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Higher order first integrals, Killing tensors and Killing-Maxwell system

Mihai Visinescu

Department Theoretical Physics, National Institute for Physics and Nuclear Engineering, Magurele MG-6, Bucharest, Romania

E-mail: mvisin@theory.nipne.ro

Abstract. Higher order first integrals of motion of particles in the presence of external gauge fields in a covariant Hamiltonian approach are investigated. The special role of Stackel-Killing and Killing-Yano tensors is pointed out. A condition of the electromagnetic field to maintain the hidden symmetry of the system is stated. A concrete realization of this condition is given by the Killing-Maxwell system and exemplified with the Kerr metric. Another application of the gauge covariant approach is provided by a non relativistic point charge in the field of a Dirac monopole. The corresponding dynamical system possessing a Kepler type symmetry is associated with the Taub-NUT metric using a reduction procedure of symplectic manifolds with symmetries. The reverse of the reduction procedure can be used to investigate higher-dimensional spacetimes admitting Killing tensors.

1. Introduction

The evolution of a dynamical system is described in the entire phase-space and from this point of view it is natural to go in search of conserved quantities to genuine symmetries of the complete phase-space, not just the configuration one. Such symmetries are associated with higher rank symmetric Stackel-Killing (SK) tensors which generalize the Killing vectors. These higher order symmetries are known as *hidden symmetries* and the corresponding conserved quantities are quadratic, or, more general, polynomial in momenta. Another natural generalization of the Killing vectors is represented by the antisymmetric Killing-Yano (KY) tensors which in many respects are more fundamental than the SK tensors.

The conformal extension of the SK tensor equation determines the conformal Stackel-Killing (CSK) tensors which define first integrals of motion of the null geodesics. Investigations of the hidden symmetries of the higher dimensional space-times have pointed out the role of the conformal Killing-Yano (CKY) tensors to generate background metrics with black hole solutions.

In the study of motion of charged particles in external gauge fields it has been proved that a gauge covariant Hamiltonian framework [1] is more appropriate. The covariant approach is also useful to investigate the possibility for a higher order symmetry to survive when the electromagnetic interactions are taken into account. A concrete realization of this possibility is given by the Killing-Maxwell (KM) system [2].

The motion of a free point particle in a (pseudo)-Riemannian space is determined by the kinetic energy and the trajectories are geodesics corresponding to the metric tensor of the configuration space. More general than the free mechanical system, in the case of a conservative

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holonomic dynamical system whose kinetic energy is modified by the addition of a potential, the trajectories are not geodesics of the metric tensor.

In many cases it is preferably to represent the trajectories as geodesics of a related metric. For example the Jacobi metric [3], conformally related to the original one, is obtained rescaling the potential and the total energy. The drawback of the conformally related metric is that its geodesics describe the trajectories of a fixed energy.

An attractive alternative is represented by the Eisenhart lift or oxidation [4] of a dynamical system which permits to put into correspondence the trajectories of a mechanical system with the geodesics of a configuration space extended in dimension.

In the case of a symplectic manifold on which a group of symmetries acts symplectically, it is possible to reduce the original phase space to another symplectic manifold in which the symmetries are divided out. Such a situation arises when one has a particle moving in an electromagnetic field [5].

On the other hand the reverse of the reduction procedure can be used to investigate complicated systems. It is possible to use a sort of unfolding of the initial dynamics by imbedding it in a larger one which is easier to integrate [6]. Sometimes the equations of motion in a higher dimensional space are quite transparent, e. g. geodesic motions, but the equations of motion of the reduced system appear more complicated [7].

As an illustration of the reduction of a symplectic manifold with symmetries and the opposite procedure of oxidation of a dynamical system we shall consider the principal bundle $\pi : \mathbb{R}^4 - \{0\} \to \mathbb{R}^3 - \{0\}$ with structure group $U(1)$. The Hamiltonian function on the cotangent bundle $T^\star(\mathbb{R}^4 - \{0\})$ is invariant under the $U(1)$ action and the reduced Hamiltonian system proves to describe the three-dimensional Kepler problem in the presence of a centrifugal potential and Dirac's monopole field. Moreover this reduction procedure is also relevant for many other problems like the geodesic flows of the generalized Taub-NUT metric, conformal Kepler system, MIC-Kepler system, etc.

Concerning the unfolding of the reduced Hamiltonian system we shall perform it by stages. In a first stage of unfolding we use an opposite procedure to the reduction by an $U(1) \simeq S^1$ action to a four-dimensional generalized Kepler problem. Finally we resort to the method introduced by Eisenhart who added one or two extra dimensions to configuration space to represent trajectories by geodesics.

The plan of the paper is as follows. In Section 2 we establish the generalized Killing equations in a covariant framework including external gauge fields and scalar potentials. In Section 3 we describe the KM system and exemplify it with the Kerr metric. In Section 4 we make a brief review of the Hamiltonian reduction of symplectic manifolds with symmetries pointing out the Hamiltonian systems defined on the cotangent bundle $T^\star(\mathbb{R}^4 - \{0\})$ with standard symplectic form. Next Section is devoted to the reverse of the reduction procedure associated with an S^1 action to a four dimensional generalized Kepler problem. We discuss the Eisenhart procedure for the oxidation limiting ourselves to dynamical systems and scalar potentials which do not involve time. Conclusions and open problems are discussed in the last Section.

2. Symmetries and conserved quantities

The classical dynamics of a point charged particle subject to an external electromagnetic field F_{ij} expressed (locally) in terms of the potential 1-form A_i

$$F = dA\,, \tag{1}$$

is derived from a Hamiltonian

$$H = \frac{1}{2} g^{ij}(p_i - A_i)(p_j - A_j) + V(x)\,. \tag{2}$$

We also added an external scalar potential $V(x)$ for generality and $\mathbf{g}$ is the metric of a (pseudo-) Riemmanian n-dimensional manifold $\mathcal{M}$.

In terms of the canonical phase-space coordinates (x^i, p_i) the conserved quantities commute with the Hamiltonian in the sense of Poisson brackets. The disadvantage of the traditional approach is that the canonical momenta p_i and implicitly the Hamilton equations of motion are not manifestly gauge covariant. This inconvenience can be removed using van Holten's receipt [1] by introducing the gauge invariant momenta:

$$\Pi_i = p_i - A_i \,. \tag{3}$$

The Hamiltonian (2) becomes

$$H = \frac{1}{2} g^{ij} \Pi_i \Pi_j + V(x) \,, \tag{4}$$

and equations of motion are derived using the Poisson bracket appropriate for the gauge covariant approach

$$\{P, Q\} = \frac{\partial P}{\partial x^i}\frac{\partial Q}{\partial \Pi_i} - \frac{\partial P}{\partial \Pi_i}\frac{\partial Q}{\partial x^i} + qF_{ij}\frac{\partial P}{\partial \Pi_i}\frac{\partial Q}{\partial \Pi_j} \,. \tag{5}$$

A first integral of degree p in the momenta Π is of the form

$$K = K_0 + \sum_{k=1}^{p} \frac{1}{k!} K^{i_1 \cdots i_k}(x) \Pi_{i_1} \cdots \Pi_{i_k} \,, \tag{6}$$

and has vanishing Poisson bracket (5) with the Hamiltonian, $\{K, H\} = 0$, which implies the following series of coupled equations:

$$K^i V_{,i} = 0 \,, \tag{7a}$$

$$K_0{}^{,i} + F_j{}^i K^j = K^{ij} V_{,j} \,. \tag{7b}$$

$$K^{(i_1 \cdots i_l ; i_{l+1})} + F_j{}^{(i_{l+1}} K^{i_1 \cdots i_l)j} = \frac{1}{(l+1)} K^{i_1 \cdots i_{l+1} j} V_{,j} \,, \quad \text{for} \quad l = 1, \cdots (p-2) \,, \tag{7c}$$

$$K^{(i_1 \cdots i_{p-1} ; i_p)} + F_j{}^{(i_p} K^{i_1 \cdots i_{p-1})j} = 0 \,, \tag{7d}$$

$$K^{(i_1 \cdots i_p ; i_{p+1})} = 0 \,. \tag{7e}$$

Here a semicolon denotes the covariant differentiation corresponding to the Levi-Civita connection and round brackets denote full symmetrization over the indices enclosed.

The last uncoupled equation (7e) is the defining equation of a SK tensor of rank p. The SK tensors represent a generalization of the Killing vectors and are responsible for the hidden symmetries of the motions, connected with conserved quantities of the form (6) polynomials in momenta. The rest of the equations (7) mixes up the terms of K with the gauge field strength F_{ij} and derivatives of the potential $V(x)$. Several applications using van Holten's covariant framework [1] are given in [8, 9, 10, 11, 12].

KY tensors are a different generalization of Killing vectors which can be studied on a manifold. They were introduced by Yano [13] from a purely mathematical perspective and later on it turned out they have many interesting properties relevant to physics. In recent years the KY tensors are related to a multitude of different topics such supersymmetries, index theorems, supergravity theories, and so on [14].

A KY tensor $Y_{\mu_1 \cdots \mu_p}$ is antisymmetric satisfying the following equation:

$$Y_{\mu_1 \cdots \mu_{p-1}(\mu_p ; \nu)} = 0 \,. \tag{8}$$

These two generalizations SK and KY of the Killing vectors could be related. Let $Y_{\mu_1\cdots\mu_p}$ be a KY tensor, then the symmetric tensor field

$$K_{\mu\nu} = Y_{\mu\mu_2\cdots\mu_p} Y_{\nu}^{\ \mu_2\cdots\mu_p} \,, \tag{9}$$

is a SK tensor and it sometimes refers to this SK tensor as the associated tensor with $Y_{\mu_1\cdots\mu_p}$.

Having in mind the special role of null geodesic for the motion of massless particles, it is convenient to look for conformal generalization of KY tensor. Let us mention also the remarkable role of the CKY tensors in the study of the properties of higher dimensional black holes (see e. g. [15, 16] and the cites contained therein). In what follows we limit ourselves to CKY tensors of rank 2 which satisfy [17]

$$Y_{\mu\nu;\lambda} + Y_{\lambda\nu;\mu} = \frac{2}{n-1}\left(g_{\lambda\mu} Y^{\sigma}_{\ \nu;\sigma} + g_{\nu(\mu} Y_{\lambda)}{}^{\sigma}{}_{;\sigma}\right) . \tag{10}$$

For what follows it is necessary to mention an interesting construction involving CKY tensors of rank 2 in 4 dimensions. Let us consider equation (10) for this particular case:

$$Y_{i(j;k)} = -\frac{1}{3}\left(g_{jk} Y^{l}_{\ i;l} + g_{i(k} Y_{j)}{}^{l}{}_{;l}\right) , \tag{11}$$

and let us denote

$$Y_k := Y^{l}_{\ k;l} \,. \tag{12}$$

The trace in ij in equation (11) leads to the following result [17]:

$$Y_{(i;j)} = \frac{3}{2} R_{l(i} Y_{j)}{}^{l} \,. \tag{13}$$

It is obvious that in a Ricci flat space ($R_{ij} = 0$) or in an Einstein space ($R_{ij} \sim g_{ij}$), Y_k is a Killing vector and we shall refer to it as the *primary Killing vector*. In Carter's construction [2] of a primary Killing vector it is used a CKY tensor which in turn is the dual of an ordinary KY tensor.

3. KM system

Returning to the system of equations (7) we should like to find the conditions of the electromagnetic tensor field F_{ij} to maintain the hidden symmetry of the system. More precisely, we are looking for favorable conditions under which the terms $F_j^{\ (i_l} K^{i_1\cdots i_{l-1})j}$ do not contribute to equations (7) regulating the conserved quantities. To make things more specific, let us assume that the system admits a hidden symmetry encapsulated in a SK tensor K_{ij} of rank 2 associated with a KY tensor Y_{ij} according to (9). The sufficient condition of the electromagnetic field to preserve the hidden symmetry is [8]

$$F_{k[i} Y_{j]}{}^{k} = 0 \,. \tag{14}$$

where the indices in square bracket are to be antisymmetrized.

We mention that this condition appeared in many different contexts as conformal Killing spinors [18], pseudo-classical spinning point particles models [19], Dirac-type operators that commute with the standard Dirac operator [20].

A concrete realization of (14) is presented by the KM system [2]. In Carter's construction a primary Killing vector (12) is identified, modulo a rationalization factor, with the source current j^i of the electromagnetic field

$$F^{ij}_{\ \ ;j} = 4\pi j^i \,. \tag{15}$$

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Therefore the KM system is defined assuming that the electromagnetic field F_{ij} is a CKY tensor which, in addition, is a closed 2-form (1). Its Hodge dual

$$Y_{ij} = *F_{ij}\,, \tag{16}$$

is a KY tensor and the KM system possesses a hidden symmetry associated with this KY tensor. It is quite simple to observe that $F_{ij}Y_k{}^j \sim F_{ij} * F_k{}^j$ is a symmetric matrix (in fact proportional with the unit matrix) and therefore condition (14) is fulfilled.

3.1. The Kerr metric

To exemplify the results presented previously, let us consider the Kerr solution to the vacuum Einstein equations which in the Boyer-Lindquist coordinates (t, r, θ, ϕ) has the form

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\phi)^2 + \frac{\sin^2\theta}{\rho^2}[(r^2+a^2)d\phi - adt]^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2\,, \tag{17}$$

where $\Delta = r^2 + a^2 - 2mr\,, \rho^2 = r^2 + a^2\cos^2\theta\,$. This metric describes a rotating black hole of mass m and angular momentum $J = am$.

As was found by Carter [21], the Kerr space admits a SK tensor, in addition to the metric tensor g_{ij}, associated with the KY tensor [22, 23]

$$Y = r\sin\theta d\theta \wedge [-adt + (r^2+a^2)d\phi] + a\cos\theta dr \wedge (dt - a\sin^2\theta d\phi). \tag{18}$$

The dual tensor

$$*Y = a\sin\theta\cos\theta d\theta \wedge [-adt + (r^2+a^2)d\phi] + rdr \wedge (-dt + a\sin^2\theta d\phi), \tag{19}$$

is a CKY tensor and represents the electromagnetic field of KM system (16). Also one can verify that the KM four-potential one-form is

$$A = \frac{1}{2}(a^2\cos^2\theta - r^2)dt + \frac{1}{2}a(r^2+a^2)\sin^2\theta d\phi\,, \tag{20}$$

and the current is to be identified with the primary Killing vector (12), namely $Y^l\partial_l = 3\partial_t$.

4. Hamiltonian reduction

It is simplest to work with the Hamiltonian formulation in order to see how the reduction and the oxidation of a dynamical system affect constants of motion.

The general setting for reduction of symplectic manifolds with symmetries is presented in [3, 5]. Here we confine ourselves to the $U(1)$ reduction of a Hamiltonian system defined on the cotangent bundle $T^\star(\mathbb{R}^4 - \{0\})$ with standard symplectic form. The reduced phase space is not symplectomorphic to the cotangent bundle $T^\star(\mathbb{R}^3 - \{0\})$ with standard symplectic form. It proves that the reduced symplectic form on $T^\star(\mathbb{R}^3 - \{0\})$ contains a two-form describing Dirac's monopole field beside the standard symplectic form.

Let us start to consider the principal fiber bundle $\pi : \mathbb{R}^4 - \{0\} \to \mathbb{R}^3 - \{0\}$ with structure group $U(1)$ whose action is given by [24]

$$x \mapsto T(t)x\,, \quad x \in \mathbb{R}^4\,, \quad t \in \mathbb{R}\,, \tag{21}$$

where

$$T(t) = \begin{pmatrix} R(t) & 0 \\ 0 & R(t) \end{pmatrix}\,, \quad R(t) = \begin{pmatrix} \cos\frac{t}{2} & -\sin\frac{t}{2} \\ \sin\frac{t}{2} & \cos\frac{t}{2} \end{pmatrix}\,. \tag{22}$$

The $U(1)$ action is lifted to a symplectic action on $T^\star(\mathbb{R}^4 - \{0\})$

$$(x, y) \to (T(t)x, T(t)y), \quad (x, y) \in (\mathbb{R}^4 - \{0\}) \times \mathbb{R}^4 \,. \tag{23}$$

Let $\Psi : T^\star(\mathbb{R}^4 - \{0\}) \to \mathbb{R}$ be the moment map associated with the $U(1)$ action (23)

$$\Psi(x, y) = \frac{1}{2}(-x_2 y_1 + x_1 y_2 - x_4 y_3 + x_3 y_4)\,. \tag{24}$$

The reduced phase-space P_μ is defined through

$$\pi_\mu : \Psi^{-1}(\mu) \to P_\mu := \Psi^{-1}(\mu)/U(1)\,, \tag{25}$$

which is diffeomorphic with $T^\star(\mathbb{R}^3 - \{0\}) \cong (\mathbb{R}^3 - \{0\}) \times \mathbb{R}^3$.

The coordinates $(q_k, p_k) \in (\mathbb{R}^3 - \{0\}) \times \mathbb{R}^3$ are given by the Kustaanheimo-Stiefel transformation [25].

The phase-space $T^\star(\mathbb{R}^4 - \{0\})$ is equipped with the standard symplectic form

$$d\Theta = \sum_1^4 dy_j \wedge dx_j\,, \quad \Theta = \sum_1^4 y_j \wedge dx_j\,. \tag{26}$$

Let $\iota_\mu : \Psi^{-1}(\mu) \to T^\star(\mathbb{R}^4 - \{0\})$ be the inclusion map. The reduced symplectic form ω_μ is determined on P_μ by

$$\pi^\star_\mu \omega_\mu = \iota^\star_\mu d\Theta\,, \tag{27}$$

namely

$$\omega_\mu = \sum_{k=1}^3 dp_k \wedge dq_k - \frac{\mu}{r^3}(q_1\, dq_2 \wedge dq_3 + q_2\, dq_3 \wedge dq_1 + q_3\, dq_1 \wedge dq_2)\,. \tag{28}$$

ω_μ consists of the standard symplectic form on $T^\star(\mathbb{R}^3 - \{0\})$ and in addition a term corresponding to the Dirac's monopole field

$$\vec{B} = -\mu \frac{\vec{q}}{r^3}\,, \tag{29}$$

of strength $-\mu$.

The reduced Hamiltonian is determined by

$$H \circ \iota_\mu = H_\mu \circ \pi_\mu\,. \tag{30}$$

For the purpose of the present paper, we shall be concerned with the reduction of the dynamical system associated with the geodesic flows of the generalized Taub-NUT metric on $\mathbb{R}^4 - \{0\}$. This metric is relevant for (conformal) Coulomb problem [24], MIC-Zwanziger system [26, 27], Euclidean Taub-NUT [28, 29, 30] and its extensions [31, 32], etc. The generalized Taub-NUT metric is

$$ds_4^2 = f(r)(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)) + g(r)(d\psi + \cos\theta d\phi)^2\,, \tag{31}$$

where the curvilinear coordinates (r, θ, ϕ, ψ) are

$$\begin{aligned} x_1 &= \sqrt{r}\cos\frac{\theta}{2}\cos\frac{\psi+\phi}{2}\,, \quad & x_2 &= \sqrt{r}\cos\frac{\theta}{2}\sin\frac{\psi+\phi}{2}\,, \\ x_3 &= \sqrt{r}\sin\frac{\theta}{2}\cos\frac{\psi-\phi}{2}\,, \quad & x_4 &= \sqrt{r}\sin\frac{\theta}{2}\sin\frac{\psi-\phi}{2}\,, \end{aligned} \tag{32}$$

and $r = \sum_1^4 x_j^2 = \sqrt{\sum_1^3 q_k^2}$.

In what follows we consider the Hamiltonian on the cotangent bundle $T^\star(\mathbb{R}^4 - \{0\})$

$$H = \frac{1}{2f(r)}p_r^2 + \frac{1}{2r^2 f(r)}p_\theta^2 + \frac{(p_\phi - p_\psi \cos\theta)^2}{2r^2 f(r)\sin^2\theta} + \frac{p_\psi^2}{2g(r)} + V(r)\,, \tag{33}$$

where, to make things more specific, we consider a potential $V(r)$ function of the radial coordinate r.

The Hamiltonian function is invariant under the $U(1)$ action with the infinitesimal generator $\frac{\partial}{\partial\psi}$ so that the conserved momentum is

$$\mu = p_\psi = \Theta(\frac{\partial}{\partial\psi})\,, \tag{34}$$

where the canonical one-form Θ could be expressed in curvilinear coordinates

$$\Theta = p_r dr + p_\theta d\theta + p_\phi d\phi + p_\psi d\psi\,, \tag{35}$$

on the cotangent bundle $T^\star(\mathbb{R}^4 - \{0\})$.

The reduced Hamiltonian (30) has the form

$$H_\mu = \frac{1}{2f(r)}\sum_{k=1}^3 p_k^2 + \frac{\mu^2}{2g(r)} + V(r)\,, \tag{36}$$

and the reduced symplectic form is

$$d\Theta_\mu = dp_r \wedge dr + dp_\theta \wedge d\theta + dp_\phi \wedge d\phi\,. \tag{37}$$

Now the search of conserved quantities of motion in the 3-dimensional curved space in the presence of the potential $V(r)$ plus the contribution of the monopole field proceeds in standard way. In some cases the system admits additional constants of motion polynomial in momenta. Some notable cases are (conformal) Coulomb problem, MIC-Zwanziger system, (generalized) Euclidean Taub-NUT space, every admitting a Runge-Lenz type conserved vector. Other examples could be found in [10, 31, 33].

5. Unfolding

It is interesting to analyze the reverse of the reduction procedure which can be used to investigate difficult problems [6]. For example the equations of motion for the dynamical system (28), (36) look quite complicated. Using a sort of *unfolding* of the 3-dimensional dynamics imbedding it in a higher dimensional space the conserved quantities are related to the symmetries of this manifold.

5.1. *Unfolding of the gauge symmetry*

To exemplify let us start with the reduced Hamiltonian (36) written in curvilinear coordinates (32). At each point of $T^\star(\mathbb{R}^3 - \{0\})$ we define the fiber S^1, the group space of the gauge group $U(1)$. On the fiber we consider the motion whose equation is

$$\frac{d\psi}{dt} = \frac{\mu}{g(r)} - \frac{\cos\theta}{r^2 f(r)\sin^2\theta}(p_\phi - \mu\cos\theta)\,. \tag{38}$$

The metric on $\mathbb{R}^4$ defines horizontal spaces orthogonal to the orbits of the circle - this is a connection on the principal bundle [34]. Using the above trivialization, we have the coordinates (r, θ, ϕ, ψ) with the horizontal spaces annihilated by the connection

$$d\psi + \cos\theta d\phi\,. \tag{39}$$

The metric on $\mathbb{R}^4$, which admits a circle action leaving invariant the symplectic form (37), can be written in the form

$$ds_4^2 = \sum_{i,j=1}^{4} g_{ij} dq^i dq^j = f(r)(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)) + h(r)(d\psi + \cos\theta d\phi)^2\,, \tag{40}$$

with the natural symplectic form (35) on $T^\star(\mathbb{R}^4 - \{0\})$.

Considering the geodesic flow of ds_4^2 and taking into account that ψ is a cycle variable

$$p_\psi = h(r)(\dot\psi + \cos\theta\dot\phi)\,, \tag{41}$$

is a conserved quantity. To make contact with the Hamiltonian dynamics on $T^\star(\mathbb{R}^3 - \{0\})$ we must identify

$$h(r) = g(r)\,. \tag{42}$$

Otherwise the resulting Hamiltonian dynamics projected onto $T^\star(\mathbb{R}^3 - \{0\})$ is that from the Hamiltonian H_μ choosing (42).

The reverse of the reduction proves to be useful since the equations of motion for the Hamiltonian

$$H = \frac{1}{2} g^{ij} p_i p_j + V\,, \tag{43}$$

are quite simple and transparent, but the equations of the quotient system appear more complicated [32]. The corresponding differential equations of the trajectories admit the first integral of motion

$$\frac{1}{2} g_{ij} \dot q^i \dot q^j + V = T + V = E\,, \tag{44}$$

where E is the conserved energy.

5.2. Eisenhart lift

In many concrete problems, after the unfolding of the gauge symmetry, one ends up with a dynamical system on an extended phase space and an Hamiltonian (43) with a "residual" scalar potential.

In the final stage of the oxidation of the dynamical system described by the Hamiltonian (43) we shall apply the Eisenhart's lift [4] (see also [35, 36]). In the general case of the Eisenhart's lift when the time enters in the constraints and in the potential function, the dynamics of a mechanical system with an n dimensional configuration space is related to a system of geodesics in an $(n + 2)$ spacetime [4, 37, 38]. In order to simplify the problem, we shall assume that the constraits of the dynamical system and the potential V do not involve time. In this simplified case it is adequately to consider a Riemannian space with $n + 1$ (in our particular case $4 + 1$) dimensions with the metric

$$ds_5^2 = \sum_{i,j=1}^{4} g_{ij} dq^i dq^j + A du^2\,, \tag{45}$$

where it is assumed that A does not involve u. The lifted system is equivalent to geodesic motion on the enlarged spacetime (45), the coordinate u being related to the action by

$$u = -2\int T dt + 2(E+b)t\,, \tag{46}$$

where b is a constant and

$$\frac{1}{2A} = V + b\,. \tag{47}$$

The Hamiltonian on the enlarge phase space (45) is

$$H_5 = \frac{1}{2}\sum_{i,j=1}^{4} g^{ij}p_i p_j + \frac{1}{2}\frac{1}{A}p_u^2\,, \tag{48}$$

where A is given by (47), p_i, p_u are the conjugate momenta and the new symplectic form is

$$\omega' = dp_i \wedge dq^i + dp_u \wedge du\,. \tag{49}$$

Let us assume that the Hamiltonian (43) on $T^\star(\mathbb{R}^4 - \{0\})$ has a constant of motion polynomial in momenta of the form (6). We lift K to the extended space

$$\mathcal{K} = \sum_{i=0}^{s} p_u^{s-i} K^{(i)}\,. \tag{50}$$

It could be easily verified that $\mathcal{K}$ is a constant along geodesics on the enlarge phase space (45) iff K is a constant of motion for the original system [38]. In fact $\mathcal{K}$ is a homogeneous polynomial in momenta corresponding to a Killing tensor of the metric (45).

6. Concluding remarks

In general the explicit and hidden symmetries of a spacetime are encoded in the multitude of Killing vectors and higher order SK tensors respectively. The inclusion of gauge fields and scalar potentials affects the geodesic conserved quantities in a nontrivial way.

When we have a symplectic manifold with symmetries, it is possible to reduce the phase space to another symplectic manifold in which the symmetries are divided out. Such a situation arises when one has a particle moving in a gauge field. In the usual applications, applying the method of reduction simplifies the equations of motion. However, in some cases, the reverse of the reduction might be useful, namely the equations of motion on the extended phase space are quite transparent, but the equations of motion of the quotient system appear more complicated. Applying an oxidation of a dynamical system with constants of motion polynomial in momenta, one may obtain spacetimes admitting SK tensors of higher rank.

The systems considered in this paper present hidden symmetries described by SK tensors of rank 2. However there are several examples of integrable systems admitting integrals of motion of higher order in momenta. Recently it has been introduced [39] a new superintegrable Hamiltonian as a generalization of the Keplerian one with three terms preventing the particle crossing the principal planes. A generalization of the Runge-Lenz vector is found and also independent isolating integrals quartic in the momenta are identified. An investigation of the Kepler problem on $N-$dimensional Riemannian spaces of non constant curvature was done [40] in order to obtain maximally superintegrable classical systems.

To conclude let us discuss shortly some problems that deserve a further attention. More elaborate examples working in a N-dimensional curved space-time and involving higher ranks ($k > 2$) SK tensors will be presented elsewhere. It would be interesting to investigate relations between KY tensors and hidden symmetries in the context of Hamilton reduction and oxidation. The concept of generalized (C)KY symmetry in the presence of a skew-symmetric torsion is more widely applicable and may become very powerful [41].

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