Gauge invariant second order Boltzman equation

Atsushi Naruko[1](a), Cyril Pitrou[2](b), Koyama Kazuya[3](b), and Misao Sasaki[4](a)

(a) *Yukawa Institute, Kyoto University, Kyoto 606-8502*
(b) *Institute of Cosmology and Gravitation, University of Portsmouth, Dennis Sciama Building Burnaby Road, Portsmouth PO1 3FX, UK*

Abstract

In this talk, we show the second order Boltzmann equation with polarisation without specifying any gauges. In order to describe a polarisation of photon, we introduce a tensor-valued distribution function. We derive the gauge transformation rules for metric, momentum and distribution function. We correct the previous results for gauge transformtion rule of distribution function in which some terms were missed. The tensor-valued distribution function is a tensor on two dimensional polarisation surface, therefore, we have to consider the change of polarisation surface as well as its transformation rules as a tensorial quantity in the gauge transformation. For a consistency, we check the gauge invariance of the derived equation. Finally, we give a short comment on the observed temperature.

1 Introduction

The non-Gaussianity in the Cosmic Microwave Background is the one of the hottest topics in Cosmology because the non-Gaussianity can be a new window for the information of primordial universe. It is often said that the simplest inflation model predicts the very tiny non-Gaussianity, therefore if non-Gaussianity is detected, this simplest model can be ruled out. From the recent studies, we know a lot of models which predict large non-Gaussianity. Therfore, from the observations of non-Gaussianity, we may constrain inflationary models.

On the other hand, the non-linear evolution of perturbations generates the so-called late time non-Gaussianity. This becomes the noise for primordial non-Gaussianity. Fortunately, if cosmological parameters are given, we can calculate this late time non-Gaussianity without ambiguity by solving second order Boltzmann equation. Then, in principle, they can be completely subtracted.

Along this direction, the second order Boltzmann equation was written down in the Poisson gauge in [1, 2]. In [3], this equation was solved numerically and they reported that $f_{\rm NL}^{\rm local} \sim 5$ can be generated. We have to consider this late time non-Guassianity seriously and estimate it correctly. Although their calculation were performed in the Poisson gauge, at a linear level, the synchronous gauge was used for the numerical calculation. Therefore there could be some inconsistency in such a calculation.

For this purpose, we write down the second order Boltzmann equation without specifying gauges. We derive the gauge transformation rules for metric, momentum and distriabution function. Finally we check the gauge invariance of the derived equation.

2 Preparations

We shall use the ADM formalism to write down the expression of the perturbed metric. In general, the metric can be decomposed in this formalism as

$$ds^2 = a^2(\eta)\left[-N^2 d\eta^2 + \gamma_{ij}(dx^i + N^i d\eta)(dx^j + N^j d\eta)\right] \tag{1}$$

[1]Email address: naruko@yukawa.kyoto-u.ac.jp
[2]Email address: Cyril.Pitrou@port.ac.uk
[3]Email address: kazuya.koyama@port.ac.uk
[4]Email address: misao@yukawa.kyoto-u.ac.jp

where N is a lapse, N^i is a shift and γ_{ij} is a spatial metric. We consider perturbations around the flat Friedmann-Robertson-Walker spacetime and we then define the perturbations as

$$N = 1 + \alpha, \qquad \gamma_{ij} = \delta_{ij} + 2h_{ij}. \tag{2}$$

Any perturbation X will be expanded into a first-order and a second order parts as

$$X = X^{(1)} + \frac{1}{2}X^{(2)} \tag{3}$$

The photon momentum satisfies the null condition $P^\mu P_\mu = 0$, thus there are only three independent components. Using the components of the momentum in the local inertial frame $p^{(a)} \equiv e^{(a)}_\mu P^\mu$, we consider the conformal energy q and the direction $n^{(i)}$ of the photon as independent variables;

$$q \equiv \sqrt{a^2 \delta_{(i)(j)} p^{(i)} p^{(j)}} = ap^{(0)}, \qquad n^{(i)} \equiv \frac{p^{(i)}}{p^{(0)}}, \qquad \delta_{(i)(j)} n^{(i)} n^{(j)} = 1, \tag{4}$$

$$e^{(0)}_\mu = a(1+\alpha)\delta^0{}_\mu, \qquad e^{(i)}_\mu = a\left[\left(N^i + h^i{}_k N^k\right)\delta^0{}_\mu + \left(\delta^i{}_j + h^i{}_j - \frac{1}{2}h^{ik}h_{kj}\right)\delta^j{}_\mu\right]. \tag{5}$$

We introduce the tensor-valued distribution function $f_{\mu\nu}$ to express the polarisation. Since the distribution function will be defined on a two-dimensional polarisation surface, it has only four degrees of freedom. These four degrees of freedom can be extracted by decomposing $f_{\mu\nu}$ into a trace part, a symmetric traceless part and an anti-symmetric part as

$$f_{\mu\nu}(x^\mu, q, n^{(i)}) \equiv \frac{1}{2}I(x^\mu, q, n^{(i)})S_{\mu\nu} + P_{\mu\nu}(x^\mu, q, n^{(i)}) + \frac{1}{2}V(x^\mu, q, n^{(i)})\epsilon_{\rho\mu\nu\sigma}e^\rho_0 d^\sigma \tag{6}$$

where $S_{\mu\nu}$ is a screen projector and its definition is

$$S_{\mu\nu}(P^\mu) \equiv g_{\mu\nu} + e^{(0)}_\mu e^{(0)}_\nu - d_\mu d_\nu, \qquad d^\mu \equiv (g^\mu{}_\nu + e^{(0)\mu}e^{(0)}_\nu)P^\nu \tag{7}$$

And also, the anti-symmetric tensor is defined by

$$\epsilon_{\alpha\beta\gamma\delta} = \epsilon_{[\alpha\beta\gamma\delta]}, \qquad \epsilon_{0123} = \sqrt{-g}. \tag{8}$$

Here I is the intensity and V the degree of circular polarization. As for $P_{\mu\nu}$, it encodes the two degrees of linear polarisation (so called Q and U in Stokes parameter).

3 Boltzmann equation

The distribution matrix satisfies the Boltzmann equation

$$\frac{\mathcal{D}f_{\alpha\beta}}{\mathcal{D}\lambda} = C_{\alpha\beta} \tag{9}$$

where $\mathcal{D}/\mathcal{D}\lambda$ is the covariant derivative along a photon trajectory $x^\mu(\lambda)$ and $C_{\mu\nu}$ is the collision term. The explicit form of $\mathcal{D}f_{\alpha\beta}/\mathcal{D}\lambda$ is

$$\frac{\mathcal{D}f_{\alpha\beta}}{\mathcal{D}\lambda} = \nabla_\mu f_{\alpha\beta}\frac{dx^\mu}{d\lambda} + \frac{\partial f_{\alpha\beta}}{\partial q}\frac{dq}{d\lambda} + \frac{\partial f_{\alpha\beta}}{\partial n^{(i)}}\frac{dn^{(i)}}{d\lambda} \tag{10}$$

where ∇_μ is the covariant derivative. Projecting the equation with the screen projector, we can show that the physical distribution matrix satisfies the following equation

$$S_\rho{}^\mu S_\sigma{}^\nu \frac{\mathcal{D}f_{\mu\nu}}{\mathcal{D}\lambda} = c_{\mu\nu} \tag{11}$$

where $c_{\mu\nu} = S_\mu{}^\alpha S_\nu{}^\beta C_{\alpha\beta}$. After short calculation, the left hand side is decomposed into the I, V and $P_{\mu\nu}$ parts similarly to Eq. (6);

$$S_\rho{}^\mu S_\sigma{}^\nu \frac{\mathcal{D}f_{\mu\nu}}{\mathcal{D}\lambda} = S_\rho{}^\mu S_\sigma{}^\nu \left(\frac{1}{2}\frac{\mathcal{D}I}{\mathcal{D}\lambda} S_{\mu\nu} + \frac{\mathcal{D}P_{\mu\nu}}{\mathcal{D}\lambda} + \frac{1}{2}\frac{\mathcal{D}V}{\mathcal{D}\lambda}\epsilon_{\gamma\mu\nu\delta} e^\gamma_{(0)} d^\delta \right). \tag{12}$$

We also decompose the collision term in the same way as

$$c_{\mu\nu}(x^\mu, q, n^{(i)}) \equiv \frac{1}{2}C^I(x^\mu, q, n^{(i)})S_{\mu\nu} + C^P_{\mu\nu}(x^\mu, q, n^{(i)}) + \frac{1}{2}C^V(x^\mu, q, n^{(i)})\epsilon_{\rho\mu\nu\sigma} e^\rho_{(0)} d^\sigma \tag{13}$$

We can write down the second order Boltzmann equation for intensity as

$$\begin{aligned}\frac{\mathcal{D}I}{\mathcal{D}\lambda} &= \frac{q}{a^2 N}\frac{\partial I}{\partial \eta} + \frac{q}{a^2}\left(n^{(i)} - \delta^{ik}h_{kj}n^{(j)} - \frac{N^i}{N} + \frac{3}{2}\delta^{ij}h_{jk}\delta^{kl}h_{lm}n^{(m)} \right)\frac{\partial I}{\partial x^i} \\ &+ \frac{q^2}{a^2}\Big\{ -\alpha_{,i}n^{(i)} + (\delta_{jk}N^k{}_{,i} - \dot{h}_{ij})n^{(i)}n^{(j)} + \alpha(\alpha_{,i} - \delta_{jk}N^k{}_{,i}n^{(j)} + \dot{h}_{ij}n^{(j)})n^{(i)} \\ &\qquad + (\alpha_{,i} - \delta_{lm}N^m{}_{,i}n^{(l)} + 2\dot{h}_{il}n^{(l)})\delta^{ik}h_{jk}n^{(j)} + (N^k{}_{,i}h_{jk} + N^k h_{ij,k})n^{(i)}n^{(j)} \Big\}\frac{\partial I}{\partial q} \\ &+ \frac{q}{a^2}(n^{(i)}n^{(j)} - \delta^{ij})\left\{ \alpha_{,j} - \delta_{kl}N^l{}_{,j}n^{(k)} + \dot{h}_{jk}n^{(k)} + (h_{jl,k} - h_{kl,j})n^{(k)}n^{(l)} \right\}\frac{\partial I}{\partial n^{(i)}} = C^I\end{aligned} \tag{14}$$

where $\dot{}$ means the derivative with rigard to η. We can write down the equation for $P_{\mu\nu}$ and V, although we don't give the expression here.

4 Gauge dependence

One of our goals is to check the gauge invariance of the Boltzmann equation. For this, we first derive the transformation rules under the gauge transformation. The gauge transformation up to the second order is given by

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu \qquad \xi^\mu = \xi^{(1)\mu} + \frac{1}{2}\xi^{(2)\mu} \tag{15}$$

From now on, we derive the gauge transformation rules for metric, momentum, and the distribution function. We will write the component of ξ^μ as $\xi^\mu = (T, L^i)$.

Gauge transformation of metric at second order are given by

$$g_{\alpha\beta} \to \tilde{g}_{\alpha\beta} = g_{\alpha\beta} - \mathcal{L}_\xi g_{\alpha\beta} + \frac{1}{2}\mathcal{L}_\xi(\mathcal{L}_\xi g_{\alpha\beta}) + \frac{1}{2}\mathcal{L}_{\xi\xi} g_{\alpha\beta} \tag{16}$$

$\xi\xi$ means $\xi^\mu{}_{,\nu}\xi^\nu$. Up to the second order, the ADM variables transform as

$$\begin{aligned}\tilde{\alpha} &= \alpha - \mathcal{H}T - \dot{T} + \frac{1}{2}(\mathcal{H}^2 + \dot{\mathcal{H}})T^2 + \mathcal{H}(2\dot{T}T + T_{,i}L^i) - \mathcal{H}\alpha T \\ &\quad - \alpha\dot{T} - \dot{\alpha}T - \alpha_{,i}L^i + N^i T_{,i} + \ddot{T}T + \dot{T}^2 + \dot{T}_{,i}L^i + \frac{1}{2}\delta^{ij}T_{,i}T_{,j}\end{aligned} \tag{17}$$

$$\begin{aligned}\tilde{N}^i &= N^i + \delta^{ij}T_{,j} - \dot{L}^i + 2\delta^{ij}\alpha T_{,j} - N^i\dot{T} - \dot{N}^i T + N^j L^i{}_{,j} - N^i{}_{,j}L^j - 2\delta^{ik}\delta^{jl}h_{jk}T_{,l} \\ &\quad - \delta^{ij}(2T_{,j}\dot{T} + T\dot{T}_{,j}) + T\ddot{L}^i + \dot{T}\dot{L}^i + \delta^{jk}T_{,j}L^i{}_{,k} - \delta^{ij}T_{,jk}L^k + \dot{L}^i{}_{,j}L^j\end{aligned} \tag{18}$$

$$\begin{aligned}2\tilde{h}_{ij} &= 2h_{ij} - 2\mathcal{H}T\delta_{ij} - (\delta_{ik}L^k{}_{,j} + \delta_{jk}L^k{}_{,i}) - 4\mathcal{H}h_{ij}T + (2\mathcal{H}^2 + \dot{\mathcal{H}})\delta_{ij}T^2 + 2\mathcal{H}\delta_{ij}(\dot{T}T + T_{,k}L^k) \\ &+ 2\mathcal{H}T(\delta_{ik}L^k{}_{,j} + \delta_{kj}L^k{}_{,i}) - N^k(\delta_{ik}T_{,j} + \delta_{jk}T_{,i}) - 2\dot{h}_{ij}T - 2h_{ij,k}L^k - 2(h_{ik}L^k{}_{,j} + h_{jk}L^k{}_{,i}) \\ &- T_{,i}T_{,j} + T(\delta_{ik}\dot{L}^k{}_{,j} + \delta_{kj}\dot{L}^k{}_{,i}) + (\delta_{ik}T_{,j} + \delta_{kj}T_{,i})\dot{L}^k \\ &+ \delta_{kl}L^k{}_{,i}L^l{}_{,j} + (\delta_{ik}L^k{}_{,jl} + \delta_{kj}L^k{}_{,il})L^l + (\delta_{ik}L^l{}_{,j} + \delta_{kj}L^l{}_{,i})L^k{}_{,l}\end{aligned} \tag{19}$$

where $\mathcal{H}$ is conformal hubble parameter defined by $\mathcal{H} \equiv \dot{a}/a$.

Then using the transformation rules for P^μ and metric, we obtain the transformation rule for q;

$$\tilde{q} = q + \delta q = q\left\{1 + \mathcal{H}T + T_{,i}n^{(i)} + \frac{1}{2}(\dot{\mathcal{H}} + \mathcal{H}^2)T^2 + (\mathcal{H}T - \dot{T})T_{,i}n^{(i)} + \frac{1}{2}\delta^{jl}T_{,j}T_{,l} + T_{,i}(\alpha n^{(i)} - \delta^{ik}h_{kj}n^{(i)})\right\} \tag{20}$$

In the same way, the gauge transformation rule of $n^{(i)}$ is given by

$$\tilde{n}^{(i)} = n^{(i)} + \delta n^{(i)} = n^{(i)} + (\delta^{ij} - n^{(i)}n^{(j)})T_{,j} + \frac{1}{2}\delta^{ik}(\delta_{kl}L^l{}_{,j} - \delta_{jl}L^l{}_{,k})n^{(j)} \tag{21}$$

Using the transformation rules for metric and momentum, we find the gauge transformation rule for the distribution function

$$\tilde{f}_{\alpha\beta} = f_{\alpha\beta} - \mathcal{L}_\xi f_{\alpha\beta} - \delta q\frac{\partial f_{\mu\nu}}{\partial q} - \delta n^i\frac{\partial f_{\mu\nu}}{\partial n^i} - \frac{a^2}{q}T_{,i}(P_\alpha f^i{}_\beta + f_\alpha{}^i P_\beta) \tag{22}$$

The last two terms were missed in [1]. These terms come from the gauge transformation of projection tensor $S_{\mu\nu}$ and physically means the change of screen.

Although we don't show the detailed calculation, using the derived gauge transformation rules for metric, momentum and distribution function, we can show the gauge invariance of the second order Boltzmann equation. This fact gives the consistency check for the derived gauge transformation rule.

5 comment on observed temperature

In all the literature, the second order temperature anisotropies are calculated in the Poisson gauge with a specific choice of the local inertial frame. Strictly speaking this is not the temperature anisotopies that we observe. Thus one needs to change the local inertial frame or perform the gauge transformation. At the first order, this is not an issue. Since the first order gauge transformation rule of temperature is given by;

$$\Theta \to \Theta + \mathcal{H}T + T_{,i}n^{(i)} \tag{23}$$

the change of the gauge and the local inertial frame only affects the monopole $\ell = 0$ and dipole $\ell = 1$ if we expands the temperature aniostropies into multipole components. Thus the $\ell \geq 2$ modes are not affected by the change of observers. However this is no longer the case at the second order. In the second order gauge transformation, there are terms that are convolutions of the first order temperature anisotorpies and the gauge transformation functions. These terms affect the observed temperatures even for the $\ell > 2$ modes. Thus a care must be taken when we compare theoretical predictions to observations.

References

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