

Tadpole contribution to magnetic photon-graviton conversion

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Photon-graviton conversion in a magnetic field is a process that is usually studied at tree level, but the one-loop corrections due to scalars and spinors have also been calculated. Differently from the tree-level process, at one-loop one finds the amplitude to depend on the photon polarization, leading to dichroism. However, previous calculations overlooked a tadpole contribution of the type that was considered to be vanishing in QED for decades but erroneously so, as shown by H. Gies and one of the authors in 2016. Here we compute this missing diagram in closed form, and show that it does not contribute to dichroism.

Keywords: Photon-graviton; tadpole; Einstein-Maxwell

1. Introduction: Photon-graviton conversion

Einstein-Maxwell theory contains a tree-level vertex for photon-graviton conversion in a constant electromagnetic field:

$$\frac{1}{2} \kappa h_{\mu\nu} \left(F^{\mu\alpha} f^\nu_\alpha + f^\mu_\alpha F^{\nu\alpha} \right) - \frac{1}{4} \kappa h^\mu_\mu F^{\alpha\beta} f_{\alpha\beta}. \quad (1)$$

Here $h_{\mu\nu}$ denotes the graviton, $f_{\mu\nu}$ the photon, $F^{\mu\nu}$ the external field, and κ the gravitational coupling constant.

This interaction leads to photon-graviton oscillations similar to the better-known neutrino or photon-axion oscillations^{1–11} (see¹² for a recent application to gravitational waves).

In momentum space, this vertex becomes

$$\Gamma^{(\text{tree})}(k, \varepsilon; F) = \epsilon_{\mu\nu} \varepsilon_\alpha \Pi_{(\text{tree})}^{\mu\nu, \alpha}(k; F), \quad \Pi_{(\text{tree})}^{\mu\nu, \alpha}(k; F) = -\frac{i\kappa}{2} C^{\mu\nu, \alpha} \quad (2)$$

with

$$C^{\mu\nu, \alpha} = F^{\mu\alpha} k^\nu + F^{\nu\alpha} k^\mu - (F \cdot k)^\mu \delta^{\nu\alpha} - (F \cdot k)^\nu \delta^{\mu\alpha} + (F \cdot k)^\alpha \delta^{\mu\nu}. \quad (3)$$

Since in a constant field the four-momentum is preserved, k^μ here is the four-momentum of the photon as well as of the graviton.

For the photon polarizations, as is customary we will use the Lorentz frame where \mathbf{E} and \mathbf{B} are collinear, and choose the polarization basis $\varepsilon_\perp^\alpha, \varepsilon_\parallel^\alpha$, where $\varepsilon_\parallel^\alpha$ lies in the plane spanned by k and \mathbf{B} or \mathbf{E} , and ε_\perp^α is perpendicular to it. For convenience we construct also the graviton polarization tensor using the same vectors:

$$\varepsilon^{\oplus\mu\nu} = \varepsilon^\perp\mu \varepsilon^\perp\nu - \varepsilon^\parallel\mu \varepsilon^\parallel\nu, \quad \varepsilon^{\otimes\mu\nu} = \varepsilon^\perp\mu \varepsilon^\parallel\nu + \varepsilon^\parallel\mu \varepsilon^\perp\nu. \quad (4)$$

From the CP-invariance of Einstein-Maxwell theory one can then derive the following selection rules:

- For a purely magnetic field ε^\oplus couples only to ε^\perp and ε^\otimes only to ε^\parallel .
- For a purely electric field ε^\oplus couples only to ε^\parallel and ε^\otimes only to ε^\perp .

2. One-loop photon-graviton vacuum polarization

In^{13,14} the worldline formalism was used along the lines of^{15–17} to study the one-loop corrections to this amplitude due to a scalar or spinor loop, see Fig. 1.



Fig. 1. One-loop correction to the photon-graviton amplitude in a constant field.

Here we employ the usual double-line notation for the full propagator in the external electromagnetic field, Fig. 2.

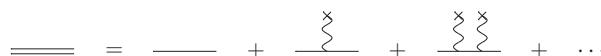


Fig. 2. Full scalar or spinor propagator in a constant field.

These calculations lead to the same type of two-parameter integrals as for the well-studied case of the one-loop photon vacuum polarisation in a constant field.^{18–29}

Both the scalar and the spinor-loop amplitudes are UV divergent, but multiplicatively renormalizable. For example, the scalar-loop contribution in dimensional regularization displays the pole

$$\Pi_{\text{scal,div}}^{\mu\nu,\alpha}(k) = \frac{ie^2\kappa}{3(4\pi)^2} \frac{1}{D-4} C^{\mu\nu,\alpha}$$

where $C^{\mu\nu,\alpha}$ is the tree level vertex (3).

For studying the relative importance of the one-loop amplitudes it is useful to normalize them by the tree-level amplitude (the “bar” on Π denotes renormalization)

$$\hat{\Pi}_{\text{scal,spin}}^{Aa}(\hat{\omega}, \hat{B}, \hat{E}) \equiv \frac{\bar{\Pi}_{\text{scal,spin}}^{Aa}(\hat{\omega}, \hat{B}, \hat{E})}{-\frac{i}{2}\kappa C^{Aa}} \quad (5)$$

where $A = \oplus, \otimes$ and $a = \perp, \parallel$, $\hat{\omega} = \frac{\omega}{m}$, $\hat{B} = \frac{eB}{m^2}$, $\hat{E} = \frac{eE}{m^2}$.

It then becomes obvious that the one-loop amplitudes become quantitatively relevant only for field strengths close to the critical ones $\hat{B}, \hat{E} \approx 1$. Macroscopic fields of this magnitude are known to exist only for the magnetic case, so that we will set $\hat{E} = 0$ in the following. In¹⁴ it was shown that for this purely magnetic case the parameter integrals allow for a straightforward numerical evaluation as long as the photon energy $\hat{\omega}$ is below the pair-creation threshold $\hat{\omega}_{\text{cr}}$,

$$\begin{aligned} \hat{\omega}_{\text{cr,scal}}^{\oplus\perp} &= \hat{\omega}_{\text{cr,scal}}^{\otimes\parallel} = 2\sqrt{1 + \hat{B}}, \\ \hat{\omega}_{\text{cr,spin}}^{\oplus\perp} &= 1 + \sqrt{1 + 2\hat{B}}, \\ \hat{\omega}_{\text{cr,spin}}^{\otimes\parallel} &= 2. \end{aligned} \quad (6)$$

There also a number of special cases were studied that allow for more explicit representations:

- (1) The case of small B and arbitrary ω leads to single - parameter integrals over Airy functions.
- (2) In the large B limit one finds a logarithmic growth in the field strength,

$$\hat{\Pi}_{\text{scal}}^{Aa}(\hat{\omega}, \hat{B}) \xrightarrow{\hat{B} \rightarrow \infty} -\frac{\alpha}{12\pi} \ln(\hat{B}), \quad (7)$$

$$\hat{\Pi}_{\text{spin}}^{Aa}(\hat{\omega}, \hat{B}) \xrightarrow{\hat{B} \rightarrow \infty} -\frac{\alpha}{3\pi} \ln(\hat{B}). \quad (8)$$

- (3) In the limit of vanishing photon energy the amplitudes can be related to the corresponding one-loop effective Lagrangians:

$$\hat{\Pi}_{\text{scal,spin}}^{\oplus\perp}(\hat{\omega} = 0, \hat{B}) = -\frac{2\pi\alpha}{m^4} \left(\frac{1}{\hat{B}} \frac{\partial}{\partial \hat{B}} + \frac{\partial^2}{\partial \hat{B}^2} \right) \mathcal{L}_{\text{scal,spin}}^{\text{EH}}(\hat{B}), \quad (9)$$

$$\hat{\Pi}_{\text{scal,spin}}^{\otimes\parallel}(\hat{\omega} = 0, \hat{B}) = -\frac{4\pi\alpha}{m^4} \frac{1}{\hat{B}} \frac{\partial}{\partial \hat{B}} \mathcal{L}_{\text{scal,spin}}^{\text{EH}}(\hat{B}). \quad (10)$$

Here $\mathcal{L}_{\text{scal,spin}}^{\text{EH}}(\hat{B})$ denotes the one-loop effective Lagrangian in a constant magnetic field, obtained for the spinor QED case by Heisenberg and Euler³⁰ and for scalar QED by Weisskopf³¹:

$$\mathcal{L}_{\text{scal}}^{\text{EH}}(\hat{B}) = -\frac{m^4}{16\pi^2} \int_0^\infty \frac{d\hat{s}}{\hat{s}^3} e^{-i\hat{s}} \left[\frac{\hat{B}\hat{s}}{\sin(\hat{B}\hat{s})} - \frac{(\hat{B}\hat{s})^2}{6} - 1 \right], \quad (11)$$

$$\mathcal{L}_{\text{spin}}^{\text{EH}}(\hat{B}) = \frac{m^4}{8\pi^2} \int_0^\infty \frac{d\hat{s}}{\hat{s}^3} e^{-i\hat{s}} \left[\frac{\hat{B}\hat{s}}{\tan(\hat{B}\hat{s})} + \frac{(\hat{B}\hat{s})^2}{3} - 1 \right]. \quad (12)$$

3. Dichroism

For realistic parameters, the one-loop corrections turn out to be small compared to the tree-level amplitudes. However, there is also a qualitative difference to the tree-level amplitude. As has been emphasized in,⁹ the tree level photon-graviton conversion does, contrary to the better known photon-axion case, not lead to a dichroism effect for photon beams. This is because both photon polarization components have equal conversion rates. This symmetry gets broken by the loop corrections. Although this effect is, of course, tiny and hardly measurable in the near future, an exhaustive analysis by Ahlers et al.³² has shown that it is still the leading contribution to magnetic dichroism in the standard model (including standard gravity)!

4. Quantum electrodynamics in external fields

In vacuum QED, there is, of course, no one-photon amplitude because of Furry's theorem. In the presence of an external field, however, one-photon tadpole diagrams such as the one in Fig. 3 will in general be nonzero.

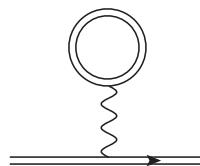


Fig. 3. Full scalar or spinor propagator in a constant field.

If the external field is constant, then the one-photon amplitude Fig. 4 can still be shown to vanish.



Fig. 4. One photon amplitude in a constant field.

The argument goes as follows:

- (1) A constant field emits only photons with zero energy-momentum, thus there is a factor of $\delta(k)$.
- (2) Because of gauge invariance, this diagram in a momentum expansion starts with the term linear in momentum.
- (3) $\delta(k)k^\mu = 0$.

Since the tadpole vanishes, it has been assumed for decades that also any diagram containing it can be discarded. For example, in the book “Quantum Electrodynamics with Unstable Vacuum” by E.S. Fradkin, D.M. Gitman and S.M. Shvartsman³⁴ it is stated (on page 225) even more generally that, “*Thus, in the constant and homogeneous external field combined with that of a plane-wave, all the diagrams containing the causal current $\mathcal{J}^\mu(x)$ (i.e., those containing tadpoles having a causal propagator $S^c(x, y)$), are equal to zero.*”

Sometimes also additional arguments have been given; in the book “Effective Lagrangians in Quantum Electrodynamics” by Dittrich and Reuter³³ it is argued that the “handcuff” diagram of Fig. 5 vanishes because of Lorenz invariance.



Fig. 5. Handcuff diagram in a constant field.

However, in 2016 H. Gies and one of the authors³⁵ noted that such diagrams can give finite values because of the infrared divergence of the connecting photon propagator. In dimensional regularization, the key integral is

$$\int d^D k \delta^D(k) \frac{k^\mu k^\nu}{k^2} = \frac{\eta^{\mu\nu}}{D}. \quad (13)$$

Applying this integral to the handcuff diagram one finds a non-vanishing result, which can be expressed in the following simple way in terms of the one-loop Euler-Heisenberg Lagrangian³⁶

$$\mathcal{L}_{\text{spin}}^{\text{1PR}} = \frac{1}{2} \frac{\partial \mathcal{L}_{\text{spin}}^{(EH)}}{\partial F^{\mu\nu}} \frac{\partial \mathcal{L}_{\text{spin}}^{(EH)}}{\partial F_{\mu\nu}}$$

(the superscript “1PR” stands for “one-particle reducible”). This adds on to the standard diagram for the two-loop Euler-Heisenberg Lagrangian, studied by V.I. Ritus³⁹ half a century ago:

$$\mathcal{L}_{\text{spin}}^{(EH)\text{2-loop}} = \quad \text{Diagram with a wavy line and a circle} \quad + \quad \text{Handcuff diagram} \quad (14)$$

Along the same lines, it was found in^{37,38} that the one-loop tadpole contribution Fig. 3 to the scalar or spinor propagator in a constant field is also non-vanishing,

and given by

$$S^{1\text{PR}}(p) = \frac{\partial S(p)}{\partial F_{\mu\nu}} \frac{\partial \mathcal{L}^{(EH)}}{\partial F^{\mu\nu}}$$

where $S(p)$ denotes the tree-level propagator in the field.

5. Tadpole contribution to the photon-graviton amplitude

Returning to the photon-graviton amplitude in a constant field, the point of the present talk is that this amplitude, too, has a previously overlooked tadpole contribution, shown in Fig. 6.

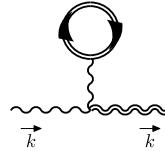


Fig. 6. Tadpole contribution to photon-graviton conversion.

Using the integral (13) it is easy to show that its contribution to the one-loop amplitudes with a scalar or spinor loop can be written as

$$\begin{aligned} \Gamma_{\text{scal}}^{(\text{tadpole})}(k^\alpha, \varepsilon_\beta; \epsilon_{\mu\nu}; F_{\kappa\lambda}) &= -i \frac{\alpha}{8\pi} \kappa \left(\varepsilon \cdot F \cdot \epsilon \cdot k + \varepsilon \cdot \epsilon \cdot F \cdot k \right) \\ &\times \int_0^\infty \frac{dz}{z} e^{-\frac{m^2}{eB}z} \frac{\coth(z) - 1/z}{\sinh(z)}, \end{aligned} \quad (15)$$

$$\begin{aligned} \Gamma_{\text{spin}}^{(\text{tadpole})}(k^\alpha, \varepsilon_\beta; \epsilon_{\mu\nu}; F_{\kappa\lambda}) &= i \frac{\alpha}{4\pi} \kappa \left(\varepsilon \cdot F \cdot \epsilon \cdot k + \varepsilon \cdot \epsilon \cdot F \cdot k \right) \\ &\times \int_0^\infty \frac{dz}{z} e^{-\frac{m^2}{eB}z} \frac{\coth(z) - \tanh(z) - 1/z}{\tanh(z)}. \end{aligned} \quad (16)$$

Expanding out the tadpole in powers of the external field we see that the leading term, which is linear in the field (Fig. 7), is removed by the photon wave function renormalization.

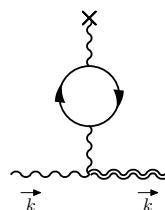


Fig. 7. Expanding the tadpole to lowest order in the field.

This gives the renormalized amplitudes

$$\begin{aligned}\Gamma_{\text{scal,ren}}^{(\text{tadpole})}(k^\alpha, \varepsilon_\beta; \epsilon_{\mu\nu}; F_{\kappa\lambda}) &= -i \frac{\alpha}{8\pi} \kappa \left(\varepsilon \cdot F \cdot \epsilon \cdot k + \varepsilon \cdot \epsilon \cdot F \cdot k \right) \\ &\quad \times \int_0^\infty \frac{dz}{z} e^{-\frac{m^2}{eB}z} \left[\frac{\coth(z) - 1/z}{\sinh(z)} - \frac{1}{3} \right], \\ \Gamma_{\text{spin,ren}}^{(\text{tadpole})}(k^\alpha, \varepsilon_\beta; \epsilon_{\mu\nu}; F_{\kappa\lambda}) &= i \frac{\alpha}{4\pi} \kappa \left(\varepsilon \cdot F \cdot \epsilon \cdot k + \varepsilon \cdot \epsilon \cdot F \cdot k \right) \\ &\quad \times \int_0^\infty \frac{dz}{z} e^{-\frac{m^2}{eB}z} \left[\frac{\coth(z) - \tanh(z) - 1/z}{\tanh(z)} + \frac{2}{3} \right].\end{aligned}$$

6. Comparison with the main diagram

These amplitudes are of a structure similar to what we got from the main diagram Fig. 1 in the limit of low photon energy ω , eqs. (9), (10).

However, they do not contribute to dichroism since the polarizations are still bound up in the tree-level vertex $(\varepsilon \cdot F \cdot \epsilon \cdot k + \varepsilon \cdot \epsilon \cdot F \cdot k)$. Thus the above-mentioned analysis of Ahlers et al.³² remains unaffected by the presence of the tadpole diagram.

7. Summary and outlook

- We have presented the first example of a non-vanishing diagram in Einstein-Maxwell theory involving a tadpole in a constant field.
- This diagram does not contribute to magnetic dichroism.
- A more quantitative analysis is in progress.
- In the ultra strong-field limit, the tadpoles have been shown even to dominate the (multi-loop) effective action in QED.⁴⁰ It would be interesting to extend this analysis to the Einstein-Maxwell case.

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