f$; I will take $f \gtrsim 1.5$ TeV.

If $\langle\Sigma\rangle = \mathbb{1}$, the $[SU(2) \times U(1)]^2$ gauge symmetry is spontaneously broken to the diagonal group $SU(2) \times U(1)$, which is identified with the SM gauge group. However, assume the Composite Higgs paradigm, the vacuum misaligns and prefers the less symmetric ground state

$$\langle\Sigma\rangle = \begin{pmatrix} c_u & & s_u & \\ & c_d & & s_d \\ -s_u & & c_u & \\ & -s_d & & c_d \end{pmatrix} \qquad \begin{array}{c} c_i = \cos\frac{v_i}{f} \\ \\ s_i = \sin\frac{v_i}{f} \end{array} \tag{2.16}$$

which breaks the SM gauge group in the conventional way for a two Higgs doublet model; v_i are the vevs of two Higgs doublets. The SM gauge couplings g, g' are related to the $\mathcal{G}_a \times \mathcal{G}_b$ couplings by

$$g_a = \frac{g}{\cos\gamma_2},\;\; g'_a = \frac{g'}{\cos\gamma_1},\;\; g_b = \frac{g}{\sin\gamma_2},\;\; g'_b = \frac{g'}{\sin\gamma_1} \tag{2.17}$$

and the exotic gauge bosons have TeV scales masses

$$M_{Z'} = M_{W'} = gf\csc 2\gamma_2 \quad M_{Z''} = g'f\csc 2\gamma_1 \tag{2.18}$$

with freedom to set the angles $\gamma_{1,2}$. The gauge bosons masses will be constrained once leptons are added.

	$SU(2)_L$	$SU(2)_R$	$U(2)_f$	$SU(4)_L$	$SU(4)_R$
$\mathcal{Y}_L$	2	1	1	4	1
$\mathcal{Y}_R$	1	2	1	4	1
Δ	1	1	$1 \oplus 3$	1	15
ϵ	1	1	$1 \oplus 3$	10	1

Table 2.1: Transformation properties of spurions under the global approximate symmetry, where $\mathcal{Y}_{L,R}$ are given in Eq. 2.19 and ϵ in Eq. 2.31.

Up to this point quark masses cannot be generated, even with $SU(2) \times U(1)$ breaking, as the quarks and Higgs transform under independent symmetries. Assuming that additional particles that couple to the quark and Σ fields, such as the heavy fermions of Ref. [109], have been integrated out at a scale $M < \Lambda$ allows for the introduction of additional spurions $\mathcal{Y}_{L,R}$. Their transform properties are given in Table 2.1. These spurions take the form of rectangular 4×2 matrices both transforming on the left by $SU(4)_L$ of the Higgs sector, and on right by either $SU(2)_L$ ($\mathcal{Y}_L$) or $SU(2)_R$ ($\mathcal{Y}_R$) of the quark sector:

$$\mathcal{Y}_L = \begin{pmatrix} y_q & 0 \\ 0 & y_q \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \ \mathcal{Y}_R = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ y_u & 0 \\ 0 & y_d \end{pmatrix}, \tag{2.19}$$

where $y < M/\Lambda < 1$. M also has to be larger than the heaviest bosons and so we take $M = \mathcal{O}(5\text{ TeV})$. At tree level, matching operators proportional to $y^2\Lambda^2/M^2$ will be induced in the EFT below the scale M. In operators proportional to y^2, derivatives appear in the ratio D/M. Radiative corrections within the EFT below M do not yield higher powers of Λ/M, since the cutoff in this theory is M and the scale Λ only appears in inverse powers as $1/\Lambda \sim 1/4\pi f$. The analysis is simplified by assuming $y \ll M/\Lambda$ and working to quadratic order in y; without this assumption, the bounds on the model are much less straightforward and thus will be left for future work.

Assume first that the heavy particles all have a universal mass M, invariant under $SU(4)_b \times U(2)_f$. The EFT now contains operators of the form

$$\mathcal{L}_{y^2} = \frac{\Lambda^2}{M^2}\left[\bar{q}_L \mathcal{Y}_L^\dagger \Sigma \not{D} \Sigma^\dagger \mathcal{Y}_L\, q_L + L \leftrightarrow R\right] \tag{2.20}$$

plus operators with two or more derivatives, where the RH quark fields are written as a doublet, $q_R = \{u_R, d_R\}$; the factor of Λ^2/M^2 appears due to the power counting discussed above. For clarity, here and throughout, the $\mathcal{O}(1)$ model-dependent Wilson coefficients are ignored. This term contains new couplings of the quarks to SM and exotic gauge bosons, which result in non-canonical kinetic terms and will require a finite wavefunction renormalization.

The lowest dimension operator that couples q_L to q_R requires at least two derivatives and is of the form $\bar{q}_L \mathcal{Y}_L^\dagger (\Sigma D^2 \Sigma^\dagger)\mathcal{Y}_R q_R$. Hence, despite the fact that Σ breaks the weak interactions, there are no quark mass terms at $\mathcal{O}(y^2)$. The necessity of $SU(4)_b$ violation to generate fermion masses is the essence of the Little Flavor mechanism. The leading contributions appear at $\mathcal{O}(y^2\alpha_b/4\pi)$ and give a $U(2)_f$ invariant common mass to the quarks:

$$\mathcal{L}_{y^2\alpha_b} = \frac{\Lambda^2}{M}\frac{\alpha_b^i}{4\pi}\bar{q}_L \mathcal{Y}_L^\dagger \left(\Sigma X_b^i X_b^i \Sigma^\dagger\right)\mathcal{Y}_R\, q_R + \text{h.c.} \tag{2.21}$$

where i is summed over $0,\ldots,3$ with $i = 0$ corresponding to the $U(1)_b$ and $i = 1,2,3$ corresponding to $SU(2)_b$. The generators $X^i = \{gT^a, g'Y\}$ are defined in Eq. 2.15.

In order to generate nondegenerate masses, the heavy particle mass, M cannot be universal. Replace $M \to M(1-\Delta)$ everywhere with

$$\Delta = \begin{pmatrix} \boldsymbol{\delta}_u + \boldsymbol{\delta}_d & & & \\ & \boldsymbol{\delta}_u + \boldsymbol{\delta}_d & & \\ & & \boldsymbol{\delta}_d - 3\boldsymbol{\delta}_u & \\ & & & \boldsymbol{\delta}_u - 3\boldsymbol{\delta}_d \end{pmatrix} \tag{2.22}$$

where $SU(4)_R$ space is shown, and $\boldsymbol{\delta}_{u,d}$ are two independent matrices in the flavor $U(2)_f$ space. The matrix Δ can be treated as a spurion transforming as in Table 2.1. Since it transforms nontrivially under $SU(4)_R$ it is now possible to construct mass terms at $\mathcal{O}(y_L y_R \Delta)$ of the form

$$\mathcal{L}_{y^2\Delta} = \frac{\Lambda^2}{M}\bar{q}_L \mathcal{Y}_L^\dagger \Sigma\Delta\Sigma^\dagger \mathcal{Y}_R\, q_R + \text{h.c.} \tag{2.23}$$

The up-type quark mass matrix is then given by the sum of terms

$$M_u = -v_u \frac{\eta_q \eta_u M}{f}\left(\frac{3\alpha_b - \alpha_b'}{16\pi} + 4\boldsymbol{\delta}_u\right) \tag{2.24}$$

where $\eta_i \equiv y_i \Lambda/M$ and the sign of Δ determines the relative sign between the two contributions. The down-type quark mass matrix is the same with subscripts u, d interchanged. To avoid fine tuning, the maximal size of the universal $\mathcal{O}(y^2\alpha_b/4\pi)$ term should be $\mathcal{O}(m_u)$. Therefore, the charm mass arises mainly from the $\boldsymbol{\delta}_u$ term, as $m_c \gg m_u$, and so one of the eigenvalues of $4\eta_q\eta_u M\boldsymbol{\delta}_u/f$ must be λ_c. Since Δ has to be perturbative ($3\boldsymbol{\delta}_{u,d} < 1$), $\gamma_{1,2}$ are bounded via

$$\frac{1}{3} \gtrsim \frac{\lambda_c}{\lambda_u}\frac{3\alpha_b - \alpha_b'}{64\pi}\ . \tag{2.25}$$

This bound is easier to satisfy in the down sector than in the up sector, as the amount of fine tuning is determined by the mass splitting. By having the two contributions partially cancel each other in the up sector, we can account for the physical up quark mass at the cost of moderate fine tuning.

Introducing Δ into single derivative operators,

$$\mathcal{L}_{y^2\Delta D} = \frac{\Lambda^2}{M^2}\bar{q}_L \mathcal{Y}_L^\dagger \Sigma\Delta \not{D}\Sigma^\dagger \mathcal{Y}_L\, q_L + L \leftrightarrow R + \text{h.c.} \tag{2.26}$$

leads to flavor off-diagonal couplings for the exotic gauge bosons. Focusing on the neutral $\Delta S = 1$ currents, to leading order in $(v/f)^2$ these have the form

$$\begin{aligned}\mathcal{L}_{\Delta S=1} &= \frac{\eta_d^2 M_{Z''}}{f}\bar{d}_R \not{Z}'' \left[R_d^\dagger\left(\boldsymbol{\delta}_u - 3\boldsymbol{\delta}_d\right)R_d\right]_{12} s_R \\ &+ \frac{\eta_q^2 M_{Z'}}{f}\bar{d}_L \not{Z}' \left[L_d^\dagger\left(\boldsymbol{\delta}_u + \boldsymbol{\delta}_d\right)L_d\right]_{12} s_L + \text{h.c.}\end{aligned} \tag{2.27}$$

The $\Delta S = 2$ currents is bound from above by

$$\mathcal{L}_{\Delta S=2} \lesssim \frac{\eta_q^2 \lambda_c^2}{4\eta_u^2 M^2} \left[\bar{d}_L \gamma_\mu s_L \bar{d}_L \gamma^\mu s_L + \frac{\eta_d^4}{\eta_q^4} \bar{d}_R \gamma_\mu s_R \bar{d}_R \gamma^\mu s_R \right] \tag{2.28}$$

where the bound is saturated when assuming all relevant mixing angles are $\mathcal{O}(1)$. If we also assume $\eta_q \sim \eta_{u,d}$, then we require $M \gtrsim 10$ TeV.

2.3 Adding Leptons

Dirac neutrino masses are generated in the same way as quark masses, which allows the model to be constrained with respect to Z' phenomenology and rare leptonic decays. The universal $\mathcal{O}(\alpha_b)$ lepton mass can be made small without any fine-tuning, unlike in the quark sector, as the mass ratio μ/e is significantly smaller than c/u.

The bounds on the $M_{Z'}$ and $M_{Z''}$ are estimated by rescaling the bounds on Z'_{SSM} [126], where the Sequential Standard Model (SSM) assumes that the Z'_{SSM} has the same gauge coupling at the SM Z boson. For $\eta_{\ell,\nu,e} \ll 1$ the Z' and Z'' couplings are $g \tan\gamma_2 T_3$ and $g' \tan\gamma_1 Q$, respectively, where T_3 is the neutral gauge generator of the SM $SU(2)$ and Q is the electric charge generator. Taking $f = 1.5$ TeV, the mass bounds are

$$M_{Z'} \gtrsim 2.0 \text{ TeV} \quad M_{Z''} \gtrsim 1.6 \text{ TeV} \quad \gamma_{1,2} \leq \frac{\pi}{4} \tag{2.29}$$

For rare lepton decays, the neutral currents that change lepton flavor are completely analogous to the $\Delta F = 1$ currents in the quark sector. The branching fraction, $\mathbf{B}$, for $\mu \to 3e$ can be bounded from above by:

$$\mathbf{B}_{\mu\to 3e} \lesssim \frac{\lambda_\mu^2}{2G_F^2 f^2 M^2} \left(\frac{45\eta_e^2 \sin^4\gamma_1}{\eta_l^2} + \frac{\eta_l^2 \sin^4\gamma_2}{\eta_e^2} \right) \tag{2.30}$$

where the bound is saturated for $\mathcal{O}(1)$ mixing angles and we assume the neutrino contribution is negligible. For $\gamma_{1,2}$ that saturate the bounds on $M_{Z'}$ and $M_{Z''}$ and $M \sim 5$ TeV the branching ratio is a tenth of the experimental bound, if $\eta_l \sim \eta_e$. It is worth noting that the large mass scale M in the lepton sector should be on the same order as the mass scale in the quark sector.

It is interesting to consider Majorana neutrinos instead of Dirac and implement a Seesaw mechanism [127–130]. There is a Majorana mass term at $\mathcal{O}(y^2\Delta^2\epsilon)$ of the form

$$\begin{aligned} \mathcal{L}_{\epsilon y^2\Delta^2} &= \frac{\Lambda^3}{M^2}\left[\left(\Sigma\Delta\Sigma^\dagger\right)\mathcal{Y}_L\ell\right]^T\epsilon\left[\left(\Sigma\Delta\Sigma^\dagger\right)\mathcal{Y}_L\ell\right]+\text{h.c.}\ , \\ \epsilon &= \frac{\Lambda}{M_N}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{\delta}_\nu & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{2.31}$$

where ϵ is a new $\Delta L = 2$ spurion transforming as in Table 2.1 and M_N is the large Majorana mass. There are additional contributions with Δ replaced by gauge generators. The analysis of a model with Majorana neutrinos is more complicated and is left for future work, along with a full treatment of three families and CP violation. Majorana neutrinos were also the subject of Ref. [131].

Chapter 3

THE ROLE OF THE ELECTRON MASS IN DAMPING CHIRAL PLASMA INSTABILITY IN SUPERNOVA AND NEUTRON STARS[1]

The inference of extreme surface magnetic fields $B_S \simeq 10^{14} - 10^{15}$ G from observations of a class of neutron stars called magnetars [132] raises many questions about how and when such fields are generated. In the conventional scenario, they are expected to arise either due to strong hydrodynamical or magnetohydrodynamic instabilities during core-collapse supernova, or during the early evolution of the proto-neutron star [133–135]. Other mechanisms, which rely on a spontaneous magnetization of the ground state of strongly interacting matter at extreme density, have also been proposed; these remain speculative due to large theoretical uncertainties. Recently in [136], it was suggested that Chiral Plasma Instability (CPI) [137] could be used to generate large fields. In this intriguing scenario, a net chiral charge is produced during core collapse of the progenitor star. As matter is compressed during core collapse, left-handed electrons are captured by protons due to the weak interaction, which results in an imbalance between the Fermi energies of left-handed and right-handed electrons. This imbalance triggers an instability that equilibrates the two chiralities and the released energy drives the growth of a coherent magnetic field. Key to this analysis is the assumption that the electron mass, which explicitly violates chirality, can be neglected [136]. The authors argue that this is a reasonable approximation because the electron mass $m_e = 0.51$ MeV is much smaller than the typical electron Fermi momentum $p_{\text{Fe}} \simeq 100$ MeV encountered in supernova and neutron stars. This assumption is revisited here and found to be false; the electron mass cannot be neglected as it leads to chiral charge equilibration much faster than

[1]This work has been previously published in Ref. [2]

the weak interactions can create an asymmetry, and that therefore this mechanism does not lead to astrophysical interesting magnetic fields.

3.1 Magnetic Field and Chiral Chemical Potential

The Chiral Plasma Instability for massless electrons with only electromagnetic interactions will be reviewed first. In this case, chiral symmetry is only violated by quantum effects (the anomaly), and at the classical level left and right handed electron numbers are separately conserved. Absent the chiral anomaly, inverse beta decay during core collapse of the neutron star progenitor leads to a net chiral charge in the resultant neutron star. Already in 1980, Vilenkin had realized that a net chiral charge density in the plasma would result in an anomalous current of the form

$$\vec{J} = \frac{2\alpha}{\pi}\mu_5\vec{B}\,, \tag{3.1}$$

where $\mu_5 = \mu_R - \mu_L$ is the chemical potential associated with the chiral charge density, and μ_R and μ_L are the chemical potentials associated with the right and left handed massless particles, $\vec{B}$ is the magnetic field, and $\alpha = e^2/4\pi$ is the fine structure constant [137]. Vilenkin, and soon afterwards, Rubakov and Tavkhelidze also recognized that the ground state with a chiral charge density would be unstable, and that this instability would resolve by generating a magnetic field [137–139]. More recently, there has been renewed interest in similar phenomena in the context of relativistic heavy-ion collisions, and is now widely known as the Chiral Magnetic Effect [140].

The current given in Eq. 3.1 is referred to as the chiral magnetic current; it can be derived from a parity violating effective action for the gauge fields (in the plasma rest frame) of the form

$$\int d^4x\, d^4y\, g(x-y)\epsilon^{0ijk}A_i(y)\partial_j A_k(x)$$

where $g(x-y)$ is in general nonlocal and proportional to μ_5 with the chiral plasma current arising from the leading term in a derivative expansion of g. [139] The origin of the chiral magnetic current is easy to understand: in a constant magnetic field electrons occupy Landau levels, where each Landau level can be viewed as a 1+1 dimensional Dirac fermion traveling along the direction of the magnetic field; the excited levels contain electrons of both spin polarizations, while the lowest Landau level only contains electrons with spin anti-aligned with the field. At nonzero μ_5 it follows that there is a difference between the density of particles in the lowest Landau level moving parallel to the magnetic field (LH chirality) versus antiparallel (RH chirality), and hence there exists an electric current in the direction of the magnetic field, $\vec{B}$. It is given by the $1+1$ dimensional current density in the magnetic field direction, $(e\mu_5/2\pi)$, times the transverse density of the lowest Landau orbits, (eB/π) (see derivation in [94], for example). Nonzero μ_5 also forces a chiral asymmetry in the excited Landau levels, but as these levels contain electrons of both polarizations they do not contribute to the electric current. No mention has been made of the anomaly, but the Landau level picture of the anomaly in 3+1 dimensions shows that the two are intimately related. [2]

When modified to incorporate the chiral magnetic current, Eq. 3.1, and finite electrical conductivity, Maxwell's equations read

$$\frac{\partial \vec{B}}{\partial t} = -\nabla\times\vec{E} \tag{3.2}$$

$$\nabla\times\vec{B} - \frac{\partial \vec{E}}{\partial t} = \sigma\ \vec{E} + \frac{2\alpha}{\pi}\mu_5\ \vec{B} \equiv \vec{J}\,, \tag{3.3}$$

where σ is the electrical conductivity, scaling as μ_e/α. The normal component of the electric current is given by $J = \sigma\ \vec{E}$ because, as will be show below, the electron mean free path is short compared to other the length scales in the problem. Assuming constant σ and μ_5, and ignoring the $\partial\vec{E}/\partial t$ term (justified below) the above equations are combined to give

$$\frac{\partial \vec{B}}{\partial t} = \frac{1}{\sigma}\ \nabla^2\vec{B} + \frac{2\alpha}{\pi\sigma}\ \mu_5\nabla\times\vec{B}\,, \tag{3.4}$$

which describes the time evolution of $\vec{B}$ in the presence of the Chiral Plasma current. The

[2] See J. Preskill's lecture notes on Quantum Chromodynamics at `http://www.theory.caltech.edu/~preskill/notes.html`, pp. 3.43-3.45, or else the derivation in Ref. [94].

unstable modes are characterized by the vector potential

$$\vec{A}_{\pm} = (\hat{x} \pm i\hat{y})\, e^{(ikz - i\omega t)} \ , \tag{3.5}$$

which corresponds to electric fields $\vec{E}_{\pm} = i\omega \vec{A}_{\pm}$ and magnetic fields $\vec{B}_{\pm} = \pm k \vec{A}_{\pm}$, where the $\pm$ subscript denotes the helicity of the fields for positive k. The wavenumber k and the frequency $\Re[\omega]$ are constants. Eq. 3.3 has exponentially growing solutions, whose helicity depend on the sign of μ_5, with amplitude

$$B_k(t) = B_k(0) e^{tk(2k_\star - k)/\sigma} \ , \qquad k_\star = \frac{\alpha\mu_5}{\pi} \tag{3.6}$$

for $0 < k < 2k_\star$, where $B_k(0)$ is the initial magnetic field – either a thermal fluctuation, or the field inherited from the progenitor star. Note that the terms kept in Eq. 3.4 are proportional to $k_\star^2/\sigma$, while the term $\partial\vec{E}/\partial t$ neglected in going from Eq. 3.3 to Eq. 3.4 is smaller by a factor of $\omega/k = k_\star/\sigma = O(\alpha^2)$; similarly the neglected plasma frequency of the photon has a negligible effect on the growing mode solution. The maximally unstable mode occurs for $k = k_\star$, with that mode growing as

$$B_\star(t) = B_\star(0) \exp\left(\Gamma_{\rm CPI}\, t\right) \ , \qquad \Gamma_{\rm CPI} = \frac{k_\star^2}{\sigma} = \frac{\alpha^2\mu_5^2}{\pi^2\sigma} \ . \tag{3.7}$$

For a recent discussion of this instability in the context of the high temperature plasmas and matter encountered in the early universe see Ref. [141–143].

The local evolution of the chiral charge density is described by the anomaly equation

$$\partial_\mu j_5^\mu = -\frac{\alpha}{2\pi} F_{\alpha\beta}\tilde{F}^{\alpha\beta} = -\partial_\mu K^\mu \ , \qquad K^\mu = \frac{\alpha}{\pi}\epsilon^{\mu\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma \ . \tag{3.8}$$

Integrating over space and assuming fields vanish at spatial infinity yields the conservation law

$$\frac{d}{dt}\left(n_5 + \frac{\alpha}{\pi} H\right) = 0 \ , \qquad H = \frac{1}{V}\int d^3x\, \vec{A}\cdot\vec{B} \ , \tag{3.9}$$

where $n_5 = N_5/V$ is the average chiral charge density, V is the volume, and H is the gauge invariant “helicity density". Note that a time-dependent helicity implies a nonzero electric field, and thus the above equation can be simply understood as the conventional effect of an electric field changing the momenta of electrons in the lowest Landau level.

Since the field Eq. 3.5 has nonzero helicity, the growth of the unstable mode converts electron chiral charge density n_5 into electromagnetic helicity H at a rate

$$\frac{\partial n_5}{\partial t} = -\frac{\alpha}{\pi}\frac{dH}{dt} = -\frac{2\alpha\Gamma_{\rm CPI}}{\pi k_\star}\, B_\star(t)^2 = -\frac{2\alpha^2\mu_5}{\pi^2\sigma}\, B_\star(t)^2 \equiv -\Gamma_B n_5\,, \tag{3.10}$$

where $B_\star(t)$ is given in Eq. 3.7. The free energy in the magnetic field is supplied by the imbalance of Fermi energy between left and right handed electrons. In time, μ_5 is driven to zero locally, and a global helical magnetic field that spontaneously breaks rotational symmetry is generated. This is the phenomena essential to the proposed mechanism for generating large magnetic fields during the supernova in Ref. [136]. However, one immediately sees a problem with using the CPI to directly generate large coherent magnetic fields on astrophysical scales: for long wavelength magnetic fields, $k_\star$ must be exceedingly small compared to μ_e, as must to a lesser extent $\mu_5 = \pi k_\star/\alpha$. This in turn implies both that the growth rate $\Gamma_{\rm CPI}$ would be very slow and that the total amount of electron energy available for conversion to magnetic field energy would be very small. For example, for $k_\star \sim (100\text{ m})^{-1}$ one finds $\mu_5 \sim 10^{-6}$ eV and $\Gamma_{\rm CPI} \sim (1\text{ yr})^{-1}$. While a large value of μ_5 can produce a large field with short wavelength, a subsequent mechanism is needed to convert this into a coherent field on a macroscopic length scale. Although such a mechanism, called the inverse cascade, has been proposed in Ref. [144], in what follows we show that in the supernova large values of μ_5 are unlikely.

In order to find out what actually happens, the size of μ_5 in a core collapse supernova must be estimated, and so consider massive electrons must be considered. The anomaly equation, Eq. 3.8, is modified to include explicit chiral symmetry breaking due to the electron mass

$$\partial_\mu j_5^\mu = 2im\bar{\psi}\gamma_5\psi - \frac{\alpha}{2\pi}F_{\alpha\beta}\tilde{F}^{\alpha\beta}\ . \tag{3.11}$$

It is not particularly simple to use this equation directly to compute rates in a plasma, since single particle asymptotic states are no longer eigenstates of chirality. Instead it is useful to discuss electron helicity eigenstates, as helicity is exactly conserved for any electron mass in the absence of interactions. For free massive electrons in a multi-electron state of definite helicity $|h\rangle$, the expectation value $\langle h|n_5|h\rangle$ is time-independent since $|h\rangle$ is a stationary state, despite n_5 not commuting with the Hamiltonian. In fact, this expectation value is given by the sum of helicity times the magnitude of the velocity ($|p|/E$) for each electron — a result that goes smoothly to the $m = 0$ limit, since in that limit all electrons have $|p|/E = 1$ and helicity becomes synonymous with chirality. Interactions are now turned on and the evolution of $|h\rangle$ due to electron helicity flipping interactions can be seen to lead to a time dependence of the expectation value of n_5, where

$$n_5(t) = \int \frac{d^3k}{(2\pi)^3} \left[f_+(k,t) - f_-(k,t)\right] \frac{|\mathbf{k}|}{\omega_{\mathbf{k}}} \ , \tag{3.12}$$

$f_\pm(k,t)$ being the electron occupation number in a state with momentum k and $\pm$ helicity.

Assuming that deviations of $f_\pm(k,t)$ from equilibrium are small, and using linear response with

$$f_\pm(k,t) = f(k) \pm \frac{\partial f(k)}{\partial \mu} \delta\mu_5(k,t) \simeq f(k) \pm \frac{\partial f(k)}{\partial \mu} \delta\bar{\mu}_5(t) \ , \tag{3.13}$$

where $\delta\bar{\mu}_5(t)$ is k-independent and $f(k)$ is the equilibrium Fermi-Dirac distribution

$$f(k) = \frac{1}{1 + e^{-\beta(\omega_k - \mu_e)}} \ , \qquad \omega_k = \sqrt{k^2 + m_e^2} \ . \tag{3.14}$$

For the first part of Eq. 3.13, $|\delta\mu_5(k,t)| \ll \mu_e$ was assumed for all k, an approximation which will be seen to be self-consistent, as the equilibration of n_5 to zero due to electron helicity changing scattering is found to be much faster than the rate of change of n_5 arising from either the CPI or the weak interactions. For the second part of Eq. 3.13 the fact that $\partial f/\partial\mu$ only has support for $|k - \mu_e| \lesssim T$ was used, and assumed that $\delta\mu_5(k,t)$ is roughly independent of k in this region, allows for the replacement $\delta\mu_5(k,t) \to \delta\bar{\mu}_5(t)$. This latter assumption is justified by the fact that helicity preserving scattering will be fast (not suppressed by the small electron mass) and so the positive and negative helicity electrons will each be in independent approximate quasi-static thermal equilibrium. Given Eq. 3.13

n_5 can be expressed as

$$n_5(t) \simeq 2\delta\bar{\mu}_5(t) \int \frac{d^3k}{(2\pi)^3} \frac{\partial f(k)}{\partial \mu} \frac{|\mathbf{k}|}{\omega_{\mathbf{k}}} \simeq \frac{\delta\bar{\mu}_5(t)\mu_e^2}{\pi^2}, \qquad \Longrightarrow \qquad \frac{\dot{n}_5}{n_5} \simeq \frac{\delta\dot{\bar{\mu}}_5}{\delta\bar{\mu}_5}, \tag{3.15}$$

where again the fact that $\partial f(k)/\partial\mu$ is sharply peaked at $|k - \mu_e| \lesssim T$ is used, with $m_e, T \ll \mu_e$. The contribution to $\dot{n}_5/n_5$ arising from electron helicity changing scattering is referred to as $-\Gamma_m$, as this contribution must vanish at zero electron mass.

The helicity changing Rutherford scattering of electrons off the ambient protons is the dominant contribution to Γ_m. Other contributions come from electron-electron scattering and Compton scattering, but the former is expected to be suppressed relative to Rutherford scattering due to the fact that electrons are far more degenerate than protons, while the latter is relatively suppressed since the proton density scales as μ_e^3, while the ambient photon density scales as T^3, where T is the temperature and $T/\mu_e \lesssim 1/10$ during the core collapse. From the Boltzmann equation, in the approximation of Eq. 3.13,

$$\frac{\partial_t\, \delta\mu_5(k,t)}{\delta\mu_5(k,t)} = -2\frac{1}{2\omega_{\mathbf{k}}} \int \frac{d^3k'}{(2\pi)^3 2\omega_{\mathbf{k}'}} \frac{1 + e^{\beta(\omega_{\mathbf{k}} - \mu_e)}}{1 + e^{\beta(\omega_{\mathbf{k}'} - \mu_e)}} W(k,k')_{+-} \tag{3.16}$$

where

$$W(k,k')_{hh'} = \int \frac{d^3p\, d^3p'}{(2\pi)^3 2\omega_{\mathbf{p}} (2\pi)^3 2\omega_{\mathbf{p}'}} |\mathcal{M}_{hh'}|^2 (2\pi)^4 \delta^4(p + k - p' - k') f(p')\,(1 - f(p)) \tag{3.17}$$

for electron scattering with incoming and outgoing momentum and helicity $(\mathbf{k}, h)$ and $(\mathbf{k}', h')$ respectively, where p, p' are the proton momenta, and $\mathcal{M}_{hh'}$ is the Rutherford scattering amplitude averaged and summed over incoming and outgoing proton spins. Neglecting proton recoil (suppressed by μ_e/M_p),

$$|\mathcal{M}|^2_{+-} = 128\pi^2\alpha^2 \frac{E_{\mathbf{p}}^2 m^2\,(1 - \cos\theta)}{\left(2k^2(1 - \cos\theta) + q_D^2\right)^2} \tag{3.18}$$

where θ is the scattering angle, E_p is the proton energy, and the inverse Debye screening length q_D provides an infrared cutoff to the scattering process. Inserting this expression into Eq. 3.16 and evaluating at $k = \mu_e$, where $\partial f/\partial\mu$ is peaked,

$$\Gamma_m = -\left(\frac{\delta\dot{\bar{\mu}}_5}{\delta\bar{\mu}_5}\right)_{\text{Ruth.}} = -\left(\frac{\partial_t\, \delta\mu_5(k,t)}{\delta\mu_5(k,t)}\right)_{\text{Ruth.}}\Bigg|_{k=\mu_e} \simeq \frac{\alpha^2 m_e^2}{3\pi\mu_e}\left[\ln\frac{4}{x} - 1\right], \qquad x \equiv \frac{q_D^2}{\mu_e^2}. \tag{3.19}$$

Because proton degeneracy and recoil can be neglected, this result coincides with the simpler expression $\Gamma_m = n_p \sigma_{\rm R}(\mu_e)$, where n_p is the proton density and $\sigma_{\rm R}(\mu_e)$ is the Rutherford cross section for electrons on the Fermi surface. Noting that $q_D^2 = 4\pi\alpha\ \partial^2\Omega/\partial\mu_e^2$, where Ω is the total free energy of the plasma, and that at the fiducial density and temperature characteristic of the supernova, the electrons can be treated as degenerate and protons as non-degenerate and so $x = 4\alpha(1 + (\mu_e/3T))/\pi$. For $\mu_e = 100$ MeV and $T = 30$ MeV, $\Gamma_m \simeq 6 \times 10^{-8}$ MeV $\simeq 10^{14}/$ s.

The equation for the local evolution of the net helicity density including helicity flipping and electron capture rates is given by

$$\frac{1}{n_5}\frac{\partial n_5}{\partial t} = -\Gamma_B - \Gamma_m + \frac{n_e}{n_5}\Gamma_w \tag{3.20}$$

where Γ_w is the rate of depletion of the electron fraction Y_e due to electron capture via charged current interactions during core collapse. Although Γ_w is density and temperature dependent, and thus governed by complex supernova dynamics, a nearly model independent upper bound can be be derived by noting that the total change during core collapse $\delta Y_e \simeq 0.4$ occurs on a time scale that is greater than the free-fall timescale $t_{\text{free-fall}} \simeq 100$ ms. Therefore,

$$\Gamma_w = \frac{\dot{Y}_e}{Y_e} < 10 \ \ \mathrm{s}^{-1} \, . \tag{3.21}$$

However, simulations indicate that the typical value is $\Gamma_w \simeq 1\ \mathrm{s}^{-1}$ [145] and this numerical estimate is used in the following calculations. Γ_m is the equilibration rate of n_5 due to explicit chiral symmetry breaking by the electron mass, given above in Eq. 3.19, and Γ_B is the anomalous depletion rate of n_5 due the conversion of n_5 into magnetic field via the CPI. A formula for Γ_B is derived in the massless electron limit in Eq. 3.10, in the presence of a chemical potential μ_5. In the realistic case with nonzero electron mass, chirality is only approximately conserved, and there is no chemical potential for chirality. Instead there is the effective $\delta\bar{\mu}_5(t)$ computed in Eq. 3.15. However, simply substituting this into Eq. 3.10 is not valid in general, since the growing mode solution Eq. 3.7 was derived assuming a constant μ_5, which can be thought of as allowing the heat bath to provide an infinite source of energy for the growing magnetic field.

The case where it is approximately valid to use Eq. 3.10 with the substitution $\mu_5 \to \delta\bar{\mu}_5(t)$ is when the CPI effect has a negligible effect on the background chiral density n_5. This regime can be explored to show that it is in fact a self-consistent solution during core collapse. Neglecting Γ_B in Eq. 3.20, a fixed point solution is found where the slow production of n_5 from the weak interactions is balanced against the rapid equilibration of n_5 due to the nonzero electron mass:

$$n_5 = \frac{\Gamma_w}{\Gamma_m} n_e \sim 10^{-14} n_e \,. \tag{3.22}$$

Using Eq. 3.15, this steady-state density corresponds to a very small time-independent effective chemical potential

$$\delta\bar{\mu}_5 = \frac{\pi^2 n_5}{\mu_e^2} = \frac{\pi^2 n_e \Gamma_w}{\mu_e^2 \Gamma_m} \simeq \frac{\mu_e}{3}\frac{\Gamma_w}{\Gamma_m} \sim \frac{1}{3} 10^{-14} \mu_e \tag{3.23}$$

This steady state value can be used to compute the rate of magnetic field production, Γ_B, as well as the length scale of the unstable mode, $k_\star$; $k_\star$ is fond to be $k_\star^{-1} = \pi/(\alpha\delta\bar{\mu}_5) \sim 250$ m for $\mu_e \simeq 100$ MeV which is astrophysically interesting. Since $k_\star^{-1}$ is a macroscopic length scale it is much larger than the electron mean free path which justifies the use of Eq. 3.3 to calculate the growth of the CPI. Using the above expression for $\delta\bar{\mu}_5$ in Eq. 3.10 and Eq. 3.7 to compute Γ_B, we find

$$\Gamma_B(t) = \frac{2\alpha^2}{\pi^2\sigma}\frac{\delta\bar{\mu}_5}{n_5} B_\star(t)^2 = \frac{2\alpha^2}{\sigma\mu_e^2} B_\star(t)^2 = \frac{2\alpha^2}{\sigma\mu_e^2} B_\star(0)^2 e^{2t\Gamma_{\rm CPI}} \,, \qquad \Gamma_{\rm CPI} = \frac{\alpha^2 \delta\bar{\mu}_5^2}{\pi^2\sigma} \,. \tag{3.24}$$

The above expression for Γ_B is only valid to the extent that $\Gamma_B \ll \Gamma_w$, or else the fixed point solution Eq. 3.22 – obtained by ignoring the effects of Γ_B – will not be correct. Such an inequality will break down eventually due to the exponential growth of $B_\star(t)$ if the prefactor proportional to the seed field $B_\star(0)^2$ at wavenumber $k_\star$ is sufficiently large and the time scale $\Gamma_{\rm CPI}^{-1}$ is sufficiently short compared to the duration of core collapse and the concomitant electron capture.

Both the prefactor and the exponential growth rate depend on the electrical conductivity σ, which is quite high in the supernova plasma. A lower bound can be derived by assuming that protons are non-degenerate and uncorrelated,

$$\sigma \gtrsim \sigma_{\rm min} = \frac{\mu_e}{4\alpha}\left[\ln\frac{4}{x} - 1\right]^{-1} , \qquad x \equiv \frac{q_D^2}{\mu_e^2} . \tag{3.25}$$

which implies that

$$\Gamma_B(0) = \frac{2\alpha^2}{\sigma\mu_e^2} B_\star(0)^2 \lesssim \frac{8\alpha^3}{\mu_e^3}\left[\ln\frac{4}{x} - 1\right] B_\star(0)^2 \sim \Gamma_m \times \left(\frac{B_\star(0)}{5\times 10^{14}\ \mathrm{G}}\right)^2 \tag{3.26}$$

and

$$\Gamma_{\rm CPI} = \frac{\alpha^2\delta\bar{\mu}_5^2}{\pi^2\sigma} \lesssim \frac{4\alpha^3}{9\pi^2}\left[\ln\frac{4}{x} - 1\right]\left(\frac{\Gamma_w}{\Gamma_m}\right)^2 \mu_e \sim 6\times 10^{-34}\ \mathrm{MeV} \sim 10^{-12}\ \mathrm{s}^{-1} . \tag{3.27}$$

At the beginning of the collapse, the assumption that Γ_B may be neglected compared to Γ_m is justified unless the initial seed field $B_\star(0)$ is already very large, around 10^{15} G. Furthermore, for more moderate initial magnetic fields, the extremely slow growth rate means that no exponential enhancement of the magnetic field occurs during the few seconds of core collapse. Finally it should be noted that if Γ_B was ever large compared to Γ_m, that would only serve to drive n_5 smaller, slowing the process down and driving it to smaller wave number $k_\star$. It is remarkable that the relatively large value obtained for Γ_m – which is proportional to m_e^2 – is responsible for damping out the Chiral Plasma Instability. This may be the first time that the fact that the electron is not massless has been shown to play a critical role in the structure and evolution of neutron stars.

In closing, a few comments on the idea that a permanent instability could persist in cold neutron matter due to the the neutral current interaction between electrons and neutrons, proportional to $G_F(\bar{e}\gamma^\mu\gamma_5 e)(\bar{n}\gamma_\mu n)$. It has been observed that in mean field theory, this term gives an effective contribution to the electron dispersion relation that resembles a chiral chemical potential, $(G_F n)(\bar{e}\gamma^0\gamma_5 e)$, where n is the neutron density. That such a term could lead to a magnetic instability was proposed in ref. [146], and considered but discarded much earlier by Vilenkin [147], who also considered the effects of rotation. While an attractive idea for generating the large magnetic fields observed in magnetars, there is an absence of an energy source for the growing magnetic field in this scenario, making Vilenkin's conclusion that such a mechanism does not work more intuitively plausible. This is to be contrasted with the scenario considered in this paper, where the growth of the helical magnetic field is powered by gravitational energy released during core collapse and temporarily stored in fermi energy of the left handed and right hand electrons, which are temporarily out of thermal equilibrium with each other; a mechanism that fails because the electron mass does not allow them to depart very far from equilibrium. Apparently what is needed to explain magnetars is a more efficient mechanism for transferring the gravitational energy released during collapse into electromagnetic energy.

Chapter 4

SIGN PROBLEMS, NOISE, AND CHIRAL SYMMETRY BREAKING IN A QCD-LIKE THEORY[1]

The QCD Lagrangian was formulated over 40 years ago and yet its application to the properties of bulk matter at nuclear density has proven to be stubbornly intractable. An inherently nonperturbative problem, the only reliable tool available for solving QCD in this energy regime is Monte Carlo evaluation of lattice QCD. Unfortunately, lattice methods suffer from a severe sign problem in the grand canonical formulation that renders Monte Carlo techniques useless for the simulation of nuclear matter at arbitrary baryon chemical potential. In a canonical formulation, despite recent impressive progress in studying light nuclei and hypernuclei [148], there remain severe problems with signal-to-noise ratios; these problems seem to share a similar underlying cause as the sign problem, namely the lightness of meson states compared to baryon states. Continued here is the research program outlined in Refs. [149–151] where a study of the probability distributions of correlators was used to argue that the origins of the sign or noise problem lie in the dynamics and spectrum of the theory, and that it is not especially useful to think of it as a “fermion" sign problem. In the particular case of QCD, the severity of the sign problem is closely related to the phenomenon of chiral symmetry breaking and the consequential existence of a light pion.

Here, to elucidate the connection between the QCD sign problem and chiral symmetry breaking, a simpler theory – the Nambu-Jona-Lasinio (NJL) model [152] in three dimensions is studied. This formulation of the NJL model with N flavors is of particular interest because it is soluble in large N, because it exhibits chiral symmetry breaking without the complication of confinement, and because it has two complementary but equivalent path integral

[1]This work has been previously published in Ref. [3]

formulations – one of which looks very QCD-like and has a severe sign problem, while the other resembles more closely the chiral quark model [29] and has no sign problem.

After reviewing some of the features of the sign problem in QCD, an analysis of the NJL will be presented. In particular, probability distributions for correlator measurements are analytically calculated, which show how the noise spectrum in these measurements is related to the presence or absence of a sign problem in the grand canonical formulation. These results should be directly applicable to suitable lattice formulations of this theory, though all the analysis is analytic and in continuum Euclidean spacetime. [2] Lastly, this chapter will conclude with speculation on the implications of this analysis for finding a path integral formulation of QCD that would allow for numerical study of bulk matter.

4.1 The sign and noise problems in QCD and the unique role of the pion

Numerical simulations of lattice QCD involve an approximation of the Feynman path integral in Euclidean spacetime. This requires a Monte Carlo evaluation of the averages of operators of interest - such as hadron correlation functions - over an ensemble of background gauge field variables generated with probability distribution

$$\mathcal{P}(U) \propto e^{-S[U]}\Delta[U] \tag{4.1}$$

where U is a link variable for the gluon degrees of freedom, S is a suitable discretization of the Yang-Mills action, and Δ is the fermion determinant; both S and Δ are functionals of the link variable U. This program has been very successful for determining properties of the QCD spectrum in the vacuum, but it runs into an obstacle when trying to simulate matter at finite density in a grand canonical ensemble. The fermion determinant Δ is a discretized version of $\det(\not{D} - m + \mu\gamma_1)$, where D_μ is the covariant derivative, m is the quark mass, μ is the chemical potential for the quark number, and the gamma matrices $\gamma_1, \ldots, \gamma_4$ are all Hermitian. At nonzero μ the fermion operator is in general complex, since $\mu\gamma_1$ is Hermitian while $\not{D}$ is anti-Hermitian, and the two do not commute. As a result the

[2] Note that x_1, ∂_1, and γ_1 with the Euclidean time direction; γ matrices satisfy $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$.

expression in Eq. 4.1 is not a suitable probability measure for the gauge field configurations; this problem has nothing to do with discretization of the lattice action, since it is a property of the continuum functional. Therefore, the discussion here will focus on the sign problem in the continuum, as the discretization of spacetime is not directly relevant (although a poor choice of discretization can in principle introduce additional sign problems not present in the continuum theory).

At nonzero chemical potential $\Delta(A_\mu, \mu) = |\Delta(A_\mu, \mu)|e^{i\theta}$; the phase starts to fluctuate wildly with changes of the gauge field for $\mu \geq m_\pi/2$ at low temperature [153]. This behavior seems precocious, since at $T = 0$, nonzero μ can have no effect on the free energy until $\mu \geq m_N/3 > m_\pi/2$, where m_N is the nucleon mass. The phenomenon was clearly explained by Splittorff and Verbaarschot [154, 155] (SV) who noted that for two degenerate flavors, $|\Delta(A_\mu, \mu)|$ correctly describes the fermion determinant with a chemical potential μ for *isospin* - namely $+\mu$ for the up quark and $-\mu$ for the down quark. Such a system will exhibit Bose-Einstein condensation of pions, with free energy becoming rapidly negative for $\mu > m_\pi/2$, as the pion is the lightest state carrying isospin. From chiral perturbation theory they derive the estimate (in a continuum Euclidean spacetime volume V, for $\mu \geq m_\pi/2$)

$$\frac{\int [dA] \left[e^{-S_{YM}[A]} \left|\det\left(\not{\partial} - m - \mu\gamma_1\right)\right|^2 e^{2i\theta}\right]}{\int [dA] \left[e^{-S_{YM}[A]} \left|\det\left(\not{\partial} - m - \mu\gamma_1\right)\right|^2\right]} \simeq \frac{2\mu F_\pi/m_\pi^2}{\sqrt{2\pi V F_\pi^4}}\, e^{-2VF_\pi^2\mu^2(1-m_\pi^2/4\mu^2)^2} \,, \tag{4.2}$$

where F_π is the pion decay constant. This result shows that the phase $\theta[A]$ is responsible for cancellations in the integral that lead to this ratio going to zero exponentially with the volume for $\mu > m_\pi/2$. The sign problem can be thought of as being necessary to keep pions *out* of the ensemble for nonzero baryon number.

A complementary obstacle is seen when studying the baryon spectrum in a canonical formulation of lattice QCD, at zero chemical potential. A typical approach to compute the mass M_B of the ground state with baryon number B is to use Monte Carlo methods to measure the correlation function $C_B(\tau) = \langle \mathcal{O}(0,\tau;A_\mu)\rangle$, where $\mathcal{O} = G(0,\tau;A_\mu)^{3B}$, G being the quark propagator from time $t = 0$ to $t = \tau$, with color, flavor, and spin indices appropriately contracted to produce a state with the desired quantum numbers. This correlation function must behave as $C_B(\tau) \propto \exp(-M_B\tau)$ at large τ and one can therefore extract M_B as the limit of $M_B = -\ln C_B(\tau)/\tau$ as $\tau \to \infty$. This procedure seems straightforward and innocent of any sign problem since we have set $\mu = 0$ and the fermion determinant is real and positive. However, one finds that a Monte Carlo measurement of $C_B(\tau)$ is very noisy at late time. A particularly simple explanation was given by Lepage [156]: he pointed out that the variance in the measurement of C_B is given by the average $\sigma^2 = \langle G(0,\tau;A_\mu)^{3B} G^\dagger(0,\tau;A_\mu)^{3B}\rangle$, corresponding to the propagation of $3B$ quarks and $3B$ antiquarks, with a ground state consisting of $3B$ pions. Therefore one would expect $\sigma^2 \propto \exp(-3Bm_\pi\tau)$ at late time, with the signal-to-noise ratio for the Monte Carlo measurement of C_B being proportional to $\exp(-3\tau(M_B/3 - Bm_\pi/2))$, which vanishes exponentially fast since the mass of B pions is less than 2/3 the mass of the B-nucleon ground state. Thus eventually the signal will always be overwhelmed by noise, and again it is due to the pion being lighter per constituent quark than the nucleon, just as we saw in the grand canonical example. If one works at fixed baryon density ρ with $B = V\rho$, where V is the volume, one sees that the noise problem grows exponentially with the volume, as one would expect from the problem encountered in the grand canonical approach, Eq. 4.2.

Savage[3] has extended the analysis to look at higher moments of the distribution function for C_B. Even moments all involve equal numbers of quarks and antiquarks, and therefore fall off exponentially with a rate determined by the pion mass. However odd moments involve expectations of operators with a net baryon number, and fall off more quickly, relative to σ - exponentially with the nucleon mass. With odd moments vanishing faster with τ than even moments, one can conclude that the probability distribution for C_B evolves exponentially fast at late time to a symmetric distribution with vanishing mean, a manifestation of the sign problem that in either approach is due to the existence of the light pion.

There have been a number of microscopic analyses of the QCD sign problem (for example,

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[3] M. Savage, private communication, 2010.

Ref. [154]), as well as general proposals to modify the conventional approach to simulating the Feynman path integrals (such as the meron cluster approach, Ref. [157]). Here, the connection between the sign problem and chiral symmetry breaking, the origin of the pion's small mass, will be pursued further.

4.2 The large-N NJL model in three dimensions

Consider a modified version of the original NJL model [152], with N flavors and reduced from four to three Euclidean dimensions:

$$\mathcal{L} = N\left(\overline{\psi}_a(\partial\!\!\!/ - m)\psi_a - \frac{g}{2}\left[(\overline{\psi}_a\psi_a)^2 + (\overline{\psi}_a i\gamma_5\psi_a)^2\right]\right) , \tag{4.3}$$

where $a, b, \ldots$ are flavor indices summed over $1, \ldots, N$, three-dimensional (3D) coordinate indices are $i, j, \ldots$, summed over $1, 2, 3$ and Greek indices $\mu, \nu, \ldots$ are summed over four-dimensional (4D) coordinates $1, \ldots, 4$. Thus $\partial\!\!\!/ = \gamma_i\partial_i$, for example. The gamma matrices are the usual 4×4 matrices used in 4D, not the 2×2 matrices appropriate for 3D, and so the Lagrangian represents $2N$ flavors of 3D Dirac fermions. However, in general, 4D nomenclature will be used- in particular, in the limit $m \to 0$ this theory has a $U(1)$ chiral symmetry in 4D, which becomes a flavor symmetry in 3D; despite the fact that there is no notion of chiral symmetry in 3D, the global $U(1)$ symmetry will be referred to as a chiral symmetry.

In 3D, this model still exhibits the key feature of the original NJL model, the phenomenon of spontaneous chiral symmetry breaking that gives rise to a Goldstone boson which, in analogy to 4D, will be called the pion. The theory is analyzed in a $1/N$ expansion using techniques similar to those used in the Gross-Neveu model, which is formulated in two-dimensional (2D). [4] This theory has been reviewed in [158, 159] and has been studied numerically in [160].

[4] Although the theory given in Eq. 4.3 is often referred to as the Gross-Neveu model, it is not asymptotically free and chiral symmetry breaking requires a critical value for the coupling g as in the original NJL model in 4D, and unlike the original Gross-Neveu model at large-N in 2D; therefore, the model will be referred to as as the 3D NJL model at large N.

4.2.1 The σ/π formulation

The conventional treatment of the theory Eq. 4.3 is to introduce auxiliary fields σ and π to obtain a bilinear fermion action,

$$\mathcal{L} = N\left(\frac{1}{2g}\left(\sigma^2+\pi^2\right)+\overline{\psi}_a\left[\not{\partial}-m+\sigma+i\pi\gamma_5\right]\psi_a\right) \ . \tag{4.4}$$

In this formulation, the σ and π fields are singlets under the $SU(N)$ flavor symmetry, while $\phi = (\sigma + i\pi)/\sqrt{2}$ transforms linearly under the chiral $U(1)$ symmetry. With the above normalization, N counting is simple: every vertex and every fermion loop brings a factor of N, every propagator a factor of $1/N$. Loops that include scalar propagators do not give a factor of N, since the mesons do not carry the N flavors.

No sign problem

An interesting feature of this theory is that at finite chemical potential the fermion determinant is real, and for even N, it is positive. To see this, note that there exists a real symmetric charge conjugation matrix C that satisfies $C^2 = 1$, $C\gamma_i C = \gamma_i^*$ for $i = 1, 2, 3$, and $C\gamma_5 C = -\gamma_5^*$. For example, taking $\gamma_i = \sigma_1 \times \sigma_i$, $\gamma_5 = \sigma_3 \times 1$ with $C = \sigma_2 \times \sigma_2$, the fermion operator for a single flavor in the grand canonical formulation satisfies $D^* = CDC$, where $D = (\not{\partial} - m + \sigma + i\pi\gamma_5 + \mu\gamma_1)$. It follows that complex eigenvalues of D come in conjugate pairs, while real eigenvalues can be unpaired. Thus $\det D$ for the N-flavor theory, Eq. 4.4, equals a real number raised to the power N and is positive for even N. This implies that there is no sign problem obstacle to simulating this theory at finite density using Monte Carlo methods [161].

Chiral symmetry breaking

Integrating out the fermions gives rise to the effective action $S_{\text{eff}} = NS(\sigma,\pi)$, where

$$S[\sigma,\pi] = \int \mathrm{d}^3x \left[\frac{1}{2g}(\sigma(x)^2+\pi(x)^2) - \mathrm{Tr}\ \ln(\not{\partial} - m + \sigma(x) + i\pi(x)\gamma_5)\right] \ . \tag{4.5}$$

The large-N expansion is equivalent to the semiclassical expansion, and the vacuum is characterized by the classical solution that minimizes the action S. The action with constant σ, $S(\sigma, 0)$, can be computed using using dimensional regularization and minimal subtraction (MS). [5] Up to an irrelevant additive constant

$$S(\sigma, 0) = VT \left[\frac{\sigma^2}{2g} + \frac{\left[(\sigma - m)^2\right]^{3/2}}{3\pi} \right] , \tag{4.6}$$

where the system is in a box of spacetime volume VT. Defining the chiral symmetry breaking minimum to be at $\langle\sigma\rangle = f$, when $m = 0$,

$$\left. \frac{\partial S(\sigma, 0)}{\partial \sigma} \right|_{\substack{m=0 \\ \sigma=f}} = 0 \quad \Longrightarrow \quad g = -\frac{\pi}{f} , \tag{4.7}$$

with $f > 0$. With this value for g (which is renormalization scheme dependent) and for nonzero quark mass $m > 0$,

$$S(\sigma, 0) = VT \left[-\frac{\sigma^2 f}{2\pi} + \frac{\left[(\sigma - m)^2\right]^{3/2}}{3\pi} + \mathcal{S}_0 \right] , \tag{4.8}$$

with minimum at

$$\langle\sigma\rangle = \frac{f}{2} \left[1 + \sqrt{1 + 4\eta} + 2\eta \right] , \qquad \eta \equiv \frac{m}{f} , \tag{4.9}$$

which shows how the chiral symmetry breaking minimum depends on the explicit quark mass. The constant, $\mathcal{S}_0$, is chosen such that the action vanishes in the chiral symmetry breaking vacuum:

$$\mathcal{S}_0 = \frac{f^3}{12\pi} \left[(1 + 4\eta)^{3/2} + (1 + 6\eta + 6\eta^2) \right] . \tag{4.10}$$

Next expanding the effective action S to second order in spacetime dependent fluctuations of the σ and π fields,

$$S(\sigma, \pi) = S(\langle\sigma\rangle, 0) + \frac{1}{2} V^2 T^2 \int \frac{d^3k}{(2\pi)^3} \left[D_\sigma(k) \delta\sigma(k) \delta\sigma(-k) + D_\pi(k) \delta\pi(k) \delta\pi(-k) \right] + .. \tag{4.11}$$

[5] In this theory minimal subtraction MS means no subtraction, as there are no logarithmic divergences and hence no $1/(d-3)$ poles

where D_σ and D_π are readily computed in the MS scheme from one-loop diagrams in the background $\langle\sigma\rangle$. The constituent quark mass M is given by

$$M \equiv \langle\sigma\rangle - m = \frac{f}{2}\left[1+\sqrt{1+4\eta}\right] \ . \tag{4.12}$$

Since there is no confinement in this theory, M is the mass of the lightest fermionic excitation and m is the current quark mass. With the fermion propagator, $G(p) \equiv (-i\not{p} + M)^{-1}$,

$$\begin{aligned} D_\sigma(k) &= \frac{1}{g} + \int \frac{d^3q}{(2\pi)^3} \mathrm{Tr}\left[G(q+k/2)G(q-k/2)\right] = -\frac{f}{\pi} + \frac{(4M^2+k^2)\cot^{-1}(2M/k) + 2Mk}{2\pi k} \ , \\ D_\pi(k) &= \frac{1}{g} - \int \frac{d^3q}{(2\pi)^3} \mathrm{Tr}\left[\gamma_5 G(q+k/2)\gamma_5 G(q-k/2)\right] = \frac{2M - 2f + k\cot^{-1}(2M/k)}{2\pi} \ . \end{aligned} \tag{4.13}$$

To leading order in $1/N$, these functions determine the σ and π dispersion relations, and their masses are defined by the location of zeros in D_σ and D_π, respectively. In the chiral limit, $m_\sigma^2 = (2f)^2$. However, the pole sits at the beginning of the two-fermion branch cut, for $m > 0$ we find $m_\sigma > 2M$, and so the σ field is unstable. Only at subleading order in $1/N$ would one see the branch cut appear in D_σ for $\sigma \to \pi\pi$ decay, as that entails an additional quark loop. A chiral expansion for the pion mass yields

$$m_\pi^2 = 4fm\left(1 - \frac{1}{3}\eta + \frac{31}{45}\eta^2 - \frac{1654}{945}\eta^3 + \dots\right) \ , \tag{4.14}$$

and near the pion pole, $k^2 = -m_\pi^2$, the pion propagator is

$$\frac{1}{ND_\pi(k)} \simeq \frac{1}{N}\frac{Z_\pi}{k^2+m_\pi^2} \ , \qquad Z_\pi = 4\pi f\left(1 + \frac{1}{3}\eta - \frac{4}{15}\eta^2 + \dots\right) \ . \tag{4.15}$$

The positivity of m_σ^2 and m_π^2 for $m > 0$ shows that the correct (e.g. stable) vacuum has been found.

4.2.2 The A/V formulation

An alternative formulation of the theory follows from using the Fierz identity

$$\left[\delta_{ij}\delta_{kl} - (\gamma_5)_{ij}(\gamma_5)_{kl}\right] = \tfrac{1}{2}\left[(\gamma_\mu)_{il}(\gamma_\mu)_{kj} - (\gamma_\mu\gamma_5)_{il}(\gamma_\mu\gamma_5)_{kj}\right] \tag{4.16}$$

to rearrange the four-fermion interaction in Eq. 4.3 before introducing auxiliary fields. The Lagrangian becomes

$$\mathcal{L} = N\left(\overline{\psi}_a(\partial\!\!\!/ - m)\psi_a + \frac{g}{4}\left[(\overline{\psi}_a\gamma_\mu\psi_b)(\overline{\psi}_b\gamma_\mu\psi_a) - (\overline{\psi}_a\gamma_\mu\gamma_5\psi_b)(\overline{\psi}_b\gamma_\mu\gamma_5\psi_a)\right]\right) \tag{4.17}$$

where $\partial\!\!\!/ = \gamma_i\partial_i$ with i summed over $i = 1, 2, 3$, while the γ_μ matrices are summed over $\mu = 1, \ldots, 4$. This formulation invites the introduction of $N \times N$ matrix valued vector and axial vector auxiliary fields V and A, giving the equivalent theory

$$\mathcal{L} = N\left(\frac{1}{g}\,\mathrm{Tr}\,\left(V_\mu V_\mu + A_\mu A_\mu\right) + \overline{\psi}\left[\partial\!\!\!/ - m + iV\!\!\!/ + A\!\!\!/\gamma_5\right]\psi\right) \quad . \tag{4.18}$$

N counting in this theory is different from the σ/π formulation, since the A and V meson fields are $N \times N$ matrices. In fact, the N counting here is identical to that of large-N QCD, and it is convenient to employ 't Hooft's double-line notation for the mesons. As in QCD, the order of a graph without external legs is given by N^χ, where χ is the Euler characteristic of the surface defined by the graph, and so to leading order only planar graphs with a minimal number of closed fermion loops need to be considered. However this class of graphs is much simpler than in QCD, since the A and V mesons have no cubic or quartic interactions, unlike gluons.

A QCD-like sign problem

In this A/V formulation, the fermion matrix at finite chemical potential is given by $D(\mu) = \left[\partial\!\!\!/ - m + iV\!\!\!/ + A\!\!\!/\gamma_5 + \mu\gamma_1\right]$, which is similar in structure to the QCD Dirac matrix with nonzero chemical potential μ (with the N flavors playing the role of color) and its determinant is similarly complex. In fact, as in QCD, an analogous SV argument [154, 155] can be made about the phase of the fermion determinant by considering two degenerate families (e.g. $2N$ fermions) so that the chiral symmetry is enlarged from $U_L(1) \times U_R(1)$ to $U_L(2) \times U_R(2)$. The fermion determinant in the case of a quark number chemical potential is $(\det D(\mu))^2$ while with an isospin chemical potential it is $|\det D(\mu)|^2$, the difference between the two being the phase $e^{2i\theta}$. In the latter case there is a transition to a pion-condensed state at $\mu = m_\pi/2$ just as in QCD, and so the SV argument leads to a similar formula as in Eq. 4.2.

It is remarkable that a theory with a sign problem so similar to that of QCD is known to have a formulation that has no sign problem, the σ/π formulation of the previous section.

Chiral symmetry breaking

Chiral symmetry breaking in the A/V formulation cannot be seen in a mean field formalism, as there is no fundamental field with the right quantum numbers to play the role of the $\overline{\psi}\psi$ condensate - again, as in QCD. Instead one has to consider the Schwinger-Dyson equation for the fermion propagator, which is exact in leading order in $1/N$ since vertex and meson propagator corrections occur at higher order. Taking the full canonically normalized fermion propagator to equal

$$G(p) = \frac{1}{-i\not{p}Z(p) + M(p)} \ , \tag{4.19}$$

which satisfies the integral equation

$$\begin{aligned} -i\not{p}Z(p) + M(p) &= -i\not{p} - m - \frac{g}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\left[-\gamma_\mu\left(\frac{1}{-i\not{k}Z(k)+M(k)}\right)\gamma_\mu \right. \\ &+ \left. \gamma_\mu\gamma_5\left(\frac{1}{-i\not{k}Z(k)+M(k)}\right)\gamma_\mu\gamma_5\right] \\ &= -i\not{p} - m + 4g\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{M(k)}{k^2Z(k)^2+M(k)^2} \ . \end{aligned} \tag{4.20}$$

From this one finds (using the MS renormalization scheme as before) that $Z(k) = 1$ and $M(k) = M$ is a constant satisfying

$$M = -m + 4g\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{M}{k^2+M^2} = -m - \frac{gM^2}{\pi} \ . \tag{4.21}$$

which has the solution

$$M = \frac{\pi}{2g}\left(-1 \pm \sqrt{1 - \frac{4gm}{\pi}}\right) \ . \tag{4.22}$$

Using the renormalization condition that the dynamical fermion mass equals f in the the chiral limit, $m = 0$, gives $g = -\pi/f$, as in Eq. 4.7 and the solution for M

$$M = \frac{f}{2}\left(1 + \sqrt{1+4\eta}\right) \qquad \eta \equiv \frac{m}{f} \ , \tag{4.23}$$

which is the same as the value derived in the σ/π formulation for the dynamical fermion mass, Eq. 4.12.

In the A/V formulation the σ and π mesons appear as a fermion and antifermion pair bound together by strong A_μ and V_μ vector meson exchange, much the same way as mesons arise in large-N QCD. Specifically, they can be seen by computing the connected four-point function $\mathcal{M}$ for a fermion/antifermion pair of flavors ℓ, k, each with 3-momentum $k/2$, to scatter into a fermion/antifermion pair of flavors i, j with 3-momenta $k/2 \pm p$. To leading order in $1/N$, $\mathcal{M}$ obeys the integral equation, shown graphically in Fig. 4.1,

$$\begin{aligned}\mathcal{M}_{ij;kl}(k,p) &= \frac{gN}{2}\Big\{[-(\gamma_\mu)_{il}(\gamma_\mu)_{kj} + (\gamma_\mu\gamma_5)_{il}(\gamma_\mu\gamma_5)_{kj}] + N[-(\gamma_\mu)_{ia}(\gamma_\mu)_{dj} + (\gamma_\mu\gamma_5)_{ia}(\gamma_\mu\gamma_5)_{dj}] \\ &\quad \int \frac{d^3q}{(2\pi)^3} G(q+k/2)_{ab}\mathcal{M}_{bc;kl}(k,q)G(q-k/2)_{cd}\Big\} \\ &= -gN\Big\{[\delta_{ij}\delta_{kl} - (\gamma_5)_{ij}(\gamma_5)_{kl}] - N\int \frac{d^3q}{(2\pi)^3}[\delta_{ij}\delta_{da} - (\gamma_5)_{ij}(\gamma_5)_{da}] \\ &\quad G(q+k/2)_{ab}\mathcal{M}_{bc;kl}(k,q)G(q-k/2)_{cd}\Big\} \ .\end{aligned} \tag{4.24}$$

where $G(p)$ is the free fermion propagator with dynamical mass M, and a Fierz identity replaces vector and axial vector gamma matrices by scalar and pseudoscalar. The solution to this equation is

$$\begin{aligned}\mathcal{M}_{ij;kl}(k) &= -N\left(\frac{\delta_{ij}\delta_{kl}}{D_\sigma(k)} + \frac{(i\gamma_5)_{ij}(i\gamma_5)_{kl}}{D_\pi(k)}\right) \\ &= -N^2\left(\delta_{ij}\delta_{kl}G_\sigma(k) + (i\gamma_5)_{ij}(i\gamma_5)_{kl}G_\pi(k)\right) \ ,\end{aligned} \tag{4.25}$$

where the label p has been drooped, as the solution is independent of p, $D_{\sigma,\pi}$ are given in Eq. 4.13, and $G_{\sigma,\pi} = N/D_{\sigma,\pi}$ are the full meson propagators. Thus, up to a sign, the interaction between valence fermion and antifermion via t-channel exchange of A_μ and V_μ mesons is exactly equivalent to an annihilation diagram in the σ/π formulation, with a single meson in the s-channel; similarly, the interaction between two valence fermions or two valence antifermions in the A/V theory is equivalent to a single meson exchange in the u channel in the σ/π theory (Fig. 4.2). This equivalence allow the cumulants of the A/V theory to be calculated in the σ/π theory.

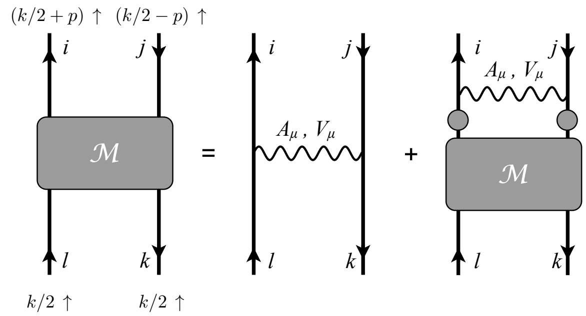

Figure 4.1: A graphical representation of the integral equation, Eq. 4.24, for the four-point correlator in the A/V formulation for an incoming fermion/antifermion pair of one flavor and an outgoing pair of another. Dirac indices are labeled.

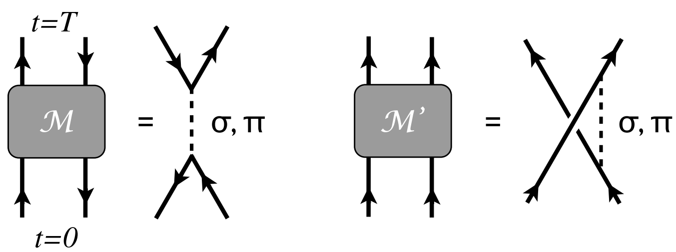

Figure 4.2: In the A/V formulation at leading order in $1/N$, interaction between a valence fermion/antifermion pair ($\mathcal{M}$) or a valence fermion pair ($\mathcal{M}'$) is equivalent to s- or u-channel exchange respectively of single σ and π mesons in the σ/π formulation, where the fermions have mass M arising from nonzero $\langle\sigma\rangle$.

4.3 *The probability distribution of the fermion propagator*

If $X[\phi]$ is a functional of a stochastic field ϕ corresponding to an observable — such as a correlation function —the normalized probability density function (pdf) for X is defined to be the path integral

$$\mathcal{P}(x) = \mathcal{N} \int [\mathrm{d}\phi]\, e^{-S[\phi]}\, \delta(X[\phi] - x) \ . \tag{4.26}$$

where $S[\phi]$ is assumed real. If one were to sample an ensemble of ϕ configurations according to the distribution $e^{-S[\phi]}$, the values of $X[\phi]$ would be distributed according to $\mathcal{P}$, making its relevance to Monte Carlo simulations evident. For an accurate estimate of $\langle X\rangle$ from a reasonable number of samples, $\mathcal{P}(x)$ should be sharply peaked around its mean. However, a broad distribution centered about a mean close to zero is also possible, which would make finding an accurate estimate of the mean, without a huge number of samples, very difficult. This is the case with baryon propagators in QCD. Alternatively, a heavy-tailed distribution for which very rare events make a significant contribution to the mean, results in very noisy and often misleading estimates of $\langle X \rangle$ from a finite sample. This latter situation is indicative of what is called "an overlap problem," which occurs when $e^{-S[\phi]}$ is peaked far from the field configurations that provide support for nonzero $X(\phi)$. With some knowledge about the nature of the tail of the distribution, it may be possible to use statistical methods to greatly improve the determination of $\langle X \rangle$ [150].

The pdf given in Eq. 4.26 is a difficult quantity to analyze using field theoretic methods because of the singular nature of the delta function; instead, consider the characteristic function (cf) $\Phi_X(s)$, which is just the Fourier transform of the pdf:

$$\Phi_X(s) = e^{-W(s)} = \mathcal{N} \int [\mathrm{d}\phi] \, e^{-S[\phi]+isX[\phi]} \ . \tag{4.27}$$

This is a useful formulation because $W(s) = -\ln \Phi_X(s)$ is on the one hand given by the connected Feynman diagrams [6] of the modified action $S[\phi]-isX[\phi]$, while, up to an irrelevant additive constant, it is also the generating function for the cumulants of X:

$$W(s) = -\sum_{n=1}^{\infty} \frac{(is)^n}{n!} \kappa_n \ . \tag{4.28}$$

Here, κ_n is the nth cumulant, with $\kappa_1 = \langle X \rangle \equiv \mu$ being the mean of $\mathcal{P}(x)$, $\kappa_2 = (\langle X^2 \rangle - \langle X \rangle^2) \equiv \sigma^2$ being the variance, etc.

This procedure has to be modified slightly when dealing with a complex observable, where Eq. 4.27 is replaced by

$$\Phi_X(s, \bar{s}) = e^{-W(s,\bar{s})} = \mathcal{N} \int [\mathrm{d}\phi] \, e^{-S[\phi]+i(sX[\phi]+\bar{s}\bar{X}[\phi])} \ . \tag{4.29}$$

[6] Since the functional $X[\phi]$ is typically nonlocal, by connected Feynman diagrams we mean neither conventional Feynman diagrams, nor conventionally connected. For example, expanding $X[\phi]$ to second order in ϕ might take the form $\left(\int d^3x F(x)\phi(x)\right)^2$ for some function F; this is treated as a fundamental 2-point vertex in our discussion.

where s is complex and the bar indicates complex conjugation. Now, $\ln W(s,\bar{s})$ has a double expansion in both s and $\bar{s}$,

$$W(s,\bar{s}) = -\sum_{m,n=1}^{\infty} \frac{(is)^m (i\bar{s})^n}{m!\,n!}\,\kappa_{m,n} \ . \tag{4.30}$$

with $\kappa_{n,m} = \bar{\kappa}_{m,n}$. For example,

$$\kappa_{1,0} = \langle X\rangle \ , \qquad \kappa_{1,1} = \langle |X|^2\rangle - |\langle X\rangle|^2 \ , \qquad \kappa_{2,0} = \langle X^2\rangle - \langle X\rangle^2 \ , \qquad \ldots \tag{4.31}$$

The cumulants of the correlation function of a single fermion with zero 2-momentum for a time extent τ can be calculated; this is the sort of measurement one would perform in a Monte Carlo simulation in order to determine the mass of the fermion. In particular taking for $X[\phi]$

$$Y_\Gamma = \ln\left[\frac{1}{V}\langle \mathbf{p}=0, t=\tau/2\,|\mathrm{Tr}\,\Gamma\left(\frac{1}{\not{\partial} - m + \sigma + i\pi\gamma_5}\right)|\mathbf{p}=0, t=-\tau/2\rangle\right] \qquad \sigma/\pi \text{ formulation,} \tag{4.32a}$$

$$X_\Gamma = \frac{1}{V}\langle \mathbf{p}=0, t=\tau/2\,|\mathrm{Tr}\,\Gamma\left(\frac{1}{\not{\partial} - m + i\not{V} + \not{A}\gamma_5}\right)|\mathbf{p}=0, t=-\tau/2\rangle \qquad A/V \text{ formulation,} \tag{4.32b}$$

where Γ is some Dirac matrix, freely chosen. The choice to look at the log of the propagator in the σ/π formulation makes the later analysis simpler. Note that in the definitions of the observable X_Γ and Y_Γ, canonically normalized fermion propagators are used, without the factor of $1/N$.

Measuring the expectation value of this correlator is a procedure for determining the mass m_f of the lightest fermion state through the formula

$$\lim_{\tau\to\infty} -\frac{1}{\tau}\ln\langle X_\Gamma\rangle = \lim_{\tau\to\infty} -\frac{1}{\tau}\ln\langle e^{Y_\Gamma}\rangle = m_f \ , \tag{4.33}$$

provided that Γ does not project out this state. Of course, it is already known from analytic calculations that $m_f = M$, with M given in Eq. 4.12, but the calculation of the cumulants for this observable establishes how difficult it would be to determine the fermion mass by numerical Monte Carlo methods using the two formulations of the NJL model, and why. In particular it will be shown that in the QCD-like A/V formulation, the pdf for X_Γ, at late time, looks as would be expected from the Lepage-Savage picture: a broad distribution that is nearly symmetric about zero with an exponentially small mean. In contrast, the pdf for the physically equivalent σ/π formulation looks heavy-tailed and close to log-normal at late time. Thus, a Monte Carlo study of this theory, without a sign problem, still faces an overlap problem and significant numerical challenges, but is perhaps amenable to a cumulant expansion as introduced for a similar system in Refs. [149–151]. This theory has also been successfully investigated recently using the “fermion bag approach" [162–164].

4.3.1 Noise distribution in the A/V formulation and the Lepage-Savage analysis

The $\kappa_{1,0}$ and $\kappa_{1,1}$ cumulants

First, compute the cumulants for measurements of the fermion propagator X_Γ in the A/V formulation; since this observable is complex, the formalism in Eq. 4.29 and 4.31 must be used. From the above discussion, the $\kappa_{mn} = \bar{\kappa}_{nm}$ cumulant for X_Γ is given by the sum of connected graphs with m copies of X_Γ and n copies of its complex conjugate, $\bar{X}_\Gamma$. These are referred to as valence fermion and valence antifermion propagators, respectively; at leading order in $1/N$ there are no sea quark loop contributions. With the definition of X, there are no factors of $1/N$ from valence fermion propagators, nor factors of N from meson coupling to valence fermions, nor is annihilation of a valence fermion with a valence antifermion allowed. The computation of $\kappa_{m,n}$ involves graphs with net fermion number $(m-n)$, and according to the Lepage-Savage analysis, it is expected that $\kappa_{m,n}$

$$\kappa_{m,n} \propto e^{-[(m-n)M+nm_\pi]\tau} \qquad (m \geq n) \ . \tag{4.34}$$

It will be shown that this expectation is born out in the A/V formulation, which is the one with a QCD-like sign problem at nonzero chemical potential.

Instead of the graphs contributing to the valence fermion self-energy, the nonperturbative solution to the Schwinger-Dyson equation should be used, replacing the mass term $(-m)$ in the fermion propagator by M from Eq. 4.23. It is convenient to have this propagator in a mixed $\{t, \mathbf{p}\}$ representation:

$$\langle \mathbf{p}', t' | \frac{1}{\partial\!\!\!/ + M} | \mathbf{p}, t \rangle \equiv (2\pi)^2 \delta^2(\mathbf{p}' - \mathbf{p}) \widetilde{G}(\mathbf{p}, t' - t) \ , \tag{4.35}$$

with

$$\widetilde{G}(\mathbf{p}, t) = \int \frac{\mathrm{d}\omega}{2\pi} \frac{e^{-i\omega t}}{-i\omega\gamma_1 - i\mathbf{p}\cdot\gamma + M} = e^{-\omega_p |t|} \frac{\omega_p \epsilon(t)\gamma_1 + i\mathbf{p}\cdot\gamma + M}{2\omega_p} \ , \tag{4.36}$$

where $\omega_p = \sqrt{|\mathbf{p}|^2 + M^2}$. Note that

$$\widetilde{G}(\mathbf{0}, t) = e^{-M|t|} \left(\frac{1 + \epsilon(t)\gamma_1}{2} \right) \ . \tag{4.37}$$

is proportional to a projection operator. Since the dynamical mass M includes all nonperturbative contributions to the fermion self-energy, it follows that the first cumulant for X_Γ is just

$$\kappa_{1,0} = \mathrm{Tr} \left[\Gamma \widetilde{G}(\mathbf{0}, \tau) \right] = z e^{-M\tau} = \langle X_\Gamma \rangle \ , \tag{4.38}$$

where $z \equiv \mathrm{Tr}\left[\Gamma\left(\frac{1+\gamma_1}{2}\right)\right]$ is the wave-function overlap of our chosen observable with the physical fermion state. Thus a Monte Carlo simulation that correctly estimates the value of $\langle X_\Gamma \rangle$ will correctly determine the fermion mass to equal M by means of the formula Eq. 4.33, provided that Γ is chosen so that z is nonvanishing.

To determine how difficult a Monte Carlo determination of $\langle X_\Gamma \rangle$ might be, the variance $\kappa_{1,1}$ needs to be calculated. At leading order in $1/N$, the sum of diagrams for $\kappa_{1,1}$ is given by attaching $\widetilde{G}$ propagators at zero spatial momentum to the legs in the first diagram in Fig. 4.1, with ends k, l at time $t = -\tau/2$ and ends i, j at time $\tau/2$, and contracting the Dirac indices of each valence fermion line with Γ. Using the result for the four-point function $\mathcal{M}$ in Eq. 4.25 (but dividing by N^2, since the X_Γ is the propagator for a canonically normalized fermion), the D_σ part of $\mathcal{M}$ is killed by the Dirac trace and only the D_π part contributes, yielding

$$\begin{aligned}
\kappa_{1,1} &= \frac{1}{VN}\int_{-\infty}^{\infty}\mathrm{d}t_1\int_{-\infty}^{\infty}\mathrm{d}t_2\int\frac{\mathrm{d}\omega}{2\pi}\,\frac{e^{-i\omega(t_2-t_1)}}{D_\pi(\omega)}\\
&\times\ Tr\Gamma\widetilde{G}(\mathbf{0},\tau/2-t_2)\gamma_5\widetilde{G}(\mathbf{0},t_2-\tau/2)\Gamma\widetilde{G}(\mathbf{0},-\tau/2-t_1)\gamma_5\widetilde{G}(\mathbf{0},t_1+\tau/2)\\
&= \int_{-\infty}^{\infty}\mathrm{d}t_1\int_{-\infty}^{\infty}\mathrm{d}t_2\,e^{-M(|\tau-2t_2|+|\tau+2t_1|)}\\
&\times\ \mathrm{Tr}\,\Gamma\left(\frac{1+\epsilon(\frac{\tau}{2}-t_2)\gamma_1}{2}\right)\gamma_5\Gamma\gamma_5\left(\frac{1+\epsilon(t_1+\frac{\tau}{2})\gamma_1}{2}\right)\bar{G}_\pi(t_2-t_1)\ , \qquad (4.39)
\end{aligned}$$

where

$$\bar{G}_\pi(t)\equiv\int\frac{d^2x}{V}\,G_\pi(\mathbf{x},t)=\frac{1}{VN}\int\frac{\mathrm{d}\omega}{2\pi}\,\frac{e^{-i\omega(t)}}{D_\pi(\omega)}\simeq\frac{Z_\pi}{2m_\pi VN}e^{-m_\pi|t|} \qquad (4.40)$$

with D_π, Z_π given in Eq. 4.13 and Eq. 4.15, respectively, and $D_\pi^{-1}(\omega)$ is approximated by the pion pole contribution, Eq. 4.15, ignoring the branch cut at $k^2<-4M^2$. If $\Gamma=1$,

$$\begin{aligned}
\kappa_{1,1} &\simeq \frac{Z_\pi}{2m_\pi VN}\int_{-\infty}^{\infty}\mathrm{d}t_1\int_{-\infty}^{\infty}\mathrm{d}t_2\,e^{-2M(|\frac{\tau}{2}-t_2|+|t_1+\frac{\tau}{2}|)}\left(1+\epsilon\left(\frac{\tau}{2}-t_2\right)\epsilon\left(t_1+\frac{\tau}{2}\right)\right)e^{-m_\pi|t_2-t_1|}\\
&\simeq \frac{2\pi}{m_\pi fVN}e^{-m_\pi\tau}\ , \qquad (4.41)
\end{aligned}$$

where (i) it is assumed that the theory near the chiral limit with $m_\pi\ll f,M$ and (ii) only the late time behavior, $\tau\gg 1/M$ is of interest here. Note that near the chiral limit $\kappa_{1,0}/\sqrt{\kappa_{1,1}}\ \propto\ \exp[(-M+m_\pi/2)\tau]\ll 1$ at late time, indicating a severe signal-to-noise problem, in agreement with the Lepage analysis.

It is interesting to note that if instead $\Gamma=(1\pm\gamma_1)/2$ then $\kappa_{1,1}$ vanishes at this order in $1/N$. This choice of Γ kills the pion contribution to the variance, and so the measurement would seem to be noise-free. This result is not expected to persist in subleading order in $1/N$, nor that in real QCD baryon observables can be decoupled from pions so easily. It does seem worth exploring whether correlating initial and final Dirac indices of baryon operators (as with this trace with $(1\pm\gamma_1)/2$ on valence quark lines) might be able to improve the signal-to-noise problem in real QCD computations.

Power counting for higher $\kappa_{m,n}$ cumulants

Higher cumulants can be computed for the A/V formulation using the equivalent σ/π diagrams as discovered in Sec. 4.2.2 and shown in Fig. 4.2. This is *not* to say that the same diagrams give the cumulants for the A/V and the σ/π theories. Fig. 4.3 shows the diagrams for several of the lower cumulants, where solid lines are the fermion propagators $1/(-i\not{p}+M)$, and dashed lines are the meson propagators G_π and G_σ (both mesons contributing in general) with couplings 1 or $i\gamma_5$, respectively, at the vertices. Again, all incoming and outgoing 2-momenta to be zero, and there is one $\Gamma_{ii'}$ contracted with each pair of like indices in the graph. As the canonically normalized fermion propagator with one particular flavor is the observable, there are no factors of N at the meson vertices, nor are there any loops giving rise to factors of N. However, each meson propagator costs a factor of $1/N$, and so $\kappa_{m,n} \propto (1/N)^{m+n-1}$, as a minimum of $(m+n-1)$ mesons is needed to make a connected graph. Furthermore, one can see by cutting the graphs at a fixed time that the minimum mass state that can possibly propagate in a graph for $\kappa_{m,n}$ with $m \geq n$ consists of $(m-n)$ fermions with mass M and n pions. Therefore generically the cumulants can be expected to scale as

$$\kappa_{m,n} \sim \frac{e^{-(m-n)M\tau} e^{-nm_\pi \tau}}{N^{m+n-1}} \qquad (m \geq n) \ . \tag{4.42}$$

This scaling could be violated if Γ can be chosen so that the pion does not couple, as discussed in the calculation of $\kappa_{1,1}$. Then m_π is replaced by $2M$, the mass of a fermion/antifermion pair. Note that at late time, the $1/N$ expansion breaks down in the sense that $\kappa_{2,0}$ becomes smaller than $\kappa_{2,2}$, for example, so long as one is near enough to the chiral limit that $m_\pi < M$. This is despite the fact that $\kappa_{2,2}$ is parametrically smaller by $1/N^2$. However, this breakdown of the $1/N$ expansion will not lead to qualitatively different results because contributions to $\kappa_{m,n}$ which are subleading in N counting will not lead to lighter intermediate states than the leading calculation, unless there is some fortuitous exclusion of the pion at leading order due to the choice of Γ that does not persist at higher order.

The above scaling implies that the distribution for the real part of the fermion propagator

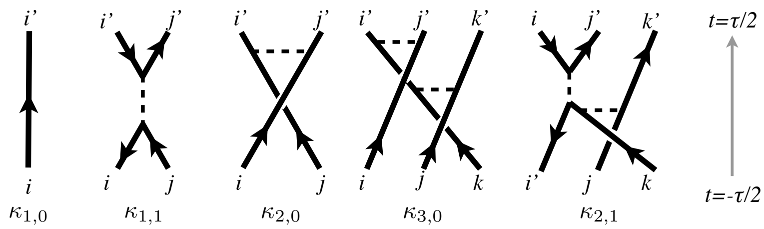

Figure 4.3: Contributions to several low cumulants for the A/V theory to leading order in $1/N$, using the σ/π method to compute them as developed in Sec. 4.2.2. Solid lines represent fermion propagators, and dashed lines are mesons. The end points are to be contracted with $\Gamma_{ii'}$, $\Gamma_{jj'}$, $\Gamma_{kk'}$. The lightest intermediate state that can appear in a graph for $\kappa_{m,n}$ with $m \geq n$ consists of $(m-n)$ fermions and n pions, and the sum of their masses determines the τ dependence of the cumulant.

near the chiral limit becomes highly symmetric about zero at late time because odd moments (for which $(m-n) \neq 0$) are seen to fall off much more quickly than even moments. This is completely consistent with the Lepage-Savage picture for baryon propagator distributions in QCD. As in QCD, it will be very difficult to use Monte Carlo methods for the A/V formulation to determine the ground state energy with a fixed large fermion number. This is not surprising because, like QCD, this theory also has a severe sign problem in the grand canonical formulation for studying states with nonzero fermion density.

4.3.2 Noise distribution in the σ/π formulation and long-tailed distributions

A graphical expansion for cumulants of $Y_\Gamma(\sigma, \pi)$

Turning to the task of computing the cumulants for the log of the fermion correlator, Y_Γ in Eq. 4.32a in the σ/π formulation, the cumulants are given by the connected graphs derived from the action

$$S_Y = NS[\sigma, \pi] - isY_\Gamma[\sigma, \pi] \tag{4.43}$$

where $S[\sigma, \pi]$ is given in Eq. 4.5. The N counting in this formulation is quite different from the A/V case since the σ and π mesons are singlets under the $U(N)$ symmetry rather than

$N \times N$ matrices. It is convenient to associate the expansion of $NS[\sigma, \pi]$ in meson fields with vertices labeled by black dots; see Fig. 4.4. There are no black tadpoles, as the chiral symmetry breaking vacuum has been solved; furthermore there are no explicit black two-point vertices as these are accounted for by using the full meson propagators. The expansion of $isY_\Gamma[\sigma, \pi]$ in powers of the meson fields is represented as white dots, the kth term in the expansion drawn as a white vertex connecting k meson lines; white vertices occur with any number of meson lines, starting with zero, and each is associated with a power of is.

The nth cumulant κ_n is then given by $n!$ times the sum of connected graphs with n insertions of white vertices, since each white vertex brings a factor of (is) and the expansion Eq. 4.28. Expanding these graphs in powers of $1/N$ is simple: each black vertex entails a power of N, while each meson propagator gives a factor of $1/N$. Loops do not give factors of N since the σ/π mesons do not carry $U(N)$ flavor quantum numbers. White dots also do not give factors of N. Thus a contribution to κ_n arising from a graph with n white vertices, b black vertices, p meson propagators and ℓ loops is of order N^{b-p}. Since every such graph satisfies $p-(b+n)=\ell-1$, the order of the graph can be rewritten as $N^{-(n-1+\ell)}$. It follows that the leading contribution to κ_n must come from the sum of tree diagrams ($\ell=0$) with n white vertices. The branches of these connected tree diagrams must end on white tadpoles, and are quite limited in number; the first few leading diagrams are shown in Fig. 4.5.

The sum of tree diagrams can be regarded as the solution to a classical theory, which in the present case is nothing other than the statement that the $1/N$ expansion is equivalent to solving for the generator of cumulants, $\ln \Phi_Y(s)$, as a saddle-point approximation dominated by the classical solution that minimizes the action S_Y (see App. A for more details). Before tackling this calculation it is worthwhile to note several simplifying features of these tree diagrams:

i. By choosing a Γ that is neither γ_5 nor $\gamma_\mu\gamma_5$, only the σ meson can couple to the white tadpole. Since parity implies that vertices conserve pion number mod 2, and all tree diagrams end on white tadpole vertices (Fig. 4.5), it follows that the only mesons prop-

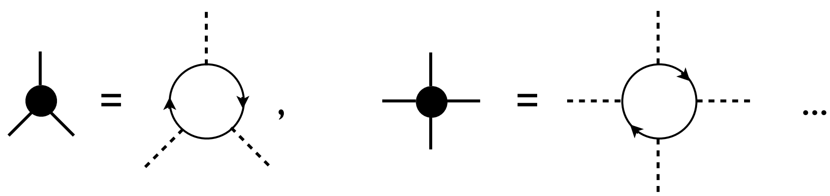

Figure 4.4: Black vertices arise in the expansion of the fermion loop, $NS(\sigma,\pi)$ of Eq. 4.5 in external meson fields. The tadpole was eliminated by vacuum minimization, and the two-point function gives the exact meson propagators, so these vertices start with the three-point function.

agating in these tree graphs are σ mesons.

ii. Since the observable Y_Γ in Eq. 4.32a is defined to be the log of the propagator of a quark with initial and final 2-momentum $\mathbf{p} = 0$, the meson at the white tadpole vertex must have $\mathbf{p} = 0$ flowing through it as well. Then, because all vertices conserve momentum and the leading graphs in Fig. 4.5 are all tree diagrams whose branches end on white tadpoles, it follows that *all* of the internal meson lines in the graph must be at $\mathbf{p} = 0$, with only nonzero energy flowing through the lines.

iii. As will be shown below, the large τ behavior of the cumulants arises from graphs with zero energy flowing through the white tadpole; thus, because of conservation of

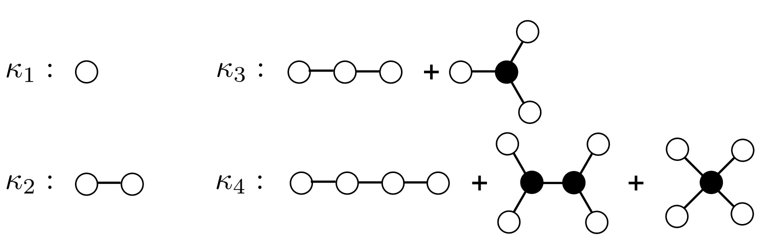

Figure 4.5: Leading contributions in the $1/N$ expansion of κ_n for $n = 1, \ldots, 4$ in the σ/π formulation. Black vertices are given in Fig. 4.4; white vertices are determined by the expansion of $isY[\sigma,\pi]$. Lines represent the exact meson propagators $G_{\sigma,\pi}$ derived from Eq. 4.13.

3-momentum at all vertices, the asymptotic τ behavior of the cumulants is given by the graphs with 3-momentum p vanishing everywhere within the graph.

To demonstrate the last point, that the white tadpole enforces zero energy flow through the diagram at late time, compute the tadpole, assuming energy k_1 and two momentum $\mathbf{k}$ flowing out of the meson line:

$$\begin{aligned}
\text{tadpole} &= \frac{-\frac{1}{V}\int_{-\infty}^{\infty} d\tau' \,\mathrm{Tr}\,\Gamma\widetilde{G}(\mathbf{0},\tau'+\tau/2)\delta\sigma(\mathbf{0},\tau')\widetilde{G}(\mathbf{0},\tau/2-\tau')}{\frac{1}{V}(2\pi)^2\delta^2(0)\mathrm{Tr}\,\Gamma\widetilde{G}(\mathbf{0},\tau/2)} \\
&= -\frac{1}{V}\int_{-\tau/2}^{\tau/2} d\tau' \int \frac{d^3k}{(2\pi)^3}(2\pi)^2\delta^2(\mathbf{k})e^{-ik_1\tau'}\delta\sigma(k) \\
&= -\frac{1}{V}\int \frac{d^3k}{(2\pi)^3}(2\pi)^2\delta^2(\mathbf{k})\frac{2\,\sin\,(k_1\tau/2)}{k_1}\delta\sigma(k) \\
&\xrightarrow[\tau\to\infty]{} -\frac{1}{V}\int \frac{d^3k}{(2\pi)^3}(2\pi)^3\delta^3(k)\delta\sigma(k)\ ,
\end{aligned} \tag{4.44}$$

where $\widetilde{G}(\mathbf{0},\tau)$ is given in Eq. 4.37, provided that

$$z = \mathrm{Tr}\left[\Gamma\left(\frac{1+\gamma_1}{2}\right)\right] \neq 0\ . \tag{4.45}$$

The $\delta(k_1)$ factor justifies computing the tree graphs in Fig. 4.5 with zero 3-momentum flowing through it, as long as the interest is only in the $\tau \to \infty$ behavior of the distribution of propagators. However, note that at finite τ the factor $\sin(k_1\tau/2)/k_1$ acts as a filter that still allows $k_1 \lesssim 1/\tau$ or, equivalently, which still allows the σ field to be time dependent on time scales $\gtrsim \tau$. This will be relevant to the next section.

The generator of cumulants from mean field theory

As mentioned previously, the sum of tree graphs is nothing other than the effective action $S_Y = NS - isY_\Gamma$ evaluated at its minimum, a classical solution for the meson fields (see App. A for further discussion). It has been shown that for an appropriate choice of Γ, this solution will in general have a vanishing pion field and a spatially constant, but time dependent, $\sigma(t)$ field. Finally, it has also been shown that the large-τ behavior is given by an even simpler solution, with a σ field that is constant in the two spatial dimensions and over time $\lesssim \tau$.

It is straightforward to compute $S_Y(\sigma)$ for a constant σ field. Note that σ is defined relative to its vacuum value $\langle\sigma\rangle = (M+m)$, where M is the actual “constituent" mass of the fermion, while m is the “current" mass appearing in the Lagrangian. For the $NS(\sigma)$ part of S_Y only the expression Eq. 4.8 evaluated at $\sigma_S = \sigma - \langle\sigma\rangle$ is needed; up to an overall additive constant:

$$NS(\sigma_S) = -NVT\left(\frac{3f\sigma_S^2 + 6\sigma_S M^2 + 2M^3 - 2\left((\sigma_S+M)^2\right)^{3/2}}{6\pi}\right) \tag{4.46}$$

where V is the spatial volume and T is the time extent of the box. The second part of S_Y is given by

$$isY_\Gamma = is\ln\left[\int\frac{d\omega}{2\pi}\,e^{-i\omega\tau}\mathrm{Tr}\,\Gamma\frac{1}{-i\omega\gamma_1 + M + \sigma_S}\right] = -is\left((M+\sigma_S)\tau - \ln z\right)\ . \tag{4.47}$$

The equation for the minimum of S_Y is therefore given by

$$\left[\sqrt{(M+\sigma_S)^4} - (M^2 + f\sigma_S) + \frac{i\pi s\tau}{NVT}\right]_{\sigma_{S0}} = 0 \tag{4.48}$$

with solution

$$\sigma_{S0} = \frac{1}{2}\left((f-2M) + \sqrt{(f-2M)^2 - 4\pi is\tau/(NVT)}\right) \tag{4.49}$$

where the solution that vanishes as $s \to 0$ is chosen, as it recovers the correct chiral symmetry breaking vacuum. Plugging this solution back into the effective action, the generator of cumulants of Y_Γ, to leading order in the $1/N$ expansion and at late time is

$$\ln\Phi_Y(s) = -S_Y(\sigma_{S0}) = is\mu + (2M-f)\tau\,\frac{6is\zeta - 1 + (1-4is\zeta)^{3/2}}{12\zeta}\ , \tag{4.50}$$

with

$$\zeta = \frac{\pi\tau}{NVT(2M-f)^2}\ , \qquad \mu \equiv \ln z - M\tau\ . \tag{4.51}$$

Expanding $\ln\Phi_Y(s)$ in powers of s yields the cumulants κ_n for Y_Γ:

$$\kappa_1 = \mu\ , \qquad \kappa_{n\geq 2} = \frac{(2(n-2))!}{(n-2)!}\,(2M-f)\tau\,\zeta^{n-1} \tag{4.52}$$

The volume factor in the denominator of ζ is easy to understand, arising from the normalization of the one particle states; the factor of T is puzzling though, arising from the assumptions of a mean field solution that is constant over the entire spacetime volume. This does not make sense, since there is no need for σ_S to adjust from its chiral symmetry breaking minimum long before or long after the correlator has acted. As pointed out in the discussion below Eq. 4.45, σ_S should not be constant over time scales $\gtrsim \tau$, and in fact the mean field should relax to its vacuum value for $t \lesssim -\tau/2$ and $t \gtrsim \tau/2$. Therefore the factor of T in ζ — the temporal size of the box — should be replaced by $\sim \tau$, the time scale of the correlator, and

$$\zeta \simeq \frac{\pi}{NV(2M-f)^2} \tag{4.53}$$

which is independent of τ. Perhaps one can compute a more accurate, time-dependent mean field solution, but this is not pursued here. In an analogous calculation of the Polyakov loop distribution at finite temperature, a strictly time independent mean field solution is probably exact in the large N limit, due to the homogeneity in time of the operator being measured.

Note that in the particular limit

$$N \to \infty \ , \qquad \tau \to \infty \ , \qquad \tau\zeta = \text{fixed} \tag{4.54}$$

all cumulants κ_n vanish for $n \geq 3$ and Y_Γ assumes a normal distribution,

$$\mathcal{P}(x) = \frac{\sqrt{2\pi}}{\Sigma} e^{\left[-\frac{(x-\mu)^2}{\Sigma^2}\right]} \left(1 + \frac{\zeta(x-\mu)}{3\Sigma^4}[(x-\mu)^2 - 3\Sigma^2] + \mathcal{O}\left(\zeta^2\right)\right) \ , \tag{4.55}$$

where $\Sigma^2 = (2M-f)\tau\zeta$, which is to say, a Monte Carlo simulation of the fermion propagator will be sampling a log-normal distribution. With Σ^2 growing linearly with time and a skewness that grows exponentially with Σ, this distribution will eventually become very heavy-tailed. Standard simulation methods would fail for such a distribution, but one could use a cumulant expansion of the Monte Carlo data to obtain an accurate measure of the fermion mass, as described in Ref. [150]. However, it should be noted that this limit requires $\tau \gtrsim NV$, which is unlikely to be reached in practical lattice simulations of this model. For the more realistic limit $\Sigma^2 \ll 1$ one may use standard statistical techniques and the signal-to-noise ratio in this case will be $\approx 1/\sqrt{N_{\rm cfg}\Sigma^2} \sim \tau^{-1/2}$, where $N_{\rm cfg}$ is the number of gauge field configurations sampled. This power-law dependence on time is far less severe than the exponential falloff of the signal-to-noise ratio in the A/V case [see Eq. 4.42].

4.4 Discussion

The motivation for returning to the well-worn Nambu-Jona-Lasinio model is to elucidate connections between chiral symmetry breaking and the sign problem in lattice QCD at finite chemical potential, without having to deal with the complications of asymptotic freedom and confinement. What is particularly attractive about the large-N, three-dimensional version of the theory is that (i) it is analytically tractable to compute features of the probability distribution of a fermion correlator, and (ii) the theory has two equivalent formulations, one without a sign problem, and one with a QCD-like, exponentially severe sign problem.

For the QCD-like A/V formulation, the fermion determinant is complex and that a Splittorff-Verbaarschot argument [154,155] can be made to show that the phase of the fermion determinant has to fluctuate wildly for $\mu > m_\pi/2$, with an expectation value exponentially small in the spatial volume. When looking at fermion correlators, the distribution evolves to have an exponentially small mean relative to its width, implying a severe signal-to-noise ratio when sampling the correlator using Monte Carlo methods. Furthermore, the severity of the problem is controlled by the difference between the fermion constituent mass M (playing the role of the baryon mass in QCD) and the much lighter pion mass m_π. This follows the Lepage-Savage scaling argument that has even cumulants of the distribution diminishing as a power of $\exp(-m_\pi\tau)$, while the odd cumulants - including the mean - fall off proportional to $\exp(-M\tau)$. It is interesting that in three dimensions one can choose an observable for measuring the fermion mass [by a particular choice of the matrix Γ in Eq. 4.32b] which eliminates the coupling of the fermion/antifermion pair to the pion, and thereby eliminates the problem of noise in the measurement of the fermion mass. Such a trick might be a useful way to reduce the noise in simulations of QCD in four dimensions, even if it cannot eliminate it.

In contrast, the σ/π formulation with even N has no sign problem at nonzero μ, and the correlator distribution - the cumulants of which can be analytically computed - is, in a certain limit, log-normal. For extremely long times the distribution is heavy-tailed, which can pose challenges to Monte Carlo sampling, but this sort of problem seems to be less severe than the exponential falloff of the signal-to-noise ratio in the A/V formulation as seen with the cumulant expansion analysis of Refs. [150, 165, 166]. For more moderate times, standard statistical methods should apply, and the signal-to-noise ratio in this case is found to have only power-law suppression with time. It should be noted that such distributions have been seen in QCD for intermediate times, before any asymptotic pion noise sets in. It has been hypothesized that these distributions are related to elastic scattering between particles [167]; the volume factors in our expressions for the cumulants, Eqs. 4.52 and 4.53, give support to this picture.

The analysis should make it clear that the sign problem encountered in QCD at nonzero chemical potential is not a fermion problem, but instead a consequence of interactions. In particular, if the particles being studied can exist in a tightly bound state of valence fermions, there is going to be a sign problem - a generic feature of a theory with dynamical chiral symmetry breaking, in which a light composite pion emerges as a Goldstone boson. This is what happens in the A/V formulation of the NJL model studied here: the fact that the A and V fields will bind a fermion/antifermion pair into a light or massless pion implies that studies of the fermion correlator will be noisy, and that at nonzero chemical potential for the fermion there will be a sign problem. In the σ/π formulation without a sign problem, the pion exists as a fundamental field and not as a bound state.

The lessons learned from this model raise the question: is it possible to introduce a mean field for the pion into the formulation of lattice QCD (without changing the theory) so that the pion does not appear as a bound state of a valence quark/anti-quark pair, $\bar{Q}Q$? For example, one might add and subtract a four-fermion interaction to the QCD Lagrangian; the attractive one could be introduced by σ and π auxiliary fields, while the repulsive interaction could be derived by means of auxiliary A and V fields. Then a valence $\bar{Q}Q$ pair would feel the usual gluon attraction, but A/V repulsion, and so would not bind to form a light pion. Instead, pions would appear as fundamental fields that could be created through $\bar{Q}Q$ annihilation, but would not appear as bound states of valence quarks. It is expected that the conventional sign problem could be ameliorated in such a theory - but this example probably introduces other sign problems, a cure perhaps as devastating as the disease.

Nevertheless, it seems plausible that inventing a way to introduce the pions into QCD as fundamental fields could be an important step toward solving the QCD sign problem and beginning to study the properties of ordinary and dense matter from first principles.

Chapter 5

A NONPERTURBATIVE REGULATOR FOR CHIRAL GAUGE THEORIES?[1]

There is a fundamental tension between taming the ultraviolet behavior of a chiral gauge theory and maintaining gauge invariance. There is no perturbative regulator known to work to all orders, and constructing a nonperturbative lattice regulator for chiral gauge theories in $d = 2, 4$ dimensions has been a daunting problem for decades [168]. This is of particular interest since the Standard Model is a chiral gauge theory. The lack of a regulator may be a purely technical problem, but it might indicate that something is missing from the Standard Model. Naive lattice fermions are always Dirac in structure, leading to unwanted mirror fermions; in the case of chiral gauge theories, decoupling these mirror fermions by means of a large mass would entail explicitly breaking the gauge symmetry. Attempts to solve the problem generally fall into three classes: (i) the gauge symmetry is broken spontaneously [169] or explicitly [170] and mirror fermions are decoupled from the gauge fields, with a procedure to recover gauge symmetry in the continuum large-volume limit, (ii) the mirror fermions are given exotic gauge-invariant strong interactions intended to induce a mass gap in that sector [171–176], or (iii) the mirror fermions are projected out of the theory [177, 178]. In the first category, the spontaneous breaking of gauge symmetry fails to yield a chiral fermion spectrum [179], while to date the explicit breaking approach has only been shown to recover continuum gauge invariance in perturbation theory. The strategy of decoupling mirror fermions via exotic interactions is a nontrivial dynamical question. In [97] the gauge fields were given space-dependent couplings, but it is thought that the construction is unlikely to have a continuum limit [180]. Certain other cases of exotic interactions have been closely

[1]This work has been previously published in Ref. [4]

examined and do not appear to possess the expected mass gap [181–183]. In the third category, where mirror fermions are projected out, one must show that the resulting fermion contribution to the gauge measure is analytic in the gauge fields and can be derived from a local fermion action. Examples in this category include the ansatz [177] based on overlap fermions [101] which is analytic, but not obviously derivable from a local fermion action, and the work of Ref. [178] which starts from the chirally projected overlap operator [103] and provides an implicit construction of a local and analytic measure in the case of $U(1)$ gauge symmetry, but which has not been generalized to non-Abelian gauge symmetries. Any nonperturbative solution is expected to agree with low order perturbation theory, including a path to failure for the case of anomalous fermion representations, and so this will be the criteria for a successful nonperturbative regulator.

The problem of how to realize global chiral symmetries on the lattice with the correct anomalies was resolved in Ref. [97]. In this construction Dirac fermions in $2n+1$ dimensions are introduced; the theory possesses a mass gap in the bulk and massless chiral modes localized on the "domain walls", which are the $2n$-dimensional surfaces of the space. The number of such modes is a topological invariant of the bulk fermion dispersion relation [100] and the theory is an example of what condensed matter physicists currently refer to as a topological insulator. Chiral symmetry becomes exact in the limit of infinite extra dimension, in which case the effective $2n$-dimensional description of these modes is the overlap fermion [101,103]. The geometry of domain wall fermions naturally suggests that by localizing gauge interactions near one surface of the extra dimension one might obtain a continuum chiral gauge theory while mirror fermions on the distant surface decouple; this has been a starting point for many of the attempts to construct chiral gauge theories cited above. There is some reason to be optimistic about a solution involving an extra dimension [184], even though particular dynamical realizations have not been successful. Here I discuss a propose for a gauge invariant solution based on domain wall fermions that works on a new principle: gauge fields are extended into the extra dimension via "gradient flow", as solutions to a gauge covariant parabolic differential equation. The effect is that the mirror fermions are endowed with nonlocal but gauge invariant couplings which allow them to decouple in perturbation theory, leaving behind a local d-dimensional chiral gauge theory in the infrared when (and only when) the theory is gauge anomaly free. A striking feature of this proposal is that even in the anomaly-free case the mirror fermions are still sensitive to topological features of the gauge field and can therefore have nonlocal and nonperturbative interactions with ordinary matter. For all we know, such interactions could be a necessary intrinsic feature of non-Abelian chiral gauge theories, implying exotic and yet to be discovered phenomenology in the Standard Model that is not apparent in Feynman diagrams.

5.1 Definition of the Chiral Measure

Euclidean Green functions in a gauge theory can be expressed as path integral averages of functionals of gauge fields with respect to a measure $e^{-S(A)}\Delta(A)$ where S is the Maxwell or Yang-Mills action and Δ is the fermion contribution to the measure. In a vector-like gauge theory Δ is just given by a product of one determinant of the Dirac operator for each fermion flavor; in a chiral gauge theory $|\Delta|^2$ must equal a product of Dirac determinants, but the problem is how to define the phase of Δ such that it is analytic in A_μ and follows from a local fermion action. It is known that if the fermion representation is in an anomalous representation of the gauge symmetry, this phase will be gauge variant and the theory ill-defined. The proposal for Δ for a single Weyl fermion in the continuum with dimension $d = 2, 4$ is

$$\frac{\det\left[\not{D}^{(R)}_{d+1} - \Lambda\epsilon(s)\right]}{\det\left[\not{D}^{(R)}_{d+1} - \Lambda\right]}\ . \tag{5.1}$$

In this expression $\not{D}^{(R)}_{d+1}$ is the $(d+1)$-dimensional Dirac operator in the gauge group representation R for the fermion, where the extra dimension denoted by coordinate $s \in [-L, L]$ is a circle with circumference $2L$, $\epsilon(s) = \mathrm{sign}(s)$, and Λ is a real mass scale whose sign is the fermion chirality. The scale $|\Lambda|$ can be thought of as the ultraviolet cutoff of the theory and will be equated with the inverse lattice spacing in a discretized version of the theory, with

$|\Lambda| L \to \infty$.

So far the expression for Δ would just seem to describe the fermion determinants of a normal domain wall fermion, with zeromodes of chirality $\mp\text{sign}(\Lambda)$ localized on the mass defects at $s = 0$ and $s = \pm L$ respectively, with a Pauli-Villars field of constant mass Λ which cancels off unwanted effects of the heavy bulk fermions. What differs in the present formulation is the gauge field in $\not{D}_{d+1}$. In the usual application of domain wall fermions to lattice QCD the d-dimensional gauge fields are independent of the coordinate s. In contrast here an s-dependent, d-dimensional gauge field $\bar{A}_\mu(x, s)$ is used; it solves the gradient flow equation

$$\partial_s \bar{A}_\nu = \frac{\xi \epsilon(s)}{|\Lambda|} D_\mu \bar{F}_{\mu\nu} \ , \quad \mu, \nu = 1, \ldots, d \ , \tag{5.2}$$

with boundary condition $\bar{A}_\mu(x, 0) = A_\mu(x)$, where $A_\mu(x)$ is the integration variable in the path integral. In the above equation D_μ and $\bar{F}_{\mu\nu}$ are the covariant derivative and Yang-Mills (or Maxwell) field strength respectively constructed out of $\bar{A}_\mu(x, s)$, where the indices run to d, not $d + 1$. The parameter ξ is dimensionless and can be set to unity for applications, but it is useful for the discussion to keep its value general, allowing us to interpolate between the conventional application of domain wall fermions with $\xi = 0$ and s-independent gauge fields, and the case $\xi \gtrsim 1$ where the gauge flow is rapid. Note that Eq. 5.2 is covariant under s-independent gauge transformations and has fixed points at solutions to the classical Yang-Mills (Maxwell) equations of motion. When linearized for small fluctuations about a stable classical solution it behaves like the heat equation, damping out the fluctuations away from $s = 0$. An analogue called Ricci flow was introduced by mathematicians over 50 years ago to smooth out metric fields while preserving diffeomorphism invariance [185–187] and was subsequently applied to gauge fields [188]. The extension to quantum field theory employed here was developed in refs. [189–191], with precursors in refs. [192, 193], and has found a variety of useful applications in lattice QCD (see for example [194–196]).

The effect of the flow equation Eq. 5.2 is illustrated by considering a $U(1)$ gauge field in two Euclidean spacetime dimensions, flowing into a third dimension. The field A_μ can be decomposed as

$$A_\mu = \partial_\mu \omega + \epsilon_{\mu\nu}\partial_\nu \lambda, \tag{5.3}$$

and extend ω, λ to $\bar{\omega}$, $\bar{\lambda}$ with boundary conditions $\bar{\omega}(p,s)|_{s=0} = \omega(p)$, $\bar{\lambda}(p,s)|_{s=0} = \lambda(p)$ and solutions

$$\bar{\omega}(p,s) = \omega(p)\ , \qquad \bar{\lambda}(p,s) = e^{-\xi p^2 |s|/\Lambda}\lambda(p)\ . \tag{5.4}$$

Evidently the gauge degree of freedom $\bar{\omega}$ is constant over the entire extra dimension and can interact with the zeromodes at either domain wall if their individual gauge currents are not conserved, while the physical gauge field $\bar{\lambda}$ – which is invariant under gauge transformations – interacts at full strength with the chiral zeromodes at $s = 0$, but interacts with the mirror fermions at $s = \pm L$ with a Gaussian damping factor $\exp(-p^2/\mu^2)$, where $\mu \equiv \sqrt{\Lambda/\xi L}$ is an IR scale with $\mu/\Lambda \to 0$ as $L \to \infty$.

It is helpful to regard this damping factor not as a property of the gauge field, but rather as a property of the fermions which behave as large objects with a Gaussian form factor, incapable of reacting to gauge bosons attempting to transfer momentum $p \gg \mu$. In fact, if there is an infrared cutoff on the d-dimensional space so that all gauge boson modes satisfy $p \gg \mu$, where μ can be made arbitrarily small, then in the appropriate sequence of limits of vanishing μ and infinite volume, the mirror fermions can be made to decouple from the physical gauge field plane waves completely. The mirror fermions are referred to as “fluff" because of their soft form factor.[2]

Even if the fluff is decoupled, the fermions in the bulk may make contributions to the action which do not look d-dimensional. Naively it would seem that the heavy bulk fermions and Pauli-Villars fields would decouple, only contributing local operators to the effective action suppressed by powers of their mass Λ, but that is incorrect; from the analysis of Callan and Harvey [98] it is known that these modes generate a Chern-Simons term when

[2] In the lattice theory, exact symmetry of the continuum formulation that ensures massless chiral surface modes at finite L is broken by the Wilson term, and the limit $\Lambda L \to \infty$ must also be taken in order to eliminate residual $O(e^{-\Lambda L})$ chiral symmetry breaking couplings between fermions and fluff.

integrated out, a marginal operator that depends on the sign of Λ but not its magnitude. This operator in the presence of an arbitrary $d+1$-dimensional background gauge field $\bar{A}_\mu$ is, for $(d+1)=3,5$:

$$\begin{aligned} S_3^{\rm bulk} &= c_3 \frac{\Lambda}{|\Lambda|} \int (\epsilon(s)-1)\, {\rm Tr}\, \left(\bar{F}\bar{A} - \tfrac{1}{3}\bar{A}^3\right) \;, \\ S_5^{\rm bulk} &= c_5 \frac{\Lambda}{|\Lambda|} \int (\epsilon(s)-1) \\ &\qquad \times {\rm Tr}\, \left(\bar{F}^2\bar{A} - \tfrac{1}{2}\bar{F}\bar{A}^3 + \tfrac{1}{10}\bar{A}^5\right). \end{aligned} \tag{5.5}$$

where p-form notation is used, with $\bar{A} = \bar{A}_\mu^a T^a dx_\mu$, $\bar{F} = d\bar{A} + \bar{A}^2$, $\mu = 1,\ldots,d+1$, and

$$c_{2n+1} = \frac{i^n}{2^{n+1}\pi^n(n+1)!} \;, \tag{5.6}$$

and the convention for γ matrices is ${\rm Tr}\,\gamma_1\cdots\gamma_{2n+1} = (i)^n$. Restrict these gauge fields $\bar{A}_\mu$ to be the solution of Eq. 5.2, with vanishing component in the $d+1$ direction, in which case only terms that involve one derivative with respect to s contribute to the above expression. Note that as $(-{\rm sign}\,\Lambda)$ is the chirality of the zeromode at $s=0$ and T^a are generators in the same representation of the gauge group as the zeromode, the sum of contributions to S_{d+1} will cancel under the same algebraic condition as the vanishing of the d-dimensional gauge anomaly among the zeromodes at $s=0$.

The variation of the above operators under gauge transformations are total derivatives with respect to s, and integration over s yields the consistent anomaly on the surfaces $s=0$ and $s=L$ after integration by parts, using the fact that $\partial_s\epsilon(s) = 2[\delta(s)-\delta(s-L)]$. In particular, for a gauge transformation $\Omega = \exp i\varepsilon(x)$,

$$\begin{aligned} \frac{\partial S_3^{\rm bulk}}{\partial\varepsilon} &= ic_3\frac{\Lambda}{|\Lambda|}\epsilon_{\mu\nu}\partial_\mu\bar{A}_\nu\Big|_{s=0}^{s=L} \\ \frac{\partial S_5^{\rm bulk}}{\partial\varepsilon} &= ic_5\frac{\Lambda}{|\Lambda|}\epsilon_{\mu\nu\rho\sigma}\,\partial_\mu\left[\bar{A}_\nu\bar{A}_\rho\bar{A}_\sigma + 2\bar{A}_\nu\partial_\rho\bar{A}_\sigma\right]\Big|_{s=0}^{s=L} \end{aligned} \tag{5.7}$$

which has exactly the right structure to cancel the anomalies of the chiral modes at $s=0$ and at $s=\pm L$, each with the correct local gauge field $\bar{A}_\mu(s)$, which is precisely what is needed to account for the overall gauge invariance of Δ.

The existence of the Chern-Simons operators Eq. 5.5 in the effective action precludes interpreting the theory Eq. 5.1 as a local d-dimensional field theory. To illustrate this, return to the simple $U(1)$ example in $d = 2$ given in Eq. 5.4. In this case

$$\begin{aligned} S_3^{\text{bulk}} &\propto \int d^2x \int ds\, (1 - \epsilon(s))\, \epsilon_{abc} \bar{A}_a \partial_b \bar{A}_c \\ &= 2 \int d^2x d^2y \left(\frac{\partial_\mu \partial_\alpha}{\Box} A_\alpha(x) \right) \Gamma(x - y) \left(\frac{\partial_\mu \partial_\beta}{\Box} \epsilon_{\beta\gamma} A_\gamma(y) \right) \end{aligned} \tag{5.8}$$

with $\Gamma(r) = (\delta^2(r) - \frac{\mu^2}{4\pi} e^{-\mu^2 r^2/4})$, where the decomposition Eq. 5.3 and solutions Eq. 5.4 are used. Because of the inverse Laplacian factors, the only way that the effective action can behave like a local 2-dimensional theory is if either Γ or the prefactor of S_3 vanishes. In the limit $\xi \to 0$, the gradient flow is turned off and $\mu = \sqrt{\Lambda/\xi L} \to \infty$, $\frac{\mu^2}{4\pi} e^{-\mu^2 r^2/4} \to \delta^2(r)$, and Γ vanishes. In this limit the gauge field has neither an s-component nor s-dependence and so cannot contribute to the Chern-Simons action. This is the limit in which one recovers the conventional application of domain wall fermions to $d = 2$ QED: one has a local, 2-dimensional theory of a massless Dirac fermion and vanishing Chern-Simons action. However, in the case of ntereste here, $\xi = 1$ and then Γ does not vanish in the limits $\mu = \sqrt{\Lambda/L} \to 0$ as $L \to \infty$, and the only way to recover an effective action with a local 2-dimensional description is to have the contributions of the various species of bulk fermions to the Chern-Simons action cancel, which is precisely equivalent to requiring the fermion representation of the target chiral gauge theory in two dimensions to be free of gauge anomalies. With the Chern-Simons operator vanishing, the remaining bulk fermion contributions to the effective action are suppressed by powers of Λ and irrelevant. This argument holds for the construction of chiral gauge theories in four dimensions as well.

Up to this point only a continuum model has been discussed. There are no apparent obstacles to discretizing the theory using an action similar to the one commonly used for domain wall fermions [197], only with gauge fields defined by Eq. 5.2. The extra dimension and the required large volume limits will pose computational challenges, but the biggest obstacle will likely be the sign problem that is generic for chiral gauge theories due to the phase of the integration measure in the continuum [198].

5.2 Topology

Up to this point the analysis has focused on gauge field flow to the trivial fixed point of Eq. 5.2, where $\bar{A}_\mu$ is pure gauge. In general, any classical solution to the d-dimensional Yang-Mills (Maxwell) action is a fixed point, although it is plausible that the only attractive fixed points in each topological sector are the exact multi-instanton solutions. For example, an arbitrary 4-dimensional Yang-Mills gauge field configuration $A_\mu(x,0)$ with winding number k could be described by a field with fluctuations about n instantons and $\bar{n} = (n-k)$ anti-instantons, which would flow to a configuration at $s=\infty$ with k instantons and no anti-instantons. This would allow nontrivial correlation functions between ordinary fermions and fluff through 't Hooft interactions [40]. For example, with one flavor of Dirac fermion ψ and its fluff partner χ, one would find a nonzero expectation value in such a gauge configuration for $(\overline{\psi}_L\psi_R)^n(\overline{\psi}_R\psi_L)^{\bar{n}}(\overline{\chi}_L\chi_R)^{n-\bar{n}}$. In a weakly coupled theory this operator would be proportional to $\exp(-(n+\bar{n})S_0)$, where $S_0 \propto 1/\alpha$ is the large action for a single instanton, and the effect of the 't Hooft vertex would be negligible. In a strongly coupled theory one would not expect topological effects to be suppressed; however it seems plausible that a generic configuration in volume V with a large number of instantons and anti-instantons $n, \bar{n} \propto V$ but a much smaller net topological charge, $(n-\bar{n}) \propto \sqrt{V}$, the spatial locations of the $(n-\bar{n})$ instantons that survive the flow to $s=\infty$ would not be highly correlated spatially with the $s=0$ gauge field configuration. Thus the 't Hooft operator that received an expectation value would be of the form $\int d^4x\,\mathcal{O}(x)\int d^4y\,\mathcal{O}'(y)$, where $\mathcal{O}$ is comprised of fermions and $\mathcal{O}'$ of fluff. Such a vertex is nonlocal, but does not allow momentum to be transmitted between the two worlds at $s=0$ and $s=\infty$, and may not even be an extensive contribution to the action. Thus its phenomenology may prove to be benign. A natural first step toward better understanding the proposal would be to investigate the nature of this topological interaction in a vector-like gauge theory (where both matter and fluff are in a real representation of the gauge group) which would not suffer a sign problem. Other features of the theory could also be explored, such as whether colored fluff is confined.

If compatible with Standard Model phenomenology, the existence of fluff with its topological interactions could conceivably be a necessary, if unexpected, nonperturbative feature of chiral gauge theories. If one indeed takes such a view, then one must speculate whether the construction outlined here is more than a prescription for the nonperturbative regularization of chiral gauge theories, or a technical approach for their numerical simulation. Could it actually be realized in nature? In this scenario the Standard Model possesses fluff quarks and leptons which have resisted discovery due to their infinitely soft form factors under gauge interactions (and possibly gravitational interactions, due to Ricci flow). The prospect that fluff could decouple from propagating gauge fields yet participate in the topological structure of the vacuum is intriguing, perhaps allowing a massless fluff quark to solve the strong CP problem without being easily seen. The phenomenology and cosmology of such matter is under investigation.

Chapter 6
CONCLUDING REMARKS

Predictions of the Standard Model agree beautifully with experimental observation and with the discovery of the Higgs at the Large Hadron Collider in 2012, the particle content and interactions of the Standard Model are complete. But this does not mean that our knowledge of physics is complete. For one, the Standard Model offers only a partial picture of physics above the TeV scale, due to the observation of neutrino masses, the question of the particle nature of Dark Matter and the lack of knowledge about quantum gravity. With its rich and complicated phenomenology, our understanding of phenomena both within and beyond the Standard Model is incomplete.

In this thesis, I have discussed how chiral symmetries and nonperturbative phenomena have resulted in the complex structure of the Standard Model that is not yet fully understood. For example, we cannot map out the full phase diagram of QCD, as finite density nuclear matter is not amenable to numerical simulation. This prohibits us from fully understanding, from first principles, phenomena ranging from the Early Universe to heavy ion collisions to neutron stars. One of the great mysteries of particle physics is how such a simple Lagrangian – QCD – can give rise to such rich low energy structure. Nor do we understand how the flavor structure of the Standard Model is generated; all that we know for certain is that it exquisitely hard to write down BSM models that have both non-trivial flavor structure and are testable. Perhaps most deeply, we are not even sure if the Standard Model is complete. Without a nonperturbative regulator, we cannot say definitively what is the basic particle content of a chiral gauge theory, and thus the Standard Model.

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Appendix A

CONNECTION BETWEEN MEAN FIELD CALCULATION AND TREE LEVEL CUMULANT DIAGRAMS

In Sec. 4.3.2, it was asserted that the sum of tree graphs contributing to the cumulants of Y_Γ is equal to the effective action S_Y, evaluated at its minimum. Here, this equivalence is shown in more detail.

The generating function, $W(s)$ has two different representations. The first is in terms of a functional integral:

$$Z = e^{-W(s)} = \int [d\phi]\, e^{-NS(\phi)+isY(\phi)} \ . \tag{A.1}$$

The second representation is as the generating function for cumulants:

$$W(s) = \text{const} - \sum_{n=1}^{\infty} \frac{(is)^n}{n!} \kappa_n \ . \tag{A.2}$$

Changing variables, $s = Nr$,

$$e^{-W(r)} = \int [d\phi]\, e^{-N\mathcal{A}(r,\phi)} \ , \tag{A.3}$$

with $\mathcal{A}(r,\phi) = S(\phi) + irY(\phi)$, and

$$W(r) = \text{const} - N \sum_{n=1}^{\infty} \frac{(ir)^n}{n!} \left(N^{n-1}\kappa_n\right) \ . \tag{A.4}$$

Compute W in a large-N expansion is equivalent to a mean field expansion:

$$\mathcal{A}(r,\phi) = \mathcal{A}^{(0)} + \sum_{n=2}^{\infty} \frac{\mathcal{A}^{(n)}}{n!} \delta\phi^n \ , \tag{A.5}$$

where

$$\mathcal{A}^{(n)} = \left.\frac{\delta^n \mathcal{A}}{\delta\phi^n}\right|_{\phi=\phi_0} \ , \qquad \mathcal{A}^{(1)} = 0 \ , \tag{A.6}$$

with ϕ_0 being the classical solution that minimizes $\mathcal{A}$ and $\delta\phi = (\phi - \phi_0)$. Therefore

$$\begin{aligned}
e^{-W(r)} &= e^{-N\mathcal{A}^{(0)}} \int [d\delta\phi]\, e^{-N\sum\limits_{n=2}^{\infty} \frac{\mathcal{A}^{(n)}}{n!}\delta\phi^n} \\
&= e^{-N\mathcal{A}^{(0)}} \left[e^{-N\sum\limits_{n=3}^{\infty} \frac{\mathcal{A}^{(n)}}{n!}\frac{\delta^n}{\delta J^n}} \int [d\delta\phi]\, e^{-\frac{N}{2}\mathcal{A}^{(2)}\delta\phi^2 + J\delta\phi} \right]_{J=0} \\
&= \frac{\text{const}}{\sqrt{\det N\mathcal{A}^{(2)}}} \times e^{-N\mathcal{A}^{(0)}} \times \left[e^{-N\sum\limits_{n=3}^{\infty} \frac{\mathcal{A}^{(n)}}{n!}\frac{\delta^n}{\delta J^n}}\, e^{-\frac{J^2}{N\mathcal{A}^{(2)}}} \right]_{J=0} . \qquad \text{(A.7)}
\end{aligned}$$

To determine the leading contributions to κ_n, the terms that are both leading in N and r-dependent must be found.

1. The constant factor in Eq. A.7 is independent of r and does not contribute to κ_n.

2. The factor in brackets in Eq. A.7 is the sum of connected diagrams whose propagators scale as $1/N$ and vertices as N. Thus, these diagrams scale as

$$N^{V-P} = N^{1-L} , \qquad \text{(A.8)}$$

   using the topological invariant $L + V - P = 1$, where V, P, and L are the numbers of vertices, propagators, and loops respectively. These diagrams do not contain tadpoles, as there are no terms linear in ϕ. Therefore, this quantity only contributes to $W(r)$ at one loop and higher and is thus $\mathcal{O}(N^0)$ and subleading.

3. The determinant factor in Eq. A.7 contributes to $W(r)$ a term $\frac{1}{2}\text{Tr}\,\ln N\mathcal{A}^{(2)} = \text{const} + \frac{1}{2}\text{Tr}\,\ln \mathcal{A}^{(2)}$. The constant is N dependent, but r independent, whereas the second term is r dependent, but N independent. Therefore, the determinant factor comes in at $\mathcal{O}(N^0)$ and is also subleading.

4. The remaining term is $e^{-N\mathcal{A}^{(0)}}$: this gives $N\mathcal{A}^0$, the classical action at its minimum, as the leading contribution to W, at $\mathcal{O}(N)$.

Now the action at the classical minimum,$\mathcal{A}^{(0)}$, must be related to the diagrams in Fig. 4.5. From above, $W(r)$ can be expanded in powers of N:

$$W(r) \quad = \quad \text{const} + N\left[W_0(r) + N^{-1}W_1(r) + N^{-2}W_2(r) + \ldots\right] \ , \tag{A.9}$$

where $W_0(r) = N\mathcal{A}^{(0)}$. We can also write $W(r)$ in terms of cumulants,

$$W(r) = \text{const} - N\sum_{n=1}^{\infty}\frac{(ir)^n}{n!}\left(N^{n-1}\kappa_n\right) \ . \tag{A.10}$$

From this, κ_n may be expanded as

$$\kappa_n = \frac{1}{N^{n-1}}\sum_{p=0}^{\infty}\frac{k_{n,p}}{N^p} \ , \tag{A.11}$$

with

$$W_p(r) = -\sum_{n=1}^{\infty}\frac{(ir)^n}{n!}\,k_{n,p} \ . \tag{A.12}$$

Previously sum of tree graphs with n white vertices was identified as the leading contribution to the nth cumulant, $k_{n,0}$. Thus,

$$W_0 = -\sum_{n=1}^{\infty}\frac{(ir)^n}{n!}\,k_{n,0} \ , \tag{A.13}$$

which proves the assertion that the sum of tree graphs in Fig. 4.5 is equal to the effective action $S_Y = NS(\phi) - isY(\phi)$, evaluated at its minimum.